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DEFECTS, TOPOLOGY, AND THE GEOMETRIC PHASE IN CONDENSED MATTER PHYSICS

BY

ABHISHEK ROY

DISSERTATION

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Doctoral Committee:

Assistant Professor Shinsei Ryu, Chair
Professor Michael Stone, Director of Research
Associate Professor Nadya Mason
Assistant Professor Taylor Hughes
Abstract

This thesis presents work on some topological applications in condensed matter physics, particularly geometric phases and defects.

The first chapter deals with pentagonal disclinations in graphene and their associated bound states. This problem had been attacked previously, using index theorems, as well as the solutions of the continuum Dirac equation. We demonstrated [1] that these two approaches as well as bare numerical computation could be made consistent once one took into account boundary conditions at the defect site. For example, the continuum model considers two pentagons and a square to be alike. However there is a physical distinction between the two, as the sublattice symmetry is locally broken in the former case.

The next chapter treats Berry phases using the Majorana representation of spin states as points on a sphere [2]. The advantages of this approach are that one has a visual representation of the evolution of a state, which automatically absorbs the gauge freedom that drops out of a geometric phase. I show how non-abelian phases can be treated in this framework.

The third chapter discusses the Kitaev toric code model and its generalizations, both to higher dimensions and richer braiding symmetries. The toric code is naturally associated to the mathematical structure of a chain complex. This leads to a unified treatment of braiding, degeneracy and effective field theory in higher dimensions [3].

The last part of this thesis is about twist defects in anyonic models. I discuss a general notion of a group defect that permutes anyons and use the toric code as well as a new honeycomb model [4], as examples. I discuss the quantum dimension, fusion and braiding of these defects. The ground state degeneracies is treated geometrically using covering spaces.

A common theme running through this work is that topological phenomena can be grounded in a lattice model. This makes them more approachable and often clarifies physical details which might otherwise be missed.
To my mother.
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Chapter 1

Topological defects in Graphene

1.1 Introduction and summary of results

Graphene is a single flat sheet of carbon atoms arranged in a hexagonal lattice. Since its discovery in 2007, it has excited great theoretical and experimental interest. Electrons in graphene have an energy-momentum relation that mimics two dimensional massless Dirac particles, with the Fermi velocity \( \approx 10^6 \text{m/s} \) playing the part of the speed of light. This opens up vast possibilities for the phenomena previously restricted to high-energy physics like zitterbewegung and the Klein paradox, to be realized in the solid state. Linear dispersion also adds new features to familiar systems e.g., the quantum Hall effect and Josephson tunneling.

In this chapter, I describe my work on pentagonal defects in graphene [1]. It was well known that the electronic spectrum for the fullerenes has six states near zero energy. Theorists had attempted to explain this phenomenon as an outcome of topology. Since the low-energy spectrum for planar graphene is described by a continuum Dirac equation, it was natural to assume that fullerenes i.e., spherical graphene be solved likewise but in curved space. An electron that encircles a disclination, picks up one phase from the rotation of its frame due to the Frank angle. It also find that its four labels attached to the two Fermi points and sublattices, have been switched since these are no longer defined globally on the entire lattice. The two effects are accounted for in the continuum, by a spin connection and non-abelian gauge field, respectively. It was stated in [6] that the zero modes were a consequence of an index theorem [7, 8] that ensures a one such state for each unit of flux, which turns out to correspond to a pair of pentagons. A fullerene has exactly 12 pentagons and hence 6 zero modes.

However, it was also seen that the numerically computed zero modes do not quite behave as predicted. Not only are they not at zero energy, but they also vary in the presence of a Kekule distortion \(^1\) and depend on the distance between pentagons, both of which are contrary to the prediction of the index theorem.

By a careful analysis of the bound state solutions of the Dirac equation, I concluded that the resolution lay in a choice of boundary conditions exactly the defect sites. The continuum model sees no difference between

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\(^1\)A pattern of single and double bonds that shows up like a mass in the Dirac equation and opens a gap
a pair of pentagons and a square — both contain the same flux. However, there is a significant difference
at the lattice level, in that the sublattice symmetry $E \leftrightarrow -E$ is broken in a line joining two pentagons but
preserved exactly by a square. Once the boundary conditions took this difference into account, I found good
qualitative agreement between tightbinding calculations and exact solutions. We also explained that the
fact that the square defect has \textit{two} exact zero modes from the special nature of a $\pi/2$ flux through a single
plaquette (see concluding remarks).

1.2 Graphene

Experimentally, the main electronic features of planar graphene are:

1. The valence and conduction bands touch at two distinct points on the Brillouin zone, $\mathbf{K}$ and $\mathbf{K'}$.

2. Graphene is a semi-metal and its conduction properties are determined by the linear dispersion near
these two ‘Dirac points’, which are related both by inversion and time-reversal symmetry (since $\mathbf{K} =
- \mathbf{K'}$ up to a reciprocal lattice vector).

3. The bipartite structure of the hexagonal lattice provides an additional ‘valley’ degeneracy and a ‘sub-
   lattice symmetry’.

I begin with a review of the well-known tightbinding model for planar graphene. Despite its simplicity,
it has proved quite effective in describing a wide range of electronic phenomena including even the integer
Hall effect.

In the nearest-neighbour approximation each Carbon atom has a single free $\pi$ electron that can hop to
one its three adjacent atoms with amplitude $t$. Note that the bipartite structure implies that electrons on
the $A$ sublattice only hop to the $B$ sublattice and vice versa. This has an important consequence.

\textbf{Sublattice symmetry} \quad \text{Let the } \Gamma \text{ be the operation that multiplies the amplitudes on the } B \text{ lattice sites
by } -1 \text{ while leaving the } A \text{ sites untouched. From the definition of nearest neighbour hopping, it is evident
that } \Gamma \text{ takes a state of energy } E \text{ to one with energy } -E. \quad 2 \text{ Since states come in pairs and each is twice
degenerate from the spin, it is immediate that the Fermi level resides at } E = 0.

By writing the wave function in Bloch form,

$$\Psi(\mathbf{r}) = \sum_{\mathbf{k}} \psi_A(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} + \sum_{\mathbf{k}} \psi_B(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}$$

\textsuperscript{2}This symmetry is broken by next-nearest neighbour hopping $t'$ that connects atoms on the same sublattice. However the
experimentally estimated magnitude of $t'$ is small compared to $t$.  

2
Figure 1.1: The zero-energy reference states $|A+\rangle$, $|B+\rangle$, $|A-\rangle$ and $|B-\rangle$. The red and blue vertices are the $A$ and $B$ sublattice, respectively. $|A\pm\rangle$ is supported only on the $A$ sites with amplitudes 1 and $\omega$ and $\omega^2$ ($\omega \equiv e^{2\pi i/3}$) arranged as shown. The action of the Hamiltonian makes them hop to the $B$’s and cancel exactly giving a zero energy state. Of course the same holds for $|B\pm\rangle$. The heavy lines indicate a Kekule-Fries structure of “double” bonds, whose hopping will increase from $t$ to $t + \delta t$ when a Peierls distortion occurs.
where \( k \) ranges over the Brillouin zone, we could easily solve for the energy spectrum and eigenstates.

However, contrary to usual practice, let us stay in position space and look at the wavefunctions at zero energy fig. 1.1. The pair \(|A\pm\rangle\) is supported on the \(A\) sublattice while the other \(|B\pm\rangle\) lies on the \(B\) sites. This is the ‘valley degeneracy’. The two states in each pair are related by a mirror reflection.

Near the \( E = 0 \) points, any wave function may be written as a linear combination of one of the above states multiplies with an envelope function,

\[
\Psi(r) = f_{A+}(r)|r\rangle_{A+} + f_{B+}(r)|r\rangle_{B+} + f_{A-}(r)|r\rangle_{A-} + f_{B-}(r)|r\rangle_{B-},
\]

where the \( f \)'s vary slowly at the scale of the lattice. To Taylor expand an \( f \) to its neighbouring lattice points, it is useful to use complex coordinates,

\[
f(z + a) = f(z) + a\partial_z f + \bar{a}\partial_{\bar{z}} f + \cdots \quad \partial_z \equiv \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} \equiv \frac{1}{2}(\partial_x + i\partial_y)
\]

Consider what the Hamiltonian does to a wave function \( f(r)|A_+\rangle \). A particular \( B \) site, which initially had zero amplitude now has,

\[
f(r) \rightarrow t\left(a(\partial_z f + \partial_{\bar{z}} f) + \omega(\omega\partial_z f + \bar{\omega}\partial_{\bar{z}} f) + \omega^2(\omega^2\partial_z f + \bar{\omega}^2\partial_{\bar{z}} f)\right)
\]

\[= 3ta\partial_{\bar{z}} f.
\]

where \( a \) is the lattice spacing and \( \omega = e^{2\pi i/3} \). Checking the reference states in fig. 1.1 we find that,

\[
\mathcal{H}(f_{A+}(r)|r\rangle_{A+}) = 3at(\partial_z f_{A+}(r)|r\rangle_{B+})
\]

Doing the same for the other states we find,

\[
\begin{pmatrix}
    f_{A+} \\
    f_{B+} \\
    f_{B-} \\
    f_{A-}
\end{pmatrix}
= 3at
\begin{pmatrix}
    -\partial_z \\
    \partial_{\bar{z}} \\
    -\partial_{\bar{z}} \\
    -\partial_z
\end{pmatrix}
\begin{pmatrix}
    f_{A+} \\
    f_{B+} \\
    f_{B-} \\
    f_{A-}
\end{pmatrix}.
\]

This is the low energy Dirac Hamiltonian for graphene. Note that it is formally Hermitian.
Kekule Distortion

In one-dimensional polyacetylene, the well known Peierls distortion lowers the energy of a system by moving alternate pairs of carbon atoms closer together. There is a competition between the (negative) hopping energy and the interatomic repulsion. At equilibrium the chain breaks up into alternating ‘single’ and ‘double’ bonds.

A similar phenomenon might occur in graphene - here we call it Kekule distortion - where each atom now has a one double bond and two single bonds. In momentum space this is a coupling between the $K$ and $K'$ Fermi points, that opens a gap and pushes the Fermi surface down. In planar graphene, the density of states vanishes at the Fermi level and hence the magnitude of the gap (which would be proportional to the DOS) is too small to be measureable. However the effect has been observed in fullerenes e.g. in $C_{60}$, double bonds have length .139 nm while the single bonds are .143 nm.

For our purpose, it is convenient to consider a Kekule mass term (whose magnitude may be made as small as we wish). The dark lines in fig. 1.1 show the double bonds with increased hopping $t + \delta t$. Note that sliding along the bonds takes $A_{\pm}$ to $B_{\mp}$ and vice-versa. The Hamiltonian is now,

$$
\mathcal{H}_\Phi = 3at \begin{pmatrix} f_{A+} \\ f_{B+} \\ f_{B-} \\ f_{A-} \end{pmatrix} = 3at \begin{pmatrix} -\partial_z & \Phi \\ \partial_z & \Phi \\ \Phi & -\partial_z \\ \Phi & \partial_z \end{pmatrix} \begin{pmatrix} f_{A+} \\ f_{B+} \\ f_{B-} \\ f_{A-} \end{pmatrix}, \tag{1.3}
$$

where $\Phi = \delta t/3at$.

There are two slightly altered bases that will be convenient later. The first is,

$$
\mathcal{H}_\Phi = 3at \begin{pmatrix} if_{A+} \\ f_{B+} \\ if_{B-} \\ f_{A-} \end{pmatrix} = 3at \begin{pmatrix} -i\partial_{\bar{z}} & \Phi \\ \Phi & \partial_{\bar{z}} \\ \Phi & i\partial_{\bar{z}} \\ \Phi & i\partial_{\bar{z}} \end{pmatrix} \begin{pmatrix} if_{A+} \\ f_{B+} \\ if_{B-} \\ f_{A-} \end{pmatrix}. \tag{1.4}
$$
and the second,

\[
\begin{pmatrix}
  f_{A+} \\
  f_{B+} \\
  f_{B-} \\
  -f_{A-}
\end{pmatrix}
= 3at
\begin{pmatrix}
  \partial \Phi \\
  -\partial \Phi \\
  \Phi \\
  -\Phi
\end{pmatrix}
\begin{pmatrix}
  f_{A+} \\
  f_{B+} \\
  f_{B-} \\
  -f_{A-}
\end{pmatrix}.
\]

(1.5)

Omitting the unimportant factor \(3at/2\) these may be abbreviated as,

\[
\begin{pmatrix}
  \sigma \cdot p \\
  \Phi
\end{pmatrix}
= \tau_3 \otimes \sigma \cdot p + \Phi \tau_1 \otimes 1
\]

(1.6)

\[
\begin{pmatrix}
  \tilde{\sigma} \cdot p \\
  \Phi \sigma_3
\end{pmatrix}
= 1 \otimes \tilde{\sigma} \cdot p + \Phi \tau_1 \otimes \sigma_3.
\]

(1.7)

Here we have defined,

\[
\sigma \cdot p = \sigma_1 \frac{\partial x}{i} + \sigma_2 \frac{\partial y}{i}
\]

(1.8)

\[
\tilde{\sigma} \cdot p = \sigma_2 \frac{\partial x}{i} - \sigma_1 \frac{\partial y}{i}.
\]

(1.9)

The \(\tau\) and \(\sigma\)'s are \(2 \times 2\) Pauli matrices that act on the \(\pm\) index (which also distinguishes \(K\) and \(K'\)) and the \(A/B\) sublattice index respectively.

I should mention here that spin degeneracy simply adds an additional identical block to the above equations and has hence been omitted in this work.

**Sublattice symmetry**: From fig. 1.1 it is easily seen that the sublattice operator that flips the amplitude on the \(B\) sublattice, should be,

\[
\Gamma = \begin{pmatrix}
  \sigma_3 \\
  -\sigma_3
\end{pmatrix} = \sigma_3 \otimes \tau_3
\]

(1.10)

In fact \(\Gamma\) is unaffected by the above change of basis and anticommutes with both \(\mathcal{H}\) and \(\tilde{\mathcal{H}}\) (eqs. (1.6) and (1.7)).
1.3 Geometric defects and effective gauge fields

We want to create pentagonal disclinations and observe their effect on the wave functions. Such a defect may be created by cutting out a 60° wedge and rejoining the severed bonds fig. 1.2 to create a cone with a pentagon at its tip. It is important to observe that if the original hexagon at the apex had only single bonds, then cutting and pasting preserves the Kekule pattern throughout the lattice.

We also wish to work with the same continuum treatment that applies to flat graphene. One might argue that this is quite reasonable in the long wavelength limit, this is quite reasonable since the wave function cannot ‘see’ any lattice distortion at the tip. As we shall see later, this is not quite true. Nevertheless we
proceed, first by taking care of the boundary conditions at the seam,

\[
\begin{pmatrix}
    f_{A+} \\
    f_{B+} \\
    f_{B-} \\
    -f_{A-}
\end{pmatrix}_R
= 
\begin{pmatrix}
    \omega^2 & -\omega \\
    -\omega & \omega^2 \\
    \omega^2 & -\omega \\
    -\omega & \omega^2
\end{pmatrix}
\begin{pmatrix}
    f_{A+} \\
    f_{B+} \\
    f_{B-} \\
    -f_{A-}
\end{pmatrix}_L.
\] (1.11)

Here we are working in the $\hat{\mathcal{H}}$ basis, and the subscripts refer to the left and right side of the seam. We can abbreviate the boundary condition,

\[
\Psi_R = -i\tau_1 \otimes \exp\{-i\pi\sigma_3/6\} \Psi_L \\
= \exp\{-i\pi\tau_1/2\} \otimes \exp\{-i\pi\sigma_3/6\} \Psi_L. \tag{1.12}
\]

Similarly, cutting a $120^\circ$ wedge creates a square at the tip of the cone and the boundary condition is,

\[
\Psi_R = -1 \otimes \exp\{-i\pi\sigma_3/3\} \Psi_L \\
= \exp\{-i\pi\tau_1\} \otimes \exp\{-i\pi\sigma_3/3\} \Psi_L. \tag{1.13}
\]

where the appropriate unitary matrix is just the square of the previous one, as it should be. Also, there is a freedom in choosing the $\tau_1$ in the last line - it might be replaced by $\tau_2, \tau_3$ or even $\mathbb{1}$. I shall return to this issue later.

### 1.3.1 Introducing the gauge field

Clearly the seam is not physical at the level of the lattice and the physical wavefunction is continuous across it. We introduce,

\[
\Psi(r, \theta) = \exp\left\{i\frac{\pi}{2}\left(\frac{3\theta}{5\pi}\right)\right\} \otimes \exp\left\{i\frac{\pi}{6}\sigma_3\left(\frac{3\theta}{5\pi}\right)\right\} \Psi(r, \theta), \tag{1.14}
\]

for the for the $60^\circ$ wedge and,

\[
\Psi(r, \theta) = \exp\left\{i\pi\tau_1\left(\frac{3\theta}{5\pi}\right)\right\} \otimes \exp\left\{i\frac{\pi}{3}\sigma_3\left(\frac{3\theta}{4\pi}\right)\right\} \Psi(r, \theta). \tag{1.15}
\]
for the $120^\circ$ one. The coefficients in front of $\theta$ have been chosen so that the new field $\tilde{\Psi}(r, \theta)$ is continuous across the reconnected seam once we excise the appropriate wedge from the range of $\theta$ viz.,

$$0 < \theta < (1 - \frac{1}{6})2\pi = \frac{5}{3}\pi$$  for the $60^\circ$ wedge  
$$0 < \theta < (1 - \frac{2}{6})2\pi = \frac{4}{3}\pi$$  for the $120^\circ$ wedge. \hspace{1cm} (1.16) \hspace{1cm} (1.17)

The boundary unitary matrices in eqs. (1.14) and (1.15) can be split up neatly into two parts. There is an effective gauge field coming from the fact that an electron that encircles the disclination, finds that its labels have been changed. There is also a spin connection that I shall now explain.

Let us take the original Hamiltonian and move to a rotating frame $e_r, e_\theta$, with the origin at the tip of the cone.

$$\mathcal{H}_\Phi = \tilde{\sigma} \cdot \mathbf{p} + \Phi \tau_1 \otimes \sigma_3$$

$$= -i\{\tilde{\sigma}_1 \cos \theta + \tilde{\sigma}_2 \sin \theta\} \frac{\partial}{\partial r} - i\{-\tilde{\sigma}_1 \sin \theta + \tilde{\sigma}_2 \cos \theta\} \frac{1}{r} \frac{\partial}{\partial \theta} + \Phi \tau_1 \otimes \sigma_3$$

$$= e^{-\frac{i}{2} \sigma_3 \theta} \left( -i\tilde{\sigma}_1 \frac{\partial}{\partial r} - i\tilde{\sigma}_2 \frac{1}{r} \left( \frac{\partial}{\partial \theta} - \frac{i}{2} \tau_1 \right) + \Phi \tau_1 \otimes \sigma_3 \right) e^{i \frac{1}{2} \sigma_3 \theta}.$$ \hspace{1cm} (1.18)

The $\sigma_{r, \theta} \equiv \tilde{\sigma}_{1,2}$ operators are now attached to the frame. The $-i\sigma_3/2$ is required to cancel the effect of taking the frame-rotation matrix $\exp\{i\sigma_3\theta/2\}$ through the $\theta$ derivative. This is the spin connection. \hspace{1cm} \footnote{An unfortunate nomenclature, since this has nothing to do with the physical spin of the electron}

In planar graphene we would introduce a rotated field $\chi(r, \theta) = e^{i \frac{3}{2} \sigma_3 \theta} \Psi(r, \theta)$, but due to the deleted wedge eq. (1.17) we must take,

$$\chi(r, \theta) = \exp \left\{ \frac{i}{2} \sigma_3 \frac{6}{5} \theta \right\} \tilde{\Psi}(r, \theta)$$  for the $60^\circ$ wedge  \hspace{1cm} (1.19)

$$\chi(r, \theta) = \exp \left\{ \frac{i}{2} \sigma_3 \frac{3}{2} \theta \right\} \tilde{\Psi}(r, \theta),$$  for the $120^\circ$ wedge. \hspace{1cm} (1.20)

In all cases $\chi$ is anti-periodic. Note that the definition exactly cancels the $\sigma_3$ part of eqs. (1.14) and (1.15), demonstrating that the spin connection is independent of the wedge angle.

Now that the coefficients of the partial derivatives are (matrix-valued) constants, we can finally write attack the eigenvalue problem.

$$\left( -i\tilde{\sigma}_1 \frac{\partial}{\partial r} - i\tilde{\sigma}_2 \frac{1}{r} \left( \frac{\partial}{\partial \theta} - \frac{i}{2} \sigma_3 + \frac{3i}{10} \tau_1 \right) + \Phi \tau_1 \sigma_3 \right) \chi(r, \theta) = E\chi(r, \theta)$$  for the $60^\circ$ wedge (pentagon) \hspace{1cm} (1.21)

$$\left( -i\tilde{\sigma}_1 \frac{\partial}{\partial r} - i\tilde{\sigma}_2 \frac{1}{r} \left( \frac{\partial}{\partial \theta} - \frac{i}{2} \sigma_3 + \frac{3i}{4} \tau_1 \right) + \Phi \tau_1 \sigma_3 \right) \chi(r, \theta) = E\chi(r, \theta)$$  for the $120^\circ$ wedge (square). \hspace{1cm} (1.22)
Here the \( i\tau_1 (3/10) \) gauge field term in eq. (1.21) comes from taking the \( \exp \{ i\tau_1 (3/10)\theta \} \) matrix appearing in eq. (1.14) through the \( \theta \) derivative, and similarly for eq. (1.22).

We make a couple of cosmetic changes. First we rotate the \( \tilde{\sigma} \) matrices back to \( \sigma \), which is harmless since the the \( \sigma_3 \) and \( \tau \)'s are unaffected. Next, we deal with the range of \( \theta \) by defining a new angle \( \phi = (6/5)\theta \) and \( \phi = (3/2)\theta \) for the two cases. \( \phi \) goes from 0 to 2\( \pi \) and is the physical angle measured from the apex of the cone.

\[
\begin{align*}
-i\sigma_1 \left( \frac{\partial}{\partial r} + \frac{1}{2r} \right) - i\sigma_2 \frac{1}{r} \left( \frac{6}{5} \frac{\partial}{\partial \phi} + \frac{3i}{10} \tau_1 \right) + \Phi \tau_1 \sigma_3 \right) \chi(r, \phi) = E\chi(r, \phi) & \quad \text{for the 60° wedge (pentagon)} \quad (1.23) \\
-i\sigma_1 \left( \frac{\partial}{\partial r} + \frac{1}{2r} \right) - i\sigma_2 \frac{1}{r} \left( \frac{3}{2} \frac{\partial}{\partial \phi} + \frac{3i}{4} \tau_1 \right) + \Phi \tau_1 \sigma_3 \right) \chi(r, \phi) = E\chi(r, \phi) & \quad \text{for the 120° wedge (square).} \quad (1.24)
\end{align*}
\]

**Solutions**

Separating radial and angular parts, we have \( \chi(r, \phi) = e^{ij\phi} \chi(r) \), where \( j \) takes half-integer values,

\[ j = \ldots, -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}, \ldots, \]

since \( \chi \) is antiperiodic under \( \theta \to \theta + 2\pi \).

Since \( \tau_1 \) commutes with the Hamiltonian, we can index solutions by its eigenvalue \( \pm 1 \),

\[
\chi_+(r, \phi) = \begin{pmatrix} u(r) \\ v(r) \\ u(r) \\ v(r) \end{pmatrix} e^{ij\phi}, \quad \chi_-(r, \phi) = \begin{pmatrix} u(r) \\ v(r) \\ -u(r) \\ -v(r) \end{pmatrix} e^{ij\phi}.
\]

The functions \( u(r), v(r) \) then satisfy,

\[
\begin{align*}
-\frac{i}{2r} \frac{d}{dr} + \frac{1}{2r} + \frac{1}{r} \frac{j + n/4}{1 - n/6} \right) v + \tau\Phi u &= Eu, \\
-\frac{i}{2r} \frac{d}{dr} + \frac{1}{2r} - \frac{1}{r} \frac{j + n/4}{1 - n/6} \right) u - \tau\Phi v &= Ev.
\end{align*}
\]

(1.25)

Here \( n = 1 \) for the 60° pentagon and \( n = 2 \) for the 120° square and \( \tau = \pm 1 \) is the eigenvalue of \( \tau_1 \).
Scattering States

Setting \( \nu = \frac{j + \tau n/4}{1 - n/6} \), in the above equation, we recognize the classical recurrence relations for the Bessel function,

\[
\frac{\alpha}{x} J_\alpha(x) + J'_\alpha(x) = J_{\alpha-1}(x)
\]

\[
\frac{\alpha}{x} J_\alpha(x) - J'_\alpha(x) = J_{\alpha+1}(x),
\]

and find eigenfunctions with \( u, v \) of the form

\[
\begin{pmatrix}
  u(r) \\
  v(r)
\end{pmatrix}
= \begin{pmatrix}
  (\epsilon + \tau \Phi) J_{\nu - 1/2}(kr) \\
  i k J_{\nu + 1/2}(kr)
\end{pmatrix}
\quad E = \epsilon \equiv +\sqrt{k^2 + \Phi^2},
\]

\[
\begin{pmatrix}
  u(r) \\
  v(r)
\end{pmatrix}
= \begin{pmatrix}
  i k J_{\nu - 1/2}(kr) \\
  (\epsilon + \tau \Phi) J_{\nu + 1/2}(kr)
\end{pmatrix}
\quad E = -\epsilon,
\]

and also

\[
\begin{pmatrix}
  u(r) \\
  v(r)
\end{pmatrix}
= \begin{pmatrix}
  (\epsilon + \tau \Phi) J_{-(\nu - 1/2)}(kr) \\
  - i k J_{-(\nu + 1/2)}(kr)
\end{pmatrix}
\quad E = \epsilon \equiv +\sqrt{k^2 + \Phi^2},
\]

\[
\begin{pmatrix}
  u(r) \\
  v(r)
\end{pmatrix}
= \begin{pmatrix}
  - i k J_{-(\nu - 1/2)}(kr) \\
  (\epsilon + \tau \Phi) J_{-(\nu + 1/2)}(kr)
\end{pmatrix}
\quad E = -\epsilon.
\]

The first set of solutions is finite at the origin when \( j > 1/2 \) and the second is finite at the origin when \( j < -1/2 \). For \( j = 1/2 \) and \( \tau \) negative, the upper component of the first set of solutions diverges at the origin, but no faster than \( r^{-1/2} \), so it still normalizable. Similarly the second set is locally normalizable for \( j = -1/2 \).

There is a special case: for \( n = 2 \), and \( j = 1/2, \tau = -1 \) or \( j = -1/2, \tau = +1 \), we have \( \nu = 0 \). The scattering solutions with \( E = \pm \sqrt{k^2 + \Phi^2} \) contain \( J_{1/2}(kr) = \sqrt{2\pi/kr} \sin kr \) and \( J_{-1/2}(kr) = \sqrt{2\pi/kr} \cos kr \), both of which are normalizable near the singular point at the origin. Additional boundary conditions must be imposed to select a complete, linearly independent, set of solutions. I defer a detailed discussion of this to next section. \(^4\)

\(^4\)The same issue with boundary conditions occurs in the work of Yamagishi [9] on the Schroedinger equation for a charge-monopole system.
Bound States

The $n = 2$, 120° wedge differs fundamentally from the $n = 1$, 60° wedge because the cutting and sewing of the graphene sheet in the former case preserves the A-B bipartite structure of the lattice. There should therefore be some operator that anti-commutes with the Hamiltonian. Tracing back our change of basis to eq. (1.15), we find

$$\Gamma = \sigma_3 \otimes \tau_3 e^{i\frac{1}{2} \phi} = \sigma_3 \otimes \left( \tau_1 \cos \phi - \tau_2 \sin \phi \right)$$

which makes $\Gamma$ a function of $\phi$. Note that $\Gamma$ is periodic under $\phi \rightarrow \phi + 2\pi$ and hence well-defined on the whole lattice. If we tried eq. (1.14) instead, we would find that $\Gamma$ has a discontinuity i.e.a single pentagon breaks sublattice symmetry, as we expect.

To explore the bound states it is more convenient to have a constant $\Gamma$ and to that end we make a (last!) change of basis, replacing the gauge field with $\tau_1$ in eq. (1.15) with a $\tau_3$. Then equation (1.15) defining the single-valued field $\tilde{\Psi}(r, \theta)$ becomes

$$\Psi(r, \theta) = \exp \left\{ i \tau_3 \left( \frac{3\theta}{4\pi} \right) \right\} \otimes \exp \left\{ i \frac{\pi}{3} \sigma_3 \left( \frac{3\theta}{4\pi} \right) \right\} \tilde{\Psi}(r, \theta),$$

and the eigenvalue equation (1.22) is replaced by

$$\left( -i\sigma_1 \left( \partial_r + \frac{1}{2r} \right) - i\sigma_2 \frac{3}{r} \left( \frac{3}{2} \partial_{\phi} + \frac{3i}{4} \tau_3 \right) + \Phi \sigma_3 (\tau_1 \cos \phi + \tau_2 \sin \phi) \right) \chi(r, \theta) = E\chi(r, \theta), \quad (1.26)$$

again with antiperiodic $\chi(r, \theta)$. Now the $\Phi$ term is angle dependent but we can take care of this by writing eq. (1.26) as,

$$e^{-i\tau_3 \frac{3\theta}{4\pi}} \left( -i\sigma_1 \left( \partial_r + \frac{1}{2r} \right) - i\sigma_2 \frac{13}{r} \left( \frac{3}{2} \partial_{\phi} + \frac{3i}{4} \tau_3 \right) + \Phi \sigma_3 \tau_1 \right) e^{+i\tau_3 \frac{3\theta}{4\pi}} \chi(r, \theta) = E\chi(r, \theta) \quad (1.27)$$

$$\Rightarrow \left( -i\sigma_1 \left( \partial_r + \frac{1}{2r} \right) - i\sigma_2 \frac{13}{r} \left( \frac{3}{2} \partial_{\phi} + \Phi \sigma_3 \tau_1 \right) \right) e^{+i\tau_3 \frac{3\theta}{4\pi}} \chi(r, \theta) = E e^{+i\tau_3 \frac{3\theta}{4\pi}} \chi(r, \theta) \quad (1.28)$$
Again \( \tau_1 \) commutes with the operator acting on \( \chi \), so we may separate solutions by its eigenvalue,

\[
e^{+i\tau_3 \phi} \chi_+(r, \phi) = \begin{pmatrix} u(r) \\ v(r) \\ u(r) \\ v(r) \end{pmatrix} e^{ij\phi}, \quad e^{-i\tau_3 \phi} \chi_-(r, \phi) = \begin{pmatrix} u(r) \\ v(r) \\ -u(r) \\ -v(r) \end{pmatrix} e^{ij\phi}.
\]

where \( j \) now take integer valuues. The \( u(r), v(r) \) satisfy,

\[
-i\left( \frac{d}{dr} + \frac{1}{r} \left( \frac{3}{2} j + \frac{1}{2} \right) \right) v + \Phi \tau u = E\tau u - i\left( \frac{d}{dr} - \frac{1}{r} \left( \frac{3}{2} j - \frac{1}{2} \right) \right) u - \Phi \tau v = E\tau v.
\]

The above equations are very similar to eq. (1.25) but we are now looking for bound states with \( E^2 - \Phi^2 < 0 \). As we might guess, the solutions are modified Bessel functions, but imposing normalizability at the origin restricts us to \( K_{\pm \frac{1}{2}}(\kappa r) \) and \( j = 0 \). Explicitly, we have

\[
\Psi_{E,1} = \begin{pmatrix} (E + \Phi)e^{i\phi/2} \\ iKe^{i\phi/2} \\ (E + \Phi)e^{-i\phi/2} \\ iKe^{-i\phi/2} \end{pmatrix} \frac{1}{\sqrt{r}} e^{-\kappa r}, \quad \Psi_{E,2} = \begin{pmatrix} (E - \Phi)e^{i\phi/2} \\ iKe^{i\phi/2} \\ (E - \Phi)e^{-i\phi/2} \\ iKe^{-i\phi/2} \end{pmatrix} \frac{1}{\sqrt{r}} e^{-\kappa r} \quad (1.29)
\]

and

\[
\Psi_{E,3} = \begin{pmatrix} (E + \Phi)e^{i\phi/2} \\ -iKe^{i\phi/2} \\ (E + \Phi)e^{-i\phi/2} \\ -iKe^{-i\phi/2} \end{pmatrix} \frac{1}{\sqrt{r}} e^{+\kappa r}, \quad \Psi_{E,4} = \begin{pmatrix} (E - \Phi)e^{i\phi/2} \\ iKe^{i\phi/2} \\ (E - \Phi)e^{-i\phi/2} \\ iKe^{-i\phi/2} \end{pmatrix} \frac{1}{\sqrt{r}} e^{+\kappa r} \quad (1.30)
\]

Here \( \kappa = \sqrt{\Phi^2 - E^2} \). \( \Gamma \) acts on these solutions to give a state with energy \(-E\).
We are especially interested in exact zero modes. When \( E = 0 \), the states can be eigenvectors of \( \Gamma \),

\[
\begin{align*}
\Psi^A_+ &= \begin{pmatrix} e^{-i\phi/2} \\ 0 \\ 0 \\ i e^{i\phi/2} \end{pmatrix} \frac{1}{\sqrt{r}} e^{-\Phi r}, & \Psi^B_- &= \begin{pmatrix} 0 \\ i e^{-i\phi/2} \\ e^{i\phi/2} \\ 0 \end{pmatrix} \frac{1}{\sqrt{r}} e^{-\Phi r}, \\
\Psi^A_- &= \begin{pmatrix} i e^{-i\phi/2} \\ 0 \\ 0 \\ e^{i\phi/2} \end{pmatrix} \frac{1}{\sqrt{r}} e^{\Phi r}, & \Psi^B_+ &= \begin{pmatrix} 0 \\ e^{-i\phi/2} \\ i e^{i\phi/2} \\ 0 \end{pmatrix} \frac{1}{\sqrt{r}} e^{\Phi r}.
\end{align*}
\]

Only one pair is normalizable, depending on the sign of \( \Phi \).

### 1.3.2 Boundary conditions

We need to impose some boundary conditions at \( r = 0 \) to restrict the space of these solutions to a complete linearly-independent set. First, observe that the Hamiltonian is hermitian with respect to the inner product

\[
\langle \Psi_1 | \Psi_2 \rangle = \int_0^{2\pi} \int_0^\infty \Psi_1^\dagger \Psi_2,
\]

because

\[
\left( -i \sigma_1 \frac{\partial}{\partial r} - \frac{i}{2r} \sigma_1 \right) = -i \sigma_1 \frac{1}{r} \frac{\partial}{\partial r} r + \frac{i}{2r} \sigma_1 = -i \sigma_1 \frac{\partial}{\partial r} - \frac{i}{2r} \sigma_1. \tag{1.31}
\]

Note that the crucial contribution of the spin connection term \( i \sigma_1/2r \) (I thank M. Stone for explaining this confusing point).

However, hermiticity is not the same as self-adjointness. To impose the latter, we must carefully consider the boundary terms that were produced when we differentiated by parts in eq. (1.31). Let \( \Psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T \) and \( \Xi = (\chi_1, \chi_2, \chi_3, \chi_4)^T \) be functions that are square integrable on \([0, \infty)\) with the measure \( r \, dr \), so they fall off faster than \( r^{-2} \) at infinity. We find,

\[
\langle \Psi | H X \rangle - \langle H \Psi | X \rangle = [-ir(\psi_1^* \chi_2 + \psi_2^* \chi_1 + \psi_3^* \chi_4 + \psi_4^* \chi_3)]_0^\infty.
\]
For the right side to vanish, the sum inside the parentheses must vanish also (in the limiting sense since \( \Psi(r) \) and \( \Xi(r) \) may diverge as \( r^{-1/2} \) at the origin. If we require boundary conditions,

\[
\begin{pmatrix}
\psi_1 \\
\psi_3
\end{pmatrix} = \begin{pmatrix} a & b \\
 c & d \end{pmatrix} \begin{pmatrix} \psi_2 \\
 \psi_4 \end{pmatrix}, \quad r \to 0
\]

so that,

\[
\lim_{r \to 0} [\psi_2^* (a^* \chi_2 + c^* \chi_4 + \chi_1) + \psi_4^* (b^* \chi_2 + \chi_3 + d^* \chi_4)] = 0
\]

for any \( \Psi_2, \Psi_4 \), then the corresponding conditions on \( \Xi \) are

\[
\begin{pmatrix} \chi_1 \\
 \chi_3 \end{pmatrix} = \begin{pmatrix} -a^* & -c^* \\
 b^* & -d^* \end{pmatrix} \begin{pmatrix} \chi_2 \\
 \chi_4 \end{pmatrix}, \quad r \to 0.
\]

The rule of thumb is that self-adjointness is equivalent to symmetric boundary conditions, so we need,

\[
\begin{pmatrix} a & b \\
 c & d \end{pmatrix} = \begin{pmatrix} -a^* & -c^* \\
 -b^* & -d^* \end{pmatrix}, \quad r \to 0.
\]

We conclude that the set of boundary conditions is encoded in four real parameters \( A, B, C, D \) where,

\[
\begin{pmatrix}
\psi_1 \\
\psi_3
\end{pmatrix} = \begin{pmatrix} iA & B + iC \\
 -B + iC & iD \end{pmatrix} \begin{pmatrix} \psi_2 \\
 \psi_4 \end{pmatrix}, \quad r \to 0. \tag{1.32}
\]

We can now use our explicit solutions for exponentially localized states eq. (1.29). Setting,

\[
\Psi = \alpha \Psi_{E,1} + \beta \Psi_{E,2}.
\]

and imposing the boundary condition (1.32) at \( r = 0 \) leads to a pair of linear equations

\[
\begin{align*}
(a + \beta)E + (a - \beta)\Phi &= -A\kappa(\alpha + \beta) - (C - iB)\kappa(\alpha - \beta), \\
(a - \beta)E + (a + \beta)\Phi &= -D\kappa(\alpha - \beta) - (C + iB)\kappa(\alpha + \beta),
\end{align*}
\]
which must be satisfied for all $\alpha$ and $\beta$. In other words the matrix,

$$
\begin{pmatrix}
E + A\kappa & \Phi + (C - iB)\kappa \\
\Phi + (C + iB)\kappa & E + D\kappa
\end{pmatrix}
$$

is degenerate, or $F \equiv \det(f)$ is zero. Recall that $\kappa = +\sqrt{\Phi^2 - E^2}$.

By inspection, the function $F(E)$ has the following properties:

1. $F(E)$ is real in the range $-|\Phi| \leq E \leq |\Phi|$, and always has two zeros at $E = \pm|\Phi|$ corresponding to the edges of the upper and lower continuum respectively. Additional zeros in that range correspond to bound states, and since $F$ is quartic in $E$, there can at most be two.

2. $F$ is a function of $B^2 + C^2$, so we may multiply $B + iC$ by a phase to set $B = 0$.

3. $F$ is symmetric in $A$ and $D$.

We have not yet mentioned the role of sublattice symmetry. From the definition in eq. (1.32), both $A$ and $D$ couple the sublattices to themselves and break the $E \leftrightarrow -E$ spectral symmetry, $\Gamma$.

**Numerical computations** were done by diagonalizing tightbinding Hamiltonian matrices. I wrote a Python program that constructs a large hexagonal lattice and cuts out wedges - giving a list of vertices and edges that specifies whether a given edge is a ‘single’ or ‘double’ bond. Then I imported this large adjacency matrix into Mathematica and found the eigenvalues as a function of $h = (t + \delta t)/t$ where $\delta t$ is the change in hopping amplitude on the double bounds. Since the lattice has open boundary conditions, one unavoidably encounters modes that live on the truncated edge and show up as spurious midgap states. However, it is not hard to distinguish these from the true localized states that live near the defects.

I now compare the zeros of $F$ to lattice calculations.

Let us start with the case where the $120^\circ$ wedge is cut out at a hexagon, resulting in a square. Since this preserves $\Gamma$, we set $A = D = 0$ and find,

$$
F(E) = 0 \longrightarrow C = -\text{sgn}(\Phi).
$$

A plot of $F(E)$ for this case is shown in fig. 1.3. There is a doubly degenerate state exactly at $E = 0$, which disappears when $\Phi$ goes from negative to positive. This exact matches with the lattice fig. 1.5, where the two zero modes delocalize as $\Phi$ changes sign.
Figure 1.3: The left-hand plot shows $F(E)$ for boundary-condition parameters $A = B = D = 0$, $C = 1$, and $\Phi = -1$ which possesses two degenerate zero modes. The right-hand plot shows $F(E)$ for the same boundary parameters, but with $\Phi$ having changed sign from negative (reduced hopping in the double bonds) to positive (enhanced hopping on the double bonds). There are now no bound states.

Figure 1.4: The left-hand figure shows $F(E)$ for $A = B = 0$, $C = 1$, $D = .5$ and $\Phi$ negative. There is a zero mode and a bound state with $E \approx -0.5|\Phi|$. In the right-hand figure we have $A = D = -1.0$, $B = 0$, $D = \sqrt{2}$, and $\Phi$ negative. There are two degenerate bound states at $E = |\Phi|/\sqrt{2}$. In both cases, the bound states cease to exist as soon as $\Phi$ changes sign from negative to positive.

We proceed to the case where the two wedges are cut from two different hexagons, giving a pair of pentagons. Sublattice symmetry is now broken in a line joining the pentagons (of course the position of this line is arbitrary). In our boundary parameters, we now let $A$ and $D$ take non-zero values. Two representative cases are shown in fig. 1.4. The plot on the right has a a doubly degenerate bound state for some $E \neq 0$ (contrast with fig. 1.3. In lattice spectrum in fig. 1.6 also has a pair of almost degenerate states for $\Phi < 0$ ($h < 1$). As $\Phi$ crosses 0, the pair mixes, splits in energy, and delocalize.

There is hence, an excellent qualitative agreement for bound states in the case of the square as well as the pair of pentagons.
Figure 1.5: The low-lying part of the energy spectrum plotted versus $h$ (the ratio of double bond hopping to single bond hopping) for a square defect created by excising a $120^\circ$ wedge. The horizontal axis has been displaced vertically so as to uncover the doubly-degenerate exact zero mode. The “within-gap” modes peeling off from the upper and lower continua for $h > 1$ (i.e. $\Phi > 0$) are “anti-localized” edge states, whose exact form depends on how we truncate the lattice on its outer boundary.

Figure 1.6: The low-lying energy spectrum plotted versus $h$ for a pair of nearby pentagons. The left-hand figure is for an $(n,m) = (1,1)$ cone in the language of ref [5], and the right-hand figure is for an $(n,m) = (0,3)$ cone. In both cases the $60^\circ$ wedges have their apices in single-bond hexagons so as to preserve the global Kekulé structure. For $h < 1$ there is a pair of nearly degenerate bound states lying just below the upper continuum. The “below gap” modes at $h > 1$ are uninteresting edge states localized at the outer boundary.
1.4 Conclusions

The continuum Dirac hamiltonian provides a good account of the long-wavelength electronic excitations on an infinite sheet of graphene. The model is still useful for cones induced by disclinations — but it must be supplemented by non-trivial boundary conditions at the tip of the cone. Although the low energy states have too long a wavelength to resolve the fine details of lattice disruption at the tip, they experience phase shifts and mode mixing that have a significant effect on the eigenstates.

For two separated pentagons, the scrambling of the A-B bipartite lattice structure along a seam joining the pentagons sufficiently violates the $E \leftrightarrow -E$ spectral symmetry as to allow bound states at non-zero $E$. In the case of two coincident pentagons (i.e. a square) the $E \leftrightarrow -E$ symmetry is preserved. However the resultant half-unit of flux through the square plaquette is too large to be approximated by a spread-out gauge field. A continuum gauge field with this flux would have bound a single state whose eigenvalue of $\Gamma$ is determined by the sign of the flux. The lattice spectrum must be unchanged, however, by the insertion of an integer flux-quantum through the square. Such an insertion can reverse the sign of the flux and so no particular sign of $\Gamma$ can be favoured. The lattice hamiltonian compromises by producing two bound states, one with each sign of $\Gamma$. All these effects can be reproduced in the continuum model by suitable choices of the parameters in the self-adjoint boundary conditions. The predictions [6] of the Jackiw-Rossi-Weinberg index theorem do not survive in a simple form.
Chapter 2

Berry Phases in the Majorana Representation

2.1 The Majorana parametrization

Early in quantum mechanics, we learn the decomposition of tensor products of angular momentum states,

\[ \text{spin } \frac{1}{2} \otimes \text{spin } \frac{1}{2} \cdots = \text{spin } j \oplus \text{spin } j - 1 \oplus \cdots \oplus \text{spin } 0 \]

It is also a familiar fact that the top state on the right side, spin \( j \) is symmetric in the coefficients on the left. In fact,

\[ \text{spin } j = \text{spin } \frac{1}{2} \otimes \text{spin } \frac{1}{2} \cdots \]

Here the ‘\( \otimes \)’ denotes the symmetrized tensor product which is defined such that,

\[ e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{2j}} = \frac{1}{(2j)!} \sum_{\pi \in S_{2j}} e_{i_{\pi(1)}} \otimes e_{i_{\pi(2)}} \otimes \cdots \otimes e_{i_{\pi(2j)}} \]

where we sum over the \( (2j)! \) elements of the permutation group \( S_{2j} \). For example,

\[ a \oplus a \oplus b = \frac{1}{3} (a \otimes a \otimes b + a \otimes b \otimes a + b \otimes a \otimes a) \]

In [10] Majorana showed that an arbitrary spin-\( j \) state \( \Psi \) can be decomposed as

\[ \Psi = \psi_{i_1} \otimes \cdots \otimes \psi_{i_{2j}}, \quad (2.1) \]

where the factors,

\[ \psi_i = \alpha_i e_1 + \beta_i e_2 \]

are a set of \( 2j \) spin-1/2 spinors with \( e_1 \equiv |\uparrow\rangle, \ e_2 = |\downarrow\rangle \). Note that the right side of eq. (2.1) is really a
single product and not a sum of products, as would be typical in a tensor product. Majorana’s result was rediscovered independently by both Schwinger [11] and Rabi [12]. Before diving into details, let us do a quick consistency check on eq. (2.1): on the right side there are 2 complex parameters for each $\psi_i$ but by the properties of a tensor product we can factor one out and swap it among the rest. This leaves $2j + 1$ free parameters, which is exactly the dimension of spin-j.

Let us begin with the inner product of two symmetrized states. Introducing a shorthand,

$$e_1 \otimes \cdots \otimes e_1 \otimes e_2 \otimes \cdots \otimes e_2 = e_1^{j+m}e_2^{j-m}.$$  

we find,

$$\langle e_1^{j+m}e_2^{j-m}|e_1^{j+m'}e_2^{j-m'}\rangle = \delta_{mm'}\frac{(j + m)!(j - m)!}{(2j)!} = \delta_{mm'}/\binom{2j}{j - m}.$$  

(2.2)

where we have simply extended the natural inner product of the spin-1/2 states and counted the number of terms in the tensor product. A $|j,m\rangle$ state ought to be identified by its eigenvalue under the the $J_3$ operator. Since,

$$J_3 e_1^{j+m}e_2^{j-m} = \frac{1}{2}\{(j + m) - (j - m)\} e_1^{j+m}e_2^{j-m} = m e_1^{j+m}e_2^{j-m},$$  

(2.3)

we find,

$$|j, m\rangle = \sqrt{(2j)!/((j + m)!(j - m)!)} e_1^{j+m}e_2^{j-m}.$$  

(2.4)

To confirm the identification, observe that $J_+ e_1 = e_2$, $J_- e_2 = 0$, and so

$$J_-|j, m\rangle = \sqrt{(2j)!/((j + m)!(j - m)!)} J_- (e_1^{j+m}e_2^{j-m})$$

$$= \sqrt{(2j)!/((j + m)!(j - m)!)} (j + m)e_1^{j+m-1}e_2^{j-m+1}$$

$$= \sqrt{(2j)!/((j + m)!(j - m)!)} (j + m)\sqrt{(j + m - 1)!(j - m + 1)!/(2j)!}\langle j, m - 1\rangle$$

$$= \sqrt{(j + m)(j - m + 1)}|j, m - 1\rangle$$

as it should.
Now let $\psi_i = \alpha_i e_1 + \beta_i e_2$. Set $z_i = -\beta_i / \alpha_i$, and expand out a decomposable state as

$$\Psi \equiv \psi_1 \odot \cdots \odot \psi_{2j} = \left( \prod_{i=1}^{2j} \alpha_i \right) \sum_{m=-j}^{m=j} a_{j-m} e_1^m e_2^{j-m},$$

where $a_{j-m}$ is the $(j - m)$-th elementary symmetric function of the $-z_i$ i.e.,

$$a_1 = -\sum z_i$$
$$a_2 = +\sum_{i < j} z_i z_j$$
$$\vdots$$
$$a_{2j} = +\prod z_i.$$

By the Viète formulae for the coefficients of a polynomial in terms of its zeros,

$$\sum_{n=0}^{2j} z^{2j-n} a_n = \prod_{i=1}^{2j} (z - z_i).$$

We therefore have

$$\langle j, m | \Psi \rangle = \left( \prod_{i=1}^{2j} \alpha_i \right) a_{j-m} \sqrt{\left( j + m \right)! \left( j - m \right)! \left( 2j \right)!}.$$

or

$$a_{j-m} = \langle j, m | \Psi \rangle \sqrt{\frac{(2j)!}{(j + m)! (j - m)! \left( \prod_{i=1}^{2j} \alpha_i \right)^{-1}}}.$$ (2.5)

The common factor $\prod_{i=1}^{2j} \alpha_i = \langle j, j | \Psi \rangle$ does not affect the location of the zeros, so we can drop it with the understanding that an overall phase remains at the end. We have thus motivated . . .

**Majorana’s Recipe:**

1. Given a spin-$j$ state

$$\Psi \equiv | \Psi \rangle = \sum_m | j, m \rangle \langle j, m | \Psi \rangle,$$

construct a set of $2j + 1$ coefficients,

$$c_{j-m} = \langle j, m | \Psi \rangle \sqrt{\frac{(2j)!}{(j + m)! (j - m)!}}, \quad m = -j, -j + 1, \ldots, j - 1, j$$ (2.6)
2. Find the zeros \( z_i \) of the degree 2\( j \) polynomial equation

\[
P_\Psi(z) = \sum_{n=0}^{2j} z^{2j-n} c_n = 0. \tag{2.7}
\]

3. The \( z_i \) are related to the coefficients of the spin-1/2 factors by

\[
z_i = -\beta_i / \alpha_i. \tag{2.8}
\]

We can extract the individual coefficients by normalizing so that \(|\alpha_i|^2 + |\beta_i|^2 = 1\). A choice of phase remain for each \( \psi_i \) but again, these can be swapped amongst the factors leaving only an overall phase to be matched with that of \( \Psi \).

It may happen that one or more of the leading coefficients \( c_n \) are zero. The polynomial equation then has degree less than 2\( j \) and possesses fewer than 2\( j \) roots. We deal with this by extending the complex plane to the Riemann sphere, and regard the missing roots as being at infinity. Their corresponding \( \alpha_i \) are then zero.

### 2.1.1 Geometric Picture

The preceding algebraic manipulations might have obscured the geometric interpretation of the Majorana representation, which is really quite simple. Upto phase, a spin-\( j \) state can be represented as 2\( j \) points on the Riemann (or Bloch) sphere. The points, which are allowed to coincide, are the zeros produced by the above recipe. More physically, they are the respective directions of each of the spin-\( \frac{1}{2} \) \( \Psi_i \) (see appendix). A \(|j,m\rangle \) state corresponds to \( j + m \) points on the north pole and \( j - m \) points at the south pole. A rotation of the state \( \Psi \) is simply a rigid rotation of the entire sphere.

In the rest of the chapter, I will use a notation that denotes a state by its zeros directly. I define,

\[
|\{w_i\}\rangle \equiv |w_1, w_2, \ldots, w_{2j}\rangle_N
\]

\[
= \frac{1}{N} |w_1 \odot w_2 \odot \ldots \odot w_{2j}\rangle
\]

\[
= \frac{1}{N} \left( \frac{1}{\bar{w}_1} \odot \frac{1}{\bar{w}_2} \odot \ldots \odot \frac{1}{\bar{w}_{2j}} \right) \tag{2.9}
\]

\[
= \frac{1}{N} \left( \frac{1}{\bar{w}_1} \odot \frac{1}{\bar{w}_2} \odot \ldots \odot \frac{1}{\bar{w}_{2j}} \right) \tag{2.10}
\]

to be a normalized spin-\( j \) state with its 2\( j \) spin-1/2 components pointing in the directions \( w_i \). \(^1\) Here, \( N \) is

\(^1\)The conjugation of the \( w_i \) is a technicality. See appendix.
a $w_i$ dependent normalization factor given by

$$N^2 = \langle \{w_i\} \{w_i\} \rangle = \frac{1}{(2j)!} \sum_{\pi \in S_{2j}} \prod_{i=1}^{2j} (1 + w_i \bar{w}_{i(\pi)}). \quad (2.11)$$

## 2.2 Berry phases in the Majorana representation

Under an adiabatic evolution of the Hamiltonian $H(t)$, the change in a spin-$j$ state $\Psi(t)$ is traced out by the evolution of its Majorana points. Under a cyclic evolution $H(T) = H(0)$, the state acquires a Berry phase. In this section we will determine how the Berry phase (both Abelian and non-Abelian) may be written as functionals of Majorana paths. For the latter, we will concentrate on the physically important case of Kramers’ degeneracy.

### 2.2.1 Abelian case

The Abelian case was worked out by Hannay [13]. We begin by differentiating an arbitrary state,

$$d\langle \{w_i\} \{w_i\} \rangle = \frac{1}{(2j)!} \sum_{\pi \in S_{2j}} \sum_i |w_{\pi(1)}\rangle \otimes \cdots (d|w_{\pi(i)}\rangle) \cdots \otimes |w_{\pi(2j)}\rangle \quad (2.12)$$

where, from $|w_i\rangle = \bar{w}_i |\uparrow\rangle + |\downarrow\rangle$ we have $d|w_i\rangle = d\bar{w}_i |\uparrow\rangle$, and

$$\mathcal{A} = i \text{Im} \left( \frac{\langle \{w_i\}|d\{w_i\}\rangle}{\langle \{w_i\}\{w_i\}\rangle} \right). \quad (2.13)$$
we obtain

$$\mathcal{A} = i \text{Im} \left[ \frac{\prod_k (1 + w_k \bar{w}_{\pi(k)}) \sum_k (w_k d \bar{w}_{\pi(k)}/(1 + w_k \bar{w}_{\pi(k)})]}{\prod_k (1 + w_k \bar{w}_{\pi(k)})} \right].$$

(2.14)

For \( j = 1/2 \), for example, we have

$$\mathcal{A} = i \text{Im} \left( \frac{\langle w | d | w \rangle}{\langle w | w \rangle} \right) = \frac{1}{2} \frac{wd \bar{w} - \bar{w}dw}{1 + \bar{w}w}, \quad d\mathcal{A} = \frac{dw \wedge d\bar{w}}{(1 + \bar{w}w)^2}. \quad (2.15)$$

The expression for \( d\mathcal{A} \) is \( i \) times one-half the area form on the two-sphere, and so we find that familiar result that in a cyclic evolution the accumulated phase is one-half of the area swept out by the spin.

For a general value of \( j \) consider the evolution of the \( E = m \) eigenstate of \( H = J \cdot \hat{n} \) as the unit vector \( \hat{n} \) varies. In the Majorana parametrisation, \( j + m \) of the \( w_i \)'s lie at the point \( w \) corresponding to the unit vector \( \hat{n} \) and \( j - m \) of the \( w_i \) lie at the antipodal point \( w' = -1/\bar{w} \). Thus equation (2.14) becomes,

$$\mathcal{A} = \frac{1}{2} (j + m) \frac{wd \bar{w} - \bar{w}dw}{1 + \bar{w}w} + \frac{1}{2} (j - m) \frac{w' \bar{w}' - \bar{w}'dw}{1 + \bar{w}'w'}. \quad (2.16)$$

For a cyclic evolution the accumulated phase is \( [(j + m) - (j - m)]/2 = m \) times the area swept out by the point \( w \) — again a familiar result [14].

**Note:** The advantage of using stereographic coordinates is evident if one contrasts eq. (2.14) with Hannay’s work[13]. If one insists on working with the unit vectors \( \mathbf{n} \) on the sphere, the resulting expressions are needlessly complicated. Hannay does end by quoting the result in complex space (suggested by Berry!).

### 2.2.2 Non-Abelian case

Non-abelian Berry transport is a generalization of the usual Berry adiabatic transport to the case of degenerate energy eigenvalues [15]. Consider the adiabatic evolution of a state

$$|\psi(t)\rangle = a_n(t)|n,t\rangle \quad (2.17)$$

that lies in a subspace \( V(t) \) spanned by an orthonormal frame \( |n,t\rangle, n = 1, \ldots, N \), of degenerate eigenstates of a Hamiltonian \( H(t) \). If \( H(t) \) varies slowly, the resulting adiabatic evolution leads to the coefficients being given by

$$a_n(t) = U_{nm}(t)a_m(0) \exp \left\{ -i \int_0^t E(t) dt \right\}, \quad (2.18)$$

...
where \( E(t) \) is the common eigenvalue, and

\[
U_{nm}(t) = \left[ \mathcal{P} \exp \left\{ - \int_0^t A \right\} \right]_{nm}
\]  

(2.19)

is a unitary matrix that generalizes the usual Berry phase. The symbol \( \mathcal{P} \) indicates a path-ordered integral and the non-abelian Berry connection

\[
A_{nm} = \langle n,t|d|m,t \rangle = -\bar{A}_{nm}
\]  

(2.20)

is a skew-hermitian matrix-valued one-form. In the special case that degenerate subspace is one-dimensional it reduces to the abelian Berry connection.

Now consider the non-abelian extension of Hanay’s formula eq. (2.14) in which the Berry connection becomes a one-form \( A \) that takes its values in the Lie algebra \( \mathfrak{u}(n) \). We can evaluate its skew-hermitian matrix elements in the Majorana parametrisation by extending (2.14). After some work we find that the off diagonal elements of the connection are

\[
A_{rs} = \frac{\langle \{w_r^\dagger\}|d|\{w_s^\dagger\} \rangle}{\sqrt{\langle \{w_r^\dagger\}|\{w_r^\dagger\} \rangle \sqrt{\langle \{w_s^\dagger\}|\{w_s^\dagger\} \rangle}}} \quad (r \neq s) \\
= \frac{\sum \pi \prod_k (1 + w_k^r \bar{w}_k^s \pi(k)) \sum_k (w_k^r d \bar{w}_k^s \pi(k)) / (1 + w_k^r \bar{w}_k^s \pi(k))}{\sqrt{\sum \pi \left[ \prod_k (1 + w_k^r \bar{w}_k^r \pi(k)) \right] \sqrt{\sum \pi \left[ \prod_k (1 + w_k^s \bar{w}_k^s \pi(k)) \right]}}}.
\]  

(2.21)

and the diagonal elements are

\[
A_{rr} = \frac{i \text{Im} \langle \{w_r^\dagger\}|d|\{w_r^\dagger\} \rangle}{\langle \{w_r^\dagger\}|\{w_r^\dagger\} \rangle} \\
= \frac{\sum \pi \text{Im} \left[ \prod_k (1 + w_k^r \bar{w}_k^r \pi(k)) \sum_k (w_k^r d \bar{w}_k^r \pi(k)) / (1 + w_k^r \bar{w}_k^r \pi(k)) \right]}{\sum \pi \left[ \prod_k (1 + w_k^r \bar{w}_k^r \pi(k)) \right]}. 
\]  

(2.22)

The expressions in equations (2.21) and (2.22) are, in general, rather complicated. Our interest, however, is in the evolution of Kramers-degenerate pairs. We therefore restrict ourselves to the time-reversal invariant Hamiltonians, and to half-integer \( j \) for which the Kramers pair of states \( |\Psi\rangle \) and \( T|\Psi\rangle \) are degenerate and mutually orthogonal.

The time reversal operator on a general spin state is

\[
T = \exp\{-i\pi J_2\} \mathcal{K}
\]  

(2.23)
where $K$ denotes complex conjugation. On a spin-$1/2$ coherent state it acts as

$$\mathcal{T}|w\rangle = |w\rangle - 1/\bar{w}\rangle$$

(2.24)

and this action extends to the general Majorana-parametrisation tensor product state.

Using the states $|\Psi\rangle$, $\mathcal{T}|\Psi\rangle$ as basis, the Berry connection becomes a two-by-two traceless skew-hermitian matrix,

$$\mathcal{A} = \begin{pmatrix} A_0 & -A_1 \\ -\bar{A}_1 & -A_0 \end{pmatrix}.$$  

(2.25)

The tracelessness of $\mathcal{A}$ is a consequence of time-reversal; it implies that the diagonal Abelian phases for a Kramers pair are complex-conjugates of each other.

In the Majorana parametrisation

$$\mathcal{A}_1 = \frac{\langle\{w_i\}|d|\mathcal{T}\{w_i\}\rangle}{\sqrt{\langle\{w_i\}|\mathcal{T}\{w_i\}\rangle}} = \frac{\sum_{\pi} \left[ \prod_k \frac{(w_{\pi(k)} - w_k) \sum_k dw_k/(w_{\pi(k)} - w_k)}{\sum_k \bar{w}_k w_{\pi(k)}} \right]}{\sum_{\pi} \left[ \prod_k (1 + \bar{w}_k w_{\pi(k)}) \right]}$$

(2.26)

For eigenstates of $H = J \cdot \hat{n}$ (2.26) simplifies considerably. Again, there are $j + m$ Majorana points $w$ and $j - m$ points at $w' = -1/\bar{w}$. The numerator contains the factor $\left[ \prod_{k,k\neq j} (w_{\pi(k)} - w_k) \right]$, which can only be non-zero for permutations that swap the $w$ and $w'$ except at most a single point that may map to itself (recall that $2j$ is odd). As a consequence the off-diagonal term is non-zero only for $m = 1/2$ states. When $m \neq 1/2$ the Berry transport is purely Abelian, in the sense that there exists a fixed basis whose states are only multiplied by a phase for any loop in Hamiltonian space.

Consider, for example rotations of $\hat{n}$ about the $z$ axis. We have $w_i \rightarrow w_i \exp i\phi$ and $dw_i = iw_id\phi$ and equation 2.26) becomes

$$\mathcal{A}_1 = \frac{\sum_{\pi} \left[ \prod_k (w_{\pi(k)} - w_k) \sum_k w_k / (w_{\pi(k)} - w_k) \right]}{\sum_{\pi} \left[ \prod_k (1 + \bar{w}_k w_{\pi(k)}) \right]} i d\phi.$$  

(2.27)

For a $m = 1/2$ state with Majorana points at $w = \tan(\theta/2)\exp\{i\phi\}$ and $w' = \cot(\theta/2)\exp\{-i\phi\}$ we get

$$\mathcal{A}_1 = \frac{(n + 1)!}{n! (n + 1)!} \frac{(1 + w w')^{2n} / w^{2n}}{(1 + w w')^{n+1} / (w w')^n} w(i d\phi)$$

$$\mathcal{A}_1 = \frac{(n + 1) w^n}{w^{2n} 1 + w w'} w(i d\phi)$$

$$\mathcal{A}_1 = \exp \left[ i (2n + 1) \phi \right] \frac{(n + 1)}{2} \sin(\theta)(i d\phi)$$

(2.28)

where $2n + 1 = 2j$ is the total number of Majorana points. We may redefine the basis states so as to absorb
the phase factor so that
\[ A_1 = \frac{(n + 1)}{2} \sin(\theta)(id\phi) = \frac{1}{2} \left( j + \frac{1}{2} \right) \sin(\theta)(id\phi), \] (2.29)
and this expression agrees with a result in [16].

2.3 Appendix: Coherent States

The Majorana polynomial \( P_\Psi(z) \) for a given \( |\Psi\rangle \) has a physical interpretation as the holomorphic spin-coherent-state wavefunction representation of \( |\Psi\rangle \). This wavefunction is obtained by first defining a family of spin coherent states
\[ |z\rangle = \exp(\pi J_+)(j, -j), \quad \langle z| = \langle j, -j| \exp(zJ_-) \equiv |z|^\dagger. \] (2.30)

These states are not normalized, but have the advantage that the \( \langle z| \) are holomorphic in the parameter \( z \) — i.e., they depend on \( z \) but not on \( \pi \). The coherent-state wavefunction \( \psi(z) \) \( \overset{\text{def}}{=} \langle z|\psi\rangle \) corresponding to a state \( |\psi\rangle \) is then a holomorphic function. The label \( z \) can be identified as the complex stereographic co-ordinate of the spin direction on the two-sphere, with \( z = 0 \) corresponding to the south pole (spin down) and \( z = \infty \) to the north pole (spin up). (Strictly speaking \( |\infty\rangle \) is undefined, and one needs two families of coherent states with a transition function to capture every spin direction. This technicality is not relevant for what follows however.)

The inner product of two states \( |\psi\rangle \) and \( |\chi\rangle \) may be evaluated in terms of their wavefunctions as
\[ \langle \psi|\chi\rangle = \frac{2j + 1}{2\pi i} \int_C \frac{dz \wedge dz}{(1 + |z|^2)^{2j+1}} \psi(z)\chi(z). \] (2.31)
Normalizable wavefunctions are therefore polynomials in \( z \) of degree less than or equal to \( 2j \). In particular
\[ \langle z|j, m\rangle = \sqrt{(2j)! \over (j - m)!(j + m)!} z^{j+m}, \] (2.32)
and the wavefunction of coherent state \( |\zeta\rangle \) is the polynomial
\[ \langle z|\zeta\rangle = (1 + z\zeta)^{2j}. \] (2.33)
When acting on the holomorphic wavefunctions then \( su(2) \) generators \( J_3 \) and \( J_\pm = J_1 \pm iJ_2 \) become

\[
\begin{align*}
J_+ & \to -z^2 \frac{\partial}{\partial z} + 2jz \\
J_- & \to \frac{\partial}{\partial z}, \\
J_3 & \to z \frac{\partial}{\partial z} - j.
\end{align*}
\]

(2.34)

It is straightforward to verify that with respect to the inner product (2.31) we have \( J^\dagger_3 = J_3 \) and \( J^\dagger_\pm = J^-_\pm \).

Now from (2.6) and (2.32) we have

\[
\Psi(z) \equiv \langle z | \Psi \rangle = \sum_m \langle z | j, m \rangle \langle j, m | \Psi \rangle = \sum_{m=-j}^j c_{j-m} z^{j+m} = P\Psi(z).
\]

The Majorana polynomial is therefore precisely the coherent state wavefunction. For each zero \( z_i \) of the coherent state wavefunction, its antipode \(-1/\bar{z}_i\) is the stereographic coordinate of the direction of one of the component spin-1/2’s. These antipodes are known as Majorana points.

In the following we sometimes find it convenient to use normalized coherent states

\[
|z\rangle_N = \frac{1}{(1+|z|^2)^{j+1/2}} \exp(zJ_+)|j, -j\rangle, \quad N\langle z| = \frac{1}{(1+|z|^2)^{j+1/2}} \exp(zJ_-)|j, -j\rangle = |z\rangle_N^\dagger
\]

(2.35)

and the corresponding non-holomorphic coherent-state wavefunctions

\[
\psi(z, \bar{z}) = N\langle z| \psi \rangle.
\]

In terms of these non-holomorphic wavefunctions the inner product becomes

\[
\langle \psi | \chi \rangle = \frac{2j+1}{2\pi i} \int_C \frac{d\bar{z} \wedge dz}{(1+|z|^2)^2} \overline{\psi(z, \bar{z})} \chi(z, \bar{z}).
\]

(2.36)

The integral now involves only the usual area form on the two-sphere expressed in stereographic coordinates. The normalized wavefunction \( \psi(z, \bar{z}) \) is therefore the probability amplitude for finding the spin-\( j \) pointing in the direction \( z \).

We can describe a spin-\( j \) state up to an overall phase by either specifying the zeros \( z_i \) of its spin coherent
state wavefunction, or by specifying the Majorana points, \( i.e. \) the directions on the Riemann sphere of its component spin-1/2 factors. Because the spin directions are the antipodes \( w_i = -1/\bar{z}_i \) of the zeros, the normalized spin-1/2 state that points in the direction \( w \) has wavefunction

\[
\psi_w(z, \bar{z}) = \frac{1}{\sqrt{1 + |z|^2}} \frac{1}{\sqrt{1 + |w|^2}} (1 + z\bar{w}).
\]  \( (2.37) \)

This is just the wavefunction for the state \(|w\rangle_N\). Similarly we define

\[
|\{w_i\}\rangle_N \equiv |w_1, w_2, \ldots, w_{2j}\rangle_N
= \frac{1}{N} |w_1\rangle \odot |w_2\rangle \odot \ldots \odot |w_{2j}\rangle
\]  \( (2.38) \)

to be a normalized spin-\( j \) state with its \( 2j \) spin-1/2 components pointing in the directions \( w_i \). Here, \( N \) is a \( w_i \) dependent normalization factor given by

\[
N^2 = \langle \{w_i\}|\{w_i\} \rangle = \frac{1}{(2j)!} \sum_{\pi \in S} \prod_{i=1}^{2j} (1 + w_i w_{\pi(i)}).
\]  \( (2.39) \)

The corresponding wavefunction is

\[
N\langle z|w_1, w_2, \ldots, w_{2j}\rangle_N = (1 + |z|^2)^{-j} N^{-1} \prod_{i=1}^{2j} (1 + z\bar{w}_i).
\]  \( (2.40) \)
Chapter 3

The Abelian Toric code and Generalizations

3.1 Introduction and summary of results

A striking feature of topological order in two dimensions, is the possible existence of particles that are neither fermions nor bosons but anyons. When a pair of such particles is exchanged adiabatically, the system acquires a complex phase. Theoretically, an ideal gas of anyons obeys neither Bose nor Fermi-Dirac statistics but something in between [1].

Anyons entered the physical realm via the fractional quantum hall effect. After Laughlin’s wave function was published [17], it was quickly realized that the elementary excitations would be anyonic in character. The Laughlin states exist at filling fractions $1/n$ where $n$ is odd, the quasiparticles have a fractional charge that is $1/n$ of the electron and the exchange phase is $e^{\pi i/n}$. The next major development was the Moore-Read state [18] proposed for the mysterious plateau at filling fraction $5/2$. In this case, the ground state in the presence of quasiparticles is degenerate and the phase produced by adiabatic exchange is actually a unitary matrix.

At the lattice level, Kitaev’s abelian toric code [19] was a remarkable achievement. Although it is a special case of the older $Z_2$ gauge [20], the toric code is very useful in practice because it is exactly (and easily) solvable. The familiar features of long-range entanglement, e.g., degenerate ground states, anyonic excitations and braiding, can be seen quite easily.

In this chapter, I present my work [3] on the $Z_k$ version of the toric code and its higher dimensional generalizations. I employ the mathematical language of simplicial chain complexes. It turns out that the exact solvability of the Kitaev hamiltonian rest on the familiar mathematical fact that the (co)boundary of a (co)boundary is zero. I introduce a simple operator correspondence that takes chains and cochains to the charge and flux (or $e$ and $m$) strings in the Kitaev model, and cycles and cocycles to closed Wilson loops.

A benefit of this abstract approach is that it carries over easily to higher dimensions. The toric code is flexible enough to be put on any triangulable manifold. Topological properties of the manifold determine physical quantities. For example, the intersection form determines ground state degeneracy and Poincare
duality translates into a statement about dual models. I present some examples in three and four dimensions and find that working with the $\mathbb{Z}_k$ theory provides more information in the presence of torsion in the (co)homology. Finally, I present the higher dimensional version of the topological field theory for the toric code and prove that it provides the right braiding amplitudes. This “BF-theory” is well-known in other contexts [21].

3.2 Simplicial chain and cochain complexes

I begin with a short review of the properties of simplicial chain complexes and their homology and cohomology groups. For concreteness, let us restrict ourselves to two dimensions for the moment. There, a simplicial complex consists of a finite collection of vertices, edges, and faces. For the most part, we will only be concerned with chain complexes that are formed from triangulations of a manifold. Such triangulations have the property that two faces share at most one edge and two edges share at most a single vertex. We form three vector spaces $C_0, C_1$ and $C_2$ consisting of formal sums,

$$\sum_{v_i} \alpha_{v_i} v_i \in C_0, \quad \sum_{e_j} \alpha_{e_j} e_j \in C_1, \quad \sum_{f_k} \alpha_{f_k} f_k \in C_2$$

where $v, e, f$ are linearly independent vertices, edges and faces, respectively, and the coefficients $\alpha$ live in $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$. These sums are called chains.

![Figure 3.1: A simplicial complex](image)

The boundary of a face is an oriented sum of edges, the boundary of an edge, contains its two endpoints, while the boundary of a point is defined to be zero always. For example, in figure 3.2,

$$\partial f_3 = e_3 - e_6 - e_7 - e_5, \quad \partial e_5 = v_3 - v_6.$$  

\footnote{More correctly, these are $\mathbb{Z}_k$ modules}
The boundary operator $\partial$ is extended linearly over $C_*$, and together they form a chain complex

$$
C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0
$$

if we add that condition that the boundary of a boundary, vanish i.e. $\partial^2 = 0$. For example,

$$
\partial \circ \partial (f_3) = \partial(e_3 - e_6 - e_7 - e_5)
= (v_5 - v_6) - (v_5 - v_4) - (v_4 - v_3) - (v_3 - v_6) = 0.
$$

Homology groups are the quotients

$$
H_* = \frac{\ker(\partial : C_* \to C_{*-1})}{\operatorname{im}(\partial : C_{*-1} \to C_*})
$$

usually read as ‘cycles modulo boundaries’. They provide purely topological information that is independent of the details of the simplicial decomposition. For example, $H_1$ counts the number of ‘holes’ i.e. loops that do not bound a sum of faces.

Every chain complex has a dual, the cochain complex - which consists of linear maps

$$
C^k \equiv \{ u_k : C_k \to \mathbb{Z}_k \}. 
$$

We define a dual basis for the vertices, edges and faces in our complex, satisfying the natural relations

$$
\langle v^i, v_j \rangle = \delta^i_j, \quad \langle e^i, e^j \rangle = \delta^i_j, \quad \langle f^k, f^l \rangle = \delta^k_l,
$$

where the $\langle, \rangle$ indicates the pairing between a map and its argument e.g. the covertex $v^3$ is a function that takes the vertex $v_3$ to 1 and the rest of the vertices to zero. Finite sums of these $v^i, e^i$ or $f^l$ are called cochains.

The coboundary operator $\delta$ is defined on cochains, to be compatible with the boundary operator on chains. Given a cochain $c^*$ and chain $c_{*-1}$

$$
\langle \delta c^*, c_* \rangle = \langle c^*, \partial c_{*-1} \rangle
$$

where $\langle, \rangle$ indicates the pairing between a map and its argument e.g. the covertex $v^3$ is a function that takes the vertex $v_3$ to 1 and the rest of the vertices to zero. Finite sums of these $v^i, e^i$ or $f^l$ are called cochains.

The coboundary of a co-vertex contains the co-edges incident on it with the appropriate sign, the coboundary of a co-edge has the co-faces bordering it, and the coboundary of a co-face is defined to be zero. In figure
\[ \delta v^6 = -e^2 - e^3 - e^5, \quad \delta e^5 = f^2 - f^3. \]

The requirement in eq. (3.1) implies that, like the boundary, the coboundary also squares to zero,

\[ C_2 \xleftarrow{\delta} C_1 \xleftarrow{\delta} C_0 \xleftarrow{\delta} 0. \]

and

\[ \delta \circ \delta(v^6) = \delta(-e^2 - e^3 - e^5) \]
\[ = -(f^2 + f^1) - (f^3 - f^1) - (f^2 - f^3) = 0 \]

Note that while the boundary decrements the ‘degree’ by one, the coboundary does exactly the opposite.

Cohomology groups are given by the quotients,

\[ H^* = \frac{\ker(\delta : C^* \to C^{*+1})}{\im(\delta C^{*-1} \to C^*)}, \]

read as ‘cocycles modulo coboundaries’.

In two dimensions, it is well known result that all possible orientable manifolds may formed by attaching handles to a sphere. \(^2\) The orientable surface \(\Sigma_g\) has genus (number of handles) \(g\). Its homology and cohomology groups are,

\[ H_*(\Sigma_g, \mathbb{Z}) = H^*(\Sigma_g, \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{for } * = 0 \\
\mathbb{Z}_k^{2g} & \text{for } * = 1 \\
\mathbb{Z}_k & \text{for } * = 2 
\end{cases} \]

As stated earlier, these groups are topological and do not depend on how a space is divided up into simplices.

All of the above may be generalized to higher dimensions, where the chain complex is expanded to include \(i\)-cells, \(i = 1 \cdots d\) and likewise for the cochains. The maps \(\partial\) and \(\delta\) are defined as above, as are the homology/cohomology groups \(H_*\) and \(H^*\).

**Poincare Duality**

For a \(n\)-dimensional closed orientable manifold \(M\), **Poincare duality** is an important isomorphism between homology and cohomology. It says,

\[ H^k(M) \cong H_{n-k}(M) \]

\(^2\)The most direct proof of this fact is Conway’s ZIP (zero irrelevancy proof!) [22]
and holds for all choices of coefficients. For example, the respective groups of the n-holed torus $\Sigma_g$ (footnote 2) clearly satisfy this relation.

Classically, Poincare duality was thought of in terms of the dual lattice. In two dimensions one puts vertices at the center of the faces and connects them with edges. $i$-chains in the dual complex are in one to one correspondence with $2 - i$ cochains in the original complex e.g. vertices are matched to dual faces. There is, however, a technicality that the dual of a triangulation is not a triangulation in $d > 2$ (a subdivision is needed). We will avoid this issue by only using cubic lattices in where required in three and higher dimensions.

It bears repeating that Poincare duality too, is independent of simplicial details. In fact, it does not even require a triangulation, applying even to non-triangulable manifolds like the four-dimensional $E_8$.

### 3.3 The Hamiltonian

Having set up the machinery of chain complexes, we can now put it to work. We claim that:

*The toric code model is a correspondence between the data of a chain complex and a set of operators.*

and spend the rest of this section trying to justify this, rather abstract, statement. As in the previous section, we start with two spatial dimensions.

The $\mathbb{Z}_k$ rotors, $\sigma$ and $\tau$ are unitary matrices such that,

$$
\tau \sigma = w \sigma \tau, \quad w \equiv e^{2\pi i/k}
$$

$$
\tau^k = \sigma^k = 1
$$

(3.2)

The first relation also implies

$$
\tau \sigma^\dagger = \bar{w} \sigma^\dagger \tau
$$

$$
\tau^\dagger \sigma^\dagger = w \sigma^\dagger \tau^\dagger.
$$

(3.3)

Note that $\sigma^\dagger = \sigma^{-1} = \sigma^{k-1}$ and similarly for the $\tau$.

One possible representation is given by,

$$
\sigma = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & w & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & w^{k-1}
\end{pmatrix}, \quad \tau = \begin{pmatrix}
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
1 & 0 & \ldots & 0
\end{pmatrix}
$$

35
but we will only require the relations in eqs. (3.2) and (3.3). The original Kitaev model [19] took the case $k = 2$ where $\sigma$ and $\tau$ were the Pauli matrices $\sigma_z$ and $\sigma_x$ respectively. In two dimensions, we attach a $k$ dimensional space (the ‘spins’) to every edge in the chain complex. The total Hilbert space is the tensor product of these vector spaces and has dimension $k^E$ for $E$ edges. Our operator correspondence $O$ takes 1-chains and 1-cochains to products of operators,

$$O \left( \sum_j \alpha_j e_j \right) = \bigotimes_j \tau_j^{\alpha_j}, \quad O \left( \sum_j \beta_j e^j \right) = \bigotimes_j \sigma_j^{\beta_j}$$  \hspace{1cm} (3.4)$$

Here $e_i$ are the individual edges, $e^i$ their duals, and the coefficients $\alpha_i$ lie in $\mathbb{Z}_k$. The operators $\tau_j$ and $\sigma_j$ act on the vector space associated to the edge $j$. In the following, we omit the tensor symbol, it being understood that operators on different edges commute and those on the same edge obey eq. (3.2).

The pairing between chains and cochains, eq. (3.1), is mirrored by the commutation of the corresponding operators.

$$O(c_1)O(c^2) = e^{i\phi} O(c^2)O(c_1), \quad \phi = \frac{2\pi}{k} (c^1, c_1).$$  \hspace{1cm} (3.5)$$

To understand eq. (3.10), observe that on the right side $\langle c^2, c_1 \rangle$ counts the edges that are common to $c_1$ and $c^2$ with the appropriate orientation. These common edges are exactly where a $\sigma$ goes past a $\tau$ and produces a phase according to eqs. (3.2) and (3.3).

Next, we define vertex and face operators,

$$O(\delta v_i) = \prod_j \sigma_j^{\epsilon_{ij}}, \quad O(f_i) = \prod_j \tau_j^{\epsilon_{ji}}.$$  \hspace{1cm} (3.6)$$

For a vertex $v_i$, the product contains the the edges in its coboundary i.e. all the edges that are incident to the $v_i$. We use the convention that $\epsilon_{ij} = \pm 1$ as the edge $i$ points in or out into the vertex $j$. Similarly the operator for a face $f_i$ contains edges bounding $f_i$, with $\epsilon_{ji} = 1$ if the edge $i$ is consistent with the counterclockwise orientation of face $j$ and $-1$ otherwise.

For example, in figure 3.2

$$O(\delta v^6) = \sigma_2^{-1} \sigma_3^{-1} \sigma_5, \quad O(\partial f_3) = \tau_3 \tau_5^{-1} \tau_6^{-1} \tau_7^{-1}$$  \hspace{1cm} (3.7)$$
The Hamiltonian is a sum over all faces and vertices in the chain complex,

\[
\mathcal{H} = - \sum_{v \in \text{vertices}} O(\delta v) - \sum_{k \in \text{faces}} O(\partial f_k) + \text{h.c.} \tag{3.8}
\]

The hermitian conjugate is necessary for \( k > 3 \), and ensures that the orientation convention is irrelevant.

Finally, we have a geometric interpretation for the essential relation that makes the Kitaev model exactly solvable. Given any vertex \( v \in C^0 \) and face \( f \in C^2 \), the vanishing of the pairing

\[
\langle \delta v, \partial f \rangle = \langle v, \partial^2 f \rangle = 0 \tag{3.9}
\]

means that

\[
O(\delta v)O(\partial f) = O(\partial f)O(\delta v) \tag{3.10}
\]

All vertex and face operators commute with each other (check the example in eq. (3.7)).

It is clear from equations (3.8) and (3.10), that the ground state must be a simultaneous eigenstate of each vertex and face operator with eigenvalue 1 i.e.

\[
O(\partial f)|GS\rangle = |GS\rangle \quad \text{and} \quad O(\delta v)|GS\rangle = |GS\rangle
\]

for all faces \( f \) and vertices \( v \). In other words, it is invariant under multiplication by boundaries and coboundaries.

From eq. (3.10), it follows that one such state is,

\[
|GS\rangle = \prod_i \left( \sum_{m=0}^{k-1} O(\partial f_i)^m \right) \left| \sigma_j = 1, \forall j \right\rangle = \sum_{c \in C^2} O(\partial c)\left| \sigma_j = 1, \forall j \right\rangle. \tag{3.11}
\]

The state being acted on, is the tensor product of \(+1\) eigenstates of the \( \sigma \) operator on each edge. In the second sum, \( \partial c \) ranges over all boundaries — this is the familiar “sum over contractible loops” that is common to several models of topological order [23].

We could also write down a ground state as sum over coboundaries.

\[
|GS\rangle = \prod_i \left( \sum_{m=0}^{k-1} O(\delta v)^m \right) \left| \tau_j = 1, \forall j \right\rangle = \sum_{c^0 \in C^0} O(\delta c^0)\left| \tau_j = 1, \forall j \right\rangle. \tag{3.12}
\]

In the presence of degeneracy, eq. (3.11) and eq. (3.12) are just two possible ground states, which are not
the same. In the following arguments, one could make either choice, switching \( \tau \) and \( \sigma \) as needed.

**Wilson Algebra** Let \( c_i \) and \( c^j \) be a closed chain and cochain respectively i.e. \( \partial c_i = 0 \) and \( \delta c^j = 0 \). Their corresponding operators commute with the Hamiltonian, since for each term in eq. (3.8),

\[
\langle \delta v, c_i \rangle = \langle v, \partial c_i \rangle = 0, \quad \langle c^j, \partial f \rangle = \langle \delta c^j, f \rangle = 0
\]

Ground state degeneracy arises from the action of terms that are *not* boundaries or coboundaries, being non-trivial classes in the first homology or cohomology, \( H_1 \) or \( H^1 \), respectively. Note that these are not included in the products in eq. (3.11) and eq. (3.12) and in contrast with eq. (3.9), they might intersect i.e. \( \langle c_i, c^j \rangle \neq 0 \). Then the operator correspondence says,

\[
O([c_i])O([c^j]) = w^{\delta [c^j] - [c_i]} O([c^j])O([c_i]),
\]

(3.13)

where the square brackets indicate that only the homology or cohomology *class* matters for the intersection i.e. \( [c_i + \partial c] = [c_i] \) and \( [c^j + \delta c'] = [c^j] \).

It is a familiar fact from quantum mechanics that a set of operators that commute with the Hamiltonian, but not with each other, leads to degeneracy. Define \( \Omega_i \equiv O([c_i]) \) and \( \Omega^j \equiv O([c^j]) \). Since these are the analogues of Wilson loops and co-loops in field theory, we refer to them and their commutation relations, as the *Wilson Algebra*.

**Note:** When a system has a continuous symmetry, then the associated generators commute with the Hamiltonian and can be exponentiated to give the action of the group. However our Wilson loop operators defined above, have finite order, \( \Omega^k = 1 \), so we clearly cannot get a continuous group. This shows that we are working with a \( \mathbb{Z}_k \), not \( U(1) \), gauge theory.

**Ground state degeneracy** : The ground state Hilbert space forms a *representation* of the Wilson Algebra; the dimension of this representation is the ground state degeneracy.

**Example:** Torus with genus \( g - \Sigma_g \). There are \( 2g \) generators in \( H_1(\Sigma_g) \) and \( H^1(\Sigma_g) \), denoted by \( \Omega_i \) and \( \tilde{\Omega}_i \) respectively, which obey the relations

\[
\Omega_i \tilde{\Omega}_j = w^{\delta \Omega_j \Omega_i} \quad i, j = 1 \ldots 2g
\]
A faithful representation for $\Sigma_1$ is given by,

$$\Omega_1 = \sigma \otimes 1, \quad \Omega_2 = 1 \otimes \sigma$$

(3.14)

and has dimension $k^2$. For the $g$ holed torus, we just take the $g$-fold tensor product of the above representation. The ground state degeneracy is hence $k^{2g}$, a well-known result [19].

### 3.3.1 Excitations and Braiding Statistics

There are two kinds of excitations, corresponding to chains and cochains that are not closed. Their operators do not commute with the Hamiltonian eq. (3.8). Instead we have,

$$O(\delta v)O(c_i) = w^{(v, \partial c_i)}O(c_i)O(\delta v), \quad O(\partial f)O(c^j) = w^{(c^j, f)}O(c^j)O(\partial f)$$

(compare with eq. (3.10)).

If we take,

$$|c_i\rangle = O(c_i)|\text{GS}\rangle \quad |c^j\rangle = O(c^j)|\text{GS}\rangle$$

to be the excited states created by their action on (any) ground state, then the energies are given by,

$$H|c_i\rangle = \left[ E_0 + 2 \sum_{v} \cos \left( \frac{2\pi (v, \partial c_i)}{k} \right) \right] |c_i\rangle$$

(3.16)

$$H|c^j\rangle = \left[ E_0 + 2 \sum_{f} \cos \left( \frac{2\pi (\delta c^j, f)}{k} \right) \right] |c^j\rangle$$

(3.17)

where $E_0$ is the energy of the ground state. Every non-zero term in the above sums arise from a charge that sits on the end of a string. In the literature, chain excitations that end on a vertex operators, have been called $e$ (for electric) and the cochain excitations, $m$ (for magnetic). I adopt the same nomenclature here. However it should be kept in mind that these are purely formal terms — there is an exact duality between the charges and no external gauge field couples to them.

Given a state with some configuration of charges and attached strings, we can apply an adiabatic transformation $^3$ by multiplying chain/cochain operators $b_{i_a}$ in a discrete, time-ordered sequence. We want to focus

---

$^3$Here adiabatic means that quantum numbers (total numbers of $e$ and $m$) stay constant from one time step to the next. The energy might jump e.g. by single $\tau$ operator creates an $e, -e$ pair on the ends of the edge, with total energy $4 \cos 2\pi / k$. 

---
Figure 3.2: Examples of excitations. On the left is a chain or $e$ type, on the right a cochain or $m$ type (note the direction of the blue arrows). With the choice of orientation for the edge shown at top left, the $\sigma$ and $\tau$ operators for each edge have been marked. The strings end with charges on the vertices/faces marked in green.

on transformations that return a configuration to itself so that the pieces add up to a (co)-boundary,

$$\partial \left( \sum_n b_1(t_n) \right) = 0, \quad \delta \left( \sum_n b^1(t_n) \right) = 0.$$ (3.18)

In operator form we have

$$U(t_N) = \left[ \mathcal{O}(b_1(t_N)) \mathcal{O}(b^1(t_N)) \right] \left[ \mathcal{O}(b_1(t_{N-1})) \mathcal{O}(b^1(t_{N-1})) \right] \ldots \left[ \mathcal{O}(b_1(t_1)) \mathcal{O}(b^1(t_1)) \right]$$ (3.19)

where we have chosen the convention that at each time step the chain operator acts after the cochain, and is hence written to the left.

(3.19) can be rearranged to

$$U(t_N) = w^{\text{link}(b_1(t), b^1(t))} \mathcal{O} \left( \sum_n b_1(t_n) \right) \mathcal{O} \left( \sum_n b^1(t_n) \right).$$ (3.20)

From (3.18), the operators in the above expression must act trivially on the ground state. We are then left with a Berry phase,

$$U(t_N)|GS\rangle = w^{\text{link}(b_1(t), b^1(t))}|GS\rangle$$

At the lattice level, $\text{link}(b_1(t), b^1(t))$ counts the number of times a $\sigma$ operator has to cross a $\tau$ when disas-
Figure 3.3: An $m$ charge (blue string) encircles an $e$ (vertex marked in green). The strings intersect at the circled edge and we pick up a phase $w = e^{2\pi i/k}$.

Sociate them into loops,

$$\text{link}(b_1, b_1^l) = \sum_n \sum_{m>n} \langle b_1(t_m), b_1(t_n) \rangle$$  \hspace{1cm} (3.21)

Topologically it may be interpreted as the *linking number* of the one-dimensional loops $b_1(t)$ and $b_1^l(t)$ in $2 + 1$ dimensions. It is clear that, the Berry phase from the cyclic evolution of a set of purely $e$ or $m$ charges must be trivial, since their operators commute. Fig. 3.3 shows an example of an $m$ charge circling around an $e$. The same phase, $w$ when an $e$ goes around an $m$ in the same direction — physically this is just a change of reference frame which can’t affect geometric phases. Thus, we have established the following:

Now we consider the exchange of two identical particles (exchange of unlike particles is undefined). Of course this is trivial for two $e$’s or two $m$’s. But, as fig. 3.5 shows, exchanging two $\psi = e^{\mu}$ particles does result in a phase $w$. In other words, $\psi$ has fractional statistics (it is a fermion for $k = 2$).

### 3.4 The toric code in higher Dimensions

In dimensions $d > 2$, we have the freedom to put our “spins” on simplices of dimension $1 \leq n \leq (d - 1)$. Even with the same underlying manifold, each $n$ gives rise to a distinct model. For $d = 2$, we were forced to
Figure 3.4: Braiding rules for anyons in the toric code. Individually $e$ and $m$ are bosons, but they obey *semionic* statistics as shown.

Figure 3.5: On the left: a pair of $\psi$ particles have been exchanged and the loops closed thereafter. Since each $e,m$ pair is to be regarded as a single unit, we time-order so that $\psi_2$ comes before (below) $\psi_1$. On the right: after a rearrangement the linking between the red loop and blue loop becomes clear.
choose \( n = 1 \) - we notate this by \( H_1^2 \). In three dimensions we have two choices,

\[
H_1^3 = - \sum_{v \in \text{vertices}} O(\delta v) - \sum_{f \in \text{faces}} O(\partial f) + h.c. \tag{3.22}
\]

\[
H_2^3 = - \sum_{e \in \text{edges}} O(\delta e) - \sum_{b \in \text{volumes}} O(\partial b) + h.c. \tag{3.23}
\]

The first is just a direct generalization of the 2D model. In the second, the degrees of freedom are placed on 2-simplices (faces), and the Hamiltonian contains both edge and volume terms. These two models are actually dual to each other — in general this is true of the pair \( H_n^d \) and \( H_{d-n}^d \) and has the following consequences.

1. In the dual lattice construction defined earlier, \( n \)-simplices intersect \((d-n)\)-simplices and the respective spins can be identified. However, as mentioned previously, this only works in general for cubic lattices.

2. For any lattice, Poincare duality implies that \( H_n^d \) and \( H_{d-n}^d \) have the same ground state degeneracy for any manifold (proved below).

3. The two models have excitations of the same dimension, but with the charge and flux labels reversed (see next section).

In even dimensions, a self-dual model may be constructed if we put spins on \( d/2 \)-simplices. This is the only case that we will consider in \( d = 4 \).

\[
H_2^4 = - \sum_{e \in \text{edges}} O(\delta e) - \sum_{b \in \text{volumes}} O(\partial b) + h.c. \tag{3.24}
\]

### 3.4.1 Wilson Algebra

Our formalism of (co)-chain complexes and corresponding operators might have seemed like overkill in two dimensions, but it begins to show its worth in \( d > 2 \). Consider a general model \( H_n^d \) defined on a simplicial complex \( M \), which contains boundaries of \( n+1 \)-chains and coboundaries of \( n-1 \) cochains. We have a direct analogue of the ground state defined in eq. (3.11),

\[
|GS\rangle = \prod_i \left( \sum_{m=0}^{k-1} O(\partial b_i)^m \right) |\sigma_j = 1, \forall j\rangle = \sum_{c \in C^{n+1}} O(\partial c) |\sigma_j = 1, \forall j\rangle. \tag{3.25}
\]
Figure 3.6: Curve A intersects both curve B and curve C at the marked points. The intersection numbers are 1 and 3 respectively.

and similarly for eq. (3.12). Next, we take the non-trivial closed chains in $H_n(M)$ and cochains in $H^n(M)$. The pairing between the two determines the the Wilson algebra,

$$
O([c_1])O([c_2]) = w([c_1],[c_2])O([c_2])O([c_1]), \quad \text{for } [c_1] \in H_n(M) \text{ and } [c_2] \in H^n(M)
$$

with the ground state Hilbert space forming an irreducible representation, as before. We will adapt the notation $\Omega^i, \Omega^j$ and generically call them Wilson surfaces.

**Poincare Duality and Intersection**

Since we have used the term ‘intersection’, it is natural to ask what curves or surfaces, if any, are intersecting each other. Recall that Poincare duality sends a cohomology class $[c^1] \in H^n(M)$ to a homology class $[c_1] \in H_{d-n}(M)$. Take two representatives $c_1$ and $c_2$ in these two classes — they are $n$ and $d-n$ dimensional surfaces respectively. Then after perturbing $c_1$ and $c_2$ if necessary, they intersect transversely at discrete points (transverse means that their tangent planes don’t intersect). The intersection number counts these points with a sign that depends on orientation. Duality says that, in the absence of torsion (to be discussed later) the intersection and the pairing are one and the same. Formally we have a commutative diagram,

$$
\begin{array}{ccc}
H_n(M,\mathbb{Z}) \otimes H^n(M,\mathbb{Z}) & \xrightarrow{\text{pairing}} & \mathbb{Z} \\
\downarrow & & \downarrow \\
H_n(M,\mathbb{Z}) \otimes H_{d-n}(M,\mathbb{Z}) & \xrightarrow{\text{intersection}} & \mathbb{Z} \\
\end{array}
$$

with $\mathbb{Z}_k$. 


3.4.2 Ground State Degeneracy in three and four dimensions

**Example:** $S^2 \times S^1$ This is the simplest non-trivial example in 3D. The algebra runs,

$$\Omega \tilde{\Omega} = w \tilde{\Omega} \Omega$$

and is represented irreducibly by,

$$\Omega = \sigma, \quad \tilde{\Omega} = \tau$$

so the ground state degeneracy is $k$.

**Example:** $T^3$ This is the space obtained by identifying opposite faces of a cube. The algebra is,

$$\Omega_i \tilde{\Omega}_j = w^{\delta_{ij}} \tilde{\Omega}_j \Omega_i \quad i, j = 1 \ldots 3$$

and has the irreducible representation,

$$\Omega_1 = \sigma \otimes \mathbb{1} \otimes \mathbb{1}, \quad \Omega_2 = \mathbb{1} \otimes \sigma \otimes \mathbb{1}, \quad \Omega_3 = \mathbb{1} \otimes \mathbb{1} \otimes \sigma$$

$$\tilde{\Omega}_1 = \tau \otimes \mathbb{1} \otimes \mathbb{1}, \quad \tilde{\Omega}_2 = \mathbb{1} \otimes \tau \otimes \mathbb{1} \quad \tilde{\Omega}_3 = \mathbb{1} \otimes \mathbb{1} \otimes \tau$$

(3.27) (3.28)

The degeneracy is $k^3$ in agreement with [24].

**Example:** $\mathbb{R}P^3$ The three-dimensional real projective space is formed by taking the solid ball in $\mathbb{R}^3$ and identifying antipodal points to be identical. It is also the underlying topological space of the 3D rotation group $SO(3)$. Since the space is orientable we apply Poincare duality and find,

$$H_1(\mathbb{R}P^3, \mathbb{Z}_k) = H^2(\mathbb{R}P^3, \mathbb{Z}_k) = \mathbb{Z}_2$$

for even $k$

$$= 0$$

for odd $k$

(3.29) (3.30)

There is no degeneracy for odd $k$ and a twice degenerate ground state for even $k$.

**Example:** $L^p_q$ Take the solid ball $B^3$ and divide its equator into $p$ equal arcs mark longitudes joining these to the northern and southern hemisphere. Now identify the lower and upper hemisphere by rotating the lower by an angle $2\pi q/p$, where $p$ and $q$ are coprime, and reflecting it in the equator. The result is the **Lens Space** $L^p_q$. For example $L^2_1$ is just the projective space in the previous example.
It can be shown from the universal coefficient theorem [25], that for prime $p$

$$H_1(L^p_q,\mathbb{Z}_k) = H^2(L^p_q,\mathbb{Z}_k) = \mathbb{Z}_p \text{ if } k \text{ divides } p \quad (3.31)$$

$$= 0 \quad \text{otherwise} \quad (3.32)$$

It follows that the ground state degeneracy is $p$ or 0 for the two cases.

**Four Dimensions:** In 4D we only consider the self-dual model $\mathcal{H}_4^4$ defined above. For a closed manifold $M$, ground state degeneracy is determined by the pairing of $H_2(M)$ and $H^2(M)$, also known as the *intersection form*. Poincare duality implies that these two groups are the same. Intuitively they are comprised of two dimensional surfaces that do not bound three dimensional volumes.

**Example:** $S^2 \times S^2$ There are two generators $S^2 \times \text{pt. and pt.} \times S^2$ which intersect once. Clearly this is the same algebra $T^2 = S^1 \times S^1$ and

$$\Omega_i \tilde{\Omega}_j = w^{ij} \tilde{\Omega}_j \Omega_i.$$ 

The degeneracy is $k^2$.

**Example:** $\mathbb{C}P^2 \mathbb{C}P^2$ is formed by taking $\mathbb{C}^3 - 0$ and identifying points that differ by a rescaling,

$$(z_1, z_2, z_3) \sim (\lambda z_1, \lambda z_2, \lambda z_3).$$

Unlike the previous examples, there is a single homologically non-trivial 2D surface, but it *self-intersects* once. The algebra is

$$\Omega \tilde{\Omega} = w \tilde{\Omega} \Omega$$

and as above, this space also has degeneracy $k$.

### 3.4.3 Topological Field Theory for the toric code

The 2D toric code has been described by a doubled Chern-Simons theory, with two fields $a_{1,2}$ and the action,

$$S_0 = \frac{k}{2\pi} \int a_1 \wedge da_2 = \frac{k}{2\pi} \int d^3x \epsilon^{\mu\nu\rho} a_1^\mu(x) \partial_\nu a_2^\rho(x) \quad \mu, \nu, \rho = 0, 1, 2 \quad (3.33)$$

In higher dimensions, it is natural to extend this to a BF theory,

$$S_0 = \frac{k}{2\pi} \int a_1 \wedge da_2 \quad (3.34)$$
where $a_1$ are $a_2$ are $p$ forms and $(n - p - 1)$ forms respectively. To make this connection precise, we must prove that the above action produces the correct amplitude for the braiding of excitations.

We start with the $(2 + 1)$D case where the excitations are both pointlike. The fields couple to currents,

$$S = \int \epsilon_{\mu\nu\rho} a_1^\mu \partial_\nu a_2^\rho + J_\mu a_1^\mu + K_\mu a_2^\mu \quad \mu, \nu, \rho = 0, 1, 2$$

where $J$ and $K$ describe the worldlines of the two particles. A gauge is chosen such that $\partial_\mu J_\mu = \partial_\mu K_\mu = 0$, and the currents are conserved.

Carrying out the path integral formally,

$$Z = \int D[a_1] D[a_2] \exp(iS) \quad (3.35)$$

$$= Z_0 \exp(-i \int d^3x d^3y J_\mu(x) A_{\mu\nu}^{-1}(x, y) K_\nu(y)) \quad (3.36)$$

where the operator $A$ is defined as,

$$A_{\mu\nu} = \frac{k}{2\pi} \epsilon_{\mu\nu\rho} \partial_\rho$$

Note that $A$ is not positive-definite and consequently we cannot do without the factor of $i$ in the path integral.

With our choice of gauge the Green function is,

$$A_{\mu\nu}^{-1}(x, y) = \frac{2\pi}{k} \frac{1}{4\pi} \epsilon_{\mu\nu\rho} \frac{1}{|x - y|} \quad (3.37)$$

Now we need an explicit form for the currents. Let the two world lines $C_1$ and $C_2$. We define the currents,

$$J_\mu(x) = \int_{C_1} dy_\mu \delta^3(x - y), \quad K_\mu(x) = \int_{C_2} dy_\mu \delta^3(x - y)$$

Putting everything into eq. 3.36,

$$Z = Z_0 \exp \left(-\frac{2\pi}{k} \int d^3x d^3y J_\mu(x) \frac{1}{4\pi} \epsilon_{\mu\nu\rho} \left( \partial_\nu \frac{1}{|x - y|} \right) K_\nu(y) \right) \quad (3.38)$$

$$= Z_0 \exp \left(-\frac{2\pi}{k} \int_{C_1} dx^\mu \int_{C_2} dy^\nu \frac{1}{4\pi} \epsilon_{\mu\nu\rho} \frac{(x - y)^\rho}{|x - y|^3} \right) \quad (3.39)$$
In the integrand, we recognize the expression for linking number of the two closed loops $C_1$ and $C_2$,

$$Z = Z_0 \exp \left( -i \frac{2\pi}{k} \text{Link}(C_1, C_2) \right)$$  \hspace{1cm} (3.40)

**Linking Number**

**Motivation:** Consider two loops carrying unit current, $C_1$ and $C_2$. By the Biot-Savart law, the magnetic field from the first loop is,

$$B(x) = \frac{1}{4\pi} \oint_{C_1} \frac{(x - y) \times dy}{|x - y|^3}.$$  

Integrating around the second loop,

$$\oint_{C_2} B(x) \cdot dx = \frac{1}{4\pi} \oint_{C_1} \oint_{C_2} \frac{(y - x)}{|y - x|^3} \cdot (dx \times dy)$$  \hspace{1cm} (3.41)

But by Ampere’s law, this counts the number of times the loop $C_1$ intersects a surface bounded by $C_2$, which implies that integral in eq. 3.41 must be an integer.

A topological view of eq. 3.41 is that it gives the degree of the map $f$,

$$f : C_1 \times C_2 \Rightarrow S^2, \quad f(x, y) = \frac{x - y}{|x - y|}$$  \hspace{1cm} (3.42)

The degree counts the number of times the image of $f$ wraps around the sphere as the arguments are varied. Observing the area form in the integrand in eq. 3.41, we may guess the expression for the linking number in $n$ dimensions. Given surfaces of $p$ and $n - p - 1$ dimensions,

$$\text{Link}(C_1^p, C_2^{n-p-1}) = \frac{1}{\text{Vol}(S^{n-1})p!(n - p - 1)!} \int dx^{\mu_1} \cdots dx^{\mu_p} \int dy^{\nu_1} \cdots dy^{\nu_{n-p-1}} \epsilon_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_{n-p-1}} \frac{(y - x)^p}{|y - x|^n}.$$  \hspace{1cm} (3.43)

Returning to our BF theory, we first recast the currents as antisymmetric tensors,

$$\tilde{J}_{\mu \nu} = \epsilon_{\mu \nu \rho} J^\rho \quad \tilde{K}_{\mu \nu} = \epsilon_{\mu \nu \rho} K^\rho,$$

which have an obvious generalization to higher dimensions. The Green function becomes,

$$A_{\mu \nu}^{-1}(x, y) = \frac{2\pi}{k} \frac{1}{\text{Vol}(S^{n-1})p!(n - p - 1)!} \epsilon_{\mu \nu \cdots \rho} \frac{1}{|x - y|^{n-2}}$$

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and we get the expression for the amplitude (eq.3.39) with the higher dimensional linking number (eq.3.43).
Chapter 4

Twist Defects

A twist defect acts on abelian anyons by permuting their labels. We will only be concerned with defects that may be created on the lattice. The first examples of these were dislocations in the Wen chessboard model (equivalent to the toric code), introduced by Bombin [26] and later treated in more detail by Wen [27]. Attached to every dislocation is a branch cut, crossing which turns an $e$ particle into an $m$ and vice-versa. Pairs of dislocations on a sphere effectively create non-trivial Wilson loops and ground state degeneracy. Braiding defects takes the system from one ground state to another and is a non-abelian operation.

First, I describe dislocation defects in the toric code. To keep the lattice model exactly solvable, new terms have to be added along the branch cut and at the dislocation sites. I also discuss Wilson loops and degeneracy. There is an important subtlety regarding the two distinct types of “double-loops” around these defects, that has thus far been ignored in the literature.

I shall then present an attempt to generalize the idea of a group defect that permutes a set of anyons while preserving their mutual braiding. Unlike non-abelian anyons in the fractional hall effect, defects are not quantum states — there is no creation operator attached to a defect and one cannot superpose them. This also sets them apart from the notion of non-abelian fluxes in $\mathbb{Z}_k$ gauge theory. However certain properties of anyons, such as fusion and braiding, do carry over. I discuss general formulae for the intersection of two Wilson loops and the quantum dimension of a defect. I also distinguish a class of defects (including the toric code) that “twist” a particle with non-trivial exchange statistics.

Finally I describe another exactly solvable anyonic model on the honeycomb lattice. This model has a larger set of excitations and the group of defects is $S_3$ which is non-abelian. The defects can be realized explicitly at the lattice level and provide a concrete instance of the general theory. I close with some remarks on the notion of defects as a probe to resolve finer details of an anyonic system.
4.1 Twists in the toric code — Dislocations

For this section, we put the toric code on a square lattice. A dislocation fig. 4.1 creates a branch cut that mixes terms in the chain complex. At the operator level, we find that appropriate terms (defined below) need to be added to the hamiltonian, such that they all commute and the system is still exactly solvable.

![Figure 4.1: A pair of dislocations on a square lattice, each one displaces the lattice vertically by half a spacing, in opposite directions.](image)

**Terms along the branch cut:** As shown in the figure, along the cut some vertical edges (shown dashed) are missing. We view them as having been deleted and rotated by $\pi/2$ anti-clockwise, and impose the rule that a $\tau$ becomes a $\sigma$. Thus the original face terms are now hybrids *viz. *,

\[
\begin{align*}
O(\partial f_1) &= \tau_7\tau_6\tau_1\sigma_5 + \text{h.c.} \\
O(\partial f_2) &= \tau_5\tau_8\tau_9\sigma_5 + \text{h.c.} \\
O(\partial f_3) &= \tau_1\tau_{10}\tau_7\sigma_9 + \text{h.c.} \\
O(\partial f_4) &= \tau_9\tau_{13}\tau_{12}\sigma_{11} + \text{h.c.}
\end{align*}
\] (4.1)
At the ends of the cut sit a pair of dislocations that have special 6 operator terms,
\[
P_1 = \tau_2 \tau_3 \tau_4 \tau_5 \tau_1 \sigma_1^\dagger + \text{h.c.}
\]
\[
P_2 = \tau_1^\dagger \tau_4^\dagger \tau_5 \tau_{11} \tau_{13} \sigma_{13} + \text{h.c.}
\]
(4.2)

Note the difference between (4.1) and (4.2). Terms in the first set still have a single operator on each edge but in the second, both terms have a \(\sigma\) and \(\tau\) together on some edge (\(e_{11}\) and \(e_{13}\) respectively). Using our commutation relations
\[
\sigma \tau = w \tau \sigma \Rightarrow \sigma \tau^\dagger = \bar{w} \tau^\dagger \sigma
\]
\[
\Rightarrow \sigma^\dagger \tau^\dagger = w \tau^\dagger \sigma^\dagger, \quad w = e^{2\pi i / k}.
\]
it is easy to check that eqs. (4.1) and (4.2) all commute with each other as well as with the other face and vertex terms surrounding them.

**Ground State**  Analagous to eq. (3.11) we can try to write down a ground state
\[
|\text{GS}\rangle = \prod_j \left( \sum_{m=0}^{k-1} (P_j)^m \right) \left( \sum_{m=0}^{k-1} O(\partial f_i)^m \right) |\sigma_j = 1, \forall j\rangle
\]
(4.3)

where the face terms \(O(\partial f_i)\) range over all faces as well the hybrids in eq. (4.1) and we have also multiplied by the projectors for the end plaquettes \(P_j\). Since we need, \(P_j |\text{GS}\rangle = |\text{GS}\rangle\), from the above expression we also require \((P_j)^k = 1\). But
\[
(P_j)^k = (\sigma \tau)^k = w^{k(k-1)/2} \sigma^k \tau^k = w^{-k/2} = -1.
\]

However, as noted by Wen [27], we can fix this by changing the definition of \(P_j\) to \(e^{\phi}P_j\) where,
\[
\phi = \frac{2\pi i}{k} \times \begin{cases} 
\text{half-integer}, & \text{if } k \text{ even.} \\
\text{integer}, & \text{if } k \text{ odd.}
\end{cases}
\]
(4.4)

Of course this does not change the commutation relations, and the ground state state (4.3) is now valid.

Dislocations transmute charges. When an \(e\) charge enters the branch cut, commutation with (4.1) forces it to change to an \(m\). We conclude that the charge and flux labels are now not defined defined globally, although locally the distinction between them as well as the braiding rules in fig. 3.4 are still valid.
Ground state degeneracy  Let us start with the sphere $S^2$ and introduce dislocations in pairs, since we want the branch cuts to terminate. For more than two defects, we find Wilson loops which intersect non-trivially with each other and hence do not commute.

fig. 4.3 illustrates one choice of basis for Wilson loops. Let the loop $W_i$ enclose the defects $i, i+1$ as shown, where $1 \leq i \leq 2N-2$ for $2N$ defects. The range of $i$ is explained by the fact that we are on a sphere, so that a loop enclosing all the defects can be contracted ‘at the back’ and is thus trivial. This implies that $W_{N-1}$ and $W_N$ can be deformed to a sum of the other loops.

From the figure it is clear that only adjacent loops intersect. The intersection is given by,

$$I(W_i, W_j) = \begin{cases} 
1, & j = i + 1 \\
-1, & j = i - 1 \\
0, & \text{otherwise}.
\end{cases}$$

By definition the Wilson Algebra is given by operators $O(W_i)$ such that,

$$O(W_i)O(W_j) = \exp \left\{ \frac{2\pi i}{k} I(W_i, W_{i+j}) \right\} O(W_j)O(W_i).$$ (4.5)

We can find a common eigenstate of the odd $W_i$, which form a maximally commuting subset. Then acting by arbitrary powers of the even $W_i$ gives a total of $k^{N-1}$ states which is the degeneracy.
Figure 4.4: The two double loops that cannot be deformed into each other. They have different \textit{writhes} \(\pm 1\) and the eigenvalues of the operators differ by a factor of \(w\).

Alternatively, by manipulating the basis for the first few \(N\), it becomes clear that \(I\) can be brought into a block diagonal form so that members of each block only intersect each other with number \(\pm 1\). For example if \(2N = 8\), we can take \((W_1, W_2), (W_1 + W_2, W_4)\) and \((W_6, W_7)\). This is just the intersection form for a \(2N - 2\) holed torus and suggests that we can treat fig. 4.3 as a double-sheeted Riemann surface of genus \(2N - 2\). We also have the same degeneracy count once we remember that there is only one kind of excitation in the covering space. Since the gluing procedure has already been described by Wen and the next chapter describes a more complicated version, I omit further details here.

For non-abelian anyons, the quantum dimension \(D\) is defined such that the degeneracy goes as \(D^M\) when the number of anyons \(M\) is large. Applying the same definition here, one finds that the quantum dimension of a defect to be \(\sqrt{k}\).

\textbf{Twists} It is interesting to consider what happens when a charge encircles a defect twice and closes back on itself. We haven’t counted such double loops in our Wilson algebra because, being contractible to an arbitrarily small neighbourhood of a defect, it is obvious that they can’t intersect anything. Hence the ground state may always be taken to be an eigenstate of the associated operator, but there is no reason for the eigenvalue to be 1. In fact, working at the lattice level we see that the eigenvalue is exactly the phase \(e^{i\phi}\) or \(we^{i\phi}\) depending on which loop we take (\(\phi\) was defined in eq. (4.4)).

Recall that we defined the anyon \(\psi\) to be a composite of \(e\) and \(m\). Encircling a defect fixes the constituents of \(\psi\). However to close back on itself, we must twist the ‘ribbon’ into a Mobius strip — these are exactly the two choices in fig. 4.4.

We are now able to justify the name \textit{twist defect}. 
4.2 General theory of twist defects

In this section I formalize certain notions that should be common to topological defects in a system with abelian anyons. The basic algebraic ingredients are:

1. A $\mathbb{Z}_k$ module $V$ of dimension $n$. Physically, we have $n$ independent anyons and any $k$ of one kind add to zero.

2. A symmetric bilinear form $\mathbb{I}$ on $V$, which describes the braiding of our anyons. Unless we specify otherwise, we will take $\mathbb{I}$ to be non-degenerate i.e. we drop any particles that braid trivially around everything.

3. A finite group $\mathcal{G}$ along with an $n$-dimensional representation $\rho$ over $\mathbb{Z}_k$. We require $\mathbb{I}$ to be $\mathcal{G}$-invariant i.e.

\[
\mathbb{I}(\rho(g)(v_1), \rho(g)(v_2)) = \mathbb{I}(v_1, v_2), \quad \forall g \in \mathcal{G} \text{ and } v_1, v_2 \in V
\]

Physically, braiding of anyons has a symmetry group $\mathcal{G}$.

Each topological defect is labelled by a group element $g \in \mathcal{G}$. An anyon $v$ that encircles the defect gets transmuted to $\rho(g)(v)$; henceforth we will shorten this to $g(v)$. Let us put $N$ defects on a sphere, at points $\{p_i\}$ with the corresponding group labels $\{g_i\}$, $i = 1 \ldots N$. Topologically, we treat the $p_i$ as punctures i.e. paths are not allowed to cross them. We also impose the condition that a loop enclosing all the defects be contractible so,

\[
g_1g_2 \cdots g_N = 1. \quad (4.6)
\]

Picking a basepoint $p$ on the sphere, define the loop $\gamma_i$ to encircle the $i$’th defect in the counterclockwise orientation as shown. $\gamma_1, \cdots, \gamma_{N-1}$ generate the fundamental group of the punctured sphere i.e. $\pi_1(S^2 - \{p_1 \cdots p_N\})$. Loops are multiplied by concatenation.
The anyonic path $\gamma_i(v)$ represents the anyon $v$ starting at $p$ and travelling along $\gamma_i$. Note that is not a loop but an open path, since $v$ returns as $g_i(v)$. A composition of paths is defined similarly, $\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_m}(v)$ where $v$ is always the starting label at $p$. A Wilson loop is a path for which the anyon can bite its own tail,

$$W(v) = \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_m}(v) \quad \text{such that } g_{i_1}g_{i_2}\cdots g_{i_m}(v) = v. \quad (4.7)$$

Note: We include the possibility that the product of the group elements is not the identity but fixes a particular anyon $v$.

The loop $W$ is also labelled piecewise by elements of $V$, and the labels change discontinuously when the path crosses a branch cut. We are allowed to deform strands and the labels on coincident strands simply add. Any two Wilson loops can be deformed so that they intersect only at finitely many points that do not fall on branch cuts. The intersection, which we will also call $I$, is defined as a sum over these points,

$$I(W(u), W(v)) = \sum_{q_i \in W^1 \cap W^2} \pm I(W^1(u)|_{p_i}, W^2(v)|_{p_i}) \quad (4.8)$$

where the sign of each term is determined by fig. 4.6.

If $\{g_i\}$ are a set of twist defects then we may conjugate by any $h \in \mathcal{G}$ to get another set $\{hg_ih^{-1}\}$ that also satisfies eq. (4.6). If $W(u)$ is a Wilson loop in the original set then it is easy to check that $W(h(u))$ is a corresponding loop in the conjugate. We can now state our first result.

**Theorem:** (Covariance of the intersection) Let $\mathbb{I}$ and $\mathbb{I}$ be the respective intersection forms for $\{g_i\}$ and $\{hg_ih^{-1}\}$. We claim that,

$$\mathbb{I}(W^1(h(u)), W^2(h(v))) = \mathbb{I}(W^1(u), W^2(v)). \quad (4.9)$$
Proof: In every term in the sum from eq. (4.8) we will find an equality of the form \( \Pi(h(u'), h(v')) = \Pi(u', v') \) (the signs are untouched). The result follows.

**Wilson Algebra**: A representation \( \mathcal{O} \), of Wilson loops as unitary operators is called a Wilson Algebra if it satisfies,

\[
\mathcal{O}(W(v)) = \mathcal{O}(W(-v))^\dagger
\]

\[
\mathcal{O}(W_1(u))\mathcal{O}(W_2(v)) = e^{2\pi i/k}(\Pi(W_1, W_2)\mathcal{O}(W_2(v))\mathcal{O}(W_1(u))).
\]

However, we must be careful in defining \( \mathcal{O}(W(u + v)) \) since \( W(u) \) and \( W(v) \) might intersect (this does not occur in the toric code). However if we fix a convention for the time-ordering of anyons within \( V \) (compare eq on the lattice), this can still be done. Note that we will never have to consider addition of operators.

It is useful to try and write down a general formula for the intersection of two Wilson loops \( W_2 \) and \( W_2 \). Let us number the strands of each Wilson loop by the defect that they encircle as shown in the figure. This is not the order in which the defects are visited, and we must be careful since the various strands of a single Wilson loop might intersect. However, after fixing this time-ordering, we can ignore crossings within the two loops and just count those between them. From the figure,

\[
\Pi(W_1(u), W_2(v)) = \sum_{i=1}^{N} \left( \sum_{j=1}^{i} \Pi(u_i - g_i(u_i), v_j - g_j(v_j)) \right) + \Pi(u_i - g_i(u_i), g_j(v_j)). \tag{4.10}
\]

### 4.2.1 Degeneracy and quantum dimension

In general, a defect \( g \) might fix certain anyons \( v \) so that \( g(v) = v \). These will give rise to Wilson loops that only circle one defect. Hence they are contractible to an arbitrarily small neighbourhood and cannot intersect with any other loops. In the Wilson algebra, they are in the center (commute with everything) and we shall assume that in the irreducible representation, they are pure phases. Call each such loop \( S_g(v) \) and the corresponding phase \( \alpha_g(v) \). If we have more than one \( g \) defect, there is no reason that the phases should be equal so we should add another index \( \alpha_i^g(v) \) to distinguish them. When we consider fusion it will be useful to introduce identity defects, which fix everything and still allow them to have non-trivial \( \alpha_1(v) \).

I now treat the special case of a chain of similar defects, each \( g_i = g \) and the total number \( N \) must be a multiple of the order of \( g \) to satisfy the constraint (4.6). We can count the \( N \) open loops encircling the
Figure 4.7: Two loops $W_{1,2}$ with their basepoints displaced a bit and indicated by rectangles, which also include any crossing from reordering within each loop. The diagram shows the strand $u_i$ from $W_1$ that encircles the $i$'th defect, returns as $g_i(u_i)$ and intersects strands $1 \cdots i$ of $W_2$ along the way. The total intersection is a sum over $i$ defects (fig. 4.7), separately since they may be visited in any order. To form a loop we must have,

$$ \sum_{i=1}^{N} v_i = g \left( \sum_{i=1}^{N} v_i \right). \quad (4.11) $$

If $g$ does not fix any non-zero anyons then the left and right sides of the above equation must each be zero as elements in $V$. This reduces our space of loops from $V^N$ to $W = V^{N-2}$. To compute ground state degeneracy, we need a maximal commuting set i.e. the largest subspace such that none of its members intersect pairwise. Here we make a further assumption of duality — the dimension of this subspace $W_{\text{comm}}$ (which is not unique) is half that of the total space of loops. \footnote{Such a duality does hold for a set of like defects in the toric code as well as the $S_3$ honeycomb model. For the quantum dimension, we only need $\dim(W_{\text{comm}}) \sim N/2$ for large $N$.} Just as in the toric we start with an initial ground state $|GS_0\rangle$ on which all of $\mathcal{O}(W_{\text{comm}})$ acts with eigenvalue 1, the rest of the states being of the form $\mathcal{O}(W_i)|GS_0\rangle$ for $W_i \in W^c$. It only remains to count the number of elements in $W^c$. Since $V \cong (\mathbb{Z}_k)^n$, we find that the degeneracy is $k^{n(N-2)/2}$. The quantum dimension $d$ is defined so that the degeneracy goes as $d^N$ for large $N$, so we have $d = k^{n/2}$. 

More generally, the defect $g$ might fix a subspace $V_g \in V$, so that $g(v) = v$ for $v \in V_g$. Let us assume that powers of $g$ fix the same anyons and no more, $V_g^n = V_g$. Then we must quotient out by $V_g$ in eq. (4.11)
but the rest of the arguments go through exactly. The final result is,

\[ d_g = \left( \frac{k^n}{|V_g|} \right)^{1/2} \text{ quantum dimension of defect } g \]  

(4.12)

where \(|V_g|\) is number of elements in \(V_g\).

### 4.2.2 Fusion

Fusion is a well known characteristic of anyons. In standard terminology,

\[ a \times b = \sum_c N_{ab}^c \]

denotes the fusion of anyons \(a\) and \(b\) into a number of “channels” \(c\) (compare to the decomposition of a tensor product of two angular momentum representations). One requires a unit 0 that fuses trivially with all anyons, \(a \times 0 = 0\), and every \(a\) to have an antiparticle \(\bar{a}\) so that,

\[ a \times \bar{a} = 0 + \sum_{c \neq 0} N_{ab}^c \].

Even though defects are not non-abelian anyons, fusion is still well-defined since it is simply the physical notion of ‘zooming out’ so we lose the distinction between two defects. Any Wilson loops that pass between them are lost and the original Hilbert space is now reducible, decomposing into blocks. It is easily guessed that \(g_1\) and \(g_2\) fuse to \(g_1 g_2\) so fusion does not commute, though it is still associative. To find the number of channels, we use the formula for quantum dimension (4.12).

\[ g_1 \times g_2 = \frac{k^n}{(|V_{g_1}| |V_{g_2}|)^{1/2}} \frac{|V_{g_1 2}|}{k^{n/2}} \frac{1/2}{(g_1 g_2) = k^{n/2} \sqrt{\frac{|V_{g_1 g_2}|}{|V_{g_1}| |V_{g_2}|}} (g_1 g_2). \]  

(4.13)

For example, setting two \(g_2 = g_1^{-1}\) we find \((k^n/|V_g|)\) where the identity defect indicates that we have created multiple abelian anyons.

### 4.2.3 Braiding

Braiding is central to the concept of a non-abelian particle. Even though it appears to be a purely local operation, exchanging two defects will take us from one ground state to another. Our definition of braiding centers around its action on Wilson loops that are deformed but do not pass through defects.

Again, we take the chain of like defects \(g\). Let us also assume that \(g\) does not fix any anyons. Forming
the “figure-eight” around neighbouring pairs gives a basis for Wilson loops. We define the braid generator \( B_i \) to be the exchange of the defects \( i \) and \( i + 1 \) in the clockwise direction — see fig. 4.8. Denoting the loop starting at defect \( i \) to be \( W_i \), the operation \( B_i \) affects the loops \( W_i \) (shown in figure) and \( W_{i\pm1} \).

We do the rest of the computation algebraically as it is less error-prone (and saves chalk!). In terms of the open loops \( \gamma_i \) defined earlier \( W_i = \gamma_i^{-1}\gamma_i \) where loops are concatenated right to left. It is easy to check that

\[
B_i(\gamma_i) = \gamma_i^{-1}\gamma_{i+1}\gamma_i \tag{4.14}
\]

\[
B_i(\gamma_{i+1}) = \gamma_i \tag{4.15}
\]

and the rest are untouched. Then \( B_i(W_j) = B_i(\gamma_{i+1}^{-1}\gamma_i) = \gamma_i^{-2}\gamma_{i+1}\gamma_i = \gamma_i^{-1}(v)W_i(g(v))\gamma_i(v) = W_i(g(v)), \) where I have added the anyon labels for clarity and the last equality can be checked by a drawing (there are no intersections in the way). Doing the rest similarly we find the braiding rules,

\[
B_i(W_j(v)) = \begin{cases} 
W_i(g(v)) & j = i \\
W_{i-1}(g(v))W_i(g(v)) & j = i - 1 \\
W_{i+1}(v)W_i(v) & j = i + 1 \\
W_j(v) & \text{otherwise}
\end{cases} \tag{4.16}
\]

If \( g \) only fixes anyons with trivial exchange statistics then the above arguments go through once we quotient out by the subspace \( V_g \).
With some labour, one can show that eq. (4.16) is consistent with the Artin Braid relations,

\[ B_i B_{i+1} B_i = B_{i+1} B_i B_{i+1} \]
\[ B_i B_j = B_j B_i \quad |i - j| > 1. \]  

(4.17)

illustrated below,

Artin proved the (hard) theorem that these are the only relations for the braid group. We have thus produced an algorithm that generates a braid representation given a group \( G \) and \( \mathbb{Z}_k \) valued representation \( \rho \). \(^2\) Adding the intersection form should allow us to construct a unitary representation (though it might only be projective).

**Braiding of unlike defects**: It turns out that braiding unlike defects *changes* their group label. In fact this is not surprising in light of eq. (4.15), once we remember that the defects were defined via the open loops \( \gamma_i \). Explicitly we have,

\[ B_1(g_1, g_2) = B_1(g_2, g_2^{-1} g_1 g_2) \]  

and so on. Note that the product of the two labels is invariant under the braid operation, \( g_1 g_2 \). In particularly, this also means that the constraint in eq. (4.6) is unaffected. Finally, plugging the above into eq. (4.17) it is not hard to check that the Artin relations are satisfied.

**Sphere Braid group**: Recall that we been placing our defects on a sphere. This adds another relation to the braid group,

\[ B_1 B_2 \cdots B_{N-1} B_{N-1} B_{N-2} \cdots B_1 = 1 \quad \text{Fadell-Neuwirth.} \]  

(4.19)

\(^2\)This is probably well-known in the mathematics literature but I have been unable to find a name for this type of representation.
Geometrically the above operation is the act of taking the first defect on a large loop that encloses all the others.

Imagine that the strands start and end on the surface of two concentric spheres, and the vertical/radial direction is time. Then the red strand on the left side can be dragged behind the inner sphere and brought back to the initial configuration on the right. (4.19) is the only independent equation added to the Artin relations on a sphere. Taken together, they imply another one that is worth stating. Define the “full-twist” operator,

$$F = (B_1B_2 \cdots B_{N-1})^{N-1}. \quad (4.20)$$

If the initial set of vertical strands is viewed as a ribbon then the full-twist is a $2\pi$ rotation. The relation says that,

$$F^2 = 1 \quad (4.21)$$

which is immediate from the fact that a $4\pi$ rotation is homotopic to the identity if we are allow to drag the ribbon around the back of the sphere — this is just the Dirac trick.

Since the Artin relations held for our braid action (eq. (4.16)), one might expect (4.19) to follow out of a tedious calculation. But here there is a surprise. In the example of a chain of threefold defects in the $S_3$ honeycomb model (see next section) one finds that the operator $M$ on the left side of (4.19) is not the identity matrix! Instead, $M^3 = 1$ but $M$ does commute with all the other $B_i$ thus forming a central extension of the braid group.
4.3 Honeycomb Model

I now introduce a new model which is similar to the toric code, but has four types of abelian anyons instead of two, and a richer set of twist defects. It will provide a non-trivial example of the results of the previous section with the group \( G = S_3 \).

Our motivation came from the Wen chessboard model [27]. The idea is to put spins on the vertices of a square lattice and have a single term for every square plaquette (see [27] for details). The model is known to be equivalent to the toric code, and has the same excitations and braiding. However, in return for being restricted to a specific lattice, we gain a simple picture of duality. The \( e \) and \( m \) particles travel between diagonal squares of the same colour, like the two bishops in chess. Duality is just a \( \pi/2 \) rotation about a vertex that switches the the pattern of light and dark squares.

The Wen model relies on two features of the square lattice on the plane: the faces have a uniform bi-colouring (squares sharing an edge are always of different colours) and the vertices are bipartite (they can be separated into two sets \( A \) and \( B \) such that adjacent vertices have different labels). It is natural then, to consider uniform tilings where:

- The vertices are bipartite.
- The faces have a uniform tri-colouring. We also want vertex-transitivity — any two \( A \) vertices are related by a symmetry operation and the same is true of the \( B \)'s. Roughly speaking, any two vertices “look” the same.

It is easily seen that these two conditions are true for a hexagonal lattice (compare fig. 1.1) but there are other examples e.g.a tiling comprised of octagons and squares.

Putting the spins on the vertices of a hexagonal lattices, we find that there must be two terms per face for the number of stabilizers (constraints) to match the degrees of freedom. \(^3\) The terms are shown in fig. 4.9. Calling them \( P_A \) and \( P_B \), the hamiltonian is

\[
\mathcal{H} \sum_P \left( P_A + P_B + \text{h.c.} \right).
\]

The ground state is defined analagously to the toric code. It is an eigenstate of every plaquette operator with eigenvalue 1. Again we have excitations on vertices that terminate open strings. They are tabulated in fig. 4.10. The “colour” \( \chi \) can take values 1,2, or 3 (shown as blue,red or green). The other label \( \bullet/\circ \) comes from the two sublattices.

\(^3\)Otherwise the ground state degeneracy grows with the size of the lattice and physically we would have a gapless system.
Figure 4.9: The two plaquette terms in the Hamiltonian for each hexagon.

Figure 4.10: Open strings in the honeycomb model. (figure by J. Teo)
Only two of the three colours are independent. As shown in fig. 4.11, the third one (say green) is the sum of the red and blue, with a negative sign. There are are total of four independent anyon types. The final piece of information we need is the intersection form, which tells us how these anyons braid around each other. This can be worked out from fig. 4.10 and we find,

$$\Pi(u, v) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}$$

$$u = (u_{1\bullet}, u_{2\bullet}, u_{1\circ}, u_{2\circ})^T, v = (v_{1\bullet}, v_{2\bullet}, v_{1\circ}, v_{2\circ})^T.$$

(4.22)

Only strings with different colour and $\bullet$/circ labels intersect. In the Chern-Simons description, (4.22) would be the corresponding K-matrix.

We notice that the Hamiltonian is indifferent to the colour of the plaquettes and the two sublattices. Thus it invariant under a threefold rotation and mirror reflection. However these operations clearly change
the anyon type (fig. 4.10). In the same basis as eq. (4.22) we have,

\[
\begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & -1
\end{bmatrix}
\quad \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

(4.23)

The important observation is that these are symmetries of \(I\),

\[
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

(4.24)

From the identities, \(t^3 = s^2 = 1\) and \(sts = t^{-1}\), we establish that the group is \(S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2\). The \(\rtimes\) denotes a the semidirect product and implies that the two subgroups do not commute. In other words, this model is not just the addition of a \(\mathbb{Z}_3\) symmetry to the original \(\mathbb{Z}_2\) duality of the toric code.

The symmetry group immediately suggests that the apparatus of the previous section could be applied if we could find defects corresponding to \(t\) and \(s\). This can indeed be done, although I will not go into the details of the lattice construction here (they are discussed extensively in [4]). Instead, I work out a couple of applications of the previous formulae.

First we look at the space of anyons fixed by a defect \(g\), \(V_g\), defined earlier. The \(s\) defect is analogous to the toric code, in that it “twists” an anyon that has non-trivial exchange statistics i.e. \(s(v) = v\) and \(\mathbb{I}(v, v) \neq 0\). From the form of the matrix for \(s\), we find that \(V_g\) is generated by \((1, 0, 0, 1)\) and \((0, 1, 1, 0)\). Using our formula for quantum dimension, we find \(d_s = k\). The \(t\) defect will not fix anything and has dimension \(d_t = k^2\) with one caveat. If \(3\) divides \(k\), we find that \(v = (k/3, 2k/3, 0, 0)\) and \(v = (0, 0, k/3, 2k/3)\) are fixed which reduces the quantum dimension to \(k^2/3\).

Fusion rules are also straightforward. A couple of examples are,

\[
\begin{align*}
{s \times s} &= t \\
{t \times t} &= k^2 t^2
\end{align*}
\]

(assuming that \(k \nmid 3\)).

**Concluding Remarks** : The intersection form in eq. (4.22) looks quite similar to that of two copies of the toric code. In fact, computing the ground state degeneracy on a \(g\) holed torus (the method is identical), we find it to be \(k^{4g}\) which is just as if each model was giving us \(k^{2g}\). It turns out that the same is true for
the entanglement entropy. What these coarse measures fail to detect, is the $S_3$ symmetry in the braiding of anyons. That finer structure is probed by the defects and justifies our investigation.

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### 4.4 Covering spaces

We have seen previously that while the toric code on a sphere has a unique ground state, putting it on a surface of genus $g$ creates degeneracy $k^{2g}$. On the other hand, twist defects and their associated Wilson loops, also create degeneracy. Covering spaces provide the geometric link between the two cases.

The idea is similar to the classical description of a Riemann surface. We have a multivalued function (the anyon label) that jumps discontinuously across branch cuts. We construct a covering space with multiple sheets, each representing one of the labels, sewn together at the branch cuts. On the covering space there is a single continuous label everywhere but at the defect sites, which are analogues of ramification points.

I illustrate the above process with a couple of examples. Since the toric code has already been treated in the literature [27], I confine myself to the honeycomb model. First, let us take three $t$ (threefold) defects on a sphere. There are two branch cuts connecting them, shown as wavy lines in figure 4.4. Even though only two colours are independent, we need a total of six sheets, the order of the symmetry group $S_3$. The glueing convention in the figure is forced once we recall that the two branch cuts are different — the first one is the cyclic permutation $\Lambda = (G,R,B)$ while the second is $\Lambda^2$. After glueing we get a torus with two types of anyons, since the $\bullet/\circ$ label remains. The ground state degeneracy is exactly that of the Kitaev toric code i.e. $k^2$.

A useful formula for counting the genus of the covering space $\Sigma$ over the sphere $S^2$, is the **Riemann-Hurwitz** theorem,

$$\chi(\Sigma) = d\chi(S^2) - \sum_i (e_p - 1),$$

(4.25)

where $\chi$ is the Euler characteristic which in turn determines the genus $g$ for a connected component by $\chi = 2 - 2g$ and $d$ is the number of sheets. $e_p$ is the **ramification index** of the point $p$ and is defined so that $p$ lifts to $d - (e_p - 1)$ copies in the cover. Ordinary points lift to $d$ copies, one in each sheet, so $e_p = 1$. A point defect “pinches” some of the sheets together and hence has $e_p > 1$. For the $S_3$ model, $d = 3, e_p = 3$ for a set of threefold defects. For twofold defects (see below), we must take a full six sheeted cover so $d = 6, e_p = 4$. 

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Applying the formula to the first example, $\chi = 0$ and $g = 1$, as we have seen from figure. For $N$ threefold defects, we would get $\chi = 6 - 2N$ so $g = N - 2$. The ground state degeneracy is $k^2g$ (two anyon labels remain) or $k^{2N-4}$.

For the next example we consider a set of non-commuting twofold defects, a pair each of $s_Y, s_R$ and $s_B$. Overall, no color is fixed and the $\bullet/\circ$ label is swapped as well, so we need a six-sheeted covering space which will have a single type of anyon. Each of the six covering spheres has three connecting tubes (4.13), corresponding to the three brach cuts. Eq. 4.25 gives a genus $g = 4$ which would seem to imply that there are eight independent Wilson loops. However, the key difference from the previous example, is that some non-contractible Wilson loops in the cover are actually trivial in the base space. We cannot read off the degeneracy simply from the genus, but must first quotient out these trivial loops which arise from the non-commutativity of the defects and the color fusion $Y_{\bullet/\circ} \times R_{\bullet/\circ} \times B_{\bullet/\circ} = 1$. This leaves four loops and the degeneracy is $k^2$ (there is only one label in the cover).
Figure 4.13: Six twofold defects, two of each color, on a sphere. The covering space has genus 4. However, there are 4 loops that are non-contractible in the cover but trivial in the base space — the sum of the small dashed loops, the longer solid line, and 60° rotations of these two.
References


