\texttt{\texttt{L}_1} ADAPTIVE OUTPUT-FEEDBACK CONTROL ARCHITECTURES

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ABSTRACT

This research focuses on development of $\mathcal{L}_1$ adaptive output-feedback control. The objective is to extend the $\mathcal{L}_1$ adaptive control framework to a wider class of systems, as well as obtain architectures that afford more straightforward tuning.

We start by considering an existing $\mathcal{L}_1$ adaptive output-feedback controller for non-strictly positive real systems based on piecewise constant adaptation law. It is shown that $\mathcal{L}_1$ adaptive control architectures achieve decoupling of adaptation from control, which leads to bounded away from zero time-delay and gain margins in the presence of arbitrarily fast adaptation. Computed performance bounds provide quantifiable performance guarantees both for system output and control signal in transient and steady state. A noticeable feature of the $\mathcal{L}_1$ adaptive controller is that its output behavior can be made close to the behavior of a linear time-invariant system. In particular, proper design of the lowpass filter can achieve output response, which almost scales for different step reference commands. This property is relevant to applications with human operator in the loop (for example: control augmentation systems of piloted aircraft), since predictability of the system response is necessary for adequate performance of the operator.

Next we present applications of the $\mathcal{L}_1$ adaptive output-feedback controller in two different fields of engineering: feedback control of human anesthesia, and ascent control of a NASA crew launch vehicle (CLV). The purpose of the feedback controller for anesthesia is to ensure that the patient’s level of sedation during surgery follows a prespecified profile. The $\mathcal{L}_1$ controller is enabled by anesthesiologist after he/she achieves sufficient patient sedation level by introducing sedatives manually. This problem formulation requires safe switching mechanism, which avoids controller initialization transients. For this purpose, we used an $\mathcal{L}_1$ adaptive controller with special output predictor initialization routine, which provides bumpless transient during switches.

For the second application, our objective was to design a single controller without parameter scheduling, which would cover the whole flight envelope of the first stage of the CLV. This approach has the potential of reducing the design costs by reducing the number of necessary wind tunnel tests. One of the main challenges we encountered was variability of the parameters of the CLV. Both aerodynamic and inertia parameters change dramatically during the CLV operation. The fact that CLV inertia significantly reduces with time allows for demanding faster controller response and more agile CLV behavior as time flows. This inspired us to develop an $\mathcal{L}_1$ adaptive controller, which would take into account for changes in the desired control specifications without the need for switching the control laws. This
is achieved by employing linear time varying (LTV) state predictor, which results in LTV reference system.

Further we present $\mathcal{L}_1$ adaptive output-feedback controller for minimum phase systems with gradient minimization type adaptation laws. This controller uses a special structure for its reference system. The stability conditions are more intuitive and can be systematically verified using classical control methods.

For completeness, we also consider an extension of the $\mathcal{L}_1$ adaptive controller to a class of nonlinear output-feedback systems. We derive a stability proof and also the performance bounds for passive nonlinear systems with implicit output equation.
To my wife Irina and my parents Elena and Rafael with love and gratitude
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CHAPTER 1

Introduction

The first results of $L_1$ adaptive control theory appeared in 2006 [1, 2], with subsequent developments culminating in [3]. Development of $L_1$ adaptive control theory was mainly motivated by the emerging need to certify advanced adaptive flight critical systems with a more affordable validation and verification process. The key feature of $L_1$ adaptive control architectures is the decoupling of adaptation (learning) and control (acting) loops, which leads to \textit{guaranteed robustness in the presence of fast adaptation}. In this context, fast adaptation indicates that the adaptation rate in $L_1$ architectures is to be selected so that the time scale of the adaptation process is faster than the time scales associated with plant parameter variations and the underlying closed-loop dynamics. Robust adaptation indicates that, despite fast adaptation in $L_1$ architectures, the robustness properties of the closed-loop adaptive system can be adjusted independently of the adaptation rate. The transient performance of the closed-loop $L_1$ adaptive system is quantified both for the system output and the input by performance bounds with respect to an $L_1$ reference system, which incorporates a lowpass filter [4]. The performance bounds can be arbitrarily reduced by increasing the adaptation gain without introducing or requiring persistence of excitation and without resorting to high-gain feedback. With $L_1$ adaptive control architectures, large learning gains appear to be beneficial both for performance and robustness, while the tradeoff between the two is resolved by selecting the underlying filter structure. The latter is a linear problem and can be addressed using conventional methods from classical and robust control. These features of $L_1$ adaptive control theory have been verified in flight tests and in various mid-to-high fidelity simulation environments. For a more extensive review we refer the reader to [5].

Most of the real world applications use models with some of the states unmeasured. Even the applications, which historically use state-feedback control, often are converted to output-feedback framework, when stricter design specifications require using higher fidelity design models. For instance in aerospace applications, resorting to low cost hardware may require to take into account the actuator dynamics in design, which leads to a system with at least one or two unmeasured states [5]. Also increasing vehicle velocity as well as other design specifications lead to increased influence of such phenomena with unmeasured states like flexible dynamics [6]. Inspired by growing demand for solutions in this area, we focus our dissertation on $L_1$ adaptive output-feedback developments.
1.1. Related Work

The initial results in adaptive control were inspired by system identification [7], which led to an architecture consisting of an on-line parameter estimator combined with automatic control design [8, 9]. Two architectures of adaptive control emerged: the direct method, where only controller parameters were estimated, and the indirect method, where process parameters were estimated and the controller parameters were obtained using some design principle.

The progress in systems theory led to fundamental theory for development of stable adaptive control architectures [10–20]. This was accompanied by several examples, including Rohrs’ example, challenging the robustness of adaptive controllers in the presence of unmodeled dynamics, [21–23]. With this example Rohrs brought up an important point: the available adaptive control algorithms to that date were unable to adjust the bandwidth of the closed-loop system and guarantee its robustness. In [24–29] the authors analyzed the causes of instability and also proposed damping-type modifications of adaptive laws to prevent them. The basic idea of all the modifications was to limit the gain of the adaptation loop and to eliminate its integral action. All these modifications solved the important problem of parameter drift, however they did not directly address the architectural problem identified by Rohrs. We notice that lack of robustness of adaptive controllers has been analyzed also in robust control literature [30]. An incomplete overview of robustness and stability issues of adaptive controllers can be found in [23].

The fundamental results [31–35] provided sufficient conditions on the bounds of uncertainties and initial conditions, which would guarantee that, with the given adaptive feedback architecture, the signals in the feedback loop remain bounded. Though very important, when dealing with practical applications, boundedness, ultimate boundedness, or even asymptotic convergence, are weak properties for nonlinear (adaptive) feedback systems. On one hand, unmodeled dynamics, latencies, and noise require precise quantification of the robustness and the stability margins of the underlying feedback loop. On the other hand, performance requirements in real applications necessitate a predictable response for the closed-loop system, dependent upon the changes in system dynamics. In adaptive control, the nature of the adaptation process plays a central role in both robustness and performance. Ideally, one would like adaptation to correctly respond to all the changes in initial conditions, reference inputs, and uncertainties, by quickly identifying a set of control parameters that would provide a satisfactory system response. This, of course, demands fast estimation schemes with high adaptation rates, and as a consequence, leads to the fundamental question of determining the upper bound on the adaptation rate that would not result in poor robustness.
characteristics. We notice that the results available in the literature consistently limited the rate of variation of uncertainties, by providing examples of destabilization due to fast adaptation [33, p. 549], while the transient performance analysis was continually reduced to persistency of excitation-type assumption, which, besides being a highly undesirable phenomenon, cannot be verified a priori.

The $\mathcal{L}_1$ adaptive control theory addressed this issue by introducing an architecture which decouples adaptation from robustness [3]. The speed of adaptation in these architectures is limited only by the available CPU (hardware), while robustness is resolved via conventional methods from classical and robust control, without persistence of excitation, gain scheduling, and high-gain feedback. Some recent findings in [36] establish connections between the $\mathcal{L}_1$ adaptive control theory and the internal model control, which are further exploited in [37] towards applying methods from robust control to the filter design of the $\mathcal{L}_1$ adaptive controller.

$\mathcal{L}_1$ adaptive control theory is developed for a wide range of classes of systems including systems with constant and time varying unknown parameters, systems with unknown nonlinearities, systems with unmodeled input dynamics as well as system with unmodeled nonlinear internal dynamics. There are also results available for $\mathcal{L}_1$ adaptive controllers with linear time varying reference systems, which are useful for applications with changing desired control specifications over time or operational envelope. All these developments are systematically presented in [3]. $\mathcal{L}_1$ adaptive controller for nonlinear systems is presented in [38,39].

Despite numerous $\mathcal{L}_1$ adaptive control architectures developed for a wide range of state-feedback systems there are only few $\mathcal{L}_1$ adaptive controllers available for output-feedback systems [40–43]. The architecture in [40] is limited to first order linear time invariant (LTI) desired systems. The $\mathcal{L}_1$ adaptive controller in [41,42] overcomes this limitation by employing a novel piecewise constant adaptation law. This law involves desired system inversion in discrete time for obtaining the estimate of the uncertainty. Similar to the other $\mathcal{L}_1$ adaptive controller architectures, reducing the sampling rate results in smaller performance bounds. While discrete time representation of this adaptation law facilitates the implementation in practice, it requires computation of the inverse of the desired system state transition matrix, which is affordable for linear systems, but hard to determine for nonlinear systems. The $\mathcal{L}_1$ adaptive controller in [43] further extends these ideas to a control law with two lowpass filters: each for terms with matched and unmatched adaptive estimates.
1.2. Dissertation Outline and Contributions

This dissertation is organized in 8 chapters for which a brief overview and contributions are given below.

- Chapter 2 considers existing $\mathcal{L}_1$ adaptive output-feedback control architecture for non-strictly positive real (SPR) LTI systems with piecewise constant adaptation laws [3,41]. We rely on time-domain analysis of a scalar example to illustrate some of the key properties of the estimation loop with piecewise constant adaptation laws, which we use in one of our controllers later.

- In Chapter 3, we apply $\mathcal{L}_1$ adaptive controller to an ascent control system of NASA Crew Launch Vehicle (CLV). The main challenge in this work is the statical instability of the vehicle, and the presence of severe flexible dynamics, which are excited by the control input and are directly affecting sensor measurements. We show that a single $\mathcal{L}_1$ adaptive controller, designed without controller parameter scheduling, is able to satisfy all given performance specifications despite significant changes in the inertia, aerodynamics of the CLV, and severe flexible modes, which change with time.

- Chapter 4 presents application of $\mathcal{L}_1$ adaptive controller to drug delivery for human anesthesia. The results show robustness of the adaptive controller to model parameter variations and adequate disturbance attenuation. Observed consumption of the sedative drug (isoflurane) is comparable to measured values during clinical trials. In this chapter, we also develop a switching mechanism for $\mathcal{L}_1$ adaptive control law to eliminate possible undesired initialization transients.

- In Chapter 5, we develop an $\mathcal{L}_1$ adaptive control architecture, which achieves performance specifications defined by a linear time-varying (LTV) reference system, which is critical in applications covering a wide range of operating conditions. A typical example of time-varying reference system would be the one resulting from a gain-scheduled baseline controller over an entire flight envelope. Relevant results have been reported in [44–49].

- Next we notice that the verification of the sufficient condition for stability for the $\mathcal{L}_1$ adaptive control architecture for non-SPR system in [3] is not straightforward, and for certain classes of systems may be impossible, as observed in [50]. In Chapter 6, we develop an $\mathcal{L}_1$ adaptive output-feedback control architecture for minimum phase systems in the presence of time and output dependent nonlinearities. The solution proposed in here, under mild assumptions on system dynamics, provides much simpler
form of the stability condition. The obtained $\mathcal{L}_1$-norm stability condition has two separate terms: the first term is responsible for feedback stability of the $\mathcal{L}_1$ controller with chosen parameters, and the second term is responsible for the effect of the uncertainty, which helps to make the tuning procedure more systematic. The $\mathcal{L}_1$ adaptive control architecture in this chapter is a modification of the architecture of Monopoli from [51], which includes the augmented error signal. As compared to [51], the architecture presented here relies on system inversion, and hence cannot be used in the presence of non-minimum phase zeros. However, it contains more parameters for tuning than the $\mathcal{L}_1$ controller in [41]. In particular, it gives an opportunity to set the estimation dynamics by tuning the poles of the observer polynomial.

• In Chapter 7, we consider output-feedback systems with control objective specified via a nonlinear desired system. This chapter extends the state-feedback $\mathcal{L}_1$ adaptive controller for nonlinear reference systems, presented in [39] to output-feedback case. Under passivity assumption and using the special structure of the system output equation, we were able to obtain sufficient conditions for the closed-loop system stability and compute the performance bounds. We also showed that the bound on the prediction error in the presence of nonzero initial conditions consists of two terms. The first term asymptotically decreases with time to zero; and the second term can be arbitrarily reduced by increasing the adaptation gain. The structure of the bound is identical to one computed for LTI systems in Section 2.2.4 of [3].

• Finally, Chapter 8 lists open problems, possible directions of future research and concludes this dissertation.
CHAPTER 2

$\mathcal{L}_1$ Adaptive Output-feedback Architecture with Piecewise-constant Adaptation Law

In this Chapter we present the $\mathcal{L}_1$ adaptive output-feedback control architecture, which achieves performance specifications defined by a non-SPR reference system [3, 41]. This extension is possible by invoking the piecewise-constant adaptive law, which allows to obtain the performance bounds between the $\mathcal{L}_1$ reference system and the closed-loop $\mathcal{L}_1$ adaptive system. These bounds can be rendered arbitrarily small by reducing the sampling rate of the adaptation law, which can be set according to the available sampling rate of the central processing unit (CPU).

2.1. Problem Formulation

Consider the following single-input single-output (SISO) system:

$$y(s) = A(s)(u(s) + d(s)), \quad (2.1)$$

where $u(t) \in \mathbb{R}$ is the input; $y(t) \in \mathbb{R}$ is the system output; $A(s)$ is a strictly-proper unknown transfer function of unknown relative degree $n_r$, for which only a known lower bound $1 < d_r \leq n_r$ is available; $d(s)$ is the Laplace transform of the time-varying uncertainties and disturbances $d(t) = f(t, y(t))$, while $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an unknown map, subject to the following assumption:

**Assumption 2.1 (Global Lipschitz continuity and boundedness).** There exist constants $L > 0$ and $L_0 > 0$, such that

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|, \quad |f(t, y)| \leq L|y| + L_0$$

hold uniformly in $t \geq 0$.

Let $r(t) \in \mathbb{R}$ be a given bounded continuous reference input signal. The control objective is to design an adaptive output-feedback controller $u(t)$ such that the system output $y(t)$ tracks the reference input $r(t)$ according to the control specifications given by a desired system

$$y_{id}(s) = M(s)r(s), \quad (2.2)$$
where $M(s)$ is a minimum phase stable transfer function of relative degree $d_r$.

### 2.2. $\mathcal{L}_1$ Adaptive Control Architecture

In this section we first provide all necessary definitions and notations before we proceed with $\mathcal{L}_1$ adaptive control architecture.

#### 2.2.1. Definitions and $\mathcal{L}_1$-norm Stability Condition

We start by rewriting the system in (2.1) as:

$$y(s) = M(s)(u(s) + \sigma(s)), \quad (2.3)$$

$$\sigma(s) = \frac{(A(s) - M(s))u(s) + A(s)d(s)}{M(s)}.$$

Let $(A_m, b_m, c_m^\top)$ be a minimal realization of $M(s)$. Then the system in (2.3) can be rewritten as:

$$\dot{x}(t) = A_m x(t) + b_m(u(t) + \sigma(t)), \quad x(0) = 0,$$

$$y(t) = c_m^\top x(t). \quad (2.4)$$

The design of the $\mathcal{L}_1$ adaptive controller proceeds by considering a stable lowpass filter $C(s)$ of relative degree greater or equal to $d_r$, with unit dc-gain $C(0) = 1$. Further the selection of $C(s)$ and $M(s)$ must ensure that

$$H(s) \triangleq A(s)M(s) \quad (2.5)$$

is stable, and the following $\mathcal{L}_1$-norm condition holds:

$$\|G(s)\|_{\mathcal{L}_1} L < 1, \quad (2.6)$$

where $G(s) \triangleq H(s)(1 - C(s))$.

Further, since $A_m$ is Hurwitz, there exists $P = P^\top > 0$ that satisfies the algebraic Lyapunov equation

$$A_m^\top P + PA_m = -Q, \quad \text{for arbitrary} \quad Q = Q^\top > 0.$$

Given the vector $c_m^\top(\sqrt{P})^{-1}$, let $D \in \mathbb{R}^{(n-1)\times n}$ be a matrix that contains the null-space of
\[ c_m^{\top}(\sqrt{P})^{-1}, \quad \text{that is} \quad D(c_m^{\top}(\sqrt{P})^{-1})^{\top} = 0, \]

and further let
\[
\Lambda \triangleq \begin{bmatrix} c_m^{\top} \\ D \sqrt{P} \end{bmatrix}.
\] (2.7)

Define \( T_s \in \mathbb{R}^+ \) as an arbitrary positive constant, which can be associated with the sampling rate of the available CPU. Further, let \( \Phi(T_s) \in \mathbb{R}^{n \times n} \) be given by
\[
\Phi(T_s) \triangleq \Lambda A_m^{-1}(e^{A_m T_s} - \mathbb{I}).
\]

Next, let
\[
H_0(s) \triangleq \frac{A(s)}{C(s)A(s) + (1 - C(s))M(s)}, \quad H_1(s) \triangleq \frac{(A(s) - M(s))F(s)}{C(s)A(s) + (1 - C(s))M(s)}, \quad H_2(s) \triangleq \frac{H(s)C(s)}{M(s)}, \quad H_3(s) \triangleq -\frac{M(s)C(s)}{(C(s)A(s) + (1 - C(s))M(s))}.
\]

Also, let
\[
\Delta \triangleq \left( \|H_1(s)\|_{L_1} \right) \|r\|_{L_{\infty}} + \|H_0(s)\|_{L_1}(L\rho_r + L_0)
\]
\[
+ \left( \|H_1(s)\|_{L_1} + \|H_0(s)\|_{L_1} \frac{\|H_2(s)\|_{L_1}}{1 - \|G(s)\|_{L_1}} L \right) \gamma_0,
\] (2.8)

where \( \gamma_0 \in \mathbb{R}^+ \) is an arbitrary constant.

Let
\[
1_1^{\top} e^{A_m A_m^{-1} t} = [\eta_1(t), \quad \eta_2^{\top}(t)],
\] (2.9)

where \( 1_1 = [1, \ 0, \ldots, \ 0]^{\top} \in \mathbb{R}^n \), \( \eta_1(t) \in \mathbb{R} \) and \( \eta_2(t) \in \mathbb{R}^{n-1} \) contain the first and the 2-to-\( n \) elements of the row vector \( 1_1^{\top} e^{A_m A_m^{-1} t} \). Then define
\[
\kappa(T_s) \triangleq \int_0^{T_s} |1_1^{\top} \Lambda e^{A_m(T_s - \tau)} b_m| \ d\tau.
\] (2.10)
Also, let $\varsigma(T_s)$ be defined as

$$
\varsigma(T_s) \triangleq \|\eta_2(T_s)\| \sqrt{\frac{\alpha}{\lambda_{\text{max}}(P_2)} + \kappa(T_s)\Delta},
$$

$$
\alpha \triangleq \lambda_{\text{max}}(\Lambda^{-\top} P \Lambda^{-1}) \left( \frac{2\Delta \|\Lambda^{-\top} P \beta_m\|}{\lambda_{\text{min}}(\Lambda^{-\top} Q \Lambda^{-1})} \right)^2.
$$

Next, we introduce the following functions

$$
\beta_1(T_s) \triangleq \max_{t \in [0, T_s]} |\eta_1(t)|, \quad \beta_2(T_s) \triangleq \max_{t \in [0, T_s]} \|\eta_2(t)\|,
$$

and also

$$
\beta_3(T_s) \triangleq \max_{t \in [0, T_s]} \eta_3(t), \quad \beta_4(T_s) = \max_{t \in [0, T_s]} \eta_4(t),
$$

where

$$
\eta_3(t) \triangleq \int_0^t |1^\top \Lambda \Lambda^{-1}(t-\tau)\Lambda^{-1}(T_s) e^{\Lambda A_m \Lambda^{-1} T_s} 1| \, d\tau,
$$

$$
\eta_4(t) \triangleq \int_0^t |1^\top \Lambda \Lambda^{-1}(t-\tau)\Lambda \beta_m| \, d\tau.
$$

The following lemma introduces a positive constant $p_1$ and a positive definite matrix $P_2$, which can be computed from the constructive proof in [3].

**Lemma 2.1.** For arbitrary $\xi \triangleq [y \ z^\top]^\top \in \mathbb{R}^n$, where $z \in \mathbb{R}^{n-1}$, there exist $p_1 \in \mathbb{R}^+$ and positive definite $P_2 \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$
\xi^\top (\Lambda^{-1})^\top P \Lambda^{-1} \xi = p_1 y^2 + z^\top P_2 z.
$$

Finally, we define

$$
\gamma_0(T_s) \triangleq \beta_1(T_s)\varsigma(T_s) + \beta_2(T_s)\sqrt{\frac{\alpha}{\lambda_{\text{max}}(P_2)}} + \beta_3(T_s)\varsigma(T_s) + \beta_4(T_s)\Delta.
$$

### 2.2.2. $\mathcal{L}_1$ Adaptive Control Architecture

The $\mathcal{L}_1$ adaptive controller consists of an output predictor, an update law and the control law, which involves a lowpass filter $C(s)$ that together with the choice of $M(s)$ needs to satisfy the $\mathcal{L}_1$-norm stability condition (2.6).
We consider the following output-predictor:

\[
\begin{align*}
\dot{x}(t) &= A_m \hat{x}(t) + b_m u(t) + \hat{\sigma}(t), \quad \hat{x}(0) = 0 , \\
\hat{y}(t) &= c_m^\top \hat{x}(t),
\end{align*}
\] (2.11)

where \( \hat{\sigma}(t) \in \mathbb{R}^n \) is the vector of adaptive parameters. Notice that while \( \sigma(t) \in \mathbb{R} \) in (2.4) is matched, the uncertainty estimation \( \hat{\sigma}(t) \in \mathbb{R}^n \) in (2.11) is unmatched.

Letting \( \tilde{y}(t) \triangleq \hat{y}(t) - y(t) \), the update law for \( \hat{\sigma}(t) \) is given by

\[
\begin{align*}
\dot{\hat{\sigma}}(t) &= \hat{\sigma}(iT_s), \quad t \in [iT_s, (i+1)T_s), \\
\hat{\sigma}(iT_s) &= -\mu(T_s)\tilde{y}(iT_s), \quad i = 0, 1, 2, \ldots,
\end{align*}
\] (2.12)

where

\[
\mu(T_s) = \Phi^{-1}(T_s)e^{A_mA^{-1}T_s}1_1 = (e^{A_mT_s} - I)^{-1}A_m^\top e^{A_mT_s}A^{-1}1_1 .
\]

The control signal is defined as follows:

\[
u(s) = F(s)r(s) - C(s) c_m^\top(sI - A_m)^{-1}b_m c_m^\top(sI - A_m)^{-1}\hat{\sigma}(s),
\] (2.13)

where \( C(s) \) was first introduced in (2.5), and \( F(s) \) is a bounded-input, bounded-output (BIBO) stable proper transfer function with \( F(0) = 1 \). The block diagram of the closed-loop \( \mathcal{L}_1 \) adaptive control system is given in Figure 2.1.
2.3. Analysis of the $\mathcal{L}_1$ Adaptive Controller

The analysis of $\mathcal{L}_1$ adaptive control architecture proceeds with definition of $\mathcal{L}_1$ reference system, which is used for computation of the performance bounds.

2.3.1. $\mathcal{L}_1$ Reference System

Consider the following $\mathcal{L}_1$ reference system:

$$y_{\text{ref}}(s) = M(s)(u_{\text{ref}}(s) + \sigma_{\text{ref}}(s)),$$

$$u_{\text{ref}}(s) = F(s)r(s) - C(s)\sigma_{\text{ref}}(s),$$

where

$$\sigma_{\text{ref}}(s) = \frac{(A(s) - M(s))u_{\text{ref}}(s) + A(s)d_{\text{ref}}(s)}{M(s)},$$

and $d_{\text{ref}}(t) \triangleq f(t, y_{\text{ref}}(t))$.

We notice that the $\mathcal{L}_1$ reference system contains uncertainty and it cannot be used in the implementation of the control system. Therefore we use it only for analysis.

Remark 2.1. We notice that the following ideal control signal

$$u_{\text{id}}(t) = r(t) - \sigma_{\text{ref}}(t)$$

is the one that leads to desired system response

$$y_{\text{id}}(s) = M(s)r(s)$$

by canceling the uncertainties exactly. Thus, the reference system in (2.14)-(2.15) has a different response as compared to the ideal one. It only cancels the uncertainties within the bandwidth of $C(s)$, which can be selected compatible with the control channel specifications. This is exactly what one can hope to achieve with any feedback in the presence of uncertainties.

The fact that the $\mathcal{L}_1$ reference system involves uncertainty implies that there are no guarantees that the reference system will be stable. The next lemma establishes stability of the $\mathcal{L}_1$ reference system.

Lemma 2.2. Let $C(s)$ and $M(s)$ verify the $\mathcal{L}_1$-norm condition in (2.6). Then, the closed-loop reference system in (2.14)-(2.15) is BIBO stable.
The proof of the lemma can be found in [3].

2.3.2. Transient and Steady-state Performance

The next theorem establishes the stability and the performance bounds of the closed-loop adaptive control system.

**Theorem 2.1.** Consider the system in (2.1) and the $L_1$ adaptive controller in (2.11), (2.12), and (2.13), subject to the $L_1$-norm condition in (2.6). If we choose $T_s$ to ensure

$$\gamma_0(T_s) < \bar{\gamma}_0,$$

where $\bar{\gamma}_0$ is an arbitrary positive constant introduced in (2.8), then

$$\|\hat{y}\|_{L_\infty} \leq \bar{\gamma}_0,$$

$$\|y_{\text{ref}} - y\|_{L_\infty} \leq \gamma_1,$$

$$\|u_{\text{ref}} - u\|_{L_\infty} \leq \gamma_2,$$

(2.16)

with

$$\gamma_1 \triangleq \frac{\|H_2(s)\|_{L_1}}{1 - \|G(s)\|_{L_1} L} \bar{\gamma}_0,$$

$$\gamma_2 \triangleq \frac{\|H_2(s)\|_{L_1} L \gamma_1 + \frac{\|H_3(s)\|_{L_1}}{M(s)} \|}{\bar{\gamma}_0}.$$

Before we proceed to the analysis of the performance bounds, consider the following proposition, the proof of which can be found in [3].

**Proposition 2.1.** The following limiting relationship is true:

$$\lim_{T_s \to 0} \gamma_0(T_s) = 0.$$

Thus, the tracking error between $y(t)$ and $y_{\text{ref}}(t)$, as well between $u(t)$ and $u_{\text{ref}}(t)$, is uniformly bounded by a constant depending on $T_s$. The proposition 2.1 implies that during the transient phase, one can achieve arbitrary improvement of tracking performance by uniformly reducing $T_s$.

**Remark 2.2.** Notice that the parameter $T_s$ is the fixed time-step in the definition of the adaptive law. The adaptive parameters in $\hat{\sigma}(t) \in \mathbb{R}^n$ take constant values during $[iT_s, (i + 1)T_s)$ for every $i = 0, 1, \ldots$. Reducing $T_s$ imposes hardware (CPU) requirements, and Theorem 2.1 further implies that the performance limitations are consistent with the hardware limitations. This fact in turn is consistent with the other $L_1$ adaptive control architectures presented in [3], where improvement of the transient performance was achieved by increasing the adaptation rate in the projection-based adaptive laws.
2.4. Intuition Behind the Piecewise Constant Adaptation Law

In this section we perform time-domain analysis of the $L_1$ adaptive controller with piecewise constant update laws. We use a first order system and we place the emphasis on the piecewise constant adaptive laws. The purpose of this section is to help the reader to get insights into these new adaptive laws. From the practical implementation standpoint, these update laws can be directly related to the sampling rate of the available CPU.

For the purpose of analysis, consider a scalar version of the system (2.1):

$$\dot{x}(t) = -a x(t) + b(u(t) + d(t)),$$  \hspace{1cm} (2.17)

where $x(t) \in \mathbb{R}$ is the measured system output (in scalar case same as the system state); $a \in \mathbb{R}$ is the unknown system parameter with known conservative bounds, $b \in \mathbb{R}^+$ is a known system parameter; $d(t) \in \mathbb{R}$ is the input disturbance. The adaptive controller presented above ensures that the system output $x(t)$ tracks a bounded reference signal $r(t)$ with the desired transient specifications given by the ideal system similar to (2.2):

$$\dot{x}_m(t) = -a_m x_m(t) + b k_r r(t), \quad x_m(0) = x_0,$$

where $a_m \in \mathbb{R}^+$ is a known system parameter, $k_r = a_m/b$.

The system in (2.17) can be written as

$$\dot{x}(t) = -a_m x(t) + b(u(t) + \sigma(t)),$$  \hspace{1cm} (2.18)

where

$$\sigma(t) \triangleq \frac{a_m - a}{b} x(t) + d(t).$$

The output predictor (2.11) for this example will take the following form

$$\dot{x}(t) = -a_m \hat{x}(t) + b(u(t) + \hat{\sigma}(t)).$$

The adaptation law (2.12) specifies to

$$\dot{\sigma}(t) = -\frac{1}{b} \Phi^{-1}(T_s) \mu(iT_s), \quad t \in [iT_s, (i+1)T_s), \quad i = 1, 2, 3, \ldots.$$  \hspace{1cm} (2.19)
The parameter $\Phi(T_s)$ is given by

$$
\Phi(T_s) = \int_0^{T_s} e^{-a_m(T_s-\tau)} d\tau = \left[ \frac{1}{a_m} e^{-a_m(T_s-\tau)} \right]_0^{T_s} = \frac{1}{a_m} (1 - e^{-a_mT_s}) .
$$

(2.20)

The function $\mu(iT_s)$ in (2.19) is given by

$$
\mu(iT_s) = e^{-a_m T_s \hat{x}(iT_s)},
$$

where $\hat{x}(t) \triangleq \hat{x}(t) - x(t)$. The control law (2.13) reduces to:

$$
u(s) = F(s) r(s) - C(s) \hat{\sigma}(s),
$$

where for simplicity of presentation, we set

$$
F(s) = C(s) = \frac{\omega}{s + \omega}.
$$

Next we demonstrate the role of the sampling and the inversion of $\Phi(T_s)$ in the update law. The system in (2.18) can be rewritten in frequency domain as

$$
x(s) = \frac{b}{s + a_m} (u(s) + \sigma(s)) ,
$$

(2.21)

where $\sigma(s)$ is the Laplace transform for $\sigma(t)$. The state predictor similarly can be rewritten as

$$
\hat{x}(s) = \frac{b}{s + a_m} (u(s) + \hat{\sigma}(s)) ,
$$

Subtracting (2.21) from the state predictor equation, we obtain the prediction error dynamics

$$
\tilde{x}(s) = \frac{b}{s + a_m} (u(s) + \hat{\sigma}(s)) - \frac{b}{s + a_m} (u(s) + \sigma(s)) = \frac{b}{s + a_m} (\hat{\sigma}(s) - \sigma(s)) .
$$

(2.22)

To understand the purpose of the sampled update law, assume that during $t \in [0, t_0)$ for some $t_0 > 0$ the system was affected by the disturbance $\sigma(t)$, which resulted in $\tilde{x}(t_0) = \tilde{x}_0 \neq 0$. Next we check how the system reacts to the accumulated prediction error $\tilde{x}_0$. In order to see the “clean” control system response, i.e. not affected by the disturbance, we set the disturbance on the next time interval $t \in [t_0, t_0 + T_s]$ to zero. The solution to (2.22) on the
time interval \( t \in [t_0, t_0 + T_s] \) for \( \sigma(t) = 0 \) is given by

\[
\tilde{x}(t) = \tilde{x}_0 e^{-a_m (t-t_0)} + \frac{b}{a_m} \hat{\sigma}(t_0) \left( 1 - e^{-a_m (t-t_0)} \right).
\]

At the end of the sampling interval we have

\[
\tilde{x}(t_0 + T_s) = \tilde{x}_0 e^{-a_m T_s} + \frac{b}{a_m} \hat{\sigma}(t_0) \left( 1 - e^{-a_m T_s} \right).
\]

Substituting the value of the estimate given by (2.19) yields

\[
\tilde{x}(t_0 + T_s) = \tilde{x}_0 e^{-a_m T_s} - \frac{1}{a_m} \Phi^{-1}(T_s) \mu(t_0) \left( 1 - e^{-a_m T_s} \right)
\]

\[
= \tilde{x}_0 e^{-a_m T_s} \left( 1 - e^{-a_m T_s} \right) \tilde{x}(t_0) \left( 1 - e^{-a_m T_s} \right)
\]

(2.23)

Thus we see that the update law on each sampling time generates an estimate, which completely compensates for the prediction error accumulated on the previous sampling period. However, in reality, the disturbance \( \sigma(t) \) is not zero during \( t \in [t_0, t_0 + T_s] \). Therefore the value of \( \tilde{x}(t_0 + T_s) \) usually is not zero as the error dynamics are also affected by the additive disturbance (see (2.22)). Setting the sampling time \( T_s \) small enough one can keep the value of \( \tilde{x}(t) \) small and achieve arbitrary performance improvement in the presence of disturbances. The rigorous proof of this result is given in [3].

Next, we note that (2.22) yields

\[
C(s)\tilde{\sigma}(s) = \frac{\omega(s + a_m)}{b(s + \omega)} \tilde{x}(s),
\]

which further leads to

\[
\|\tilde{\sigma}_f\|_{L_\infty} \leq \left\| \frac{\omega(s + a_m)}{b(s + \omega)} \right\|_{L_1} \|\tilde{x}\|_{L_\infty},
\]

where \( \tilde{\sigma}_f(s) \triangleq C(s)\tilde{\sigma}(s) \) is a filtered version of \( \tilde{\sigma}(t) \). From this bound we see that small values of the output prediction error lead to smaller values of filtered disturbance estimation error. This means that the fast estimation loop of the \( L_1 \) adaptive controller estimates the input disturbance of the system. Large adaptation gain achieves small estimation error of the low frequency content of the disturbance.

Remark 2.3. Notice that \( \Phi(T_s) \), defined in (2.20), is equal to the system state at the time \( T_s \) for a constant unit control signal, zero initial conditions, and in the absence of the
disturbance, i.e.

\[ \Phi(T_s) = x(T_s), \quad u(t) \equiv 1, \quad \sigma \equiv 0, \quad x(0) = 0. \]

From (2.23) we see that \( \Phi^{-1}(T_s) \) inverts the error dynamics discretely. Namely, it cancels the effect of the system dynamics at fixed sampling times \( iT_s, i = 1, 2, 3, \ldots \), making the system response equal to

\[ \tilde{x}((i + 1)T_s) = \tilde{x}(iT_s)e^{-a_mT_s} - \mu(iT_s). \]

The role of \( \mu(iT_s) \) is to compensate for the prediction error accumulated since the previous sample period.

**Remark 2.4.** Notice that for every number \( T_s > 0 \) the quantity \( \Phi(T_s) \) in (2.20) is always bounded away from zero. This implies that the inverse of \( \Phi(T_s) \) always exists, and the sampled update law can be applied to any system. The same holds also for systems of arbitrary dimension, including non strictly positive real systems, as proved in [3,41], where \( \Phi(T_s) \) is a non-singular matrix of a more general structure. However, notice that \( \Phi(T_s) \) approaches zero as \( T_s \to 0 \). This implies that the inversion of \( \Phi(T_s) \) is possible only in discrete-time setting. Despite that, the control signal is continuous, as it is defined via an output of a lowpass filter.

### 2.4.1. Simulation Results

To demonstrate the interpretation of the \( L_1 \) adaptive controller with piecewise constant adaptation laws given above we perform numerical simulations of the closed-loop control system. Consider the plant in (2.17). Let the system parameters be given by \( a = a_m = b = 1 \). In this case, the objective is to compensate only for the disturbance \( d(t) \).

We set the sampling time \( T_s = 0.01 \) s, which results in the following

\[ \Phi(T_s) = \frac{1}{a_m} \left( 1 - e^{-a_mT_s} \right) = 0.01, \]

and

\[ \mu(iT_s) = e^{-a_mT_s}\tilde{x}(iT_s) = 0.99\tilde{x}(iT_s). \]

For the control design we choose the following first order lowpass filter:

\[ C(s) = \frac{1}{0.1s + 1}. \]
Let the disturbance be given by

\[ d(t) = 100(u(t - 0.01) - u(t - 0.02)) + (300 + 500 \sin(500t))(u(t - 0.04) - u(t - 0.05)) + (500 \sin(1000t) - 500)(u(t - 0.07) - u(t - 0.08)), \]

where \( u(t) \) denotes a step function. This function is zero everywhere except for 3 intervals of the length \( T_s \). The distance between these intervals is \( 2T_s \). On the first interval the disturbance is constant, while on the second and the third intervals it contains sine waves with different frequencies and bias. Such shape allows for separation of the system response to the error produced by the disturbance from the intervals, where the system is affected by the disturbance. This separation occurs because the length and the starting time of the disturbance intervals coincide with the sampling period and the time of samples of the system. In particular, this means that during the first disturbance interval the sampled update law will generate zero estimate. It will respond to the effect of the disturbance only in the second sample period, when the disturbance is zero. Thus, we will be able to observe the system transient resulting from the disturbance and the system response to the resulting error separately.

Figure 2.2 shows the simulation results for the system without the plant uncertainties and the disturbance \( d(t) \). The plot of the prediction error given by Figure 2.2c along with the plot for the parameter estimates given by Figure 2.2d show that the \( \mathcal{L}_1 \) controller does not respond to the disturbance during the sampling period, when the disturbance occurs. However, at the beginning of the following sampling period the controller generates a parameter estimate, which completely cancels the prediction error, accumulated during this sampling period. During other sampling periods both the prediction error and the parameter estimate remain zero.

The perfect cancellation of the error caused by the disturbance \( d(t) \) in one sample step as observed in Figure 2.2c, is possible, if the plant pole is the same as the pole of the desired dynamics, i.e. \( a = a_m \). Otherwise, if \( a \neq a_m \), the ideal plant dynamics used by the inversion based update law are significantly different from the real plant dynamics, which results in additional prediction error. This error is treated by the \( \mathcal{L}_1 \) controller the same way as the error resulting from the disturbance.

Figure 2.3 shows the simulation results for the unstable plant with the parameters \( a = -0.5, b = 0.5 \) and the same \( \mathcal{L}_1 \) controller. Notice that while the controller still tends to cancel the prediction error, it is not able to compensate for it completely due to the uncertain plant
Figure 2.2: System response for disturbance $d(t)$. 

dynamics. However, a small value of the sampling time ($T_s = 0.01$ s) makes this error small, as predicted in [3, 41].
Figure 2.3: Uncertain system response for disturbance $d(t)$. 
CHAPTER 3

**Application to Ares I Crew Launch Vehicle Model**

Ares-I is a two stage rocket with a solid propellant first stage derived from the Shuttle Reusable Solid Rocket Motor and an upper stage that uses engines based on the Saturn V. Among numerous technical challenges in building a CLV is the ascent flight control system. Problems in vehicle control arise because long, slender launch vehicles, such as Saturn V and Ares-I, cannot be considered rigid but must be treated as flexible structures. Similar to flexible aircraft, the resulting dynamics are highly coupled with significant interactions between rigid body dynamics and structural modes. Since the structure possesses low damping, oscillatory bending modes of considerable amplitude can be produced, thus, subjecting control sensors to these large amplitudes at their particular location. If not properly accounted for, the local sensor readings are interpreted as describing the total vehicle behavior which may cause self-excitation and instability of the control system. A description of the particular challenges associated with the Ares-I Crew Exploration Vehicle and the ascent control system design goals are presented in [52].

The control challenges associated with an Ares-I CLV and the potential of $L_1$ adaptive control theory motivated this work. We explore the $L_1$ adaptive output feedback control architecture described in Chapter 2 to achieve the tracking objective and guarantee stability and robustness in the presence of uncertain dynamics, such as changes in flexible mode characteristics, and unexpected failures.

In what follows we describe a generic flexible crew launch vehicle model, then present the implementation of the $L_1$ adaptive controller for the generic CLV and show the simulation results and analysis of the designed control system.

### 3.1. Generic Crew Launch Vehicle Model

A nonlinear comprehensive model of a generic CLV, obtained by amalgamation of several legacy vehicles exhibiting the desired characteristics of a flexible space launch vehicle, was obtained from NASA Marshal Space Flight Center. This publicly released generic crew launch vehicle model has been distributed in a SAVANT Matlab/Simulink based tool [53,54].

In the CLV model used for $L_1$ adaptive controller design, the control system commanded trajectory $r(t)$ is generated by a guidance system and is represented by quaternions that define the desired position of vehicle’s body frame with respect to an inertial frame. The
guidance system is not modeled in the simulation: instead the commanded trajectory is taken from a file provided with the model. The feedback signal from the plant \( y(t) \) is the vehicle’s angular position in the body reference frame expressed in quaternions. The input \( e(t) \) into the \( L_1 \) adaptive controller is the attitude tracking error in roll \( \phi \), pitch \( \theta \), and yaw \( \psi \). A control conversion block is used to compute the three dimensional error vector between the four-dimensional commanded trajectory and the output quaternions. The control input signal consists of three components: one is the commanded thrust for the Reaction Control System (RCS), which controls body roll axis only, and the other two components represent the commanded thrust vector gimbal angles for the Solid Rocket Booster (SRB) in pitch and yaw directions. The only actuator dynamics present in the model are those associated with the SRB control of the pitch and yaw axes.

The plant model simulates the kinematics and the dynamics of the vehicle and takes into account the following:

- CLV aerodynamic forces and torques,
- SRB engine dynamics,
- Gravity model,
- Nozzle engine inertia effects (Tail-Wags-Dog),
- Slosh in fuel tanks of the second stage,
- Flexible body dynamics,
- Actuator dynamics for the SRB control system.

In the following we describe the fundamental equations, on which the launch vehicle dynamics are based.

### 3.1.1. Kinematic and Dynamic Equations for the CLV

The simulation model uses three reference frames to define all angular and translational coordinates of the launch vehicle. These frames are shown in Figure 3.1. The \( \Upsilon \) is a global inertial frame without considering heliocentric-rotation, which is connected with the Earth center. The \( Z \) axis is directed to the north gyro-pole. The local frame has its origin connected to Earth center and rotates with the Earth. The \( Z_l \) axes of the local frame coincides with the \( Z_\Upsilon \). The body frame has its origin at the vehicle center of gravity and the \( X_b \) axis is directed along the center line towards the nose of the rocket.
The equations of angular motion are given by

\[ \dot{\omega}(t) = \epsilon(t) , \]
\[ \dot{Q}_{Ib}(t) = \frac{1}{2}Q_{Ib}(t) \begin{bmatrix} \omega(t) \\ 0 \end{bmatrix} , \]

where

\[ \epsilon(t) = I^{-1}(t)(M_a(P, \rho, v, Q_{Ib}, t) + M_{rcs} + M_r(P, \theta_N, \psi_N, t) \\
+ M_{TWD}(\ddot{\theta}_N, \ddot{\psi}_N) + M_{sl}(a_b, Q_{Ib}, \epsilon, \omega, t) - \omega(t) \times (I(t)\omega(t))) . \]

The equations of translational motion are given by

\[ \dot{v}(t) = a(t) , \]
\[ \dot{p}(t) = v(t) , \]

where

\[ a(t) = Q^*_I a_b(t) Q_I + g(p) , \]
\[ a_b(t) = \frac{F_a(P, \rho, v, Q_{Ib}, t) + F_r(P, \theta_N, \psi_N, t) + F_{sl}(a_b, Q_{Ib}, \epsilon, \omega, t)}{m(t)} . \]
In the above equations, the system states are given via the following variables:

- \( \omega(t) \) is the vector of angular rates of CLV in the body frame,
- \( Q_{Ib}(t) \) is the quaternion of translation from the \( \Upsilon \) frame to the body frame,
- \( v(t) \) is the linear velocity vector presented in the \( \Upsilon \) frame,
- \( p(t) \) is the position of CLV’s center of mass in the \( \Upsilon \) frame.

The control input variables are:

- \( \theta_N \) is the nozzle position corresponding to the pitch angle,
- \( \psi_N \) is the nozzle position corresponding to the yaw angle,
- \( M_{rcs} \) is the torque applied by RCS engine.

The angular acceleration in the body frame and the translational acceleration in the \( \Upsilon \) frame are denoted by \( \dot{\epsilon}(t) \) and \( \dot{a}(t) \) respectively. Further, \( a_b(t) \) is the translational acceleration, without gravity, in the body frame, and \( g(P) \) denotes the gravity acceleration. In the equation of angular motion (3.1), the following torques are taken into account: \( M_a(P, \rho, v, Q_{Ib}, t) \) is the torque induced by aerodynamic effects, \( M_r(P, \theta_N, \psi_N, t) \) is the rocket engine torque, \( M_{TWD}(\ddot{\theta}_N, \ddot{\psi}_N) \) is the torque due to engine nozzle inertia effect, \( M_{sl}(a_b, Q_{Ib}, \epsilon, \omega, t) \) is the slosh induced torque. In the equations of translational motion (3.2), the following forces are considered: \( F_a(P, \rho, v, Q_{Ib}, t) \) is the aerodynamic force, \( F_r(P, \theta_N, \psi_N, t) \) is the main rocket engine force, \( F_{sl}(a_b, Q_{Ib}, \epsilon, \omega, t) \) is the slosh induced force. Finally, \( I(t) \) denotes the inertia tensor, \( m(t) \) is the mass of the vehicle, and \( P \) and \( \rho \) are the static pressure and the atmospheric density, respectively, at the vehicle’s current position.

### 3.1.2. CLV Aerodynamics

The aerodynamic model consists of three parts: flight conditions model, aerodynamic coefficients lookup tables, and computation of aerodynamic forces and torques. The first part performs calculations of altitude, Mach number, dynamic pressure, angle of attack and sideslip. Then these variables are used to obtain the corresponding information on aerodynamic coefficients and baseline forces from the lookup tables, which are based on wind tunnel data. The computation of forces and moments is done according to the following
equations
\[ F_a = \bar{q}SC_f + F_{\text{base}}, \]
\[ M_a = \bar{q}ScC_m + r_g \times F_a, \]
where \( \bar{q} \) is the dynamic pressure, \( S \) is the surface area, \( C_f \) and \( C_m \) are the aerodynamic coefficient matrices, \( F_{\text{base}} \) is the base force, \( c \) is the aerodynamic cord length, and \( r_g \) is the position of aerodynamic force center point with respect to the center of mass of the vehicle.

3.1.3. SRB Engine

The engine model computes the propulsive force, \( F_r(P, \theta_N, \psi_N, t) \), and the moment, \( M_r(P, \theta_N, \psi_N, t) \). First, the thrust in the vacuum corresponding to the current time is read from a table, then it is recalculated for the current value of the static pressure. The rocket engine force and moment are obtained by considering current gimbal angles and the engine location with respect to the center of mass of the vehicle.

3.1.4. Gravity Model

The non-spherical Earth effects are taken into account by the model, which is based on the J4 NASA gravity model.

3.1.5. Nozzle Engine Inertia Effects

The torque produced by the Tail-Wags-Dog (TWD) effect is calculated according to the following equation:
\[ M_{\text{TWD}} = \begin{bmatrix} 0 \\ \ddot{\theta}_N \\ \ddot{\psi}_N \end{bmatrix} I_N, \]
where \( I_N \) is the nozzle’s inertia tensor.

3.1.6. Slosh Model for the Fuel Tanks of the Second Stage

The slosh model consists of two similar models for liquid oxygen and hydrogen. The fuel slosh phenomena are modeled via a spring-damper systems. All the parameters are functions of the liquid fuel level in the tanks and are taken from the lookup tables.
3.1.7. Flexible Body Dynamics

Flexible body dynamics are linear and are based on a modal data set that contains mode shapes and frequencies. These key elements reflect the location of the sensors and the actuators. Flexible dynamics are integrated into the model as an additive component to the rigid body sensor computations of the angular position and the angular rate.

3.1.8. Actuator Models

As the roll channel has no actuator model dynamics, the command is directly transformed into thrust that results in the RCS torque applied to the plant. The pitch and yaw channels have the same second order actuator model given by the following transfer function:

$$T_{act}(s) = \frac{a_0}{b_2 s^2 + b_1 s + b_0}.$$ 

The actuator bandwidth in the provided model is roughly 3.3 Hz.

3.2. Simulation Results

Since the coupling between control channels is not desired and can be treated as system uncertainty, we design the $L_1$ adaptive controller for each channel independently using the same controller structure but with different parameters. In this framework, the inputs to the $L_1$ adaptive controller are the computed tracking errors in terms of Euler angles. Therefore, the commanded input $r(t)$ in equation (2.13) becomes $r(t) \equiv 0$, which leads to the following control law for each channel

$$u(s) = -\frac{C(s)}{M(s)}c_m^\top(sI - A_m)^{-1}\dot{s}(s).$$

The structure of the output predictor (2.11) and the adaptation law (2.12) remains the same with $y(t)$ being the measured Euler angle errors.

In the current design the following transfer functions for the desired dynamics were selected:

$$M_{\phi}(s) = K_{M\phi} \frac{1}{1/\omega_{M\phi}^2 s^2 + 2\xi_{M\phi}/\omega_{M\phi} s + 1}$$

for roll and

$$M_{\theta,\psi}(s) = K_M \frac{1}{1/\omega_{M\theta}^2 s^2 + 2\xi_{M\theta}/\omega_{M\theta} s + 1} \frac{1}{1/\omega_{M\psi}^2 s^2 + 2\xi_{M\psi}/\omega_{M\psi} s + 1}$$

for pitch and yaw, respectively.
for pitch and yaw. We can choose the same transfer functions and controller parameters for pitch and yaw channels because the CLV is symmetric and the plant properties for those channels are identical. The transfer function parameters are: $K_{M\phi} = 1$, $\omega_{M\phi} = 0.7$, $\xi_{M\phi} = 0.707$, $K_M = 2.82$, $\omega_M = 0.7$, $\xi_M = 0.7$, $\omega_{Mp} = 1.32$, $\xi_{Mp} = 0.7$, $\omega_M = 0.08$, $\xi_M = 0.604$. The Bode diagrams of $M_\phi(s)$ and $M_{\theta,\psi}(s)$ are shown in Figure 3.2. We notice that due to the nature of the system, the bandwidth of the roll channel is significantly higher than the bandwidth of the lateral channels.

![Bode Diagram](image)

**Figure 3.2:** Pitch and roll desired system frequency response.

The following lowpass filters were selected:

$$C_\phi(s) = \frac{1}{1/\omega_{C\phi}^2 s^2 + 2\xi_{C\phi}/\omega_{C\phi}s + 1} \frac{1}{1/(3\omega_{C\phi})s + 1}$$

for roll channel and

$$C_{\theta,\psi}(s) = \frac{1/\omega_{Cz}s + 1}{1/\omega_{Cp}s + 1} \frac{1}{1/\omega_{Cp}s^2 + 2\xi_{C}/\omega_{C}s + 1}$$

for pitch and yaw channels, where the following parameters were used: $\omega_{C\phi} = 7$, $\xi_{C\phi} = 1$, $\omega_{Cz} = 1$, $\omega_{Cp} = 100$, $\omega_C = 1$, $\xi_C = 5$. The frequency responses of $C_\phi(s)$ and $C_{\theta,\psi}(s)$ are shown in Figure 3.3. The sampling time for all adaptive controllers was set to $T = 0.001$ s.

Full nonlinear simulations for the closed loop system, with all modeled events included, were run to evaluate the performance of the $\mathcal{L}_1$ adaptive controller. The results were obtained for both cases, with and without flexible dynamics. For reference purposes, the guidance commanded ascent trajectory is plotted in Figure 3.4. Note that for the first 10 seconds in all plots in this section, the command is held constant. This is done to verify that the
Figure 3.3: Pitch and roll lowpass filter frequency response.

Figure 3.4: Guidance commanded trajectory.

$L_1$ adaptive controller does not produce spurious signals.

For the purpose of comparison, the results for the baseline controller are reviewed first. The baseline controller that was provided with the model is decoupled and has the same architecture in all three axes (roll, pitch, yaw). The architecture in each axis consists of a lowpass filter on the error signal coming into the controller. The main purpose of the filters is to filter the high frequencies that appear in the error signals. The filters are followed by a PID controller. The PID controller gains are scheduled on the relative velocity of the vehicle. The generated control command signal is bounded by a saturation block, which introduces
the physical control limitations of the plant.

Figures 3.5-3.6 show the results for the generic CLV, with and without flexible dynamics, with the baseline PID controller. From Figures 3.5b, 3.6b it is clear that the PID controller is unable to handle flexible dynamics without the notch filters in the loop. This implies high sensitivity to uncertainty in flexible body dynamics. Furthermore, such a controller requires accurate design with appropriate selection of notches and gain scheduling.

(a) Flexible dynamics disabled.

(b) Flexible dynamics enabled.

Figure 3.5: Tracking errors of the closed-loop system with baseline PID controller.
Figures 3.7-3.8 show performance results for the generic flexible CLV, with and without flexible dynamics, with the $\mathcal{L}_1$ adaptive output feedback controller. Note that the time response of the closed-loop system with and without flexible dynamics is almost the same, which implies that the $\mathcal{L}_1$ adaptive controller does not excite the flexible modes. Comparing the two cases, tracking errors do not increase substantially in the presence of flexible dynamics. The $\mathcal{L}_1$ adaptive controller commands in all three axes, with and without flexible dynamics, are very similar in magnitude to those of the baseline PID with rigid body dynamics only. Furthermore, the trajectory tracking performance of the closed-loop system is illustrated in Figure 3.9. We see that the $\mathcal{L}_1$ adaptive controller achieved the desired tracking performance for the transient during ascent. For completeness we also present, in Figure 3.10, the commanded and actual control signals in the pitch and yaw axes. These plots show us that the control signal produced by $\mathcal{L}_1$ adaptive controller does not exceed the available bandwidth of the control system [55].

The results clearly indicate that the system has good tracking performance with small errors. These results demonstrate that a single design of $\mathcal{L}_1$ adaptive controller is able to handle statically unstable flexible plant with large parametric variation in mass, velocity, aerodynamics properties without addition of notch filters and without retuning for different flight conditions along the first stage of the trajectory.
(a) Flexible dynamics disabled.

(b) Flexible dynamics enabled.

Figure 3.6: Closed-loop system control command with baseline PID controller.
Figure 3.7: Tracking errors of the closed-loop system with $\mathcal{L}_1$ controller.

(a) Flexible dynamics disabled.

(b) Flexible dynamics enabled.
Figure 3.8: Closed-loop system control command with $L_1$ controller.
Figure 3.9: Trajectory tracking performance of the $\mathcal{L}_1$ controller with flexible dynamics enabled.

Figure 3.10: Actuator response of the closed-loop system with $\mathcal{L}_1$ controller.
CHAPTER 4

Application to Human Anesthesia Control

In this chapter we present another application of the $L_1$ adaptive control theory. This application shows that the methods of $L_1$ control theory are not limited only to the aerospace field and can be successfully applied to a broad area of engineering and control systems design field. The specifics of the considered in this chapter problem required us to apply several known output predictor modifications as well as introduce a new control switching algorithm for improved transient performance in the presence of nonzero initialization error. Stability and performance of the proposed switching algorithm are ensured by a proof along with transient performance bounds computed for control signal. Simulations results in this chapter are based on 6 different patient models obtained by processing six experiment data sets. We also present the system identification (ID) algorithm, which was used for computation of the patient models.

4.1. Control Problem Description

During surgery, one of the main responsibilities of the attending anesthesiologist is to fill the role of a multivariable feedback controller for a complex and highly-coupled system. In this role, the anesthesiologist must continuously observe and adjust the rates and overall amounts of anesthetic agents delivered to the patient, with the fundamental goal being to maintain appropriate levels of sedation, analgesia and muscle relaxation. At the same time, the anesthesiologist must maintain ventilation parameters and monitor hemodynamic and respiratory functions, including heart rate, blood pressure, oxygen saturation and exhaled carbon dioxide (CO$_2$) levels. Our long-term goal in this research, is to develop methods for partially automating the delivery of anesthesia, allowing the anesthesiologist to focus on more critical, or potentially urgent, events that occur during surgery.

To design and implement model-based feedback control in anesthesia delivery we require (1) adequate and appropriate means of sensing a patient’s levels of sedation; and (2) relatively simple mathematical models capturing the patient’s response to anesthetic agents. Over the past two decades, the *bispectral index* (BIS), a statistical index based on phase and frequency relations between the component frequencies in EEG recordings, has been used extensively as a measure of sedation level [56–58], although it is not entirely without controversy [59]. The BIS value is a single dimensionless number ranging from 0 to 100, where 100 corresponds
to a patient being fully awake and alert, and 0 corresponds to a silent EEG. A BIS value between 60 and 40 is considered a viable level for general anesthesia, in which the patient is not aware and surgery can be performed [60]. In this chapter, our main focus will be on the design and evaluation of feedback controllers that maintain an adequate sedation level yet require a minimum amount of anesthetic, and more importantly, that demonstrate robustness to some of the external disturbances encountered in a surgical setting.

4.2. Patient Modeling and System Identification

In the control design work discussed herein, we refer to clinical trial data used in earlier studies [61–63]. The original clinical trial from which this data was collected was designed to define the relations between clinical evaluation of the state of consciousness, explicit recall, drug concentrations and BIS effects of the anesthetic agent isoflurane when administered to healthy volunteers (further called as patients) under controlled conditions. In addition to the isoflurane, a series of external stimuli (disturbances) were applied to the patients throughout the administration of anesthesia. These stimuli included: laryngeal mask insertion and removal; evoked potential evaluations; and alertness evaluations that included yelling at, shaking, and squeezing the trapezius muscle of the patient. Time-synchronized output measurements of the patients’ BIS as well as isoflurane concentrations by volume were recorded every $\tau_p = 5$ seconds. An example of a set of data taken from one subject during the clinical trial is shown in Figure 4.3. This data is fairly representative of the response expected from healthy volunteers to anesthesia and stimuli, however, as to be expected individual responses exhibit some variation. Inter-patient variability is one of the main motivations for considering the adaptive control techniques for this problem.

In our study we use gray box system ID, which allows for explicitly taking into account changes in the patient sensitivity to the drug input during the trial procedure. This is done by considering patient models of the form:

$$
\dot{x}_p(t) = A_p x_p(t) + b_p (u(t) + \alpha_{in}), \quad x_p(0) = x_0, \\
y(t) = k_a(t)(c^T_p x_p(t) + \alpha_{out}) + \delta(t),
$$

(4.1)

where $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$ are the control input represented by isoflurane concentration by volume and the measured output represented by BIS index respectively, $(A_p \in \mathbb{R}^{n \times n}, b_p \in \mathbb{R}, c^T_p \in \mathbb{R}^{1 \times n})$ are the patient dynamics matrices, $\alpha_{in} \in \mathbb{R}$ and $\alpha_{out} \in \mathbb{R}$ are the input and output mean biases, $k_a(t) \in \mathbb{R}$ is the time-varying output gain, and $\delta(t) \in \mathbb{R}$ is the output disturbance. We develop a closed-loop controller for the sedated state, and therefore we
perform identification only using data intervals with BIS < 70. Our assumption is that the attending physician performs the initial induction from the alert state to the lightly-sedated state in order to closely monitor initial patient response. Upon being lightly sedated and observed for safety reasons, the patient is then switched to the proposed automated control regime.

We start the identification process by computing the gain $k_a(t)$. Since the sensitivity of the patient to the sedative gradually decreases during the surgery, we use piecewise linear functions to model the gain, which consists of 3 intervals: initial constant gain, growth interval, and final constant. Figure 4.1 shows a typical output gain $k_a(t)$ computed for Patient 1. This choice of model of $k_a(t)$ leads to four parameters, which need to be identified in order to completely define the function $k_a(t)$: initial gain, final gain and start times and end times of gain rise. These parameters are determined by averaging the BIS signal over two steady-state regions of the clinical data. For instance, Figure 4.3 shows three steady state regions at approximately $t = 4000$ s, $t = 6000$ s and $t = 10000$ s. For our design we choose two of these regions: $t = 6000$ s and $t = 10000$ s, which result in the gain plot in Figure 4.1. Notice that we did not choose the interval around $t = 4000$ s because it is close to the initial sedation and hence might be affected by the unmodeled sedation dynamics.

![Figure 4.1: Output gain $k_a(t)$ for Patient 1.](image)

The next step of the model (4.1) ID involves identification and removal of the input $\alpha_{in}$ and output $\alpha_{out}$ biases and gain normalization. The input and output biases are determined using the Matlab function `detrend(·,0)`, and the gain normalization is performed by dividing the clinical BIS response by the gain $k_a(t)$. The resulting data for Patient 1 are shown in Figure 4.2.

Next, we inspect the normalized data and determine the intervals which contain small disturbances, show sufficiently rich dynamics and correspond to the sedated state. In Figure 4.2, these regions are colored in red. After the data for system ID are selected, we
determine the order of the model by analyzing the system Hankel singular values (Matlab function \texttt{n4sid(},1:11,\tau_p,0))\texttt{), and then perform system ID using subspace identification methods (Matlab function \texttt{n4sid(},n,\tau_p,0))\texttt{). Employing the approach described herein resulted in models of \textit{first and second orders}, which is significantly lower as compared to our earlier work based on black box system ID [64,65]. The validation results for one of the patients are shown in Figure 4.3. We see that our model captures well the patient transient and steady state behaviors during sedation.

Finally, we compute the output disturbance model $\delta(t)$. For each patient we subtract the measured BIS signal from the simulated response signal derived using the measured ISO as an input, as shown in Figure 4.3. The resulting disturbance for one of the patients is shown in Figure 4.4. We use this disturbance in the simulations to test the performance of the $\mathcal{L}_1$ adaptive control system. Notice that in this case the disturbance captures real
measurement noise, response of the patient to the external stimuli and also contains modeling error.

![Figure 4.4: Output disturbance $\delta(t)$ for Patient 1.](image)

To complete the simulations using the model given by (4.1), we assume input and output sampling intervals of time $\tau_p$, and an input saturation level of 3% concentration (by volume) of isoflurane. The block diagram of the complete patient model is shown in Figure 4.5. Our ID process is summarized in Table 4.1. We apply this process to all six patient data sets, which results in 6 different models to be used for $\mathcal{L}_1$ adaptive controller design and validation.

![Figure 4.5: Block diagram of the patient model.](image)

4.3. $\mathcal{L}_1$ Adaptive Controller Architecture

The patient dynamics resulting from system identification is given as a SISO transfer function, which implies that the output-feedback $\mathcal{L}_1$ adaptive control architecture presented in Chapter 2 is the most suitable for this application.

In this chapter we also consider the problem of safe controller switching. Since the patients’ models discussed in the preceding section are valid only during sedation (BIS < 70), the designed controller would not be used to initiate the anesthetic induction. Therefore due to safety considerations the surgery is initiated with manual open-loop control by the
1. Analyze BIS response and compute gain $k_a(t)$. Normalize BIS using this output gain.

2. Subtract mean biases $\alpha_{\text{in}}$ and $\alpha_{\text{out}}$ from the BIS data.

3. Determine data intervals that correspond to the sedated state, contain small disturbances, and contain sufficiently rich dynamics.

4. Determine appropriate model order using Hankel singular value analysis.

5. Perform system ID on preprocessed data from intervals determined in Step 3.

6. Compute the output disturbance $\delta(t)$.

7. Validate the ID model using clinical trial data.

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
<th>Matlab function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Analyze BIS response and compute gain $k_a(t)$. Normalize BIS using this output gain.</td>
<td>—</td>
</tr>
<tr>
<td>2</td>
<td>Subtract mean biases $\alpha_{\text{in}}$ and $\alpha_{\text{out}}$ from the BIS data.</td>
<td><code>detrend(:,0);</code></td>
</tr>
<tr>
<td>3</td>
<td>Determine data intervals that correspond to the sedated state, contain small disturbances, and contain sufficiently rich dynamics.</td>
<td>—</td>
</tr>
<tr>
<td>4</td>
<td>Determine appropriate model order using Hankel singular value analysis.</td>
<td><code>n4sid(:,1:11,\tau_p,0);</code></td>
</tr>
<tr>
<td>5</td>
<td>Perform system ID on preprocessed data from intervals determined in Step 3.</td>
<td><code>n4sid(:,n,\tau_p,0);</code></td>
</tr>
<tr>
<td>6</td>
<td>Compute the output disturbance $\delta(t)$.</td>
<td>—</td>
</tr>
<tr>
<td>7</td>
<td>Validate the ID model using clinical trial data.</td>
<td>—</td>
</tr>
</tbody>
</table>

**Table 4.1: System ID algorithm.**

The $L_1$ adaptive controller is engaged by a command from the anesthesiologist when the patient reaches an adequate sedated state, after which the anesthesiologist sets the desired value of BIS and the $L_1$ adaptive controller then adjusts the concentration of isoflurane automatically. We note that the anesthesiologist should also have an opportunity to override the closed-loop control mechanism and return to an open-loop control regime at any time for safety considerations. The main requirement for the $L_1$ adaptive controller during switching is to avoid undesired switching transients and bursting [66].

As noted in [3], a nonzero initialization error of the output predictor leads to exponentially decaying terms in the system transient, which may lead to undesired behavior after the $L_1$ adaptive controller is enabled. While in state-feedback architectures this problem can be resolved by proper initialization of the state-predictor using the first measurement of the system state, in the case of output-feedback this information is not available.

Therefore we propose the following switching method: the estimation loop of the $L_1$ adaptive controller should be enabled at the beginning of the surgery. However, during the period of time when open-loop control is used, the control signal generated by (2.13) (hereafter we denote it $u_{L_1}(t)$) is discarded, and both the patient and the output-predictor are fed with control signals entered by the anesthesiologist, that is $u(t) = u_a(t)$. During this regime, the initialization transients in the estimation loop converge and the prediction error is reduced to its normal value. After the anesthesiologist switches to closed-loop mode, the
control input for both patient and output predictor are switched to that generated by (2.13), that is, 
\[ u(t) = u^L_1(t) \]. This eliminates bursting due to output predictor state mismatch. This scheme also allows switching back to open-loop mode if requested by the anesthesiologist.

Using switching of the control signal requires consideration of the system (2.1) written in state space form with nonzero initial conditions:

\[
\begin{align*}
\dot{x}_p(t) &= Ax_p(t) + b(u(t) + f(t, y(t))), \quad x_p(0) = x_{p0}, \\
y(t) &= c^\top x_p(t),
\end{align*}
\]

where \( x_p(t) \in \mathbb{R}^n \) and \( y(t) \in \mathbb{R} \) are not the measured system state and the measured system output, respectively; \( A \in \mathbb{R}^{n \times n} \), \( b, c \in \mathbb{R}^n \) are unknown Hurwitz matrix and vectors such that \( A(s) = c^\top (sI - A)^{-1}b \), where \( A(s) \) was defined in (2.1); \( u(t) \in \mathbb{R} \) is a bounded exogenous system input with \( \|u\|_{L_\infty} \leq \bar{u} \), where \( \bar{u} \in \mathbb{R}^+ \) is known. The initial condition \( x_0 \) is unknown, however it is assumed to belong to a known compact set, such that \( \|x_{p0}\|_\infty \leq \rho_0 \) for a given \( \rho_0 \in \mathbb{R}^+ \); and \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is an unknown map subject to Assumption 2.1.

**Assumption 4.1.** We assume that the system in (4.2) is input-to-stable (ISS) with respect to \( u(t) \). For the system (4.2), this implies that there exit \( \alpha_A, \beta_A \in \mathbb{R}^+ \) such that

\[
\|y\|_{L_\infty} \leq \alpha_A \bar{u} + \beta_A,
\]

for all \( \|u\|_{L_\infty} \leq \bar{u} \) and all \( \|x_{p0}\|_\infty \leq \rho_0 \). We assume that some conservative knowledge of the values \( \alpha_A \) and \( \beta_A \) is available.

**Remark 4.1.** Notice that Assumption 4.1 always holds for anesthesia control, since open-loop patient’s BIS response to anesthesia input is always bounded.

For this system we use the \( L_1 \) adaptive control architecture defined in Chapter 2 with the only change in the output predictor. Since during the open-loop regime, the plant (4.2) uses an exogenous signal \( u(t) \) instead of the control input \( u^L_1(t) \), to ensure stability of the prediction error dynamics, we modify the the output predictor as following:

\[
\begin{align*}
\dot{\hat{x}}(t) &= A_m \hat{x}(t) + b_m u(t) + \hat{\sigma}(t), \quad \hat{x}(0) = \hat{x}_0, \\
\hat{y}(t) &= c_m^\top \hat{x}(t),
\end{align*}
\]

where \( \hat{x}_0 \) is chosen such that \( c_m^\top \hat{x}_0 = c^\top x_{p0} \) and \( \|\hat{x}_0\|_\infty \leq \rho_{\hat{x}_0} \), for some known \( \rho_{\hat{x}_0} \in \mathbb{R}^+ \), which can be always achieved since \( y(0) \) is measured. A block diagram illustrating this switching scheme is shown in Figure 4.6.
4.4. \( L_1 \) Adaptive Controller Analysis

Since stability and performance of the closed-loop \( L_1 \) adaptive controller was considered in Chapter 2, herein we focus on open-loop operation regime of the \( L_1 \) adaptive controller. Next we demonstrate how the proposed scheme can help to improve the closed-loop system performance. We start with stability analysis, which shows boundedness of all signals of the \( L_1 \) adaptive controller in the open-loop regime. Then we consider transients resulting from nonzero initialization error and analyze their impact on the closed-loop system performance.

4.4.1. Error Dynamics

The system (4.2) can be rewritten as

\[
y(s) = A(s)(u(s) + d(s)) + y_{in}(s),
\]

(4.4)

where \( A(s) \) is defined in (2.1), \( d(t) \triangleq f(t,y(t)) \), and \( y_{in}(t) \) is the output of the following autonomous system:

\[
\begin{align*}
\dot{x}_{in}(t) &= A x_{in}(t), \quad x(0) = x_{i0}, \\
y_{in}(t) &= c^\top x_{in}(t).
\end{align*}
\]

The system (4.4) along with the \( L_1 \) adaptive control law (2.13) can be rewritten as

\[
u_{L_1}(s) = F(s)k_y r(s) - \frac{C(s)}{c_m^\top (sI - A_m)^{-1}b_m} c_m^\top (sI - A_m)^{-1} \hat{\sigma}(s),
\]

(4.5)

where \( \hat{\sigma}(t) \) is defined in (2.12), and

\[
\sigma(s) = \frac{(A(s) - M(s))u(s) + A(s)d(s)}{M(s)}.
\]

(4.6)

Next, we rewrite (4.3) in frequency domain as

\[
\hat{y}(s) = M(s)u(s) + c_m^\top (sI - A_m)^{-1} \hat{\sigma}(s) + \hat{y}_{in}(s),
\]

(4.7)
where \( \tilde{y}_{\text{in}}(t) \) is the output of the following autonomous system:

\[
\begin{align*}
\dot{\hat{x}}_{\text{in}}(t) &= A_m \hat{x}_{\text{in}}(t), \quad \hat{x}(0) = \hat{x}_0, \\
\tilde{y}_{\text{in}}(t) &= c_m^\top \hat{x}_{\text{in}}(t).
\end{align*}
\]

Then the prediction error dynamics is computed by subtracting (4.5) from (4.7), and is given by

\[
\tilde{y}(s) = c_m^\top (sI - A_m)^{-1} \hat{\sigma}(s) - M(s) \sigma(s) + \tilde{y}_{\text{in}}(s),
\]

(4.8)

where \( \tilde{y}_{\text{in}}(s) = \tilde{y}_{\text{in}}(s) - y_{\text{in}}(s) \). This can be further simplified to

\[
\tilde{y}(s) = M(s) \hat{\sigma}(s) + \tilde{y}_{\text{in}}(s),
\]

(4.9)

where

\[
\hat{\sigma}(s) = \frac{c_m^\top (sI - A_m)^{-1} \sigma(s) - \sigma(s)}{M(s)}.
\]

(4.10)

Finally, we rewrite (4.8) in state space as

\[
\begin{align*}
\dot{\tilde{x}}(t) &= A_m \tilde{x}(t) - b_m \sigma(t) + \hat{\sigma}(t), \quad \tilde{x}(0) = \tilde{x}_0, \\
\tilde{y}(t) &= c_m^\top \tilde{x}(t),
\end{align*}
\]

(4.11)

where \( \tilde{x}_0 \in \mathbb{R}^n \) is an unknown bounded initial condition of the prediction error dynamics with \( \| \tilde{x}_0 \|_\infty \leq \tilde{\rho}_0 \), where \( \tilde{\rho}_0 \) can be computed using the bound \( \rho_{p0} \) and the initial condition \( \hat{x}_0 \). Also notice that according to the definition of \( \hat{x}_0 \), we have \( \tilde{y}(0) = 0 \).

### 4.4.2. Constant Definitions and Notation

Next we define the constants and notations used in the analysis. Most of the definitions are identical to the definitions introduced in Chapter 2, with the only difference in the definitions of constants \( \Delta \) and \( \alpha \). We repeat the affected definitions here for the sake of completeness. The rest of the notation used herein is defined in Chapter 2. Let

\[
\tilde{d} \triangleq L(\alpha A \bar{u} + \beta_A) + L_0,
\]

and

\[
\Delta \triangleq \left\| \frac{A(s) - M(s)}{M(s)} \right\|_{L_1} \bar{u} + \left\| \frac{A(s)}{M(s)} \right\|_{L_1} \tilde{d},
\]

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where $L$ and $L_0$ are defined in Assumption 2.1. Also, let $\varsigma(T_s)$ be defined as

\[
\varsigma(T_s) \triangleq \frac{\alpha}{\lambda_{\text{max}}(P_2)} + \kappa(T_s)\Delta,
\]

(4.12)

\[
\alpha \triangleq \max \left\| \lambda_{\text{max}}(A^\top P A^{-1}) \left( \frac{2\Delta \|A^\top P b_m\|}{\lambda_{\text{min}}(A^\top QA^{-1})} \right)^2, n\lambda_{\text{max}}(P_2(0))\|A\|_\infty \right\|.
\]

(4.13)

Next, we introduce the following functions

\[
\beta_1(T_s) \triangleq \max_{t \in [0, T_s]} |\eta_1(t)|, \quad \beta_2(T_s) \triangleq \max_{t \in [0, T_s]} \|\eta_2(t)\|,
\]

(4.14)

and also

\[
\beta_3(T_s) \triangleq \max_{t \in [0, T_s]} \eta_3(t), \quad \beta_4(T_s) = \max_{t \in [0, T_s]} \eta_4(t),
\]

(4.15)

where

\[
\eta_3(t) \triangleq \int_0^t 1^\top e^{A_1 A_3^{-1} (t-\tau)} \Lambda \Phi^{-1}(T_s) e^{A_1 A_3^{-1} T_s} 1 d\tau,
\]

\[
\eta_4(t) \triangleq \int_0^t 1^\top e^{A_1 A_3^{-1} (t-\tau)} \Lambda b_m d\tau.
\]

Finally, we define

\[
\gamma_0(T_s) \triangleq \beta_1(T_s)\varsigma(T_s) + \beta_2(T_s)\sqrt{\frac{\alpha}{\lambda_{\text{max}}(P_2)}} + \beta_3(T_s)\varsigma(T_s) + \beta_4(T_s)\Delta.
\]

Proposition 4.1. The following limiting relationship is true:

\[
\lim_{T_s \to 0} \gamma_0(T_s) = 0.
\]

The proof of this proposition is similar to the proof of Proposition 2.1 given in [3] and therefore is omitted here.

4.4.3. Stability of the Open-loop $L_1$ Adaptive Controller

The following theorem establishes stability of the estimation loop dynamics.

Theorem 4.1. Consider the system in (4.2) and the open-loop $L_1$ adaptive controller in (4.3), (2.12) and (2.13). The following bound holds for the estimation error dynamics:

\[
\|\tilde{y}\|_\infty \leq \gamma_0(T_s).
\]

(4.16)
Proof. From Assumption 4.1 we have,
\[ \|y\|_{\mathcal{L}_\infty} \leq \alpha_A \|u\|_{\mathcal{L}_\infty} + \beta_A = \alpha_A \bar{u} + \beta_A, \]
which along with Assumption 2.1 leads to
\[ \|d\|_{\mathcal{L}_\infty} \leq L(\alpha_A \bar{u} + \beta) + L_0 = \bar{d}. \]
From the definition of \(\sigma(s)\) in (4.6), we obtain the following bound
\[ \|\sigma\|_{\mathcal{L}_\infty} \leq \left\| \frac{A(s) - M(s)}{M(s)} \right\|_{\mathcal{L}_1} \bar{u} + \left\| \frac{A(s)}{M(s)} \right\|_{\mathcal{L}_1} \bar{d} = \Delta. \] (4.17)
Next, consider the state transformation
\[ \tilde{\xi}(t) = \Lambda \tilde{x}(t). \]
It follows from (4.11) and the definition of \(\Lambda\) in (2.7) that
\[ \dot{\tilde{\xi}}(t) = \Lambda A_m \Lambda^{-1} \tilde{\xi}(t) + \Lambda \dot{\sigma}(t) - \Lambda b_m \sigma(t), \quad \tilde{\xi}(0) = \Lambda \tilde{x}_0, \] (4.18)
where \(\tilde{\xi}_1(t)\) is the first element of \(\tilde{\xi}(t)\) and \(\tilde{\xi}_1(0) = 0\). Solving this, we obtain
\[ \tilde{\xi}(iT_s + t) = e^{\Lambda A_m \Lambda^{-1} t} \tilde{\xi}(iT_s) + \int_{iT_s}^{iT_s+t} e^{\Lambda A_m \Lambda^{-1} (iT_s + t - \tau)} \Lambda \dot{\sigma}(iT_s) d\tau 
- \int_{iT_s}^{iT_s+t} e^{\Lambda A_m \Lambda^{-1} (iT_s + t - \tau)} \Lambda b_m \sigma(\tau) d\tau \]
\[ = e^{\Lambda A_m \Lambda^{-1} t} \tilde{\xi}(iT_s) + \int_0^t e^{\Lambda A_m \Lambda^{-1} (t - \tau)} \Lambda \dot{\sigma}(iT_s) d\tau 
- \int_0^t e^{\Lambda A_m \Lambda^{-1} (t - \tau)} \Lambda b_m \sigma(iT_s + \tau) d\tau. \] (4.19)
Next we break \(\tilde{\xi}(iT_s + t)\) into two components as follows
\[ \tilde{\xi}(iT_s + t) = \chi(iT_s + t) + \zeta(iT_s + t), \] (4.20)
where
\begin{align}
\chi(iT_s + t) &\triangleq e^{A\Lambda m \Lambda^{-1} t} \begin{bmatrix}
\bar{y}(iT_s) \\
0
\end{bmatrix} + \int_0^t e^{A\Lambda m \Lambda^{-1} (t-\tau)} \Lambda \hat{\sigma}(iT_s) d\tau, \\
\zeta(iT_s + t) &\triangleq e^{A\Lambda m \Lambda^{-1} t} \begin{bmatrix}
0 \\
\bar{z}(iT_s)
\end{bmatrix} - \int_0^t e^{A\Lambda m \Lambda^{-1} (t-\tau)} \Lambda b_m \sigma(iT_s + \tau) d\tau,
\end{align}
\tag {4.21}

and \( \bar{z}(t) \triangleq [\tilde{\xi}_2(t), \tilde{\xi}_3(t), \ldots, \tilde{\xi}_n(t)] \). It also follows from (4.19) that
\begin{align}
\chi(iT_s + t) &= \begin{bmatrix}
\bar{y}(iT_s + t) \\
0
\end{bmatrix}, \\
\zeta(iT_s + t) &= \begin{bmatrix}
0 \\
\bar{z}(iT_s + t)
\end{bmatrix}.
\end{align}

Next we prove by induction that for all \( iT_s \) one has
\begin{align}
|\bar{y}(iT_s)| &\leq \varsigma(T_s), \\
\bar{z}^T(iT_s) P_2 \bar{z}(iT_s) &\leq \alpha,
\end{align}
\tag {4.23}  
\tag {4.24}

where \( \varsigma(T_s) \) and \( \alpha \) are defined in (4.12)-(4.13), and \( P_2 \) is introduced in Lemma 2.1.

We start by noting that, due to the initialization procedure we have \( \bar{y}(0) = 0 \), which implies \( |\bar{y}(0)| \leq \varsigma(T_s) \). We can also show that given the definition of \( \alpha \) in (4.13) we have
\begin{align}
\bar{z}^T(0) P_2 \bar{z}(0) &\leq \lambda_{\max}(P_2) \| \bar{z}(0) \|^2 = \lambda_{\max}(P_2) \| \tilde{\xi}(0) \|^2 \\
&\leq \lambda_{\max}(P_2) \| \Lambda \tilde{x}_0 \|^2 \leq n \lambda_{\max}(P_2) \| \Lambda \|_\infty \tilde{\rho}_0^2 \leq \alpha.
\end{align}

Next, for arbitrary \( (j+1)T_s \), we prove that if
\begin{align}
|\bar{y}(jT_s)| &\leq \varsigma(T_s), \\
\bar{z}^T(jT_s) P_2 \bar{z}(jT_s) &\leq \alpha,
\end{align}
\tag {4.25}  
\tag {4.26}

then the inequalities (4.25)-(4.26) hold for \( j+1 \) as well, which would imply that the bounds in (4.23)-(4.24) hold for all \( iT_s \).

To this end, assume that (4.25)-(4.26) hold for \( j \). Then (4.20) holds, and (4.21) leads to
\begin{align}
\chi((j + 1)T_s) &= e^{A\Lambda m \Lambda^{-1} (j+1)T_s} \begin{bmatrix}
\bar{y}(jT_s) \\
0
\end{bmatrix} + \int_0^{(j+1)T_s} e^{A\Lambda m \Lambda^{-1} (T_s-\tau)} \Lambda \hat{\sigma}(jT_s) d\tau.
\end{align}
\tag {4.27}
Substituting the adaptation law from (2.12) in (4.27), we have

$$
\chi((j + 1)T_s) = 0 .
$$

(4.28)

It follows from (4.22) that $\zeta(t)$ is the solution to the following dynamics:

$$
\dot{\zeta}(t) = \Lambda A_m \Lambda^{-1} \zeta(t) - \Lambda b_m \sigma(t) ,
$$

(4.29)

$$
\zeta(jT_s) = \begin{bmatrix} 0 \\ \tilde{z}(jT_s) \end{bmatrix} , \quad t \in [jT_s, (j + 1)T_s].
$$

(4.30)

Consider now the following function

$$
V(t) = \zeta^\top(t) \Lambda^{-\top} P \Lambda^{-1} \zeta(t)
$$

over $t \in [jT_s, (j + 1)T_s]$. Since $\Lambda$ is nonsingular and $P$ is positive definite, $\Lambda^{-\top} P \Lambda^{-1}$ is positive definite and, hence, $V(t)$ is a positive definite function. It follows from Lemma 2.1 and the relationship in (4.30) that

$$
V(\zeta(jT_s)) = \tilde{z}^\top(jT_s) P_2 \tilde{z}(jT_s),
$$

which, along with the upper bound in (4.26), leads to

$$
V(\zeta(jT_s)) \leq \alpha .
$$

(4.31)

It follows from (4.29) that over $t \in [jT_s, (j + 1)T_s]$

$$
\dot{V}(t) = \zeta^\top(t) \Lambda^{-\top} P \Lambda^{-1} A_m \Lambda^{-1} \zeta(t) + \zeta^\top(t) \Lambda^{-\top} A_m^\top \Lambda^\top \Lambda^{-\top} P \Lambda^{-1} \zeta(t)
$$

$$
- 2\zeta^\top(t) \Lambda^{-\top} P \Lambda^{-1} \Lambda b_m \sigma(t)
$$

$$
= -\zeta^\top(t) \Lambda^{-\top} Q \Lambda^{-1} \zeta(t) - 2\zeta^\top(t) \Lambda^{-\top} P b_m \sigma(t) .
$$

Using the upper bound from (4.17), one can derive over $t \in [jT_s, (j + 1)T_s]$

$$
\dot{V}(t) \leq -\lambda_{\min}(\Lambda^{-\top} Q \Lambda^{-1}) \|\zeta(t)\|^2 + 2\|\zeta(t)\| \|\Lambda^{-\top} P b_m\| \Delta .
$$

(4.32)

Notice that for all $t \in [jT_s, (j + 1)T_s]$, if

$$
V(t) > \alpha ,
$$

(4.33)
we have
\[ \|\zeta(t)\| > \sqrt{\frac{\alpha}{\lambda_{\max}(\Lambda^{-\top}P\Lambda^{-1})}} = \frac{2\Delta\|\Lambda^{-\top}Pb_m\|}{\lambda_{\min}(\Lambda^{-\top}Q\Lambda^{-1})}, \]
and the upper bound in (4.32) yields
\[ \dot{V}(t) < 0. \] (4.34)

Thus, it follows from (4.31), (4.33), and (4.34) that
\[ V(t) \leq \alpha, \quad \forall \ t \in [jT_s, (j + 1)T_s], \]
and therefore
\[ V((j + 1)T_s) = \zeta^T((j + 1)T_s)(\Lambda^{-\top}P\Lambda^{-1})\zeta((j + 1)T_s) \leq \alpha. \] (4.35)

Since (4.20) holds for \((j + 1)T_s\), the bound (4.35) along with (4.28) imply
\[ \tilde{\xi}^T((j + 1)T_s)(\Lambda^{-\top}P\Lambda^{-1})\tilde{\xi}((j + 1)T_s) \leq \alpha. \]

Using the result of Lemma 2.1, one can derive
\[ \tilde{z}^T((j + 1)T_s)P_2\tilde{z}((j + 1)T_s) \leq \tilde{\xi}^T((j + 1)T_s)(\Lambda^{-\top}P\Lambda^{-1})\tilde{\xi}((j + 1)T_s) \leq \alpha, \]
which implies that the upper bound in (4.26) holds for \(j + 1\).

Next, it follows from (4.18), (4.20), and (4.28) that
\[ \tilde{y}((j + 1)T_s) = 1_1^T\zeta((j + 1)T_s), \]
and the definition of \(\zeta((j + 1)T_s)\) in (4.22) leads to the following expression:
\[ \tilde{y}((j + 1)T_s) = 1_1^Te^{\Lambda A_m\Lambda^{-1}T_s}\begin{bmatrix} 0 \\ \tilde{z}(jT_s) \end{bmatrix} - 1_1^T\int_0^{T_s} e^{\Lambda A_m\Lambda^{-1}(T_s-\tau)}\Lambda b_m\sigma(jT_s + \tau) \, d\tau. \]
The bounds in (4.26) and (4.17) yield the following upper bound:

\[
|\tilde{y}(j+1)T_s| \leq \|\eta_2(T_s)\|\|\tilde{z}(jT_s)\| + \int_0^{T_s} |\mathbf{1}^T e^{A A_m \Lambda^{-1}(T_s-\tau)} \Lambda b_m \|\sigma(jT_s + \tau)|d\tau
\]

\[
\leq \|\eta_2(T_s)\| \sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \kappa(T_s) \Delta = \varsigma(T_s),
\]

where \(\eta_2(T_s)\) and \(\kappa(T_s)\) were defined in (2.9) and (2.10), while \(\varsigma(T_s)\) was defined in (4.12). This confirms the upper bound in (4.25) for \(j+1\). Hence, (4.23)-(4.24) hold for all \(iT_s\).

For all \(iT_s + t\), where \(0 \leq t \leq T_s\), using the expression from (4.19) we can write that

\[
\tilde{y}(iT_s + t) = \mathbf{1}^T e^{A A_m \Lambda^{-1}t} \tilde{\xi}(iT_s) + \mathbf{1}^T \int_0^t e^{A A_m \Lambda^{-1}(t-\tau)} \Lambda \tilde{\sigma}(iT_s) d\tau
\]

\[
- \mathbf{1}^T \int_0^t e^{A A_m \Lambda^{-1}(t-\tau)} \Lambda b_m \sigma(iT_s + \tau) d\tau.
\]

The upper bound in (4.17), the adaptation law (2.12) and the definitions of \(\eta_1(t)\), \(\eta_2(t)\), \(\eta_3(t)\) and \(\eta_4(t)\) lead to the following upper bound:

\[
|\tilde{y}(iT_s + t)| \leq |\eta_1(t)||\tilde{y}(iT_s)| + \|\eta_2(t)\|\|\tilde{z}(iT_s)\| + \eta_3(t)|\tilde{y}(iT_s)| + \eta_4(t)\Delta.
\]

Taking into consideration (4.23)-(4.24), and recalling the definitions of \(\beta_1(T_s)\), \(\beta_2(T_s)\), \(\beta_3(T_s)\), \(\beta_4(T_s)\) in (4.14)-(4.15), for all \(0 \leq t \leq T_s\) and nonnegative integers \(i\), we have

\[
|\tilde{y}(iT_s + t)| \leq \beta_1(T_s)\varsigma(T_s) + \beta_2(T_s)\sqrt{\frac{\alpha}{\lambda_{\max}(P_2)}} + \beta_3(T_s)\varsigma(T_s) + \beta_4(T_s) \Delta = \gamma_0(T_s).
\]

Since this bound holds for all \(t \geq 0\) we obtain the bound (4.16), which completes the proof. \(\square\)

The above theorem ensures that the estimation-loop of the \(L_1\) adaptive controller remains stable while the system is operating in the open loop regime. The stability of the \(L_1\) adaptive controller in the closed-loop mode was established in Chapter 2. Since the switching signal is not state dependent and the number of switches is finite, we can conclude stability of the overall system.

Next we perform analysis of the open loop regime and show that it helps to reduce the effects with the initialization transients.
4.4.4. Control Signal and Nonzero Initialization Error Analysis

Consider the following $\mathcal{L}_1$ reference system

$$
\dot{x}_{ref}(t) = Ax_{ref}(t) + b(u(t) + f(t, x_{ref}(t))), \quad x_{ref}(0) = x_{p0},
$$
(4.36)

with the reference control signal defined as

$$
u_{ref}(s) = F(s)r(s) - C(s)\sigma_{ref}(s),
$$
(4.37)

where

$$
\sigma_{ref}(s) = \frac{(A(s) - M(s))u(s) + A(s)d_{ref}(s)}{M(s)},
$$
(4.38)

with $d_{ref}(t) = f(t, x_{ref}(t))$.

**Remark 4.2.** We note that the $\mathcal{L}_1$ reference system is obtained by replacing the unknown parameter estimates in the $\mathcal{L}_1$ control law by their actual values. Since it depends on the unknown variables, it cannot be used for implementation. However, since the $\mathcal{L}_1$ reference system does not involve the estimation loop and has simpler structure as compared to the $\mathcal{L}_1$ adaptive controller, it is useful for performance analysis.

We start our analysis by noting that the system (4.36) is identical to the system in (4.2), which implies that

$$
\|y_{ref} - y\|_{\infty} = 0.
$$

Moreover this fact also implies that $d(t) \equiv d_{ref}(t)$ and consequently the definitions (4.6) and (4.38) imply

$$
\sigma_{ref}(t) \equiv \sigma(t).
$$
(4.39)

Next we consider the control signal (2.13) and rewrite it as

$$
u_{\mathcal{L}_1}(s) = F(s)k_{g}r(s) - C(s)\sigma(s) - C(s)\tilde{\sigma}(s),
$$
(4.40)

where $\tilde{\sigma}(s)$ was defined in (4.10). Subtracting (4.40) from (4.37) and taking into account (4.39), we obtain

$$
u_{ref}(s) - \nu_{\mathcal{L}_1}(s) = -C(s)(\sigma_{ref}(s) - \sigma(s)) + C(s)\tilde{\sigma}(s) = C(s)\tilde{\sigma}(s).
$$
(4.41)
From (4.9), we obtain
\[ \tilde{\sigma}(s) = \frac{1}{M(s)} \tilde{y}(s) - \frac{1}{M(s)} \tilde{y}_\text{in}(s). \]

Next, we can rewrite (4.41) as
\[ u_{\text{ref}}(s) - u_{\mathcal{L}_1}(s) = \frac{C(s)}{M(s)} \tilde{y}(s) - \frac{C(s)}{M(s)} \tilde{y}_\text{in}(s), \tag{4.42} \]
which leads to the following bound
\[ \|u_{\text{ref}} - u_{\mathcal{L}_1}\|_{\mathcal{L}_\infty} \leq \left\| \frac{C(s)}{M(s)} \right\|_{\mathcal{L}_1} \|\tilde{y}\|_{\mathcal{L}_\infty} + \left\| \frac{C(s)}{M(s)} \right\|_{\mathcal{L}_1} \|\tilde{y}_\text{in}\|_{\mathcal{L}_\infty} \tag{4.43} \]
where \( \rho_{\tilde{y}_\text{in}} \) is computed as
\[ \rho_{\tilde{y}_\text{in}} \triangleq \sup_{\|x_0\|_\infty \leq \rho_0, \|\tilde{x}_0\|_\infty \leq \rho_{\tilde{x}_0}} \|\tilde{y}_\text{in}\|_{\mathcal{L}_\infty}. \]

We note that the bound (4.43) consists of two terms: first depends on the estimation loop dynamics and can be made arbitrarily small by reducing the sampling time \( T_s \); the second term depends on the initial conditions and as we can see from (4.42) it converges to zero exponentially fast because \( \tilde{y}_\text{in}(t) \) converges exponentially to zero and \( \frac{C(s)}{M(s)} \) is exponentially stable.

Notice that the \( \mathcal{L}_1 \) reference system (4.36) represents achievable desired behavior and does not show any controller initialization transients because the reference control law (4.37) does not depend on the plant initial conditions. Further notice that the second term in (4.42) may have slow zeros from inversion of \( M(s) \), which may produce large spikes in the beginning of the transient in the \( \mathcal{L}_1 \) adaptive control law due to initialization errors. Therefore, if we first run the \( \mathcal{L}_1 \) controller in open-loop regime (\( u(t) = u_a(t) \)) and disregard the \( \mathcal{L}_1 \) adaptive control signal \( u_{\mathcal{L}_1}(t) \), the transients due to the second term will settle without entering the plant, and we will avoid large control magnitudes, oscillations and switching transients related to initialization error of the output predictor of \( \mathcal{L}_1 \) adaptive controller.
4.5. Architecture Modifications for Improved Performance

In this chapter we also incorporate three modifications into the $\mathcal{L}_1$ adaptive control architecture, which we use to improve performance of the closed-loop system.

**Hedging:** The first modification is called hedging [67], and for $\mathcal{L}_1$ adaptive control it was introduced in [68]. Its purpose is to improve stability and performance of the system in the presence of input saturation. This modification consists of adding a saturation block $\Delta_h$ with the same limits as the patient’s input at the control input of the output predictor, which significantly reduces windup [68, 69].

**Output predictor modification:** The second modification aims to reduce noise levels due to sampling of the patient’s control signal. It consists of adding a sampler at the output-predictor’s input with the same sampling period as the patient control signal $\tau_u$. This modification relies on the decoupling property of the $\mathcal{L}_1$ adaptive architecture, which is discussed in [70].

**Filter with extended bandwidth:** Finally, as was mentioned previously, the $\mathcal{L}_1$ adaptive controller compensates for the disturbance within the lowpass filter bandwidth. This implies that a larger bandwidth will improve performance; however a larger bandwidth also reduces robustness of the closed-loop system [3]. This creates a tradeoff between performance and robustness. In Figure 4.3 we observe short intervals of time with severe disturbance, which correspond to motion of patients and external actions. Thus, if we design a single lowpass filter for the duration of the clinical procedure, we will achieve a conservative design, which will not compensate for disturbances during these short intervals of time. On the other hand, if we obtain a filter that addresses the severe disturbances, it will have weaker robustness properties and higher noise amplification. Thus, we propose to design two separate filters: $C(s)$ for the main operation time, and a filter with extended bandwidth $C_e(s)$, which has improved performance for severe disturbances. Since such disturbances can be observed by the anesthesiologist (he/she observes visually when the patient is moving and when surgical actions are applied), we allow the anesthesiologist to decide when to switch to the control law with the aggressive filter, and when to switch back. The switching mechanism between the control laws as well as the stability proofs are the same as described above for the $\mathcal{L}_1$ adaptive controller. An illustration is given in Figure 4.6.
In this section we present the simulation results for the \( L_1 \) adaptive controller defined above based on the 6 patient models. We start by explaining the controller tuning procedure and provide the controller parameters, which are used in the simulations. Then we give the simulation results for two cases: (a) nominal patient models and (b) patient models with disturbances. In the second case we compare the simulation results to the clinical data. In the end of our simulation study, we compute the total amount of consumed isoflurane for each case to ensure that the feedback controller does not lead to increased consumption of the inhaled sedative.

### 4.6.1. Tuning of the \( L_1 \) Adaptive Controller Parameters

We begin tuning the \( L_1 \) adaptive controller with the choice of the output predictor. The output predictor specifies the desired transient behavior of the closed-loop system [3]. Since our design objective is to achieve similar performance in terms of tracking and disturbance rejection to that observed in the clinical trials, we choose the desired system based on the characteristics of the transient response from clinical trial results. It is also important to note that the time constants of the output predictor should be chosen adequately by considering the time constants of the plant. For instance, if the settling time of the output predictor is chosen significantly faster than the settling time of the plant, this may lead to overly
aggressive control with large magnitude input signals and unavoidable input saturation, causing the system to leave the linear operation region. On the other hand, if the settling time of the predictor is much larger than the settling time of the plant, this may result in overly conservative design. Therefore, we consider the settling times for the patients’ models given in Table 4.2 along with the clinical BIS responses in Figures 4.10-4.15 and set the output predictor (desired) dynamics to

\[ M(s) = -p_g \frac{0.01^2}{s^2 + 2 \cdot 0.9 \cdot 0.01s + 0.01^2} \cdot \]

This transfer function has a settling time of \( \tau_s = 470 \) s, which is slightly greater than the smallest settling time among all patient models. The model gain \( p_g \) is set to \( p_g = 80 \), which corresponds to the average gain of the patient models. We note that this gain plays an important role in ensuring satisfactory performance of the system. Figure 4.7 presents a comparison of the transient response of patients 2 and 4 with \( M(s) \) given above.

\[
\begin{array}{c|ccccccc}
\text{Patient} & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
\tau_s, s & 682 & 4010 & 4320 & 419 & 1020 & 2620 \\
\end{array}
\]

**Table 4.2:** Settling time of the identified patient models.

Next, we design the lowpass filter \( C(s) \). From the reference system (2.14)-(2.15), we see that the \( \mathcal{L}_1 \) adaptive controller compensates for the uncertainty within the bandwidth of the lowpass filter \( C(s) \). Therefore the choice of the filter bandwidth needs to ensure that the frequency range for the main disturbance content falls within the bandwidth of the filter \( C(s) \). We consider the power spectrum of the output disturbance \( e(t) \) for all patients and
choose the following second order filter:

\[ C(s) = \left( \frac{0.02^2}{s^2 + 0.02s + 0.02^2} \right) \left( \frac{0.1}{s + 0.1} \right) \left( \frac{0.08}{s + 0.08} \right). \]

For illustration we show in Figure 4.8 the power spectrum of the disturbance \( e(t) \) for all patient models and the gain plot of the filter \( C(s) \). We see that the main frequency content of the disturbance is covered by the filter bandwidth, which ensures compensation of the majority of the disturbance effects on the system transient. Using the same approach, we design the second filter \( C_e(s) \) for operation under adverse conditions:

\[ C_e(s) = \left( \frac{0.05^2}{s^2 + 0.05 \cdot 1.5s + 0.05^2} \right) \left( \frac{0.1}{s + 0.1} \right) \left( \frac{0.07}{s + 0.07} \right). \]

As can be seen in Figure 4.8 the bandwidth for \( C_e(s) \) is larger than the bandwidth for \( C(s) \), which ensures better compensation for the disturbance. However, as mentioned before, \( C_e(s) \) also amplifies the noise in the control channel as compared to the filter \( C(s) \) and therefore should be enabled only in the presence of large disturbances.

![Disturbance power spectrum.](image1) ![Gain response of the lowpass filters in dB.](image2)

**Figure 4.8:** Disturbance spectrum for all patient models and the lowpass filters gain response.

Since the patients’ models constructed in this chapter are adequate only during the patients’ sedation (BIS < 70), the \( \mathcal{L}_1 \) adaptive controller is enabled after patient sedation, when the initial BIS transient settles. While in practice this switching is done manually by the anesthesiologist, for our simulations using real clinical data, we define a switching signal for each patient data. The switching signal takes a 0 value till the BIS falls below 70 and all initial transients are settled, and is equal to 1 subsequently. During the time interval for which the switching signal is 0, the actual clinical data are used for the control
input of the patient model and the output predictor, which simulates the manual control mode of anesthesia. When the switching signal is changed to 1, the control signal generated by the $\mathcal{L}_1$ adaptive controller is used to ensure tracking of the reference commands. This approach allows evaluation of the behavior of the closed-loop control system in the presence of switching, along with the tracking performance of the $\mathcal{L}_1$ adaptive controller.

Similarly, the filter with extended bandwidth $C_e(s)$ is enabled by a command from the anesthesiologist when he or she observes developing adverse conditions. In our simulations we consider this command as an external signal to the feedback controller, which is defined separately for each patient model by inspection of the BIS clinical response. Figure 4.9 shows an example of one BIS clinical response. The areas with adverse disturbances are shown in cyan, and green intervals represent the command to enable $C_e(s)$.

![Figure 4.9: Illustration of the choice of the switching intervals for the filter with extended bandwidth using Patient 6 clinical data.](image)

In order to be able to compare the transient behavior of the $\mathcal{L}_1$ adaptive controller with clinical data, the reference BIS command is obtained for each patient model by inspection of the BIS clinical response, and an estimation of the target values for BIS is maintained by the anesthesiologist.

Next we set the sampling time of the adaptation law to $T_s = 0.01$ s, which is significantly faster than the time constants for the patient dynamics and the output predictor. This choice ensures sufficiently small values of the performance bounds (2.16). Further, we set $Q = I$ and compute $D = [0.9515 \ 0.3078]$. The hedging saturation is set to $\Delta_h = 3\%$ and the control signal sampling to $\tau_u = 5$ s (this is the same as the patient model). The filter for the reference commands is set to

$$F(s) = \frac{10}{s + 10}.$$  

We note that since we assume that the patients’ dynamics are uncertain, in the above
design we use only averaged and conservative information from the models. Moreover, the controller parameters given above are used for all simulations without any retuning.

4.6.2. Simulations for Nominal Patient Models

Next we consider nominal patient models, that is without output disturbances. The simulation results for 6 patient models are shown in Figures 4.10-4.15. For these simulations we do not use the lowpass filter with extended bandwidth and we enable $\mathcal{L}_1$ adaptive feedback control at the beginning of the transient. We see that the $\mathcal{L}_1$ adaptive controller achieves similar transient specifications during control of the sedated state despite significant inter-patient model variability. In fact, the transient specifications for all cases are close to the transient specifications of the response of the ideal system shown in Figure 4.7. Note that during some intervals, for example at $t = 5000$ s in Figure 4.15, the system output has significant undershoot, which occurs immediately after the patient reaches the sedated state and initial transients have not settled. The reason for this behavior is due to the fact that the patient model, as well as the $\mathcal{L}_1$ adaptive controller, were tuned only for the sedated state after the initial transients converge and are not applicable for this time interval. This behavior illustrates the need for initial manual anesthesia control and the importance of the switching scheme.

4.6.3. Simulations Using Clinical Trial Data

In this section we consider the identified uncertain patient models with output disturbances. The simulation results for 6 patients are shown in Figures 4.16-4.22. We see that for all patients the achieved performance of the $\mathcal{L}_1$ adaptive controller is similar to the per-
formance observed in the clinical data. The single $\mathcal{L}_1$ adaptive controller demonstrates the ability to compensate for patient time-varying input gain and disturbances, as well as inter-patient uncertainty. The noise level and the magnitudes of the generated control signal are comparable to the signals used in the clinical trials. Using the extended (more aggressive) lowpass filter reduced the tracking errors, as compared to the clinical results in the presence of adverse disturbances. However, comparing the results in Figure 4.16 to the simulations performed without the use of the extended filter in Figure 4.17, we see that during the time intervals when the filter with extended bandwidth $C_e(s)$ is applied, the level of noise in the control signal is higher as compared to the results formed using the filter $C(s)$. This fact reflects a fundamental tradeoff between performance of the feedback system and its sensitivity to measurement noise. Since the control signal is sampled relatively slowly ($5$ s), the observed noise in the control signal does not cause issues for either the hardware or the patient, due to the large time scale. In this work we tune the extended filter with the noise.
level allowance to be about twice as large as the noise level for the nominal filter. Finally, we note that the proposed switching schemes, both for filters and the $L_1$ adaptive control, do not lead to noticeable switching transients or “transfer bumps”.

### 4.6.4. Evaluation of Isoflurane Consumption

One important quantity to note in anesthesia is the overall amount of sedatives consumed during surgery. In Table 4.3 we provide computed amounts of the consumed isoflurane in liters both during simulations and the clinical trials. We see that, while there are no clear trends, the $L_1$ adaptive feedback controller leads to similar consumption of isoflurane as that resulting from manual control in the clinical trials; this validates our design against the clinical trial data.
Figure 4.15: $\mathcal{L}_1$ controller performance for Patient 6 without output disturbance.

Figure 4.16: $\mathcal{L}_1$ controller performance for Patient 1.

Figure 4.17: $\mathcal{L}_1$ controller performance for Patient 1 without use of the extended filter.
Figure 4.18: $\mathcal{L}_1$ controller performance for Patient 2.

Figure 4.19: $\mathcal{L}_1$ controller performance for Patient 3.

Figure 4.20: $\mathcal{L}_1$ controller performance for Patient 4.
Patient 1 2 3 4 5 6
w/o disturbance, l 0.849 0.650 0.649 0.673 0.662 0.636
w disturbance, w/o $C_e(s)$, l 0.845 0.641 0.653 0.732 0.623 0.666
w disturbance, l 0.849 0.643 0.651 0.736 0.624 0.666
Clinical ISO, l 0.832 0.651 0.638 0.796 0.614 0.647

Table 4.3: Isoflurane consumption.
CHAPTER 5
Extension to Linear Time-varying Reference System

In this Chapter we present an extension of $L_1$ adaptive controller with a time-varying reference system to a class of nonlinear uncertain output-feedback systems [71]. In our derivations we consider nonzero initialization error and derive performance bounds between the closed-loop adaptive system and the $L_1$ reference system for both system state and control input. The performance bounds can be arbitrarily reduced by decreasing the sampling time and increasing the inversion filter bandwidth. The $L_1$-norm stability condition in this case is similar to the $L_1$-norm condition for state-feedback architecture with unmatched uncertainties [3, Section 3.2], and in the absence of the uncertainty in the system input vector $b(t)$, it has identical structure, which is simpler as compared to the stability condition for the output-feedback architecture presented [3, Section 4.2].

5.1. Problem Formulation

Consider the following class of nonlinear systems:

$$
\begin{align*}
\dot{x}(t) &= A(t)x(t) + b(t)u(t) + f(t, x(t)), \quad x(0) = x_0, \\
y(t) &= c_m(t)x(t),
\end{align*}
$$

(5.1)

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}$ are the system state (not measured) and the system output (measured), respectively; $A(t) \in \mathbb{R}^{n \times n}$ and $b(t) \in \mathbb{R}^n$ are unknown time-varying matrix and a vector, respectively; $c_m(t) \in \mathbb{R}^n$ is a known time-varying vector; $u(t) \in \mathbb{R}$ is the control input. The initial condition $x_0$ is unknown, however it is assumed to belong to a known set such that $\|x_0\|_\infty \leq \rho_0 < \infty$, for a given $\rho_0 \in \mathbb{R}^+$; and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is an unknown map subject to the following assumption:

**Assumption 5.1** (*Semi-global Lipschitz continuity and boundedness*). For all $\delta \in \mathbb{R}^+$ there exist constants $L_0(\delta) \in \mathbb{R}^+$ and $B(\delta) \in \mathbb{R}^+$, such that for all $x, x_1, x_2 \in \mathbb{R}^n$ with $\|x\|_\infty \leq \delta$, $\|x_1\|_\infty \leq \delta$, $\|x_2\|_\infty \leq \delta$ the following bounds

$$
\|f(t, x_1) - f(t, x_2)\|_\infty \leq L_0(\delta)\|x_1 - x_2\|_\infty,
$$

$$
\|f(t, x)\|_\infty \leq L_0(\delta)\|x\|_\infty + B(\delta)
$$

hold uniformly for $t \geq 0$. 
Let $r(t) \in \mathbb{R}$ be a given bounded reference input signal. Similar to Chapter 2, the control objective is to design an adaptive output-feedback controller, which ensures that the system output $y(t)$ tracks the reference input $r(t)$ according to a desired model given by

$$
\begin{align*}
\dot{x}_{id}(t) &= A_m(t)x_{id}(t) + b_m(t)r_g(t), & x_{id}(0) &= x_0, \\
y_{id}(t) &= c_m^\top(t)x_{id}(t),
\end{align*}
$$

(5.2)

where $A_m(t) \in \mathbb{R}^{n \times n}$ is a known Hurwitz time-varying matrix; $b_m(t) \in \mathbb{R}^n$ is a known time-varying vector such that the relative degree of the desired model is less or equal to $d_r \geq 1$ for all $t \geq 0$; and

$$
r_g(t) \triangleq k_g(t)r(t), \quad k_g(t) \triangleq \frac{-1}{c_m(t)A_m^{-1}(t)b_m(t)}.
$$

Let the systems above verify the following assumptions:

**Assumption 5.2.** There exist constants $\Delta_1$ and $\Delta_2$, such that

$$
\| (A(t) - A_m(t)) \|_\infty < \Delta_1, \quad \| (b(t) - b_m(t)) \|_1 < \Delta_2, \quad \forall t \geq 0.
$$

**Assumption 5.3.** There exist constants $\mu_b, \mu_{bm}, \mu_c \in \mathbb{R}^+$, such that $\| b(t) \|_\infty \leq \mu_b$, $\| b_m(t) \|_\infty \leq \mu_{bm}$, and $\| c_m(t) \|_1 \leq \mu_c$.

**Assumption 5.4.** $A_m(t), b_m(t)$, and $c_m(t)$ are at least $d_r$ times continuously differentiable.

**Assumption 5.5.** The pairs $(A_m(t), b_m(t))$ and $(A(t), b(t))$ are strongly controllable, and the pairs $(A_m(t), c_m^\top(t))$ and $(A(t), c_m^\top(t))$ are strongly observable [72].

**Assumption 5.6 (Stability of the desired system).** There exist positive constants $\mu_A > 0$, $d_A > 0$, and $\mu_\lambda > 0$, such that for all $t \geq 0$, $\| A_m(t) \|_\infty \leq \mu_A$, $\| \dot{A}_m(t) \|_\infty \leq d_A$, and $\text{Re}[\lambda_i(A_m(t))] \leq -\mu_\lambda$, $\forall i = 1, \ldots, n$, where $\lambda_i(A_m(t))$ is a point-wise eigenvalue of $A_m(t)$. Further, for all $t \geq 0$, the equilibrium of the state equation

$$
\dot{x} = A_m(t)x(t),
$$

is exponentially stable, and the solution of

$$
A_m^\top(t)P(t) + P(t)A_m(t) = -\mathbb{I}
$$

satisfies $P(t) = P^\top(t) > 0$ and $\| \dot{P}(t) \|_\infty \leq \epsilon_P < 1$.

**Remark 5.1.** We notice that the above assumption is standard for LTV control theory. Sometimes it is referred to as stability of slowly-varying systems [73].
5.2. \( L_1 \) Adaptive Control Architecture

5.2.1. Definitions and \( L_1 \)-norm Sufficient Condition for Stability

We can rewrite the system in (5.1) as

\[
\begin{align*}
\dot{x}(t) &= A_m(t)x(t) + b_m(t)u(t) + \sigma(t), \quad x(0) = x_0, \\
y(t) &= c^\top_m(t)x(t), 
\end{align*}
\]

(5.3)

where

\[
\sigma(t) \triangleq (A(t) - A_m(t))x(t) + (b(t) - b_m(t))u(t) + f(t, x(t)).
\]

(5.4)

Let \( x_{in}(t) \) and \( \tilde{y}_{in}(t) \) be the initial condition response (zero-input response) of the systems

\[
\begin{align*}
\dot{x}_{in}(t) &= A_m(t)x_{in}(t), \quad x_{in}(0) = x_0, \\
\dot{\tilde{x}}_{in}(t) &= A_m(t)\tilde{x}_{in}(t), \quad \tilde{x}_{in}(0) = \hat{x}_0 - x_0, \\
\tilde{y}_{in}(t) &= c^\top_m(t)\tilde{x}_{in}(t),
\end{align*}
\]

where \( \hat{x}_0 \) needs to verify \( c^\top_m(0)\hat{x}_0 = y(0) \). Then, let \( \rho_{in} \) and \( \tilde{\rho}_{in} \) be defined as

\[
\rho_{in} \triangleq \max_{\|x_0\|_\infty \in \rho_0} \|x_{in}\|_{L_\infty}, \quad \tilde{\rho}_{in} \triangleq \max_{\|x_0\|_\infty \in \rho_0, \|\hat{x}_0\|_\infty \in \rho_0} \|\tilde{y}_{in}\|_{L_\infty}.
\]

Next, let \( \mathcal{H}_m \) and \( \mathcal{H}_{xm} \) be the maps from \( u(t) \) to \( y(t) \) and \( u(t) \) to \( x(t) \), respectively, and \( \mathcal{H}_{xum} \) be the map of the system in (5.3) from \( \sigma(t) \) to \( x(t) \) with initial conditions equal to zero. Also, let \( \mathcal{H}_{yum} \) be the map from \( \sigma(t) \) to \( y(t) \) in (5.3) with initial conditions set to zero as well. The design of the \( L_1 \) adaptive controller proceeds by considering a strictly proper system \( C(s) \) with relative degree greater or equal to \( d_r \) and \( C(0) = 1 \). Further, the selection of \( C(s) \) must ensure that there exists a constant \( \rho_r \in \mathbb{R}^+ \) such that

\[
\mathcal{H}_\omega \triangleq (\mathbb{I} + (b - b_m)C\mathcal{H}_m^{-1}\mathcal{H}_{yum})^{-1}
\]

(5.5)

is stable and the following \( L_1 \)-norm condition is satisfied:

\[
\|G_{um}\|_{L_1} \leq \frac{\rho_r - \|\mathcal{H}_{xm}C\mathcal{H}_m^{-1}\mathcal{F}\|_{L_1}\tilde{\rho}_{in} - \|\mathcal{H}_{xm}C\|_{L_1}\|r_y\|_{L_\infty} - \rho_{in}}{L_\rho\rho_r + \Delta_2\|C\|_{L_1}\|r_y\|_{L_\infty} + \Delta_2\|\mathcal{H}_m^{-1}\mathcal{F}\|_{L_1}\tilde{\rho}_m + B(\rho_r)},
\]

(5.6)
where $C$ is the input-output map of the lowpass filter transfer function $C(s)$, $\mathcal{F}$ is the input-output map of
\[ F(s) = \frac{1}{\sum_{i=0}^{d_r} a_i s^i}, \quad a_0 = 1, \quad (5.7) \]
which has real poles, and
\[ \mathcal{G}_{um} \triangleq \left( \mathbb{I} - \mathcal{H}_m \mathcal{C} \mathbf{H}_{m}^{-1} \mathbf{c}_{m}^T \right) \mathcal{H}_x m \mathcal{H}_x, \quad (5.8) \]
and $L_\rho$ is defined as
\[ L_\rho \triangleq \Delta_1 + L_0(\rho), \quad (5.9) \]
where
\[ \rho \triangleq \rho_r + \bar{\gamma}_1, \quad (5.10) \]
with $\bar{\gamma}_1$ being an arbitrary (small) positive constant.

**Remark 5.2.** The definition and the procedure of computing the $\mathcal{L}_1$-norms for LTV systems can be found in [3].

**Remark 5.3.** Notice that the structure of the right hand side of the $\mathcal{L}_1$-norm stability condition (5.6) is similar to the structure of the $\mathcal{L}_1$-norm condition for state-feedback systems with unmatched uncertainties given in [3, Section 3.2]. Similar to the Section 3.2 the term $\rho_r$ defines the bound on the state of the $\mathcal{L}_1$ reference system; the terms $L_\rho \rho_r$ and $B(\rho_r)$ follow from the system nonlinearity, and the term $\rho_m$ is due to the initial condition response. The other terms in (5.6) are not present in the $\mathcal{L}_1$-norm condition from Section 3.2 and are the result of generalization of the class of systems. Namely, the term $\Delta_2 ||\mathcal{C}||_{\mathcal{L}_1} ||r_g||_{\mathcal{L}_\infty}$ is due to uncertainty in the structure of the system input vector $b(t)$; and the terms $\Delta_2 ||\mathcal{C} \mathbf{H}_m^{-1} \mathcal{F}||_{\mathcal{L}_1} \tilde{\rho}_m$ and $||\mathcal{H}_x m \mathcal{C} \mathbf{H}_m^{-1} \mathcal{F}||_{\mathcal{L}_1} \tilde{\rho}_m$ are due to the mismatch in the initial condition of the plant and our choice of the initial conditions of the state predictor $\hat{x}_0$. Notice that this term becomes relevant only in the output-feedback architecture, for which we cannot directly measure the initial conditions of the plant. Therefore the initialization error can be significant as compared to the state-feedback architecture, in which we can always initialize the state predictor using the system state measurement.

**Remark 5.4.** Consider the structure of $\mathcal{G}_{um}$ in (5.6). In many state feedback architectures the $\mathcal{L}_1$-norm condition can always be satisfied by choosing the filter with sufficiently large bandwidth. However, we notice that the $\mathcal{L}_1$-norm stability condition (5.6) cannot be satisfied simply by increasing the bandwidth of the lowpass filter $C(s)$, since $||\mathcal{G}_{um}||_{\mathcal{L}_1}$ does not necessarily decrease if the bandwidth of the filter is increased. Similar to [41], the condi-
tion (5.6) requires appropriate filter tuning. Filter design can be done using some of the methods presented in [74].

Next, define

$$\rho_{ur} \triangleq \|C\|_{L_1} r_g \|_{L_\infty} + \|CH_m^{-1}\|_{L_1} \left(\|H_{yum} H_\omega\|_{L_1} (\Delta_2 (\|C\|_{L_1} r_g) \|_{L_\infty}
+ \|CH_m^{-1} F\|_{L_1} \tilde{\rho}_m) + L_\rho \rho_r + B(\rho_r) + \|F\|_{L_1} \tilde{\rho}_m\right),$$

(5.11)

and let

$$\rho_u \triangleq \rho_{ur} + \bar{\gamma}_2,$$

(5.12)

where $\bar{\gamma}_2 \in \mathbb{R}^+$ is an arbitrary (small) constant. Further, define $\Delta$ as

$$\Delta \triangleq L_\rho \rho + \Delta_2 \rho_u + B(\rho).$$

(5.13)

Further, let

$$\gamma_1 \triangleq \frac{1}{1 - \|G\|_{L_1} L_\rho} \left(\|G (b - b_m) CH_m^{-1}\|_{L_1} (\|F\|_{L_1} \bar{\gamma}_0 + \|C \|_{L_1} \Delta)
+ \|H_{xm} CH_m^{-1} F\|_{L_1} \bar{\gamma}_0 + \|H_{xm} CH_m^{-1} (1 - F) H_{yum}\|_{L_1} \Delta)\right) + \beta,$$

(5.14)

where the values of $\bar{\gamma}_0 \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$ are arbitrarily small positive constants. Note that the denominators in (5.14) are nonsingular due to (5.6). We also define

$$\gamma_2 \triangleq \|CH_m^{-1} H_{yum} H_\omega\|_{L_1} (L_\rho \gamma_1 + \|C \|_{L_1} \Delta)
+ \|CH_m^{-1} F\|_{L_1} \bar{\gamma}_0 + \|CH_m^{-1} (1 - F) H_{yum}\|_{L_1} \Delta + \beta.$$

(5.15)

The choice of $\bar{\gamma}_0$, $\beta$, and $F(s)$ must ensure that

$$\gamma_1 < \bar{\gamma}_1, \quad \gamma_2 < \bar{\gamma}_2.$$

**Remark 5.5.** The $L_1$-norm given by $\|H_{xm} CH_m^{-1} (1 - F) H_{yum}\|_{L_1}$ exists since $H_{xm}$, $H_m^{-1}$, and $H_{yum}$ are stable maps and the relative degree of $C(s)$ is greater than or equal to the relative degree of $H_m$. Further, the value of this $L_1$-norm can be made arbitrarily small by increasing the bandwidth of $F(s)$ in a similar manner as it is shown in Lemma 2.1.5 in [3].
Using Assumption 5.6 and the properties of \( P(t) \), it follows that there exits a nonsingular \( \sqrt{P(t)} \) for all \( t \geq 0 \) such that
\[
P(t) = \left( \sqrt{P(t)} \right)^\top \sqrt{P(t)}.
\]

Let \( D(t) \in \mathbb{R}^{n-1 \times n} \) contain the basis of the null-space of \( c_m^\top(t) \left( \sqrt{P(t)} \right)^{-1} \), that is
\[
D(t) \left( c_m^\top(t) \left( \sqrt{P(t)} \right)^{-1} \right)^\top = 0,
\]
for all \( t \geq 0 \); and further let
\[
\Lambda(t) \triangleq \begin{bmatrix} c_m^\top(t) \\ D(t) \sqrt{P(t)} \end{bmatrix}, \tag{5.16}
\]
Notice that
\[
\Lambda(t) \left( \sqrt{P(t)} \right)^{-1} = \begin{bmatrix} c_m^\top(t) \left( \sqrt{P(t)} \right)^{-1} \\ D(t) \end{bmatrix}
\]
is full rank, and hence \( \Lambda^{-1}(t) \) exists \( \forall t \in [0, \infty) \).

**Lemma 5.1.** For arbitrary \( \xi(t) \triangleq [y(t) \ z(t)]^\top \in \mathbb{R}^n \), where \( y(t) \in \mathbb{R} \) and \( z(t) \in \mathbb{R}^{n-1} \), there exist \( p_1(t) \in \mathbb{R}^+ \) and a positive definite \( P_2(t) \in \mathbb{R}^{n-1 \times n-1} \) such that
\[
\xi^\top(t) \Lambda^{-\top}(t) P(t) \Lambda^{-1}(t) \xi(t) = p_1(t) y^2(t) + z^\top(t) P_2(t) z(t)
\]
for all \( t \geq 0 \).

The proof of the lemma is similar to the proof of Lemma 4.2.1 in [3] and therefore is omitted.

Further, let \( T_s \in \mathbb{R}^+ \) be an arbitrary constant that can be associated with the sampling rate of the available CPU, and let \( \phi_{\xi}(i, T_s) \in \mathbb{R}^{n \times n} \) be given by
\[
\phi_{\xi}(i, T_s) \triangleq \int_{iT_s}^{(i+1)T_s} \Phi_{\xi}((i + 1)T_s, \tau) \Lambda(\tau)d\tau, \tag{5.17}
\]
where \( \Phi_{\xi}(\cdot, \cdot) \) represents the state transition matrix for the autonomous system with state matrix given by \( \left( \Lambda(t) A_m(t) - \frac{d}{dt} \Lambda(t) \right) \Lambda^{-1}(t) \).

Next, define \( 1_1 = [1, \ 0, \ \ldots, \ 0]^\top \in \mathbb{R}^n \), and let
\[
1_1^\top \Phi_{\xi}(iT_s + t, iT_s) = [\eta_1(i, T_s, t), \ \eta_2^\top(i, T_s, t)], \tag{5.18}
\]

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where $\eta_1(i, T_s, t) \in \mathbb{R}$ and $\eta_2(i, T_s, t) \in \mathbb{R}^{n-1}$ contain the first and the 2-to-$n$ elements of the row vector $1_1^T \Phi_\xi(iT_s + t, iT_s)$. Next let

$$
\kappa(i, T_s) \triangleq \int_{iT_s}^{(i+1)T_s} \|1_1^T \Phi_\xi((i + 1)T_s, \tau)\Lambda(\tau)\|_1 d\tau.
$$

(5.19)

Also, let $\zeta(i, T_s)$ and $\alpha$ be defined as $\zeta(0, T_s) \triangleq 0$, and for $i \in \mathbb{N}$:

$$
\zeta(i, T_s) \triangleq \|\eta_2(i - 1, T_s, T_s)\| \sqrt{\frac{\alpha}{\lambda_{P_{2 \max}}} + \kappa(i - 1, T_s)\Delta},
$$

(5.20)

$$
\alpha \triangleq \max \left\{ \lambda_{P_{\max}} \left( \frac{2\lambda_{P_{\max}}\Delta \mu_P \sqrt{n}}{\lambda_{P_{\min}}(1 - \epsilon_P)} \right)^2, 4n\lambda_{\max}(P_2(0))\|\Lambda(0)\|_2^2P_0^2 \right\},
$$

(5.21)

where

$$
\lambda_{P_{2 \max}} \triangleq \sup_{t \in [0, \infty), i = 1 \ldots n - 1} \lambda_i(P_2(t)), \quad \lambda_{P_{\max}} \triangleq \sup_{t \in [0, \infty), i = 1 \ldots n} \lambda_i(P(t)),
$$

$$
\lambda_{P_{\min}} \triangleq \inf_{t \in [0, \infty), i = 1 \ldots n} \lambda_i(P(t)),
$$

and

$$
\mu_P \triangleq \sup_{t \in [0, \infty)} \|P(t)\|_\infty.
$$

(5.22)

Next, we introduce the following constants

$$
\beta_1(i, T_s) \triangleq \max_{t \in [iT_s, (i+1)T_s]} |\eta_1(i, T_s, t)|, \quad \beta_2(i, T_s) \triangleq \max_{t \in [iT_s, (i+1)T_s]} \|\eta_2(i, T_s, t)\|,
$$

(5.23)

$$
\beta_3(i, T_s) \triangleq \max_{t \in [iT_s, (i+1)T_s]} \eta_3(i, T_s, t), \quad \beta_4(i, T_s) \triangleq \max_{t \in [iT_s, (i+1)T_s]} \eta_4(i, T_s, t),
$$

(5.24)

where

$$
\eta_3(i, T_s, t) \triangleq \int_{iT_s}^{iT_s+t} \left|1_1^T \Phi_\xi(iT_s + t, \tau)\Lambda(\tau)\phi_\xi^{-1}(i, T_s)\Phi_\xi((i + 1)T_s, iT_s)1_1^T \right|d\tau,
$$

$$
\eta_4(i, T_s, t) \triangleq \int_{iT_s}^{iT_s+t} \|1_1^T \Phi_\xi(iT_s + t, \tau)\Lambda(\tau)\|_1d\tau.
$$
Finally, let
\[
\gamma_0(i, T_s) \triangleq \beta_1(i, T_s) \varsigma(i, T_s) + \beta_2(i, T_s) \sqrt{\frac{\alpha}{\lambda_{P_2 \text{max}}}} + \beta_3(i, T_s) \varsigma(i, T_s) + \beta_4(i, T_s) \Delta ,
\]
(5.25)
and
\[
\gamma_\tilde{x}(T_s) \triangleq \sup_{i \in \mathbb{N} \cup \{0\}} \gamma_0(i, T_s).
\]
The following lemma shows that the value of \( \gamma_\tilde{x}(T_s) \) can be made arbitrarily small by reducing the value of \( T_s \).

**Lemma 5.2.** The following limiting relationship is true:
\[
\lim_{T_s \to 0} \gamma_\tilde{x}(T_s) = 0.
\]
The proof of the lemma follows from the proof of Lemma 4.2.2 in [3], given the property \( \Phi_\tilde{\xi}(iT_s, iT_s) = 0 \).

### 5.2.2. \( L_1 \) Adaptive Control Architecture

The \( L_1 \) adaptive controller is comprised of the following elements:

**Output Predictor:**
\[
\begin{align*}
\dot{x}(t) &= A_m(t)\dot{x}(t) + b_m(t)u(t) + \dot{\sigma}(t) , \\
\dot{y}(t) &= c_m^\top(t)\dot{x}(t),
\end{align*}
\]
(5.26)
where \( \dot{\sigma}(t) \in \mathbb{R}^n \) is the vector of adaptive parameters updated by the following piecewise-constant adaptation laws:

**Adaptation Laws:**
\[
\dot{\sigma}(t) = -\mu(i, T_s)\tilde{y}(iT_s) , \quad t \in [iT_s, (i + 1)T_s) , \quad i = 0, 1, 2, \ldots ,
\]
(5.27)
where \( \tilde{y}(t) \triangleq \hat{y}(t) - y(t) \), and
\[
\mu(i, T_s) = \phi_\tilde{\xi}^{-1}(i, T_s)\Phi_\tilde{\xi}((i + 1)T_s, iT_s)1_1 ,
\]
with \( \phi_\tilde{\xi}(i, T_s) \) and \( \Phi_\tilde{\xi}((i + 1)T_s, iT_s) \) being defined in (5.17).

**Remark 5.6.** The matrix \( \mu(i, T_s) \) consists of time dependent gains that may be computed
off-line. For the numerical computation of the state transition matrix for time varying systems can one can use the Peano-Baker Series [75].

**Control Law:**

\[
    u = Cr_g - CH_{m}^{-1}FHy_{um}\sigma. \tag{5.28}
\]

Next we present an algorithm for on-line computation of \( H_{m}^{-1}F \).

### 5.2.3. Computation of \( H_{m}^{-1}F \)

Let

\[
    \nu_{\text{out}} = H_{m}^{-1}F\nu_{\text{in}},
\]

where \( \nu_{\text{out}}(t) \in \mathbb{R} \) and \( \nu_{\text{in}}(t) \in \mathbb{R} \) are the output and the input of \( H_{m}^{-1}F \) in (5.28) respectively. The mapping for \( H_{m} \) may be represented in Byrnes-Isidori form [76] through a coordinate transformation \( U(t) \in \mathbb{R}^{n \times n} \) leading to

\[
    \dot{\psi}(t) = \begin{bmatrix}
        0 & 1 & 0 & \cdots & 0 \\
        0 & 0 & 1 & \cdots & \vdots \\
        \vdots & \ddots & \ddots & \ddots & \vdots \\
        0 & 0 & \cdots & 0 & 1
    \end{bmatrix}
    \psi(t) + \begin{bmatrix}
        0 \\
        0 \\
        \vdots \\
        0
    \end{bmatrix}
    \theta(t) + \begin{bmatrix}
        \nu_{\text{out}}(t),
    \end{bmatrix}
\]

\[

\dot{\theta}(t) = \begin{bmatrix}
    E(t) & 0 & \cdots & 0
\end{bmatrix}
\psi(t) + G(t)\theta(t),
\]

where \( \psi(t) \in \mathbb{R}^{d_r} \) is the system state, \( \theta(t) \in \mathbb{R}^{n-d_r} \) are the zero dynamics state, \( \nu_{\text{in}}F = F\nu_{\text{in}} \), and \( R(t) \in \mathbb{R}^{d_r \times d_r} \), \( S(t) \in \mathbb{R}^{1 \times n-d_r} \), \( J(t) \in \mathbb{R} \), \( E(t) \in \mathbb{R}^{n-d_r \times 1} \), and \( G(t) \in \mathbb{R}^{n-d_r \times n-d_r} \) are defined in [76]. Notice that

\[
    \psi(t) = \begin{bmatrix}
        \psi_1(t) \\
        \psi_2(t) \\
        \vdots \\
        \psi_{d_r}(t)
    \end{bmatrix}
    = \begin{bmatrix}
        \nu_{\text{in}}F(t) \\
        \nu_{\text{in}}^{(1)}F(t) \\
        \vdots \\
        \nu_{\text{in}}^{(d_r-1)}F(t)
    \end{bmatrix}.
\]

This implies that

\[
    \nu_{\text{in}}^{(d_r)}F(t) = S(t)\theta(t) + J(t)\nu_{\text{out}}(t) + R_1(t)\nu_{\text{in}}F(t) + R_2(t)\nu_{\text{in}}^{(1)}F(t) + \ldots + R_{d_r}(t)\nu_{\text{in}}^{(d_r-1)}F(t).
\]
Therefore, \( \nu_{out}(t) \) may be computed as follows

\[
\nu_{out}(t) = \frac{\nu_{inF}^{(d_r)}(t) - S(t)\theta(t) - R_1(t)\nu_{inF}(t) - R_2(t)\nu_{inF}(t) - \ldots - R_{d_r}(t)\nu_{inF}^{(d_r-1)}(t)}{J(t)},
\]

(5.30)

where \( J(t) \neq 0 \) due to Assumption 5.5. The derivatives \( \nu_{inF}^{(d_r)} \) to \( \nu_{inF}^{(d_r-1)} \) can be computed using a fast filter \( F(s) \) defined in (5.7), which gives us

\[
\nu_{inF}^{(i)}(s) = \frac{s^i}{a_{d_r}s^{d_r} + \ldots + a_1s + 1} \nu_{in}(s) = F(s)s^i\nu_{in}(s), \quad i = 1 \ldots d_r,
\]

where the relative degree of \( F(s) \) is \( d_r \). The value of \( \theta(t) \) can be computed by (5.29). Notice that the lowpass filter \( C \) in (5.28) cuts out high-frequency content produced by differentiation.

### 5.3. Analysis of the \( \mathcal{L}_1 \) Adaptive Controller

#### 5.3.1. \( \mathcal{L}_1 \) Reference System

The \( \mathcal{L}_1 \) reference system is given by

\[
\dot{x}_{ref}(t) = A_m(t)x_{ref}(t) + b_m(t)u_{ref}(t) + \sigma_{ref}(t), \quad x_{ref}(0) = x_0,
\]

(5.31)

\[
y_{ref}(t) = c_m^\top(t)x_{ref}(t),
\]

(5.32)

\[
u_{ref} = Cr_g - CH^{-1}(H_{yym}\sigma_{ref} - F\tilde{y}_{in}),
\]

(5.33)

\[
\sigma_{ref}(t) \equiv (A(t) - A_m(t))x_{ref}(t) + (b(t) - b_m(t))u_{ref}(t) + f(t, x_{ref}(t)),
\]

(5.34)

where \( \tilde{y}_{in}(t) \) is the output of the following system

\[
\dot{x}_{in}(t) = A_m(t)x_{in}(t), \quad x_{in}(0) = \hat{x}_0 - x_0,
\]

\[
\tilde{y}_{in}(t) = c_m^\top(t)x_{in}(t).
\]

We notice that the \( \mathcal{L}_1 \) reference system contains the system uncertainties and the unknown initial condition \( x_0 \). Therefore it is not implementable and is used only for the analysis purposes.

**Lemma 5.3.** For the \( \mathcal{L}_1 \) reference system in (5.31)-(5.33), subject to the \( \mathcal{L}_1 \)-norm conditio-
tion (5.6), we have

\[ \|x_{\text{ref}}\|_{\infty} < \rho_r, \tag{5.35} \]
\[ \|u_{\text{ref}}\|_{\infty} \leq \rho_{ur}, \tag{5.36} \]
\[ \|y_{\text{ref}}\|_{\infty} < \mu_c \rho_r. \tag{5.37} \]

**Proof.** Let

\[ \vartheta_{\text{ref}}(t) \triangleq (A(t) - A_m(t)) x_{\text{ref}}(t) + f(t, x_{\text{ref}}(t)). \tag{5.38} \]

Then, from (5.34) it follows that \( \sigma_{\text{ref}}(t) = (b(t) - b_m(t)) u_{\text{ref}}(t) + \vartheta_{\text{ref}}(t) \). Substituting the reference control law (5.33) and taking into account the definition in (5.5), we obtain

\[
\sigma_{\text{ref}} = (b - b_m) (C_r - CH_m^{-1} (H_{y\text{um}} \sigma_{\text{ref}} - F_{\bar{y}_{\text{in}}})) + \vartheta_{\text{ref}}
\]
\[
= -(b - b_m) CH_m^{-1} H_{y\text{um}} \sigma_{\text{ref}} + (b - b_m) (C_r + CH_m^{-1} F_{\bar{y}_{\text{in}}}) + \vartheta_{\text{ref}}
\]
\[
= (I + (b - b_m) CH_m^{-1} H_{y\text{um}})^{-1} ((b - b_m) (C_r + CH_m^{-1} F_{\bar{y}_{\text{in}}}) + \vartheta_{\text{ref}})
\]
\[
= H_o ((b - b_m) (C_r + CH_m^{-1} F_{\bar{y}_{\text{in}}}) + \vartheta_{\text{ref}}).
\]

The system in (5.31) can be written as

\[ x_{\text{ref}} = H_{xm} u_{\text{ref}} + H_{xum} \sigma_{\text{ref}} + x_{in}. \]

Then, by substituting \( u_{\text{ref}}(t) \) (5.33), we obtain

\[ x_{\text{ref}} = H_{xm} (C_r - CH_m^{-1} (H_{y\text{um}} \sigma_{\text{ref}} - F_{\bar{y}_{\text{in}}})) + H_{xum} \sigma_{\text{ref}} + x_{in}. \]

The above equation can be rearranged to obtain the following

\[ x_{\text{ref}} = (H_{xum} - H_{xm} CH_m^{-1} H_{y\text{um}}) \sigma_{\text{ref}} + H_{xm} CH_m^{-1} F_{\bar{y}_{\text{in}}} + H_{xm} C_r + x_{in}. \]

Next, using the fact that \( H_{y\text{um}} = {c_m^\top H_{xum}} \), (5.39), and the definition in (5.8), we obtain

\[
x_{\text{ref}} = (I - H_{xm} CH_m^{-1} c_m^\top) H_{xum} \sigma_{\text{ref}} + H_{xm} CH_m^{-1} F_{\bar{y}_{\text{in}}} + H_{xm} C_r + x_{in}
\]
\[
= G_{um} ((b - b_m) (C_r + CH_m^{-1} F_{\bar{y}_{\text{in}}}) + \vartheta_{\text{ref}}) + H_{xm} CH_m^{-1} F_{\bar{y}_{\text{in}}} + H_{xm} C_r + x_{in}.
\]

Next, we use a contradictive argument to prove the bound in (5.35). For this purpose, we assume that (5.35) does not hold. Since \( x_{\text{ref}}(t) \) is continuous and \( \|x_{\text{ref}}(0)\|_{\infty} = \|x_0\|_{\infty} \leq \)
\( \rho_0 < \rho_r \), then there exists time \( \tau > 0 \), such that

\[
\| x_{\text{ref}}(t) \|_\infty \leq \rho_r, \quad \forall t \in [0, \tau),
\]

\[
\| x_{\text{ref}}(\tau) \|_\infty = \rho_r,
\]

(5.41)

which implies \( \| x_{\text{ref}} \|_\infty = \rho_r \). Using Assumptions 5.2 and 5.1 and (5.9), we obtain the following bound from (5.38):

\[
\| \theta_{\text{ref}} \|_\infty \leq \Delta_1 \rho_r + L_0(\rho_r) \rho_r + B(\rho_r) \leq L_{\rho} \rho_r + B(\rho_r),
\]

(5.42)

where we use the fact that \( L_{\rho} \geq L_{\rho_r} \). This allows us, using (5.40) and Assumption 5.2, to obtain the following bound:

\[
\| x_{\text{ref}}(\tau) \|_{L_\infty} \leq \| G_{\text{um}} \|_{L_1} \| C \|_{L_1} \| r_g \|_{L_\infty} + \| CH_m^{-1} F \|_{L_1} \| \tilde{y}_{\text{in}} \|_{L_\infty} + \| \theta_{\text{ref}} \|_{L_\infty} \\
+ \| H_{xm} CH_m^{-1} F \|_{L_1} \| \tilde{y}_{\text{in}} \|_{L_\infty} + \| H_{xm} C \|_{L_1} \| r_g \|_{L_\infty} + L_{\rho} \rho_r \\
+ B(\rho_r)),
\]

(5.43)

Notice that (5.6) can be rewritten as

\[
\| G_{\text{um}} \|_{L_1} \left( L_{\rho} \rho_r + \Delta_2 \| C \|_{L_1} \| r_g \|_{L_\infty} + \| CH_m^{-1} F \|_{L_1} \| \tilde{y}_{\text{in}} \|_{L_\infty} + \| \theta_{\text{ref}} \|_{L_\infty} \right) \\
+ \| H_{xm} CH_m^{-1} F \|_{L_1} \| \tilde{y}_{\text{in}} \|_{L_\infty} + \| H_{xm} C \|_{L_1} \| r_g \|_{L_\infty} + \rho_{\text{in}} < \rho_r,
\]

which along with (5.43), implies

\[
\| x_{\text{ref}} \|_{L_\infty} < \rho_r.
\]

This fact contradicts (5.41), and hence the bound in (5.35) is proven. The bound in (5.37) follows immediately from Assumption 5.3.

To prove the bound in (5.36), we substitute (5.39) in (5.33) to obtain

\[
u_{\text{ref}} = C r_g - CH_m^{-1} (H_{ym} H_{\omega} ((b - b_m) (C r_g + CH_m^{-1} F \tilde{y}_{\text{in}}) + \theta_{\text{ref}}) - F \tilde{y}_{\text{in}}),
\]

which using the bound in (5.42), results in

\[
\| u_{\text{ref}} \|_{L_\infty} \leq \| C \|_{L_1} \| r_g \|_{L_\infty} + \| CH_m^{-1} \|_{L_1} \left( \| H_{ym} H_{\omega} \|_{L_1} \left( \| C \|_{L_1} \| r_g \|_{L_\infty} \\
+ \| CH_m^{-1} F \|_{L_1} \| \tilde{y}_{\text{in}} \|_{L_\infty} + L_{\rho} \rho_r + B(\rho_r) + \| F \|_{L_1} \| \tilde{y}_{\text{in}} \|_{L_\infty} \right) \right).
\]

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Taking into account the definition of $\rho_{ur}$ in (5.11), we conclude that

$$\|u_{ref}\|_{\mathcal{L}_\infty} \leq \rho_{ur},$$

which completes the proof. \qed

5.3.2. Transient and Steady-State Performance

We will now proceed with the derivation of the performance bounds. Towards this end, let $\tilde{x}(t) \triangleq \hat{x}(t) - x(t)$ and $\tilde{\sigma}(t) \triangleq \hat{\sigma}(t) - \sigma(t)$. Then, the error dynamics between (5.26) and (5.3) are given by

$$
\begin{align*}
\dot{\tilde{x}}(t) &= A_m(t)\tilde{x}(t) + \tilde{\sigma}(t), \quad \tilde{x}(0) = \hat{x}_0 - x_0, \\
\tilde{y}(t) &= c^\top_m(t)\tilde{x}(t).
\end{align*}
$$

(5.44)

Notice that due to initialization of the output predictor, we have $\tilde{y}(0) = 0$.

Next, consider the state transformation

$$\tilde{\xi}(t) = \Lambda(t)\tilde{x}(t).$$

It follows from (5.44) and the definition of $\Lambda(t)$ in (5.16) that

$$
\begin{align*}
\dot{\tilde{\xi}}(t) &= \left(\Lambda(t)A_m(t)\Lambda^{-1}(t) - \frac{d}{dt}(\Lambda(t))\Lambda^{-1}(t)\right)\tilde{\xi}(t) + \Lambda(t)\tilde{\sigma}(t), \quad \tilde{\xi}(0) = \Lambda(0)\tilde{x}_0, \\
\tilde{y}(t) &= \tilde{\xi}_1(t),
\end{align*}
$$

(5.45)

where $\tilde{\xi}_1(t)$ is the first element of $\tilde{\xi}(t)$ and $\tilde{\xi}_1(0) = 0$. The next lemma derives the bound on the output prediction error.

**Lemma 5.4.** Consider the system in (5.1) and the $\mathcal{L}_1$ adaptive controller in (5.26), (5.27), and (5.28) subject to the $\mathcal{L}_1$-norm condition in (5.6). If we choose $T_s$ to ensure

$$\gamma_{\tilde{x}}(T_s) < \bar{\gamma}_0,$$

(5.46)

where $\bar{\gamma}_0$ is an arbitrary positive constant introduced in (5.14), and if for an arbitrary $\tau \geq 0$ the following bounds hold:

$$\|x_\tau\|_{\mathcal{L}_\infty} < \rho, \quad \|u_\tau\|_{\mathcal{L}_\infty} < \rho_u,$$

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then
\[ \| \tilde{y}_T \|_{L_\infty} < \bar{\gamma}_0. \] (5.47)

**Proof.** We prove the bound in (5.47) by a contradiction argument. Since \( \tilde{y}(0) = 0 \) and \( \tilde{y}(t) \) is continuous, then assuming that (5.47) does not hold, implies that there exists \( t' \in (0, \tau] \), such that
\[ |\tilde{y}(t)| < \gamma_0, \quad \forall \ t \in [0, t'), \]
\[ |\tilde{y}(t')| = \gamma_0, \] (5.48)
which leads to
\[ \| \tilde{y}_T \|_{L_\infty} = \gamma_0 . \]

The following bound can be produced from (5.4) using Assumptions 5.1, 5.2 and Definition (5.13):
\[ \| \sigma' \|_{L_\infty} \leq \Delta_1 \rho + \Delta_2 \rho_u + L_0(\rho) \rho + B(\rho) = L \rho \rho + \Delta_2 \rho_u + B(\rho) = \Delta. \] (5.49)

It follows from (5.45) that
\[ \tilde{\xi}(iT_s + t) = \Phi \tilde{\xi}(iT_s + t, iT_s) \tilde{\xi}(iT_s) + \int_{iT_s}^{iT_s+t} \Phi (iT_s + t, \tau) \Lambda(\tau) \tilde{\sigma}(iT_s) d\tau \]
\[ - \int_{iT_s}^{iT_s+t} \Phi (iT_s + t, \tau) \Lambda(\tau) \sigma(\tau) d\tau. \] (5.50)

Since
\[ \tilde{\xi}(iT_s + t) = \begin{bmatrix} \tilde{y}(iT_s + t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{z}(iT_s + t) \end{bmatrix}, \] (5.51)
where \( \tilde{z}(t) \triangleq [\tilde{\xi}_2(t), \tilde{\xi}_3(t), \ldots, \tilde{\xi}_n(t)]^T \), it follows from (5.50) that \( \tilde{\xi}(iT_s + t) \) can be decomposed as
\[ \tilde{\xi}(iT_s + t) = \chi(iT_s + t) + \zeta(iT_s + t), \] (5.52)
where
\[ \chi(iT_s + t) \triangleq \Phi \tilde{\xi}(iT_s + t, iT_s) \begin{bmatrix} \tilde{y}(iT_s) \\ 0 \end{bmatrix} + \int_{iT_s}^{iT_s+t} \Phi (iT_s + t, \tau) \Lambda(\tau) \tilde{\sigma}(iT_s) d\tau, \]
\[ \zeta(iT_s + t) \triangleq \Phi \tilde{\xi}(iT_s + t, iT_s) \begin{bmatrix} 0 \\ \tilde{z}(iT_s) \end{bmatrix} - \int_{iT_s}^{iT_s+t} \Phi (iT_s + t, \tau) \Lambda(\tau) \sigma(\tau) d\tau. \] (5.53)
Next we prove by induction that for all $i$ such that $iT_s \leq t'$ one has

$$|\tilde{y}(iT_s)| \leq \varsigma(i, T_s),$$  \hspace{1cm} (5.54)

$$\tilde{z}^T(iT_s)P_2(iT_s)\tilde{z}(iT_s) \leq \alpha,$$  \hspace{1cm} (5.55)

where $\varsigma(i, T_s)$ and $\alpha$ were defined in (5.20)-(5.21). We start by noting that, since $\tilde{y}(0) = 0$, we have $|\tilde{y}(0)| \leq \varsigma(0, T_s)$. We can also show that given the definition of $\alpha$ in (5.21), we have

$$\tilde{z}^T(0)P_2(0)\tilde{z}(0) \leq \lambda_{\text{max}}(P_2(0))\|\tilde{z}(0)\|_2^2 \leq 4n\lambda_{\text{max}}(P_2(0))\|\Lambda(0)\|_\infty\rho_0^2 \leq \alpha.$$  \hspace{1cm} (5.56)

Next, we prove that if (5.54)-(5.55) hold for arbitrary $i$, such that $(i+1)T_s \leq t'$, then

$$|\tilde{y}((i + 1)T_s)| \leq \varsigma(i + 1, T_s),$$  \hspace{1cm} (5.57)

hold as well. To this end, assume that (5.54)-(5.55) hold for $i$, and in addition that $(i+1)T_s \leq t'$. Then, it follows from (5.52) that

$$\tilde{\xi}((i + 1)T_s) = \chi((i + 1)T_s) + \zeta((i + 1)T_s),$$  \hspace{1cm} (5.58)

where

$$\chi((i + 1)T_s) = \Phi_{\tilde{\xi}}((i + 1)T_s, iT_s) \begin{bmatrix} \tilde{y}(iT_s) \\ 0 \end{bmatrix} + \int_{iT_s}^{(i+1)T_s} \Phi_{\tilde{\xi}}((i + 1)T_s, \tau)\Lambda(\tau)\hat{\sigma}(iT_s)d\tau,$$  \hspace{1cm} (5.59)

$$\zeta((i + 1)T_s) = \Phi_{\tilde{\xi}}((i + 1)T_s, iT_s) \begin{bmatrix} 0 \\ \tilde{z}(iT_s) \end{bmatrix} - \int_{iT_s}^{(i+1)T_s} \Phi_{\tilde{\xi}}((i + 1)T_s, \tau)\Lambda(\tau)\sigma(\tau)d\tau.$$  \hspace{1cm} (5.60)

Substituting the adaptive law from (5.27) in (5.59), we have

$$\chi((i + 1)T_s) = 0.$$  \hspace{1cm} (5.61)
On the other hand, it follows from (5.53) that $\zeta(t)$ is the solution to the following dynamics:

\[
\dot{\zeta}(t) = \left( \Lambda(t)A_m(t) - \frac{d}{dt}\Lambda(t) \right) \Lambda^{-1}(t)\zeta(t) - \Lambda(t)\sigma(t),
\]
\[
\zeta(iT_s) = \begin{bmatrix} 0 \\ \bar{z}(iT_s) \end{bmatrix}, \quad t \in [iT_s, (i+1)T_s].
\]

Consider now the following function

\[
V(\zeta(t)) = \zeta^T(t)\Lambda^{-T}(t)P(t)\Lambda^{-1}(t)\zeta(t),
\]
\[
(5.63)
\]

over $t \in [iT_s, (i+1)T_s]$. Lemma 5.1 implies that $\Lambda^{-T}(t)P(t)\Lambda^{-1}(t)$ is positive definite and, hence, $V(\zeta)$ is a positive definite function. Further, it follows from (5.53), Lemma 5.1 and the fact (5.61) that

\[
V(\zeta(iT_s)) = \bar{z}^T(iT_s)P_2(iT_s)\bar{z}(iT_s),
\]

which, along with the upper bound in (5.55), leads to

\[
V(\zeta(iT_s)) \leq \alpha.
\]

Next we perform a reverse state transformation for the system in (5.62) with $\bar{\zeta}(t) \triangleq \Lambda^{-1}(t)\zeta(t)$ to obtain the following system

\[
\dot{\bar{\zeta}}(t) = A_m(t)\bar{\zeta}(t) - \sigma(t).
\]

The function $V(\zeta)$ in (5.63) now takes the form

\[
V(\bar{\zeta}(t)) = \bar{\zeta}^T(t)P(t)\bar{\zeta}(t).
\]

(5.66)

Taking the time derivative of (5.66) along the trajectories (5.65) over $t \in [iT_s, (i+1)T_s]$, and using Assumption 5.6, we obtain

\[
\dot{V}(t) = \bar{\zeta}^T(t)P(t)\bar{\zeta}(t) + \bar{\zeta}^T(t)\dot{P}(t)\bar{\zeta}(t) + \bar{\zeta}^T(t)P(t)\dot{\bar{\zeta}}(t)
\]
\[
= \left( \bar{\zeta}^T(t)A_m^T(t) - \sigma^T(t) \right) P(t)\bar{\zeta}(t) + \bar{\zeta}^T(t)\dot{P}(t)\bar{\zeta}(t) + \bar{\zeta}^T(t)P(t) \left( A_m(t)\bar{\zeta}(t) - \sigma(t) \right)
\]
\[
= \bar{\zeta}^T(t) \left( A_m^T(t)P(t) + P(t)A_m(t) + \dot{P}(t) \right) \bar{\zeta}(t) - \sigma^T(t)P(t)\bar{\zeta}(t) - \bar{\zeta}^T(t)P(t)\sigma(t)
\]
\[
= -\bar{\zeta}^T(t) \left( I - \dot{P}(t) \right) \bar{\zeta}(t) - 2\bar{\zeta}^T(t)P(t)\sigma(t),
\]

which, using the the facts that $\|\sigma(t)\|_{\infty} \leq \|\sigma_{tv}\|_{L_\infty}, \| \cdot \|_1 \leq \sqrt{n} \cdot \| \cdot \|_2$, the definition (5.22),
and the bound in (5.49), can be bounded as follows

\[
\dot{V}(t) \leq -\zeta^T(t) \left( I - \dot{P}(t) \right) \zeta(t) + 2\|\zeta^T(t)P(t)\sigma(t)\|_\infty \\
\leq -\zeta^T(t) \left( I - \dot{P}(t) \right) \zeta(t) + 2\|\zeta(t)\|_1\|P(t)\|_\infty\|\sigma(t)\|_\infty \\
\leq -\zeta^T(t) \left( I - \dot{P}(t) \right) \zeta(t) + 2\sqrt{n}\|\zeta(t)\|_1\|\mu\Delta\|P\Delta\] .
\]

Assumption 5.6, along with the fact that 

\[
|\zeta^T(t)\dot{P}(t)\zeta(t)| \leq \|\zeta(t)\|^2\rho(\dot{P}(t)) \leq \|\zeta(t)\|^2\|\dot{P}(t)\|_\infty,
\]

implies

\[
\zeta^T(t) \left( I - \dot{P}(t) \right) \zeta(t) \geq \frac{1 - \epsilon_P}{\lambda_{P_{\text{max}}}} \zeta^T(t)P(t)\zeta(t) \geq \frac{1 - \epsilon_P}{\lambda_{P_{\text{max}}}} \|\zeta(t)\|^2 .
\]

This results in

\[
\dot{V}(t) \leq -\frac{1 - \epsilon_P}{\lambda_{P_{\text{max}}}} \|\zeta(t)\|^2 + 2\sqrt{n}\|\zeta(t)\|\mu_P\Delta .
\]

(5.67)

Notice that for any \( t \in [iT_s, (i + 1)T_s] \), if

\[
V(t) > \alpha ,
\]

we have

\[
\|\zeta(t)\| > \sqrt{\frac{\alpha}{\lambda_{P_{\text{max}}}}} \geq \frac{2\lambda_{P_{\text{max}}}\Delta \mu_P\sqrt{n}}{\lambda_{P_{\text{min}}} (1 - \epsilon_P)} ,
\]

and the upper bound in (5.67) yields

\[
\dot{V}(t) < 0 .
\]

Thus, it follows from (5.64) that

\[
V(t) \leq \alpha , \quad \forall \ t \in [iT_s, (i + 1)T_s] .
\]

Taking into account the relationship (5.58), along with (5.61), we can rewrite (5.63) as

\[
V((i + 1)T_s) = \tilde{\zeta}^T((i + 1)T_s)(\Lambda^{-\top}((i + 1)T_s)P((i + 1)T_s)\Lambda^{-1}((i + 1)T_s))\tilde{\zeta}((i + 1)T_s) \leq \alpha .
\]

Using the result of Lemma 5.1, one can derive

\[
\tilde{\zeta}^T((i + 1)T_s)P_2(t)\tilde{\zeta}((i + 1)T_s) \leq
\tilde{\xi}^T((i + 1)T_s)(\Lambda^{-\top}((i + 1)T_s)P((i + 1)T_s)\Lambda^{-1}((i + 1)T_s))\tilde{\xi}((i + 1)T_s) \leq \alpha ,
\]

which implies that the upper bound in (5.57) holds. Next, it follows from (5.58), (5.61),
and (5.51) that
\[ \tilde{y}((i + 1)T_s) = 1_1^T \zeta((i + 1)T_s) , \]
and (5.60) leads to the following expression:
\[ \tilde{y}((i + 1)T_s) = 1_1^T \Phi_\xi((i + 1)T_s, iT_s) \begin{bmatrix} 0 \\ \tilde{z}(iT_s) \end{bmatrix} - 1_1^T \int_{iT_s}^{(i+1)T_s} \Phi_\xi((i + 1)T_s, \tau) \Lambda(\tau) \sigma(\tau) d\tau . \]

The upper bounds in (5.57) and (5.49) yield the following upper bound:
\[ |\tilde{y}((i + 1)T_s)| \leq \|\eta_2(i, T_s, T_s)\||\tilde{z}(iT_s)|| + \int_{iT_s}^{(i+1)T_s} \|1_1^T \Phi_\xi((i + 1)T_s, \tau) \Lambda(\tau)\|_1 \|\sigma(\tau)\|_\infty d\tau \]
\[ \leq \|\eta_2(i, T_s, T_s)\| \sqrt{\alpha \lambda_{P_{2\max}}} + \kappa(i, T_s) \Delta = \varsigma(i + 1, T_s) , \]
where \( \eta_2(i, T_s) \) and \( \kappa(i, T_s) \) were defined in (5.18) and (5.19), while \( \varsigma(i, T_s) \) was defined in (5.20). This confirms the upper bound in (5.56). Hence, (5.54)-(5.55) hold for all \( i \) such that \( iT_s \leq t' \).

For all \( iT_s + t \leq t' \), where \( 0 \leq t \leq T_s \), using the expression from (5.50), we can write that
\[ \tilde{y}(iT_s + t) = 1_1^T \Phi_\xi(iT_s + t, iT_s) \tilde{\zeta}(iT_s) + 1_1^T \int_{iT_s}^{iT_s + t} \Phi_\xi(iT_s + t, \tau) \Lambda(\tau) \hat{\sigma}(iT_s) d\tau - 1_1^T \int_{iT_s}^{iT_s + t} \Phi_\xi(iT_s + t, \tau) \Lambda(\tau) \sigma(\tau) d\tau . \]

The upper bound in (5.49) and the definitions of \( \eta_1(t), \eta_2(t), \eta_3(t) \) and \( \eta_4(t) \) lead to the following upper bound:
\[ |\tilde{y}(iT_s + t)| \leq |\eta_1(i, T_s, t)| |\tilde{y}(iT_s)| + \|\eta_2(i, T_s, t)\||\tilde{z}(iT_s)|| + \eta_3(i, T_s, t) |\tilde{y}(iT_s)| + \eta_4(i, T_s, t) \Delta . \]

Taking into consideration (5.54)-(5.55), and recalling the definitions of \( \beta_1(i, T_s) \), \( \beta_2(i, T_s) \), \( \beta_3(i, T_s) \), \( \beta_4(i, T_s) \) in (5.23)-(5.24), for all \( 0 \leq t \leq T_s \) and for arbitrary non-negative integer \( i \) subject to \( iT_s + t \leq t' \), we have
\[ |\tilde{y}(iT_s + t)| \leq \beta_1(i, T_s) \varsigma(i, T_s) + \beta_2(i, T_s) \sqrt{\frac{\alpha}{\lambda_{P_{2\max}}}} + \beta_3(i, T_s) \varsigma(i, T_s) + \beta_4(i, T_s) \Delta . \]
Since the right hand side coincides with the definition of $\gamma_0(i, T_s)$ in (5.25), for all $t \in [0, t']$ we have the following bound

$$|\tilde{y}(t)| \leq \gamma_0(i, T_s), \quad \forall i \in \mathbb{N} \cup \{0\},$$

which along with (5.46) yields

$$\|\tilde{y}\|_{L_\infty} \leq \gamma_{\tilde{x}}(T_s) < \tilde{\gamma}_0.$$

This clearly contradicts the statement in (5.48). Therefore (5.47) holds and the proof is completed. □

The next theorem states the main result of the Chapter.

**Theorem 5.1.** Given the closed-loop system with the $L_1$ adaptive controller defined via (5.1), (5.26), (5.27), (5.28), subject to the $L_1$-norm condition in (5.6), and the $L_1$ reference system in (5.31)-(5.33), if we choose $T_s$ to ensure

$$\gamma_{\tilde{x}}(T_s) < \tilde{\gamma}_0,$$

where $\tilde{\gamma}_0$ is an arbitrary positive constant introduced in (5.14), we have

$$\|x\|_{L_\infty} \leq \rho, \quad (5.68)$$
$$\|u\|_{L_\infty} \leq \rho_u, \quad (5.69)$$
$$\|x_{\text{ref}} - x\|_{L_\infty} < \gamma_1, \quad (5.70)$$
$$\|y_{\text{ref}} - y\|_{L_\infty} < \mu_c \gamma_1, \quad (5.71)$$
$$\|u_{\text{ref}} - u\|_{L_\infty} < \gamma_2, \quad (5.72)$$

where $\gamma_1$ and $\gamma_2$ are defined in (5.14) and (5.15) respectively.

**Proof.** To accomplish the proof, we use contradictory argument. Assume that the bounds in (5.70) and (5.72) do not hold (either one of them or both simultaneously). Then, since

$$\|x_{\text{ref}}(0) - x(0)\|_\infty = 0 < \gamma_1, \quad \|u_{\text{ref}}(0) - u(0)\|_\infty = 0 < \gamma_2,$$

and $x(t)$, $x_{\text{ref}}(t)$, $u(t)$, and $u_{\text{ref}}(t)$ are continuous, there exists time $\tau \in \mathbb{R}^+$ such that

$$\|x_{\text{ref}}(\tau) - x(\tau)\|_\infty = \gamma_1, \quad \text{or} \quad \|u_{\text{ref}}(\tau) - u(\tau)\|_\infty = \gamma_2,$$
while
\[ \|x_{\text{ref}}(t) - x(t)\|_{\infty} < \gamma_1, \quad \text{and} \quad \|u_{\text{ref}}(t) - u(t)\|_{\infty} < \gamma_2, \]
for all \( t \in [0, \tau) \). This implies that the following equalities hold
\[ \| (x_{\text{ref}} - x)_{\tau} \|_{L_{\infty}} \leq \gamma_1, \quad \| (u_{\text{ref}} - u)_{\tau} \|_{L_{\infty}} \leq \gamma_2. \] (5.74)

From Lemma 5.3 we obtain
\[ \|x_{\text{ref}}\|_{L_{\infty}} \leq \rho_r, \quad \|u_{\text{ref}}\|_{L_{\infty}} \leq \rho_u, \]
which along with the definitions of \( \rho \) and \( \rho_u \) in (5.10) and (5.12) allows to derive from (5.74)
\[ \|x_{\tau}\|_{L_{\infty}} \leq \rho_r + \gamma_1 < \rho, \quad \text{and} \quad \|u_{\tau}\|_{L_{\infty}} \leq \rho_u + \gamma_2 < \rho_u. \] (5.75)

Let
\[ \vartheta(t) \triangleq (A(t) - A_m(t)) x(t) + f(t, x(t)), \quad \vartheta_{\text{ref}}(t) \triangleq (A(t) - A_m(t)) x_{\text{ref}}(t) + f(t, x_{\text{ref}}(t)). \]

Then using Assumptions 5.1, 5.2 and the bounds (5.74)-(5.75) we obtain
\[ \| (\vartheta_{\text{ref}} - \vartheta)_{\tau} \|_{L_{\infty}} \leq \Delta_1 \| (x_{\text{ref}} - x)_{\tau} \|_{L_{\infty}} + L_0(\rho) \| (x_{\text{ref}} - x)_{\tau} \|_{L_{\infty}} = L_\rho \| (x_{\text{ref}} - x)_{\tau} \|_{L_{\infty}}, \] (5.76)
\[ \| \sigma_{\tau} \|_{L_{\infty}} \leq \Delta_1 \rho + \Delta_2 \rho_u + L(\rho) \rho + B(\rho) = L_\rho \rho + \Delta_2 \rho_u + B(\rho) = \Delta, \] (5.77)
where \( L_\rho \) was defined in (5.9). Notice that from (5.4) and (5.34) it follows that
\[ \sigma_{\text{ref}}(t) - \sigma(t) = \vartheta_{\text{ref}}(t) - \vartheta(t) + (b(t) - b_m(t))(u_{\text{ref}}(t) - u(t)). \] (5.78)

Adding and subtracting \( \mathcal{C} \mathcal{H}_m^{-1} \mathcal{F} \mathcal{H}_{\text{yum}} \sigma \) and \( \mathcal{C} \mathcal{H}_m^{-1} \mathcal{H}_{\text{yum}} \sigma \) from (5.28) results in
\[ u = \mathcal{C} r_g - \mathcal{C} \mathcal{H}_m^{-1} \mathcal{F} \mathcal{H}_{\text{yum}} \tilde{\sigma} + \mathcal{C} \mathcal{H}_m^{-1} (1 - \mathcal{F}) \mathcal{H}_{\text{yum}} \sigma - \mathcal{C} \mathcal{H}_m^{-1} \mathcal{H}_{\text{yum}} \sigma. \]

Next, subtracting this result from (5.33) yields
\[ u_{\text{ref}} - u = -\mathcal{C} \mathcal{H}_m^{-1} \mathcal{H}_{\text{yum}} (\sigma_{\text{ref}} - \sigma) + \mathcal{C} \mathcal{H}_m^{-1} \mathcal{F} \tilde{y}_{\text{in}} + \mathcal{C} \mathcal{H}_m^{-1} \mathcal{F} \mathcal{H}_{\text{yum}} \tilde{\sigma} - \mathcal{C} \mathcal{H}_m^{-1} (1 - \mathcal{F}) \mathcal{H}_{\text{yum}} \sigma. \]

Further, we rewrite (5.44) as \( \tilde{y} = \mathcal{H}_{\text{yum}} \tilde{\sigma} + \tilde{y}_{\text{in}} \), which leads to
\[ u_{\text{ref}} - u = -\mathcal{C} \mathcal{H}_m^{-1} \mathcal{H}_{\text{yum}} (\sigma_{\text{ref}} - \sigma) + \mathcal{C} \mathcal{H}_m^{-1} \mathcal{F} \tilde{y} - \mathcal{C} \mathcal{H}_m^{-1} (1 - \mathcal{F}) \mathcal{H}_{\text{yum}} \sigma. \] (5.79)
Substituting this in (5.78) and using the definition (5.5), we obtain
\[
\sigma_{ref} - \sigma = \vartheta_{ref} - \vartheta + (b - b_m)(-\mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{ym} \sigma)
+ \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\tilde{y} - \mathcal{C}\mathcal{H}_m^{-1}(1 - \mathcal{F})\mathcal{H}_{ym} \sigma
+ \mathcal{H}_\omega(\vartheta_{ref} - \vartheta) + \mathcal{H}_\omega(b - b_m)\mathcal{C}\mathcal{H}_m^{-1}(\mathcal{F}\tilde{y} - (1 - \mathcal{F})\mathcal{H}_{ym} \sigma).
\] (5.80)

The systems in (5.3) and (5.31) can be written as
\[
x = \mathcal{H}_x m u + \mathcal{H}_x um \sigma + x_m,
\]
\[
x_{ref} = \mathcal{H}_x m u_{ref} + \mathcal{H}_x um \sigma_{ref} + x_m,
\]
which leads to
\[
x_{ref} - x = \mathcal{H}_x m (u_{ref} - u) + \mathcal{H}_x um (\sigma_{ref} - \sigma).
\]
Substituting (5.79), yields
\[
x_{ref} - x = \mathcal{H}_x m (-\mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{ym} (\sigma_{ref} - \sigma) + \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\tilde{y} - \mathcal{C}\mathcal{H}_m^{-1}(1 - \mathcal{F})\mathcal{H}_{ym} \sigma) + \mathcal{H}_x um (\sigma_{ref} - \sigma)
= (\mathcal{H}_x um - \mathcal{H}_x m \mathcal{C}\mathcal{H}_m^{-1}\mathcal{H}_{ym} (\sigma_{ref} - \sigma) + \mathcal{H}_x m \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\tilde{y} - \mathcal{H}_x m \mathcal{C}\mathcal{H}_m^{-1}(1 - \mathcal{F})\mathcal{H}_{ym} \sigma,
\]
which after substituting (5.80) and using the definition (5.8) takes the following form
\[
x_{ref} - x = \mathcal{G}_{um} (\vartheta_{ref} - \vartheta) + \mathcal{G}_{um} (b - b_m)\mathcal{C}\mathcal{H}_m^{-1}(\mathcal{F}\tilde{y} - (1 - \mathcal{F})\mathcal{H}_{ym} \sigma)
+ \mathcal{H}_x m \mathcal{C}\mathcal{H}_m^{-1}\mathcal{F}\tilde{y} - \mathcal{H}_x m \mathcal{C}\mathcal{H}_m^{-1}(1 - \mathcal{F})\mathcal{H}_{ym} \sigma.
\]

Notice that due to (5.75), the assumptions of Lemma 5.4 are satisfied. Therefore, taking into account (5.76) and (5.77), we obtain the following bound
\[
\| (x_{ref} - x)_{\tau} \|_{\mathcal{L}_\infty} \leq \| \mathcal{G}_{um} \|_{\mathcal{L}_1} \| (\vartheta_{ref} - \vartheta)_{\tau} \|_{\mathcal{L}_\infty} + \| \mathcal{G}_{um} (b - b_m)\mathcal{C}\mathcal{H}_m^{-1} \|_{\mathcal{L}_1} \| \mathcal{F} \|_{\mathcal{L}_1} \| \tilde{y}_{\tau} \|_{\mathcal{L}_\infty} + \| \mathcal{H}_x m \mathcal{C}\mathcal{H}_m^{-1}(1 - \mathcal{F})\mathcal{H}_{ym} \|_{\mathcal{L}_1} \| \tilde{y} \|_{\mathcal{L}_\infty}
\leq \| \mathcal{G}_{um} \|_{\mathcal{L}_1} \| L_{\rho} \| (x_{ref} - x)_{\tau} \|_{\mathcal{L}_\infty} + \| \mathcal{G}_{um} (b - b_m)\mathcal{C}\mathcal{H}_m^{-1} \|_{\mathcal{L}_1} \| \mathcal{F} \|_{\mathcal{L}_1} \tilde{\gamma}_0
+ \| \mathcal{H}_x m \mathcal{C}\mathcal{H}_m^{-1}(1 - \mathcal{F})\mathcal{H}_{ym} \|_{\mathcal{L}_1} \| \tilde{\gamma} \|_{\mathcal{L}_\infty}
\leq \frac{1}{1 - \| \mathcal{G}_{um} \|_{\mathcal{L}_1} \| L_{\rho} \|} \left( \| \mathcal{G}_{um} (b - b_m)\mathcal{C}\mathcal{H}_m^{-1} \|_{\mathcal{L}_1} \| \mathcal{F} \|_{\mathcal{L}_1} \tilde{\gamma}_0 + \| \mathcal{H}_x m \mathcal{C}\mathcal{H}_m^{-1}(1 - \mathcal{F})\mathcal{H}_{ym} \|_{\mathcal{L}_1} \Delta \right)
\leq \gamma_1.
\]
This contradicts to the first equality in (5.73). It remains to show that the second equality also is not true. Towards this end, we substitute (5.80) into (5.79), and taking into account the bounds in (5.76), (5.77), and Lemma 5.4, we obtain

\[
\| (u_{\text{ref}} - u)_{\tau} \|_{\mathcal{L}_\infty} \leq \| \mathcal{C} \mathcal{H}_m^{-1} \mathcal{H}_{\gamma \mu} \|_{\mathcal{L}_1} (L_\rho \gamma_1 + \| (b - b_m) \mathcal{C} \mathcal{H}_m^{-1} \|_{\mathcal{L}_1} (\| \mathcal{F} \|_{\mathcal{L}_1} \gamma_0 \\
+ \| (1 - \mathcal{F}) \mathcal{H}_{\gamma \mu} \|_{\mathcal{L}_1} \Delta)) + \| \mathcal{C} \mathcal{H}_m^{-1} \mathcal{F} \|_{\mathcal{L}_1} \gamma_0 + \| \mathcal{C} \mathcal{H}_m^{-1} (1 - \mathcal{F}) \mathcal{H}_{\gamma \mu} \|_{\mathcal{L}_1} \Delta < \gamma_2 ,
\]

which contradicts to the second equation in (5.73). Thus the bounds in (5.70), (5.72) hold. The bound in (5.71) follows from the fact that \( y_{\text{ref}}(t) - y(t) = c_m^\top(t) (x_{\text{ref}}(t) - x(t)) \). The results (5.68) and (5.69) follow directly from the bounds in (5.75).

\[ \square \]

**Remark 5.7.** We notice that from the definitions of \( \gamma_1 \) and \( \gamma_2 \), defined in (5.14) and (5.15), and Lemmas 5.2 and 5.4, it follows that by reducing the sampling time \( T_s \) and the bandwidth of the lowpass filter \( F(s) \) one can achieve arbitrarily small performance bounds (5.70)-(5.72).

### 5.4. Simulation Results

For verification of the theoretical results and illustration of the performance of the proposed \( \mathcal{L}_1 \) adaptive controller we consider the desired model (5.2) with the following parameters:

\[
A_m(t) = \begin{bmatrix} 0 & 1 \\ -w^2(t) & -1.4w(t) \end{bmatrix}, \quad b_m(t) = \begin{bmatrix} 1 \\ w^2(t) + 0.1w^2(t) \sin(0.2\pi t) \end{bmatrix},
\]

\[
c_m(t) = [1 - 0.2\cos(\pi t/20) \quad 0],
\]

where

\[
w(t) = \begin{cases} 1 + 10(1 - e^{-0.01t}), & 0 \leq t \leq 15, \\ 11 - 10e^{-0.15} + 0.5\sin(0.2e^{-0.15}(t - 15)), & t > 15. \end{cases}
\]

For the simulation of the uncertainty in the plant (5.1), we consider two sets of system matrices \( A(t) \) and \( b(t) \), given by

\[
A_I(t) = \begin{bmatrix} 0 & 1 + 0.5\sin(0.5t) \\ -w^2(t) & -2 \end{bmatrix}, \quad b_I = \begin{bmatrix} 0.7 \\ 1.5b_m(t) \end{bmatrix},
\]

\[83\]
Figure 5.1: Time history of the system’s parameters during the time interval $t \in [0, 20]$.

and

$$A_{II}(t) = \begin{bmatrix} 0 & 1 + t/40 \\ -0.8w^2(t) & -3 \end{bmatrix}, \quad \text{for } t \leq 20,$$

$$A_{II}(t) = \begin{bmatrix} 0 & 1.5 \\ -0.8w^2(t) & -3 \end{bmatrix}, \quad \text{for } t > 20,$$

with

$$b_{II}(t) = \begin{bmatrix} 1.3 \\ 0.8b_{m2}(t) \end{bmatrix}, \quad \forall t \geq 0,$$

where $b_{m2}(t)$ is the second component of the vector $b_{m}(t)$. Figure 5.1 shows the time history of the system’s parameters. As we can see from changes of $\omega(t)$, the desired system becomes faster with time, which makes the performance requirements stricter. The input gain of the plant increases monotonically with time.

For the system nonlinearity $f(t, x)$ we use the following functions:

$$f_{I}(x, t) = \begin{bmatrix} -0.2x_1^2(t) + 2 \sin(x_1(t))x_2(t) \\ x_2(t) - 5 \ln(|x_1(t)| + 1)x_2(t) + 0.5 \sin(0.3t) \end{bmatrix},$$

$$f_{II}(x, t) = \begin{bmatrix} 0.2x_2^2(t) - 0.7\sqrt{|x_1(t)|} - 0.1 \sin(0.2t) \\ 3x_1(t) - x_1^2(t) + 0.3 \cos(0.3t) - 0.2 \end{bmatrix}.$$

These two functions along with two sets of the plant’s system matrices allows us to set four scenarios of uncertainty for simulation purposes:

1. set $A(t) = A_{I}(t)$, $b(t) = b_{I}(t)$, and $f(t, x) = f_{I}(t, x)$;

2. set $A(t) = A_{I}(t)$, $b(t) = b_{I}(t)$, and $f(t, x) = f_{II}(t, x)$;
3. set $A(t) = A_{II}(t)$, $b(t) = b_{II}(t)$, and $f(t, x) = f_I(t, x)$;
4. set $A(t) = A_{II}(t)$, $b(t) = b_{II}(t)$, and $f(t, x) = f_{II}(t, x)$.

For the $L_1$ adaptive controller we choose a first order lowpass filter

$$C(s) = \frac{20}{s + 20},$$

the sampling rate $T_s = 0.001$ s, and $Q = I$. The adaptation gain $\mu(i, T_s)$ is computed off-line, and its values are stored in the table. For computation of the system inverse (5.30), we set the filter $F(s)$ to

$$F(s) = \frac{1}{0.0005s + 1},$$

and using [76], we compute the following functions:

$$R_1(t) = w^2(t) + 0.1w^2(t)\sin(0.2\pi t) + \frac{0.01\pi \sin(\pi t/20)}{1 - 0.2\cos(\pi t/20)},$$

$$S(t) = 1 - 0.2\cos(\pi t/20),$$

$$J(t) = 1 - 0.2\cos(\pi t/20),$$

$$E(t) = -1.4w(t) - w^2(t) + 0.1w^2(t)\sin(0.2\pi t),$$

$$G(t) = \frac{-w^2(t) - (w^2(t) + 0.1w^2(t)\sin(0.2\pi t))^2 - (w^2(t) + 0.1w^2(t)\sin(0.2\pi t))1.4w(t)}{1 - 0.2\cos(\pi t/20)} - \frac{0.02\pi w^2(t)\cos(0.2\pi t)}{1 - 0.2\cos(\pi t/20)}.$$

Figure 5.2 shows the closed-loop system performance for the first scenario in the presence of zero initial conditions. From Figures 5.2a and 5.2b we see that the $L_1$ adaptive controller compensates for the uncertainty and achieves tracking of the reference command according to the selected desired system. Moreover, both the system output and the control signal of the $L_1$ adaptive controller almost coincide with the corresponding signals of the $L_1$ reference system (5.31)-(5.33). The prediction error $\tilde{y}(t)$ remains small, while the adaptive estimate $\hat{\sigma}(t)$ changes rapidly to compensate for the system uncertainty and nonlinearity. Notice that the desired settling time for the second step command is smaller than for the first command due to changes in the system matrix $A_m(t)$. Also notice that the control signal remains free of high frequency oscillations and has a reasonable behavior for its derivative.

The transient response of the closed-loop system with $L_1$ adaptive controller for the four scenarios of the system uncertainty is shown in Figure 5.3. We see that the system output for all scenarios remains close to the desired model response and does not vary.
Figure 5.2: Performance of the $\mathcal{L}_1$ adaptive controller for scenario 1.
Figure 5.3: Response of the $\mathcal{L}_1$ adaptive controller for different scenarios of system uncertainty and nonlinearity.

Figure 5.4: Response of the $\mathcal{L}_1$ adaptive controller to step commands of different magnitude for scenario 1.

significantly, which agrees with the theoretical results. An important observation is that the transient of the system output for all scenarios is predictable and close to the linear (desired model) response. However, the control signal noticeably changes for different scenarios to compensate for different types of uncertainties and nonlinearities.

To illustrate this feature further, we consider the closed-loop system response for step reference commands with different amplitudes $r_1(t) = 0.5$, $r_2(t) = 1$, $r_3(t) = 1.5$. The simulation results in Figure 5.4 appear consistent with the results in Figure 5.3, and show that despite the nonlinearity in the plant, the output of the closed-loop system behaves similar to the linear system, for which the output response scales with the amplitude of the step input.

Next, we consider nonzero initial conditions $x_0 = [0.5 \ 1]^T$ and $\dot{x}_0 = [0.5 \ 0]^T$. Figure 5.5
Figure 5.5: Response of the $L_1$ adaptive controller in the presence of nonzero initialization error for scenario 1.

Figure 5.6: Simulation results for $L_1$ adaptive controller in the presence of time delay $\tau = 50$ ms.

shows the simulation results for the first scenario. The transient response of the system remain predictable and consistent with results in Figure 5.4. The nonzero initialization error between the output predictor and the plant does not affect the system transient significantly.

Finally, we check robustness of the closed-loop system to input time delays. Figure 5.4 shows the simulation results for the input time delay $\tau = 50$ ms. We see that the system remains stable without significant changes in the system output response. The control signal shows some oscillations, which result from closeness of the injected time delay to the marginal time delay $T = 70$ ms, at which the system loses stability.
CHAPTER 6

$\mathcal{L}_1$ Adaptive Output-feedback Architecture with Monopoli’s Augmented Error Approach

In this Chapter we present an $\mathcal{L}_1$ adaptive output-feedback controller for a class of uncertain nonlinear systems in the presence of time and output dependent unknown nonlinearities. As compared to the $\mathcal{L}_1$ adaptive output-feedback control architecture in Chapter 2, the architecture proposed below relies on system inversion, and is therefore limited to minimum phase systems. Similar to prior solutions in $\mathcal{L}_1$ adaptive control theory, the feedback structure is comprised of the three main elements, involving predictor, adaptation laws and lowpass filter, with the only difference that the predictor here is an input predictor and not an output predictor. Whereas in the $\mathcal{L}_1$ adaptive output-feedback architecture from Section 2, the verification of the sufficient condition for stability, written in terms of $\mathcal{L}_1$ norm of cascaded systems, is not straightforward, the solution proposed in this section, under mild assumptions on system dynamics, provides much simpler form of the stability condition. The obtained $\mathcal{L}_1$-norm stability condition has two separate terms: the first term is responsible for feedback stability of the $\mathcal{L}_1$ controller with chosen parameters, and the second term is responsible for the effect of the uncertainty, which helps to make the tuning procedure more systematic. Similar to the architecture in Chapter 2, the closed-loop system achieves arbitrarily close tracking of the input and the output signals of the reference system.

6.1. Problem Formulation

Consider the system given by

$$y(s) = W(s)(u(s) + \sigma(s)), \quad W(s) \triangleq k_0 \frac{B(s)}{A(s)} ,$$  \hspace{1cm} (6.1)

where

- $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ are the system input and the output respectively.

- $\sigma(s)$ is the Laplace transform of $\sigma(t) \triangleq f(t, y(t))$, and $f(t, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the unknown nonlinearity, which represents unmodeled system nonlinearities and disturbances. Assume that for arbitrary $y_1, y_2 \in \mathbb{R}$ there exists a known constant $L \in \mathbb{R}^+$, such that:

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|, \quad \forall \ t \geq 0 .$$  \hspace{1cm} (6.2)
Also assume that there exists a known constant $L_0 \in \mathbb{R}^+$, such that

$$|f(t, 0)| \leq L_0, \quad \forall \ t \geq 0. \quad (6.3)$$

- $A(s)$ and $B(s)$ are relatively prime unknown monic polynomials in $s$, subject to the following assumptions:
  - $B(s)$ is Hurwitz.
  - $n_A > n_B$, where $n_A \triangleq \deg(A(s))$ and $n_B \triangleq \deg(B(s))$ are the unknown degrees of the polynomials. The upper bound $n > n_A$ and the relative degree of $W(s)$ given by $n_r \triangleq n_A - n_B$ are known.
  - Conservative information about location of the polynomial coefficients is available.
- $k_0 \in \mathbb{R}$ is the unknown high frequency gain of the system, with known sign and known bounds. Without loss of generality, let $k_m < k_0 < k_M$, where $k_m, k_M \in \mathbb{R}^+$ are known.

The control objective is to design a control law $u(t)$, which ensures that the system output $y(t)$ tracks a bounded reference signal $r(t)$ with performance specifications, given by the following desired (ideal) system:

$$y(s) = k_g M(s) r(s), \quad M(s) \triangleq \frac{N(s)}{D(s)}, \quad (6.4)$$

where

- $r(t) \in \mathbb{R}$ is uniformly bounded by

$$\|r\|_{L_{\infty}} \leq \bar{r}, \quad (6.5)$$

where $\bar{r} \in \mathbb{R}^+$ is given;
- $N(s)$ and $D(s)$ are arbitrary known monic Hurwitz polynomials:

$$N(s) \triangleq s^{n-m} + \alpha_1^N s^{n-m-1} + \cdots + \alpha_n^N,$$
$$D(s) \triangleq s^{n-m} + \alpha_1^D s^{n-m-1} + \cdots + \alpha_n^D,$$

with $n_D \triangleq \deg(D(s))$ and $n_N \triangleq \deg(N(s))$ such that $n < n_N$;
- The relative degree of $M(s)$ is equal to $n_r$, i.e. $n_D - n_N = n_r$;
• \( k_g \in \mathbb{R}^+ \) is the high-frequency gain of the system, given by
\[
k_g \triangleq \frac{\alpha_{n-m}^D}{\alpha_{n-m}^N},
\]
which ensures that \( y(t) \) tracks constant \( r(t) \) with zero steady state error.

### 6.2. System Parametrization

The next lemma shows that the system (6.1) can be represented in the form more suitable for the design and analysis of the \( \mathcal{L}_1 \) adaptive controller.

**Lemma 6.1 (System parametrization).** The system in (6.1) can be rewritten as follows:
\[
y(s) = M(s) \left( k_0 u(s) + h^\top \phi_u(s) + k^\top \phi_y(s) + w(s) \right),
\]
where

- \( h, k \in \mathbb{R}^n \) are unknown system parameters;
- \( \phi_u(s) \triangleq P(s)u(s), \phi_y(s) \triangleq P(s)y(s) \) are computable signals with
\[
P(s) \triangleq \frac{\lambda(s)}{p(s)}, \quad p(s) = N(s)p^*(s),
\]
where \( \lambda(s) \triangleq [1 \ s \ s^2 \cdots s^{n-1}]^\top \), and \( p^*(s) \) is an arbitrary monic Hurwitz polynomial of degree \( n_{p^*} \triangleq n - n_N \);
- \( w(s) \triangleq (k_0 + h^\top P(s))\sigma(s) \) represents the system nonlinearity and disturbance.

**Proof.** Consider the following control law
\[
k_0 u(s) = -h^\top \phi_u(s) - k^\top \phi_y(s) - w(s) + k_g r(s),
\]
where the values of \( h \) and \( k \) will be defined later in the proof. Notice that for any possible values of \( k \) and \( h \), the transfer functions \( h^\top P(s) \) and \( k^\top P(s) \) are stable and proper, and the signal \( w(t) \) is bounded, as it is the output of a stable and proper system with bounded input. Thus, the control signal is well defined for all the values of \( h \) and \( k \), and can be further rewritten as
\[
k_0 u(s) = -h^\top P(s)u(s) - k^\top P(s)y(s) - w(s) + k_g r(s).
\]
Isolating $u(s)$, we obtain
\[
\left( k_0 + \frac{h^\top \lambda(s)}{p(s)} \right) u(s) = -\frac{k^\top \lambda(s)}{p(s)} y(s) - w(s) + k_g r(s),
\]
which leads to
\[
u(s) = -\frac{k^\top \lambda(s) y(s) + p(s) w(s) - p(s) k_g r(s)}{k_0 p(s) + h^\top \lambda(s)}.
\]
Substituting this into (6.1) leads to
\[
y(s) = k_0 \frac{B(s)}{A(s)} \left( -\frac{k^\top \lambda(s) y(s) + p(s) w(s) - p(s) k_g r(s)}{k_0 p(s) + h^\top \lambda(s)} + \sigma(s) \right).
\]
Isolating $y(s)$, we obtain
\[
\left( 1 + k_0 \frac{B(s)}{A(s)} \frac{k^\top \lambda(s)}{k_0 p(s) + h^\top \lambda(s)} \right) y(s) = k_0 \frac{B(s)}{A(s)} \left( \frac{-p(s) w(s) - p(s) k_g r(s)}{k_0 p(s) + h^\top \lambda(s)} + \sigma(s) \right),
\]
which leads to
\[
y(s) = \frac{k_0 B(s) p(s)(k_g r(s) - w(s)) + k_0 B(s)(k_0 p(s) + h^\top \lambda(s)) \sigma(s)}{A(s)(k_0 p(s) + h^\top \lambda(s)) + k_0 B(s) k^\top \lambda(s)}.
\]
Let
\[
q(s) \triangleq p(s) + \frac{h^\top}{k_0} \lambda(s) - s^n.
\]
Notice that $\deg(p(s)) = n$, $\deg(h^\top \lambda(s)) = n - 1$ and therefore $\deg(h^\top \lambda(s)) = \deg(p(s) - s^n)$. This implies that by proper choice of the vector $h$ arbitrary polynomial $q(s)$ with the degree $\deg(q(s)) \leq n - 1$ can be obtained. Using this notation, we rewrite (6.9) as follows
\[
y(s) = \frac{B(s) p(s)(k_g r(s) - w(s)) + B(s)(k_0 p(s) + h^\top \lambda(s)) \sigma(s)}{A(s) s^n + A(s) q(s) + B(s) k^\top \lambda(s)}.
\]
From the fact that $A(s)$ and $B(s)$ are relatively prime, Bezout identity implies that there exist unique polynomials $q(s)$ and $k^\top \lambda(s)$ such that for arbitrary polynomial $q^*(s)$ of degree $n_{q^*} \triangleq n + n_A - 1$ the following equality holds
\[
q^*(s) = A(s) q(s) + B(s) k^\top \lambda(s).
\]
This leads to
\[
y(s) = \frac{B(s) p(s)(k_g r(s) - w(s)) + k_0 B(s) q(s) \sigma(s)}{A(s) s^n + q^*(s)}.
\]
By proper choice of the vector parameters $h$ and $k$, we obtain

$$q^*(s) = B(s)p^*(s)D(s) - A(s)s^n.$$  

Notice that $\deg(B(s)p^*(s)D(s)) = n + n_A$. However, because all the polynomials on the right hand side are monic, $\deg(q^*(s)) = n + n_A - 1$, which leads to cancellation of the highest power upon subtraction. This further leads to

$$y(s) = \frac{B(s)p(s)(k_g r(s) - w(s)) + B(s)(k_0 p(s) + h^\top \lambda(s))\sigma(s)}{B(s)p^*(s)D(s)},$$

and

$$y(s) = \frac{N(s)}{D(s)}(k_g r(s) - w(s)) + \frac{k_0 p(s) + h^\top \lambda(s)}{p^*(s)D(s)} \sigma(s)$$

$$= M(s)(k_g r(s) - w(s)) + \frac{N(s)}{D(s)} \left(k_0 + h^\top \frac{\lambda(s)}{p(s)}\right) \sigma(s)$$

$$= M(s) k_g r(s).$$

From (6.8) it follows that

$$k_g r(s) = k_0 u(s) + h^\top \phi_u(s) + k^\top \phi_y(s) + w(s).$$

Substituting this into the equation for $y(t)$, we finally obtain

$$y(s) = M(s) \left(k_0 u(s) + h^\top \phi_u(s) + k^\top \phi_y(s) + w(s)\right).$$

The system representation, given by (6.6), can be further rewritten in more convenient form:

$$y(s) = M(s) \left(k_m u(s) + (k_0 - k_m) u(s) + h^\top \phi_u(s) + k^\top \phi_y(s) + w(s)\right)$$

$$= M(s) \left(k_m u(s) + \theta^\top \phi(s) + w(s)\right),$$

(6.10)

where

$$\theta \triangleq \begin{bmatrix} k_0 - k_m & h^\top & k^\top \end{bmatrix}^\top \in \mathbb{R}^{2n+1},$$

$$\phi(s) \triangleq \begin{bmatrix} u(s) & \phi_u^\top(s) & \phi_y^\top(s) \end{bmatrix}^\top \in \mathbb{R}^{2n+1}.$$  

Using the conservative information about the location of the coefficients of $A(s)$ and $B(s)$ one can compute some conservative bounds for the vectors $h$ and $k$ by solving the Bezout identity.
Using this information one can further obtain the conservative set where the parameter \( \theta \) is located. We denote this set as \( \Theta \).

6.3. \( L_1 \) Adaptive Control Architecture

The \( L_1 \) adaptive output feedback controller is based on system inversion. A lowpass filter at the system output is used to make the system transfer function proper and the inversion well defined. The inverted signal is then compared to the signal produced by the input predictor to generate the prediction error. The standard gradient descent adaptation law uses the prediction error augmented by an auxiliary error. The control law generates the control signal via the output of a low-pass filter.

6.3.1. Definitions and \( L_1 \)-norm Stability Condition

Consider the following 3 filters

\[
C_G(s) \triangleq \frac{\omega_G}{s + \omega_G}, \quad C_E(s) \triangleq \frac{(\omega_E)^l}{(s + \omega_E)^l}, \quad C_0(s),
\]

where \( C_0(s) \) is a stable strictly proper transfer function with unit DC gain \( C_0(0) = 1 \); \( l \geq n_r \) is the order of the lowpass filter; and \( \omega_E \in \mathbb{R}^+ \), \( \omega_G \in \mathbb{R}^+ \) are the parameters of the filters. Next, define

\[
C_H(s) \triangleq C_G(s)C_E(s), \quad C_F(s) \triangleq C_H(s)C_0(s).
\]

Let the choice of the filters in (6.12) satisfy the following \( L_1 \) norm condition:

\[
\|H_{yy}(s)\|_{L_1} + \|H_{yw}(s)\|_{L_1} < 1,
\]

where the constant \( L \) is defined in (6.2), and

\[
H_{yy}(s) \triangleq \frac{k_m(1 - C_F(s))}{k_m + C_F(s)(k_0 - k_m)} \left(1 - \frac{A(s)N(s)}{B(s)D(s)}\right),
\]

\[
H_{yw}(s) \triangleq \frac{k_0k_m(1 - C_F(s))}{k_m + C_F(s)(k_0 - k_m)} \frac{N(s)}{D(s)}.
\]

Next, define

\[
\rho_{yret} \triangleq \frac{\|H_{yw}(s)\|_{L_1}L_0 + \|H_{yr}(s)\|_{L_1}}{1 - \|H_{yy}(s)\|_{L_1} + \|H_{yw}(s)\|_{L_1}L}.
\]
where $L_0$ is defined in (6.3), $\bar{r}$ is given by (6.5), and
\[
H_{yr}(s) \triangleq \frac{N(s)}{D(s)} \frac{k_0k_g}{k_m + C_F(s)(k_0 - k_m)}.
\]
(6.18)

Further, let $\gamma_y' \in \mathbb{R}$ be an arbitrary (small) constant, and let
\[
\rho_y \triangleq \rho_{y_{\text{ref}}} + \gamma_y'.
\]

Finally, let
\[
\bar{w}_0 \triangleq L\rho_y + L_0.
\]
(6.19)

and
\[
\bar{w} \triangleq \left\| k_0 + h^\top P(s) \right\|_{\mathcal{L}_1} \bar{w}_0.
\]
(6.20)

6.3.2. $\mathcal{L}_1$ Adaptive Controller

Next we present the elements of the $\mathcal{L}_1$ adaptive controller.

**Filtered Inversion of the System.** Consider the system given by (6.10). Define the filtered inverse of the system as
\[
\nu(s) = \frac{C_H(s)}{M(s)} y(s).
\]
(6.21)

Notice that this inverse corresponds to the filtered total system input corrupted with the uncertainties and the disturbance:
\[
\nu(s) = C_H(s) \left( k_m u(s) + \theta^\top \phi(s) + w(s) \right).
\]
(6.22)

**Input Predictor.** Consider the following input predictor, which mimics the structure of the filtered system input in (6.22):
\[
\hat{\nu}(s) = C_H(s) (k_m u(s) + \hat{\mu}_\phi(s)) + C_G(s) \hat{w}(s),
\]
(6.23)

where $\hat{\nu}(t) \in \mathbb{R}$ is the estimate of the total system input, $\hat{\mu}_\phi(s)$ is the Laplace transform of
\[
\hat{\mu}_\phi(t) \triangleq \hat{\theta}^\top(t) \phi(t),
\]
and $\hat{\theta}(t) \in \mathbb{R}^{2n+1}$, $\hat{w}(t) \in \mathbb{R}$ are the adaptive estimates of the unknown parameters. Notice, that the adaptive estimate $\hat{w}(t)$ gets filtered by $C_G(s)$. 

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**Augmented Error.** Let $\tilde{\nu}(t) \triangleq \nu(t) - \hat{\nu}(t)$ be the prediction error. We consider the following auxiliary error:

\[ \eta(s) \triangleq C_H(s)\hat{\mu}_\phi(s) - C_G(s)\hat{\mu}_X(s), \tag{6.24} \]

where $\hat{\mu}_X(s)$ is the Laplace transform of

\[ \hat{\mu}_X(t) \triangleq \hat{\theta}^\top(t)X(t), \]

and $X(t)$ is the filtered version of $\phi(t)$ given by

\[ X(s) \triangleq C_E(s)\phi(s). \]

Using this auxiliary error, we define the augmented error, which will be further used in the adaptation laws:

\[ \varepsilon(t) = \tilde{\nu}(t) + \eta(t). \tag{6.25} \]

**Adaptation Laws.** The adaptive estimates are governed by the following adaptation laws:

\[ \dot{\hat{\theta}}(t) = \Gamma \text{Proj} \left( \hat{\theta}(t), \omega_G \varepsilon(t) X(t) \right), \quad \hat{\theta}(0) = \hat{\theta}_0, \tag{6.26} \]

\[ \dot{\hat{w}}(t) = \Gamma \text{Proj} \left( \hat{w}(t), \omega_G \varepsilon(t) \right), \quad \hat{w}(0) = \hat{w}_0, \]

where $\Gamma \in \mathbb{R}^+$ is the adaptation gain, and the projection bounds are set to ensure $\hat{\theta}(t) \in \Theta$, and $\hat{w}(t) \in \Delta \triangleq [-\bar{w}, \bar{w}]$ for all $t \geq 0$, where $\bar{w}$ was defined in (6.20).

**Control Law.** The $L_1$ adaptive control law is given by

\[ u(s) = \frac{1}{k_m} \left( k_g \hat{r}(s) - C_F(s) \left( \hat{\mu}_\phi(s) + \hat{w}(s) \right) \right), \tag{6.27} \]

where the filter $C_F(s)$ is defined in (6.13).

Following the development above, the $L_1$ controller consists of the system given by (6.1), the inversion law in (6.21), the input predictor in (6.23), the adaptation laws defined in (6.26) along with (6.24)-(6.25), and the control law given by (6.27). The block diagram of the closed-loop system is given in Figure 6.1. In this block diagram the block with $\Phi : (u(t), y(t)) \rightarrow \phi(t)$ is defined according to (6.11).

**Remark 6.1.** For the purpose of implementation, the output inversion law, the input pre-
dictor and the auxiliary error are combined and implemented according to the following relationship:

\[ \varepsilon(s) = \tilde{\nu}(s) + \eta(s) = \nu(s) - \hat{\nu}(s) + \eta(s) \]

\[ = \frac{C_H(s)}{M(s)} y(s) - C_H(s) (k_m u(s) + \hat{\mu}_\phi(s)) - C_G(s) \hat{w}(s) + \eta(s) \]

\[ = \frac{C_H(s)}{M(s)} y(s) - C_H(s) k_m u(s) - C_G(s) \hat{\mu}_X(s) - C_G(s) \hat{w}(s). \]

### 6.4. Analysis of the \( \mathcal{L}_1 \) Adaptive Controller

#### 6.4.1. Closed-loop Adaptive System Representation

We start the analysis of the closed-loop adaptive system by rewriting it in a more convenient form. Namely, the next lemma shows that the closed-loop system can be split into two parts. The first part does not contain adaptive estimates, and therefore we further
refer to it as nonadaptive. The second part involves the estimation errors, and we refer to it as adaptive part.

**Lemma 6.2 (Closed-loop adaptive system representation).** The closed-loop adaptive system shown in Figure 6.1 can be equivalently represented as

\[ y(s) = H_{yy}(s)y(s) + H_{yw}(s)\sigma(s) + H_{yr}(s)r(s) + H_{ye}(s)e(s), \quad (6.29) \]

\[ u(s) = H_{uy}(s)y(s) + H_{uw}(s)\sigma(s) + H_{ur}(s)r(s) + H_{ue}(s)e(s), \quad (6.30) \]

where

\[ H_{ye}(s) \triangleq \frac{N(s)}{D(s)} \frac{k_0 C_0(s)}{k_m + C_F(s)(k_0 - k_m)}, \quad (6.31) \]

\[ H_{uy}(s) \triangleq \frac{C_F(s) D(s)}{k_m + C_F(s)(k_0 - k_m)} \frac{D(s)}{N(s)} \left( \frac{A(s)N(s)}{B(s)D(s)} - 1 \right), \quad (6.32) \]

\[ H_{uw}(s) \triangleq -\frac{k_0 C_F(s)}{k_m + C_F(s)(k_0 - k_m)}, \quad (6.33) \]

\[ H_{ur}(s) \triangleq \frac{k_g}{k_m + C_F(s)(k_0 - k_m)}, \quad (6.34) \]

\[ H_{ue}(s) \triangleq \frac{C_0(s)}{k_m + C_F(s)(k_0 - k_m)}, \quad (6.35) \]

the transfer functions \( H_{yy}(s), H_{yw}(s), H_{yr}(s) \) are defined in (6.15), (6.16), (6.18) respectively, and

\[ e(s) \triangleq C_H(s) (\tilde{\mu}_\phi(s) + \tilde{w}(s)), \quad (6.36) \]

\[ \tilde{\mu}_\phi(t) \triangleq \tilde{\theta}^\top(t) \phi(t), \]

\[ \tilde{\theta}(t) \triangleq \theta - \hat{\theta}(t), \]

\[ \tilde{w}(t) \triangleq w(t) - \hat{w}(t). \]

**Proof.** From the system in (6.1) it follows that

\[ k_0 u(s) = \frac{A(s)}{B(s)} y(s) - k_0 \sigma(s). \]

On the other hand, (6.10) can be written as

\[ y(s) = \frac{N(s)}{D(s)} \left( k_0 u(s) - (k_0 - k_m) u(s) + \theta^\top \phi(s) + w(s) \right). \]
Substituting the first equation in the second, gives us
\[ y(s) = \frac{N(s)}{D(s)} \left( \frac{A(s)}{B(s)} y(s) - k_0 \sigma(s) - (k_0 - k_m) u(s) + \theta^\top \phi(s) + w(s) \right). \]

Isolating \( y(s) \), we obtain
\[ \frac{D(s)}{N(s)} \left( 1 - \frac{N(s) A(s)}{D(s) B(s)} \right) \theta^\top \phi(s) + w(s) - k_0 \sigma(s) - (k_0 - k_m) u(s), \]

which leads to
\[ \theta^\top \phi(s) + w(s) = \frac{D(s)}{N(s)} \left( 1 - \frac{N(s) A(s)}{D(s) B(s)} \right) y(s) + k_0 \sigma(s) + (k_0 - k_m) u(s). \tag{6.37} \]

Consider the control law given in (6.27). Using (6.37), it can be rewritten as follows:
\[
\begin{align*}
    k_m u(s) &= k_g r(t) - C_F(s) \left( \theta^\top \phi(s) + w(s) \right) + C_F(s) (\bar{\mu} \phi(s) + \bar{w}(s)) \\
    &= k_g r(s) - C_F(s) (\theta^\top \phi(s) + w(s)) + C_0(s) e(s) \\
    &= k_g r(s) + C_0(s) e(s) - C_F(s) (k_0 - k_m) u(s) \\
    &\quad - C_F(s) \left( \frac{D(s)}{N(s)} \left( 1 - \frac{N(s) A(s)}{D(s) B(s)} \right) y(s) + k_0 \sigma(s) \right).
\end{align*}
\]

Isolating \( u(s) \), we obtain
\[
\begin{align*}
    k_m u(s) &= \left( 1 + C_F(s) \frac{k_0 - k_m}{k_m} \right)^{-1} \left( k_g r(s) + C_0(s) e(s) \\
    &\quad - C_F(s) \left( \frac{D(s)}{N(s)} \left( 1 - \frac{N(s) A(s)}{D(s) B(s)} \right) y(s) + k_0 \sigma(s) \right) \right)
\end{align*}
\]
\[
\begin{align*}
    &= \frac{k_m}{k_m + C_F(s)(k_0 - k_m)} (k_g r(s) + C_0(s) e(s)) - \frac{k_m k_0 C_F(s)}{k_m + C_F(s)(k_0 - k_m)} \sigma(s) \\
    &\quad - \frac{k_m C_F(s)}{k_m + C_F(s)(k_0 - k_m)} \frac{D(s)}{N(s)} \left( 1 - \frac{A(s) N(s)}{B(s) D(s)} \right) y(s) \\
    &= k_m H_{wy}(s) y(s) + k_m H_{uw}(s) \sigma(s) + k_m H_{uer}(s) r(s) + k_m H_{uer}(s) e(s),
\end{align*}
\]

which proves the equality in (6.30).
To prove the equality for \( y(s) \) we substitute the control law given by (6.27) into (6.10):

\[
y(s) = \frac{N(s)}{D(s)} (k_g r(s) - C_F(s) (\bar{m}_\phi(s) + \bar{w}(s)) + \theta^T \phi(s) + w(s))
\]

\[
= \frac{N(s)}{D(s)} k_g r(s) + \frac{N(s)}{D(s)} C_0(s) e(s) + \frac{N(s)}{D(s)} (1 - C_F(s)) \left( \theta^T \phi(s) + w(s) \right).
\]

Further, substituting (6.37) into this equation, leads to

\[
y(s) = \frac{N(s)}{D(s)} k_g r(s) + \frac{N(s)}{D(s)} C_0(s) e(s) + (1 - C_F(s)) \left( 1 - \frac{N(s) A(s)}{D(s) B(s)} \right) y(s)
\]

\[
+ \frac{N(s)}{D(s)} (1 - C_F(s)) k_0 \sigma(s) + \frac{N(s)}{D(s)} (1 - C_F(s)) (k_0 - k_m) u(s).
\]

Finally, substituting (6.38), we obtain

\[
y(s) = \frac{N(s)}{D(s)} k_g r(s) + \frac{N(s)}{D(s)} C_0(s) e(s) + (1 - C_F(s)) \left( 1 - \frac{N(s) A(s)}{D(s) B(s)} \right) y(s)
\]

\[
+ \frac{N(s)}{D(s)} (1 - C_F(s)) k_0 \sigma(s) + \frac{N(s)}{D(s)} (1 - C_F(s)) (k_0 - k_m) u(s)
\]

\[
- \frac{N(s)}{D(s)} (1 - C_F(s)) k_0 \sigma(s) + \frac{N(s)}{D(s)} \left( 1 - \frac{A(s) N(s)}{B(s) D(s)} \right) y(s)
\]

\[
- \frac{N(s)}{D(s)} \left( \frac{k_0 - k_m}{k_m + C_F(s)} \right) \left( \frac{k_0 - k_m}{k_m + C_F(s)} \right) \left( 1 - \frac{A(s) N(s)}{B(s) D(s)} \right) y(s)
\]

\[
= \frac{N(s)}{D(s)} k_g k_0 r(s) + \frac{N(s)}{D(s)} k_0 C_0(s)
\]

\[
+ \frac{k_m (1 - C_F(s))}{k_m + C_F(s) (k_0 - k_m)} \left( 1 - \frac{A(s) N(s)}{B(s) D(s)} \right) y(s)
\]

\[
+ \frac{k_0 k_m}{k_m + C_F(s) (k_0 - k_m)} \frac{N(s)}{D(s)} (1 - C_F(s)) \sigma(s)
\]

\[
= H_{y r}(s) y(s) + H_{y w}(s) \sigma(s) + H_{y r}(s) r(s) + H_{y e}(s) e(s),
\]

which completes the proof. \( \Box \)

**Remark 6.2.** Notice that all the transfer functions \( H_{**}(s) \) are proper. Therefore the closed-loop adaptive system representation in (6.29), (6.30) is well defined.

Let \( \mathcal{E} \) be the map, which generates the error \( e(t) \) given by (6.36):

\[
\mathcal{E} : (t, r(t), u(t), y(t), \phi(t)) \rightarrow e(t).
\]
Notice that $\mathcal{E}$ denotes the adaptive part of the system, which generates the error $e(t)$. Figure 6.2 shows the block diagram of the closed-loop adaptive system given by (6.29)-(6.30).

### 6.4.2. $\mathcal{L}_1$ Reference System

Consider the following $\mathcal{L}_1$ reference system:

$$
\begin{align*}
y_{\text{ref}}(s) & = H_{yy}(s)y_{\text{ref}}(s) + H_{yw}(s)\sigma_{\text{ref}}(s) + H_{yr}(s)r(s), \\
u_{\text{ref}}(s) & = H_{uy}(s)y_{\text{ref}}(s) + H_{uw}(s)\sigma_{\text{ref}}(s) + H_{uer}(s)r(s),
\end{align*}
$$

where $\sigma_{\text{ref}}(s)$ is the Laplace transform of $\sigma_{\text{ref}}(t) \triangleq f(t, y_{\text{ref}}(t))$. The block diagram is given in Figure 6.3. Notice that this system is equal to the closed-loop adaptive system given by (6.29), (6.30) without the error $(e(t) \equiv 0)$.

**Remark 6.3.** Recall that the transfer functions $H_{e*}(s)$ depend upon unknown system parameters, which render this reference system non-implementable. This system is used only for the analysis purposes.

The $\mathcal{L}_1$ reference system has feedback structure involving the design parameters and the system uncertainties. The next lemma derives sufficient conditions for stability of the $\mathcal{L}_1$ reference system.

**Lemma 6.3 (Stability of the reference system).** If the $\mathcal{L}_1$ norm condition in (6.14) is satisfied, then the $\mathcal{L}_1$ reference system in (6.40) is BIBO stable, and the following uniform bounds
Figure 6.3: Block diagram of the $\mathcal{L}_1$ reference system.

hold:

\[
\|y_{ref}\|_{\mathcal{L}_\infty} \leq \rho_{y_{ref}}, \\
\|u_{ref}\|_{\mathcal{L}_\infty} \leq \rho_{u_{ref}},
\]

where $\rho_{y_{ref}}$ is defined in (6.17), and

\[
\rho_{u_{ref}} \triangleq \|H_{uy}(s)\|_{\mathcal{L}_1} \rho_{y_{ref}} + \|H_{uw}(s)\|_{\mathcal{L}_1} (L\rho_{y_{ref}} + L_0) + \|H_{uer}(s)\|_{\mathcal{L}_1} \bar{\rho}.
\]

**Proof.** The equation in (6.40) leads to the following upper bound, valid for arbitrary $\tau \geq 0$

\[
\|y_{ref,\tau}\|_{\mathcal{L}_\infty} \leq \|H_{yy}(s)\|_{\mathcal{L}_1} \|y_{ref,\tau}\|_{\mathcal{L}_\infty} + \|H_{yw}(s)\|_{\mathcal{L}_1} \|\sigma_{\tau}\|_{\mathcal{L}_\infty} + \|H_{yr}(s)\|_{\mathcal{L}_1} \bar{\rho}.
\]

Notice that from (6.2) and (6.3) it follows that

\[
|f(t, y_{ref}(t))| \leq L|y_{ref}(t)| + L_0, \quad \forall \ t \geq 0,
\]

which further leads to

\[
\|y_{ref,\tau}\|_{\mathcal{L}_\infty} \leq \|H_{yy}(s)\|_{\mathcal{L}_1} \|y_{ref,\tau}\|_{\mathcal{L}_\infty} + \|H_{yw}(s)\|_{\mathcal{L}_1} (L\|y_{ref,\tau}\|_{\mathcal{L}_\infty} + L_0) + \|H_{yr}(s)\|_{\mathcal{L}_1} \bar{\rho}.
\]

This upper bound can be rewritten as

\[
\|y_{ref,\tau}\|_{\mathcal{L}_\infty} \leq \frac{\|H_{yw}(s)\|_{\mathcal{L}_1} L_0 + \|H_{yr}(s)\|_{\mathcal{L}_1} \bar{\rho}}{1 - \|H_{yy}(s)\|_{\mathcal{L}_1} - \|H_{yw}(s)\|_{\mathcal{L}_1} L}.
\]
The $\mathcal{L}_1$-norm condition in (6.14) ensures that the RHS of this bound is positive. The fact that the RHS is independent of $\tau$ yields the uniform bound in (6.41).

The uniform boundedness of $y_{\text{ref}}(t)$, and hence $\sigma_{\text{ref}}(t)$, lead to the following upper bound for the control signal of the reference system, given by (6.40):

$$
\| u_{\text{ref}} \|_{\mathcal{L}_\infty} \leq \| H_{uy}(s) \|_{\mathcal{L}_1} \| y_{\text{ref}} \|_{\mathcal{L}_\infty} + \| H_{uw}(s) \|_{\mathcal{L}_1} \| \sigma_{\text{ref}} \|_{\mathcal{L}_\infty} + \| H_{uer}(s) \|_{\mathcal{L}_1} \bar{r} \\
= \rho u_{\text{ref}}.
$$

\[\square\]

**Remark 6.4 (Reference system vs. Ideal system).** To see the connection between the reference system in (6.40) and the ideal system defined in (6.4), we consider the limiting case of filters with $\omega_G, \omega_E \to \infty$ and $C_0(s) \to 1$. This leads to $C_F(s) \to 1$. Further, the transfer functions in (6.31)-(6.35), (6.15), (6.16), and (6.18) reduce to the following:

$$
H_{yy}(s) = 0, \quad H_{uy}(s) = \frac{1}{k_0} \frac{D(s)}{N(s)} \left( \frac{A(s)N(s)}{B(s)D(s)} - 1 \right),
$$

$$
H_{yw}(s) = 0, \quad H_{uw}(s) = -1,
$$

$$
H_{yr}(s) = k_g \frac{N(s)}{D(s)}, \quad H_{ur}(s) = \frac{k_g}{k_0}.
$$

We see that the output of the reference system is identical to the output of the ideal system, when $C_F(s) \to 1$. However, the control input of the reference system is not implementable, as $H_{uy}(s)$ is now improper.

Next, we recall that in the adaptive control law, given by (6.27), the parameter estimations are passed through the low-pass filter $C_F(s)$, which limits the frequency range of the generated control signal and, as a result, the possibility of the adaptive controller to compensate for high-frequency content of the uncertainties. This is consistent with the philosophy of the $\mathcal{L}_1$ adaptive control theory in a sense that the $\mathcal{L}_1$ reference system assumes only partial compensation of the uncertainties within the bandwidth of the control channel, given by the lowpass filter $C_F(s)$. The response of the reference system can be made arbitrarily close to the one of the ideal system by increasing the bandwidth of the low-pass filter.

### 6.4.3. Error Dynamics

Consider the augmented error given by (6.28). Notice that the transfer function $C_H(s)/M(s)$ is strictly proper and stable by definition. Multiplying the system in (6.10)
from both sides by $C_H(s)/M(s)$, we obtain

$$\frac{C_H(s)}{M(s)} y(s) = C_H(s) \left( k_m u(s) + \theta^T \phi(s) + w(s) \right).$$

Substituting this back into (6.28), we obtain the error dynamics

$$\varepsilon(s) = C_H(s) \theta^T \phi(s) - C_G(s) \hat{\mu}_X(s) + C_H(s) w(s) - C_G(s) \hat{w}(s)$$

$$= C_G(s) \left( \theta^T X(s) - \hat{\mu}_X(s) \right) + C_G(s) (C_E(s) w(s) - \hat{w}(s))$$

$$(6.42)$$

where $\hat{\mu}_X(s)$ is the Laplace transform of $\hat{\mu}_X(t) \triangleq \hat{\theta}(t) X(t)$, $\hat{w}_E(t) \triangleq w_E(t) - \hat{w}(t)$, and $w_E(s) \triangleq C_E(s) w(s)$.

Using the definition of $C_G(s)$ we can write the error dynamics in the state space form as follows:

$$\dot{\varepsilon}(t) = -\omega_G \varepsilon(t) + \omega_G \left( \hat{\theta}(t) X(t) + \hat{w}_E(t) \right), \quad \varepsilon(0) = 0.$$  

$$(6.43)$$

**Lemma 6.4 (Boundedness of the error dynamics).** Consider the error dynamics given by (6.43). If for some $\tau \geq 0$

$$\|w_{\tau}\|_{L_\infty} \leq \bar{w},$$

then the augmented error is bounded:

$$\|\varepsilon_{\tau}\|_{L_\infty} \leq \frac{\bar{\varepsilon}}{\sqrt{\Gamma}},$$  

$$(6.44)$$

where

$$\bar{\varepsilon} \triangleq \sqrt{\frac{2\bar{w}\bar{w}_E d}{\omega_G}} + 4 \left( \max_{\theta \in \Theta} \|\theta\|^2 + \bar{w} \right),$$

and

$$\bar{w}_Ed \triangleq \|sC_E(s)\|_{L_1} \bar{w}.$$  

Moreover, the derivative of the parameter estimation error is also bounded as follows

$$\left\| \dot{\hat{\theta}}(t) \right\|_{\infty} \leq \sqrt{\Gamma} \omega_G \bar{\varepsilon} \|X(t)\|_{\infty}, \quad \forall \ t \in [0, \tau].$$  

$$(6.45)$$

**Proof.** Consider the following Lyapunov function candidate:

$$V(\varepsilon, \hat{\theta}, \hat{w}_E) = \frac{1}{2} \varepsilon^2 + \frac{1}{2\Gamma} \left( \hat{\theta}^T \hat{\theta} + \hat{w}_E^2 \right).$$  

$$(6.46)$$
Taking the time derivative along the trajectories we obtain

\[ \dot{V}(t) = \varepsilon(t)\dot{\varepsilon}(t) + \frac{1}{\Gamma} \left( \tilde{\theta}^\top(t)\dot{\theta}(t) + \tilde{w}_E(t)\dot{\tilde{w}}_E(t) \right) \]
\[ = -\omega_G\varepsilon^2(t) + \omega_G\varepsilon(t) \left( \tilde{\theta}^\top(t)X(t) + \tilde{w}_E(t) \right) \]
\[ + \frac{1}{\Gamma} \left( -\tilde{\theta}^\top(t)\dot{\theta}(t) + \tilde{w}_E(t)\dot{\tilde{w}}_E(t) - \tilde{w}_E(t)\dot{\tilde{w}}(t) \right) \]
\[ = -\omega_G\varepsilon^2(t) + \frac{1}{\Gamma}\tilde{w}_E(t)\dot{\tilde{w}}_E(t) \]
\[ + \tilde{\theta}^\top(t) \left( \omega_G\varepsilon(t)X(t) - \text{Proj} \left( \dot{\tilde{\theta}}(t), \omega_G\varepsilon(t)X(t) \right) \right) \]
\[ + \tilde{w}_E(t) \left( \omega_G\varepsilon(t) - \text{Proj} \left( \dot{\tilde{w}}(t), \omega_G\varepsilon(t) \right) \right). \]

Using the properties of the projection operator we conclude that

\[ \dot{V}(t) \leq -\omega_G\varepsilon^2(t) + \frac{1}{\Gamma}\tilde{w}_E(t)\dot{\tilde{w}}_E(t). \]

Thus, have \( \dot{V}(t) \leq 0 \), if

\[ |\varepsilon(t)| \geq \sqrt{\frac{1}{\omega_G\Gamma} |\tilde{w}_E(t)\dot{\tilde{w}}_E(t)|}. \]

From the definition of \( C_E(s) \) it follows that \( \|C_E(s)\|_{\mathcal{L}_1} = 1 \). From the boundedness of \( w(t) \) on the time interval \( t \in [0, \tau] \), we get that \( \|w_E(t)\|_{\mathcal{L}_\infty} \leq \bar{w} \), which, along with the projection bounds, implies that \( \|\tilde{w}_E(t)\|_{\mathcal{L}_\infty} \leq 2\bar{w} \). Also, notice that \( \|\dot{w}_E(t)\|_{\mathcal{L}_\infty} \leq \bar{w}_E \). Thus, \( \dot{V}(t) \leq 0 \) for \( \forall t \in [0, \tau] \), if

\[ |\varepsilon(t)| \geq \sqrt{\frac{2\bar{w}\bar{w}_E}{\omega_G\Gamma}}, \quad \forall t \in [0, \tau]. \]  

From (6.46) it follows that along the trajectories

\[ V(t) \leq \frac{1}{2}\varepsilon^2(t) + \frac{2}{\Gamma} \left( \max_{\theta \in \Theta} \|\theta\|^2 + \bar{w} \right), \quad \forall t \in [0, \tau]. \]

From (6.47) it follows that if

\[ V(t) \geq V_{\text{max}} = \frac{\bar{w}\bar{w}_E}{\omega_G\Gamma} + \frac{2}{\Gamma} \left( \max_{\theta \in \Theta} \|\theta\|^2 + \bar{w} \right), \quad \forall t \in [0, \tau], \]
then $\dot{V}(t) \leq 0$. From the fact that $\varepsilon(0) = 0$ it follows that
\[
V(0) \leq \frac{2}{\Gamma} \left( \max_{\theta \in \Theta} \|\theta\|^2 + \bar{w} \right) \leq V_{\text{max}}.
\]
Hence, for all $t \in [0, \tau]$ we have $V(t) \leq V_{\text{max}}$, and from the fact that $V(t) \geq 1/2\varepsilon^2(t)$, we obtain the following upper bound:
\[
|\varepsilon(t)| \leq \frac{1}{\sqrt{\Gamma}} \sqrt{\frac{2\bar{w}\bar{w}_E \delta_d}{\omega_G}} \left( \max_{\theta \in \Theta} \|\theta\|^2 + \bar{w} \right) = \frac{\bar{\varepsilon}}{\sqrt{\Gamma}}, \quad \forall \ t \in [0, \tau],
\]
which gives the upper bound in (6.44).

To prove the second upper bound we first notice that $\|\dot{\theta}(t)\|_{\infty} = \|\dot{\hat{\theta}}(t)\|_{\infty}$, and then from the adaptation laws it follows that
\[
\left\| \dot{\hat{\theta}}(t) \right\|_{\infty} \leq \Gamma \omega_G \|e_\tau\|_{L_\infty} \|X(t)\|_{\infty} \leq \sqrt{\Gamma} \omega_G \bar{\varepsilon} \|X(t)\|_{\infty}, \quad \forall \ t \in [0, \tau].
\]

6.4.4. Boundedness of the Adaptation Error

In this section we study the properties of the map $E$ defined in (6.39), and shown in Figure 6.3, which generates the error $e(t)$. The next lemma establishes the boundedness result for this error.

**Lemma 6.5 (Boundedness of the error $e(t)$).** For the closed-loop adaptive system shown in Figure 6.1, if for some $\tau \geq 0$ the signal $w(t)$ is bounded:
\[
\|w_\tau\|_{L_\infty} \leq \bar{w},
\]
then the error $e(t)$, defined in (6.36), is bounded as follows
\[
\|e_\tau\|_{L_\infty} \leq \bar{e}_0(\Gamma, \omega_E) + \bar{e}_\phi(\Gamma, \omega_E) \|\phi_\tau\|_{L_\infty}^2,
\]  
(6.48)

where
\[
\bar{e}_0(\Gamma, \omega_E) \triangleq \frac{\bar{\varepsilon}}{\sqrt{\Gamma}} + \|C_G(s)(1 - C_E(s))\|_{L_1} \bar{w},
\]  
(6.49)
\[
\bar{e}_\phi(\Gamma, \omega_E) \triangleq \frac{\sqrt{2n + 1} \sqrt{\Gamma} \omega_G}{\omega_E} \sum_{k=0}^{t-1} \frac{\bar{\varepsilon}}{k!}.
\]  
(6.50)
Proof. The error in (6.36) can be rewritten as

\[ e(s) = C_H(s) (\bar{\mu}_\phi(s) + \bar{w}(s)) \]
\[ = C_H(s) (\bar{\mu}_\phi(s) + \bar{w}(s)) - C_H(s)\mu(s) + C_G(s)\bar{\mu}_X(s) + \eta(s) \]
\[ = C_G(s) (\bar{\mu}_X(s) + \bar{w}_E(s)) + \eta(s) + C_G(s) (1 - C_E(s)) \bar{w}(s) \]
\[ = \varepsilon(s) + \eta(s) + \eta_w(s), \tag{6.51} \]

where \( \varepsilon(s) \) is the augmented error from (6.42), \( \eta(s) \) is the auxiliary error defined in (6.24), and

\[ \eta_w(s) \triangleq C_G(s) (1 - C_E(s)) \bar{w}(s). \]

Next we prove boundedness of each of the signals \( \varepsilon(t) \), \( \eta(t) \), and \( \eta_w(t) \) on the time interval \( t \in [0, \tau] \).

**Boundedness of \( \varepsilon(t) \).** Application of Lemma 6.4 immediately leads to

\[ \| \varepsilon \|_{L_\infty} \leq \frac{\bar{\varepsilon}}{\sqrt{1}}. \tag{6.52} \]

**Boundedness of \( \eta(t) \).** The auxiliary error can be rewritten as

\[ \eta(s) = C_G(s)\eta_E(s), \quad \eta_E(s) \triangleq C_E(s)\hat{\mu}_\phi(s) - \hat{\mu}_X(s). \tag{6.53} \]

Notice that

\[ \eta_E(s) = C_E(s)\hat{\mu}_\phi(s) - \hat{\mu}_X(s). \]

Let \( c_E(t) \) be the impulse response for \( C_E(s) \). Then

\[ \eta_E(t) = \int_0^t c_E(t - \xi)\hat{\theta}^\top(\xi)\phi(\xi)d\xi - \int_0^t c_E(t - \xi)\hat{\theta}^\top(t)\phi(\xi)d\xi. \tag{6.54} \]

Substituting the following

\[ \hat{\theta}(t) = \hat{\theta}(0) + \int_0^t \hat{\theta}(\lambda)d\lambda \]
into (6.54), we obtain

\[
\eta_E(t) = \int_0^t c_E(t - \xi) \left( \hat{\omega}^\top(0) + \int_0^\xi \hat{\omega}^\top(\lambda)d\lambda \right) \phi(\xi)d\xi \\
= \int_0^t c_E(t - \xi) \left( \hat{\omega}^\top(0) + \int_0^t \hat{\omega}^\top(\lambda)d\lambda \right) \phi(\xi)d\xi \\
= \int_0^t c_E(t - \xi) \int_0^\xi \hat{\omega}^\top(\lambda)d\lambda \phi(\xi)d\xi - \int_0^t c_E(t - \xi) \int_0^t \hat{\omega}^\top(\lambda)d\lambda \phi(\xi)d\xi \\
= \int_0^t c_E(t - \xi) \int_0^\xi \hat{\omega}^\top(\lambda)d\lambda \phi(\xi)d\xi - \int_0^t c_E(t - \xi) \int_0^t \hat{\omega}^\top(\lambda)d\lambda \phi(\xi)d\xi \\
= - \int_0^t \int_\xi^t c_E(t - \xi) \hat{\omega}^\top(\lambda)\phi(\lambda)d\lambda d\xi = - \int_0^t \hat{\omega}^\top(\lambda) \int_0^\lambda c_E(t - \xi)\phi(\xi)d\xi d\lambda \\
= - \int_0^t \hat{\omega}^\top(\lambda) \int_{\xi - \lambda}^t c_E(\xi)\phi(t - \xi)d\xi d\lambda.
\]

From the definition of \(C_E(s)\) it follows that \(\|C_E(s)\|_{\mathcal{L}_1} = 1\). Therefore on \(t \in [0, \tau]\) we have \(\|X(t)\|_{\infty} \leq \|\phi_r\|_{\mathcal{L}_\infty}\), and the upper bound in (6.45) can be rewritten as

\[
\left\| \hat{\omega}(t) \right\|_{\infty} \leq \sqrt{T}\omega G \Xi \|X(t)\|_{\infty} \leq \sqrt{T}\omega G \Xi \|\phi_r\|_{\mathcal{L}_\infty}, \quad \forall \ t \in [0, \tau].
\]

This upper bound yields

\[
|\eta_E(t)| \leq \sqrt{2n + 1} \left( \int_0^t \left\| \hat{\omega}^\top(\lambda) \right\|_{\infty} \int_{\xi - \lambda}^t |c_E(\xi)|d\xi d\lambda \right) \|\phi_r\|_{\mathcal{L}_\infty} \\
\leq \sqrt{2n + 1}\sqrt{T}\omega G \Xi \left( \int_0^t \int_{\xi - \lambda}^\infty |c_E(\xi)|d\xi d\lambda \right) \|\phi_r\|^2_{\mathcal{L}_\infty}, \quad \forall \ t \in [0, \tau]. \quad (6.55)
\]

Consider the integral in the parenthesis. Since \(C_E(s) = (\omega_E)^l/(s + \omega_E)^l\), then

\[
\int_{\xi - \lambda}^\infty |c_E(\xi)|d\xi = \int_{\xi - \lambda}^\infty c_E(\xi)d\xi = \int_{\xi - \lambda}^\infty \left( \omega_E \right)^l \xi^{l-1}e^{-\omega_E\xi}d\xi = \frac{1}{(l-1)!} \int_{\omega_E(\xi - \lambda)}^\infty \xi^{l-1}e^{-\xi}d\xi \\
= \sum_{k=0}^{l-1} \frac{(-1)^k}{k!} e^{-\omega_E(t-\lambda)}.
\]

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Using this equation, we obtain the following upperbound
\[
\int_0^t \int_{t-\lambda}^\infty |c_E(\xi)| d\xi d\lambda = \int_0^t \sum_{k=0}^{l-1} \frac{(\omega_E(t-\lambda))^k}{k!} e^{-\omega_E(t-\lambda)} d\lambda = \int_0^t \sum_{k=0}^{l-1} \frac{(\omega_E \lambda)^k}{k!} e^{-\omega_E \lambda} d\lambda
\]
\[
\leq \int_0^\infty \sum_{k=0}^{l-1} \frac{(\omega_E \lambda)^k}{k!} e^{-\omega_E \lambda} d\lambda = \frac{1}{\omega_E} \int_0^\infty \sum_{k=0}^{l-1} \frac{\lambda^k}{k!} e^{-\lambda} d\lambda
\]
\[
= \frac{1}{\omega_E} \sum_{k=0}^{l-1} \frac{1}{k!} \left[ -\sum_{j=0}^k \lambda^j e^{-\lambda} \right]_0^\infty = \frac{1}{\omega_E} \sum_{k=0}^{l-1} \frac{1}{k!} \left[ -e^{-\lambda} \right]_0^\infty = \frac{1}{\omega_E} \sum_{k=0}^{l-1} \frac{1}{k!}.
\]
Substituting this into (6.55), we obtain
\[
|\eta_E(t)| \leq \left( \sqrt{2n+1} \sqrt{\Gamma \omega_G} \sum_{k=0}^{l-1} \frac{\bar{\varepsilon}}{k!} \right) \|\phi_T\|^2_{L_\infty}, \quad \forall \ t \in [0, \tau].
\]
From (6.53) and the fact that \(\|C_G(s)\|_{L_1} = 1\), it follows that \(\|\eta_r\|_{L_\infty} \leq \|\eta_r\|_{L_\infty}\), which yields
\[
\|\eta_r\|_{L_\infty} \leq \left( \sqrt{2n+1} \sqrt{\Gamma \omega_G} \sum_{k=0}^{l-1} \frac{\bar{\varepsilon}}{k!} \right) \|\phi_T\|^2_{L_\infty}. \tag{6.56}
\]
Boundedness of \(\eta_w(t)\). From the definition of \(\eta_w(t)\) it immediately follows that
\[
\|\eta_w\|_{L_\infty} \leq \|C_G(s)(1 - C_E(s))\|_{L_1} \|\hat{w}_r\|_{L_\infty}
\]
\[
\leq \|C_G(s)(1 - C_E(s))\|_{L_1} \bar{w}. \tag{6.57}
\]
Combining the bounds in (6.52), (6.56) and (6.57) according to (6.51), we obtain the following upper bound on \(e(t)\):
\[
\|e_r\|_{L_\infty} \leq \frac{\bar{\varepsilon}}{\sqrt{\Gamma}} + \left( \sqrt{2n+1} \sqrt{\Gamma \omega_G} \sum_{k=0}^{l-1} \frac{\bar{\varepsilon}}{k!} \right) \|\phi_T\|^2_{L_\infty} + \|C_G(s)(1 - C_E(s))\|_{L_1} \bar{w}
\]
\[
= \bar{e}_0(\Gamma, \omega_E) + \bar{e}_\phi(\Gamma, \omega_E) \|\phi_T\|^2_{L_\infty}. \tag{□}
\]

Remark 6.5. For the bound in (6.48), notice that for any fixed value of \(\omega_G\), if we set \(\omega_E = \Gamma\),
then
\[
\lim_{\Gamma \to \infty} \bar{e}_0(\Gamma, \omega_E = \Gamma) = \lim_{\Gamma \to \infty} \left( \frac{\bar{\varepsilon}}{\sqrt{\Gamma}} + \|C_G(s)(1 - C_E(s))\|_{L_1} \bar{w} \right) = 0,
\]
\[
\lim_{\Gamma \to \infty} \bar{e}_\phi(\Gamma, \omega_E = \Gamma) = \lim_{\Gamma \to \infty} \left( \frac{\sqrt{2n + 1}\sqrt{\Gamma} \omega_G}{\omega_E} \sum_{k=0}^{l-1} \frac{\bar{\varepsilon}}{k!} \right) = 0.
\]

Thus, for arbitrary bounded signal \( \phi(t) \) over an interval \( t \in [0, \tau] \), the output of the system \( \mathcal{E} \) can be made arbitrarily small by setting the values of the controller parameters as above and increasing the adaptation gain.

6.4.5. Stability and Performance Bounds of the Closed-loop Adaptive System

Consider the closed-loop system in (6.29), (6.30). Figure 6.2 shows that it consists of two parts: nonadaptive part, identical to the \( L_1 \) reference system, and the adaptive part \( \mathcal{E} \). The simplified block diagram of this system is shown in Figure 6.4. The next theorem proves stability of the closed-loop system and establishes the uniform performance bounds between the closed-loop adaptive system and the reference system.

**Theorem 6.1.** Let the filter \( C_F(s) \) satisfy the \( L_1 \) norm stability condition in (6.14). For any fixed \( \omega_G > 0 \) and arbitrary (small) constant \( \varepsilon \in \mathbb{R}^+ \), if we set \( \omega_E \geq \Gamma \) and set the adaptive gain large enough to satisfy the following conditions

\[
\|C_G(s)(1 - C_E(s))\|_{L_1} < \frac{\varepsilon}{2\bar{\phi}_w \bar{w}},
\]

\[
\gamma_y < \gamma'_y,
\]

and

\[
\Gamma > \max \left\{ (2n + 1) \left( \frac{\varepsilon \omega_G (\bar{\phi}_0 + 2\varepsilon \bar{\phi}_0 \sum_{k=0}^{l-1} \frac{1}{k!}}{\bar{w}} \right)^2 ; \left( \frac{2\bar{\phi}_w \bar{w}}{\varepsilon} \right)^2 \right\},
\]

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then the closed-loop system is stable, and the following uniform performance bounds hold

\[
\|y_{\text{ref}} - y\|_{\mathcal{L}_\infty} < \gamma_y, \quad (6.60)
\]
\[
\|u_{\text{ref}} - u\|_{\mathcal{L}_\infty} < \gamma_u, \quad (6.61)
\]

where

\[
\gamma_e \triangleq \bar{e}_0(\Gamma, \omega_E) + \bar{e}_\phi(\Gamma, \omega_E) (\bar{\phi}_0 + 2\epsilon)^2,
\]
\[
\gamma_y \triangleq \frac{\|H_{ye}(s)\|_{\mathcal{L}_1}}{1 - \|H_{yy}(s)\|_{\mathcal{L}_1} - \|H_{yw}(s)\|_{\mathcal{L}_1} L \gamma_e},
\]
\[
\gamma_u \triangleq (\|H_{uy}(s)\|_{\mathcal{L}_1} + \|H_{uw}(s)\|_{\mathcal{L}_1} L) \gamma_y + \|H_{ue}(s)\|_{\mathcal{L}_1} \gamma_e,
\]

and

\[
\bar{u}_0 \triangleq \left\| \frac{H_{uy}(s)H_{yw}(s)}{1 - H_{yy}(s)} + H_{uw}(s) \right\|_{\mathcal{L}_1} \bar{w}_0 + \left\| \frac{H_{uy}(s)H_{yr}(s)}{1 - H_{yy}(s)} + H_{ur}(s) \right\|_{\mathcal{L}_1} \bar{r},
\]
\[
\bar{u}_e \triangleq \left\| \frac{H_{uy}(s)H_{ye}(s)}{1 - H_{yy}(s)} + H_{ue}(s) \right\|_{\mathcal{L}_1},
\]
\[
\bar{y}_0 \triangleq \left\| \frac{P(s)H_{yw}(s)}{1 - H_{yy}(s)} \right\|_{\mathcal{L}_1} \bar{w}_0 + \left\| \frac{P(s)H_{yr}(s)}{1 - H_{yy}(s)} \right\|_{\mathcal{L}_1} \bar{r},
\]
\[
\bar{y}_e \triangleq \left\| \frac{P(s)H_{ye}(s)}{1 - H_{yy}(s)} \right\|_{\mathcal{L}_1}.
\]

**Proof.** We prove the bounds in (6.60), (6.61) using a contradiction argument. Assume that (6.60) and/or (6.61) do not hold. Then continuity of \( y(t), y_{\text{ref}}(t), u(t), \) and \( u_{\text{ref}}(t) \) along with the fact that \( y(0) = y_{\text{ref}}(0) = u(0) = u_{\text{ref}}(0) = 0 \) implies that there exists time \( \tau > 0 \), such that either or both of the following conditions hold

\[
|y_{\text{ref}}(t) - y(t)| < \gamma_y, \quad |y_{\text{ref}}(\tau) - y(\tau)| = \gamma_y, \quad (6.62)
\]
\[
|u_{\text{ref}}(t) - u(t)| < \gamma_u, \quad |u_{\text{ref}}(\tau) - u(\tau)| = \gamma_u. \quad (6.63)
\]

Consider the case when (6.62) holds. Using the upper bound in (6.41) from Lemma 6.3 we obtain

\[
\|y_r\|_{\mathcal{L}_\infty} \leq \rho_{y_{\text{ref}}} + \gamma_y < \rho_{y_{\text{ref}}} + \gamma'_y = \rho_y.
\]

Notice that from (6.2) and (6.3) it follows that

\[
|f(t, y(t))| \leq L |y(t)| + L_0,
\]

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which further leads to
\[ \| f_r \|_{\mathcal{L}_\infty} \leq L \rho_y + L_0 = \bar{w}_0, \]
where \( f \) stands for \( f(t, y(t)) \), and \( \bar{w}_0 \) was defined in (6.19). Using the definition of \( w(t) \) in Lemma 6.1, we obtain
\[ \| w_r \|_{\mathcal{L}_\infty} \leq \| k_0 + h^T P(s) \|_{\mathcal{L}_1}(L \rho_y + L_0) = \bar{w}. \]
The closed-loop system in (6.29) can be written as
\[ y(s) = \frac{H_{yw}(s)\sigma(s) + H_{yr}(s)r(s) + H_{ye}(s)e(s)}{1 - H_{yy}(s)} . \tag{6.64} \]
Substituting it into (6.30), we obtain
\[ u(s) = \left( \frac{H_{uy}(s)H_{yu}(s)}{1 - H_{yy}(s)} + H_{uw}(s) \right) \sigma(s) + \left( \frac{H_{uy}(s)H_{yr}(s)}{1 - H_{yy}(s)} + H_{ur}(s) \right) r(s) \]
\[ + \left( \frac{H_{uy}(s)H_{ye}(s)}{1 - H_{yy}(s)} + H_{ue}(s) \right) e(s) . \]
The stability of the reference system implies that all the transfer functions in this equation as well as in (6.64) are stable. Recall that \( P(s) \) defined in (6.7) is also stable. Therefore the signals \( u(t) \) and \( \phi_y(t) \) can be upper bounded as follows
\[ \| u_r \|_{\mathcal{L}_\infty} \leq \left\| \frac{H_{uy}(s)H_{yu}(s)}{1 - H_{yy}(s)} + H_{uw}(s) \right\|_{\mathcal{L}_1} \bar{w}_0 + \left\| \frac{H_{uy}(s)H_{yr}(s)}{1 - H_{yy}(s)} + H_{ur}(s) \right\|_{\mathcal{L}_1} \bar{\tau} \]
\[ + \left\| \frac{H_{uy}(s)H_{ye}(s)}{1 - H_{yy}(s)} + H_{ue}(s) \right\|_{\mathcal{L}_1} \| e_r \|_{\mathcal{L}_\infty} = \bar{u}_0 + \bar{u}_e \| e_r \|_{\mathcal{L}_\infty} , \]
\[ \| (\phi_y)_r \|_{\mathcal{L}_\infty} \leq \frac{\| P(s)H_{yu}(s) \|}{1 - H_{yy}(s)} \bar{w}_0 + \frac{\| P(s)H_{yr}(s) \|}{1 - H_{yy}(s)} \bar{\tau} + \frac{\| P(s)H_{ye}(s) \|}{1 - H_{yy}(s)} \| e_r \|_{\mathcal{L}_\infty} \]
\[ = \bar{y}_0 + \bar{y}_e \| e_r \|_{\mathcal{L}_\infty} . \]
Next, using (6.11) we obtain the following upper bound
\[ \| \phi_r \|_{\mathcal{L}_\infty} \leq \max \{ \| u_r \|_{\mathcal{L}_\infty}, \| (\phi_u)_r \|_{\mathcal{L}_\infty}, \| (\phi_y)_r \|_{\mathcal{L}_\infty} \} \]
\[ \leq \max \{ \frac{\| P(s) \|_{\mathcal{L}_1}}{1 - H_{yy}(s)} \| u_r \|_{\mathcal{L}_\infty}, \| (\phi_y)_r \|_{\mathcal{L}_\infty} \} \]
\[ \leq \max \{ \frac{\| P(s) \|_{\mathcal{L}_1}}{1 - H_{yy}(s)} \bar{u}_0, \bar{y}_0 \} + \max \{ \frac{\| P(s) \|_{\mathcal{L}_1}}{1 - H_{yy}(s)} \bar{u}_e, \bar{y}_e \} \| e_r \|_{\mathcal{L}_\infty} \]
\[ = \bar{\phi}_0 + \bar{\phi}_e \| e_r \|_{\mathcal{L}_\infty} . \tag{6.65} \]
Let $\bar{\phi} \triangleq \bar{\phi}_0 + 2\epsilon$. Next, we show that

$$\| \phi_\tau \|_{L_\infty} < \bar{\phi}. \quad (6.66)$$

We use a contradiction to prove this upper bound. Notice that $\phi(t)$ is a continuous function, and that $u(0) = 0$, $\phi(0) = 0$. Thus, if (6.66) is not true, then there exists some time $\tau_1 \in [0, \tau]$, such that

$$\| \phi(\tau_1) \|_{L_\infty} = \bar{\phi},$$

$$\| \phi(t) \|_{L_\infty} < \bar{\phi}, \quad t < \tau_1. \quad (6.67)$$

Substituting (6.59) into (6.50) and setting $\omega_E \geq \Gamma$, we obtain

$$\bar{\epsilon}_\phi \leq \frac{\sqrt{2n + 1} \omega_G}{\sqrt{\Gamma}} \sum_{k=0}^{l-1} \frac{\bar{\epsilon}}{k!} \epsilon < \frac{\epsilon}{(\bar{\phi}_0 + 2\epsilon)^2 \phi_e}.$$

Using the definition of $\bar{\phi}$, this upper bound can be written as

$$\bar{\epsilon}_\phi < \frac{\epsilon}{\bar{\phi}^2 \phi_e},$$

which can be equivalently represented as

$$\bar{\phi}^2 \phi_e \bar{\epsilon}_\phi < \epsilon. \quad (6.68)$$

Next, consider (6.49). Substituting (6.58) and (6.59) leads to

$$\bar{\epsilon}_0 = \frac{\bar{\epsilon}}{\sqrt{\Gamma}} + \| C_E(s) (1 - C_E(s)) \|_{L_1} \bar{\epsilon} < \frac{\epsilon}{\phi_e},$$

which further yields

$$\bar{\phi}_e \bar{\epsilon}_0 < \epsilon. \quad (6.69)$$

Finally, we compute the upper bound for $\phi(t)$, using the previously obtained bound in (6.65) along with the result of Lemma 6.5, given by (6.48). Substituting (6.48) into (6.65) leads to

$$\| \phi_{\tau_1} \|_{L_\infty} \leq \bar{\phi}_0 + \bar{\phi}_e \bar{\epsilon}_0 + \bar{\phi}_e \bar{\epsilon}_\phi \| \phi_{\tau_1} \|_{L_\infty}^2.$$

According to our assumption, we have $\| \phi_{\tau_1} \|_{L_\infty} = \bar{\phi}$, and therefore one can write

$$\| \phi_{\tau_1} \|_{L_\infty} \leq \bar{\phi}_0 + \bar{\phi}_e \bar{\epsilon}_0 + \bar{\phi}_e \bar{\epsilon}_\phi \bar{\phi}^2.$$
The previously obtained upper bounds in (6.68) and (6.69) yield
\[ \| \phi_{\tau_1} \|_{L_\infty} < \bar{\phi}_0 + 2\epsilon = \bar{\phi}, \]
which gives a contradiction to the assumption in (6.67). Since all used upper bounds are valid for arbitrary \( \tau_1 \in [0, \tau] \), (6.66) holds.

Now we proceed with construction of the contradiction to the claim in (6.62). Substituting (6.66) into (6.48), we obtain
\[ \| e_{\tau} \|_{L_\infty} < \bar{e}_0 + \bar{e}_0 \bar{\phi}^2 = \bar{e}_0 + \bar{e}_0 (\bar{\phi}_0 + 2\epsilon)^2 = \gamma_e. \] (6.70)

Subtracting \( y_{\text{ref}}(t) \), given by (6.40), from \( y(t) \), given by (6.29) and (6.30), respectively, we obtain
\[ y(s) - y_{\text{ref}}(s) = H_{yy}(s)(y(s) - y_{\text{ref}}(s)) + H_{yw}(s)(\sigma(s) - \sigma_{\text{ref}}(s)) + H_{ye}(s)e(s). \]

This equation leads to the following upper bound
\[ \| (y - y_{\text{ref}})_{\tau} \|_{L_\infty} \leq \| H_{yy}(s) \|_{L_1} \| (y - y_{\text{ref}})_{\tau} \|_{L_\infty} + \| H_{yw}(s) \|_{L_1} \| (\sigma - \sigma_{\text{ref}})_{\tau} \|_{L_\infty} + \| H_{ye}(s) \|_{L_1} \| e_{\tau} \|_{L_\infty}. \]

Taking into account the Lipschitz condition in (6.2), we obtain
\[ \| (y - y_{\text{ref}})_{\tau} \|_{L_\infty} \leq \| H_{yy}(s) \|_{L_1} \| (y - y_{\text{ref}})_{\tau} \|_{L_\infty} + \| H_{yw}(s) \|_{L_1} L \| (y - y_{\text{ref}})_{\tau} \|_{L_\infty} + \| H_{ye}(s) \|_{L_1} \| e_{\tau} \|_{L_\infty}. \]

Using the upper bound in (6.70), this bound can be written as
\[ \| (y - y_{\text{ref}})_{\tau} \|_{L_\infty} \leq \frac{\| H_{ye}(s) \|_{L_1}}{1 - \| H_{yy}(s) \|_{L_1} - \| H_{yw}(s) \|_{L_1} L} \| e_{\tau} \|_{L_\infty} \]
\[ < \frac{\| H_{ye}(s) \|_{L_1}}{1 - \| H_{yy}(s) \|_{L_1} - \| H_{yw}(s) \|_{L_1} L} \gamma_e \]
\[ = \gamma_y, \]
which contradicts (6.62), and hence proves the performance bound in (6.60).

Next, subtracting \( u_{\text{ref}}(t) \), given by (6.40), from \( u(t) \), given by (6.30), we obtain
\[ u(s) - u_{\text{ref}}(s) = H_{uy}(s)(y(s) - y_{\text{ref}}(s)) + H_{uw}(s)(\sigma(s) - \sigma_{\text{ref}}(s)) + H_{ue}(s)e(s). \]
This leads to the following bound
\[
\|(u - u_{\text{ref}})\|_{L_{\infty}} \leq \|H_{uy}(s)\|_{L_{1}}\|(y - y_{\text{ref}})\|_{L_{\infty}} + \|H_{uw}(s)\|_{L_{1}}\|\sigma - \sigma_{\text{ref}}\|_{L_{\infty}} + \|H_{ue}(s)\|_{L_{1}}\|e\|_{L_{\infty}}
\]
\[
\leq (\|H_{uy}(s)\|_{L_{1}} + \|H_{uw}(s)\|_{L_{1}}) \|(y - y_{\text{ref}})\|_{L_{\infty}} + \|H_{ue}(s)\|_{L_{1}}\|e\|_{L_{\infty}} + \|H_{uw}(s)\|_{L_{1}}\|e\|_{L_{\infty}}
\]
\[
= \gamma_u,
\]
which contradicts (6.63) and completes the proof. \(\square\)

**Remark 6.6.** Consider the definition of scalars \(\gamma_e, \gamma_y,\) and \(\gamma_u\) in Theorem 6.1. From Remark 6.5 it follows that \(\gamma_e\) is decreasing as the adaptation gain is increasing. In the limiting case this leads to
\[
\lim_{\Gamma \to \infty} \gamma_e = \lim_{\Gamma \to \infty} \bar{e}_0(\Gamma, \omega_E = \Gamma) + (\tilde{\phi}_0 + 2\epsilon)^2 \lim_{\Gamma \to \infty} \bar{e}_\phi(\Gamma, \omega_E = \Gamma) = 0.
\]
This implies that \(\gamma_y\) and \(\gamma_u\) are decreasing with the adaptation gain. In the limiting case we obtain:
\[
\lim_{\Gamma \to \infty} \gamma_y = \lim_{\Gamma \to \infty} \frac{\|H_{ye}(s)\|_{L_1}}{1 - \|H_{yy}(s)\|_{L_1} - \|H_{yw}(s)\|_{L_1} L} \lim_{\Gamma \to \infty} \gamma_e = 0,
\]
\[
\lim_{\Gamma \to \infty} \gamma_u = (\|H_{uy}(s)\|_{L_1} + \|H_{uw}(s)\|_{L_1} L) \lim_{\Gamma \to \infty} \gamma_y + \|H_{ue}(s)\|_{L_1} \lim_{\Gamma \to \infty} \gamma_e = 0.
\]

Notice that the adaptation laws are defined for arbitrary positive finite adaptation gains. However, the observed behavior of the performance bounds implies that they can be arbitrarily reduced by increasing the adaptation gain \(\Gamma\). This means that the transient of the closed-loop adaptive system can be made arbitrarily close to the transient of the reference system by setting sufficiently large adaptation gain.

### 6.5. Simulations

To validate the theoretical results presented above, consider the system in (6.1) with the following parameters: \(k_0 = 5, B(s) = s + 0.2,\) and \(A(s) = s^2 + s + 2\). These parameters lead to the following second order plant:
\[
W_1(s) \triangleq k_0 \frac{B(s)}{A(s)} = 5 \frac{s + 0.2}{s^2 + s + 2}.
\]
Next, let $N(s) = s + 20$, $D(s) = s^2 + 12s + 20$, which require $k_g = 1$ for tracking step reference signals. These parameters lead to the following ideal system

$$M(s) \triangleq k_g \frac{N(s)}{D(s)} = \frac{s + 20}{s^2 + 12s + 20}.$$ 

Assume that our conservative knowledge of the unknown plant (6.71) leads to the following minimum value of the plant high frequency gain $k_m = 1$, and the following parameter sets $\Theta = \{\theta \in \mathbb{R}^{2n+1} : \|\theta\| \leq 4\}$, $\Delta = [-1000, 1000]$. Next we choose the adaptation gain to be $\Gamma = 1000$, and the filter parameters to be $\omega_G = 100$, $\omega_E = 100000$, and $C_0(s) = \frac{20}{s + 20}$.

These parameters lead to the following filter

$$C_F(s) = \frac{20}{s + 20} \frac{100000}{s + 100000} \frac{100}{s + 100}.$$ 

Notice that according to the theory we need to keep $\omega_E$ high enough to ensure that the closed-loop adaptive system is close to the reference system. The above parameters result in the following values of the $L_1$-norms in (6.14)

$$\|H_{yy}(s)\|_{L_1} = 0.239, \quad \|H_{yy}(s)\|_{L_1} = 0.172,$$

which along with the resulting value of $L = 4$, gives us (from (6.14)):

$$0.239 + 0.172 \cdot 4 = 0.927 < 1.$$ 

Thus our choice of parameters satisfy the $L_1$-norm sufficient condition for any uncertainty with $L \leq 4$.

Further, choose the remaining parameters of the adaptive controller as follows:

$$p = N(s)(s + 5) = s^2 + 25s + 100.$$ 

We set the projection bounds to be equal to $\Theta$ and $\Delta$, defined in the beginning of this section.

First we test the closed-loop system performance for step reference signals. Figure 6.5 shows the simulation results in the absence of the nonlinearities ($f(t, y) \equiv 0$). We see that the transient of the closed-loop adaptive system almost coincides with the transient of the
reference system both for the system output and the control signal. The augmented error remains small, and the parameter estimations remain bounded.

Next, consider Figure 6.5d. Notice that for \( t \in [0, 1) \), when the reference command is zero, the controller does not produce any control signal, and the parameter estimates do not change. At the time instant \( t = 1 \), when the step reference signal is introduced, we observe that the adaptive estimates almost instantaneously jump to the parameter values, which render the response of the closed-loop system very close to the one of the reference system. Notice that the rapid jump of the parameter estimates and the large adaptation gain do not cause high gain control. The values of the control signal remain within reasonable bounds, and there are no large oscillations in the control signal.

Figure 6.6 shows the simulation results in the presence of the input disturbance

\[
f_1(t, y) = \sin y - 0.5 + \sin(0.3t) .
\]

Notice that at \( t = 0 \) the disturbance is nonzero and it results in the Lipschitz constant \( L = 1 \).
Figure 6.6: Step response of the closed-loop adaptive system in the presence of the disturbance.

The adaptive estimates in this case have 2 jumps. The first one happens at the initial time and results in generating the adequate control signal to compensate for the disturbance. The second one happens at the time instant of the step command. Notice that compared to the previous case, the system outputs remain close, while the control signals are different. In Figure 6.6 the control signal also contains a component for compensation of the disturbance.

Figure 6.7 shows the system response to the step reference commands of different amplitudes in the presence of input disturbance. We see the predictable response of the closed-loop adaptive system. The system responses scale with the reference commands.

Figure 6.8 shows the simulation results for the sinusoidal reference command \( r(t) = \cos(0.5t) \). These plots show good tracking performance.

Next we consider two more input disturbances:

\[
\begin{align*}
    f_2(t, y) &= |y| + 0.5 + 0.1 \sin(2t), \\
    f_3(t, y) &= -\cos y + 0.5|y| + 0.5y + 1 + 0.5 \sin(0.5t) + 0.2 \sin t.
\end{align*}
\]
Figure 6.7: Closed-loop system response to the step signals of different size.

Figure 6.8: Closed-loop system response to the sinusoidal reference command.

The simulation results for all three cases of the input disturbances are shown in Figure 6.9. The simulations are done without any retuning of the controller parameters. We see that all three system responses are close.

Next, we consider a different plant given by

\[ W_2(s) \triangleq k_0 \frac{B(s)}{A(s)} = 10 \frac{s + 0.1}{s^2 + s + 4}. \]

Figure 6.10 shows the simulation results for both plants. The simulations are done with the same parameters of the controller without any retuning. We see that the system outputs almost coincide for both plants, while the control signals are different. This indicates that the performance of the controller, defined by the reference system, is not significantly affected by the uncertainties in the system dynamics.

The important feature of the \( \mathcal{L}_1 \) adaptive control architecture is that it has bounded
Figure 6.9: Closed-loop system response for the disturbances $w_0^1(t)$, $w_0^2(t)$, $w_0^3(t)$.

Figure 6.10: Closed-loop system response for the plants $W_1(s)$ and $W_2(s)$.

away from zero time-delay margin in the presence of fast adaptation, [4]. Finally, we perform a test of the closed-loop system robustness to time delays. Figure 6.11 shows the simulation results of the closed loop system for different values of the adaptation gains in the presence of time delay of 12 ms. We also change the value of $\omega_E$ with the adaptation gain according to $\omega_E = 100\Gamma$. Figure 6.11 also illustrates that increasing the adaptation gain does not affect neither the system output nor the control signal.
Figure 6.11: Closed-loop system response for different adaptation gains in the presence of the time delay $\tau = 12$ ms.
7.1. Problem Formulation

Consider the following SISO nonlinear system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + f(t, x) + g(t, y)(u(t) + d(t, x)), \quad x(0) = x_0, \\
y(t) &= h_x(t, x),
\end{align*}
\]

(7.1)

where \(y(t) \in \mathbb{R}\) is the measured system output; \(u(t) \in \mathbb{R}\) is the control signal; \(x(t) \in \mathbb{R}^n\) is the unmeasured system state; \(x_0 \in \mathbb{R}^n\) is the unknown initial condition with known bound \(\|x_0\| \leq \rho_0\); \(A \in \mathbb{R}^{n \times n}\) is a known Hurwitz matrix; \(f(t, x) \in \mathbb{R}^n\), \(g(t, y) \in \mathbb{R}^n\) and \(h_x(t, x) \in \mathbb{R}\) are known nonlinear functions; and \(d(t, x) \in \mathbb{R}\) is an unknown nonlinear function representing the system uncertainty/disturbance. This system is subject to the following assumptions:

**Assumption 7.1.** There exists a function \(\psi(t, y) \in \mathbb{R}^{1 \times n}\), such that

\[
\psi(t, y)g(t, y) \equiv 1.
\]

**Remark 7.1.** We notice that for existence of \(\psi(t, y)\) it is sufficient to have at least one of the elements of vector \(g(t, y)\) being nonzero for all time \(t\) and for all output values \(y\).

**Assumption 7.2** (Lipschitz continuity). For all \(x_1 \in \mathbb{R}^n, x_2 \in \mathbb{R}^n,\) and \(y_1 \in \mathbb{R}, y_2 \in \mathbb{R}\) such
that $\|x_1\| \leq \delta_x \in \mathbb{R}^+$, $\|x_2\| \leq \delta_x$, $\|y_1\| \leq \delta_y \in \mathbb{R}^+$, $\|y_2\| \leq \delta_y$ the following bounds

$$
\|f(t, x_1) - f(t, x_2)\| \leq L_{\delta_x}\|x_1 - x_2\|,
$$
$$
\|g(t, y_1) - g(t, y_2)\| \leq L^g(\delta_y)\|y_1 - y_2\|,
$$
$$
\|h_x(t, x_1) - h_x(t, x_2)\| \leq L^h(\delta_x)\|x_1 - x_2\|
$$

hold uniformly in time, where $L_{\delta_x}$, $L^g(\delta_y)$, $L^h(\delta_x) \in \mathbb{R}^+$ are positive nondecreasing functions of $\delta_x$ and $\delta_y$.

**Assumption 7.3 (Boundedness of system nonlinearities).** For all $x \in \mathbb{R}^n$ and for all $y \in \mathbb{R}$ such that $\|x\| \leq \delta_x \in \mathbb{R}^+$ and $\|y\| \leq \delta_y \in \mathbb{R}^+$, the following bounds

$$
\|f(t, x)\| \leq L_{\delta_x}\|x\| + L_0,
$$
$$
\|g(t, y)\| \leq g(\delta_y),
$$
$$
\|\psi(t, y)\| \leq \bar{\psi}(\delta_y),
$$
$$
|h_x(t, x)| \leq \bar{h}(\delta_x),
$$
$$
|d(t, x)| \leq \bar{d}(\delta_x)
$$

hold uniformly in time, where $L_{\delta_x}$, $g(\delta_y)$, $\bar{h}(\delta_x)$, $\bar{d}(\delta_x) \in \mathbb{R}^+$ are positive nondecreasing functions of $\delta_x$ or $\delta_y$, $L_0 \in \mathbb{R}^+$ are constants.

**Assumption 7.4 (Boundedness of partial derivatives).** For all $x \in \mathbb{R}^n$ and for all $y \in \mathbb{R}$ such that $\|x\| \leq \delta_x \in \mathbb{R}^+$ and $\|y\| \leq \delta_y \in \mathbb{R}^+$, the following bounds

$$
\left| \frac{\partial}{\partial t} d(t, x) \right| \leq d'_d(\delta_x),
$$
$$
\left| \frac{\partial}{\partial x} d(t, x) \right| \leq d'_{d_a}(\delta_x),
$$
$$
\left| \frac{\partial}{\partial t} \psi(t, y) \right| \leq d'_{\psi}(\delta_y),
$$
$$
\left| \frac{\partial}{\partial y} \psi(t, y) \right| \leq d'_{\psi}(\delta_y),
$$
$$
\left| \frac{\partial}{\partial t} h_x(t, x) \right| \leq d'_{h}(\delta_x),
$$
$$
\left| \frac{\partial}{\partial x} h_x(t, x) \right| \leq d'_{h}(\delta_x),
$$

hold uniformly in time, where $d'_d(\delta_x)$, $d'_{d_a}(\delta_x)$, $d'_{\psi}(\delta_y)$, $d'_{\psi}(\delta_y)$, $d'_{h}(\delta_x)$, $d'_{h}(\delta_x) \in \mathbb{R}^+$ are positive nondecreasing functions of $\delta_x$ or $\delta_y$.

**Assumption 7.5 (Passivity type assumption).** Let $P = P^T > 0$ be the solution to the following algebraic Lyapunov equation

$$
A^T P + PA = Q, \quad Q = Q^T > 0,
$$

(7.2)
Also let \( h(t, x, y) \in \mathbb{R} \) be a function, such that

\[
y = h(t, x, y)
\]  

(7.3)

has a unique solution \( y \), given by \( y = h_x(t, x) \). Then we assume that for some \( Q \) and for all \( \delta \in \mathbb{R} \) and all \( x \in \mathbb{R}^n \) the following relationship holds

\[
x^\top Pg(t, \delta) = h(t, x, \delta).
\]

(7.4)

**Remark 7.2.** By substituting \( \delta \equiv y \) in (7.4), we see that Assumption 7.5 implies passivity of the system (7.1). Thus, Assumption 7.5 is a stronger version of passivity\(^1\) requirement on the system (7.1).

**Remark 7.3.** We also note that the assumption (7.5) restricts the class of systems to one with implicit structure for the output equation. Some examples of systems with implicit output equation are systems with *dynamic vision* [78, 79]. However, the systems in these examples are multi-input, multi-output (MIMO), while in our work the problem is formulated for a class of SISO systems. Therefore we plan to extend the problem formulation (7.1) to MIMO case in our future research.

The control objective is to design the control signal \( u(t) \), which compensates for the low-frequency content of the system uncertainty and disturbance \( d(t, x) \) and achieves control specifications given via the following ideal system

\[
\dot{x}_{id}(t) = Ax_{id}(t) + f(t, x_{id}), \quad x_{id}(0) = x_0, \\
y_{id}(t) = h_x(t, x_{id}),
\]

(7.5)

where \( y_{id}(t) \in \mathbb{R} \) is the desired system output; and \( x_{id}(t) \in \mathbb{R}^n \) is the system state.

### 7.2. \( L_1 \) Adaptive Controller Architecture

In this section we first provide all necessary definitions and notations before we proceed with \( L_1 \) adaptive controller architecture. We also impose constraints on some of the constants, which are crucial for closed-loop system stability.

\(^1\)The system (7.1) is passive if there exist a positive definite solution \( P \) to the algebraic Lyapunov equation (7.2) such that \( x^\top Pg(t, h_x(t, x)) = h_x(t, x) \), [77]
7.2.1. Definitions and Stability Conditions

We start by defining a lowpass filter $C(s)$, which is used in the control law. In this chapter, for simplicity of derivations, we consider a first order filter

$$ C(s) = \frac{\omega}{s + \omega}, $$

where $\omega \in \mathbb{R}^+$ is subject to stability conditions presented below. Next, let $\rho_r \in \mathbb{R}^+$ be an arbitrary constant, which satisfies

$$ \beta_{x_{ref}} \triangleq \lambda_{\min}(Q) - 2\|P\|L_{\rho_r} > 0, \quad (7.6) $$

and is lower bounded by

$$ \rho_r > \frac{\|P\|L_0}{\alpha_{ref}\lambda_{\min}(P)} + \sqrt{\left(\frac{\|P\|L_0}{\alpha_{ref}\lambda_{\min}(P)}\right)^2 + \frac{\lambda_{\max}(P)\rho_0^2}{\lambda_{\min}(P)}}, \quad (7.7) $$

where

$$ \alpha_{ref} \triangleq \frac{\beta_{x_{ref}}}{\lambda_{\max}(P)}. $$

Also, we define

$$ \bar{\rho}_r \triangleq \sqrt{\frac{\lambda_{\max}(P)\rho_0^2}{\lambda_{\min}(P)}} + \frac{2\rho_r\|P\|L_0 + 2\rho_r\bar{g}(\bar{h}(\rho_r))\|P\|\bar{\nu}_{ref}}{\alpha_{ref}\lambda_{\min}(P)}, \quad (7.8) $$

where

$$ \bar{\nu}_{ref} \triangleq d_{\bar{d}}(\rho_r) + d_{\bar{d}}(\rho_r)(\|A\|\rho_r + L_{\rho_r}\rho_r + 2\bar{g}(\bar{h}(\rho_r))\bar{d}(\rho_r)). $$

The choice of the lowpass filter bandwidth needs to ensure that

$$ \bar{\rho}_r < \rho_r. \quad (7.9) $$

Remark 7.4. We notice that (7.7) solves the following inequality:

$$ \rho_r > \sqrt{\frac{\lambda_{\max}(P)\rho_0^2}{\lambda_{\min}(P)}} + \frac{2\rho_r\|P\|L_0}{\alpha_{ref}\lambda_{\min}(P)}, $$

which implies that there always exists sufficiently large $\omega$, such that (7.9) is satisfied.

Next, we define

$$ \rho_{ur} \triangleq \bar{d}(\rho_r). \quad (7.10) $$
Also, let $\gamma_{\bar{x}}^{\text{max}} \in \mathbb{R}^+$ be an arbitrary constant such that
\[
\gamma_{\bar{x}}^{\text{max}} > 2\rho_0 \sqrt{\frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)}},
\]
and define
\[
\rho \triangleq \rho_r + \gamma_x,
\]
where $\gamma_x \in \mathbb{R}^+$ is a constant subject to stability conditions given below. Further, let
\[
\Delta \triangleq d(\rho),
\]
and
\[
d_x \triangleq \|A\|\rho + L_\rho \rho + L_0 + \bar{g}(\bar{h}(\rho))2\Delta,
\]
\[
d_{x_{\text{ref}}} \triangleq \|A\|\rho_r + L_\rho \rho_r + L_0 + \bar{g}(\bar{h}(\rho_r))2\Delta.
\]
Then, define
\[
d_{\sigma_{\text{ref}}} \triangleq d_\rho \rho_{\text{ref}} d_x + d_{\rho} \rho_{\text{ref}},
\]
\[
d_a \triangleq d_\rho \rho d_x + d_{\rho} \rho,
\]
\[
d_{e_a} \triangleq d_\rho \rho d_x + d_{\rho} \rho + d_\rho \rho_{\text{ref}} d_{x_{\text{ref}}} + d_\rho \rho_{\text{ref}},
\]
Next, let
\[
\bar{\rho} \triangleq \rho + \gamma_{\bar{x}}^{\text{max}},
\]
and
\[
d_\psi \triangleq \tilde{d}_\psi (\bar{h}(\rho)) + d_{\psi} (\bar{h}(\rho)) (d_\rho (\rho) + d_{\rho} (\rho) d_x).
\]
Also, let
\[
\gamma_\tilde{\eta} \triangleq (2\omega + \|A\| + L_\tilde{\rho} \tilde{\psi}(\bar{h}(\rho)) + d_\psi,
\]
and
\[
\beta_e \triangleq \lambda_{\text{min}}(Q) - 2L_\rho \|P\| > 0,
\]
\[
\alpha_e \triangleq \frac{\beta_e}{\lambda_{\text{min}}(P)},
\]
which allow to define

\[ \bar{\gamma}_x \triangleq \sqrt{\frac{2\gamma_x \| P \| (L^g(h(p))L^h(p)\frac{1}{\omega}d_{\text{rel}} + \ddot{g}(\hat{h}(p))\frac{1}{\omega}d_{e_s} + \ddot{g}(\hat{h}(p))\gamma_{\hat{\eta}}\gamma_{\hat{\eta}}^{\text{max}}}{\alpha_\epsilon \lambda_{\text{max}}(P)}}}. \]  

(7.23)

We assume that there exists \( \omega, \gamma_x, \) and \( \gamma_{\hat{\eta}}^{\text{max}} \), which ensure

\[ \bar{\gamma}_x < \gamma_x. \]  

(7.24)

**Remark 7.5.** Notice that the first two terms in (7.23) can be arbitrarily reduced by increasing the lowpass filter bandwidth, while the third term depends on the system’s initial conditions bound and cannot be arbitrarily reduced; in fact, its value grows as \( \omega \) is increased. We notice that if the value of \( \rho_0 \) is large enough, then it may be possible that the condition (7.24) cannot be satisfied for any value of \( \omega, \gamma_x, \) and \( \gamma_{\hat{\eta}}^{\text{max}} \). In this case, the analysis given in this chapter cannot guarantee stability of the closed-loop system. To understand the reason of this limitation, we point to (7.20), which defines the bound on the filtered estimation error in the control law. This bound results in a coefficient proportional to \( \omega \) in (7.23). This relationship is similar to the one observed in high-gain observers [77], where it was referred to as peaking phenomenon. While in simulations we illustrate this phenomenon, it is worth mentioning that this phenomenon can be addressed by applying techniques similar to Chapter 4.

Next, let

\[ \gamma_u \triangleq d(\rho_{\text{rel}}) + d(\rho) + \gamma_{\hat{\eta}}\gamma_{\hat{\eta}}^{\text{max}}. \]  

(7.25)

Further, we define

\[ \beta_{\hat{\eta}} \triangleq \lambda_{\text{min}}(Q) - 2\| P \| L_{\hat{\rho}}, \]

where we assume that the choice of \( \hat{\rho} \) ensures

\[ \beta_{\hat{\eta}} > 0. \]  

(7.26)

Finally, let

\[ \alpha_{\hat{\eta}} \triangleq \frac{\beta_{\hat{\eta}}}{\lambda_{\text{max}}(P)}, \]

and

\[ \gamma_{\hat{\eta}}(t) \triangleq \sqrt{\left( \frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)} \frac{4u_0^2}{\nu_{\alpha_{\hat{\eta}}\lambda_{\text{min}}(P)}} - \frac{4\Delta d_\sigma}{\Gamma\alpha_{\hat{\eta}}\lambda_{\text{min}}(P)} \right) e^{-\alpha_{\hat{\eta}}t} + \frac{4\Delta^2}{\Gamma\lambda_{\text{min}}(P)} + \frac{4\Delta d_\sigma}{\Gamma\alpha_{\hat{\eta}}\lambda_{\text{min}}(P)}}. \]  

(7.27)
where $\Gamma \in \mathbb{R}^+$ is the adaptation gain, which needs to ensure

$$
\gamma_{\hat{x}}^{\text{max}} > \sup_{t \in [0, \infty)} \gamma_{\hat{x}}(t).
$$

(7.28)

**Remark 7.6.** We notice that (7.11) implies that if the adaptation gain $\Gamma$ is chosen sufficiently large, it is always possible to satisfy the condition in (7.28).

**Remark 7.7.** Notice that if (7.26) is satisfied, both (7.6) and (7.21) also hold. Thus, only three conditions from the above stated ones restrict the class of systems. These conditions are (7.9), (7.24), (7.26). Condition (7.9) imposes a constraint on a minimal lowpass filter bandwidth, which is required to ensure stability of the $L_1$ reference system. Condition (7.24) is related to the initialization error and peaking phenomenon and was discussed above. Finally, (7.26) restricts the rate of growth of system nonlinearity $f(t, x)$.

### 7.2.2. Controller Architecture

The $L_1$ adaptive controller, similar to all presented above architectures, is comprised of an output predictor, an adaptation law and a control law, which involves the lowpass filter $C(s)$. Below we define each of these components.

We consider the following **output predictor**:

$$
\dot{\hat{x}}(t) = A\hat{x}(t) + f(t, \hat{x}) + g(t, y)(u(t) + \hat{\sigma}(t)), \quad \hat{x}(0) = \hat{x}_0,
$$

$$
\hat{y}(t) = h(t, \hat{x}, y),
$$

(7.29)

where $\hat{y}(t) \in \mathbb{R}$ and $\hat{x}(t) \in \mathbb{R}^n$ are the predictor output and state respectively; $\hat{\sigma}(t) \in \mathbb{R}$ is the uncertainty estimate updated by the following **adaptation law**:

$$
\dot{\hat{\sigma}}(t) = \Gamma \text{Proj}(\hat{\sigma}(t), -\hat{y}(t)), \quad \hat{\sigma}(0) = \hat{\sigma}_0,
$$

(7.30)

where $\hat{y}(t) \triangleq \hat{y}(t) - y(t)$; $|\hat{\sigma}_0| \leq \Delta$; and Proj$(\cdot)$ operator ensures that the estimate does not leave the ball $|\hat{\sigma}(t)| \leq \Delta$, where $\Delta$ was defined in (7.13).

The adaptive **control law** is given by:

$$
u(s) = -C(s)\hat{\sigma}(s),
$$

(7.31)

where $u(s)$ and $\hat{\sigma}(s)$ are the Laplace transforms of $u(t)$ and $\hat{\sigma}(t)$ respectively.
7.3. Analysis of the $L_1$ Adaptive Controller

We start our analysis with defining the $L_1$ reference system, for which we prove stability. Then we consider the estimation loop and derive a bound on the prediction error and filtered estimation error. Finally using these results, we prove stability of the closed-loop adaptive system and compute the performance bounds.

7.3.1. $L_1$ Reference System

Consider the following $L_1$ reference system

\[
\begin{align*}
\dot{x}_{\text{ref}}(t) &= Ax_{\text{ref}}(t) + f(t, x_{\text{ref}}) + g(t, y_{\text{ref}})(u_{\text{ref}}(t) + d(t, x_{\text{ref}})), \quad x_{\text{ref}}(0) = x_0, \\
y_{\text{ref}}(t) &= h(t, x_{\text{ref}}, y_{\text{ref}}),
\end{align*}
\]  

(7.32)

where $y_{\text{ref}}(t) \in \mathbb{R}$ is the measured system output; $u_{\text{ref}}(t) \in \mathbb{R}$ is the reference control signal; $x_{\text{ref}}(t) \in \mathbb{R}^n$ is the reference system state. The reference control law is given by

\[
u_{\text{ref}}(s) = -C(s)\sigma_{\text{ref}}(s),
\]

(7.33)

where $\sigma_{\text{ref}}(s)$ is the Laplace transform of $\sigma_{\text{ref}}(t) \triangleq d(t, x_{\text{ref}}(t))$.

We note that this reference system depends on the system uncertainty and therefore is used only for analysis. Also note that, similar to the previously presented $L_1$ adaptive controllers, this reference system assumes partial compensation of the uncertainty within the frequency range specified by the bandwidth of the lowpass filter $C(s)$. Stability of the $L_1$ reference system (7.32)-(7.33) is established by the following lemma.

**Lemma 7.1.** Consider the $L_1$ reference system (7.32). Let all the parameters satisfy the conditions in Section 7.2.1. Then the following bounds hold:

\[
\|x_{\text{ref}}\|_{L_\infty} \leq \rho_r,
\]

(7.34)

\[
\|u_{\text{ref}}\|_{L_\infty} \leq \rho_{ur},
\]

(7.35)

where $\rho_r$ and $\rho_{ur}$ were defined in (7.8) and (7.10).

**Proof.** Consider the following Lyapunov function

\[
V(x_{\text{ref}}) = x_{\text{ref}}^T P x_{\text{ref}}.
\]

(7.36)
Its derivative along the system trajectories is given by
\[
\dot{V}(t) = -x_{\text{ref}}^\top(t)Qx_{\text{ref}}(t) + 2x_{\text{ref}}^\top P(f(t, x_{\text{ref}}) + g(t, y_{\text{ref}})(u_{\text{ref}}(t) + d(t, x_{\text{ref}}))) \\
\leq -\lambda_{\text{min}}(Q)\|x_{\text{ref}}(t)\|^2 + 2\|x_{\text{ref}}(t)\|\|P\|\|f(t, x_{\text{ref}})\| \\
+ 2\|x_{\text{ref}}(t)\|\|P\|\|g(t, y_{\text{ref}})\|\|u_{\text{ref}}(t) + d(t, x_{\text{ref}})\|.
\] (7.37)

Next we prove it by contradiction. Towards this end assume that (7.34) does not hold and due to the fact that \(x_{\text{ref}}(t)\) is continuous and \(\|x_{\text{ref}}(0)\| \leq \rho_0 < \rho_r\), there exists a time \(\tau \in \mathbb{R}^+\), such that
\[
\|x_{\text{ref}}(\tau)\| = \rho_r, 
\] (7.38)
and
\[
\|x_{\text{ref}}(t)\| < \rho_r, \quad \forall t \in [0, \tau). 
\]
Thus \(\|x_{\text{ref}}\|_{\infty} \leq \rho_r\), and using Assumption 7.3, we obtain
\[
\|y_{\text{ref}}\|_{\infty} \leq \bar{h}(\rho_r),
\]
which further allow to rewrite (7.37) as
\[
\dot{V}(t) \leq -\lambda_{\text{min}}(Q)\|x_{\text{ref}}(t)\|^2 + 2\|x_{\text{ref}}(t)\|\|P\|L_{\rho_r} + 2\rho_r\|P\|L_0 \\
+ 2\rho_r\|P\|\bar{g}(\bar{h}(\rho_r))\|u_{\text{ref}}(t) + d(t, x_{\text{ref}})\| \\
= -\beta_{x_{\text{ref}}}\|x_{\text{ref}}(t)\|^2 + 2\rho_r\|P\|L_0 + 2\rho_r\bar{g}(\bar{h}(\rho_r))\|P\|\|u_{\text{ref}}(t) + d(t, x_{\text{ref}})\|,
\]
where \(\beta_{x_{\text{ref}}}\) is defined in (7.6). Also notice that due to (7.6) we have \(\beta_{x_{\text{ref}}} > 0\). Next from (7.36), we obtain
\[
\|x_{\text{ref}}(t)\| \geq \frac{V(t)}{\lambda_{\text{max}}(P)},
\]
which leads to
\[
\dot{V}(t) \leq -\frac{\beta_{x_{\text{ref}}}}{\lambda_{\text{max}}(P)}V(t) + 2\rho_r\|P\|L_0 + 2\rho_r\bar{g}(\bar{h}(\rho_r))\|P\|\|u_{\text{ref}}(t) + d(t, x_{\text{ref}})\| \\
= -\alpha_{\text{ref}}V(t) + 2\rho_r\|P\|L_0 + 2\rho_r\bar{g}(\bar{h}(\rho_r))\|P\|\|u_{\text{ref}}(t) + d(t, x_{\text{ref}})\|. 
\] (7.39)

Next, we denote \(u_{\text{ref}}(t) \triangleq u_{\text{ref}}(t) + d(t, x_{\text{ref}})\), which further can be rewritten as
\[
\nu_{\text{ref}}(s) = \frac{(1 - C(s))\sigma_{\text{ref}}(s)}{s + \omega} = \frac{s}{s + \omega}\sigma_{\text{ref}}(s) = \frac{1}{s + \omega}s\sigma_{\text{ref}}(s). \] (7.40)
This leads to the following bound
\[ \| \nu_{\text{ref}} \|_{L^\infty} \leq \frac{1}{\omega} \left\| \frac{\omega}{s + \omega} \right\|_{L^1} \| \sigma_{\text{ref}} \|_{L^\infty} = \frac{1}{\omega} \| \sigma_{\text{ref}} \|_{L^\infty}. \]

From (7.32) and (7.33), we obtain
\[ \| \dot{x}_{\text{ref}} \|_{L^\infty} \leq \| A \| \rho_r + L_{\rho_r} \rho_r + \bar{g}(\bar{h}(\rho_r)) \| 1 - C(s) \|_{L^1} \bar{d}(\rho_r) \]
\[ \leq \| A \| \rho_r + L_{\rho_r} \rho_r + 2\bar{g}(\bar{h}(\rho_r)) \bar{d}(\rho_r), \]
which results in
\[ \| \sigma_{\text{ref}} \|_{L^\infty} \leq \left\| \left( \frac{\partial}{\partial \tau} d(t, x_{\text{ref}}) \right) \right\|_{L^\infty} + \left\| \left( \frac{\partial}{\partial x_{\text{ref}}} d(t, x_{\text{ref}}) \right) \right\|_{L^\infty} \| \dot{x}_{\text{ref}} \|_{L^\infty} \]
\[ \leq \dot{d}_d(\rho_r) + \dot{d}_d(\rho_r) \| A \| \rho_r + L_{\rho_r} \rho_r + 2\bar{g}(\bar{h}(\rho_r)) \bar{d}(\rho_r), \]
and consequently we obtain
\[ \| \nu_{\text{ref}} \|_{L^\infty} \leq \frac{1}{\omega} \left( \dot{d}_d(\rho_r) + \dot{d}_d(\rho_r) \| A \| \rho_r + L_{\rho_r} \rho_r + 2\bar{g}(\bar{h}(\rho_r)) \bar{d}(\rho_r) \right) = \frac{\bar{v}_{\text{ref}}}{\omega}. \]

Substituting this into (7.39), we obtain
\[ \dot{V}(t) \leq -\alpha_{\text{ref}} V(t) + 2\rho_r \| P \| L_0 + 2\rho_r \bar{g}(\bar{h}(\rho_r)) \| P \| \frac{\bar{v}_{\text{ref}}}{\omega}. \]

Next we consider the following ODE:
\[ \dot{z}(t) = -\alpha_{\text{ref}} z(t) + b, \quad z(0) = V(0). \]
Its solution is given by
\[ z(t) = e^{-\alpha_{\text{ref}} t} V(0) + \frac{b}{\alpha_{\text{ref}}} \left( 1 - e^{-\alpha_{\text{ref}} t} \right). \]
Next we note that \( \dot{V}(t) \leq \dot{z}(t) \) for all \( t \in [0, \tau] \) and apply comparison lemma to (7.49) and
obtain
\[ V(t) \leq e^{-\alpha_{\text{ref}}t}V(0) + \frac{b}{\alpha_{\text{ref}}} (1 - e^{-\alpha_{\text{ref}}t}) \]
\[ \leq V(0) + \frac{b}{\alpha_{\text{ref}}} \leq \lambda_{\text{max}}(P)\|x_0\|^2 + \frac{2\rho_r\|P\|L_0 + 2\rho_r\bar{g}(\rho_r))\|P\|\rho_{\text{ref}}}{\alpha_{\text{ref}}}. \]

From the definition of \( V(t) \) in (7.36), we obtain
\[ \|x_{\text{ref}}\|_{\infty} \leq \sqrt{\frac{V(t)}{\lambda_{\text{min}}(P)}} \leq \sqrt{\frac{\lambda_{\text{max}}(P)\rho_0^2}{\lambda_{\text{min}}(P)}} + \frac{2\rho_r\|P\|L_0 + 2\rho_r\bar{g}(\rho_r))\|P\|\rho_{\text{ref}}}{\alpha_{\text{ref}}\lambda_{\text{min}}(P)} = \bar{\rho}_r. \]

Hence, (7.9) leads to
\[ \|x_{\text{ref}}\|_{\infty} < \rho_r, \]
which contradicts our assumption (7.38) and thus completes the proof of (7.34).

In order to prove (7.35), we use (7.33) to obtain
\[ \|u_{\text{ref}}\|_{\infty} \leq \|C(s)\|_{\infty}\|d(t, x_{\text{ref}})\|_{\infty} = \|d(t, x_{\text{ref}})\|_{\infty}, \]
which along with Assumption 7.3 leads to
\[ \|u_{\text{ref}}\|_{\infty} \leq \bar{d}(\rho_r) = \rho_{ur}, \]
and we complete the proof. \( \square \)

### 7.3.2. Transient and Steady-state Performance

Subtracting (7.1) from (7.29) and taking into account (7.3), we obtain the following prediction error dynamics:
\[ \dot{x}(t) = A\hat{x}(t) + f(t, \hat{x}) - f(t, x) + g(t, y)\hat{\sigma}(t), \quad \hat{x}(0) = \bar{x}_0, \]
\[ \hat{y}(t) = h(t, \hat{x}, y) - h(t, x, y), \]
where \( \bar{x}(t) = \hat{x}(t) - x(t), \hat{\sigma}(t) = \hat{\sigma}(t) - \sigma(t), \) and \( \bar{x}_0 \triangleq \hat{x}_0 - x_0. \)

The following lemma is an intermediate result, which is used for the boundedness proof of the prediction error \( \bar{x}(t). \)
Lemma 7.2. Consider the system in (7.1). If there exists time $\tau \in \mathbb{R}^+$, such that
\[
\|x_\tau\|_{L_\infty} \leq \rho ; \tag{7.42}
\]
then
\[
\|\dot{x}_\tau\|_{L_\infty} \leq d_x , \\
\|\dot{\sigma}_\tau\|_{L_\infty} \leq d_\sigma , \\
\|\dot{\psi}_\tau\|_{L_\infty} \leq d_\psi , 
\]
and also
\[
\|\dot{\sigma}_{ref}\|_{L_\infty} \leq d_{\sigma_{ref}} , \\
\|\dot{(\sigma_{ref} - \sigma)}\|_{L_\infty} \leq d_{e_{\sigma}} ,
\]
where $\rho$ is defined in (7.12), and $d_x, d_\sigma, d_\psi, d_{\sigma_{ref}}, d_{e_{\sigma}}$ are defined in (7.14), (7.16), (7.19), (7.15), (7.17) respectively.

Proof. Notice that (7.42) along with Assumption 7.3 implies
\[
\|(d(t, x(t)))_\tau\|_{L_\infty} \leq \bar{d}(\rho) = \Delta , \\
\|(f(t, x(t)))_\tau\|_{L_\infty} \leq L_\rho \rho + L_0 , \\
\|(h_x(t, x(t)))_\tau\|_{L_\infty} \leq \bar{h}(\rho) ,
\]
which leads to
\[
\|(g(t, y(t)))_\tau\|_{L_\infty} \leq \bar{g}(\bar{h}(\rho)) .
\]
Next, the control law (7.31) along with the projection bound on the adaptive estimate gives us
\[
\|u_\tau\|_{L_\infty} \leq \|C(s)\|_{L_1} \Delta ,
\]
From (7.1), using these results, we obtain
\[
\|\dot{x}_\tau\|_{L_\infty} \leq \|A\|\rho + L_\rho \rho + L_0 + \bar{g}(\bar{h}(\rho))(\|C(s)\|_{L_1} + 1)\Delta \\
\leq \|A\|\rho + L_\rho \rho + L_0 + \bar{g}(\bar{h}(\rho))2\Delta = d_x .
\]
To show the second bound, we note that
\[
\dot{\sigma}(t) = \frac{\partial}{\partial x} d(t, x) \dot{x}(t) + \frac{\partial}{\partial t} d(t, x),
\]
which along with Assumption 7.4 results in
\[
\|\dot{\sigma}_\tau\|_{L_\infty} \leq d_x^\sigma(\rho) + d_t^\sigma(\rho) = d_\sigma.
\]
Similarly, we obtain (7.43). Next using Lemma 7.1, we can write
\[
\|\dot{\sigma}_{\text{ref}} - \dot{\sigma}_\tau\|_{L_\infty} \leq d_x^\sigma(\rho) + d_t^\sigma(\rho) + d_x^\sigma(\rho_{\text{ref}}) + d_t^\sigma(\rho_{\text{ref}}) = d_e_\sigma.
\]
In order to prove the last bound, we notice that
\[
\dot{y}(t) = \frac{\partial}{\partial t} h_x(t, x) + \frac{\partial}{\partial x} h_x(t, x) \dot{x}(t),
\]
which along with Assumption 7.4 leads to
\[
\|\dot{y}_\tau\|_{L_\infty} \leq d_t^h(\rho) + d_x^h(\rho).
\]
Similarly, we obtain
\[
\|\dot{\psi}_\tau\|_{L_\infty} \leq d_t^\psi(\bar{h}(\rho)) + d_x^\psi(\bar{h}(\rho))(d_t^h(\rho) + d_x^h(\rho)) = d_\psi,
\]
which completes the proof. \(\square\)

The following lemma derives the bound on the state prediction error.

**Lemma 7.3.** If there exists time \(\tau \in \mathbb{R}^+\), such that
\[
\|x_\tau\|_{L_\infty} \leq \rho,
\]
then
\[
\|\bar{x}(t)\| \leq \gamma_{\bar{x}}(t), \quad \forall t \in [0, \tau], \tag{7.44}
\]
\[
\|\dot{x}_\tau\|_{L_\infty} \leq \dot{\rho},
\]
where $\gamma(t)$ and $\hat{\rho}$ were defined in (7.27) and (7.18).

**Proof.** Consider the following Lyapunov function:

$$V(\tilde{x}, \tilde{\sigma}) = \tilde{x}^T P \tilde{x} + \frac{1}{\Gamma} \tilde{\sigma}^2.$$  \hfill (7.45)

Its derivative along the system trajectories for all $t \in [0, \tau]$ is given by

$$\dot{V}(t) = -\tilde{x}^T(t)Q \tilde{x}(t) + 2 \tilde{x}^T(t)P(f(t, \dot{x}) - f(t, x)) + \frac{2}{\Gamma} \tilde{\sigma}(t) \dot{\tilde{\sigma}}(t) - \frac{2}{\Gamma} \tilde{\sigma}(t) \dot{\tilde{\sigma}}(t).$$  \hfill (7.46)

Assumption 7.5 implies

$$\dot{V}(t) = -\tilde{x}^T(t)Q \tilde{x}(t) + 2 \tilde{y}(t) \tilde{\sigma}(t) + 2 \tilde{x}^T(t)P(f(t, \dot{x}) - f(t, x)) + \frac{2}{\Gamma} \tilde{\sigma}(t) \dot{\tilde{\sigma}}(t) - \frac{2}{\Gamma} \tilde{\sigma}(t) \dot{\tilde{\sigma}}(t),$$

which after substituting the adaptation law (7.30) results in

$$\dot{V}(t) \leq -\tilde{x}^T(t)Q \tilde{x}(t) + 2 \tilde{x}^T(t)P(f(t, \dot{x}) - f(t, x)) - \frac{2}{\Gamma} \tilde{\sigma}(t) \dot{\tilde{\sigma}}(t).$$

Using this inequality, we can write the following bound

$$\dot{V}(t) \leq -\lambda_{min}(Q) \| \tilde{x}(t) \|^2 + 2 \| \tilde{x}(t) \| \| P \| \| f(t, \dot{x}) - f(t, x) \| + \left\| \frac{2}{\Gamma} \tilde{\sigma}(t) \dot{\tilde{\sigma}}(t) \right\|,$$

which after using the result in Lemma 7.2 yields

$$\dot{V}(t) \leq -\lambda_{min}(Q) \| \tilde{x}(t) \|^2 + 2 \| \tilde{x}(t) \| \| P \| \| f(t, \dot{x}) - f(t, x) \| + \frac{4 \Delta_{d\sigma}}{\Gamma}.$$

Next we use a contradiction argument. Assume that $\| \dot{x}(t) \| > \hat{\rho}$ during $[0, \tau]$. Then since $\dot{x}(t)$ is continuous and $\| \dot{x}(0) \| \leq \rho_0 < \hat{\rho}$, there exists some time $\tau_1 \in [0, \tau]$, such that

$$\| \dot{x}(\tau_1) \| = \hat{\rho},$$  \hfill (7.47)

and

$$\| \dot{x}(t) \| < \hat{\rho}, \quad \forall t \in [0, \tau_1).$$
Notice that according to definition $\hat{\rho} > \rho$. Therefore, for all $t \in [0, \tau_1)$ Assumption 7.2 yields

\[
\dot{V}(t) \leq -\lambda_{\min}(Q)\|\tilde{x}(t)\|^2 + 2\|\tilde{x}(t)\|^2\|P\|L_{\hat{\rho}} + \frac{4\Delta d_{\sigma}}{\Gamma} \\
\leq -(\lambda_{\min}(Q) - 2\|P\|L_{\hat{\rho}})\|\tilde{x}(t)\|^2 + \frac{4\Delta d_{\sigma}}{\Gamma} \\
= -\beta_{\tilde{x}}\|\tilde{x}(t)\|^2 + \frac{4\Delta d_{\sigma}}{\Gamma}.
\] (7.48)

From (7.45), we obtain

\[
V(t) \leq \lambda_{\max}(P)\|\tilde{x}(t)\|^2 + \frac{4\Delta^2}{\Gamma},
\]
or

\[
\|\tilde{x}(t)\|^2 \geq \frac{V(t) - \frac{4\Delta^2}{\Gamma}}{\lambda_{\max}(P)}.
\]

Substituting it in (7.48), we obtain

\[
\dot{V}(t) \leq -\frac{\beta_{\tilde{x}}}{\lambda_{\max}(P)} V(t) + \frac{\beta_{\tilde{x}} 4\Delta^2}{\Gamma\lambda_{\max}(P)} + \frac{4\Delta d_{\sigma}}{\Gamma}.
\] (7.49)

Next we consider the following ODE:

\[
\dot{z}(t) = -\alpha_{\tilde{x}} z(t) + b, \quad z(0) = V(0).
\]

Its solution is given by

\[
z(t) = e^{-\alpha_{\tilde{x}} t} V(0) + \frac{b}{\alpha_{\tilde{x}}} (1 - e^{-\alpha_{\tilde{x}} t}).
\]

Next we note that $\dot{V}(t) \leq \dot{z}(t)$ for all $t \in [0, \tau_1]$ and apply comparison lemma to (7.49) to obtain

\[
V(t) \leq e^{-\alpha_{\tilde{x}} t} V(0) + \frac{b}{\alpha_{\tilde{x}}} (1 - e^{-\alpha_{\tilde{x}} t})
\]
\[
= \left(V(0) - \frac{b}{\alpha_{\tilde{x}}}\right) e^{-\alpha_{\tilde{x}} t} + \frac{b}{\alpha_{\tilde{x}}},
\]
or equivalently

\[
V(t) \leq \left(\lambda_{\max}(P)\|\tilde{x}_0\|^2 - \frac{4\Delta d_{\sigma}}{\Gamma\alpha_{\tilde{x}}}\right) e^{-\alpha_{\tilde{x}} t} + \frac{4\Delta^2}{\Gamma} + \frac{4\Delta d_{\sigma}}{\Gamma\alpha_{\tilde{x}}},
\]

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Next, from (7.45), we obtain
\[
\|\tilde{x}(t)\| \leq \sqrt{\frac{V(t)}{\lambda_{\min}(P)}} \leq \sqrt{\left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|\tilde{x}_0\|^2 - \frac{4\Delta d_\sigma}{\Gamma \alpha_2 \lambda_{\min}(P)}\right) e^{-\alpha_2 t} + \frac{4\Delta^2}{\Gamma \lambda_{\min}(P)} + \frac{4\Delta d_\sigma}{\Gamma \alpha_2 \lambda_{\min}(P)}} = \gamma_{\tilde{x}}(t),
\]
which is the same as (7.44). Now it remains to show that this bound holds for all \(t \in [0, \tau]\). Towards this end we notice that for all \(t \in [0, \tau_1]\), we have
\[
\|\dot{x}(t)\| \leq \|x(t)\| + \|\tilde{x}(t)\| \leq \rho + \gamma_{\tilde{x}}(t) < \rho + \gamma_{\tilde{x}}^{\max} = \hat{\rho},
\]
which contradicts (7.47). Thus, our assumption (7.47) is wrong and \(\|\dot{x}(t)\| \leq \hat{\rho}\). This completes the proof. \(\square\)

The following lemma derives the bound on the low frequency content of the estimation error. We notice that as it follows from the definition of \(\gamma_{\tilde{\eta}}\) in (7.20), this error consists of two components: one resulting from nonzero initialization error and the second due to estimation dynamics. The second component can be arbitrarily reduced by increasing the adaptation gain \(\Gamma\).

**Lemma 7.4.** If there exists time \(\tau \in \mathbb{R}^+\) such that
\[
\|x_\tau\|_{L_\infty} \leq \rho,
\]
then
\[
\|\tilde{\eta}_\tau\|_{L_\infty} \leq \gamma_{\tilde{\eta}} \gamma_{\tilde{x}}^{\max},
\]
where \(\tilde{\eta}(s) \triangleq C(s)\tilde{\sigma}(s)\), and \(\gamma_{\tilde{\eta}}\) is defined in (7.20).

**Proof.** We start the proof by multiplying the prediction error dynamics in (7.41) by \(\psi(t, y)\), which results in
\[
\psi(t, y)\dot{x}(t) = \psi(t, y)A\tilde{x}(t) + \psi(t, y)(f(t, \dot{x}) - f(t, x)) + \psi(t, y)g(t, y)\tilde{\sigma}(t),
\]
which after taking into account Assumption 7.1 can be rewritten as
\[
\tilde{\sigma}(t) = \psi(t, y) \dot{x}(t) - \dot{\psi}(t, y) A \dot{x}(t) - \psi(t, y)(f(t, \dot{x}) - f(t, x))
\]
\[
= \frac{d}{dt} (\psi(t, y) \dot{x}(t)) - \dot{\psi}(t, y) \dot{x}(t) - \psi(t, y) A \dot{x}(t) - \psi(t, y) (f(t, \dot{x}) - f(t, x)).
\]
This leads to:
\[
\tilde{\eta}(s) = C(s) \tilde{\sigma}(s) = C(s) s \mathcal{L}[\psi(t, y) \dot{x}(t)]
\]
\[
- C(s) \mathcal{L}[\dot{\psi}(t, y) \dot{x}(t) + \psi(t, y) A \dot{x}(t) + \psi(t, y) (f(t, \dot{x}) - f(t, x))].
\]
Applying Lemma 7.3, Lemma 7.2 and taking into account Assumptions 7.2, (7.3) and (7.28), we obtain
\[
\|\tilde{\eta}_r\|_{L_\infty} \leq \|C(s)s\|_{L_1} \| (\psi(t, y) \dot{x}(t))_r\|_{L_\infty}
\]
\[
+ \|C(s)\|_{L_1} \| (\dot{\psi}(t, y) \dot{x}(t) + \psi(t, y) A \dot{x}(t) + \psi(t, y) (f(t, \dot{x}) - f(t, x)))_r\|_{L_\infty}
\]
\[
\leq \|C(s)s\|_{L_1} \tilde{\eta} \gamma_{\tilde{x}} \|f\|_{\gamma_{\tilde{x}}^\max} + d_{\psi} \gamma_{\tilde{x}} \|\tilde{\eta}(\tilde{\rho})\|_{\gamma_{\tilde{x}}^\max} \|A\|_{\gamma_{\tilde{x}}^\max} + \tilde{\psi}(\tilde{h}(\tilde{\rho})) \rho \gamma_{\tilde{x}}^\max.
\]
Notice that
\[
\|C(s)s\|_{L_1} = \omega \left\| \frac{s}{s + \omega} \right\|_{L_1} \leq 2\omega,
\]
which leads to
\[
\|\tilde{\eta}_r\|_{L_\infty} \leq \left(2\omega \tilde{\psi}(\tilde{h}(\tilde{\rho})) + d_{\psi} + \tilde{\psi}(\tilde{h}(\tilde{\rho}))\|A\| + \tilde{\psi}(\tilde{h}(\tilde{\rho})) \rho \gamma_{\tilde{x}}^\max \right) \gamma_{\tilde{x}} \gamma_{\tilde{x}}^\max,
\]
and completes the proof.

Next we derive the error dynamics between the $L_1$ reference system and the closed-loop $L_1$ adaptive system. We start by rewriting (7.31) as
\[
u(s) = -C(s) \sigma(s) - C(s) \tilde{\sigma}(s) = -C(s) \sigma(s) - \tilde{\eta}(s).
\]
(7.50)
Subtracting this from (7.33), we obtain
\[
e_u(s) \triangleq u_{ref}(s) - u(s) = -C(s) e_{\sigma}(s) + \tilde{\eta}(s),
\]
(7.51)
where $e_{\sigma}(s) \triangleq \sigma_{ref}(s) - \sigma(s)$. Next we denote
\[
u(t) \triangleq u(t) + d(t, x(t)) + \tilde{\eta}(t),
\]
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which using (7.50), can be rewritten as

$$\nu(s) = (1 - C(s))\sigma(s).$$

Subtracting (7.1) from (7.32), and taking into account (7.40), we obtain

$$\dot{e}_x(t) \triangleq \dot{x}_{\text{ref}}(t) - \dot{x}(t) = Ae_x(t) + f(t, x_{\text{ref}}) - f(t, x) + g(t, y_{\text{ref}})\nu_{\text{ref}}(t) - g(t, y)(\nu(t) - \tilde{\eta}(t)), \quad e_x(0) = 0,$$

which can be further represented as

$$\dot{e}_x(t) = Ae_x(t) + f(t, x_{\text{ref}}) - f(t, x) + (g(t, y_{\text{ref}}) - g(t, y))\nu_{\text{ref}}(t) + g(t, y)(\nu_{\text{ref}}(t) - \nu(t)) + g(t, y)\tilde{\eta}(t).$$

Denoting

$$e_{\nu}(s) \triangleq \nu_{\text{ref}}(s) - \nu(s) = (1 - C(s))e_{\sigma}(s),$$

we obtain

$$\dot{e}_x(t) = Ae_x(t) + f(t, x_{\text{ref}}) - f(t, x) + (g(t, y_{\text{ref}}) - g(t, y))\nu_{\text{ref}}(t) + g(t, y)e_{\nu}(t) + g(t, y)\tilde{\eta}(t).$$

(7.52)

Next lemma derives the bound on the error dynamics under the assumption of boundedness of the system state. This assumption is further relaxed in the following theorem.

**Lemma 7.5.** Consider the error dynamics in (7.52). If there exists time $\tau \in \mathbb{R}^+$, such that

$$\|x_\tau\|_{L_\infty} \leq \rho,$$

(7.53)

then

$$\|e_{x_\tau}\|_{L_\infty} < \gamma_x,$$

(7.54)

where $\gamma_x$ was defined in (7.23).

**Proof.** Consider the following Lyapunov function

$$V(e_x) = e_x^T Pe_x.$$  

(7.55)
Its derivative along the error dynamics is given by

\[ \dot{V}(t) = -e_x^T(t)Qe_x(t) + 2e_x^T(t)P(f(t, x_{\text{ref}}) - f(t, x)) + 2e_x^T(t)P((g(t, y_{\text{ref}}) - g(t, y))\nu_{\text{ref}}(t) + g(t, y)e_\nu(t) + g(t, y)\tilde{\eta}(t)). \]

Using Assumption 7.2, taking into account Lemma 7.1 and (7.53), and using the fact that \( \rho_r < \rho \), we obtain the following bound which leads to

\[ \dot{V}(t) \leq -\lambda_{\text{min}}(Q)\|e_x(t)\|^2 + 2\|e_x(t)\|^2L_\rho\|P\| + 2e_x^T(t)P((g(t, y_{\text{ref}}) - g(t, y))\nu_{\text{ref}}(t) + g(t, y)e_\nu(t) + g(t, y)\tilde{\eta}(t)) \leq -\beta_e\|e_x(t)\|^2 + 2e_x^T(t)P((g(t, y_{\text{ref}}) - g(t, y))\nu_{\text{ref}}(t) + g(t, y)e_\nu(t) + g(t, y)\tilde{\eta}(t)), \]

where \( \beta_e \) was defined in (7.21).

Next using the definitions of \( \nu_{\text{ref}} \), we can write

\[ \nu_{\text{ref}}(s) = (1 - C(s))\sigma_{\text{ref}} = \frac{\omega}{\omega s + \omega}s\sigma_{\text{ref}}(s), \]

which along with Lemma 7.2 leads to

\[ \|\nu_{\text{ref}}\|_{L_\infty} \leq \frac{1}{\omega} \left\| \frac{\omega}{s + \omega} \right\|_{L_1} \|\dot{\sigma}_{\text{ref}}\|_{L_\infty} \leq \frac{1}{\omega}d_{\sigma_{\text{ref}}}. \]

Similarly, we obtain

\[ \|e_\nu\|_{L_\infty} \leq \frac{1}{\omega}d_{e_\sigma}. \]

Next we use a contradiction argument. Towards this end, we assume that (7.54) does not hold. Since \( e_x(0) = 0 \) and \( e_x(t) \) is continuous, there exists a time \( \tau \in \mathbb{R}^+ \) such that

\[ \|e_x(\tau)\| = \gamma_x, \quad (7.57) \]

and

\[ \|e_x(t)\| < \gamma_x, \quad \forall t \in [0, \tau). \]
This allows to rewrite (7.56) as

\[ \dot{V}(t) \leq -\beta_v \|e_x(t)\|^2 + 2\gamma_x P \| (L^g(h(\rho)))L^h(\rho) \| \nu_{\text{ref}}(t) + \bar{g}(\bar{h}(\rho)) |e_u(t)| + \bar{g}(\bar{h}(\rho)) \| \bar{\eta}(t) \| \]

\[ \leq -\beta_v \|e_x(t)\|^2 + 2\gamma_x P \| (L^g(h(\rho)))L^h(\rho) \| \frac{1}{\omega} d_{\nu_{\text{ref}}} + \bar{g}(\bar{h}(\rho)) \frac{1}{\omega} d_{e_a} + \bar{g}(\bar{h}(\rho)) \| \bar{\eta}(t) \| \]

\[ \leq -\beta_v \|e_x(t)\|^2 + 2\gamma_x P \| (L^g(h(\rho)))L^h(\rho) \| \frac{1}{\omega} d_{\nu_{\text{ref}}} + \bar{g}(\bar{h}(\rho)) \frac{1}{\omega} d_{e_a} + \bar{g}(\bar{h}(\rho)) \| \bar{\eta}(t) \| \gamma_\eta \gamma_\bar{\eta} \max, \]

where we have also used Assumption 7.2 and Lemma 7.4. From (7.55), we obtain

\[ \|e_x(t)\|^2 \leq \frac{V(t)}{\lambda_{\text{min}}(P)}, \]

which implies

\[ \dot{V}(t) \leq -\frac{\beta_v}{\lambda_{\text{min}}(P)} V(t) + 2\gamma_x P \| (L^g(h(\rho)))L^h(\rho) \| \frac{1}{\omega} d_{\nu_{\text{ref}}} + \bar{g}(\bar{h}(\rho)) \frac{1}{\omega} d_{e_a} + \bar{g}(\bar{h}(\rho)) \| \bar{\eta}(t) \| \gamma_\eta \gamma_\bar{\eta} \max \]

\[ \leq -\alpha_e V(t) + 2\gamma_x P \| (L^g(h(\rho)))L^h(\rho) \| \frac{1}{\omega} d_{\nu_{\text{ref}}} + \bar{g}(\bar{h}(\rho)) \frac{1}{\omega} d_{e_a} + \bar{g}(\bar{h}(\rho)) \| \bar{\eta}(t) \| \gamma_\eta \gamma_\bar{\eta} \max, \]

where \( \alpha_e \) was defined in (7.22). Since \( V(0) = 0 \), and \( \dot{V}(t) < 0 \), if

\[ V(t) \geq \frac{2\gamma_x P \| (L^g(h(\rho)))L^h(\rho) \| \frac{1}{\omega} d_{\nu_{\text{ref}}} + \bar{g}(\bar{h}(\rho)) \frac{1}{\omega} d_{e_a} + \bar{g}(\bar{h}(\rho)) \| \bar{\eta}(t) \| \gamma_\eta \gamma_\bar{\eta} \max}{\alpha_e \lambda_{\text{max}}(P)}, \]

we obtain

\[ \|e_x\|_{\infty} \leq \sqrt{\frac{V(t)}{\lambda_{\text{max}}(P)}} \leq \sqrt{\frac{2\gamma_x P \| (L^g(h(\rho)))L^h(\rho) \| \frac{1}{\omega} d_{\nu_{\text{ref}}} + \bar{g}(\bar{h}(\rho)) \frac{1}{\omega} d_{e_a} + \bar{g}(\bar{h}(\rho)) \| \bar{\eta}(t) \| \gamma_\eta \gamma_\bar{\eta} \max}{\alpha_e \lambda_{\text{max}}(P)}} \]

\[ = \bar{\gamma}_x < \gamma_x, \]

which contradicts (7.57) and completes the proof. \( \square \)

The next theorem proves the stability of the closed-loop adaptive system and derives the performance bounds between the \( \mathcal{L}_1 \) reference system and the closed-loop adaptive system.

**Theorem 7.1.** Consider the system (7.1) and the \( \mathcal{L}_1 \) adaptive controller (7.29)-(7.31). If Assumptions 7.1-7.5 and all conditions in Section 7.2.1 hold, then

\[ \|e_x\|_{\infty} \leq \gamma_x, \quad (7.58) \]

\[ \|e_u\|_{\infty} \leq \gamma_u, \]
where $\gamma_x$ and $\gamma_u$ are defined in (7.23) and (7.25).

**Proof.** For this proof we use a contradiction argument. Towards this end we assume that $\|x(t)\|$ is unbounded. Since $\|x_0\| < \rho$ and $x(t)$ is continuous, there exists time $\tau \in \mathbb{R}^+$, such that

$$\|x(\tau)\| = \rho,$$  

(7.59)

and

$$\|x(t)\| < \rho, \quad \forall t \in [0, \tau).$$

Application of Lemma 7.5 leads to

$$\|e_x\|_{L_\infty} < \gamma_x.$$

Next we notice that

$$\|x_\tau\|_{L_\infty} \leq \|x_{ref}\|_{L_\infty} + \|e_x\|_{L_\infty} < \|x_{ref}\|_{L_\infty} + \gamma_x,$$

which along with Lemma 7.1 leads to

$$\|x_\tau\|_{L_\infty} < \rho_r + \gamma_x = \rho.$$  

This fact contradicts the assumption in (7.59) and hence proves the bound in (7.58).

To prove the second bound, we use (7.51) to obtain

$$\|e_u\|_{L_\infty} \leq \|C(s)\|_{L_1} \|e_\sigma\|_{L_\infty} + \|\tilde{\eta}\|_{L_\infty}.$$

Taking into account Lemma 7.4 and the fact $\|C(s)\|_{L_1} = 1$, we can rewrite this as

$$\|e_u\|_{L_\infty} \leq d(\rho_{ref}) + d(\rho_{max}) \gamma_u = \gamma_u,$$

which completes the proof. $\square$

### 7.4. Simulation Results

In this section we use a second order illustrative example to verify the theoretical results presented in this chapter. Namely, let us consider the system (7.1) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix},$$

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Figure 7.1: Nonlinear function $g(t, y)$ at time $t = 0$.

and

$$f(t, x) = \begin{bmatrix} x_1^3(t) \\ x_1^2(t) + 3x_2^2(t) \sin(t) + 0.4 \sin(3t)^3 \end{bmatrix},$$

$$g(t, y) = 1 + e^{-0.5|y|} + 0.1 \sin(0.2t).$$

For the system’s output equation we choose $h_x(t, x)$, which satisfy to $y = h(t, x, y)$, with

$$h(t, x, y) = \left(0.5x_1(t) + \frac{10}{14}x_2(t)\right) \left(1 + e^{-0.5|y|} + 0.1 \sin(0.2t)\right).$$

To illustrate our choice of system parameters and give an idea of the type of nonlinearities we are using in this example, we present the plot of $g(t, y)$ for the time slice $t = 0$ in Figure 7.1 and simulation results for the ideal system (7.5) in Figure 7.2. We notice that the ideal system output shows significant influence of the nonlinearities (which in this case represent the desired behavior of the system).

Next, we notice that our choice of $A$ is Hurwitz and by letting $Q = I$, we obtain

$$P = \begin{bmatrix} 396 & 1/2 \\ 280 & 1/2 \\ 1 & 1/2 \end{bmatrix}.$$ 

It is straightforward to check that our choice satisfies Assumption 7.5. For simulations we
use three scenarios of different uncertainty/disturbances given by

\[
\begin{align*}
  d_1(t,x) &= x_1(t) + x_2^2(t) + 0.1 \sin(0.5t), \\
  d_2(t,x) &= -0.3 \sin(x_2(t)) + 2e^{-x_1^2(t)} + 0.1 \cos(0.5t), \\
  d_3(t,x) &= x_1(t) + x_2^2(t) + 0.1 \sin(20t).
\end{align*}
\]

For the \( L_1 \) adaptive controller we choose the lowpass filter

\[
C(s) = \left( \frac{30}{s+30} \right) \left( \frac{50}{s+50} \right) \left( \frac{100}{s+100} \right),
\]

the adaptation gain \( \Gamma = 100\,000 \), and the projection bound \( \Delta = 1000 \). Notice, that while theoretical results above are obtained for first order lowpass filters, in our simulations we use higher order filters to show that the presented \( L_1 \) adaptive controller is not limited only by first order filters. Moreover, we note that higher order filters provide larger number of tuning parameters and therefore better flexibility of the design.

We start our simulation study by considering a closed-loop \( L_1 \) adaptive control system with the nominal plant, that is without uncertainty \( d(t, y) \equiv 0 \). Simulation results for this case are given in Figure 7.3. We notice that \( L_1 \) adaptive controller generates almost zero control signal during all operation. This is consistent with the fact that the plant in this case has no uncertainty and therefore requires no compensation.

Next we consider the behavior of the system in the presence of uncertainty. Figure 7.4 shows the output response of the plant in open loop, that is without \( L_1 \) adaptive controller. As we can see, the system is unstable for all three scenarios of uncertainty \( d(t, x) \). Figures 7.5-7.7 show simulation results for the closed-loop \( L_1 \) adaptive control system for the same scenarios. The \( L_1 \) adaptive controller has recovered stability of the closed-loop system for all
scenarios. As we can see from Figure 7.7, $L_1$ adaptive controller has successfully compensated for most of the uncertainty $d_1(t, x)$ and achieved performance similar to the response of the ideal system. For scenario 2 we have chosen $d(t, x)$ with large initial value, which creates large initial error; however as we can see from Figure 7.6, $L_1$ adaptive controller quickly generates control input, which cancels the undesired bias in the plant dynamics. For the third scenario, we chose disturbance $d_3(t, x)$ with high-frequency component. Since $L_1$ adaptive controller aims for compensation of uncertainty in low-frequency range given via the lowpass filter $C(s)$, we can observe propagation of the high-frequency content of the disturbance to the closed-loop system output in Figure 7.7.

Next, we test performance of the $L_1$ adaptive controller in the presence of nonzero initialization error. For that we set the initial conditions of the plant to $x_0 = [-0.2 - 0.3]^\top$, while keeping zero initial conditions for the output predictor. Simulation results for nominal plant are given in Figure 7.8. As we can see nonzero initial conditions lead to initial output transient error, which quickly decays with time. Notice that the initialization error also
(a) System output.  
(b) Control history.

**Figure 7.5**: Closed-loop $\mathcal{L}_1$ adaptive control system for scenario 1.

(a) System output.  
(b) Control history.

**Figure 7.6**: Closed-loop $\mathcal{L}_1$ adaptive control system for scenario 2.

(a) System output.  
(b) Control history.

**Figure 7.7**: Closed-loop $\mathcal{L}_1$ adaptive control system for scenario 3.
Figure 7.8: $L_1$ adaptive controller in the presence of nonzero initialization error.

Figure 7.9: State response for the plant and output predictor in the presence of nonzero initial conditions.

causes a spike in the beginning of the control signal history. This behavior is similar to peaking phenomenon [77], which can be addressed by application of control signal saturation or by using control switching techniques similar to Chapter 4. Figure 7.9 shows the states response for the plant and the output predictor, which demonstrate a behavior consistent with the theoretical results. Namely, we observe convergence of the states of the output predictor to a small neighborhood of the states of the plant. This fact also suggests that the output predictor combines two roles: parameter estimator and state observer.

Finally, we test robustness of the closed-loop $L_1$ controller to input time delays. For that purpose, we add a time delay $\tau = 70$ ms at the plant input. The simulation results in Figure 7.10 show that the closed-loop system remains stable in the presence of time delay, and large adaptation gains $\Gamma$ do not hurt the robustness of the closed-loop system.
Figure 7.10: $\mathcal{L}_1$ adaptive controller in the presence of input time delay $\tau = 70$ ms.
CHAPTER 8
Conclusions and Future Research

8.1. Conclusions

This dissertation focuses on application and development of $\mathcal{L}_1$ adaptive output-feedback control methods. Prior results in this area were limited to two $\mathcal{L}_1$ adaptive controllers for LTI systems [3, 40, 41]. The first architecture [40] is based on gradient descent adaptation laws and its stability analysis relies on SPR property of the reference system. The second architecture [41] relaxes the SPR assumption and allows implementation of higher order reference system by means of a novel piecewise constant adaptation law. In Chapter 2 of this dissertation we review the architecture from [41] and use time domain analysis of a scalar example to illustrate some of the key properties of the estimation loop with piecewise constant adaptation laws.

In Chapters 3 and 4, we apply the $\mathcal{L}_1$ adaptive output-feedback control architecture with piecewise constant adaptation laws to the problems in two different areas: ascent control system of NASA CLV and control of drug delivery during human anesthesia. For the first application we design an $\mathcal{L}_1$ adaptive controller as an augmentation of baseline ascent control system for the whole flight envelope. We show that a single $\mathcal{L}_1$ adaptive controller, designed without controller parameter scheduling, is able to satisfy all given performance specifications despite significant changes in the inertia and aerodynamics of the CLV. Moreover, since CLV has tall and slender body, its dynamics features severe flexible modes, which change with time. The $\mathcal{L}_1$ controller’s architecture with filter in the control loop allows to take into account the frequency range where the flexible modes reside and provides a straightforward tuning procedure to address the problem of flexible dynamics of the vehicle.

$\mathcal{L}_1$ adaptive controller for drug delivery during human anesthesia is designed to automatically maintain prespecified BIS profile during the surgery, which is one of the tasks of the anesthesiologist during surgery. The simulation study is based on models obtained from clinical trial data from 6 volunteers. The results show robustness of the adaptive controller to model parameter variations and adequate disturbance attenuation. Observed consumption of isoflurane is comparable to measured values during clinical trials. Since the patient models are valid only for sedated state of the patient, the $\mathcal{L}_1$ adaptive controller is designed to be enabled by anesthesiologist after the patient reaches initial sedated state by manual introduction of isoflurane. For this purpose we also develop a switching mechanism for
The $\mathcal{L}_1$ adaptive control law to eliminate possible undesired initialization transients.

Next we notice that the $\mathcal{L}_1$ adaptive controller in Chapter 3 satisfies given transient specifications, which do not change with time. However, the CLV significantly changes its inertia parameters with time as fuel is consumed. This fact suggests changing the transient specifications to make the CLV more agile as the mass of the vehicle reduces. Therefore in Chapter 5, we extend the $\mathcal{L}_1$ adaptive controller to accommodate LTV reference systems, which can represent time-varying control specifications.

Our second extension of the $\mathcal{L}_1$ adaptive controller, presented in Chapter 6, is inspired by the fact that the $\mathcal{L}_1$-norm stability condition used in Chapter 2 involves a complicated feedback structure with lowpass filter. In order to address the design complexity of the $\mathcal{L}_1$ adaptive controller for non-SPR systems, we consider another alternative approach proposed by Monopoli [51]. In this case we obtain an $\mathcal{L}_1$-norm condition with separate terms of simpler structure, which give better intuition and admit application of robust control design methods.

Finally, Chapter 7 presents an $\mathcal{L}_1$ adaptive controller for a class of nonlinear output-feedback systems with implicit output equation. For this extension we use gradient minimization adaptive laws, which were previously used in $\mathcal{L}_1$ adaptive output-feedback controller for SPR systems [40]. The SPR assumption for linear systems in [40] is replaced by a more general passivity type assumption. In this work we obtain stability conditions and performance bounds, which are consistent with all previous $\mathcal{L}_1$ adaptive control architectures. Namely, in the case of zero initial conditions, the stability of the closed-loop system can be granted by selecting sufficiently large bandwidth of the lowpass filter, and the performance bounds can be arbitrarily improved by increasing the adaptation gain. Nonzero initial conditions lead to performance bounds, which are depended on the bound on the initial conditions. We also show that the bound on the prediction error in the presence of nonzero initial conditions consists of two terms. The first term asymptotically decreases with time to zero; and the second term can be arbitrarily reduced by increasing the adaptation gain. This structure of the bound is identical to the one computed for LTI systems in Section 2.2.4 of [3].

8.2. Future Research

Future research will focus on $\mathcal{L}_1$ adaptive controller analysis presented in Chapter 7 with an objective to reduce the conservatism in assumptions, as well as on further extension of the problem statement to a more general class of nonlinear systems. The main two sources of limitations for the developed $\mathcal{L}_1$ adaptive controller reside in the class of applicable
systems with implicit structure of the system output equation and passivity requirement in Assumption 7.5. The need for passivity assumption can be avoided by developing alternative form of the adaptation law. For instance, in the case of $L_1$ adaptive controller for LTI systems, the SPR assumption was relaxed by using piecewise constant adaptation laws. This adaptation law is based on the desired system inversion, as we demonstrated in Chapter 5, which makes it challenging for nonlinear reference systems, since there are no straightforward methods for computation of the system inverse in this case.

On the other hand, implicit form of the system output was necessary to decouple the parameter estimation process from the state estimation. To explain what we mean here, we notice that in the $L_1$ adaptive controller for nonlinear systems the state predictor plays simultaneously a role of parameter estimator and a state observer. Ability of the estimation loop to achieve small state estimation error plays critical role for stability. The implicit output equation in (7.1) allows to apply Assumption 7.5 to the derivative of Lyapunov function (7.46), avoiding the nonlinear error terms due to the mismatch between the predictor and the plant states, which subsequently helps to render the stability analysis of the adaptive parameter estimation independent from the state estimation errors. Therefore changing the form of the adaptation law as well as investigating alternative architectures incorporating state observer can help to address this issue and extend the results to significantly wider class of systems in the future work.

Next in our future research, we plan to extend the set of clinical trial data for a larger number of volunteers to further improve the modeling and the system ID algorithms, as well as to better validate the $L_1$ adaptive controller design presented in Chapter 4.
BIBLIOGRAPHY


