SELF IMPROVING ORLICZ-POINCARÉ INEQUALITIES

BY

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DISSERTATION

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Abstract

In [20], Keith and Zhong prove that spaces admitting Poincaré inequalities also admit a priori stronger Poincaré inequalities. We use their technique, with slight adjustments, to obtain a similar result in the case of Orlicz-Poincaré inequalities. We give examples in the plane that show all hypotheses are required and develop the theory of Orlicz-Poincaré inequalities for nondoubling Young functions to show that the ∞-Poincaré inequality does not improve to any Orlicz-Poincaré inequality.

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1 Introduction

We say the metric measure space (X, d, μ) admits a p-Poincaré inequality, $p \geq 1$, with constants $C \geq 1$ and $0 < \tau \leq 1$, if the following holds: Every ball contained in X has measure in $(0, \infty)$, and we have

$$\fint_{\tau B} |u - u_{\tau B}|\, d\mu \leq C(\text{diam } B) \left(\fint_B (\text{Lip } u)^p d\mu \right)^{\frac{1}{p}}, \tag{1.1}$$

for all balls $B \subset X$, and for every Lipschitz function $u : X \to \mathbb{R}$, with τB the dilation of the ball, $u_{\tau B}$ the average value on τB, and

$$(\text{Lip } u)(x) = \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x, y)}.$$

The geometric properties of Poincaré inequalities have played a key role in the development of analysis on metric spaces. Heinonen and Koskela ([13]) showed that spaces admitting Poincaré inequalities behave like Euclidean space for quasiconformal and quasisymmetric maps. Furthermore, Cheeger's generalization of Rademacher's theorem about the almost everywhere differentiability of Lipschitz functions requires that the space admits a Poincaré inequality ([2]).

We say X admits a (q, p)-Poincaré inequality when the L^1-norm on the left hand side of the p-Poincaré inequality is replaced by the L^q-norm. Hölder's inequality and the definition of Poincaré inequalities gives that (q_1, p_1)-Poincaré inequalities imply (q_2, p_2)-Poincaré inequalities for $q_2 < q_1$ and $p_1 < p_2$. The question of whether spaces admitting Poincaré inequalities may also admit a priori stronger Poincaré inequalities, a so-called self improvement result, has been studied in great detail. Hajlasz and Koskela ([8]) showed that under some mild restrictions $(1, p)$-Poincaré inequalities imply (q, p)-Poincaré inequalities for certain $q > 1$. Heikkinen ([9]) gave the analogous result for Orlicz-Poincaré inequalities. In [20, Theorem 1.0.1], Keith and Zhong show

Theorem 1.1 ([20, Theorem 1.0.1]). *Let $p > 1$ and let (X, d, μ) be a complete metric measure space with μ Borel and doubling, that admits a p-Poincaré inequality. Then there exists $\epsilon > 0$ such that (X, d, μ) admits a q-Poincaré inequality for every*

$q > p - \epsilon$, quantitatively.

This result answered a number of important open questions even in Euclidean space. For example, an open question in the theory of weighted Poincaré inequalities and Sobolev spaces [12] was whether or not p-admissable weights had open ended behavior. Theorem 1.1 answers that question in the affirmative. Furthermore, various definitions that generalize the first order Sobolev space $W^{1,p}$ to complete doubling metric measure spaces were known to be equivalent under the assumption of a slightly stronger Poincaré inequality (namely, a q-Poincaré inequality for $q < p$). Theorem 1.1 removes the need for the additional hypothesis, and all standard definitions for $W^{1,p}$ coincide on complete doubling metric measure spaces that admit a p-Poincaré inequality.

Self-improving results have a long history in analysis. For example, the Hardy-Littlewood Maximal operator of a given function f is in $L^1(B)$ if and only if f is in $L \log L(B)$ ([26]). Self-improving results appear in geometric analysis, for example, in the classical Sobolev inequalities and Gehring's reverse Hölder inequalities ([7]).

There are many generalizations of Poincaré inequalities on general metric spaces with no a priori differentiable structure ([11]). Poincaré inequalities have also been generalized to include Orlicz functions. The $(1, \Psi)$-Orlicz-Poincaré inequality, which we simply call a Ψ-Poincaré inequality, is essentially the classical Poincaré inequality with a general convex function replacing the power function related to the parameter p. The full definition is presented in Chapter 3. See [27] for a more detailed development of Orlicz-Poincaré inequalities. These inequalities will be the main focus of this work.

The original motivation is the following question.

Question 1.2. *Suppose that (X, d, μ) is a complete metric measure space with μ Borel and doubling which admits an ∞-Poincaré inequality. Does (X, d, μ) admit any other Orlicz-Poincaré inequality?*

This question can be interpreted two ways. The stronger formulation is to ask for improvement to a universal Orlicz-Poincaré inequality for all X. The weaker formulation is to ask if X admits any Orlicz-Poincaré inequality, where the Orlicz function may depend on the space X. Hölder's inequality gives that ∞-Poincaré inequality is the "weakest" Poincaré inequality. Durand-Cartagena, Shanmugalingam, and Williams show that the ∞-Poincaré inequality does not improve to a p-Poincaré inequality for any finite p ([6]). Nondoubling Orlicz functions, in some sense, interpolate between these two cases. We will see in Chapter 6 that the answer to either formulation of Question 1.2 is no.

Before exploring the above question, we will use the technique developed for Theorem 1.1 to study Orlicz-Poincaré inequalities for particular doubling Young functions. The argument provided by Keith and Zhong is very intricate with interdependent parameters. In some sense, Theorem 1.3 can be thought of both as a proof that Orlicz-Poincaré inequalities do improve and as a demonstration of the utility of the proof technique. Specifically, [20] gives that the set of parameters for which X admits a Poincaré inequality is open. Theorem 1.3 shows that the hypotheses can be relaxed without changing the conclusion. We focus on doubling convex functions, as they can be described by their leading order algebraic behavior. To each doubling Orlicz function, Ψ, we will associate a parameter p_Ψ, the algebraic decomposition exponent, so that $\Psi(t) = t^{p_\Psi}\phi(t)$ with strict growth conditions on ϕ (expanded on in Section 3). The critical examples will be functions of the type $\Gamma_{p,q}(t) = t^p \log^q(1+t)$ for $p > 1$, $q \in \mathbb{R}$. Theorem 1.3 shows this family of functions form, in some sense, the upper bound on the Orlicz-Poincaré inequalities that exhibit the same improvement as in the classical case.

Theorem 1.3. *Let $p > 1$ and let (X, d, μ) be a complete metric measure space with μ Borel and doubling. Suppose X admits a Ψ-Poincaré inequality for some Ψ so that*

$$\lim_{t\to\infty} \frac{\Psi(t)}{\Gamma_{p_\Psi, p_\Psi - 1}(t)} = 0. \tag{1.2}$$

Then there exists $\epsilon > 0$ such that X admits a $(p_\Psi - \epsilon)$-Poincaré inequality whose constants depend only on the original parameters.

The proof compares measures of level sets to make statements about the objects in the Poincaré inequality. Without the growth condition (1.2) we cannot make any meaningful inequalities about these level sets. We show in Chapter 5 that this growth condition is required for topological reasons as well. We could conclude the *a priori* weaker statement X admits a Φ-Poincaré Inequality such that $\lim_{t\to\infty} \frac{\Phi(t)}{t^{p_\Psi}} = 0$ and $\lim_{t\to\to} \frac{t^{p-\epsilon}}{\Phi(t)} = 0$. However, this is equivalent to the statement of Theorem 1.3 by Jensen's Inequality (3.8) and Theorem 1.1.

The paper is divided into seven chapters. Chapter 2 contains classical material on Poincaré inequalities and analysis on metric spaces. Standard results about Orlicz functions and Orlicz-Poincaré inequalities in the doubling case are found in Chapter 3. Section 4.1 contains standard notation and the technical lemmas used in the proof of Theorem 1.3 found in Section 4.2.

The necessity of the growth condition (1.2) and completeness are explored in

Chapter 5 where we will prove the following theorem.

Theorem 1.4. *There exists a compact, planar set equipped with the Euclidean metric and the Lebesgue measure that admits a $\Gamma_{2,r}$-Poincaré inequality for $r > \frac{1+\sqrt{5}}{2}$, but does not admit a $\Gamma_{2,q}$-Poincaré inequality for any $q < 1$.*

Theorem 1.3 and Theorem 1.4 show that the log scale of the family $\Gamma_{p,r}$ is the correct scale to use for investigating self-improvement of Orlicz-Poincaré inequalities. The question of what value for r is best possible will be discussed in further detail in Chapter 7.

Chapter 6 extends the results in the doubling case to general Orlicz functions. We finish with questions about the hypotheses and extensions of Theorem 1.3 in Chapter 7.

2 Analysis on Metric Spaces

In this section we discuss the equivalence of Poincaré inequalities, Gromov-Hausdorff convergence, and Lorentz spaces. These topics are related to the various questions that arise from the self-improvement of Orlicz-Poincaré inequalities. The relationship to Lorentz space and applications of stability under Gromov-Hausdorff limits will be discussed in further detail in chapter 6.

For the remaining $X = (X, d, \mu)$ will be a metric measure space with μ a Borel and doubling measure that satisfies $0 < \mu(B) < \infty$ for all balls $B \subset X$. Recall that a measure μ is *doubling* if there are constants $r, C_d > 0$ such that $\mu(B(x, 2r)) \leq C_d\mu(B(x, r))$ for every $x \in X$. Here $B(x, r) = \{y \in X : d(y, x) < r\}$ denotes the open ball in X with center $x \in X$ and radius $0 < r < \infty$. The corresponding closed ball will be denoted $\overline{B(x, r)} = \{y \in X : d(y, x) \leq r\}$. When we use tB to denote the dilation of a ball, there is an implied center and radius. This is important, as in a general metric space a ball could be described with two different choces for center which could give two different sets when dilated. We say two functions are comparable, and denote it $f \approx g$, if there exists a constant $C > 0$ so that the double inequality $\frac{1}{C}g(t) \leq f(t) \leq Cg(t)$ is satisfied. We will denote the average integral $\frac{1}{\mu(A)}\int_A u d\mu = \fint_A u d\mu = u_A$ for every $A \subset X$ such that $\mu(A) > 0$ and measurable function $u : X \to [-\infty, \infty]$.

We say that a property holds quantitatively when the constants and parameters in the claim depend only on those constants stated (or implicit) in the hypotheses.

2.1 Equivalence of Poincaré inequalities

As in the case of Sobolev spaces, Poincaré inequalities on metric spaces without the classical gradient can be abstracted in different ways. One possible definition is that of Keith and Zhong found in the introduction. It is also possible to formulate the Poincaré inequality using upper gradients and continuous (measurable) functions as in [13]. First, we will recall the definition of an upper gradient. Next, we will include the relevant material that relates the two definitions of Poincaré inequalities.

Definition 2.1 (Upper gradient)**.** Given a real-valued function u in a metric space

X, a Borel function $\rho : X \to [0, \infty]$ is said to be an *upper gradient of* u if

$$|u(x) - u(y)| \leq \int_\gamma \rho ds$$

for each rectifiable curve γ joining x and y in X.

It should be noted that every function has an upper gradient, namely $\rho \equiv \infty$. In a similar fashion, every L-Lipschitz function has $\rho \equiv L$ and Lip $u = \limsup_{y \to x} \dfrac{|u(x) - u(y)|}{d(x, y)}$ as upper gradients ([2]). We now recall the definitions of quasiconvex, geodesic, and length space.

Definition 2.2 (Quasiconvex)**.** A metric space X is C*-quasiconvex* for $C \geq 1$, if each pair of points $x, y \in X$ can be joined by a rectifiable curve γ in X such that

$$\text{length}(\gamma) \leq C d_X(x, y).$$

Definition 2.3 (Geodesic)**.** A metric space X is *geodesic* if every pair of points $x, y \in X$ can be joined by a curve γ in X such that

$$\text{length}(\gamma) = d_X(x, y).$$

Definition 2.4 (Length Space)**.** A metric space X is a *length space* if for every pair of points $x, y \in X$

$$\inf_\gamma \text{length}(\gamma) = d_X(x, y).$$

In Chapter 6 we will see that any complete metric measure space that admits an Orlicz-Poincaré inequality is quasiconvex. From the definitions we can see that a geodesic space is also a length space. It is also known that any proper length space is also geodesic ([15, Lemma 7.3.12]). See [15, Section 7.3] for more discussion on the relation between quasiconvexity and Poincaré inequalities.

Definition 2.5 (p-Poincaré inequality for Upper Gradients)**.** We say that X admits a *p-Poincaré inequality for upper gradients* if there are constants $\lambda \geq 1$ and $C \geq 1$ so that

$$⨍_B |u - u_B| \, d\mu \leq C(\text{diam } B) \left(⨍_{\lambda B} \rho^p d\mu \right)^{\frac{1}{p}} \tag{2.1}$$

for all balls B in X, for all bounded continuous functions u on B, and for all upper gradients ρ of u.

The above definition is also known as a weak $(1,p)$-Poincaré inequality due to the dilation of the ball in the integral on the right hand side of the Poincaré inequality. However, we will use the term weak in Chapter 6 in a different context. A natural question is when are Definitions 1.1 and 2.5 equivalent. This question has been investigated in [2], [14], [18] , and [19]. We present a brief exposition of the results in [14].

The equivalence of the different definitions of Poincaré inequalities utilizes a connection between Poincaré inequalities and capacity. The idea is to show that the satisfaction of a Poincaré inequality with a class of functions (measurable, continuous, Lipschitz) is necessary and sufficient for a inequality relating the measure of subsets of the ball and capacity. This is useful because we can then relate the capacities defined by Lipschitz functions and measurable functions. More precisely, given $\mathfrak{S}$ is either the class of Lipschitz, continuous, or measurable functions defined on measurable sets of X, and define the $(p,\mathfrak{S})$-capacity as

$$\mathrm{cap}_p^{\mathfrak{S}}(E,F;U) = \inf \int_U \rho^p d\mu,$$

where E and F are disjoint closed sets in U, the infimum is taken over all upper gradients ρ of functions u that belong to $\mathfrak{S}$ in U, and $u|_E \geq 1$ and $u|_F \leq 0$.

We now have the two following propositions which will show that if X is a proper and quasiconvex metric space, then Definitions 2.5 and 1.1 are equivalent.

Proposition 2.6 ([14, Proposition 2.1]). *The space X admits a $(1,p)$-Poincaré inequality for the class $\mathfrak{S}$ if and only if there exists constants $C \geq 1$ and $\lambda \geq 1$ such that*

$$\min\{\mu(E),\mu(F)\} \leq c(\mathrm{diam}\ B)^p \mathrm{cap}_p^{\mathfrak{S}}(E,F;\lambda B)$$

whenever E and F are two disjoint compact subsets of a ball B in X.

Proposition 2.7 ([14, Proposition 2.2]). *Assume that X is b-quasiconvex with $b \geq 1$. If E and F are two disjoint, compact subsets of a ball B in X, then*

$$\mathrm{cap}_p^L(E,F;B) \leq \mathrm{cap}_p^M(E,F;4bB),$$

where L and M denote the classes of Lipschitz and measurable functions, respectively.

As Definition 2.5 implies 1.1 we only need to prove that the admission of Poincaré inequalites defined with Lipschitz functions gives the admission of Poincaré inequalities with measurable functions. The equivalence can be immediately deduced from the preceding propositions as Proposition 2.7 allows the the class of Lipschitz functions in Proposition 2.6 be replaced by the class of measureable functions.

2.2 Gromov Hausdorff Convergence

We present the definitions of limits of pointed metric spaces and the stability of Poincaré inequalities under these limits shown by [2] and further developed in [18]. Gromov-Hausdorff distance is a generalization of Hausdorff distance that gives a sense of the distance between two metric spaces, as opposed to the distance between two sets. First, recall the construction of Hausdorff distance.

Definition 2.8 (ϵ- Neighborhood)**.** Let A be a nonempty set in the metric space (X, d), then for $\epsilon > 0$

$$N_\epsilon(A) = \bigcup_{x \in A} B(x, \epsilon).$$

Definition 2.9 (Hausdorff Distance)**.** The Hausdorff distance between two sets $A \subset X$ and $B \subset X$ is the smallest ϵ such that each set is in the ϵ-neighborhood of the other. More precisely,

$$d_H^X(A, B) = \inf \{\epsilon : A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A)\}$$

It should be noted that the Hausdorff distance defines a metric only on the closed and bounded sets of X. If A is any set in X then $d_H^X(A, \bar{A}) = 0$. If we removed the boundedness assumption, then the Hausdorff distance can be ∞.

We can now introduce the notions of distance between and convergence of metric spaces. The key is to view the metric spaces as sets isometrically embedded in some very large space like l^∞, and then find the Hausdorff distance. Such embeddings are guaranteed to exist via the Kuratowski embedding or the Fréchet embedding theorem.

Definition 2.10 (Gromov-Hausdorff Distance)**.** Let (X, d) and $(Y, \hat{d})$ be two separable, compact metric spaces and $i : X \to l^\infty$ and $j : Y \to l^\infty$ be two isometric embeddings. We define the Gromov-Hausdorff distance between X and Y as

$$d_{GH}(X, Y) = \inf_{i,j} d_H^{l^\infty}(i(X), j(Y)).$$

Definition 2.11 (Gromov-Hausdorff Convergence)**.** If $X, X_1, X_2, \ldots$ are metric spaces such that $\lim_{i\to\infty} d_{GH}(X_i, X) = 0$. We say that X_i Gromov-Hausdorff converges to X and write $X_i \stackrel{GH}{\to} X$.

The following alternate version of Gromov-Hausdorff convergence is more useful in noncompact settings.

Definition 2.12 (Pointed Gromov-Hausdorff Convergence)**.** A sequence of pointed separable metric spaces $(X_1, d_1, p_1), (X_2, d_2, p_2), \ldots$ is said to pointed Gromov-Hausdorff converge to a pointed separable metric space (X, d, p) if for each $r > 0$ and $\epsilon > 0$ so that $\epsilon < r$, there exists i_0 such that for $i \geq i_0$ there is a map $f_i^\epsilon : B(p_i, r) \to X$ satisfying:

(i) $f_i^\epsilon(p_i) = p$;

(ii) $|d(f_i^\epsilon(x), f_i^\epsilon(y)) - d_i(x, y)| < \epsilon$ for all $x, y \in B(p_i, r)$;

(iii) $B(p, r - \epsilon) \subset N_\epsilon(f_i^\epsilon(B(p_i, r)))$.

Definition 2.13 (Pointed measured Gromov-Hausdorff convergence)**.** Consider a sequence of compact metric measures spaces $(X_1, d_1, \mu_1), (X_2, d_2, \mu_2), \ldots$. A compact metric space (X, d, μ) is a measured Gromov-Hausdorff limit of (X_i, d_i, μ_i) if there exist isometric embeddings $\iota_i : X_i \to l^\infty$, such that $d_H^\infty(\iota_i(X_i), \iota(X)) \to 0$ and that $(\iota_i)_\sharp \mu_i$ converges weakly to $\iota_\sharp \mu$ as measures on l^∞. We say (X_i, d_i, p_i, μ_i) pointed measured Gromov-Hausdorff converges to a proper metric measure space (X, d, p, μ) if (X_i, p_i) pointed Gromov-Hausdorff converges to (X, p) and

$$(\bar{B}(p_i, r), d_i, \mu_i \lfloor \bar{B}(p_i, r)) \stackrel{GH}{\to} (\bar{B}(p, r), d, \mu \lfloor \bar{B}(p, r))$$

in the measured sense for every $r > 0$.

With the above definition we can present the following important theorem first published in [18]. We present the version from [15] that agrees with the language from the above definitions.

Theorem 2.14 ([15, Chapter 10])**.** *Let (X_i, d_i, p_i, μ_i) be a sequence of complete length spaces which converge in the pointed measured Gromov-Hausdorff sense to a complete space (X, d, p, μ). Let $1 \leq p < \infty$, C_D, $C_p < \infty$ and $\lambda \geq 1$ be fixed. If each of the measures μ_i is doubling with constant C_D and each space (X_i, d_i, μ_i) satisfies the p-Poincaré inequality with constants C_p and λ, then (X, d, μ) also satisfies the p-Poincaré with constants C_p' and λ' depending only on p, C_p, λ, and C_D.*

We will modify the theorem in Chapter 3 in the case of doubling Young functions, but we will postpone the proof until Chapter 6 when the theorem will be formulated in the language of Orlicz-Poincaré inequalities for nondoubling Young functions. The proof in the nondoubling case still follows the classical proof, but requires modifications to the statements to account for the loss of the doubling assumption. This theorem can be used to generate important counterexamples as will be shown in Chapter 6.

2.3 Lorentz Spaces

Lorentz Spaces $L^{p,q}$ are interpolating spaces between weak L^p and L^p in the sense that $L^{p,\infty}$ = weak L^p and $L^{p,p} = L^p$. In this section we present results from [17] and [29] that relate Lorentz spaces and Orlicz gauge functions.

Definition 2.15 (Distribution Function)**.** Let f be a measurable function on X. The distribution function of f, $\omega(\cdot, f)$, is defined as

$$\omega(\alpha, f) = \mu(\{x \in X : |f(x)| > \alpha\}), \qquad \alpha \geq 0.$$

From this distribution function we can define the nonincreasing arrangement of f, f^*, by

$$f^*(t) = \inf\{\alpha > 0 : \omega(\alpha, f) \leq t\}.$$

We can now define the Lorentz space $L^{p,q}$ through this nonincreasing arrangement.

Definition 2.16 (Lorentz space $L^{p,q}(X)$)**.** $L^{p,q}(X)$ is the class of all measurable functions on X for which the norm

$$\|f\|_{L^{p,q}} := \left(\int_0^{\mu(X)} (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

is finite.

A gauge is a non-increasing function $\zeta : (0, \infty) \to [0, \infty)$. As shown in [17], gauges can be used to associate Young functions to Lorentz spaces as seen in the following proposition. In the theory of Lorentz spaces, there is a fundamental difference between the spaces $L^{p,q}$ for $q > 1$ and $L^{p,1}$. For example, the continuous embedding of Sobolev functions in L^∞ depends on the membership of gradients of those functions in $L^{p,1}$. See [3, Example 1.2] and [4, Example 3.4] for the role of $\Gamma_{p,p-1}$ in Orlicz-Sobolev embeddings. In the following, we will use the class of functions $\Gamma_{p,r}$ to generate the gauges that will separate $L^{p,1}$ from $L^{p,q}$, $q > 1$. The growth condition appears as the borderline in that context as well, as seen in the corollary below.

Proposition 2.17 ([17]). *Given a positive function ζ on $(0, \infty)$, let*

$$F_\phi(s) = \begin{cases} s\zeta^{\frac{1}{p}-1}(s), & \text{if } s > 0 \\ 0, & s = 0 \end{cases},$$

then g is in $L^{p,1}(\Omega)$ if and only if there is a positive nonincreasing function $\zeta \in L^{\frac{1}{p}}((0,\infty))$ such that $\int_\Omega F_\zeta(|g|) < \infty$.

Definition 2.18 (Associated). We say a Young function $\Psi(s)$ is *associated* to $L^{p,1}(\Omega)$ if $\Psi = F_\zeta$ for some positive nonincreasing function $\zeta \in L^{\frac{1}{p}}((c,\infty))$ for all $c > 0$.

The following corollary places an explicit restriction on the r parameter necessary for $\Gamma p, r$ to be associated to $L^{p,1}$.

Corollary 2.19. *If $\Gamma_{p,r}$ is associated to $L^{p,1}$, then $r > p - 1$.*

Proof. We would like to associate $\Gamma_{p,r}$ to $L^{p,1}$ which means that

$$s\zeta^{\frac{1}{p}-1}(s) := s^p \log^r(1+s).$$

This gives that

$$\zeta(s) = s^{-p} \log^{\frac{-rp}{p-1}}(1+s).$$

We now calculate

$$\int_c^\infty s^{-1} \log^{\frac{-r}{p-1}}(1+s)ds.$$

We apply the change of variables $s = e^t$ to conclude that the integral converges if and only if $r > p - 1$. □

The results of this section are most useful to motivate the development of Orlicz-Poincaré theory for nondoubling Young functions in Chapter 6. Gromov-Hausdorff convergence is the key idea to showing the non-improvement of the ∞-Poincaré inequality. The detour to Lorentz spaces shows one example of the growth condition (1.2) in other contexts. The growth condition appears again in Chapter 7.

3 Orlicz Functions and Orlicz-Poincaré Inequalities

3.1 Orlicz Functions

In this section we recall the definition of Young functions and N-functions. Standard references are [25] and [27].

Definition 3.1 (Young Function)**.** A function $\Psi : [0,\infty) \to [0,\infty]$ is a *Young Function* if

$$\Psi(t) = \int_0^t \psi(s)ds,$$

for $\psi : [0,\infty) \to [0,\infty]$, and $\psi(0) = 0$, increasing, left continuous and neither identically zero nor identically infinite on $(0,\infty)$.

Note that such Ψ is convex, increasing, left continuous, with $\Psi(0) = 0$, and $\lim_{t\to\infty} \Psi(t) = \infty$.

Definition 3.2 (N-function)**.** A continuous Young function $\Psi : [0,\infty) \to [0,\infty]$ is an *N-function* if all of the following hold:

1) $\Psi(t) = 0$ only if $t = 0$,

2) $\frac{\Psi(t)}{t} \to \infty$ as $t \to \infty$, and

3) $\frac{\Psi(t)}{t} \to 0$ as $t \to 0$.

Definition 3.3 (Generalized Inverse)**.** For a Young function Ψ the *generalized inverse* $\Psi^{-1} : [0,\infty) \to [0,\infty]$ is given by

$$\Psi^{-1}(t) = \inf\{s : \Psi(s) > t\},$$

where the infimum of the empty set is ∞.

The generalized inverse Ψ^{-1} is right continuous and increasing. As before Ψ and Ψ^{-1} always satisfy the double inequalities

$$\Psi(\Psi^{-1}(t)) \le t \le \Psi^{-1}(\Psi(t)) \tag{3.1}$$

for all $t \geq 0$. Note that if Ψ is continuous and strictly increasing, then the generalized inverse coincides with the standard inverse.

Example 3.4. Recall the family of doubling Young functions $\Gamma_{p,r}(t) := t^p \log^r(1+t)$. If $t > c > 0$, then

$$\frac{1}{C} t^{\frac{1}{p}} \log^{\frac{-r}{p}}(1+t) \leq \Gamma_{p,r}^{-1}(t) \leq C t^{\frac{1}{p}} \log^{\frac{-r}{p}}(1+t), \tag{3.2}$$

where $\Gamma_{p,r}^{-1}$ is the generalized inverse of $\Gamma_{p,r}$. This inverse plays a role in proving the satisfaction of Orlicz-Poincaré inequalities in Chapter 5. For t bounded away from 0, it is clear that $\Gamma_{p,r}\left(t^{\frac{1}{p}} \log^{\frac{-r}{p}}(1+t)\right) \geq \frac{1}{p} t \log^{-r}(1+t) \log^r(t) \geq C(c,p,r)t$. For the other direction we have

$$\begin{aligned} \Gamma_{p,r}\left(t^{\frac{1}{p}} \log^{\frac{-r}{p}}(1+t)\right) &\leq C(c,p) t \log^{-r}(t) \log^r\left(t \log^{-r}(t)\right) \\ &\leq C(c,p) t \log^{-r}(t) \log^r(t). \end{aligned}$$

Definition 3.5 (Complementary Function)**.** Given a Young function, Ψ, the *complementary function* $\tilde{\Psi} : [0,\infty) \to [0,\infty]$ is given by

$$\tilde{\Psi}(s) = \sup\{st - \Psi(t) : t \geq 0\}.$$

Complementary functions Ψ and $\tilde{\Psi}$ satisfy the *Young inequality*

$$st \leq \tilde{\Psi}(s) + \Psi(t) \qquad \forall s, t > 0.$$

The most basic example of an N-function is $\Psi(t) = t^p$ for $p > 1$. The complementary function for $\Psi(t) = t^p$ is $\tilde{\Psi}(t) = t^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Another important example that appears in Chapter 5 is the complementary function $\tilde{\Gamma}_{p,r}$.

Example 3.6. We find $\tilde{\Psi}$ directly by finding the max of $st - \Psi(t)$. This max exists as $\Gamma_{p,r}$ is a continuous, convex function which implies that the function $f_s(t) = st - \Psi(t)$ has exactly one critical point. This calculation yields

$$\tilde{\Gamma}_{p,r}(s) = s\left[\Gamma'_{p,r}\right]^{-1}(s) - \Gamma_{p,r}\left(\left[\Gamma'_{p,r}\right]^{-1}(s)\right). \tag{3.3}$$

We use the previous calculations for the inverse to find that

$$\left[\Gamma'_{p,r}\right]^{-1}(s) \approx s^{\frac{1}{p-1}} \log^{\frac{-r}{p-1}}(1+s).$$

Substituting $s^{\frac{1}{p-1}} \log^{\frac{-r}{p-1}}(1+s)$ into (3.3) gives that

$$\tilde{\Gamma_{p,r}}(s) \approx s^q \log^{\frac{-r}{p-1}}(1+s). \tag{3.4}$$

Definition 3.7 (Doubling)**.** We call a Young function *doubling* if there is a constant C_2 such that

$$\Psi(2t) \leq C_2 \Psi(t), \tag{3.5}$$

for each $t \geq 0$. The smallest C_2 is the *doubling constant*. We say a Young function Ψ is *eventually doubling* if $\exists\, t_0 > 0$ such that (3.5) holds for all $t > t_0$.

In the theory of Orlicz functions (3.5) is also known as the Δ_2-condition (globally), see [25] and [27]. The basic example of a doubling function is at^p for $a > 0, p \geq 1$.

Proposition 3.8. *If $\Psi(t)$ is an N-function, then $\Psi^{-1}(t)$ is doubling.*

Proof. Let $s < t$. Since Ψ is convex and $\Psi(0) = 0$, we have $\Psi(s) = \Psi(\frac{s}{t}t + (1 - \frac{s}{t})0) \leq s\frac{\Psi(t)}{t}$. Thus, $\frac{\Psi(t)}{t}$ is increasing. Similarly, $\Psi(\frac{s}{t}\Psi^{-1}(t)) \leq s$ which gives $\frac{\Psi^{-1}(t)}{t} \leq \frac{\Psi^{-1}(s)}{s}$. Since $\frac{\Psi^{-1}(t)}{t}$ is decreasing we have

$$\Psi^{-1}(s+t) \leq \frac{s+t}{t}\Psi^{-1}(t) \Rightarrow \Psi^{-1}(s+t) \leq \Psi^{-1}(t) + s\frac{\Psi^{-1}(t)}{t} \leq \Psi^{-1}(t) + \Psi^{-1}(s).$$

We conclude that Ψ^{-1} is doubling with constant 2 by setting $s = t$. □

The proof gives a stronger condition than doubling, but doubling is all that is necessary in practice.

Definition 3.9 (Dominated)**.** We say a function Ψ is dominated by a polynomial if $\exists\, t_0 > 0, C > 0$ such that $\Psi(t) \leq Ct^\alpha$ for all $t > t_0$.

Proposition 3.10. *If $\Psi(t)$ is a Young function that is eventually doubling, then its derivative $\Psi'(t)$ is dominated by a polynomial.*

Proof. There exists t_0 so that $\Psi(2t) = \int_t^{2t} \Psi'(x)dx + \Psi(t) \geq t\Psi'(t) + \Psi(t)$ for all $t > t_0$. The eventually doubling condition gives $\Psi(2^k) < C(t_0, C_2)\left(2^k\right)^\alpha$ with $\alpha = \log_2 C$. Thus, there exists an $\alpha > 1$ such that $\Psi(t) \leq C(t_0, C_2)t^\alpha$ for all $t > t_0$. We apply the eventually doubling condition again to conclude $t\Psi'(t) \leq (C-1)\Psi(t) \leq C_1 t^\alpha$. Since $\alpha > 1$ we have the desired result, $\Psi'(t) \leq C_1 t^{\alpha-1}$. □

Corollary 3.11. *If $\Psi(t)$ is eventually doubling, then $\Psi(t)$ is dominated by a polynomial.*

An example of Krasnosel'skii and Rutickii ([24,25, Chapter 2.3]) shows that the converse of Corollary 3.11 is not true. Define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\varphi(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ (n-1)! & \text{if } (n-1)! \leq t \leq n!, n \geq 2 \end{cases}$$

Let Φ be the indefinite integral of φ, then $\Phi(t) \leq \frac{t^2}{2}$ for $t \geq 0$, but $\Phi(2n!) > n\Phi(n!), n \geq 1$.

Definition 3.12 (Algebraic Decomposition Exponent)**.** For any Young function Ψ we define the *algebraic decomposition exponent of* Ψ, p_Ψ, by

$$p_\Psi := \inf \left\{ p : \lim_{t\to\infty} \frac{\Psi(t)}{t^p} = 0 \right\}.$$

Corollary 3.11 says that the algebraic decomposition exponent of a doubling Young function is finite.

Definition 3.13 (Algebraic Decomposition)**.** For $\Psi(t)$ a strictly increasing doubling Young function we have the *algebraic decomposition*,

$$\Psi(t) = t^{p_\Psi} \phi(t), \tag{3.6}$$

where $\phi(t)$ satisfies the following growth properties:

$$\lim_{t\to\infty} \frac{\phi(t)}{t^\epsilon} = 0 \qquad \forall\, \epsilon > 0$$

and

$$\lim_{t\to\infty} \frac{t^{-\delta}}{\phi(t)} = 0 \qquad \forall\, \delta > 0$$

Corollary 3.11 along with the decomposition (3.6) yield the following inequality which we will exploit in order to prove our main result.

Corollary 3.14. *Let $\Psi(t) = t^{p_\Psi}\phi(t)$ be the algebraic decomposition of a doubling Young function Ψ. If $t\phi(t)$ is convex, then*

$$R_\Psi(t) := \frac{\Psi'(t)}{\Psi'(2^k t)} \leq \frac{1}{2^{k(p_\Psi - 1)}}\ \forall\, t > 0\ \forall\, k \in \mathbb{N}. \tag{3.7}$$

Corollary 3.14 plays an integral role in a change of variables argument in the proof of the main theorem. It also informs the choice of coefficients for the inequalities in the setup for the proof. We assume convexity of $t\phi(t)$ because if $\Psi(t) = t^p \log^q(1+t)$ for $q < 0$, then $1 \geq R_\Psi \geq \frac{1}{2^{k(p_\Psi - 1)}}$.

Proof of Corollary 3.14. Rewriting $R_\Psi(t)$ using the algebraic decomposition gives

$$\begin{aligned}\frac{\Psi'(t)}{\Psi'(2^k t)} &= \frac{t^{p_\Psi - 1}\left(p_\Psi \phi(t) + t\phi'(t)\right)}{2^{k(p_\Psi-1)} t^{p_\Psi - 1}\left(p_\Psi \phi(2^k t) + 2^k t \phi'(2^k t)\right)} \\ &= \frac{1}{2^{k(p_\Psi-1)}} \frac{\left(p_\Psi \phi(t) + t\phi'(t)\right)}{\left(p_\Psi \phi(2^k t) + 2^k t \phi'(2^k t)\right)}.\end{aligned}$$

The convexity of $t\phi(t)$ gives that $p_\Psi \phi(t) + t\phi'(t)$ is increasing, and we have the desired inequality. □

The restriction that $t\phi(t)$ is convex may seem restrictive, but does not have much effect in practice. The functions $\Gamma_{p,q}$ with $q < 0$ fail the convexity restriction. However, if X admits a $\Gamma_{p,q}$-Poincaré inequality with $q < 0$, then we now longer need Theorem 1.3 as X admits a p-Poincaré inequality and Theorem 1.1 applies.

We now recall Jensen's inequality.

Proposition 3.15. *If $\Psi : \mathbb{R} \to \mathbb{R}$ is convex, $u \in L^1_{Loc}(X)$, and $A \subset X$ has positive, finite measure, then*

$$\Psi\left(\fint_A |u| d\mu\right) \leq \fint_A \Psi(|u|) d\mu. \tag{3.8}$$

3.2 Orlicz-Poincaré Inequality

This section presents the foundations of Orlicz-Poincaré inequalities studied by Tuominen ([27], [28]), Heikkinen ([10]) and Björn ([1]).

Definition 3.16 (Ψ-Poincaré inequality)**.** Let $\Psi : [0,\infty) \to [0,\infty)$ be a strictly increasing Young function. Then the pair $(u, \mathrm{Lip}\ u)$ with u a Lipschitz function *satisfies a Ψ-Poincaré inequality* if there exists constants $C_\Psi > 0$ and $\tau \geq 1$ such that

$$\fint_B |u - u_B| d\mu \leq C_\Psi r \Psi^{-1}\left(\fint_{\tau B} \Psi(\mathrm{Lip}\ u) d\mu\right) \tag{3.9}$$

for each $B = B(x,r)$. If the inequality holds for every Lipschitz function $u : X \to \mathbb{R}$ with fixed constants, then X *admits a Ψ-Poincaré inequality.*

Lemma 3.17 ([27, Lemma 5.6])**.** *If the pair $(u, \mathrm{Lip}\ u)$ satisfy a Ψ_1-Poincaré inequality, then the pair satisfies a Ψ_2-Poincaré inequality for any $\Psi_2 = \varphi \circ \Psi_1$ where φ is any Young function.*

Proof. This is a direct consequence of the definition of the Ψ_1-Poincaré inequality, Proposition 3.15, and inequality (3.1). The fact that the pair $(u, \text{Lip } u)$ satisfies a Ψ_1-Poincaré inequality gives

$$\fint_B |u - u_B| d\mu \le C_{\Psi_1} r \Psi_1^{-1} \left(\fint_{\tau B} \Psi_1(\text{Lip } u) d\mu \right).$$

Apply (3.1) to introduce φ to get

$$C_{\Psi_1} r \Psi_1^{-1} \left(\fint_{\tau B} \Psi_1(\text{Lip } u) d\mu \right) \le C_{\Psi_1} r \Psi_1^{-1} \left(\varphi^{-1} \left(\varphi \left(\fint_{\tau B} \Psi_1(\text{Lip } u) d\mu \right) \right) \right).$$

Finally, apply Proposition 3.15, and

$$\begin{aligned} C_{\Psi_1} r \Psi_1^{-1} & \left(\varphi^{-1} \left(\varphi \left(\fint_{\tau B} \Psi_1(\text{Lip } u) d\mu \right) \right) \right) \\ & \le C_{\Psi_1} r \Psi_1^{-1} \left(\varphi^{-1} \left(\fint_{\tau B} \varphi \left(\Psi_1(\text{Lip } u) \right) d\mu \right) \right) \\ & \le C r \left(\varphi \circ \Psi_1 \right)^{-1} \left(\fint_{\tau B} \left(\varphi \circ \Psi_1 \right) (\text{Lip } u) d\mu \right) \end{aligned}$$

as desired. □

We will use the following lemma of Tuominen.

Lemma 3.18 ([28, Corollary 4.2])**.** *Let $\Psi_1, \Psi_2 : [0, \infty) \to [0, \infty)$ be strictly increasing Young functions such that there exist constants $C_1, C_2 > 0$ so that*

$$\frac{\Psi_1(t)}{\Psi_2(C_2 t)} \le C_1 \frac{\Psi_1(s)}{\Psi_2(C_2 s)} \tag{3.10}$$

for all $0 < s < t$. If the pair $(u, \text{Lip } u)$ satisfies a Ψ_1-Poincaré inequality, then it also satisfies a Ψ_2-Poincaré inequality.

Many useful examples satisfy (3.10) even with $C_1 = C_2 = 1$. For example, it is satisfied by $\Psi_1 = \Gamma_{p,q}$ for any $q \in (-\infty, \infty)$ and $\Psi_2 = \Gamma_{p,r}$ for $r > q$.

3.2.1 Consequences of Orlicz-Poincaré Inequalities

We begin with a pair of propositions from [27] and [28], respectively, that relate the validity of Orlicz-Poincaré inequalities to the pointwise inequality described in the classical case in [8].

Proposition 3.19 ([27, Lemma 5.15])**.** *Assume that $\Psi : [0, \infty) \to [0, \infty)$ is a strictly increasing Young function. If a pair $u \in L^1_{loc}(X)$ and a measurable function*

$g \geq 0$ satisfy a Ψ-Poincaré inequality, then for μ-almost all $x, y \in X$,

$$|u(x) - u(y)| \leq Cd(x,y) \left(\Psi^{-1}\left(M_R\Psi(\rho(x))\right) + \Psi^{-1}\left(M_R\Psi(\rho(y))\right)\right), \tag{3.11}$$

where $R = 2\tau d(x,y)$. The constant $C > 0$ depends only on the doubling constant C_d of μ and on the constant C_Ψ of the Ψ-Poincaré inequality.

The following proposition shows the converse holds with the usual definition of Orlicz-Poincaré inequality under the stronger assumption that Ψ is doubling. We will see in chapter 6 that the converse holds without assuming doubling if the definition of the Orlicz-Poincaré inequality is modified.

Proposition 3.20 ([28, Theorem 3.2])**.** *Let X be a doubling, geodesic metric space, and Ψ a doubling Young function. If $u \in L^1_{loc}(X)$, and $\sigma \geq 1$, $C > 0$ are such that the inequality*

$$|u(x) - u(y)| \leq Cd(x,y) \left(\Psi^{-1}\left(M_{\sigma d(x,y)}\Psi(\rho(x))\right) + \Psi^{-1}\left(M_{\sigma d(x,y)}\Psi(\rho(y))\right)\right), \tag{3.12}$$

holds for μ-almost all $x, y \in X$, then the pair u, g satisfies a Ψ-Poincaré inequality with $\tau = 3\sigma$. The constant $C_\Psi > 0$ depends only on the constants of μ, Ψ, and of C in (3.12).

Remark 3.21. The original statement in [28] uses the assumption that X is Q-regular. However, recall that a metric measure space is said to satisfy a *relative lower volume decay of order Q* if there is a constant $C_0 \geq 1$ such that

$$\left(\frac{s}{r}\right)^Q \leq C_0 \frac{\mu\left(B(x,s)\right)}{\mu\left(B(a,r)\right)} \tag{3.13}$$

whenever $a \in X$, $x \in B(a,r)$, and $0 < s \leq r$. This decay is essentially one half of the growth bounds guaranteed by Q-regularity, and it is the only half used in the proof of the above proposition. It is known that every doubling metric measure spaces satisfies the relative lower volume decay with $Q = \log_2 C_D$, where C_D is the doubling constant. For more discussion see, for example, [11].

We end this section with two results that are of interest independent of the main theorem. We first present them here in the case that Φ is a doulbing Orlicz function. With this assumption, the proofs are direct consequences of the classical theory. In Chapter 6, more detailed proofs will be provided as a number of modifications to not only the proofs and statements, but also the definition of Orlicz-Poincaré inequality will be made. The second is a stability result that, in the nondoubling case, ultimately answers the motivating question and plays a role in the sharpness of the main theorem. The first proposition will be used in the argument that shows that Orlicz-Poincaré inequalities are stable, but appears in other applications as well.

Proposition 3.22. *Suppose that X is a complete and doubling metric measure space and Φ a doubling Young function. Then X admits a Ψ-Poincaré inequality if and only if there exist constants $C > 0$ such that (3.9) holds for every open ball B in X, for every Lipschitz function $u : X \to \mathbb{R}$, and for every Lipschitz continuous upper gradient $\rho : X \to [0, \infty)$ of u in X. The constants in the Orlicz-Poincaré inequality depend only on each other and on the doubling constant of the measure.*

Proposition 3.23. *Let (X_i, d_i, p_i, μ_i) be a sequence of complete length spaces which converge in the pointed measured Gromov-Hausdorff sense to a complete metric measure space (X, d, p, μ). Let Ψ be a doubling Young function, C_D, $C_\Psi < \infty$ and $\lambda \geq 1$ be fixed. If each of the measures μ_i is doubling with constant C_D and each space (X_i, d_i, μ_i) admits the Ψ-Poincaré inequality with constants C_Ψ and λ, then (X, d, μ) also admits the Ψ-Poincaré with constants C'_Ψ and λ' depending only on Ψ, C_Ψ, λ, and C_D.*

This proposition is analogous to the stability result Theorem 2.14. The proofs of these propositions are essentially unchanged from the classical case. The statements and proofs, which require modifications from the classical case, will be presented in Chapter 6.

4 Improvement of Orlicz-Poincaré Inequalities

We begin by defining objects that will be used in the proof of the main theorem.

Definition 4.1 (Sharp Fractional Maximal Operator)**.** Let $E \subset X$ be an open set and $u : E \to \mathbb{R}$ be Lipschitz. For every $x \in E$, and $t > 0$ we define

$$M^{\sharp}_{E,t}u(x) = \sup_{B} \frac{1}{\text{diam } B} \fint_{B} |u - u_B| d\mu \tag{4.1}$$

with the supremum taken over all balls $tB \subset E$ that contain x. Commonly, E will be a ball in X.

It will be useful for us to measure the extent to which a Lipschitz function, u, varies from its average value on some ball.

Definition 4.2 (Level Sets of the Sharp Fractional Maximal Operator)**.** Let $u : X \to \mathbb{R}$ be Lipschitz, we define

$$U_{E,t,\lambda} = \left\{ M^{\sharp}_{E,t}u > \lambda \right\}$$

for every $\lambda > 0$.

To prove Theorem 1.3 we will look at the sharp maximal function $M^{\sharp}_{\cdot}u$ of some Lipschitz function, u. We then find bounds on the measure of level sets of this maximal operator. The maximal operator relates to the left hand side of our Poincaré inequality, and the level sets will play a role in a standard Cavalieri type repartitioning argument where we compare measures of cross sections. More explicitly, we have an inequality for the measure of the level sets which we will integrate and repartition to get a statement about the integral of the sharp maximal function and Lip u from which we can derive an improved Poincaré inequality. We will follow the argument shown in [20] adapting the lemmas as necessary to fit the context of Orlicz-Poincaré inequalities. We will provide proofs of the results which are substantially different from [20] and give a brief description of the proofs of the remaining lemmas.

4.1 Level Set Estimates for the Sharp Maximal Operator

We first find local estimates for our sharp fractional maximal function. To do this, we will follow the treatment in [20]. Recall the algebraic decomposition exponent $p_\Psi := \inf\left\{p : \lim_{t\to\infty} \frac{\Psi(t)}{t^p} = 0\right\}$. For the remainder of the chapter we will suppress the Young function Ψ and set $p_\Psi = p$. We also define $X_i := 2^{i-1}X_1$ where X_1 is some fixed ball in X.

Proposition 4.3. *Let $\alpha \in \mathbb{N}$. There exists $k_1 \in \mathbb{N}$ that depends only on C and α such that for all integers $k \geq k_1$ and every $\lambda > 0$ with*

$$\frac{1}{\operatorname{diam} X_1} \fint_{X_1} |u - u_{X_1}|\, d\mu > \lambda, \tag{4.2}$$

we have

$$\begin{aligned} |X_1| \leq 2^{kp-\alpha} \left|U_{X_4,40,2^k\lambda}\right| &+ 8^{kp-\alpha} \left|U_{X_4,40,8^k\lambda}\right| \\ &+ 8^{k(p+1)} \left|\left\{x \in X_5 : \operatorname{Lip} u(x) > 8^{-k}\lambda\right\}\right| \end{aligned} \tag{4.3}$$

This proposition is used to generate global estimates that are utilized in the proof of the main theorem. The proof is done by contradiction. The proof has four key components from [20] which will be stated with varying degrees of proof.

Proof. The negation of (4.3) gives

$$\left|U_{X_4,40,2^k\lambda}\right| < 2^{-kp+\alpha}, \tag{4.4}$$

$$\left|U_{X_4,40,8^k\lambda}\right| < 8^{-kp+\alpha}, \tag{4.5}$$

and

$$\left|\left\{x \in X_5 : \operatorname{Lip} u(x) > 8^{-k}\lambda\right\}\right| < 8^{-k(p+1)}. \tag{4.6}$$

First, we show that the function u varies from its average value by a definite amount even away from the set where its deviation is large as measured by the sharp maximal operator. There exists a $\tilde{k} \in \mathbb{Z}$ such that for all $k > \tilde{k}$ we have

$$\int_{X_2 \setminus U_{2^k}} \left|u - u_{X_2 \setminus U_{2^k}}\right| d\mu \geq \frac{1}{C} \tag{4.7}$$

To show (4.7), we use (4.4) to control the diameter of disjoint balls that cover the set where u deviates from its average value by a large amount, but neither intersect nor avoid that set too much. We apply (4.2), rescaling u by $\frac{u}{\lambda}$ to set $\lambda = 1$, and show that the contribution from the set of large deviation is small enough to guarantee a positive contribution from the set of small deviations. Next, we reduce the small deviation of u by creating a $C8^k$-Lipschitz extension f of $u|_{X_3\setminus U_{8^k}}$ to X_3 such that

$$M^{\sharp}_{X_3,1} f(x) \leq C M^{\sharp}_{X_5,1} u(x) \tag{4.8}$$

for every $x \in X_2\setminus U_{8^k}$.

This proof uses a specific extension similar to the Whitney decomposition. The sharp maximal operator is used to find the sets where the deviation from the average value is small. The fact [20, Lemma 2.3.1] that u is Lipschitz on these sets with a prescribed Lipschitz constant establishes that f is Lipschitz. Recall the algebraic decomposition of Ψ, $\Psi(t) = t^{p_\Psi}\phi(t)$, and let

$$F_s = \left\{ x \in X_2 : \ M_2^{\#} f(x) > s \right\}$$

for every $s > 0$.

Next, we need to show

$$\int_{X_3\setminus U_{8^k}} \Psi\left(\mathrm{Lip}\ f\right) d\mu \leq C 8^{-k}\phi(8^k) \tag{4.9}$$

and, for $2^{2k} < s < 2^{3k-1}$

$$|F_s| \leq \frac{C\phi\left(8^k\right)}{\Psi(s)}. \tag{4.10}$$

We will first show (4.9). We repartition $\int_{X_3\setminus U_{8^k}} \Psi\left(\mathrm{Lip}\ f\right) d\mu$ and separate the level sets of $\Psi\left(\mathrm{Lip}\ f\right)$. From [20] we have that $\mathrm{Lip}\ f \leq C8^k$ almost everywhere on $X_3\setminus U_{8^k}$ and $\mathrm{Lip}\ f = \mathrm{Lip}\ u$ on $X_3\setminus U_{8^k}$. This gives

$$\begin{aligned}\int_{X_3\setminus U_{8^k}} \Psi\left(\mathrm{Lip}\ f\right) d\mu = & \int_0^{8^{-k}} \Psi'(t)\mu\left(\{x \in X_3\setminus U_{8^k} : \ \mathrm{Lip}\ f(x) > t\}\right) dt \\ & + \int_{8^{-k}}^{C8^k} \Psi'(t)\mu\left(\{x \in X_3\setminus U_{8^k} : \ \mathrm{Lip}\ f(x) > t\}\right) dt \\ \leq & \Psi(8^{-k})|X_3\setminus U_{8^k}| + C\Psi(8^k)\left|\left\{x \in X_3 : \ \mathrm{Lip}\ f(x) > 8^{-k}\right\}\right|\end{aligned}$$

We apply (4.6) to obtain

$$\begin{aligned}\Psi(8^{-k})|X_3\backslash U_{8^k}| + C\Psi(8^k)\left|\left\{x\in X_3:\ \mathrm{Lip}\, f(x) > 8^{-k}\right\}\right| \\ \le C\Psi(8^{-k}) + C\Psi(8^k)8^{-k(p+1)} \\ \le C8^{-k}\phi(8^k)\end{aligned}$$

and (4.9) is proved.

We now show (4.10). The Ψ-Poincaré inequality gives that

$$\Psi\left(\frac{1}{\mathrm{diam}\ B}\fint_B |f-f_B|\,d\mu\right) \le C\fint_B \Psi\left(\mathrm{Lip}\ f\right)d\mu$$

This with the definition of the sharp fractional maximal operator and the uncentered Hardy-Littlewood maximal operator yields

$$\Psi\left(M_2^{\#}f\right)(x) \le CM\left(\chi|_{X_3}\Psi\left(\mathrm{Lip}\ f\right)\right)(x),$$

for every $x\in X_2$. It follows from the weak-L^1 bound on the uncentered Hardy-Littlewood maximal operator that we have the following chain of inequalities:

$$\begin{aligned}\left|\left\{x\in X_2:\ M_2^{\#}f(x) > s\right\}\right| &\le \left|\left\{x\in X_2:\ \Psi\left(M_2^{\#}f\right)(x) > \Psi(s)\right\}\right| \\ &\le \left|\left\{x\in X_2:\ CM\left(\chi|_{X_3}\Psi\left(\mathrm{Lip}\ f\right)\right)(x) > C\Psi(s)\right\}\right| \\ &\le \frac{C_1}{C\Psi(s)}\int_{X_3}\Psi\left(\mathrm{Lip}\ f\right)d\mu\end{aligned}$$

From (4.9) and (4.3) we have

$$\int_{X_3}\Psi\left(\mathrm{Lip}\ f\right) \le \Psi(8^k)\left|U_{8^k}\right| + \int_{X_3\backslash U_{8^k}}\Psi\left(\mathrm{Lip}\ f\right)d\mu \le C\phi(8^k) \tag{4.11}$$

This is true for every $s>0$, but we only apply it for $2^{2k} < s < 2^{3k-1}$. Let f_j be the McShane extension of $f|_{X_2\backslash F_{2^j}}$ to a $C2^j$-Lipschitz function on X. Using f_j we define a function h where

$$h = \frac{1}{k}\sum_{j=2k}^{3k-1} f_j.$$

The final step is to show

$$\int_{X_2} \Psi\left(\text{Lip } h\right) d\mu \geq \frac{1}{C} \tag{4.12}$$

and

$$\text{Lip } h \leq \chi|_{X_2 \backslash U_{8^k}}(x) \text{Lip } f(x) + \frac{C}{k} \sum_{j=2k}^{3k-1} 2^j \chi|_{U_{8^k} \cup F_{2^j}}(x), \tag{4.13}$$

for almost every $x \in X_2$.

We will only prove (4.12). The other statement is exactly the same in the traditional setting found in [20]. Choose $k \in \mathbb{N}$ large enough so that by (4.8) $F_{4^k} \subset U_{2^k}$ and $|X_2 \cup U_{2^k}| < \epsilon(k)$. This gives us that $f_j = u$ almost everywhere on $X_2 \backslash U_{2^k}$ for all $j \in 2k, \ldots, 3k$. By (4.7), we have

$$\begin{aligned}
\int_{X_2} |h - h_{X_2}| &\geq \int_{X_2 \backslash U_{2^k}} \left| u - u_{X_2 \backslash U_{2^k}} \right| - \int_{X_2 \backslash U_{2^k}} \left| u_{X_2 \backslash U_{2^k}} - h_{X_2} \right| \\
&= \int_{X_2 \backslash U_{2^k}} \left| u - u_{X_2 \backslash U_{2^k}} \right| \\
&\quad - \left| \frac{1}{|X_2 \backslash U_{2^k}|} \int_{X_2 \backslash U_{2^k}} u - \frac{1}{X_2} \left(\int_{X_2 \backslash U_{2^k}} u + \int_{U_{2^k}} h \right) \right| \mu\left(X_2 \backslash U_{2^k}\right) \\
&\geq \int_{X_2 \backslash U_{2^k}} \left| u - u_{X_2 \backslash U_{2^k}} \right| - \frac{|X_2 \cap U_{2^k}|}{|X_2|} \int_{X_2 \backslash U_{2^k}} u - \frac{|X_2 \backslash U_{2^k}|}{|X_2|} \int_{U_{2^k}} h \\
&\geq \frac{1}{c}
\end{aligned}$$

when $|U_{2^k}|$ is sufficiently small as determined by k. We can now apply the Ψ-Poincaré inequality to achieve the desired property.

To finish the conclusion first, note that the sets F_s are decreasing in s. Let $j(x) = \max\{j : x \in U_{8^k} \cup F_{2^j}\}$. Also note that $\chi|_{U_{8^k} \cup F_{2^j}} = 1 \Leftrightarrow j \leq j(x)$. We now have

$$\begin{aligned}
\int_{X_2} \Psi\left(\frac{1}{k} \sum_{j=2k}^{3k-1} 2^j \chi|_{U_{8^k} \cup F_{2^j}} \right) d\mu &= \int_{X_2} \Psi\left(\frac{1}{k} \sum_{i=2k}^{j(x)} 2^i \right) d\mu \\
&= \sum_{j=2k}^{3k-1} \int_{\{x \in X_2 : j(x) = j\}} \Psi\left(\frac{1}{k} \sum_{i=2k}^{j} 2^i \right) d\mu \\
&\leq \int_{X_2} \sum_{j=2k}^{3k-1} \Psi\left(\frac{1}{k} \sum_{i=2k}^{j} 2^i \right) \chi|_{U_{8^k} \cup F_{2^j}} d\mu.
\end{aligned}$$

We apply (4.10) to obtain

$$\begin{aligned}\int_{X_2} \sum_{j=2k}^{3k-1} \Psi\left(\frac{1}{k}\sum_{i=2k}^{j} 2^i\right) \chi|_{U_{8^k}\cup F_{2^j}}\, d\mu &\le C\sum_{j=2k}^{3k-1} \Psi\left(\frac{2^{j+1}}{k}\right)\frac{\phi(8^k)}{\Psi\left(2^j\right)} \\ &\le C\sum_{j=2k}^{3k-1} \frac{2^p\phi(8^k)}{k^p} = Ck^{1-p}\phi(8^k)\end{aligned} \tag{4.14}$$

Thus by (4.12) we have

$$\frac{1}{C} \le \int_{X_2} \Psi\left(\mathrm{Lip}\, h\right) d\mu.$$

We apply (4.11), (4.13), and (4.14) to obtain

$$\int_{X_2} \Psi\left(\mathrm{Lip}\, h\right) d\mu \le C8^{-k}\phi(8^k) + Ck^{1-p}\phi(8^k)$$

which gives a contradiction as (1.2) implies $\phi(8^k) = o\left(k^{p-1}\right)$. The need for both the restriction to $p > 1$ and (1.2) is clear to ensure the necessary contradiction. From this we can conclude at least one of the estimates from (4.4,4.5,4.6) must fail which implies that (4.3) holds.

□

From this local estimate we can derive a global statement with which we will finish the proof of Theorem 1.3.

Proposition 4.4. *There exists $k_1 \in \mathbb{N}$ that depends only on C such that for all integers $k \ge k_1$ and every $\lambda > 0$, we have*

$$\begin{aligned}\left|U_{\tilde{B},40,\lambda}\right| &\le 2^{kp-\alpha}\left|U_{\tilde{B},40,2^k\lambda}\right| \\ &\quad + 8^{kp-\alpha}\left|U_{\tilde{B},40,8^k\lambda}\right| + 10^{k(p+1)}\left|\left\{x\in\tilde{B} : Lip\ u(x) > 8^{-k}\lambda\right\}\right|\end{aligned} \tag{4.15}$$

The extension from the local statement to this global proposition has no dependence on the parameter in the Poincaré inequality. Therefore the proof of Proposition 4.4 is no different in our context. For details of the proof see [20].

4.2 Proof of Theorem 1.3

We begin by integrating the inequality of the level sets (4.15), against the measure $d\lambda^{p-\epsilon}$ to get the following inequality:

$$\int_0^\infty \left|U_{\tilde{B},40,\lambda}\right| d\lambda^{p-\epsilon} \leq \int_0^\infty 2^{kp-\alpha}\left|U_{\tilde{B},40,2^k\lambda}\right| d\lambda^{p-\epsilon} + \int_0^\infty 8^{kp-\alpha}\left|U_{\tilde{B},40,8^k\lambda}\right| d\lambda^{p-\epsilon}$$
$$+ 10^{kp}\int_0^\infty \left|\left\{x \in \tilde{B} : \text{Lip } u(x) > 10^{-k}\lambda\right\}\right| d\lambda^{p-\epsilon}$$

The change of variables $\lambda \to 2^k\lambda$ yields

$$\int_0^\infty \left|U_{\tilde{B},40,\lambda}\right| d\lambda^{p-\epsilon} \leq \left(\frac{2^{\epsilon k}}{2^\alpha} + \frac{8^{\epsilon k}}{8^\alpha}\right)\int_0^\infty \left|U_{\tilde{B},40,\lambda}\right| d\lambda^{p-\epsilon}$$
$$+ C(k)10^{kp}\int_0^\infty \left|\left\{x \in \tilde{B} : \text{Lip } u(x) > \lambda\right\}\right| d\lambda^{p-\epsilon}$$

We now repartition the integration of the level sets to obtain

$$\int_{\tilde{B}} \left(M^\sharp_{\tilde{B},40}u(x)\right)^{p-\epsilon} d\mu \leq \left(\frac{2^{\epsilon k}}{2^\alpha} + \frac{8^{\epsilon k}}{8^\alpha}\right)\int_{\tilde{B}} \left(M^\sharp_{\tilde{B},40}u(x)\right)^{p-\epsilon} d\mu$$
$$+ C(k)\int_{\tilde{B}} (\text{Lip } u)^{p-\epsilon}\, d\mu$$

Therefore, with any $\alpha \geq 1$ and an appropriate choice of ϵ such that $\left(\frac{2^{\epsilon k}}{2^\alpha} + \frac{8^{\epsilon k}}{8^\alpha}\right) < 1$ it follows that

$$\int_{\tilde{B}} \left(M^\sharp_{\tilde{B},40}u(x)\right)^{p-\epsilon} d\mu \leq C\int_{\tilde{B}} (\text{Lip } u)^{p-\epsilon}\, d\mu$$

To complete the proof, we need the following observation:

$$M^\sharp_{\tilde{B},40}u(x) \geq \frac{1}{\text{diam } B'} \fint_{B'} |u - u_{B'}|\, d\mu$$

for every $x \in B' = \frac{1}{40}\tilde{B}$. We apply Hölder's inequality to achieve the desired result.

5 Examples of Sharpness

We discuss two key hypotheses in the main theorem: completeness and the growth condition (1.2). The first example, taken from [22], shows the necessity of the completeness assumption. The second example uses modified Cantor diamond sets of the type developed in [23]. While the Cantor diamond sets constructed are not shown to admit the critical Orlicz-Poincaré inequality $\Gamma_{2,1}(t)$, they do show that something like the growth condition (1.2) is needed.

5.1 Necessity of Completeness

For Poincaré inequalities defined via upper gradients [13] there are examples that show the completeness of the metric space X is necessary. The example constructed by Koskela ([22]) in the planar case shows the necessity of completeness. Jensen's inequality gives that if a set K is removable for $W^{1,p}(X)$, then it is removable for $W^{1,\Psi}(X)$ where $p \leq p_\Psi$. Without completeness we can create a set that is not removable for $W^{1,q}(X)$ for any $q < p$. This is not as strong as finding a non-removable set for $W^{1,\Phi}(X)$ for any Φ such that $\Psi \circ \Phi^{-1}$ is convex, but the conclusion of the main theorem that a Ψ-Poincaré inequality improves to a $(p_\Psi - \epsilon)$-Poincaré inequality implies that the former condition is enough. A direct proof for the necessity of completeness in [16, Theorem B] shows the existence of a compact n-regular metric space that admits a $\Gamma_{p,\lambda}$-Poincaré inequality, but no $\Gamma_{p,\lambda-\epsilon}$-Poincaré inequality for any ϵ. Completeness is still necessary for Poincaré inequalities of the type (1.1), but the reason has to do with lack of density of Lipschitz functions in the appropriate Sobolev spaces as opposed to the construction of open sets with particular properties. For a more in depth discussion about the necessity of the completeness condition see [20, Section 1.1].

5.2 Necessity of Condition (1.2)

In [23], Koskela and MacManus introduce the so called Cantor diamond sets to show that quasiconformal maps may fail to preserve Poincaré inequalities above the Ahlfors regularity parameter. More specifically, if X is a Q-regular space that satisfies a p-Poincaré inequality with $p < Q$ and f is a quasiconformal mapping of X onto Y, then Y satisfies a p'-Poincaré inequality for $p' < Q$. The Cantor diamond sets, X_λ, are precisely planar 2-regular sets that satisfy p-Poincaré inequalities for all $p > \frac{2-d_\lambda}{1-d_\lambda}$ where $d_\lambda = \frac{\log 2}{\log \frac{2}{\lambda}}$ is the dimension of the complementary Cantor set, but there exists a quasiconformal image that satisfies no Poincaré inequalities. The set X_λ is inductively constructed by removing the middle subinterval of length $(1-\lambda)\frac{\lambda^{n-1}}{2^{n-1}}$ from each 2^n intervals to create a Cantor set, E_λ. Place a square whose diagonal is the removed interval in each component of the complement of E_λ. The set X_λ will be a chain of diamonds connected by the set E_λ.

In order to prove Theorem 1.4, we will construct a planar set similarly to that above. However, we will choose λ_i so that the corresponding Cantor set has dimension zero.

An important property of planar sets is annular linear local connectivity (ALLC).

Definition 5.1 (ALLC)**.** A metric space (X, μ) is *annularly linearly locally connected* if there exists a constant $C > 0$ so that for all points $a \in X$ and for all $0 \le r < R$, each pair of distinct points in $A(a, r, R)$ can be joined by a curve that is contained in the larger annulus $A(a, \frac{r}{C}, CR)$.

It is known that if a planar set is not ALLC, then it cannot satisfy a 2-Poincaré inequality. (See [13, Corollary 5.8] and [21, Theorem 3.3] for example.) Theorem 1.3 then gives that there are no Cantor Diamond sets that admit $\Gamma_{2,r}$-Poincaré inequalities for $r < 1$. If they did admit such Poincaré inequalities, then they would improve to $(2-\epsilon)$-Poincaré inequalities. This is clearly not possible by the above as Cantor Diamond sets are not ALLC. If we can construct a set that satisfies a $\Gamma_{2,r}$-Poincaré inequality for $r > 1$, then, with the above propositions, the growth condition (1.2) would give the best possible range for Ψ. On the other hand, if Theorem 1.4 is best possible, then there may be some improvement in the conclusion of Theorem 1.3.

For the proof of Theorem 1.4, we will follow the method used in [23].

Proof of Theorem 1.4. We will construct a Cantor set, E_λ, with $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$. We obtain E_λ inductively with step $n = 0$ being the interval $[0, 1]$. At each step n we remove the middle interval of length $\frac{1-\lambda_n}{2^{n-1}} \prod_{i=1}^{n-1} \lambda_i$ from each interval in the inductive construction, where $\lambda_n := 2^{-(2^{2n})}$. We obtain the planar set X_λ by replacing the complementary intervals of E_λ by squares with diagonals on the complementary intervals.

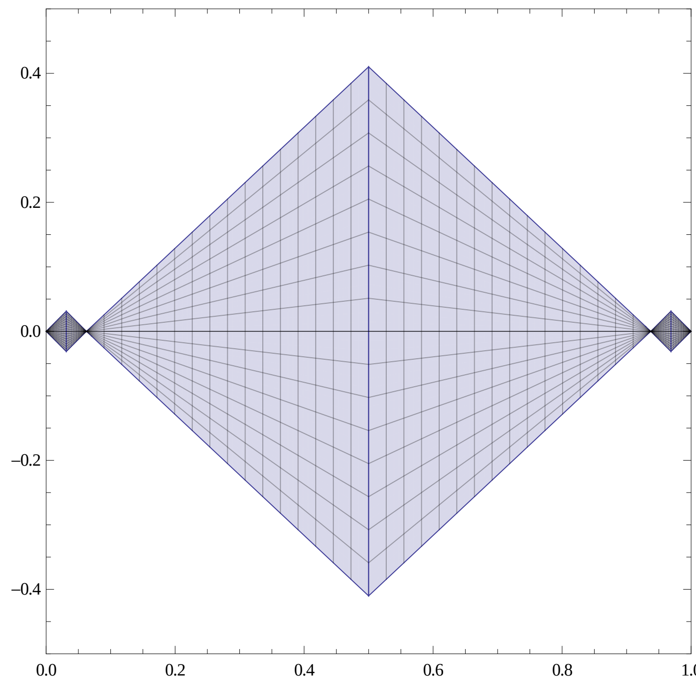

Figure 5.1: First two iterations in the construction of X_λ.

We equip E_λ with the Euclidean metric, and we denote the n-dimensional Lebesgue measure as dX if $x \in \mathbb{R}^n$. For any two points a, b in X_λ let $B_{ab} := B(a, |b-a|) \bigcup B(b, |b-a|)$, where $B(x, r)$ denotes a ball in X_λ.

We show that X_λ satisfies the desired Poincaré inequality by examining the Lipschitz map $F(x, y) = (x, y\delta(x))$ from $[0,1] \times [-1,1]$ to X_λ where $\delta(x)$ is the distance from x to E_λ. We reduce the satisfaction of the $\Gamma_{2,r}$-Poincaré inequality to the following pointwise inequality

$$|u(b) - u(a)| \leq C \log^{\frac{1-r^2}{2}} \left(\frac{1}{|b-a|} \right) \Gamma_{2,r}^{-1} \left(\int_{B_{ab}} \Gamma_{2,r}\left(g(c)\right) d\mu(c) \right). \tag{5.1}$$

By the definition of F we have

$$|u(b) - u(a)| = |u \circ F(b) - u \circ F(a)| \leq C \int_{[a,b]\times[-1,1]} g \circ F(c) dc$$

as $\|DF\|$ is comparable to 1. With a change of variables we get

$$|u(b) - u(a)| \leq C \int_{F([a,b]\times[-1,1])} \frac{g(z)}{\delta(z)} dz.$$

We apply the generalized Hölder's inequality to obtain

$$
\begin{aligned}
&|u(b)-u(a)| \\
&\qquad \leq \Gamma_{2,r}^{-1}\left(\int_{F([a,b]\times[-1,1])} \Gamma_{2,r}(g(z))dz\right) \tilde{\Gamma}_{2,r}^{-1}\left(\int_{F([a,b]\times[-1,1])} \tilde{\Gamma}_{2,r}\left(\frac{1}{\delta(z)}\right)dz\right).
\end{aligned}
$$

We first consider the case where a and b are the endpoints of a single diamond. Then we have

$$
\begin{aligned}
\int_{F([a,b]\times[-1,1])} \tilde{\Gamma}_{2,r}\left(\frac{1}{\delta(z)}\right)dz &= 2\int_a^{\frac{b+a}{2}} \tilde{\Gamma}_{2,r}\left(\frac{1}{|x-a|}\right)dx \\
&= 4\int_0^{\frac{b-a}{2}} \tilde{\Gamma}_{2,r}\left(\frac{1}{x}\right)xdx \\
&\leq C\log^{1-r}\left(\frac{2}{|b-a|}\right)
\end{aligned}
$$

From this and (3.3) we derive the desired inequality as $\tilde{\Gamma}_{2,r}^{-1}\left(C\log^{1-r}\left(\frac{2}{|b-a|}\right)\right)$ is comparable to $C\log^{\frac{1-r^2}{2}}\left(\frac{2}{|b-a|}\right)$.

If a and b are arbitrary points in E_λ, then there is some largest diamond whose endpoints lie in the interval $[a,b]$. We denote the step at which this diamond appears in the inductive process as n_0. Similarly to case one, we have

$$
\begin{aligned}
\int_{F([a,b]\times[-1,1])} \tilde{\Gamma}_{2,r}\left(\frac{1}{\delta(x)}\right)dx &= \int_{F([a,b]\times[-1,1])\setminus F(E_\lambda\times[-1,1]\cap[a,b]\times[-1,1])} \tilde{\Gamma}_{2,r}\left(\frac{1}{\delta(x)}\right)dx \\
&= C\sum_{n=n_0}^{\infty}\log^{1-r}\left(\frac{2^{n+1}}{\prod_{i=1}^{n}\lambda_i}\right)2^{n-n_0} \\
&\leq C\left(\sum_{i=1}^{n_0+1}(2^{2i}+1)^{1-r}\right) \\
&\leq C\log^{1-r}\left(\frac{1}{|b-a|}\right)
\end{aligned}
$$

and we apply (3.3) again to obtain (5.1) for all points a and b.

We now derive the appropriate Poincaré inequality from inequality (5.1). Fix an arbitrary ball B. For all points $a,b\in B$ we have $B_{ab}\subset\tau B$ for some $\tau>1$. We then integrate both sides of the inequality over B twice to obtain

$$\int_B \int_B |u(b) - u(a)| dbda \leq C \int_B \int_B \log^{\frac{1-r^2}{2}} \left(\frac{1}{|b-a|} \right) \Gamma_{2,r}^{-1} \left(\int_{B_{ab}} \Gamma_{2,r} \left(g(z) \right) d\mu(z) \right) dbda$$

Since $r > \frac{1+\sqrt{5}}{2}$, we have $1 - r^2 < -r$, and we obtain

$$C \int_B \int_B \log^{\frac{1-r^2}{2}} \left(\frac{1}{|b-a|} \right) \Gamma_{2,r}^{-1} \left(\int_{B_{ab}} \Gamma_{2,r} \left(g \right) \right) dadb \leq C \int_B \int_B \log^{\frac{-r}{2}} \left(\frac{1}{|b-a|} \right) \Gamma_{2,r}^{-1} \left(\int_{B_{ab}} \Gamma_{2,r} \left(g \right) \right) dadb.$$

On the left hand side we see that

$$\int_B \int_B |u(b) - u(a)| dbda \geq |B|^2 \fint_B |u(a) - u_B| da$$

which gives

$$\fint_B |u(a) - u_B| da \leq C \fint_B \fint_B \log^{\frac{-r}{2}} \left(\frac{1}{|b-a|} \right) \Gamma_{2,r}^{-1} \left(\int_B \Gamma_{2,r}(g) \right) dadb.$$

Since $|b - a| < |B|^{\frac{1}{2}} = (\text{diam } B)$, we can eliminate any dependence on a and b and absorb the average integrals into the constant C. Finally, we conclude

$$\begin{aligned}
\fint_B |u(a) - u_B| da &\leq C \frac{|B|^{\frac{1}{2}}}{|B|^{\frac{1}{2}}} \log^{\frac{-r}{2}} \left(\frac{1}{|B|} \right) \Gamma_{2,r}^{-1} \left(\int_B \Gamma_{2,r}(g(z)) d\mu(z) \right) \\
&\leq C |B|^{\frac{1}{2}} \Gamma_{2,r}^{-1} \left(\frac{1}{|B|} \right) \Gamma_{2,r}^{-1} \left(\int_B \Gamma_{2,r}(g(z)) d\mu(z) \right) \\
&\leq C (\text{diam } B) \Gamma_{2,r}^{-1} \left(\fint_B \Gamma_{2,r}(g(z)) d\mu(z) \right)
\end{aligned}$$

as desired. □

We have a small gap between the condition given by (1.2) and the example in Proposition 1.4. It is doubtful that the main theorem can be improved to bridge this gap, as that would suggest improvement from a $\Gamma_{1,\frac{\sqrt{5}-1}{2}}$-Poincaré inequality to 1-Poincaré inequality. It is probable that the example could be improved. The Cantor diamond sets were originally developed for the properties of their images under quasiconformal mappings. Since we do not require any control on the quasiconformal images of these sets, they might be unnecessarily complex. However, they are interesting example to use due to their historical significance. They might also be useful to study as the development of Orlicz-Poincaré inequalities and their consequences continue.

6 Orlicz-Poincaré Inequalities for Nondoubling Orlicz Functions

To answer Question 1.2, we must develop the theory of Orlicz-Poincaré inequalities for nondoubling Young functions. The main result of this section is the stability of Poincaré inequalities under Gromov-Hausdorff limits. Stability of Poincaré inequalities is the key tool in answering Question 1.2. We begin by presenting the Sierpiński strip example in [6] that immediately answers Question 1.2 in the negative modulo stability of Orlicz-Poincaré inequalities. The rest of the chapter will be devoted to developing the theory necessary to show stability in the nondoubling case. For this chapter, we will present the Poincaré inequalities using the upper gradient. This presentation more closely follows the motivating theorems and definitions found in [6], [28], and [15].

6.1 ∞-Poincaré Inequality

Definition 6.1 (∞-Poincaré Inequality)**.** The pair (u, ρ) with u a measurable function and ρ an upper gradient of u *satisfies an ∞-Poincaré inequality* if there exists constants $C_\infty > 0$ and $\tau \geq 1$ such that

$$\fint_B |u - u_B| d\mu \leq C_\infty r \, \|\rho\|_{L^\infty(\tau B)} \tag{6.1}$$

for each $B = B(x, r)$. If the inequality holds for every measureable function u and every upper gradient ρ of u with fixed constants, then X *admits an ∞-Poincaré inequality.*

It is known in the classical case that if X admits a p-Poincaré inequality for any p, then X is quasiconvex. In [5], a stronger condition was shown to hold if X admits any Poincaré inequality including the ∞-Poincaré inequality which will be introduced in Chapter 6. First, we recall the definition of thick quasiconvexity.

Definition 6.2 (Thick Quasiconvexity)**.** X is *thick quasiconvex* if there is a constant $C \geq 1$ such that for all $x, y \in X$, all $0 < \epsilon < \frac{1}{4} d(x, y)$, and all measurable sets

$E \subset B(x,\epsilon)$ and $F \subset B(y,\epsilon)$ satisfying $\mu(E)\mu(F) > 0$, we have that

$$\mathrm{Mod}_\infty\left(\Gamma\left(E,F,C\right)\right) > 0,$$

where $\Gamma\left(E,F,C\right)$ denotes the set of curves γ connecting $p \in E$ and $q \in F$ with $l(\gamma) \leq Cd(p,q)$.

For a precise description of Mod_∞ see [6, Definition 2.1, Definition 3.1].

Proposition 6.3 ([5, Theorem 4.6])**.** *Let X be a complete and doubling metric measure space. X admits the ∞-Poincaré inequality if and only if X is thick quasi-convex.*

This of course then gives that if X admits a Ψ-Poincaré inequality, then X is thick quasiconvex.

The Sierpiński strip is an example of a space that admits an ∞-Poincaré inequality, but no finite p. As we will see in the next section, it is also the example that answers Question 1.2 in the negative.

Example 6.4 (Sierpiński Strip)**.** Let Q_j be the j-th step in the iterative construction of the Sierpiński gasket in $\mathbb{R}^2$. That is $Q_0 = [0,1] \times [0,1]$. Step one is to remove the middle open square of side-length $\frac{1}{3}$ from Q_0 leaving 8 squares of side-length $\frac{1}{3}$. Step two is to remove the middle open square of side-length $\frac{1}{3}^2$ from each of the 8 squares in Q_1. We iterate this process to get that Q_j is the union of 8^j squares of side-length $\frac{1}{3}^j$. To each Q_j we renormalize the Lebesgue measure to have measure one. This gives us a measure μ_j concentrated on Q_j for each j. To get the Sierpinski strip we concatenate the Q_j's forming the metric measure space $X = \bigcup_{j=1}^{\infty}(Q_j + (j-1,0))$ with measure $\mu = \sum_j \chi_{Q_j+(j-1,0)} \cdot \mu_j$ where $(Q_j + (j-1,0))$ is the translation of Q_j to the right by $j-1$ units parallel to the x-axis. The space (X,d,μ) is thick quasiconvex which, by Proposition 6.3, gives that (X,d,μ) admits the ∞-Poincaré inequality. Now, we can immediately conclude that ∞-Poincaré inequality does not improve as the sequence of pointed spaces $\{(Q_j + (j-1,0), d_j, \mu_j, (j-1,0))\}$ converge to the metric measure space (S,d,μ) in the pointed measured Gromov-Hausdorff sense where $S = \bigcup_j Q_j$ is the Sierpiński carpet and μ is an Ahlfors regular measure that coincides with the Hausdorff measure of dimension $\log_3 8$. As we will show in the next section, Ψ-Poincaré inequalities persist through pointed measured Gromov-Hausdorff limits, and the limit space (S,d,μ) does not support any Orlicz-Poincaré inequalities. This gives that (X,d,μ) admits the ∞-Poincaré inequality and no other, and the answer to Question 1.2 is no. For more information on Poincaré inequalities and the Sierpiński strip see [6]. We conclude the chapter by showing that Orlicz-Poincaré inequalities are stable under Gromov-Hausdorff limits as claimed.

6.2 Consequences of Nondoubling Orlicz-Poincaré Inequalities

We begin by recalling the definition of Orlicz Space for a given Young function.

Definition 6.5 ($L^\Psi(\Omega)$)**.** If Ψ is a Young function and $\Omega \subset \mathbb{R}^n$ is an open set, then define *the Orlicz space* $L^\Psi(\Omega)$ by

$$L^\Psi(\Omega) = \left\{ u : \Omega \to [-\infty, \infty] : u \text{ measurable}, \int_\Omega \Psi(\alpha|u|)dx < \infty \text{ for some } \alpha > 0 \right\}.$$

This definition is part of the motivation to modify the Orlicz-Poincaré inequality in the nondoubling case.

Definition 6.6 (weak Ψ-Poincaré inequality)**.** Let $\Psi : [0, \infty) \to [0, \infty)$ be a strictly increasing Young function. Then the pair (u, ρ) with u a measurable function and ρ an upper gradient of u *satisfies a "weak" Ψ-Poincaré inequality* if there exists constants $C_\Psi > 0$, $\tau \geq 1$, and $\alpha \geq 1$ such that

$$⨍_B |u - u_B| d\mu \leq C_\Psi r \Psi^{-1} \left(⨍_{\tau B} \Psi(\alpha\rho) d\mu \right) \tag{6.2}$$

for each $B = B(x, r)$. If the inequality holds for every measurable function u and for all upper gradients ρ of u with fixed constants, then X *admits a weak Ψ-Poincaré inequality.*

Note the α in the argument of Ψ. The inclusion of this α is why the above definition is referred to as a weak Ψ-Poincaré inequality. This is different from the classical case when the term weak refers to the dilation of the ball. In the doubling case $\alpha = 1$, and we recover the original definition as the α can be removed with a controlled penalty to C_Ψ. In the nondoubling case, constants inside the arguments of nondoubling Young functions will propagate through all steps of the proof. This definition allows for the persistance of Orlicz-Poincaré inequalities through biLipschitz changes to the metric on the space X. Any biLipschitz change to the metric on X will have the effect of changing the pair (u, ρ) to $(u, \alpha\rho)$ which will appear inside the argument of the nondoubling Young function Ψ. There is no hope for such statements, nor for modifying Proposition 3.20 without modifying the original definition.

The main result of this section is the stability of Orlicz-Poincaré inequalities in full generality.

Theorem 6.7. *Let (X_i, d_i, p_i, μ_i) be a sequence of complete length spaces which converge to a complete space (X, d, p, μ) in the pointed measured Gromov-Hausdorff sense. Let Ψ be a continuous Young function, C_D, $C_\Psi < \infty$ and $\lambda \geq 1$ be fixed. If each of the measures μ_i is doubling with constant C_D and each space (X_i, d_i, μ_i) admits a weak Ψ-Poincaré inequality with constants C_Ψ and λ, then (X, d, μ) also admits a weak Ψ-Poincaré with constants C'_Ψ and λ' depending only on Ψ, C_Ψ, λ, and C_D.*

The following propositions represent the points of modification in the proof of Theorem 3.23 found in [15]. The first is the adaptation of Proposition 3.20.

Proposition 6.8. *Let X be a doubling, geodesic metric space, and Ψ a continuous Young function. If $u \in L^1_{loc}(X)$, and $\sigma \geq 1$, $C > 0$ are such that the inequality*

$$|u(x) - u(y)| \leq Cd(x,y)\left(\Psi^{-1}\left(M_{\sigma d(x,y)}\Psi(\alpha\rho(x))\right) + \Psi^{-1}\left(M_{\sigma d(x,y)}\Psi(\alpha\rho(y))\right)\right), \tag{6.3}$$

holds for μ-almost all $x, y \in X$, then the pair (u, ρ) satisfies a weak Φ-Poincaré inequality with $\tau = 3\sigma$. The constant $C_\Phi > 0$ will depend only on the constants of μ, Ψ, α, and C in (6.3).

We follow the proof found in [28].

Proof. Fix a point $x_0 \in X$ and $R > 0$, then let $B = B(x_0, R) \subset X$. Let $\tau = 3\sigma$, and $h = \rho\chi_{\tau B}$. We can modify u on a set of measure zero so that inequality (6.3) gives that

$$|u(x) - u(y)| \leq C_0 d(x,y)\left[\Psi^{-1}\left(M\Psi(\alpha h)(x)\right) + \Psi^{-1}\left(M\Psi(\alpha h)(y)\right)\right] \tag{6.4}$$

for all $x, y \in B$. If $h = 0$ almost everywhere in τB, then u is constant by (6.3) and the Orlicz-Poincaré inequality immediately follows. Thus, assume $h > 0$ on a set of positive measure in τB. Let $\hat{h} = \Psi^{-1}\left(\Psi(h) + \fint_{\tau B}\Psi(h)d\mu\right)$. It follows that

$$\hat{h} \geq \frac{1}{2}\Psi^{-1}\left(\fint_{\tau B}\Psi(\hat{h})d\mu\right) > 0 \tag{6.5}$$

as

$$\begin{aligned}
\Psi^{-1}\left(\fint_{\tau B}\Psi(\hat{h})d\mu\right) &= \Psi^{-1}\left(\fint_{\tau B}\Psi\left[\Psi^{-1}\left(\Psi(h)+\fint_{\tau B}\Psi(h)d\mu\right)\right]d\mu\right)\\
&\le \Psi^{-1}\left(\fint_{\tau B}\left[\Psi(h)+\fint_{\tau B}\Psi(h)d\mu\right]d\mu\right)\\
&= \Psi^{-1}\left(\fint_{\tau B}\Psi(h)d\mu+\fint_{\tau B}\Psi(h)d\mu\right)\\
&\le 2\Psi^{-1}\left(\fint_{\tau B}\Psi(h)d\mu\right)\\
&\le 2\Psi^{-1}\left(\fint_{\tau B}\Psi(h)d\mu+\Psi(h)\right)\\
&= 2\hat{h}.
\end{aligned}$$

This calculation also shows that the change from h to $\hat{h}$ does not affect the satisfaction of the Orlicz-Poincaré inequality. Therefore, will refer to $\hat{h}$ simply as h.

We now define the level sets which will generate our desired Orlicz-Poincaré inequality from the weak-L^1 bound on the Hardy-Littlewood maximal operator as in the setup of the main theorem. For each $k \in \mathbb{Z}$, let

$$E_k = \left\{x \in B : \Psi^{-1}\left(M\Psi(\alpha h)(x)\right) \le 2^k\right\} \quad \text{with} \quad a_k = \sup_{E_k}|u(x)|.$$

By definition $E_{k-1} \subset E_k$ which gives that $a_{k-1} \le a_k$ for every k and

$$\int_B |u-u_B|d\mu \le 2\int_B |u|d\mu \le 2\sum_{k=-\infty}^{\infty} a_k\mu(E_k\backslash E_{k-1}). \tag{6.6}$$

By (6.4), u is $C_0 2^{k+1}$-Lipschitz in E_k. This gives that

$$|u(x)| \le |u(x)-u(y)| + |u(y)| \le C_0 2^{k+1} d(x,y) + a_{k-1} \tag{6.7}$$

for every $x \in E_k$ and $y \in E_{k-1}$. We now fix $x \in E_k$ and apply a technical lemma from [28] to get $B(x,r)\cap B$ contains a ball of radius $\frac{r}{2}$ if $0 < r \le 2R$. The relative lower volume decay of X guarantees

$$\mu(B(x,r)\cap B) \ge \frac{1}{C_Q}\left(\frac{r}{2R}\right)^Q \mu(B), \tag{6.8}$$

where $Q = \log_2 C_D$ and $C_Q = 4^Q$.

If $\mu(E_{k-1}) > 0$, then choose

$$r_k = 2^{1+\frac{1}{Q}} C_Q^{\frac{1}{Q}} \mu\left(B \backslash E_{k-1}\right)^{\frac{1}{Q}} \frac{1}{\mu(B)^{\frac{1}{Q}}} R.$$

This choice of r_k along with (6.8) guarantees that

$$\mu\left(B(x, r_k) \cap B\right) > \mu\left(B \backslash E_k - 1\right).$$

We can conclude that there is a $y \in B(x, r_k) \cap E_{k-1}$. To guarantee that $r_k \leq 2R$, we must choose k large enough so that

$$2C_Q \mu\left(B \backslash E_{k-1}\right) \leq \mu(B). \tag{6.9}$$

The maximal function theorem gives that for $f \in L^1(X)$

$$\mu\left(\{x \in B : Mf(x) > \lambda\}\right) < \frac{C}{\lambda} \|f\|_1 .$$

From this, we have

$$\mu(B \backslash E_{k-1}) \leq \frac{C}{\Psi\left(2^{k-1}\right)} \int_{\tau B} \Psi(\alpha h) d\mu. \tag{6.10}$$

The choice or r_k, (6.7), and (6.10) gives the following upper bound for a_k:

$$\begin{aligned} a_k &\leq a_{k-1} + C 2^k \mu(B \backslash E_{k-1})^{\frac{1}{Q}} \frac{R}{\mu(B)^{\frac{1}{Q}}} \\ &\leq a_{k-1} + C 2^k R \mu(B)^{\frac{-1}{Q}} \Psi\left(2^{k-1}\right)^{\frac{-1}{Q}} \left(\int_{\tau B} \Psi(\alpha h) d\mu\right)^{\frac{1}{Q}} \end{aligned}$$

for E_{k-1} so that (6.9) holds.

Iterating this process, we have that for every $k > k_0$, where E_{k_0} satisfies (6.9),

$$a_k \leq a_{k_0} + C R \mu(B)^{\frac{-1}{Q}} \sum_{i=k_0+1}^{k} \frac{2^i}{\Psi\left(2^{i-1}\right)^{\frac{1}{Q}}} \left(\int_{\tau B} \Psi(\alpha h) d\mu\right)^{\frac{1}{Q}} \tag{6.11}$$

Claim 6.9. *There exists a k_0 so that (6.9) holds and a constant $C(C_Q, C_D, \tau) \geq 1$ such that*

$$\fint_{\tau B} \Psi(\alpha h) d\mu \leq 2\Psi\left(2^{k_0}\right), \tag{6.12}$$

and

$$\Psi\left(2^{k_0-1}\right) \leq 2C \fint_{\tau B} \Psi(\alpha h) d\mu. \tag{6.13}$$

Proof of claim. First, (6.5) guarantees that E_{k-1} is empty for small k. We also have that $\mu(E_k) \to \mu(B)$ as $k \to \infty$. This gives that there exists a k_0 so that

$$\mu(E_{k_0-1}) \leq \left(1 - \frac{1}{2C_Q}\right) \mu(B) \leq \mu(E_{k_0}). \tag{6.14}$$

To show (6.12), we have that $\Psi(\alpha h) \in L^1(B)$. Thus, the Lebesgue differentiation theorem gives that $M\Psi(\alpha h) \geq |\Psi(\alpha h)|$ almost everywhere in B. We now apply (6.5), the continuity of Ψ, and the fact that Ψ^{-1} is doubling with constant 2 to see that for almost every $x \in B$

$$\begin{aligned}\Psi^{-1}\left(M\Psi(\alpha h)(x)\right) &\geq \Psi^{-1}\left(\Psi(\alpha h)\right)\\ &= \alpha h\\ &\geq \frac{1}{2}\Psi^{-1}\left(\fint_{\tau B} \Psi(\alpha h) d\mu\right).\end{aligned}$$

The choice of k_0 so that E_{k_0} is nonempty gives that there exists an $x \in B$ so that

$$M\Psi(\alpha h)(x) \leq \Psi\left(2^{k_0}\right),$$

and we can conclude

$$\fint_{\tau B} \Psi(\alpha h) d\mu \leq 2\Psi\left(2^{k_0}\right). \tag{6.15}$$

To show (6.13), use (6.14) and (6.10) to obtain

$$\begin{aligned}\frac{\mu(B)}{2C_Q} &\leq \mu(B \backslash E_{k_0-1})\\ &= \mu\left(\left\{x \in B : \Psi^{-1}(M\Psi(\alpha h)(x)) > 2^{k_0-1}\right\}\right)\\ &\leq \frac{C}{\Psi\left(2^{k_0-1}\right)} \int_{\tau B} \Psi(\alpha h) d\mu.\end{aligned} \tag{6.16}$$

□

We now have a set E_{k_0} of positive measure on which u is 2^{k_0+1}-Lipschitz. We can also assume that $ess \inf_{E_{k_0}} |u| = 0$ by adding a constant to u. We apply (6.13) and the Lipschitz continuity to obtain an upper bound on a_{k_0}.

$$\begin{aligned} a_{k_0} = \sup_{E_{k_0}} |u| \le 2^{k_0-1} 8R \le \Psi^{-1}\left(\Psi\left(2^{k_0-1}\right)\right) 8R \\ \le \frac{16R}{\mu(B)} \Psi^{-1}\left(\fint_{\tau B} \Psi(\alpha h) d\mu\right). \end{aligned} \tag{6.17}$$

Define $A_k = E_k \backslash E_{k-1}$ to rewrite (6.6) as

$$\frac{1}{2}\int_B |u - u_B| d\mu \le \sum_{k=-\infty}^{\infty} a_k \mu(A_k) \tag{6.18}$$

For $k > k_0$, (6.11) guaratees that

$$\begin{aligned} \sum_{k=-\infty}^{\infty} a_k \mu(A_k) &\le \sum_{k=-\infty}^{k_0} a_{k_0}\mu(A_k) \\ &\quad + \sum_{k=k_0+1}^{\infty}\left(a_{k_0} + \frac{CR}{\mu(B)^{\frac{1}{Q}}} \sum_{i=k_0+1}^{k} \frac{2^i}{\Psi\left(2^{i-1}\right)^{\frac{1}{Q}}}\left(\int_{\tau B}\Psi(\alpha h)d\mu\right)^{\frac{1}{Q}}\right) \\ &\le \sum_{k=-\infty}^{\infty} a_{k_0}\mu(A_k) \\ &\quad + \frac{CR}{\mu(B)^{\frac{1}{Q}}}\left(\int_{\tau B}\Psi(\alpha h)d\mu\right)^{\frac{1}{Q}} \sum_{k=k_0+1}^{\infty}\sum_{i=k_0+1}^{k} \frac{2^i}{\Psi\left(2^{i-1}\right)^{\frac{1}{Q}}}\mu(B\backslash E_{k-1}). \end{aligned} \tag{6.19}$$

From (6.10) we have that

$$\begin{aligned} \frac{CR}{\mu(B)^{\frac{1}{Q}}}\left(\int_{\tau B}\Psi(\alpha h)d\mu\right)^{\frac{1}{Q}} \sum_{k=k_0+1}^{\infty}\sum_{i=k_0+1}^{k} \frac{2^i}{\Psi\left(2^{i-1}\right)^{\frac{1}{Q}}}\mu(B\backslash E_{k-1}) \\ \le \frac{CR}{\mu(B)^{\frac{1}{Q}}}\left(\int_{\tau B}\Psi(\alpha h)d\mu\right)^{1+\frac{1}{Q}} \frac{2^{k_0}}{\Psi\left(2^{k_0}\right)^{1+\frac{1}{Q}}}. \end{aligned} \tag{6.20}$$

Switching the orders of summation on the left hand side and repeatedly applying the fact that $\frac{\Psi(2^j)}{\Psi(2^{j+1})} \leq \frac{1}{2}$ for all j gives that

$$\begin{aligned}\sum_{k=k_0+1}^{\infty}\sum_{i=k_0+1}^{k}\frac{2^i}{\Psi\left(2^{i-1}\right)^{\frac{1}{Q}}}\frac{1}{\Psi\left(2^{k-1}\right)} &\leq \sum_{i=k_0+1}^{\infty}\frac{2^i}{\Psi\left(2^{i-1}\right)^{\frac{1}{Q}}}\sum_{k=i}^{\infty}\frac{1}{\Psi\left(2^{k-1}\right)}\\ &\leq \sum_{i=k_0+1}^{\infty}\frac{2^i}{\Psi\left(2^{i-1}\right)^{\frac{1}{Q}}}\frac{1}{\Psi\left(2^{i-1}\right)}\sum_{j=0}^{\infty}2^{-j}\\ &\leq \sum_{i=k_o+1}^{\infty}\frac{2^i}{2^{(i-k_o-1)(1+\frac{1}{Q})}}.\end{aligned}$$

We derive the desired Orlicz-Poincaré inequality from (6.18) by applying (6.19), (6.20), (6.15), and the doubling property of μ to get

$$\begin{aligned}\int_B |u-u_B|d\mu &\leq CR\Psi^{-1}\left(\fint_{\tau B}\Psi(\alpha h)d\mu\right)\mu(B)\\ &\quad+\frac{CR\mu(\tau B)^{1+\frac{1}{Q}}}{\mu(B)^{\frac{1}{Q}}}\left(\fint_{\tau B}\Psi(\alpha h)d\mu\right)^{1+\frac{1}{Q}}\frac{\Psi^{-1}\left(\fint_{\tau B}\Psi(\alpha h)d\mu\right)}{\Psi\left(2^{k_0}\right)^{1+\frac{1}{Q}}}\\ &\leq CR\Psi^{-1}\left(\fint_{\tau B}\Psi(\alpha h)d\mu\right)\mu(B)\\ &\quad+CR\left(\frac{2\Psi\left(2^{k_0}\right)}{\Psi\left(2^{k_0}\right)}\right)^{1+\frac{1}{Q}}\Psi^{-1}\left(\fint_{\tau B}\Psi(\alpha h)d\mu\right)\mu(B)\\ &\leq CR\Psi^{-1}\left(\fint_{\tau B}\Psi(\alpha h)d\mu\right)\mu(B).\end{aligned}$$

□

The above proposition along with Proposition 3.19 give an equivalent condition to the satisfaction of Orlicz-Poincaré inequalities. We now can prove an important theorem that allows to check only Lipschitz functions with Lipschitz continuous upper gradients when verifying that a space admits an Orlicz-Poincaré inequality.

Proposition 6.10. *Suppose that X is a complete and doubling metric measure space and Φ a doubling Young function. Then X admits a weak Ψ-Poincaré inequality if and only if there exist constants $C>0$ such that (3.9) holds for every open ball B in X, for every Lipschitz function $u: X\to\mathbb{R}$, and for every Lipschitz continuous upper gradient $\rho: X\to[0,\infty)$ of u in X. The constants in the Orlicz-Poincaré inequality depend only on each other and on the doubling constant of the measure.*

This proposition is analogous to Proposition 3.22. We provide only a summary of the proof found in [15] as the only change from the classical case is the use of (6.3) instead of the classical pointwise inequality with $\Psi(t) = t^p$.

Proof. We first show that it is sufficient to check only Lipschitz functions and their Lipschitz continuous upper gradients. By Proposition 3.19, the pointwise inequality (6.3) holds for each Lipschitz function and for each of its of Lipschitz continuous upper gradients in X. If we can prove that (6.3) holds for a fixed pair (u, ρ), where u is an measurable function and ρ some upper gradient, for all x, y in some full measure subset of a dilated ball, then we can apply Proposition 6.8. We can replace ρ with a suitable function in such way that allows us to assume that ρ is lower semicontinuous. This gives that there is an increasing sequence of Lipschitz functions ρ_i that converge pointwise to ρ. For each of these Lipschitz functions define a function u_i so that

$$u_i(z) = \inf_{\gamma} \int_{\gamma} \rho_i ds,$$

where the infimum is taken over all rectifiable curves, γ, joining z and y in some dilated ball. The quasiconvexity of X guarantees that u_i is Lipschitz for each i. We also have that ρ_i is an upper gradient of u_i with Lip $u_i(z) \leq L\rho_i(z)$. Next, we consider the cutoff function v_i of u_i that vanishes outside of some dilated ball. We then create a Lipschitz upper gradient τ_i of v_i that satisfies

$$\text{Lip } v_i(z) \leq L\tau_i(z),$$

and $L\tau_i$ is a Lipschitz continuous upper gradient of v_i. We then conclude that (6.3) holds for u_i and ρ. To finish the proof, we show by contradiction that $|u_i(x)| \geq \frac{1}{2}$ for some i. The key step is to find a sequence of rectifiable curves (γ_i) joining x to y in some dilated ball such that

$$\int_{\gamma_i} \rho_i \leq \frac{1}{2}$$

for each i. The Arzelá-Ascoli theorem guarantees that a subsequence converges uniformly to a Lipschitz map γ. The contradiction is reached by showing that

$$\int_{\gamma} \rho ds \leq \frac{1}{2},$$

and concluding that ρ cannot be an upper gradient of u. The proof is then complete as if we assume that X admits an Orlicz-Poincaré inequality then it is thick quasiconvex. This implies that Lip $u(x) \leq C\rho(x)$ for every $x \in X$ and for every continuous upper gradient ρ of u in X. $\square$

We will utilize the following proposition which is also used in the classical case.

Proposition 6.11 ([15, Chapter 10]). *Let X be a length space which is a subset of a geodesic metric space Z. Let u and ρ be bounded Lipschitz functions on X such that $\inf_X \rho > 0$ and ρ is an upper gradient of u. Fix $\delta > 0$. Then there exist Lipschitz functions $\bar{u}$ and $\bar{\rho}$ on Z which extend to u and ρ respectively. Moreover, $\bar{\rho}$ is bounded and $(1+\delta)\bar{\rho}$ is an upper gradient of $\bar{u}$ (on Z).*

We now have the major results necessary to verify Theorem 6.7.

Proof of Theorem 6.7. We reproduce the proof found in [15]. Let (X_i, d_i, μ_i, p_i) converge to (X, d, μ, p) in the pointed measured Gromov-Hausdorff sense. Proposition 6.10 gives that it suffices to verify that X admits a Ψ-Poincaré inequality for each pair (u, ρ) where u is a bounded Lipschitz function and ρ a bounded Lipschitz continuous upper gradient. We use Proposition 6.11 to show that there are Lipschitz extensions $\bar{u}$ and $\bar{\rho}$ of u and $\rho + \frac{\delta}{2}$ so that $(1+\delta)\bar{\rho}$ is an upper gradient of $\bar{u}$. Denote $B_\infty(\iota(x), r)$ and $B_\infty(\iota_i(x_i), r)$ as B^∞ and B_i^∞ respectively. (X_i, d_i, μ_i, p_i) converging to (X, d, μ, p) guarantees that there exists an N such that $2B^\infty \subset 4B_i^\infty \subset 6B^\infty$ and $2\lambda B^\infty \subset 4\lambda B_i^\infty \subset 6\lambda B^\infty$ holds for each $n \geq N$. Since X_i admits a Ψ-Poincaré inequality, we have

$$\begin{aligned}
\fint_{2B^\infty} |\bar{u} - \bar{u}_{4B_i^\infty}| d((\iota_i)_\sharp \mu_i) &\leq C_\Psi C_D r \Psi^{-1}\left(\fint_{4\lambda B_i^\infty} \Psi\left((1+\delta)\bar{\rho}\right) d((\iota_i)_\sharp \mu_i)\right) \\
&\leq C_\Psi C_D^2 r \Psi^{-1}\left(\fint_{6\lambda B^\infty} \Psi\left((1+\delta)\bar{\rho}\right) d((\iota_i)_\sharp \mu_i)\right)
\end{aligned}$$

We use the fact that $\bar{u}_{4B_i^\infty} \to \alpha$ for some $\alpha \in \mathbb{R}$, μ is also doubling for C_D^2, and that $\int_K u d\mu \leq \liminf_i \int_W u d\mu_i$ for a compact set $K \subset X$, a nonnegative continuous function u, and bounded set W with $K \subset W$ and $\mathrm{dist}(K, X \backslash W) > 0$ to obtain

$$\begin{aligned}
\fint_{B^\infty} |\bar{u} - \alpha| d((\iota)_\sharp \mu) &\leq \frac{1}{(\iota)_\sharp \mu\,(B^\infty)} \int_{2B_i^\infty} |\bar{u} - \alpha| d((\iota)_\sharp \mu) \\
&\leq C \liminf_i \fint_{2B_i^\infty} |\bar{u} - \bar{u}_{4B_i^\infty}| d((\iota_i)_\sharp \mu_i) \\
&\leq C r \limsup_i \Psi^{-1}\left(\fint_{6\lambda \bar{B}^\infty} \Psi\left((1+\delta)\bar{\rho}\right) d((\iota_i)_\sharp \mu)\right) \\
&\leq C r \Psi^{-1}\left(\fint_{7\lambda B^\infty} \Psi\left((1+\delta)\bar{\rho}\right) d((\iota)_\sharp \mu)\right).
\end{aligned}$$

Send $\delta \to 0$ to obtain

$$\fint_{B^\infty} |\bar{u} - \alpha| d((\iota)_\sharp \mu) \leq C r \Psi^{-1} \left(\fint_{7\lambda B^\infty} \Psi(\bar{\rho})\, d((\iota)_\sharp \mu) \right).$$

Finally, we use that $\fint_B |u - u_B| d\mu \leq 2 \fint_B |u - \alpha| d\mu$ to conclude that

$$\fint_B |u - u_B| d\mu \leq C r \Psi^{-1} \left(\fint_{7\lambda B} \Psi(\rho)\, d\mu \right)$$

as desired. □

While Theorem 6.7 is the key step in answering Question 2, it is also important in the development of nondoubling Orlicz-Poincaré inequalities. There are many interesting results in the classical theory that are not known to extend to the non-doubling case. Each result that extends to the nondoubling case hints that this case may be just as robust as the classical theory.

7 Remarks, Questions, and Conjectures

Theorem 1.3 shows that improvement in the vein of Keith and Zhong can be obtained for some Orlicz-Poincaré inequalities. It also raises a number of questions.

Question 7.1. *Does the growth condition (1.2) give the best possible r for the family $\Gamma_{p,r}$ in Theorem 1.3?*

We recall the discussion at the end of Chapter 5. The Cantor diamond example shows that Orlicz-Poincaré inequalities above $\Gamma_{2,\frac{\sqrt{5}+1}{2}}$-Poincaré inequalities may not improve. However, the example shows nothing for $1 < r < \frac{\sqrt{5}+1}{2}$. This may mean that condition (1.2) could be strengthened. However, the deficit is probably in the example suggesting that $r = p - 1$ may be best possible.

Question 7.2. *Are there examples that show the necessity of the growth condition (1.2) for $p > 2$?*

The cubes that constitute the Cantor Diamond sets used in the planar case can be built in higher dimensions as well. Analogous to the planar case, these sets could give examples for p equal to the dimension of the constructed cubes. A related question is to ask what happens to the r generated from these examples in other dimensions. More precisely, if we denote the parameter derived from the n-dimensional Cantor diamonds as r_n, so $r_2 = \frac{\sqrt{5}+1}{2}$, then what happens to r_n as $n \to \infty$? If $r_n \to n - 1$, then these Cantor diamond are the correct case of examples as we recover growth condition (1.2) in the limit. If $r_n \to \infty$ in some other fashion, then it may be another suggestion that we should look for another class of examples.

Question 7.3. *Is there a version of Theorem 1.3 for doubling Orlicz functions that dominate $\Gamma_{p,r}$ for $r > p - 1$?*

The Orlicz functions that dominate $\Gamma_{p,r}$ for $r > p - 1$ fail the growth condition (1.2). If we weaken the conclusion of the theorem, then we may be able to remove the growth condition. More precisely, we may find improvement on the Orlicz scale to another Orlicz function that fails the growth condition. It is unclear if this range of improvement would include the case $r = p - 1$. Improvement all the way to the $r = p - 1$ case would have implications for the next question. The growth condition appears in other contexts as well. Recall from Chapter 3 that the growth condition also serves as the divide between the Lorentz spaces $L^{p,1}$ and $L^{p,q}$ for $q > 1$ and as the divide for the optimal embedding range in the theory of Orlicz-Sobolev embeddings.

Question 7.4. *What Orlicz-Poincaré inequality other than those defined by Young functions dominating* Ψ *guaranteed by Jensen's inequality, if any, holds for the case* $p_\Psi = 1$*?*

We could see improvement of the following form: suppose X admits a $\Gamma_{1,r}$-Poincaré inequality for some $r > 0$, then X admits a Ψ-Poincaré inequality for some $\Psi(t)$ so that $\lim_{t\to\infty} \frac{\Psi(t)}{\Gamma_{1,r}(t)} = 0$. A 1-Poincaré inequality represents the case $r = p - 1 = 0$ that is not accounted for in Theorem 1.3. It would be interesting to see how close functions above the case $r = p - 1$ can improve towards that case.

Question 7.5. *If* X *admits a nondoubling Orlicz-Poincaré inequality, then does it improve to another nondoubling Orlicz-Poincaré inequality where the new function defining the inequality is dominated by the original Orlicz function?*

Removing the doubling assumption would require a complete overhaul of both the proof and the statement of Theorem 1.3.

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