GAME THEORETIC FORMULATIONS AND SOLUTION METHODS FOR MICROECONOMIC MARKET MODELS

BY

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DISSERTATION

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Abstract

Microeconomic equilibrium problems are intimately related to game theory, but the current state of knowledge for several types of microeconomic problems is limited. This dissertation rigorously addresses several of these problems from the standpoint of variational inequality theory. In addition to proving theoretical results about problem properties, important microeconomic implications are discussed when applicable. The individual chapters of this work focus on:

- equilibrium existence for perfectly competitive capacity expansion problems with risk-averse players;
- a comprehensive analysis of the application of Lemke’s method to affine generalized Nash equilibrium problems;
- the formulation and study of a unified power market model encompassing different microeconomic behavioral assumptions, capacity markets, emission permit auctions, and consumer surplus maximization;
- an investigation of differential Nash games with mixed state-control constraints.

Although the presented results are applicable to a broad class of games that satisfy certain structural properties, electricity markets represent a key application area underlying several of the chapters and are specifically addressed in Chapter 4. As a whole, this dis-
sertation should clearly illustrate how variational inequality theory can be used to analyze these applications and should provide the basis for the development of effective solution methodologies. It should also provide deeper insights into complex microeconomic games that cannot be described solely with demand and supply curves. Suggestions for additional research are provided at the end of each chapter to motivate future work in the respective domains.
Acknowledgments

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List of Symbols

Spaces
\( \mathbb{R}^n \) The real \( n \)-dimensional space
\( \mathbb{R}^n_+ \) The nonnegative orthant of \( \mathbb{R}^n \)

Matrices
\( A \) A matrix with real entries \( a_{ij} \)
\( A^T \) The transpose of matrix \( A \)
\( \det(A) \) The determinant of square matrix \( A \)
\( A^{-1} \) The inverse of square matrix \( A \)
\( \lambda_{\text{max}}(A) \) The largest eigenvalue of the symmetric matrix \( A \)
\( \|A\| \) The Euclidean norm of square matrix \( A \), \( \sqrt{\lambda_{\text{max}}(A^T A)} \)
\( \rho(A) \) The spectral radius of square matrix \( A \) (i.e., the largest magnitude eigenvalue of \( A \))
\( I \) The identity matrix
\( \epsilon I \) The identity matrix with diagonal entries of \( \epsilon \)

Vectors
\( x \) A column vector with real entries \( x_i \)
\( x^T \) The transpose of vector \( x \)
\( \{x_k\} \) The sequence of vectors \( x^1, x^2, \cdots \)
\(x^*\) The optimal value of vector \(x\)

\(|x|\) The Euclidean norm of vector \(x\), \(\sqrt{x^T x}\)

\(\text{diag}(x)\) The diagonal matrix with diagonal elements equal to the components of vector \(x\)

\(x \perp y\) Vectors \(x\) and \(y\) are perpendicular

\(x^\perp\) The orthogonal complement of vector \(x\)

\(1\) The vector of all ones

**Functions**

\(F : \mathcal{D} \to \mathcal{R}\) A mapping with domain \(\mathcal{D}\) and range \(\mathcal{R}\)

\(JF\) The Jacobian of mapping \(F\)

\(\partial F(x)\) The Clarke generalized Jacobian of mapping \(F\) at point \(x\)

\(f(x)\) The scalar function \(f\) evaluated at point \(x\)

\(f(x;y)\) The scalar function \(f(x, y)\) where \(y\) is treated as constant

\(\nabla f\) The gradient of function \(f\)

\(\nabla^2 f\) The Hessian matrix of function \(f\)

\(\partial f(x)\) The Clarke subdifferential of scalar function \(f\) at point \(x\)

\(f \in C^k\) The scalar function \(f\) is continuously differentiable of order \(k\)

\(\text{argmax } f(x)\) The set of points that maximizes the scalar function \(f\)

\(\text{argmin } f(x)\) The set of points that minimizes the scalar function \(f\)

**Sets**

\(K_1 \subset K_2\) Set \(K_1\) is a proper subset of set \(K_2\)

\(K_1 \subseteq K_2\) Set \(K_1\) is a subset of set \(K_2\)

\(K_1 \cap K_2\) The intersection of sets \(K_1\) and \(K_2\)

\(K_1 \cup K_2\) The union of sets \(K_1\) and \(K_2\)

\(K_1 \setminus K_2\) The complement of set \(K_2\) in set \(K_1\)
∪Kᵢ  The union of sets Kᵢ
ΠKᵢ  The Cartesian product of sets Kᵢ
B(F; ϵ, K)  ϵ-neighborhood of mapping F restricted to the set K, comprising all continuous functions G such that \[\sup_{y \in K} \|G(y) - F(y)\| < ϵ\]
B(x, ϵ)  The open ball with center at point x and radius ϵ
C(x; K, F)  The critical cone of the pair (K, F) at point x ∈ K
K∞  The recession cone of set K
T(x; K)  The tangent cone of set K at point x ∈ K
cl K  The topological closure of set K
∅  The empty set

VI Terminology

VI(K, F)  The variational inequality defined by set K and mapping F
SOL(K, F)  The solution set of VI(K, F)
NCP(F)  The nonlinear complementarity problem defined by mapping F
LCP(q, M)  The linear complementarity problem defined by vector q and matrix M
SOL(q, M)  The solution set of LCP(q, M)
F^n_{nat}  The natural map associated with pair (K, F)
F^n_{nor}  The normal map associated with pair (K, F)

Miscellaneous

Πaᵢ  The product of scalars aᵢ
Π_K(x)  The Euclidean projection of point x onto set K
E(X)  The expectation of random variable X
Chapter 1

Introduction

In the years since its introduction by von Neumann and Morgenstern [169] in 1944, game theory has proven to be useful in the analysis of a wide variety of problems, including those arising in microeconomics, engineering, political science, and psychology. In each of these problems, individual entities, known as “players,” interact and make decisions that affect each other. These interactions may change either the ordering of player outcome preferences or the set of feasible actions available to a player. It should be apparent that an optimization problem trivially corresponds to a game consisting of only one player.

Games are typically classified according to several distinctions, the most important of which are arguably the following:

- Cooperative or non-cooperative
  
  Each player in a non-cooperative game acts independently to achieve its best individual outcome while players in a cooperative game make decisions jointly to achieve a mutually desirable outcome. Since this dissertation focuses on microeconomic games, which are usually modeled as non-cooperative due to regulations restricting producer
collusion, cooperative game theory will not be discussed. The interested reader may refer to [96, 97, 103, 120, 162, 207] for a brief overview of research on cooperative games.

- **Zero-sum or nonzero-sum**

  Zero-sum games and nonzero-sum games differ in how player payoffs are allocated. For zero-sum games, the aggregate payoff across players is, by definition, zero so the cumulative positive payoff must equal the cumulative negative payoff. This concept can colloquially be stated as “if I win [blank], you lose [blank],” the idea behind many athletic competitions. On the other hand, a nonzero-sum game can have a nonzero net payoff, meaning that while some players may lose and other players may win, the aggregate result of the game can be a “win” or a “loss” across all players. Economic transactions are typical examples of “winning” nonzero-sum games because both parties should be better off after the exchange. This dissertation addresses nonzero-sum games; refer to [25, 51, 168, 194] for some of the only theoretical research on zero-sum games since their introduction in [169].

- **Perfect information or imperfect information**

  The information available to players can greatly affect their decisions. In a game with perfect information, all players have the exact same knowledge about current conditions. Games with imperfect information do not possess this property and therefore must be analyzed in a different manner. A large amount of research has been conducted on both of these game forms (e.g., [17, 98, 117, 191, 198, 205, 217]), but only games characterized by perfect information are addressed here.

- **Open-loop or closed-loop**

  The concepts of open-loop and closed-loop games are related to the effect of feed-
back on player decisions, a principle that is associated with the passage of time. In an open-loop game, feedback is not present; the opposite is true for closed-loop games. Because of the relationship between feedback and time, this game distinction is traditionally only applicable to differential Nash games (see Chapter 5). The relationship between these two problem forms has been studied in detail (e.g., [82, 84, 193, 203, 215]), and it has been proven that every open-loop Nash equilibrium strategy is a closed-loop Nash equilibrium strategy. The converse implication does not hold. Only open-loop linear-quadratic differential Nash games are discussed in this dissertation, but the interested reader can refer to [14, 72, 185] for a general discussion of open-loop and closed-loop properties for the problem studied in Chapter 5.

- Discrete strategy space or continuous strategy space

Many simple games involve players choosing actions from discrete, finite strategy spaces. Examples of such games include the prisoner’s dilemma and the coordination game presented in Examples 1 and 2 of Section 1.2. Games with continuous strategy spaces cannot be expressed via a payoff matrix but are more applicable to microeconomic games and are therefore examined here.

- Cournot or Bertrand [in microeconomic games]

Cournot and Bertrand competition are two fundamental forms of strategic competition in microeconomics. By definition, Cournot players compete in quantity knowing that their production affects price whereas Bertrand players compete in price. Descriptions, assumptions, and properties of these modeling methods can be found in any undergraduate-level microeconomics textbook. All of the strategic games studied in this dissertation follow the quantity competition framework of Cournot models.
1.1 Solution concepts

To formalize the concept of a non-cooperative game, it is important to understand how players make decisions and evaluate outcomes. Naturally, each player must have some method of judging possible outcomes in terms of preference. For a game with \( \{1, \cdots, F\} \) being the finite set of players, let \( K_i \subseteq \mathbb{R}^{n_i} \) denote the set of feasible decisions for player \( i \) and \( K_{-i} \triangleq \prod_{j \neq i} K_j \) denote the concatenated feasible space of all players other than player \( i \).

For player \( i \) to order all possible outcomes in terms of preference of decisions \( x \triangleq (x_i, x_{-i}) \) where \( x_i \in K_i \) and \( x_{-i} \in K_{-i} \), there must exist some objective function \( f_i : \prod_{i=1}^{F} K_i \to \mathbb{R} \) that ranks player \( i \)'s outcome preferences either ordinally or cardinally. For microeconomic games, \( f_i \) is usually profit or a monotonic transformation thereof.

In a non-cooperative setting, player \( i \) has no direct influence over rivals’ decisions so \( x_{-i} \) should be treated exogenously by player \( i \). In the setting of a minimization problem, player \( i \) wants to solve

\[
\text{minimize}_{x_i \in K_i} \quad f_i(x_i; x_{-i}) \tag{1.1}
\]

with \( x_{-i} \) exogenous in (1.1) but endogenous to the game as a whole. As introduced by John Nash [163, 165], one possible solution to the game in which each player \( i \) solves (1.1) is a tuple of feasible decisions \( x^* \triangleq (x^*_i)_{i=1}^{F} \) such that no player can improve its objective function by unilaterally changing its decision. This type of game and the associated solution concept are known as a Nash game/Nash equilibrium problem (NEP) and a Nash equilibrium (NE), respectively. Mathematically, a feasible decision vector \( x^* \) is a Nash equilibrium if and only if, for each player \( i = 1, \cdots, F \),

\[
f_i(x_i^*; x_{-i}^*) \leq f_i(x_i; x_{-i}^*) \text{ for all } x_i \in K_i. \tag{1.2}
\]
The above development of player $i$’s optimization problem assumes that the feasible region $K_i$ is independent of others’ decisions $x_{-i}$. Although this is a common phenomenon for some problems, it may happen that players’ decisions can be restricted by their rivals’ decisions. For example, the availability of a fixed number of emission permits links producers’ feasible regions together because the total number of permits purchased by or allocated to the producers cannot exceed the permit supply. In these problems, player $i$’s feasible region is defined by the set $K_i(x_{-i})$ where $x_{-i}$ is a feasible tuple of rivals’ decisions. This extended problem is known as a generalized Nash equilibrium problem (GNEP) [73, 128, 133, 135, 179]. Although the definition of a generalized Nash equilibrium (GNE) is identical to that of a Nash equilibrium except with $K_i(x_{-i}^*)$ replacing $K_i$, the identification of a GNEP equilibrium is significantly more difficult than for a standard NEP for a number of reasons. For instance, a GNE $x^*$ must satisfy the condition $x^* \in K(x^*) \triangleq \mathcal{F} \prod_{i=1}^{n} K_i(x^*_{-i})$ which by itself demands that the point-to-set map $K$ has a fixed point. Due to the inherent difficulty of solving GNEPs, considerable research into tractable GNE restrictions has been conducted (e.g., [73, 85, 197]).

The aforementioned NEPs have only dealt with static (i.e., discrete-time) games in which players make decisions by solving optimization problems in finite dimensions. The concept of a Nash equilibrium is not limited to this discrete-time case and can also be applied to differential games in which players choose continuous-time decision trajectories to solve optimal control problems. In their simplest form, optimal control problems deal with two classes of decision variables as determined by a set of ordinary differential equations and constrained by algebraic inequalities: “control” variables are not governed by ordinary differential equations while “state” variables are differential. Although the extension of Nash equilibria to this differential game may not seem difficult, the incorporation of optimal control problems makes equilibria much more difficult to find. Indeed, current deriva-
tions of necessary optimality conditions for a certain class of optimal control problems are incomplete at best (see e.g. [99]).

1.2 Simple examples

Because the Nash equilibrium concept is fundamental to this research, three simple and well-known example games are presented. These problems, as well as many others, can easily be found in introductory game theory textbooks.

**Example 1** (Prisoner’s dilemma). Two suspects are arrested on suspicion of robbing a bank. They are questioned separately by the police and cannot communicate. If neither suspect testifies against the other, both suspects receive one year in prison on a minor charge. If only one suspect testifies against the other, the silent suspect will receive five years in prison while the other goes free. If both suspects testify against each other, they both receive three years in prison. Given that the suspects want to minimize their personal prison terms (maximize their time outside of prison), the game’s payoff matrix is provided with Player 1 choosing the row, Player 2 choosing the column, and the pair (a,b) corresponding to (Player 1 payoff, Player 2 payoff). The Nash equilibrium strategy is indicated by NE in the upper left corner of the appropriate block.

Figure 1.1: Payoff matrix of the Prisoner’s dilemma game

<table>
<thead>
<tr>
<th></th>
<th>Talk</th>
<th>Silent</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>NE</strong></td>
<td>-3, -3</td>
<td>0, -5</td>
</tr>
<tr>
<td>Talk</td>
<td>-5, 0</td>
<td>-1, -1</td>
</tr>
<tr>
<td>Silent</td>
<td>-3, -3</td>
<td>0, -5</td>
</tr>
</tbody>
</table>

The unique Nash equilibrium of this game is for both prisoners to testify.

**Example 2** (Coordination game). A handshake is required to finalize a business deal. It is only possible for two people to shake hands if the same hand is offered by each person and there is only one chance to shake. The handshake will result in a $5 million profit for each party.

![Payoff matrix of the Coordination game](image)

There are two Nash equilibria for this game, thus demonstrating that Nash equilibria are not necessarily unique.

**Example 3** (Emission game). Two polluters have the choice of emitting either “Low” or “High” levels of pollution. However, regulatory limits on pollution quantity are set so that it is not possible for both polluters to emit at a “High” level. Because of pollution mitigation costs, polluter 1 loses $5 million if emitting at “Low” and loses $3 million at “High.” Polluter 2 loses $6 million at “Low” and $2 million at “High.” The payoff matrix is provided and labeled with GNE instead of NE because the polluters “share” the regulatory constraint regarding their combined emissions; the crossed-out space indicates an infeasible strategy pair due to this shared regulatory limit.
Both generalized Nash equilibria for this game result in the same total emission level.

The simplistic nature of these examples should not be construed to indicate that game theoretic problems are trivial [otherwise there would no reason for this dissertation]. Instead, these examples were presented to motivate the application of game theory to significantly more complicated problems for which conclusions cannot be drawn from such an elementary analysis.

### 1.3 Basic variational inequality theory

At a mathematical level, game theory is intimately related to the field of variational inequality (VI) problems. Therefore, this section aims to provide the reader with a basic understanding of this mathematical framework and will be crucial to most of the theoretical results of this dissertation. Only a brief review of finite-dimensional VI problems is presented here; the interested reader is referred to [75] for a comprehensive discussion of VI theory.
1.3.1 Problem definition and equivalent formulations

Before addressing the relationship between game theory and VI problems, the definition of a VI is needed. With $K \subseteq \mathbb{R}^n$ and mapping $F : K \to \mathbb{R}^n$, the variational inequality $VI(K, F)$ attempts to identify a vector $x \in K$ such that

$$(y - x)^T F(x) \geq 0 \quad \text{for all } y \in K.$$  \hfill (1.3)

Associated with $VI(K, F)$ is a [possibly empty] solution set denoted by $SOL(K, F)$. In the special case when $K \triangleq \mathbb{R}^n_+$, $VI(K, F)$ can be expressed as a nonlinear complementarity problem $NCP(F)$ which attempts to identify a vector $x \in \mathbb{R}^n_+$ such that

$$0 \leq x \perp F(x) \geq 0,$$

or equivalently,

$$
\begin{cases}
  x_j \geq 0 \\
  (F(x))_j \geq 0 \\
  x_j(F(x))_j = 0
\end{cases}
$$

Finally, if $F$ is an affine map given by $q + Mx$ for some vector $q \in \mathbb{R}^n$ and some matrix $M \in \mathbb{R}^{n \times n}$, the $NCP(F)$ becomes a linear complementarity problem $LCP(q, M)$.

In addition to these variational inequality specializations, it is important to realize that problems can be posed in either primal or primal-dual form. Assume that

$$K \triangleq \{ x \in \mathbb{R}^n \mid Ax \geq b, Cx = d \},$$  \hfill (1.4)

where $A \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{\ell \times n}$, $b \in \mathbb{R}^m$, and $d \in \mathbb{R}^\ell$. Using only primal variables, $VI(K, F)$ can be formulated as (1.3); this can viewed as the \textit{primal} variational inequality associated with the problem. If dual multipliers are introduced for each constraint, an equivalent \textit{primal-dual} variational inequality can be defined. The following well-known proposition
guarantees that the primal and the primal-dual variational inequalities solve the same problem.

**Proposition 1.1** (Proposition 1.2.1, [75]). With $K$ defined by (1.4), a vector $x$ solves $\text{VI}(K, F)$ if and only if there exist vectors $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^\ell$ that solve the primal-dual variational inequality:

\[
0 = F(x) + C^T \mu - A^T \lambda \\
0 = d - Cx \\
0 \leq \lambda \perp Ax - b \geq 0.
\]

□

**Remark 1.1.** Technically, the primal-dual VI of Proposition 1.1 is a VI specialization known as a mixed complementarity problem (MiCP). □

If $K$ is further restricted to $\{x \in \mathbb{R}^n_+ \mid Ax \geq b\}$, it is simple to see that the primal-dual variational inequality given in Proposition 1.1 can be expressed as $\text{NCP}(\tilde{F})$ in the variables $(x, \lambda)$ with

\[
\tilde{F}(x, \lambda) \triangleq \begin{pmatrix} F(x) - A^T \lambda \\ Ax - b \end{pmatrix}.
\]

If $F$ is an affine map, $\text{NCP}(\tilde{F})$ is a linear complementarity problem. This correspondence between primal and primal-dual variational inequalities is leveraged at many points in this research because the different formulations can greatly simplify the proofs of certain problem properties.

To make the connection between optimization problems, Nash equilibria, and variational
inequalities explicit, consider the optimization problem

\[
\minimize_{x \in K} f(x) \tag{1.5}
\]

where \( K \subseteq \mathbb{R}^n \) is a closed convex set and the function \( f : U \to \mathbb{R} \) is continuously differentiable on \( U \), an open superset of \( K \). A necessary optimality condition for (1.5) is \( x^* \in \text{SOL}(K, \nabla f) \). If the function \( f \) is convex, \( x^* \in \text{SOL}(K, \nabla f) \) is also sufficient for optimality and (1.5) is equivalent to \( \text{VI}(K, \nabla f) \). In the opposite direction, a \( \text{VI}(K, F) \) with mapping \( F : U \to \mathbb{R}^n \) being continuously differentiable on \( U \), an open convex superset of the closed convex set \( K \), is associated with an optimization problem when \( JF(x) \) is symmetric for all \( x \in U \). This symmetry requirement is analogous to the Hessian of a scalar function being symmetric and proves that \( F \) is a gradient map.

When \( K \) is defined by (1.4), the cited relationship between (1.5) and \( \text{VI}(K, \nabla f) \) provides a direct equivalence between the Karush-Kuhn-Tucker (KKT) optimality conditions of optimization problem (1.5) and the primal-dual VI of Proposition 1.1 Recall the definition of KKT optimality conditions provided in Theorem 1.1.

**Theorem 1.1** (First-order necessary and sufficient KKT conditions). Let \( f \) be a continuously differentiable convex function and feasible region \( K \) be defined by (1.4). The vector \( x \) is a solution of (1.5) if and only if there exist vectors \( \lambda \in \mathbb{R}^m \) and \( \mu \in \mathbb{R}^\ell \) such that

\[
0 = \nabla f(x) + C^T \mu - A^T \lambda \\
0 = d - Cx \\
0 \leq \lambda \perp Ax - b \geq 0. \quad \square
\]

**Definition 1.1** (Linear independence constraint qualification (LICQ)). If the set of active
constraint gradients is linearly independent for vector $x$, then the linear independence constraint qualification (LICQ) holds at $x$. □

It is well known that the vectors $\lambda$ and $\mu$ in Theorem 1.1 are unique if LICQ holds. It is easy to see that the KKT conditions of Theorem 1.1 are equivalent to the primal-dual VI of $\text{VI}(K, \nabla f)$, hence the terminology KKT system of the VI for a primal-dual VI. Although not stated here, both Proposition 1.1 and Theorem 1.1 can be generalized for problems in which the feasible region $K$ is not polyhedral as long as a suitable constraint qualification holds at solution $x$.

Returning to the definition of a Nash equilibrium provided in Section 1.1, assume that each player $i$ has a closed convex feasible region $K_i \subseteq \mathbb{R}^{n_i}$ and an objective function $f_i : \prod_{i=1}^{\mathcal{F}} K_i \to \mathbb{R}$ that is convex and continuously differentiable in $x_i$ on an open superset of $K_i$. By definition, a Nash equilibrium solves

$$
\text{minimize}_{x_i \in K_i} f_i(x_i; x_{-i})
$$

and an equivalent VI($K, F$) can be formulated with

$$
K \triangleq \prod_{i=1}^{\mathcal{F}} K_i \quad \text{and} \quad F(x) \triangleq (\nabla_{x_i} f_i(x))_{i=1}^{\mathcal{F}}. \tag{1.6}
$$

For the associated primal-dual VI, let each $K_i$ be defined by an $i$-specialization of (1.4) (i.e., $\{x \in \mathbb{R}^{n_i} \mid A_i x \geq b_i, C_i x \geq d_i\}$). The point $x$ is a Nash equilibrium if and only if
there exist vectors \( \lambda \triangleq (\lambda_i)_{i=1}^F \) and \( \mu \triangleq (\mu_i)_{i=1}^F \) such that

\[
\begin{align*}
0 &= \nabla_{x_i} f_i(x_i, x_{-i}) + C_i^T \mu_i - A_i^T \lambda_i \\
0 &= d_i - C_i x_i \\
0 &\leq \lambda_i \perp A_i x_i - b_i \geq 0
\end{align*}
\]

\( F \)

(1.7)

The conditions of (1.7) obviously correspond to the concatenated set of KKT conditions for each player. Therefore, a Nash equilibrium can be expressed as a solution of either \( \text{VI}(K, F) \) defined by (1.6) or a concatenation of player KKT conditions (1.7).

To establish a similar variational inequality equivalence for a GNEP, the concept of a quasi-variational inequality (QVI) \( \text{[179]} \) must be introduced because player feasible regions are no longer fixed sets. After defining \( \text{QVI}(K, F) \) analogous to \( \text{VI}(K, F) \) except with \( K \) replaced by \( K(x) \), it is simple to see that a GNEP is equivalent to \( \text{QVI}(K, F) \) where \( F(x) \) is as defined in (1.6) and \( K(x) \triangleq F \prod_{i=1}^F K_i(x_{-i}) \).

1.3.2 Solution existence and uniqueness

Natural questions that arise when studying variational inequalities relate to solution existence and uniqueness. Namely, what properties of the \( \text{VI}(K, F) \) guarantee solution existence (i.e., \( \text{SOL}(K, F) \neq \emptyset \)) and when does uniqueness hold (i.e., \( \text{SOL}(K, F) \) is a singleton)?

In convex optimization, recall that a nonempty compact feasible region guarantees the existence of an optimal solution and uniqueness follows from strict convexity of the objective function. As may be expected, similar results hold for VIs. However, the more general character of variational inequalities allows for broader existence and uniqueness results.

**Proposition 1.2** (Proposition 2.2.3, [75]). Let \( K \subseteq \mathbb{R}^n \) be a closed convex set and mapping
$F : K \to \mathbb{R}^n$ be continuous. If there exists a vector $x^{\text{ref}} \in K$ such that the set

$$L_\prec \equiv \{ x \in K \mid (x - x^{\text{ref}})^T F(x) < 0 \}$$

is bounded (possibly empty), then VI($K, F$) has a solution. □

From the form of $L_\prec$, it is apparent that $x^{\text{ref}}$ of Proposition 1.2 corresponds to a specific $y \in K$ of (1.3). Furthermore, $x^* \in \text{SOL}(K, F)$ has $F(x^*) = 0$ if $x^*$ is in the topological interior of $K$. Therefore, $L_\prec$ cannot contain any interior solutions of VI($K, F$). Unfortunately, no further geometric characterization of $L_\prec$ is readily available. For a related existence result, consider the set

$$L_\leq \equiv \{ x \in K \mid (x - x^{\text{ref}})^T F(x) \leq 0 \} \quad (1.8)$$

which is obviously a nonempty superset of $L_\prec$. Given the existence of an appropriate $x^{\text{ref}} \in K$, boundedness of $L_\leq$ guarantees that SOL($K, F$) is a nonempty compact subset of $L_\leq$. It follows that the compactness of $K$ itself gives solution existence.

**Corollary 1.1** (Corollary 2.2.5, [75]). Let $K \subseteq \mathbb{R}^n$ be a compact convex set and mapping $F : K \to \mathbb{R}^n$ be continuous. The set SOL($K, F$) is nonempty and compact. □

Simple proofs of Corollary 1.1 are available via the classical fixed-point theorems of Brouwer (for single-valued maps) and Kakutani (for set-valued maps).

**Theorem 1.2** (Brouwer Fixed-Point Theorem). Let $K \subset \mathbb{R}^n$ be a nonempty, convex, compact set. Every continuous function $F : K \to K$ has a fixed point in $K$. □

**Theorem 1.3** (Kakutani Fixed-Point Theorem). Let $K \subset \mathbb{R}^n$ be a nonempty, convex, compact set. Let $F : K \to K$ be a set-valued map such that for each $x \in K$, $F(x)$ is a nonempty, closed, convex subset of $K$. If $F$ is closed on $K$, then $F$ has a fixed point. □
For Nash equilibrium problems, existence can commonly be proven with one of these fixed-point theorems through an examination of players’ best response maps, the solution sets of the players’ optimization problems parameterized by the rivals’ decision variables.\[163\][164][165].

Although Corollary 1.1, Theorem 1.2, and Theorem 1.3 can be used to prove solution existence for many problems, compactness of K does not always hold. Therefore, Proposition 1.2, which may be applied when K is unbounded, along with the related Proposition 1.3 are applicable to a wider variety of variational inequalities.

**Proposition 1.3** (Proposition 2.2.7, [75]). Let $K \subseteq \mathbb{R}^n$ be a closed convex set and mapping $F : K \to \mathbb{R}^n$ be continuous. If there exist a vector $x_{\text{ref}} \in K$ and a scalar $\zeta \geq 0$ such that

$$\lim_{x \in K, \|x\| \to \infty} \inf \frac{(x - x_{\text{ref}})^T F(x)}{\|x\| \zeta} > 0,$$

then SOL($K, F$) is nonempty and compact. \[
\]

The concept of monotonicity is intimately related to the uniqueness of variational inequality solutions. Akin to how the strict convexity of an objective function guarantees that there is at most one solution to an optimization problem, strict monotonicity provides the same property for variational inequalities.

**Definition 1.2.** A mapping $F : K \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is

(a) **monotone** on $K$ if $(x - y)^T (F(x) - F(y)) \geq 0$ for all $x, y \in K$;

(b) **strictly monotone** on $K$ if $(x - y)^T (F(x) - F(y)) > 0$ for all $x, y \in K$ with $x \neq y$. \[
\]
**Definition 1.3.** A mapping $F(x) \triangleq Mx$ is

(a) a *P matrix* if all principal minors of $M$ are positive;

(b) a *$P_0$ matrix* if all principal minors of $M$ are nonnegative.

Establishing the monotonicity of $F$ by definition is usually difficult. Luckily, more tractable conditions can be verified when $F$ is continuously differentiable.

**Proposition 1.4** (Proposition 2.3.2, [75]). Let mapping $F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuously differentiable on the open convex set $U$. The map $F$ is

(a) monotone on $U$ if and only if $JF(x)$ is positive semidefinite for all $x \in U$;

(b) strictly monotone on $U$ if $JF(x)$ is positive definite for all $x \in U$.

In many variational inequalities, monotonicity can be established on $K$ through Proposition 1.4 when $U$ is taken to be an open convex superset of $K$. However, an important terminology clarification is in order. Traditionally, positive definite/semidefinite matrices are defined with an assumption of matrix symmetry. This symmetry requirement is by no means necessary because, for any matrix $A \in \mathbb{R}^{n \times n}$ and vector $x \in \mathbb{R}^n$, $x^T Ax = x^T \left( \frac{A + A^T}{2} \right) x$. Therefore, an asymmetric matrix $A$ is positive definite if $\frac{A + A^T}{2}$ is positive definite. This definition of positive definite/semidefinite matrices is particularly important for variational inequalities because not every $F$ is a gradient map as was discussed previously. Indeed, many of the $F$ maps derived in this dissertation have asymmetric Jacobians due to their origin in Nash equilibrium problems. With the definition of monotonicity, the desired solution uniqueness result is available.

**Theorem 1.4** (Theorem 2.3.3, [75]). Let $K \subseteq \mathbb{R}^n$ be a closed convex set and mapping $F : K \rightarrow \mathbb{R}^n$. If $F$ is strictly monotone on $K$, $VI(K, F)$ has at most one solution.
It follows from Theorem 1.4 that a strictly monotone $F$ paired with solution existence established by results such as Propositions 1.2, 1.3, or Corollary 1.1 guarantees a unique solution.

A weaker form of uniqueness deals with the map $F$ evaluated for the set of VI solutions.

**Definition 1.4.** The solution set $\text{SOL}(K,F)$ is $F$-unique if $F(\text{SOL}(K,F))$ is at most a singleton.

It is obvious that solution uniqueness implies $F$-uniqueness but that the converse does not hold. Theoretically, $F$-uniqueness allows for a concise description of all elements in $\text{SOL}(K,F)$ (refer to [75, Proposition 2.3.12]) and can be established by a variety of monotone-type properties.

**Corollary 1.2** (Corollary 2.3.7, [75]). Let $K \subseteq \mathbb{R}^n$ be a closed convex set and mapping $F : U \supset K \to \mathbb{R}^n$ be continuously differentiable on the open set $U$. If $JF(x)$ is symmetric for all $x \in K$ and $F$ is monotone on $K$, then $F(\text{SOL}(K,F))$ is a singleton.

### 1.3.3 Specialization to partitioned variational inequalities

For many variational inequalities, especially those arising from Nash equilibrium problems (1.6), the feasible region $K$ of the primal VI (1.3) can be represented as the Cartesian product

$$F \prod_{i=1}^{r} K_i.$$  \hspace{1cm} (1.9)

This special VI representation, known as a *partitioned* VI, allows for the weakening of the existence and uniqueness conditions detailed for general variational inequalities. While the existence conditions for the Cartesian $K$ will not be presented (refer to [75, Proposition 3.5.1]), the properties required for solution uniqueness when $K$ is a Cartesian product
will be used in Chapter 4 and are therefore presented. Basically, the concept of $F$ being monotone (strictly monotone) can be relaxed to $F$ being a $P_0$ ($P$) function; the term relaxation is accurate because monotonicity (strict monotonicity) implies a $P_0$ ($P$) function but the converse is not true.

**Definition 1.5.** Let $K$ be given by (1.9). A mapping $F : K \rightarrow \mathbb{R}^n$ is

(a) a $P$ function on $K$ if for all pairs of distinct vectors $x$ and $y$ in $K$,

$$\max_{1 \leq i \leq \mathcal{F}} (x_i - y_i)^T (F_i(x) - F_i(y)) > 0;$$

(b) a $P_0$ function on $K$ if for all pairs of distinct vectors $x$ and $y$ in $K$, there exists an $i \in \{1, \cdots, \mathcal{F}\}$ such that

$$x_i \neq y_i \quad \text{and} \quad (x_i - y_i)^T (F_i(x) - F_i(y)) \geq 0. \quad \square$$

It is again difficult to verify Definition 1.5 directly. Fortunately, a more tractible condition similar to Proposition 1.4 can be verified with the additional requirement that $K$ is rectangular (i.e., the Cartesian product of one-dimensional intervals), an assumption that must be treated carefully.

**Proposition 1.5** (Proposition 3.5.9, [75]). Let mapping $F : U \rightarrow \mathbb{R}^n$ be continuously differentiable on the open set $U$ containing the rectangular set $K$. The following two statements hold.

(a) If $JF(x)$ is a $P$ matrix for all $x \in U$, then $F$ is a $P$ function on $U$.

(b) If $JF(x)$ is a $P_0$ matrix for all $x \in U$, then $F$ is a $P_0$ function on $U$. \quad \square
Proposition 1.6 (Proposition 3.5.10, [75]). Let $K$ be given by (1.9). If $F$ is a continuous $P_0$ function on $K$, then $VI(K, F + \epsilon I)$ has a unique solution for every $\epsilon > 0$. Furthermore, if $F$ is a $P$ function on $K$, then $VI(K, F)$ has at most one solution.

Many of the results of this dissertation can be derived using the basic variational inequality theory of this section. Any additional required results are presented as needed.

1.4 Research summary

This dissertation addresses four different types of games arising from microeconomic problems incorporating factors such as player risk aversion, shared constraints, and continuous-time dynamics. Each of the specialized problems poses a unique challenge to characterizing game properties and identifying relevant solution methods via VI theory. In this light, the dissertation aims to present and rigorously resolve each difficulty. Unique contributions of the research include the proof of equilibrium existence for a competitive capacity expansion game, the in-depth analysis of the application of Lemke’s method to games with shared constraints, the development of a detailed and unifying power market model, and the examination of differential Nash games with mixed state-control constraints.

The dissertation is divided into four chapters with the first three chapters dealing with static Nash games and the final chapter dealing with differential Nash games. Each chapter contributes in a variety of ways to the existing literature for its respective problem. For the reader’s convenience, a brief summary of each chapter is provided here.

Chapter 2

Competitive capacity expansion models, especially those analyzing responses

\footnote{This research direction was motivated by a problem posed by Benjamin Hobbs (The Johns Hopkins University).}
to potential emission regulation scenarios (e.g., [78, 245]), may provide valuable insights into the effects that the perceived likelihood of certain regulatory policies have on markets. Because of the probabilistic nature of outcomes, individual players solve two-stage stochastic optimization problems. When the associated Nash equilibrium problem is formulated as a variational inequality, the unbounded feasible region makes the proof of solution existence nontrivial. Therefore, this chapter proves existence for a broad class of these competitive capacity expansion models by utilizing a very general existence result. As a consequence, it can be claimed that every such game, regardless of the probabilities assigned to possible uncertainty realizations by each player, has a solution. The main contributions of this research include:

- the incorporation of nonlinear utility functions as player objectives;
- the establishment of a relationship between the given utility function problems and mean-risk models;
- the proof of solution existence for multi-market problems under three different assumptions on how competitive prices are determined.

Chapter 3. Shared constraints arise in many real-world problems as was mentioned in the development of the generalized Nash equilibrium concept in Section 1.1. When these constraints are included in a GNEP where each player solves a convex quadratic optimization problem, the necessary and sufficient solution optimality conditions take the form of a linear complementarity problem as described by Proposition 1.1. The behavior of Lemke’s method [47, 138, 178], a well-known method for solving linear complementarity problems, becomes unpredictable in the presence of shared constraints, a feature not previously noted in the literature. Furthermore, every possible solution obtained via Lemke’s
method is characterized by a special Lagrange multiplier property, thereby restricting the
types of solutions that can be identified to a very specific subset of all solutions. This
work rigorously studies the given GNEP and proposes methods to alleviate the problems
it induces in Lemke’s method. The primary contributions of this research include:

- the rigorous characterization of solutions that can be obtained via Lemke’s method;
- the formulation of a modified Lemke’s method that is guaranteed to terminate at a
  solution under suitable conditions;
- the development of a problem reformulation technique and an associated solution
  method that can find solutions not previously identifiable via Lemke’s method.

Chapter \[4\] Since the deregulation of the electricity industries in many countries in the
1990s, microeconomic electricity market models have been a subject of considerable re-
search with common research subfields including market power identification (e.g., \[32\])
and regulatory policy analysis (e.g., \[245\]). Given the broad range of questions and models
associated with electricity markets, several review papers have been authored \[86, 106, 222\],
but none of these reviews unifies the vast array of existing models, a necessary undertaking
in order to better represent the complexities inherent in the market. This research for-
mulates a comprehensive market framework by creating an overarching model that can be
specialized to more common market models. For this unified market model, the chapter
provides:

- a detailed treatment of equilibrium solution and uniqueness;
- proofs that certain numerical methods will successfully identify an equilibrium under
  suitable conditions;
• extensions of existence and uniqueness results to market models with consumer surplus maximization, capacity markets, and emission permits.

Chapter 2: The possibility of electricity price spikes in deregulated markets became a regulatory concern after two separate incidents in 1998 [1] and 1999 [37]. Important characteristics of these spikes is their brief temporal nature. To study this behavior over time using game theory, it is necessary to model the problem with a sufficiently fine time discretization. In the limit, a differential Nash game is obtained where players can be modeled as individually solving optimal control problems. Unfortunately, no solution method for identifying a differential Nash equilibrium of the game is immediately available if any player’s optimal control problem includes mixed state-control constraints. Therefore, this work provides two different approaches for identifying differential Nash equilibria when mixed state-control constraints are present. These approaches are based on the following ideas:

• the equivalence of a differential Nash game to a single optimal control problem under suitable conditions;

• a Jacobi-type algorithm that converges to a solution of the differential Nash game.

Although the chapters of this dissertation are independent in the sense that each can be read separately without any loss of understanding, the Nash equilibrium and variational inequality concepts of Sections 1.1 and 1.3 are fundamental to all of the research. The dissertation concludes in Chapter 6.

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2This research direction was motivated by work with Victor Zavala during a summer internship at Argonne National Laboratory.
Chapter 2

Solution existence in perfectly competitive capacity expansion models

2.1 Introduction

In microeconomics, non-cooperative players competing in production can be modeled as either strategic or “price-taking” with respect to commodity prices; players satisfying the latter assumption are hereafter referred to as competitive players to emphasize the idea that such an assumption is associated with perfectly competitive markets (i.e., markets in which no player has sufficient market power to affect the commodity price). Whereas a strategic player knows that its production can affect price and accounts for this property when choosing production quantities to maximize profit, a competitive player assumes that prices are exogenously determined. Based on these assumptions about player behavior,

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3This chapter has been adapted from a manuscript submitted for review at Operations Research, Manuscript ID OPRE-2013-05-245 (Co-authors: Benjamin Hobbs and Jong-Shi Pang).
a large range of market outcomes that are commonly characterized by Nash equilibria (see (1.2)) may be possible. Although a large amount of research has been conducted on Nash equilibrium problems with strategic players (e.g., [76, 105, 109, 110, 123, 151, 184, 230, 238]), theoretical results for the existence of Nash equilibria for perfectly competitive market models have remained largely unchanged since the classical works [7, 56, 149, 225]. This research develops new Nash equilibrium existence results for perfectly competitive market games in which players make capacity investments under uncertainty followed by production decisions. In the capacity expansion models studied in this chapter, traditional existence results based on fixed-point theorems (e.g., [56, 149, 163, 165, 197]) are not applicable due to the unboundedness of the players’ feasible regions and the lack of explicit bounds on the price variables.

Although this research focuses on a general formulation of a capacity expansion game, the electricity industry serves as an important source of topical examples and associated regulatory questions for this problem. In this economic sector, emission regulations and incentives for new capacity investment are becoming increasingly prevalent as illustrated recently by the European Union Emissions Trading Scheme [42] and the creation of markets for generation capacity in several U.S. regions. Methods proposed for reducing greenhouse gas emissions and other air pollutants are commonly related to cap-and-trade schemes and Pigovian taxes, but the implementation of these ideas may unintentionally create disincentives to invest in certain types of new capacity or technology. An example of such unintended consequences can be found in [41] where it is demonstrated that the imposition of a carbon emissions trading program in California may only lead to a marginal decrease in total emissions due to the shifting of production and contracts. As another example, water pollution rules in California may lead to shutting down a substantial fraction of that state’s fossil-fueled capacity, which in turn is anticipated to lead to shortages in flexible
generation needed to meet reliability standards given the large amounts of wind power being built [31].

In addition to undesirable outcomes that may arise after certain regulations are implemented, prolonged uncertainty about the timing and nature of possible regulations may be equally problematic in that it can cause companies to make suboptimal decisions in anticipation of possible regulation details. For instance, it has been found that regulatory uncertainty for carbon emission regulations can lead to capacity investment choices in highly polluting technology if grandfathering of existing capacity is seen as possible [192]. A comparison of induced capacity investment decisions under three different carbon allowance allocation schemes can be found in [245].

Finally, the presence of risk aversion in electricity generator behavior may aggravate undesirable outcomes. There has been an ongoing debate in the power industry about the desirability of separate mechanisms to procure generation capacity in addition to energy commodity resources (e.g., [107]); some regulators fear that risk aversion and energy price caps will discourage needed investment. This concept has been demonstrated in several modeling studies. Using conditional value-at-risk as a risk measure, it was found in [68] that risk aversion decreases capacity investment in markets with low electricity price caps, shifting capacity investment toward generation technologies that are not capital-intensive. These results reinforce the findings from [167] which explored the interrelationships between risk aversion, capacity investment, and demand uncertainty. In [167], it was found that socially efficient capacity investment choices occur only when long-term bilateral contracts between generators and consumers are available. With respect to market mechanism design, [221] proposes a reliability contract market to encourage risk-averse firms to invest in new capacity.
The inspiration for the current work lies in capacity expansion models in the electricity power sector [78, 157, 176, 235, 245] that model price-taking producers making capacity expansion decisions prior to uncertainty realization and a set of production decisions, one for each potential uncertainty realization, to maximize an expectation-based objective function. The producers are modeled as maximizers of either expected profit (i.e., risk neutrality) or a concave function of expected profit (i.e., risk aversion) in [78], while the other papers only deal with expected profit objective functions. When framed in terms of risk preferences, utility functions are commonly employed to describe both of these objective function forms; following [78], exponential utility functions are employed in this chapter.

In this research, existence results are provided for perfectly competitive capacity expansion equilibrium problems incorporating player risk aversion and other extended considerations. While [78, 245] have both claimed Nash equilibrium existence results for their respective problems, the former existence proof is inconsistent with existing theory while the latter proof may be obfuscated by the presence of emission allowance allocation schemes and is restricted to risk-neutral objectives. In addition to providing a correct proof of the result in [78] under certain conditions, this development extends to multi-market equilibrium problems with different types of commodity pricing rules.

Significant strides have been made recently in the study of existence of a Nash equilibrium for games with exogenous pricing mechanisms; see in particular the survey paper [76] and the article [182]. While these references have analyzed games with exogenous pricing from market clearing conditions, a major difference between the models therein and the ones analyzed here arises in how prices enter into player objective functions. In the former models, the prices appear in the players’ objective functions solely as Lagrange multipliers of the side constraints (i.e., the market clearing conditions), whereas in this model, the
prices enter into the players’ risk-averse utility functions through their profits. Thus, the results in the cited references are not directly applicable; instead, it is necessary to resort to a separate analysis. Furthermore, there are no readily available existence results for perfectly competitive games where prices are determined by general inverse demand/supply functions. Starting from a fundamental existence theorem for a nonlinear complementarity problem (NCP), a self-contained study of a perfectly competitive game under various pricing schemes, uncertainty, and risk aversion is presented.

This chapter is divided into seven sections. A brief summary of stochastic optimization problems is presented in Section 2.2 followed by a discussion of the general game forms in Section 2.3. The specific capacity expansion game formulations addressed in this chapter are presented in Section 2.4, and Section 2.5 develops and proves solution existence for perfectly competitive market equilibrium problems when prices are determined by inverse supply/demand functions. Section 2.6 addresses problems with prices determined by market clearing conditions. Section 2.7 extends these results to perfectly competitive models where demands are determined by consumers solving surplus maximization problems, and the chapter concludes with Section 2.8.

2.2 A brief overview of stochastic optimization problems

As described in Section 2.1, the presence of uncertainty is an important complication in the capacity expansion game studied in this chapter. In this light, it is natural to frame each player’s problem as a stochastic optimization problem where uncertainty can arise for a variety of reasons (e.g., wind speed variation, imposition of competing federal regulations) and is assumed to be captured by a probabilistic framework. Using the given probabilistic model, a variety of questions can be posed, including which decision is optimal
on average and how should the decision be changed after the uncertainty is resolved [if changes are possible]. These two questions are addressed by *anticipative* and *adaptive* stochastic optimization problems, respectively.

The concept of stochastic optimization first arose in 1955 [53] as an extension of linear programming to problems with uncertain parameters. This stochastic problem form was further studied in [231, 232, 233] where solution set properties and an equivalence for the adaptive problem form were established. The general convex stochastic optimization problem was first explored in [58, 195, 196]; a survey of these and other results is provided in [57]. In addition to having uncertainty present in the objective function, uncertainties can also be present in the constraint set. The framing of this concept in terms of probabilistic constraints was first presented in [39]. For a comprehensive discussion of stochastic optimization problems and solution techniques, the interested reader can refer to [19, 202].

It is important to note that much of the past research on stochastic optimization problems has been focused on the adaptive framework (e.g., [16, 18, 122, 123, 134, 141, 224, 243, 244]). This focus is commonly justified by the fact that the anticipative problem is not theoretically different from a deterministic problem when uncertainty is only manifested in a random vector affecting the objective function. Interestingly, a lack of objective function linearity can cause adaptive stochastic optimization problems to superficially resemble anticipative problems as will be seen in this chapter.

Before delving into the specifics of stochastic optimization, the basic foundation of probability theory should be formally presented [94]. A probability space is defined by the tuple $(\Omega, \mathcal{G}, P)$. The term $\Omega$ represents a nonempty set known as the *sample space* for the uncertainty with $\omega \in \Omega$ being a specific *outcome*. $\mathcal{G}$ is a set of subsets of $\Omega$ (known as
events) that is a σ-algebra, meaning that

(a) \( \Omega \in \mathcal{G} \);

(b) If \( W \in \mathcal{G} \), then \((\Omega \setminus W) \in \mathcal{G}\);

(c) If \( V, W \in \mathcal{G} \), then \( V \cup W \in \mathcal{G} \). Furthermore, if \( W_1, W_2, \cdots \) is a sequence of elements in \( \mathcal{G} \), then \( \bigcup_{k=1}^{\infty} W_k \in \mathcal{G} \).

Note that \( \mathcal{G} \) has been substituted for the more standard \( \mathcal{F} \) for notational consistency. Additionally, there should formally exist a random vector \( \xi \) that maps outcomes \( \omega \in \Omega \) to certain problem parameters, but, with an abuse of notation, this mapping is ignored here. Lastly, \( P \) is a probability measure on \( \mathcal{G} \) satisfying

(a) \( P(W) \geq 0 \) for all \( W \in \mathcal{G} \);

(b) If \( V, W \in \mathcal{G} \) and if \( V \) and \( W \) are mutually exclusive, then \( P(V \cup W) = P(V) + P(W) \).

Furthermore, if \( W_1, W_2, \cdots \) is a sequence of mutually exclusive events in \( \mathcal{G} \), then

\[
P \left( \bigcup_{k=1}^{\infty} W_k \right) = \sum_{k=1}^{\infty} P(W_k);
\]

(c) \( P(\Omega) = 1 \).

The anticipative stochastic optimization problem attempts to identify a decision that is optimal on average. This averaging concept is naturally associated with the calculation of the objective function’s expectation. With \( f(x, \omega) \) as the objective function associated with decision \( x \) and outcome \( \omega \in \Omega \), let the expectation

\[
\mathbb{E}[f(x, \omega)] \triangleq \int_{\Omega} f(x, \omega)P(d\omega),
\]

which is assumed to be well-defined. Constraints on decisions are commonly expressed as either independent of \( \omega \) or dependent on \( \omega \). Therefore, a general anticipative stochastic
optimization problem is given by

\[
\begin{align*}
\text{minimize} & \quad E[f(x, \omega)] \\
\text{subject to} & \quad h_1(x) \geq 0 \\
& \quad h_2(x, \omega) \geq 0 \quad \text{(see below)}
\end{align*}
\]

(2.1)

where \( h_1 \) and \( h_2 \) map \( x \) and \( (x, \omega) \) to appropriately dimensioned vectors, respectively. Traditionally, the latter group of constraints may be enforced for either all \( \omega \in \Omega \), in expectation (i.e., \( E[h_2(x, \omega)] \geq 0 \)), or in probability (i.e., \( P(h_2(x, \omega) \geq 0) \geq 1 - \alpha \)).

Unlike an anticipative optimization problem, an adaptive optimization problem involves two [or more] decision stages: the first decision is made under uncertainty while the recourse decision is made after the uncertainty is resolved. These two problems naturally interact in that the first-stage decision is in effect during the second-stage recourse problem. Therefore, the adaptive stochastic optimization problem consists of deciding a first-stage decision and a recourse decision for each \( \omega \in \Omega \) so that the expected overall objective function is minimized. With \( h_1 \) and \( h_2 \) being appropriately dimensioned mappings, the first-stage problem is formulated as

\[
\begin{align*}
\text{minimize} & \quad f(x) + E\left[\hat{f}^*(x, \omega)\right] \\
\text{subject to} & \quad h_1(x) \geq 0,
\end{align*}
\]

(2.2)

where \( \hat{f}^*(x, \omega) \) is the optimal solution of the second-stage problem

\[
\begin{align*}
\text{minimize} & \quad \hat{f}(y, \omega) \\
\text{subject to} & \quad h_2(x, y) \geq 0.
\end{align*}
\]

(2.3)
For problems in which \( \Omega \) is finite, (2.2) and (2.3) can be combined into a single optimization problem. With \( K \) elements in \( \Omega \), this single optimization problem is

\[
\text{minimize}_{x \in \mathbb{R}^n, y_1, \ldots, y_K} \ f(x) + \sum_{k=1}^{K} \hat{f}(y_k, \omega_k)P(\omega_k)
\]

subject to

\[
h_1(x) \geq 0 \\
h_2(x, y_k) \geq 0 \quad \text{for all } k = 1, \ldots, K.
\]

(2.4)

A similar combined problem can be obtained for more general \( \Omega \) under certain conditions [211]. The adaptive idea of decisions both before and after uncertainty realization contrasts the anticipative model in which decisions are only made prior to realization of uncertainty. This distinction will become important when classifying the capacity expansion model of this chapter.

### 2.3 Different competitive game formulations

Consider a perfectly competitive problem with \( F \) non-cooperative players simultaneously making decisions to minimize their respective objective functions while treating prices and competitors’ decisions exogenously. In a specialization of (1.1), let player \( i \) solve the problem

\[
\text{minimize}_{x_i \in K_i} \ f_i(x_i; x_{-i}, p),
\]

(2.5)

where the vector \( p \in \mathbb{R}^L \) represents \( L \) commodity prices and for each \( p \), the mapping \( f_i(\bullet; x_{-i}, p) : U \to \mathbb{R} \) is a continuously differentiable, convex function defined on some open convex proper superset \( U \) of \( K_i \triangleq \{ x \in \mathbb{R}^n_i \mid A_i x \geq b_i \} \) with \( A_i \in \mathbb{R}^{m_i \times n_i} \) and \( b_i \in \mathbb{R}^{m_i} \) for all \( i \). From Theorem 1.1 the KKT conditions of (2.5) are both necessary and
sufficient for optimality:

\[ 0 \leq x_i \perp \nabla_{x_i} f_i(x_i; x_{-i}, p) - A_i^T \lambda_i \geq 0 \]

\[ 0 \leq \lambda_i \perp A_i x_i - b_i \geq 0, \]

where \( \lambda_i \) is the Lagrange multiplier vector for the constraint \( A_i x_i - b_i \geq 0 \).

In a perfectly competitive framework, players may both buy commodities (such as resources) from supply markets and sell commodities (the products) in demand markets; their purchases and sales determine the commodity prices described by the vector \( p \). Unlike the classical Arrow-Debreu general equilibrium problem, a partial equilibrium problem stipulates the way that prices are derived. One such stipulation is through an econometric model wherein prices are determined by explicit inverse demand/supply functions. With \( x_{i\ell} \) denoting player \( i \)'s production for market \( \ell \), an example of such a function is a linear inverse supply function given by

\[ p_{\ell}(x) = P_{\ell}^0 - \frac{P_{\ell}^0}{Q_{\ell}^0} \sum_{i=1}^{F} x_{i\ell}, \]  

(2.7)

where \( x \triangleq (x_i)_{i=1}^{F} \) and the scalar \( P_{\ell}^0 \) and the ratio \( \frac{P_{\ell}^0}{Q_{\ell}^0} \) represent the price-intercept and the slope of the function for commodity \( \ell \), respectively. Inverse demand functions are defined similarly. For perfectly competitive games, it is important to note that even when (2.7) is specified, it is not substituted into the objective function of (2.5) because \( p \) is taken to be exogenous by each player. Rather, the inverse supply function is substituted for \( p \) in each player’s KKT optimality conditions (2.6).

A second method of determining market prices \( p \) is related to uniform-price auction mecha-
anisms; namely, $p$ is postulated to satisfy the following market clearing condition expressed by a complementarity condition between the price and excess demand/supply:

$$0 \leq p \perp g(p; x) \geq 0,$$

(2.8)

where $g$ is a vector-valued, continuous excess demand/supply function. As such, $g(p; x)$ simply requires supply to be greater than or equal to a given demand. The concept of an auction arises because a non-binding market clearing constraint (i.e., supply is greater than demand) implies a zero market price. Furthermore, each player realizes the same commodity price if it is positive. If the function $g(\bullet; x)$ is integrable such that $g(p; x) = \nabla_p \theta(p; x)$ for some scalar function $\theta$, then (2.8) is equivalent to the first-order optimality conditions of the optimization problem in price:

$$\minimize_{p \geq 0} \theta(p; x).$$

Therefore, in this (price-integrable) case, a game with prices determined by (2.8) can be modeled as an extended game with one additional player, the price player. More generally, if this (price-) integrability condition does not hold, then the same game is an instance of the recently introduced class of (distributed) multi-agent optimization problems with equilibrium constraints (MOPECs), where the complementarity condition is generalized to a variational inequality with $p$ as the primal variable and $x$ as an exogenous variable (but endogenous to the overall MOPEC). The class of MOPECs was recently introduced by Michael Ferris without a detailed analysis. In this chapter, this framework is not explicitly addressed but rather mentioned as a direction for further research. For related MOPEC models, the interested reader is referred to [24] and the slides of a presentation available at http://www.cs.wisc.edu/~ferris/talks/chicago-mar.pdf.
When price is determined by a market clearing condition, the function \( g(p; x) \) must incorporate demand for the commodity. This demanded quantity can be specified either exogenously as in (2.8) or endogenously depending on the type of consumer behavior postulated. For the exogenous case, consumers demand a fixed amount of the commodity regardless of price, a situation of perfect price inelasticity. The third pricing mechanism considered here follows the formulation of (2.8) but replaces the exogenously specified demand in \( g(p; x) \) with an endogenous quantity. This modification is achieved by modeling consumers as choosing the demanded quantity to maximize consumer surplus, which is defined as the integral of the consumer demand function less commodity price from zero to the quantity demanded. Since perfectly competitive equilibrium models are of interest, commodity prices should be treated as exogenous in the consumer optimization problem. Thus, perfectly competitive games with endogenous demand can be modeled with \( \mathcal{F} + 2 \) players (i.e., \( \mathcal{F} \) producers, 1 consumer, and 1 price player); for a model of this type, see Section 2.7.

In summary, methods of price determination in a perfectly competitive market model can be separated into three groups as shown in Figure 2.1.

**Figure 2.1: Price determination categories**

Specializing the inequality (1.2), a Nash equilibrium for a perfectly competitive market
model is a vector of decisions $\langle x^*, p^* \rangle$ such that for each producer $i = 1, \ldots, F$,

$$f_i(x_i; x^*_{-i}, p^*) \geq f_i(x^*_i; x^*_{-i}, p^*) \quad \text{for all } x_i \in K_i$$

and, for each commodity $\ell = 1, \ldots, L$, either $p^*_\ell = p_\ell(x^*)$ (commodity price by inverse demand/supply function) or

$$p_\ell g_\ell(p_\ell; p^*_{-\ell}, x^*) \geq p^*_\ell g_\ell(p^*, x^*) = 0 \quad \text{for all } p_\ell \geq 0 \text{ (commodity price by market clearing)}.$$

Concatenating the KKT conditions of the producers' optimization problems along with the price stipulations, the following nonlinear complementarity formulations are obtained for the perfectly competitive market game:

- when all market prices are determined by inverse demand/supply functions:

$$0 \leq x_i \perp \nabla_x f_i(x_i; x_{-i}, p) \big|_{p = p(x)} - A^T_i \lambda_i \geq 0 \quad \text{for all } i = 1, \ldots, F$$

$$0 \leq \lambda_i \perp A_i x_i - b_i \geq 0 \quad \text{for all } i = 1, \ldots, F; \quad (2.9)$$

- when all market prices are determined by the market clearing complementarity condition (2.8):

$$0 \leq x_i \perp \nabla_x f_i(x_i; x_{-i}, p) - A^T_i \lambda_i \geq 0 \quad \text{for all } i = 1, \ldots, F$$

$$0 \leq \lambda_i \perp A_i x_i - b_i \geq 0 \quad \text{for all } i = 1, \ldots, F \quad (2.10)$$

$$0 \leq p \perp g(p; x) \geq 0.$$

Presumably, a mixed model can be stated wherein some prices are determined by inverse demand/supply functions while others are determined by market clearing conditions. An
extended analysis can be made for such a mixed model; for simplicity, only the above two cases wherein all prices are determined by the same method are examined here. Another remark about the two formulations (2.9) and (2.10) is that while the same notation \( f_i(x; x_{-i}, p) \) is used for player \( i \)'s objective, in the case of (2.10), this function should include some form of the constraint function \( g(p; x) \) (see Section 2.6).

A key challenge in establishing the existence of an equilibrium pair satisfying (2.9) or (2.10) is the lack of explicit bounds for the variables \( x \) and \( p \). Specialized to capacity expansion models such as those detailed in [78, 245], players choose both production levels and the amount of production capacity to install; such production decisions are captured by the vector \( x \). There are constraints enforcing that production levels are bounded by installed production capacity. In the event that this capacity has no prescribed upper bound, the compactness assumption commonly employed to prove solution existence does not hold for these problems. This is the primary mistake made in the existence proof of [78] where boundedness is claimed through an appeal to objective function properties at Nash equilibria before the existence of such equilibria is established; such a “proof” is a circular argument and needs to be corrected.

Instead of applying an argument based on explicit boundedness of the decision variables, the treatment in [245] is followed by employing a fundamental NCP existence result to analyze the problems (2.9) and (2.10). Formally, being a specialization of a variational inequality, the NCP attempts to find a vector \( z \) such that \( 0 \leq z \perp F(z) \geq 0 \), where \( F \) is a given continuous vector function. The following result is drawn from [75, Theorem 2.6.1].

**Theorem 2.1.** Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a continuous function. If there exists a constant \( C > 0 \) such that all solutions of \( 0 \leq z \perp F(z) + z\tau \geq 0 \) for \( \tau > 0 \) satisfy \( \|z\| \leq C \), then the NCP: \( 0 \leq z \perp F(z) \geq 0 \) has a solution. \( \square \)
Notice that boundedness is assumed in a special way in Theorem 2.1, namely, on the solution sets of the augmented NCPs parametrized by the scalar $\tau$. Rather than analyzing (2.9) and (2.10) in their abstract formulation, a concrete application of Theorem 2.1 is provided for the capacity expansion problem with uncertainty, extending the special cases in [78, 245] to broader contexts. Theorem 2.1 is applied by assuming that, corresponding to a sequence $\{\tau_k\}$ of positive scalars, an unbounded sequence $\{z^k\}$ exists such that for each $k$, $0 \leq z^k \perp F(z^k) + z^k \tau_k \geq 0$; a contradiction is then derived.

2.4 The capacity expansion game

To begin, the specialization of player optimization problems (2.5) to the capacity expansion framework is presented. Consider a stochastic optimization problem in which each player makes a capacity investment decision under uncertainty followed by production decisions for each possible uncertainty realization. In [78, 245], uncertainty arises from the possible implementation of different greenhouse gas emission regulations. Let $\Omega$ represent the finite scenario space (taken to contain at least 2 elements) with $\omega \in \Omega$ denoting a specific scenario realization and $\text{PR}_i\omega$ being player $i$’s assumed scenario $\omega$ probability. It is required that $\text{PR}_i\omega \in (0, 1)$ for all $i = 1, \cdots, F$ and all $\omega \in \Omega$. This player-specific probability structure is a generalization of the game in which all players assume the same probability $\text{PR}_\omega$ for each $\omega \in \Omega$.

**Remark 2.1.** Although practical methods for determining $\text{PR}_i\omega$ values based on producer responses are surely important, they are out of the scope of this research. Instead, this research proves that, under certain assumptions, an equilibrium exists for an arbitrary set of these probabilities; thus, the specific probability values and the methodology used to obtain them have no bearing on the presented results. □
For a capacity expansion problem in which multiple markets exist, there are two general
types of constraints that player \( i \) may face depending on production properties. Either the
player (a) must produce goods for each market in a different facility, or (b) can produce
goods for all markets in the same facility. The former case is relevant for conglomerates that
produce a wide variety of goods while the latter case may be applied when products are
based on closely related designs. A player may also be involved in both types of markets;
the analysis of such a mixed situation will not be dealt with here since it is similar to the
presented analysis. With \( L \) markets, player \( i \) is constrained in case (a) by

\[
g_{i\ell\omega} \leq x_{i\ell} \quad \text{for all } \ell = 1, \ldots, L \text{ and } \omega \in \Omega, \tag{2.11}
\]

where \( g_{i\ell\omega} \) is the production level for market \( \ell \) in scenario \( \omega \) and \( x_{i\ell} \) is the capacity installed
to serve market \( \ell \). For case (b), player \( i \) is constrained by

\[
\sum_{\ell=1}^{L} g_{i\ell\omega} \leq x_i \quad \text{for all } \omega \in \Omega,
\]

where \( x_i \) is the total installed production capacity. Since the existence proofs for case (a)
and case (b) are similar, only case (a) will be discussed here. Other constraints can be
included but are omitted here for simplicity.

It is assumed that player \( i \) chooses \( g_{i\ell\omega} \) and \( x_{i\ell} \) to maximize expected exponential utility

\[
\sum_{\omega \in \Omega} \left( 1 - \exp \left\{ -a_i \pi_{i\omega} \right\} \right) PR_{i\omega}, \tag{2.12}
\]

where \( a_i > 0 \) and \( \pi_{i\omega} \) is the profit earned in scenario \( \omega \). The scalar \( a_i \) describes the level
of risk aversion of player \( i \) with a larger \( a_i \) corresponding to more risk aversion. Note
that the function (2.12) is concave in \( \pi_{i\omega} \geq 0 \). An exponential utility function is chosen
because it has the appealing characteristic of constant risk aversion [24] and involves a minimal number of parameters (one). In principle, the analysis in the later sections can be generalized to other utility functions with properties similar to the exponential family.

With \( p_{\ell \omega} \) being the price of commodity \( \ell \) in scenario \( \omega \) (taken as exogenous by each player), the total revenue from player \( i \)'s sales in scenario \( \omega \) is \( \sum_{\ell=1}^{L} p_{\ell \omega} g_{i \ell \omega} \). For each unit of production of commodity \( \ell \), it is assumed that player \( i \) incurs a fixed marginal cost \( MC_{i \ell} \).

When producers also need to buy commodities from perfectly competitive markets, each unit of production entails an additional cost. Letting \( \{1, \cdots, R\} \) denote the set of commodities (resources) needed by the producers, \( p_{r \omega} \) be the price of commodity \( r \in \{1, \cdots, R\} \) in scenario \( \omega \), and \( U_{r \ell}^i \) be the positive number of units of commodity \( r \) required per unit of producer \( i \)'s production for market \( \ell \), each unit of commodity \( \ell \) production incurs a cost of \( \sum_{r=1}^{R} U_{r \ell}^i p_{r \omega} \) for a total production cost of \( MC_{i \ell} + \sum_{r=1}^{R} U_{r \ell}^i p_{r \omega} \) for market \( \ell \). In [78, 245], the commodity being purchased by producers represents emission permits required by federal regulators. Finally, capacity installation is assumed to have a fixed marginal cost \( C_{i \ell} \) so producer \( i \) pays \( C_{i \ell} x_{i \ell} \) for installing capacity.

With profit equaling revenue minus cost, producer \( i \)'s profit in scenario \( \omega \) is

\[
\pi_{i \omega} \triangleq \sum_{\ell=1}^{L} \left[ \left( p_{\ell \omega} - MC_{i \ell} - \sum_{r=1}^{R} U_{r \ell}^i p_{r \omega} \right) g_{i \ell \omega} - C_{i \ell} x_{i \ell} \right].
\]
Thus, player $i$’s optimization problem is

$$\text{maximize } \sum_{\omega \in \Omega} \left( 1 - \exp \left\{ -a_i \sum_{\ell=1}^{L} \left[ (p_{i\ell\omega} - MC_{i\ell} - \sum_{r=1}^{R} U_{i\ell}^{r} p_{r\omega}^{i} ) g_{i\ell\omega} - C_{i\ell} x_{i\ell} \right] \right\} \right) PR_{i\omega}$$

subject to

$$x_{i\ell} - g_{i\ell\omega} \geq 0 \quad (\lambda_{i\ell\omega}) \quad \text{for all } \ell = 1, \ldots, L \text{ and } \omega \in \Omega$$

$$g_{i\ell\omega}, x_{i\ell} \geq 0 \quad \text{for all } \ell = 1, \ldots, L \text{ and } \omega \in \Omega,$$

where $\lambda_{i\ell\omega}$ is the Lagrange multiplier associated with the production capacity constraint. Notice that the objective function of (2.13) is separable in the scenarios; the production variables $\{g_{i\ell\omega}\}_{\omega \in \Omega}$ are linked through the capacity constraints (2.11) with the (unknown) capacities $x_{i\ell}$ being variables determined by the overall model.

As formulated, certain elements of (2.13) resemble an anticipative optimization problem while other elements are commonly found in adaptive problems: the objective function minimizes an expectation, but a single decision is made under uncertainty and recourse decisions are made for each uncertainty realization. Therefore, although the problem structure superficially resembles (2.1), it is contended that this problem is better categorized as an adaptive optimization problem. Indeed, if a linear utility function is used instead of an exponential utility function, (2.4) is obtained immediately.

### 2.5 Inverse demand/supply function models

Let the market prices be determined by continuous inverse demand and inverse supply functions $p_{i\ell\omega} \triangleq \theta_{i\ell\omega} \left( \sum_{j=1}^{F} g_{i\ell j\omega} \right)$ for all pairs $(\ell, \omega)$ and $p_{r\omega}^{i} \triangleq \theta_{r\omega}^{i} \left( \sum_{i=1}^{F} \sum_{\ell=1}^{L} U_{i\ell}^{r} g_{i\ell\omega} \right)$ for all
pairs \((\omega, r)\), respectively. To simplify notation, denote
\[
\sum_{i=1}^{\mathcal{F}} g_{i\ell\omega} \text{ by } G_{\ell\omega} \quad \text{and} \quad \sum_{i=1}^{\mathcal{F}} \sum_{l=1}^{L} U_{il}^r g_{i\ell\omega}
\]
by \(S_{\omega}^r\).

**Theorem 2.2.** The perfectly competitive Nash equilibrium problem defined by \((2.13)\) for all producers \(i = 1, \cdots, \mathcal{F}\), the continuous functions \(\theta_{\ell\omega}(G_{\ell\omega})\) for all \((\ell, \omega)\), and the continuous functions \(\theta_{\omega}^r(S_{\omega}^r)\) for all \((\omega, r)\) has a solution if the following conditions hold:

(a) there exists a positive integer \(\mathcal{M}\) such that \(|\theta_{\ell\omega}(G_{\ell\omega})| \leq \mathcal{M}\) for all \((\ell, \omega)\) when \(G_{\ell\omega} \geq 0\);

(b) \(\limsup_{G_{\ell\omega} \to \infty} \theta_{\ell\omega}(G_{\ell\omega}) \leq 0\);

(c) \(\theta_{\omega}^r(S_{\omega}^r) \geq 0\) when \(S_{\omega}^r \geq 0\).

**Proof.** By concatenating the necessary and sufficient KKT optimality conditions of \((2.13)\) for all players \(i = 1, \cdots, \mathcal{F}\) to form \((2.9)\), Theorem 2.1 requires the examination of the solution sets of the NCP parameterized by a sequence of positive scalars \(\{\tau_k\}\):

- for all \(i, \ell, \) and \(\omega\),

\[
0 \leq g_{i\ell\omega}^k \perp \tau_k g_{i\ell\omega}^k + \lambda_{i\ell\omega}^k - \text{PR}_{i\omega} a_i \left( \theta_{\ell\omega}(G_{\ell\omega}^k) - \text{MC}_{i\ell} - \sum_{r=1}^{R} U_{il}^r \theta_{\omega}^r(S_{\omega}^{r,k}) \right) \times \\
\exp \left\{-a_i \sum_{l=1}^{L} \left( \theta_{\ell\omega}(G_{\ell\omega}^k) - \text{MC}_{il} - \sum_{r=1}^{R} U_{il}^r \theta_{\omega}^r(S_{\omega}^{r,k}) \right) g_{i\ell\omega}^k - C_{i\ell} x_{i\ell}^k \right\} \geq 0
\]

\[
0 \leq x_{i\ell}^k \perp g_{i\ell\omega}^k + \lambda_{i\ell\omega}^k \geq 0
\]
• for all $i$ and $\ell$,

$$0 \leq x^k_{i\ell} \perp \tau_{k} x^k_{i\ell} - \sum_{\omega \in \Omega} \lambda^k_{i\ell\omega} + \sum_{\omega \in \Omega} \text{PR}_{i\omega} a_i \text{C}_{i\ell \times} \left\{ \exp \left\{ -a_i \sum_{\ell' = 1}^{L} \left[ \left( \theta_{\ell'\omega} \left( G^k_{\ell'\omega} \right) - \text{MC}_{i\ell} - \sum_{r = 1}^{R} \text{U}_{i\ell} \theta_{\omega r} \left( S^r_{\omega} \right) \right) \sum_{\ell'} \text{g}^k_{i\ell'\omega} - \text{C}_{i\ell \times} \text{g}^k_{i\ell'\omega} \right]\right\} \right\} \geq 0,$$

where $G^k_{\ell'\omega} \triangleq \sum_{i = 1}^{F} g^k_{i\ell'\omega}$ and $S^r_{\omega} \triangleq \sum_{i = 1}^{F} \sum_{\ell = 1}^{L} \text{U}_{i\ell} g^k_{i\ell'\omega}$. It suffices to prove that the sequence of solution tuples $\{g^k, x^k, \lambda^k\}$ is bounded as $k \to \infty$. Boundedness is proven by variable: first $\{g^k\}$, then $\{x^k\}$, and finally $\{\lambda^k\}$.

**Boundedness of $\{g^k\}$.** To obtain a contradiction, assume that $\{g^k\}$ is unbounded so there exists some triple $(i', \ell', \omega')$ such that $\{g^k_{i'\ell'\omega'}\}$ is unbounded as $k \to \infty$. It follows that for some infinite index set $\kappa \subset \{1, 2, \cdots, \infty\}$,

$$\lim_{k(\in \kappa) \to \infty} g^k_{i'\ell'\omega'} = \infty.$$

Without loss of generality, assume that $g^k_{i'\ell'\omega'} > 0$ for all $k \in \kappa$. Define the following two index sets

$$J_1 \triangleq \left\{ \ell \in \{1, \cdots, L\} \mid \exists \ell' \text{ with } \limsup_{k(\in \kappa) \to \infty} g^k_{i\ell'\omega'} = \infty \right\} \ni \ell' \text{ and } J_2 \triangleq \{1, \cdots, L\} \setminus J_1.$$

For every $\ell \in J_1$,

$$G^k_{\ell'\omega'} = \sum_{i = 1}^{F} g^k_{i\ell'\omega'} \geq g^k_{i\ell'\omega'} \to \infty \text{ as } k(\in \kappa) \to \infty.$$

Assume without loss of generality that $\{g^k_{i\ell'\omega'}\}_{k \in \kappa}$ is bounded for every $\ell \in J_2$ and all $i$. 42
Hence, for sufficiently large $k \in \kappa$,

$$
\sum_{\ell \in J_1} \left( \theta_{\ell \omega'} \left( G_{\ell \omega'}^k \right) - MC_{i' \ell} - \sum_{r=1}^{R} U_{i' \ell}^r \theta_{\omega'}^r \left( S_{\omega'}^{r,k} \right) \right) g_{i' \ell \omega'}^k
\leq \sum_{\ell \in J_1} \left( \theta_{\ell \omega'} \left( G_{\ell \omega'}^k \right) - MC_{i' \ell} \right) g_{i' \ell \omega'}^k < 0 \quad \text{by condition (b)},
$$

and for some constant $M_1 > 0$,

$$
\sum_{\ell \in J_2} \left( \theta_{\ell \omega'} \left( G_{\ell \omega'}^k \right) - MC_{i' \ell} \right) g_{i' \ell \omega'}^k \leq \sum_{\ell \in J_2} M g_{i' \ell \omega'}^k \leq M_1.
$$

Hence,

$$
\exp \left\{ -a_{i'} \sum_{\ell=1}^{L} \left( \theta_{\ell \omega'} \left( G_{\ell \omega'}^k \right) - MC_{i' \ell} - \sum_{r=1}^{R} U_{i' \ell}^r \theta_{\omega'}^r \left( S_{\omega'}^{r,k} \right) \right) g_{i' \ell \omega'}^k - C_{i' \ell} x_{i' \ell}^k \right\}
\geq \exp \left( -a_{i'} M_1 \right).
$$

For $g_{i' \ell' \omega'}^k > 0$, complementarity necessitates that

$$
0 = \tau_k g_{i' \ell' \omega'}^k + \lambda_{i' \ell' \omega'}^k - PR_{i' \omega'} a_{i'} \left( \theta_{\ell' \omega'} \left( G_{\ell' \omega'}^{k'} \right) - MC_{i' \ell'} \right) \times
$$

$$
\exp \left\{ -a_{i'} \sum_{\ell=1}^{L} \left( \theta_{\ell \omega'} \left( G_{\ell \omega'}^{k'} \right) - MC_{i' \ell} - \sum_{r=1}^{R} U_{i' \ell}^r \theta_{\omega'}^r \left( S_{\omega'}^{r,k} \right) \right) g_{i' \ell \omega'}^{k'} - C_{i' \ell} x_{i' \ell}^{k'} \right\}.
$$

For $k$ sufficiently large, Term 1 is nonpositive by conditions (b) and (c) and Term 2 is positive by $\exp \left( -a_{i'} M_1 \right) > 0$. A contradiction is obtained and $\{g^k\}$ is therefore bounded.
Boundedness of \( \{x^k\} \). For the sake of contradiction, assume that there exists some pair \((i', \ell')\) such that for an infinite index set \( \kappa \subset \{1, 2, \cdots, \infty\} \),

\[
\lim_{k(\in \kappa) \to \infty} x^k_{i' \ell'} = \infty.
\]

It follows that for every \( \omega \in \Omega \), by the boundedness of \( \{g^k\} \),

\[
-a_i \sum_{\ell=1}^{L} \left[ \left( \theta_{\ell \omega} (G^k_{i' \ell \omega}) - MC_{i' \ell} - \sum_{r=1}^{R} U^r_{i' \ell \omega} \theta^r_{\ell \omega} \left( S^r_{\ell \omega} \right) \right) g^k_{i' \ell \omega} - C_{i' \ell} x^k_{i' \ell} \right] 
\geq -a_i \sum_{\ell=1}^{m} \mathcal{M} g^k_{i' \ell \omega} + a_i \sum_{\ell=1}^{m} C_{i' \ell} x^k_{i' \ell} \quad \text{by conditions (a) and (c)}
\]

\[
\to \infty \quad \text{as } k(\in \kappa) \to \infty.
\]

Since it may be assumed that \( x^k_{i' \ell} > 0 \) for all \( k \in \kappa \) without loss of generality, by complementarity,

\[
\sum_{\omega \in \Omega} \lambda^k_{i' \ell' \omega} = \tau_k x^k_{i' \ell'} + \sum_{\omega \in \Omega} PR_{i' \omega} a_i \times C_{i' \ell'} \times \exp \left\{ -a_i \sum_{\ell=1}^{L} \left[ \left( \theta_{\ell \omega} (G^k_{i' \ell \omega}) - MC_{i' \ell} - \sum_{r=1}^{R} U^r_{i' \ell \omega} \theta^r_{\ell \omega} \left( S^r_{\ell \omega} \right) \right) g^k_{i' \ell \omega} - C_{i' \ell} x^k_{i' \ell} \right] \right\} \to \infty
\]

as \( k(\in \kappa) \to \infty \). Therefore, there must exist an index \( \omega' \) such that

\[
\lim_{k(\in \kappa) \to \infty} \lambda^k_{i' \ell' \omega'} = \infty.
\]

Without loss of generality, assume that \( \lambda^k_{i' \ell' \omega'} > 0 \) for all \( k \in \kappa \). However, by complementarity,

\[
x^k_{i' \ell'} - g^k_{i' \ell' \omega'} + \tau_k \lambda^k_{i' \ell' \omega'} = 0,
\]

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a contradiction of the boundedness of \( \{g^k\} \). Thus, \( \{x^k\} \) is bounded.

**Boundedness of \( \{\lambda^k\} \).** Assume that there exists some triple \((i', \ell', \omega')\) such that for an infinite index set \( \kappa \subset \{1, 2, \ldots , \infty\} \),

\[
\lim_{k(\in \kappa) \to \infty} \lambda^k_{i'\ell'\omega'} = \infty,
\]

where \( \lambda^k_{i'\ell'\omega'} > 0 \) for all \( k \in \kappa \) without loss of generality. From the boundedness of \( \{x^k\} \) and \( \{g^k\} \), the equality \( x^k_{i'\ell'} - g^k_{i'\ell'\omega'} + \tau_k \lambda^k_{i'\ell'\omega'} = 0 \) implies that \( \{\tau_k \lambda^k_{i'\ell'\omega'}\} \) is bounded for all \( k \in \kappa \). As a consequence, it must hold that

\[
\lim_{k(\in \kappa) \to \infty} \tau_k = 0.
\]

Since

\[
\lambda^k_{i'\ell'\omega'} \leq \sum_{\omega \in \Omega} \lambda^k_{i'\ell'\omega} \leq \tau_k x^k_{i'\ell'} + \sum_{\omega \in \Omega} \text{PR}_{i'\omega' \ell'\omega} a_{i'\ell'} C_{i'\ell'} \times \exp \left\{ -a_{i'\ell'} \sum_{\ell = 1}^L \left[ \left( G^k_{\ell \omega'} - MC_{i'\ell'} \right) \left( S_{i'\ell\omega'} - R \sum_{r = 1}^R U_{i'\ell r} \theta_{\omega'} \left( S_{i'\ell\omega'} \right) \right) \right] \right\},
\]

a contradiction is obtained because the left-hand side tends to \( \infty \) whereas the right-hand side remains bounded as \( k(\in \kappa) \to \infty \). Hence, \( \{\lambda^k\} \) is bounded.

**Remark 2.2.** Conditions (a)-(c) of Theorem 2.2 enforce common properties of microeconomic demand and supply functions. For the demand functions, condition (a) requires that prices are bounded above for all feasible production quantities and condition (b) requires that the market price under infinite production is nonpositive. For the supply function markets, prices must be nonnegative for all feasible production quantities by condition (c).
Obviously, the validity of Theorem 2.2 is not restricted to equilibrium problems with linear demand and supply functions.

2.6 Market clearing models

To motivate the perfectly competitive model of capacity expansion under a market clearing condition, consider the situation where the aggregate production of all firms needs to fulfill a demand $D_{\ell\omega}$ in market $\ell$ under scenario $\omega$; thus the demand satisfaction constraints

$$\sum_{i=1}^{F} g_{i\ell\omega} - D_{\ell\omega} \geq 0 \quad \text{for all } \ell = 1, \cdots, L \text{ and } \omega \in \Omega \quad (2.14)$$

need to be taken into consideration by the firms in their respective profit maximization problems. One way to accomplish this is to include (2.14) in problem (2.13), which then becomes

$$\max_{g, x} \sum_{\omega \in \Omega} \left( 1 - \exp \left\{ -a_i \sum_{\ell=1}^{L} \left( p_{\ell\omega} - \text{MC}_{i\ell} - \sum_{r=1}^{R} U_{i\ell,r} P_{r\omega} \right) (g_{i\ell\omega} - C_{i\ell}x_{i\ell}) \right\} \right) \text{PR}_{i\omega}$$

subject to

$$x_{i\ell} - g_{i\ell\omega} \geq 0 \quad (\lambda_{i\ell\omega}) \quad \text{for all } \ell = 1, \cdots, L \text{ and } \omega \in \Omega$$

$$g_{i\ell\omega}, x_{i\ell} \geq 0 \quad \text{for all } \ell = 1, \cdots, L \text{ and } \omega \in \Omega$$

$$\sum_{i=1}^{F} g_{i\ell\omega} - D_{\ell\omega} \geq 0 \quad (\mu_{i\ell\omega}) \quad \text{for all } \ell = 1, \cdots, L \text{ and } \omega \in \Omega,$$

where the (possibly non-unique) multiplier $\mu_{i\ell\omega}$ of the shared constraint (2.14) may depend on the player, thus the subscript $i$. Before completing the description of the overall model,
note that the players’ optimization problems \(2.15\) with exogenous prices \(p_{\ell \omega}\) and \(p_{r \omega}'\) lead to a generalized Nash equilibrium problem (GNEP) wherein the demand constraints \(2.14\) are common to (i.e., shared by) all the players’ optimization problems. Since the pioneering paper of Rosen [197], there has been extensive research on such a game; some recent papers include [73, 85, 128, 132, 133, 135, 160, 206]. The cited papers have investigated generalized Nash equilibria of various kinds according to the stipulations of the multipliers \(\mu_{i \ell \omega}\) (i.e., shadow prices). One of the most common stipulations leads to the concept of a variational equilibrium wherein the shared constraints are postulated to have the same shadow prices \(\mu_{\ell \omega}\) (without the subscript \(i\)) for all players. Since shadow prices represent the marginal value of constraint relaxation, it is natural to relate \(\mu_{\ell \omega}\) (the shadow prices of demand satisfaction) to \(p_{\ell \omega}\) (the market prices of demanded commodities). This multiplier specification is the uniform pricing mechanism. Other generalized equilibrium concepts are discussed in the cited references. A variational equilibrium can be computed by the solution of a variational inequality that is equivalent to the problem \(2.10\).

To derive the latter problem in this context, consider a (partial) Lagrangian formulation of \(2.15\) where the common multiplier of each shared constraint is given by its market price (i.e., postulate that \(\mu_{\ell \omega} = p_{\ell \omega}\)). This results in the following problem:

\[
\begin{align*}
\text{maximize} & \quad \sum_{\omega \in \Omega} \left[ \sum_{\ell = 1}^{L} p_{\ell \omega} g_{i \ell \omega} + \left( 1 - \exp \left\{ -a_i \sum_{\ell = 1}^{L} \left[ \left( p_{\ell \omega} - \text{MC}_{i \ell} - \sum_{r = 1}^{R} U_{r \ell i} p_{r \omega} \right) g_{i \ell \omega} - C_{i \ell} x_{i \ell} \right] \right\} \right) \right] PR_{i \omega} \\
\text{subject to} & \quad x_{i \ell} - g_{i \ell \omega} \geq 0 \quad (\lambda_{i \ell \omega}) \quad \text{for all } \ell = 1, \cdots, L \text{ and } \omega \in \Omega \\
& \quad g_{i \ell \omega}, x_{i \ell} \geq 0 \quad \text{for all } \ell = 1, \cdots, L \text{ and } \omega \in \Omega,
\end{align*}
\]
where each price $p_{\ell \omega}$ is subject to the exogenously imposed complementary slackness condition

$$0 \leq p_{\ell \omega} \perp \sum_{i=1}^{\mathcal{F}} g_{i\ell \omega} - D_{\ell \omega} \geq 0.$$  

(2.17)

In terms of profit maximization, the objective function in (2.16) does not have a simple economic interpretation (it is comprised of expected revenue and expected utility from profit). Therefore, it is slightly altered so that it depends solely on profit. Formulate player $i$'s optimization problem as follows:

$$\max_{g_i, x_i} \sum_{\omega \in \Omega} \left[ b_i \left( \sum_{\ell=1}^{L} \left( p_{\ell \omega} - MC_{i\ell} - \sum_{r=1}^{R} U_{i\ell}^{r} p_{i\ell}^{r} \right) g_{i\ell \omega} - C_{i\ell} x_{i\ell} \right) + \left( 1 - \exp \left\{ -a_i \sum_{\ell=1}^{L} \left( p_{\ell \omega} - MC_{i\ell} - \sum_{r=1}^{R} U_{i\ell}^{r} p_{i\ell}^{r} \right) g_{i\ell \omega} - C_{i\ell} x_{i\ell} \right\} \right) \right] PR_{i\omega}$$  

(2.18)

subject to

$$x_{i\ell} - g_{i\ell \omega} \geq 0 \quad (\lambda_{i\ell \omega}) \quad \text{for all } \ell = 1, \cdots, L \text{ and } \omega \in \Omega$$

$$g_{i\ell \omega}, x_{i\ell} \geq 0 \quad \text{for all } \ell = 1, \cdots, L \text{ and } \omega \in \Omega,$$

where $b_i$ is a nonnegative scalar. The use of such a scalar serves two purposes: (1) a conversion factor between expected profit and expected utility, and (2) the unification of the objective function of (2.18) with that of (2.13) when $b_i = 0$.

The formulation of (2.18) is closely related to the concept of mean-risk stochastic optimization. In mean-risk optimization, the objective function consists of the mean of a function of a random variable plus some measure of the function’s risk. In the classical mean-variance approach of Markowitz [145], the risk term is modeled by the variance. Risk measures such as semideviation from a target, central semideviation, and conditional value-at-risk (CVaR) are alternatives to the variance and have been studied extensively for optimiza-
tion problems [2, 201] in recent years. In the context of games with strategic players, the reference [123] employed CVaR as a risk measure for the analysis of supply-side risk in uncertain power markets. The current treatment differs from the latter reference in that presented models are for perfectly competitive games with market clearing conditions for price.

Remark 2.3. An equivalent and possibly more standard formulation of 2.18 from a mean-risk optimization perspective can be obtained by dividing the objective function by \( b_i^2 \). This modification makes \( \frac{1}{b_i} \) the weighting factor (typically denoted as \( c \)) used to convert from the specified risk measure to a risk-neutral payoff but does not change the presented equilibrium existence results. □

The description of the overall capacity expansion model under market clearing conditions is completed with a set of complementarity conditions to be satisfied by the resource prices \( p^r_\omega \), similar to (2.17) for the product prices. Namely, these supply prices are subject to

\[
0 \leq p^r_\omega \perp \overline{R}^r - \sum_{i=1}^{\mathcal{F}} \sum_{\ell=1}^{L} U^r_{i\ell} g_{i\ell\omega} \geq 0 \quad \text{for all } \omega \in \Omega \text{ and } r = 1, \cdots, R, \quad (2.19)
\]

where \( \overline{R}^r \) is the total available quantity of commodity \( r \). The partial Lagrangian objective function form obtained when these supply constraints are enforced as shared constraints with common multipliers has already been accounted for in (2.18).

In summary, the perfectly competitive capacity expansion problem with market clearing-based prices and player risk aversion described by the exponential utility function (2.12) is to determine a tuple \( \{g, x, p, p^r\} \) such that
• each perfectly competitive player $i$ solves the optimization problem (2.18);

• prices satisfy the market clearing conditions (2.17) and (2.19).

The existence result below pertains to the case where $b \triangleq (b_i)_{i=1}^F > 0$, reflecting that all prices are determined by market clearing conditions. It is not clear whether the result will remain valid if some $b_i = 0$ while market clearing conditions are in place for the prices. Again, the challenge in the proof of the result lies in the lack of feasible region compactness.

**Theorem 2.3.** Let $b > 0$ and $D_{\ell \omega} > 0$ for all $\ell$ and $\omega$. Assume that there exists some vector $g$ satisfying

$$
\mathbb{R}^r \geq \sum_{i=1}^F \sum_{\ell=1}^L U_{i\ell}^r g_{i\ell\omega} \quad \text{and} \quad \sum_{i=1}^F g_{i\ell\omega} \geq D_{\ell \omega}
$$

(2.20)

for all $\ell$, $\omega$, and $r$. The perfectly competitive Nash equilibrium problem defined by the problems (2.18) and the market clearing conditions (2.17) and (2.19) has a solution.

**Proof.** The proof here is more complex than that of Theorem 2.2. By concatenating the necessary and sufficient KKT optimality conditions of (2.18) for all players $i = 1, \cdots, F$, examine the solutions of the nonlinear complementarity conditions parameterized by a sequence of positive scalars $\{\tau_k\}$:
• for all $i$, $\ell$, and $\omega$,

\[
0 \leq g_{i\ell\omega}^k \perp \tau_k g_{i\ell\omega}^k + \lambda_{i\ell\omega}^k - \text{PR}_{i\omega} \left[ b_i \left( p_{i\ell\omega}^k - \text{MC}_{i\ell} - \sum_{r=1}^{R} U_{i\ell \omega}^r p_{r\ell\omega}^k \right) + a_i \left( p_{i\ell\omega}^k - \text{MC}_{i\ell} \right) - \sum_{r=1}^{R} U_{i\ell \omega}^r p_{r\ell\omega}^k \right] \times \exp \left\{ -a_i \sum_{\ell=1}^{L} \left[ \left( p_{i\ell\omega}^k - \text{MC}_{i\ell} - \sum_{r=1}^{R} U_{i\ell \omega}^r p_{r\ell\omega}^k \right) g_{i\ell\omega}^k - C_{i\ell} x_{i\ell}^k \right] \right\} \geq 0
\]

\[
0 \leq \lambda_{i\ell\omega}^k \perp x_{i\ell\omega}^k - g_{i\ell\omega}^k + \tau_k \lambda_{i\ell\omega}^k \geq 0
\]

• for all $\ell$ and $\omega$,

\[
0 \leq p_{i\ell\omega}^k \perp \sum_{i=1}^{F} g_{i\ell\omega}^k - D_{i\ell\omega} + \tau_k p_{i\ell\omega}^k \geq 0
\]

• for all $\omega$,

\[
0 \leq p_{\omega}^r \perp \sum_{i=1}^{F} \sum_{\ell=1}^{L} U_{i\ell \omega}^r g_{i\ell\omega}^k + \tau_k p_{\omega}^r \geq 0
\]

• for all $i$ and $\ell$,

\[
0 \leq x_{i\ell}^k \perp \tau_k x_{i\ell}^k - \sum_{\omega \in \Omega} \lambda_{i\ell\omega}^k + \sum_{\omega \in \Omega} \text{PR}_{i\omega} C_{i\ell} \times \left( b_i + a_i \exp \left\{ -a_i \sum_{\ell=1}^{L} \left[ \left( p_{i\ell\omega}^k - \text{MC}_{i\ell} - \sum_{r=1}^{R} U_{i\ell \omega}^r p_{r\ell\omega}^k \right) g_{i\ell\omega}^k - C_{i\ell} x_{i\ell}^k \right] \right\} \right) \geq 0.
\]

It suffices to prove that the sequence of solution tuples $\{g^k, x^k, \lambda^k, p^k, p^{r,k}\}$ is bounded as $k \to \infty$.

**Boundedness of $\{g^k\}$.** Assume that $p_{i\ell\omega}^k > 0$. By complementarity,

\[
g_{i\ell\omega}^k \leq \sum_{i=1}^{F} g_{i\ell\omega}^k = D_{i\ell\omega} - \tau_k p_{i\ell\omega}^k \leq D_{i\ell\omega} \quad \text{for all} \ i.
\]
On the other hand, if \( p^k_{\ell', \omega'} = 0 \), then, for all \( i \),

\[
\lambda^k_{i\ell', \omega'} - PR_{i\omega'} \left[ b_i \left( -MC_{i\ell'} - \sum_{r=1}^{R} U^r_{i\ell'} p^r_{\omega'} \right) + a_i \left( -MC_{i\ell'} - \sum_{r=1}^{R} U^r_{i\ell'} p^r_{\omega'} \right) \right] \times \\
\exp \left\{ -a_i \sum_{\ell=1}^{L} \left[ \left( p^k_{\ell, \omega'} - MC_{i\ell} - \sum_{r=1}^{R} U^r_{i\ell} p^r_{\omega'} \right) g^k_{i\ell\omega'} - C_{i\ell} x^k_{i\ell} \right] \right\} + \tau_k g^k_{i\ell', \omega'} > 0,
\]

which implies, by complementarity, that \( g^k_{i\ell', \omega'} = 0 \) for all \( i \). The boundedness of \( \{g^k\} \) follows.

**Boundedness of \( \{x^k\} \).** For the sake of contradiction, assume that there exists some pair \((i', \ell')\) such that for an infinite index set \( \kappa \subset \{1, 2, \ldots, \infty\} \),

\[
\lim_{k(\in \kappa) \to \infty} x^k_{i'\ell'} = \infty.
\]

Without loss of generality, assume that \( x^k_{i'\ell'} > 0 \) for all \( k \in \kappa \). For every sufficiently large \( k \in \kappa \), one of the following two mutually exclusive and exhaustive cases must hold:

- \( \lambda^k_{i'\ell', \omega'} > 0 \) for some index \( \omega' \). In this case,

\[
x^k_{i'\ell'} = g^k_{i'\ell', \omega'} - \tau_k \lambda^k_{i'\ell', \omega'} \leq g^k_{i'\ell', \omega'} \leq D_{\ell', \omega'},
\]

where the last inequality follows from the boundedness proof of \( \{g^k\} \). This contradicts the assumption that \( \{x^k_{i'\ell'}\} \) is unbounded.
• \( \lambda_{i'\ell'\omega}^k = 0 \) for all \( \omega \). In this case, since \( x_{i'\ell'}^k > 0 \) for all \( k \in \kappa \),

\[
\tau_k x_{i'\ell'}^k + \sum_{\omega \in \Omega} \text{PR}_{i'\omega} C_{i'\ell'} \times \left( b_{i'} \right.
\]

\[
+ a_{i'} \exp \left\{ -a_{i'} \sum_{\ell = 1}^L \left[ \left( p_{i'\ell}^k \mu_{i'\ell} - \sum_{r = 1}^R \text{U}_{i'\ell}^r \text{P}_{\omega}^{r,k} \right) g_{i'\ell'\omega}^k - C_{i'\ell'} x_{i'\ell'}^k \right] \right\} = 0,
\]

which is also a contradiction. The boundedness of \( \{x^k\} \) follows.

**Boundedness of \( \{\lambda^k\} \).** Assume that there exists some triple \((i', \ell', \omega')\) such that for an infinite index set \( \kappa \subset \{1, 2, \cdots, \infty\} \),

\[
\lim_{k(\in \kappa) \to \infty} \lambda_{i'\ell'\omega}^k = \infty,
\]

where \( \lambda_{i'\ell'\omega}^k > 0 \) for all \( k \in \kappa \) without loss of generality. By complementarity,

\[
x_{i'\ell'}^k + \tau_k \lambda_{i'\ell'\omega}^k = g_{i'\ell'\omega}^k,
\]

which implies that

\[
\tau_k \lambda_{i'\ell'\omega}^k \leq g_{i'\ell'\omega}^k.
\]
Since \( \{g^k\} \) is bounded, it follows that \( \{\tau_k\} \to 0 \) as \( k(\in \kappa) \to \infty \).

\[
\sum_{\omega \in \Omega} \lambda_{i'\ell'\omega}^k \leq \tau_k x_{i'\ell}^k + \sum_{\omega \in \Omega} \text{PR}_{i'\omega} C_{i'\ell'} \times 
\left( b_{i'} + a_{i'} \exp \left\{ -a_{i'} \sum_{\ell=1}^L \left( p_{\ell \omega}^k - MC_{i'\ell} - \sum_{r=1}^R U_{i'\ell}^r p_{r\omega}^k \right) g_{i'\ell\omega}^k - C_{i'\ell'} x_{i'\ell}^k \right\} \right)
= \tau_k a_{i'\ell'}^k + \sum_{\omega \in \Omega} \text{PR}_{i'\omega} C_{i'\ell'} \times 
\left( b_{i'} + a_{i'} \exp \left\{ a_{i'} \sum_{\ell=1}^L \left( -p_{\ell \omega}^k + MC_{i'\ell} + \sum_{r=1}^R U_{i'\ell}^r p_{r\omega}^k \right) g_{i'\ell\omega}^k + C_{i'\ell'} x_{i'\ell}^k \right\} \right).
\]

To derive a contradiction, it suffices to show that for every \( \ell \) and \( \omega \),

\[
\left\{ \left( -p_{\ell \omega}^k + MC_{i'\ell} + \sum_{r=1}^R U_{i'\ell}^r p_{r\omega}^k \right) g_{i'\ell\omega}^k \right\}_{k \in \kappa}
\]

is bounded above. Assume the contrary; without loss of generality, by working with a proper subsequence if necessary, say that for some pair \( (\tilde{\ell}, \tilde{\omega}) \),

\[
\lim_{k(\in \kappa) \to \infty} \left[ \left( -p_{\ell \omega}^k + MC_{i'\ell} + \sum_{r=1}^R U_{i'\ell}^r p_{r\omega}^k \right) g_{i'\ell\omega}^k \right] = \infty.
\]

Hence, for all \( k \in \kappa \) sufficiently large,

\[-p_{\ell \omega}^k + MC_{i'\ell} + \sum_{r=1}^R U_{i'\ell}^r p_{r\omega}^k > 0 \quad \text{and} \quad g_{i'\ell\omega}^k > 0.\]
Therefore,

\[ \lambda^k_{i',\ell} - \text{PR}_{i'\omega} \left[ b_i' \left( p^k_{\ell,\omega} - \text{MC}_{i'\tilde{\ell}} - \sum_{r=1}^R U^r_{i'\ell} p^{r,k}_{\omega} \right) + a_i' \left( p^k_{\ell,\omega} - \text{MC}_{i'\tilde{\ell}} - \sum_{r=1}^R U^r_{i'\ell} p^{r,k}_{\omega} \right) \right] \times \exp \left\{ -a_i' \sum_{\ell=1}^L \left[ \left( p^k_{\ell,\omega} - \text{MC}_{i'\ell} - \sum_{r=1}^R U^r_{i'\ell} g^{r,k}_{i'\ell,\omega} - C_{i'\ell} x_{i'\ell}^k \right) \right] + \tau_k g^{k}_{i'\ell,\omega} = 0, \right\} \]

a contradiction because the left-hand side is positive. This establishes the boundedness of \( \{\lambda^k\} \).

**Boundedness of \( \{p^{r,k}\} \).** Assume that there exists some pair \((\omega', r')\) such that for an infinite index set \( \kappa \subset \{1, 2, \ldots, \infty\} \),

\[ \lim_{k(\in \kappa) \to \infty} p^{r',k}_{\omega'} = \infty. \]

As before, assume without loss of generality that \( p^{r',k}_{\omega'} > 0 \) for all \( k \in \kappa \). Thus, by complementarity,

\[ \tilde{R}^{r'} - \sum_{i=1}^I \sum_{\ell=1}^L U^r_{i\ell} g^{k}_{i\ell,\omega'} + \tau_k p^{r',k}_{\omega'} = 0. \]

With \( \tilde{R}^{r'} \) being a positive constant and \( \{g^k\} \) being bounded, it follows that \( \{\tau_k\} \to 0 \) as \( k(\in \kappa) \to \infty \). Consider two mutually exclusive and exhaustive cases for such a sequence of solutions:
\{p^k_{i\omega}\}_{k \in \kappa} is bounded for some \ell. In this case, for all \(k \in \kappa\) sufficiently large and for all \(i\),

\[
\lambda^k_{i\ell\omega}, - PR_{i\omega}, \left[ b_i \left( p^k_{i\ell\omega}, - MC_{i\ell} - \sum_{r=1}^{R} U_{i\ell}^r p^r_{i\omega'} \right) + a_i \left( p^k_{i\ell\omega}, - MC_{i\ell} - \sum_{r=1}^{R} U_{i\ell}^r p^r_{i\omega'} \right) \times 
\exp \left\{ -a_i \sum_{\ell=1}^{L} \left[ \left( p^k_{i\ell\omega}, - MC_{i\ell} - \sum_{r=1}^{R} U_{i\ell}^r p^r_{i\omega'} \right) g^k_{i\ell\omega}, - C_{i\ell} x^k_{i\ell} \right] \right\} + \tau_k g^k_{i\ell\omega}, > 0. \]

Hence, by complementarity, it follows that \(g^k_{i\ell\omega}, = 0\) for all \(i\). However,

\[
0 \leq \sum_{i=1}^{F} g^k_{i\ell\omega}, - D_{i\omega}, + \tau_k p^k_{i\omega}, = -D_{i\omega}, + \tau_k p^k_{i\omega}, \rightarrow -D_{i\omega}, \quad \text{as } k(\in \kappa) \rightarrow \infty. 
\]

This yields a contradiction.

\{p^k_{i\omega}\}_{k \in \kappa} is unbounded for all \ell. Without loss of generality, assume that

\[
\lim_{k(\in \kappa) \rightarrow \infty} p^k_{i\omega}, = \infty
\]

and that \(p^k_{i\omega}, > 0\) for all \(\ell\) and \(k\). For all \(i\) and \(\ell\),

\[
\lambda^k_{i\ell\omega}, - PR_{i\omega}, \left[ b_i \left( p^k_{i\ell\omega}, - MC_{i\ell} - \sum_{r=1}^{R} U_{i\ell}^r p^r_{i\omega'} \right) + a_i \left( p^k_{i\ell\omega}, - MC_{i\ell} - \sum_{r=1}^{R} U_{i\ell}^r p^r_{i\omega'} \right) \times 
\exp \left\{ -a_i \sum_{\ell=1}^{m} \left[ \left( p^k_{i\ell\omega}, - MC_{i\ell} - \sum_{r=1}^{R} U_{i\ell}^r p^r_{i\omega'} \right) g^k_{i\ell\omega}, - C_{i\ell} x^k_{i\ell} \right] \right\} + \tau_k g^k_{i\ell\omega}, \geq 0. \]
There are 2 subcases to consider, depending on whether the sequence \( \{ \hat{e}_k^i \}_{k \in \kappa} \), where

\[
\hat{e}_k^i \triangleq \exp \left\{ -a_i \sum_{\ell=1}^{L} \left( \left( \frac{p_{\ell \omega}^k}{p_{\ell \omega}^k} - MC_{it} - \sum_{r=1}^{R} U_{it}^r p_{\omega}^{r,k} \right) g_{it \omega}^k - C_{it} x_{it}^k \right) \right\},
\]

is bounded for all \( i \) or unbounded for some \( i \). Consider the former case first. Without loss of generality, assume that the sequence \( \{ \hat{e}_k^i \}_{k \in \kappa} \) converges to a limit, say \( \hat{e}_i^\infty \), which must be nonnegative. Let

\[
\hat{p}_{\ell \omega}^k \triangleq \frac{p_{\ell \omega}^k}{\sum_{\ell=1}^{L} p_{\ell \omega}^k + \sum_{r=1}^{R} p_{\omega}^{r,k}} \quad \text{and} \quad \hat{p}_{\omega}^{r,k} \triangleq \frac{p_{\omega}^{r,k}}{\sum_{\ell=1}^{L} p_{\ell \omega}^k + \sum_{r=1}^{R} p_{\omega}^{r,k}}.
\]

Without loss of generality, assume that the sequences \( \{ \hat{p}_{\ell \omega}^k \}_{k \in \kappa} \) and \( \{ \hat{p}_{\omega}^{r,k} \}_{k \in \kappa} \) converge to the limits \( \hat{p}_{\ell \omega}^\infty \) and \( \hat{p}_{\omega}^{r,\infty} \), respectively, which must be nonnegative and satisfy

\[
\sum_{\ell=1}^{L} \hat{p}_{\ell \omega}^\infty + \sum_{r=1}^{R} \hat{p}_{\omega}^{r,\infty} = 1.
\]

Furthermore, assume that \( \{ g_{it \omega}^k \}_{k \in \kappa} \) converges to a limit \( \hat{g}_{it \omega}^\infty \).

\[
\text{PR}_{t \omega} \left[ b_i + a_i \hat{e}_i^\infty \right] \left( -\hat{p}_{\ell \omega}^\infty + \sum_{r=1}^{R} U_{it}^r \hat{p}_{\omega}^{r,\infty} \right) \geq 0,
\]

which implies \( -\hat{p}_{\ell \omega}^\infty + \sum_{r=1}^{R} U_{it}^r \hat{p}_{\omega}^{r,\infty} \geq 0 \). Since the system (2.20) is feasible, it can be deduced that

\[
\sum_{r=1}^{R} \bar{R}^r \hat{p}_{\omega}^{r,\infty} \geq \sum_{\ell=1}^{L} D_{t \omega} \hat{p}_{\ell \omega}^\infty.
\]

(2.21)

It is claimed that

\[
\sum_{i=1}^{I} \sum_{\ell=1}^{L} \left[ \hat{p}_{\ell \omega}^\infty - \sum_{r=1}^{R} U_{it}^r \hat{p}_{\omega}^{r,\infty} \right] g_{it \omega}^k = 0 \text{ for all } k \in \kappa \text{ sufficiently}
\]

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large. Indeed, if this is false, then for an infinite subset \( \kappa' \) of \( \kappa \), for some \( i \) and \( \ell \),

\[
\left[ -\hat{p}_{i\ell \omega}^k, + \sum_{r=1}^{R} U_{i\ell \ell}^r \hat{p}_{r \omega}^r \right] > 0 \text{ and } g_{i\ell \omega}^k > 0 .
\]

The latter implies that

\[
0 = \sum_{\ell=1}^{L} \hat{g}_{i\ell \omega}^k \left[ \sum_{i=1}^{F} g_{i\ell \omega}^k, - D_{i\ell \omega} + \tau_k \hat{p}_{i\ell \omega}^k \right] + \sum_{r=1}^{R} \hat{p}_{r \omega}^r \left[ \sum_{i=1}^{F} \sum_{\ell=1}^{L} U_{i\ell \ell}^r \hat{g}_{i\ell \omega}^k + \sum_{\ell=1}^{L} \hat{p}_{i\ell \omega}^k \right]
\]

which in the limit as \( k(\in \kappa') \to \infty \) yields \( \left[ -\hat{p}_{i\ell \omega}^k, + \sum_{r=1}^{R} U_{i\ell \ell}^r \hat{p}_{r \omega}^r \right] = 0 \), which is a contradiction. By a similar argument, it can be deduced by (2.21) that

\[
0 = \sum_{\ell=1}^{L} \hat{g}_{i\ell \omega}^k \left[ \sum_{i=1}^{F} g_{i\ell \omega}^k, - D_{i\ell \omega} + \tau_k \hat{p}_{i\ell \omega}^k \right] + \sum_{r=1}^{R} \hat{p}_{r \omega}^r \left[ \sum_{i=1}^{F} \sum_{\ell=1}^{L} U_{i\ell \ell}^r \hat{g}_{i\ell \omega}^k + \sum_{\ell=1}^{L} \hat{p}_{i\ell \omega}^k \right]
\]

a contradiction. Consider the second subcase where \( \{ \hat{e}_{i\ell}^k \}_{k \in \kappa} \) is unbounded for some \( i \).

There must exist some \( \ell \) and an infinite index set \( \kappa' \subseteq \kappa \) such that

\[
\left[ -p_{i\ell \omega}^k, + MC_{i\ell} + \sum_{r=1}^{R} U_{i\ell \ell}^r \hat{p}_{r \omega}^r \right] g_{i\ell \omega}^k \to \infty
\]
as \( k(\in \kappa') \to \infty \). Hence,

\[
g_{i\ell \omega}^k \left[ \lambda_{i\ell \omega}^k \right. \left. - PR_{i\ell \omega} \right] \left[ b_i \left( p_{i\ell \omega}^k, - MC_{i\ell} - \sum_{r=1}^{R} U_{i\ell \ell}^r \hat{p}_{r \omega}^r \right) + a_i \left( p_{i\ell \omega}^k, - MC_{i\ell} - \sum_{r=1}^{R} U_{i\ell \ell}^r \hat{p}_{r \omega}^r \right) \right]
\]

\[
\times \exp \left\{ -a_i \sum_{\ell=1}^{L} \left[ \left( p_{i\ell \omega}^k, - MC_{i\ell} - \sum_{r=1}^{R} U_{i\ell \ell}^r \hat{p}_{r \omega}^r \right) g_{i\ell \omega}^k, - C_{i\ell} \hat{p}_{r \omega}^r \right] \right\} + \tau_k \left( g_{i\ell \omega}^k \right)^2 = 0 ,
\]

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which is a contradiction because the left-hand side is positive. This completes the proof
of the boundedness of \( \{ p^{r,k} \} \).

**Boundedness of \( \{ p^k \} \).** Assume that there exists some pair \((\ell', \omega')\) such that for an infinite index set \( \kappa \subset \{1, 2, \cdots, \infty\} \),

\[
\lim_{k(\in \kappa) \to \infty} p^k_{\ell' \omega'} = \infty.
\]

Assume without loss of generality that \( p^k_{\ell' \omega'} > 0 \) for all \( k \in \kappa \). Thus, by complementarity,

\[
\sum_{i=1}^{F} g^k_{i \ell' \omega'} + \tau_k p^k_{\ell' \omega'} = D_{\ell' \omega'}.
\]

Therefore, \( \{ \tau_k \} \to 0 \) as \( k(\in \kappa) \to \infty \). Because

\[
\lambda^k_{i \ell' \omega'} - \text{PR}_{i \omega'} \left[ b_i \left( p^k_{i \ell' \omega'} - \text{MC}_{i \ell'} - \sum_{r=1}^{R} U^r_{i \ell'} p^r_{i \omega'} \right) + a_i \left( p^k_{i \ell' \omega'} - \text{MC}_{i \ell'} - \sum_{r=1}^{R} U^r_{i \ell'} p^r_{i \omega'} \right) \right] \times \exp \left\{ -a_i \sum_{\ell=1}^{L} \left[ \left( p^k_{i \ell \omega'} - \text{MC}_{i \ell} - \sum_{r=1}^{R} U^r_{i \ell} p^r_{i \omega'} \right) g^k_{i \ell \omega'} - \text{C}_{i \ell} x^k_{i \ell} \right] \right\} + \tau_k g^k_{i \ell' \omega'} \ge 0,
\]

it follows readily that \( \{ p^k \} \) must be bounded. This contradiction completes the proof of the theorem. \( \square \)

**Remark 2.4.** The positivity of the scalars \( b_i \) is used in several places in the above proof. Essentially, the proof reveals the possibility that if some \( b_i = 0 \), the existence of an equilibrium solution to the perfectly competitive model could be in jeopardy if the market clearing conditions (2.17) and (2.19) are imposed exogenously to the players’ optimization problems and yet these conditions are not properly accounted for in the producers’ objective functions. As explained in the discussion that leads to the formulation (2.18), the requirement
that \( b > 0 \) is natural from the standpoint of Lagrangian duality in optimization.

### 2.7 An extension with consumer surplus maximization

In (2.14), it was assumed that \( D_{\ell \omega} \) was constant, thereby implying that consumers have zero price elasticity of demand. In real-world settings, consumers usually adjust their purchases based on price. When there is nonzero cross-price elasticity of demand, this type of consumer behavior is modeled as the maximization of consumer surplus (i.e., the area under the demand curve less total cost). For this stochastic problem, the consumer is assumed to maximize expected consumer surplus by choosing their consumption in market \( \ell \) and scenario \( \omega \), with \( \text{PR}^C_{\omega} \in (0, 1) \) being the probability the consumer associates with scenario \( \omega \). For simplicity, assume linear demand curves \( P_C^{\ell \omega} - \frac{P_C^{\ell \omega}}{Q_C^{\ell \omega}} q_{\ell \omega} \) for all pairs \((\ell, \omega)\), where \( P_C^{\ell \omega} \) and \( Q_C^{\ell \omega} \) are positive constants. Since the consumer is perfectly competitive with respect to market prices, the consumer’s optimization problem is

\[
\text{maximize } \sum_{\omega \in \Omega} \sum_{\ell = 1}^L \left( P_C^{\ell \omega} q_{\ell \omega} - \frac{P_C^{\ell \omega}}{2Q_C^{\ell \omega}} q_{\ell \omega}^2 - p_{\ell \omega} q_{\ell \omega} \right) \text{PR}^C_{\omega} \quad (2.22)
\]

It should be obvious that (2.22) can be expressed as \( L|\Omega| \) scalar optimization problems, one for each market \( \ell \) and each scenario \( \omega \). Letting \( D_{\ell \omega} \) denote the optimal solution of (2.22), \( D_{\ell \omega} \) must satisfy the KKT conditions

\[
0 \leq D_{\ell \omega} \perp \left[ -p_{\ell \omega} + \frac{P_C^{\ell \omega}}{Q_C^{\ell \omega}} D_{\ell \omega} + p_{\ell \omega} \right] \text{PR}^C_{\omega} \geq 0 \quad \text{for all } \ell = 1, \cdots, L \text{ and } \omega \in \Omega. \quad (2.23)
\]

Augmenting the perfectly competitive model with the above consumer conditions, the next existence result for a perfectly competitive problem with a surplus-maximizing consumer can be proven. This result requires no assumptions other than the positivity of \( b \triangleq (b_i)_{i=1}^F \).
Theorem 2.4. Let $b > 0$. The perfectly competitive Nash equilibrium problem defined by the producers’ problems (2.18), the market clearing conditions (2.17) and (2.19), and the consumer surplus maximization condition (2.23) has a solution.

Proof. The condition (2.23) yields that for $p_{\ell\omega} \geq 0$,

$$D_{\ell\omega} = \frac{Q_C^{C}}{P_C^{C}} \max \left( 0, P_C^{C} - p_{\ell\omega} \right) \leq Q_C^{C}, \quad (2.24)$$

showing that the demand $D_{\ell\omega}$ is a piecewise linear function of the price $p_{\ell\omega}$ bounded above by the constant $Q_C^{C}$. Substituting this expression for $D_{\ell\omega}$ into (2.17) reduces the game with consumer surplus maximization to the same problem in Theorem 2.3 except that $D_{\ell\omega}$ is replaced by the variable expression $\frac{Q_C^{C}}{P_C^{C}} \max \left( 0, P_C^{C} - p_{\ell\omega} \right)$. Theorem 2.1 is still applicable to the resulting NCP.

Following the proof of Theorem 2.3, the sequences of tuples $\{g^k\}$, $\{x^k\}$, and $\{\lambda^k\}$ are bounded. The boundedness of the price sequences $\{p^k, p_r^k\}$ needs to be shown.

**Boundedness of $\{p^k\}$.** With the above expression of the demand $D_{\ell\omega}$, the price $p_{\ell\omega}^k$ satisfies the complementarity condition

$$0 \leq p_{\ell\omega}^k \perp \sum_{i=1}^{F} g_{i\ell\omega}^k - \frac{Q_C^{C}}{P_C^{C}} \max \left( 0, P_C^{C} - p_{\ell\omega}^k \right) + \tau_k p_{\ell\omega}^k \geq 0. \quad (2.25)$$

If $\max \left( 0, P_C^{C} - p_{\ell\omega}^k \right) = 0$, then the above complementarity condition becomes

$$0 \leq p_{\ell\omega}^k \perp \sum_{i=1}^{F} g_{i\ell\omega}^k + \tau_k p_{\ell\omega}^k \geq 0,$$
which yields $p^k_{i\omega} = 0$, contradicting $\max(0, P^C_{i\omega} - p^k_{i\omega}) = 0$. Therefore,

$$\max\left(0, P^C_{i\omega} - p^k_{i\omega}\right) = P^C_{i\omega} - p^k_{i\omega},$$

which yields $p^k_{i\omega} \leq P^C_{i\omega}$. Thus, $\{p^k\}$ is bounded.

**Boundedness of** $\{p^{r,k}_{\omega}\}$. Assume that there exists some pair $(\omega', r')$ such that for an infinite index set $\kappa \subset \{1, 2, \cdots, \infty\}$,

$$\lim_{k(\in \kappa) \to \infty} p^{r',k}_{\omega'} = \infty. \tag{2.25}$$

As before, assume without loss of generality that $p^{r',k}_{\omega'} > 0$ for all $k \in \kappa$. Thus, by complementarity,

$$R^{r'} - \sum_{i=1}^F \sum_{\ell=1}^L U^r_{i\ell} g^k_{i\ell\omega'} + \tau_k p^{r',k}_{\omega'} = 0, \tag{2.26}$$

implying that $\{\tau_k\} \to 0$ as $k(\in \kappa) \to \infty$. By (2.25), for all $k \in \kappa$ sufficiently large, all $i$, and all $\ell$,

$$\lambda^k_{i\ell\omega'} - PR_{i\omega'} \left[ b_i \left( p^k_{i\omega'} - MC_{i\ell} - \sum_{r=1}^R U^r_{i\ell} p^{r,k}_{i\omega'} \right) + a_i \left( p^k_{i\omega'} - MC_{i\ell} - \sum_{r=1}^R U^r_{i\ell} p^{r,k}_{i\omega'} \right) \times \exp \left\{ -a_i \sum_{\ell=1}^L \left[ \left( p^k_{i\omega'} - MC_{i\ell} - \sum_{r=1}^R U^r_{i\ell} p^{r,k}_{i\omega'} \right) g^k_{i\ell\omega'} - C_{i\ell} \right] \right\} + \tau_k g^k_{i\ell\omega'} \right] > 0.$$ 

Consequently, $g^k_{i\ell\omega'} = 0$ for all $i$ and $\ell$, a contradiction of (2.26). \hfill \Box

**Remark 2.5.** Substituting the expression for the demand $D_{i\omega}$ into the complementarity condition (2.17), it can be seen that this capacity expansion game with consumer surplus maximization is an instance of a distributed multi-agent optimization problem with price
2.8 Conclusion

Existence results for general situations involving risk averse investors have become particularly useful due to the realization that risk neutrality assumptions underlying many large-scale energy market models are highly unrealistic. In actual energy (and other) markets, investors are typically characterized by short-sightedness, risk aversion, differences in beliefs, and the inability to hedge against many risks due to incomplete markets. As a consequence, policy or investment recommendations based upon models that assume low discount rates and either perfect foresight or risk neutrality combined with a common set of beliefs about scenario probabilities might be unwise.

This chapter has established the existence of a Nash equilibrium for more realistic multi-stage market models that account for risk aversion and player-specific beliefs about the future. Additionally, three distinct pricing mechanisms were examined and a well-posed model with risk aversion and market clearing conditions was developed (see (2.16) and (2.18)). Directions of future work include (a) investigating whether the analytic tools of [75] can establish equilibria uniqueness properties, and (b) selecting a particular Nash equilibrium to optimize a secondary (e.g., system-wide) objective through the methodology developed in [77]. It is conjectured that equilibrium uniqueness will not hold in general, but the more realistic nature of this problem should bolster confidence in this type of market model.
Chapter 3

Affine generalized Nash equilibria and Lemke’s method

3.1 Introduction

As was briefly discussed in Section 1.1, Nash equilibrium problems in which players’ decisions are restricted by their rivals’ decisions are known as generalized Nash equilibrium problems. In general, GNEPs are difficult to solve without imposing restrictions on either the problem form or the desired solution properties; this research focuses on a tractable subset of GNEPs characterized by convex quadratic player objective functions and affine constraint sets. With this problem specification, the necessary and sufficient KKT optimality conditions (Theorem 1.1) can be represented by a linear complementarity problem, thus the naming convention “affine” GNEP (AGNEP). This research analyzes the difficulties that arise when applying Lemke’s method, a well-known pivoting algorithm for linear

\[ \text{This chapter is slightly adapted from [206] (Co-authors: Jong-Shi Pang and Uday Shanbhag). Springer and Mathematical Programming, DOI 10.1007/s10107-012-0558-3 (2012), original copyright is given to the publication in which the material was originally published, by adding; with kind permission from Springer Science and Business Media. License # 3156660758413.} \]
complementarity problems (LCPs), to AGNEPs and proposes modifications that address these issues.

Constraints that link player feasible regions in GNEPs take one of two forms: coupled or shared. Both constraint types involve the joint variables \((x_i, x_{-i})\), but coupled constraints may be different for each player while shared constraints must be identical across all players. This chapter only deals with shared constraints; the interested reader is referred to [178] for an analysis of problems with coupled constraints. Although the concept of AGNEPs with shared constraints may seem abstract, equilibrium problems of this form arise in many applications. Several such examples are listed below, the first three of which are studied in greater detail subsequently:

- A (static) river basin pollution problem first discussed in [100] and subsequently extended to a dynamic setting in [128] where shared constraints model locational pollution limits set by a regulatory authority and respected by the players (see also [45]);

- A rate allocation game in which a set of players compete for bandwidth over a communication network [4, 240] with shared constraints enforcing the network routing capacity limits;

- A bilateral electricity market game with piecewise linear prices, transmission network constraints, and shared constraints representing regional sales caps [110]. More general nonsmooth Nash games are examined by Pang and Sun in [184];

- An electricity market model with regulated transmission prices [230] where shared constraints model limits on net energy flows;
A multiclearing electricity market game \cite{28} where the shared constraints model minimum generation requirements in various markets.

The natural KKT conditions associated with the shared constraints of an AGNEP are

\[
0 \leq \lambda_i^s \perp \sum_{j=1}^{\mathcal{F}} A_j x_j - b \geq 0 \\
\vdots \\
0 \leq \lambda_{\mathcal{F}}^s \perp \sum_{j=1}^{\mathcal{F}} A_j x_j - b \geq 0,
\]

where \( A_j \in \mathbb{R}^{m_s \times n_j} \) for all \( j \) and \( b \in \mathbb{R}^{m_s} \). These duplicated constraints may lead to degeneracy and may therefore make the complete set of KKT conditions difficult to solve if the presence of \( \mathcal{F} \) potentially different Lagrange multiplier vectors \( \lambda_i^s \) associated with the same shared constraints is not taken into account. To address this degeneracy, much past research has been devoted to specialized problem forms in which the \( \mathcal{F} \lambda^s_i \)-vectors are not independent. In his pioneering paper \cite{197}, Rosen introduced the important concept of a normalized Nash equilibrium to treat the GNEP. Based on optimality theory in nonlinear programming, a normalized NE is an equilibrium characterized by proportional multipliers of the shared constraints in all players’ optimization problems. While this is a restrictive requirement, it nevertheless has certain advantages that render the computation of a GNE more tractable than the general case where there is no such stipulation on the multipliers. Furthermore, such a requirement appears to have relevance when these Lagrange multipliers are interpreted as prices with normalized NEs being viewed as instances of uniform pricing.

A different GNE solution concept, known as a variational equilibrium (VE), was coined in \cite{73} to describe a NE with common multipliers corresponding to the shared constraints. The adjective “variational” signifies that such an equilibrium can be obtained as a solution.
of a VI. This research takes a Lemke’s method-specific approach to address this degeneracy issue and can obtain solutions that are neither normalized Nash equilibria nor variational equilibria.

**Remark 3.1.** Because of their relation to variational inequalities, there is a considerable amount of research studying the relationship between variational equilibria and other GNE forms. One such relationship question is whether the existence of a GNE (with no stipulation on consistency across $\lambda_i$-vectors) implies the existence of a VE. In economic parlance, if true, the VE is a refinement of the GNE. The notion of refinement has been explored extensively: trembling hand perfect [209] and proper [159] equilibria are refinements of mixed strategy Nash equilibria in static finite strategy games [158] while the subgame-perfect Nash equilibrium is a refinement of the Nash equilibrium of a dynamic game (see [171, Chapter 3.8]). In [133], conditions are provided under which the VE is indeed a refinement of a GNE.

This research fills an important gap in the recent surge of interest in computing GNE (e.g. [61, 74, 85, 104, 122, 123, 129, 130, 132, 133, 135, 160, 179]). For the most part, the existing literature as exemplified by these references deals with the GNEP in general and not the AGNEP; moreover, the algorithms presented therein are generally infinite in nature: their convergence, if at all, occurs only in the limit and the conditions for convergence are fairly abstract. For instance, [179] introduces a quasi-variational inequality (QVI) formulation of a GNEP and a penalty-based solution method. For an AGNEP, the QVI is equivalent to an LCP, but the proposed penalty-based solution method relies on an implication similar to the Mangasarian-Fromovitz constraint qualification for convergence that seems unnecessary in the affine case. In contrast, as an LCP, the finitely terminating Lemke’s method can in principle be applied to compute a GNE. There is a host of literature
on this topic, ranging from the classic works of Eaves on the LCP of a special kind [63] that includes the AGNEP, on computing a VE of a polymatrix game [64], on computing stationary points of affine maps [66, 67] that include the VE, and on solving equations using piecewise linear homotopies [65], to the more contemporary treatment of [33, 34] and the unpublished manuscript [178] that presents sufficient conditions to ensure the successful termination of Lemke’s method for an AGNEP. There is also the publicly available computer software path [59, 81]; while being a fairly robust solver for general LCPs, a major drawback of this solver for computing equilibria of AGNEPs is that there is no theoretical guarantee of success even if a solution is known to exist. Thus, the gap filled by the present work is that it offers a rigorous study on the application of a finitely terminating algorithm to the AGNEP with a focus on its applicability for realistic instances of such games.

Of the aforementioned papers, [178, Section 6] presents some of the most general research on AGNEPs: the coupled constraints are all assumed to be particular to the individual players and there is no stipulation on the Lagrange multipliers of any coupled constraints that happen to be shared constraints. However, no special attention is paid to the application of Lemke’s method to solve the AGNEP. This is the departure of the present work; namely, much of the focus herein is on how shared constraints and their multipliers affect the success or failure of Lemke’s method. As will be discussed later, Lemke’s method finitely terminates with either a solution or along an infeasible ray (see Section 3.3). Thus, understanding the $\lambda_i^s$-multiplier properties associated with successful termination of Lemke’s method (i.e., identification of a solution) is of obvious importance.

The focus on the $\lambda_i^s$-vectors in this research is justified because the Lagrange multiplier of a constraint is economically interpreted as a “shadow price,” the change in objective function value that would arise from a marginal relaxation of the constraint. In particu-
lar, common multipliers for shared constraints may be interpreted as the uniform auction price of the constraint resource. Players possessing different Lagrange multiplier values for the same constraint can be interpreted as having contrasting valuations for the same good. This situation may arise, for instance, when there is an asymmetry in preferences, existence of market power, incompleteness in information, or incompleteness in markets; pay-as-bid auctions capture precisely these valuation disparities. A subsequent result, Proposition 3.4, establishes the $\lambda_i^s$-multiplier structure of solutions obtained from a traditional implementation of Lemke's method for an AGNEP. In all such equilibria, the binding status of a shared constraint implies the existence of a single player who obtains all of the benefit from a marginal unit of the shared resource, a display of market power. This rather interesting solution property is an unexpected consequence of the way that Lemke’s method works. It is of prime importance in market design to examine all possible equilibria that may arise, and this special $\lambda_i^s$-multiplier structure provides a key motivation for the present work. To what extent and how each such equilibrium is of practical value depends on the application and other issues beyond the scope of this chapter.

The main contributions of this research are as follows:

(a) It is proven that a successful application of Lemke’s method to the natural LCP formulation of an AGNEP (i.e., without any special stipulation of the $\lambda_i^s$-vectors) will compute a GNE of a particular kind.

(b) The notion of a partial variational equilibrium is introduced; this equilibrium is characterized by the property that the Lagrange multipliers of some shared constraints are forced to be common across all players while Lagrange multipliers associated with other shared constraints are not so restricted. Using this notion in conjunction with a modified
Lemke’s method (refer to (c)), it is possible to compute generalized Nash equilibria that do not satisfy the property specified in (a). A brief sketch of a further generalization of this kind of NE, a coalitional equilibrium, is also presented for which each shared constraint is associated with a partition of the player set into coalitions (i.e., mutually disjoint subsets whose union is the whole set of players) such that the players within each such coalition are required to have common multipliers for the given shared constraint.

(c) A modification of Lemke’s method is introduced and is shown to compute a solution of the AGNEP under certain conditions. The solution may be of the form referred to in (a) or (b). Computational results demonstrate that the proposed modification of Lemke’s method can compute generalized Nash equilibria that may not be obtained by the standard version of the method due to random degenerate pivots. Furthermore, a performance analysis suggests that the algorithm scales roughly linearly with the number of shared constraints.

(d) It is shown that specific regularizations of the natural LCPs arising from AGNEPs produce trajectories that are guaranteed to converge to a variational equilibrium if one exists; more generally, it is shown that a certain regularization scheme of the LCP formulation of an AGNEP will produce in the limit a broad class of generalized Nash equilibria that generalizes Rosen’s normalized NE concept. Similar to Rosen’s normalized NE, the generalization provided in this paper is a special case of the restricted GNE of [85] corresponding to equilibria in which shared constraint multipliers lie in a cone. Such a regularization scheme can in turn be combined with the idea of a coalitional equilibrium to further expand the solution types that can be found by Lemke’s method.

(e) A constraint reformulation and parametrization idea is introduced that will produce yet another kind of generalized Nash equilibria that is characterized by a different $\lambda^i$-multiplier
property. Solutions with this multiplier property are not readily computable without the problem reformulation.

The remainder of the chapter is organized in 9 sections. In Section 3.2, the examined AGNEP is explicitly formulated and its associated natural LCP is provided. After a summary of Lemke’s method in Section 3.3, the classes of partial variational equilibria and coalitional equilibria are introduced in Section 3.4. The modified Lemke method is presented in Section 3.5 and its successful termination is analyzed therein. In Section 3.6, the identification of alternate generalized Nash equilibria is demonstrated by introducing a novel problem reformulation. A regularization technique that leads to generalizations of the traditional normalized equilibria of Rosen is discussed in Section 3.7. Section 3.8 addresses the aforementioned application problems while the performance of the modified Lemke method is presented in Section 3.9, together with a description of alternate equilibria that can be obtained through coalitional and reformulation techniques. Section 3.10 provides concluding remarks on this research.

### 3.2 Problem formulation

In this section, the canonical form of the AGNEP studied in this work is introduced along with its LCP formulation. Following the notation of Chapter 1, the AGNEP $G$ consists of $F$ non-cooperative players, each of whom solves a convex quadratic program whose objective function and constraints contain the rival players’ decision variables. Specifically, taking the rivals’ decision variable vector $x_{-i} \triangleq (x_j)_{j \neq i}$ as exogenous, player $i$’s optimization problem is:

$$
\text{minimize}_{x_i \in K_i(x_{-i})} f_i(x_i; x_{-i}) \triangleq \frac{1}{2} x_i^T H_{ii} x_i + x_i^T \sum_{j \neq i} H_{ij} x_j + h_i^T x_i,
$$
where, for a given $x_{-i}$,

$$K_i(x_{-i}) \triangleq \left\{ x_i \in \mathbb{R}_+^{n_i} \mid B_i x_i \geq r_i, \sum_{j=1}^F A_j x_j \geq b \right\}.$$ 

Here, $H_{ii}$ is an $n_i \times n_i$ symmetric positive semidefinite matrix, $H_{ij} \in \mathbb{R}^{n_i \times n_j}$, $h_i \in \mathbb{R}^{n_i}$, $B_i \in \mathbb{R}^{m_i \times n_i}$, $r_i \in \mathbb{R}^{m_i}$, $A_j \in \mathbb{R}^{m_s \times n_j}$, and $b \in \mathbb{R}^{m_s}$. The inequality $B_i x_i \geq r_i$ describes player $i$’s private constraints, whereas $\sum_{j=1}^F A_j x_j \geq b$, which is common to all players’ problems, represents the shared constraints that couple all players’ variables. Throughout the paper, the superscripts $p$ and $s$ signify “private” and “shared”, respectively, with respect to constraint Lagrange multipliers.

The above definition of the AGNEP is not of the most general kind. For instance, coupled [but non-shared] constraints like $\sum_{j=1}^F E_{ij} x_j \geq g_i$ for player $i$ have been omitted. There are several reasons for not dealing with coupled constraints of the latter type. One, without them, this research can focus on understanding the impact of the shared constraints on the solution of the AGNEP by Lemke’s method. Two, there have not been many realistic problems reported in the literature that possess such non-shared, coupled, affine constraints. Three, a preliminary analysis of the AGNEP with non-shared coupled constraints can be found in [178]; the results obtained therein have thus far remained abstract due to the lack of reported applied problems of this kind to motivate further developments.

By introducing multipliers $\lambda^p_i \in \mathbb{R}_+^{n_i}$ for player $i$’s private constraint set $B_i x_i \geq r_i$ and $\lambda^s_i \in \mathbb{R}_+^{m_s}$ for the same player’s shared constraint set $\sum_{j=1}^F A_j x_j \geq b$, and by concatenating all players’ necessary and sufficient KKT optimality conditions, it is easy to see that a
GNE is characterized as a solution to the LCP\((q, M)\):

\[
0 \leq z \perp w \triangleq q + Mz \geq 0,
\]  

(3.1)

where the vectors \(z, w, \) and \(q\) are all of dimension \(L \triangleq \sum_{i=1}^{F} n_i + \sum_{i=1}^{F} m_i + Fm_s\) and given by

\[
\begin{align*}
\begin{bmatrix} x \end{bmatrix} & \triangleq \begin{bmatrix} x_1 \end{bmatrix} \quad \begin{bmatrix} \lambda^p \end{bmatrix} & \triangleq \begin{bmatrix} \lambda^p_1 \end{bmatrix} \quad \begin{bmatrix} \lambda^s \end{bmatrix} & \triangleq \begin{bmatrix} \lambda^s_1 \end{bmatrix} \\
\vdots & \vdots & \vdots & \vdots \\
\begin{bmatrix} x_F \end{bmatrix} & \begin{bmatrix} \lambda^p_F \end{bmatrix} & \begin{bmatrix} \lambda^s_F \end{bmatrix}
\end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix} w^x \end{bmatrix} & \triangleq \begin{bmatrix} w^x_1 \end{bmatrix} \quad \begin{bmatrix} w^p \end{bmatrix} & \triangleq \begin{bmatrix} w^p_1 \end{bmatrix} \quad \begin{bmatrix} s \end{bmatrix} & \triangleq \begin{bmatrix} s_1 \end{bmatrix} \\
\vdots & \vdots & \vdots & \vdots \\
\begin{bmatrix} w^x_F \end{bmatrix} & \begin{bmatrix} w^p_F \end{bmatrix} & \begin{bmatrix} s_F \end{bmatrix}
\end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix} h \end{bmatrix} & \triangleq \begin{bmatrix} h_1 \end{bmatrix} \quad \begin{bmatrix} r \end{bmatrix} & \triangleq \begin{bmatrix} r_1 \end{bmatrix} \quad \begin{bmatrix} b \end{bmatrix} & \triangleq \begin{bmatrix} -b \end{bmatrix} \\
\vdots & \vdots & \vdots & \vdots \\
\begin{bmatrix} h_F \end{bmatrix} & \begin{bmatrix} r_F \end{bmatrix} & \begin{bmatrix} b \end{bmatrix}
\end{bmatrix},
\end{align*}
\]

(3.2)
and

\[
\begin{bmatrix}
H_{11} & \cdots & H_{1F} & -B_1^T & -A_1^T \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
H_{F1} & \cdots & H_{FF} & -B_F^T & -A_F^T \\
B_1 & & & \ddots & \\
& & & & B_F \\
A_1 & \cdots & A_F & & \\
\vdots & \cdots & \vdots & & \\
A_1 & \cdots & A_F & & \\
\end{bmatrix}
\in \mathbb{R}^{L \times L}. \quad (3.3)
\]

The solution set of the LCP\((q, M)\) (3.1) is denoted by \(\text{SOL}(q, M)\). Throughout this chapter, the components of each \(x_i\)-variable are indicated by \(x_{ik}\) for \(k = 1, \cdots, n_i\), similarly for the \(\lambda^p_i\) and \(\lambda^s_i\) variables. Note that the players’ slack variable vectors \(s_i \triangleq \sum_{j=1}^{F} A_j x_j - b\) are all equal to the same \(s\) for all \(i\).

The primary goal of this work is to perform a comprehensive investigation of how the shared constraints \(\sum_{j=1}^{F} A_j x_j \geq b\) affect the solvability of the AGNEP \(\mathcal{G}\) via Lemke’s method, focusing on the types of equilibria that can be obtained by the algorithm and its proposed variants by providing sufficient conditions to rule out method failure (i.e., ray termination; see Section 3.3). The premise of this investigation is that without the shared constraints, the game can be formulated as an “affine” variational inequality (AVI) \(75\) defined by the mapping \(F(x) \triangleq h + H_0 x\) and the Cartesian product of polyhedra \(K \triangleq \prod_{i=1}^{F} K_i\) where \(H_0\)
is the players’ “Jacobian matrix” given by

\[ H_0 \triangleq \begin{bmatrix} H_{11} & \cdots & H_{1F} \\ \vdots & \ddots & \vdots \\ H_{F1} & \cdots & H_{FF} \end{bmatrix} \]

and \( K_i \triangleq \{ x_i \in \mathbb{R}^{n_i}_{+} \mid B_i x_i \geq r_i \} \). Several previous works have already extensively examined the solution of such an AVI by Lemke’s method [33, 34, 63, 64, 66, 67].

### 3.3 Lemke’s method: The traditional version

Lemke’s method [138] attempts to solve the LCP(\(q, M\)) by following an almost complementarity solution path of the augmented LCP:

\[
0 \leq z \perp w \triangleq q + dz_0 + Mz \geq 0 \tag{3.4}
\]

\[
0 \leq z_0 \perp w_0 \triangleq q_0 - d^T z \geq 0 \tag{3.5}
\]

where \( q_0 \) is a sufficiently large constant, the positive vector \( d \) is called the covering vector (typically 1), the scalar \( z_0 \) is called the artificial variable, and an almost complementary solution is a tuple \((z, z_0)\) for which all but one of the complementarity conditions are satisfied. For Lemke’s method, the unsatisfied complementarity condition is always (3.5); if (3.5) becomes satisfied, the sufficiently large nature of \( q_0 \) guarantees that \( z_0 = 0 \) and that a solution of LCP(\(q, M\)) has been obtained.

For technical reasons, the augmented LCP (3.4)–(3.5) can be studied purely via (3.4). To follow a feasible almost complementary path, Lemke’s method starts with the feasible point of (3.4), the most obvious of which is \( z = 0 \) and \( z_0 = \max_k \left\{ \frac{-q_k}{d_k} \right\} \). After this ini-
tialization, Lemke’s method performs successive simple pivotal exchanges of variables [47, Section 2.3] where the entering (driving) variable is the complement of the previous exiting (blocking) variable [for (3.4), $z_k$ is complementary to $w_k$] and the current blocking variable is determined by a minimum ratio test. Notationally, each simple pivot exchange of Lemke’s method is represented as (Blocking variable, Driving variable). The following Lemke’s method tableau example [47, Example 4.4.7] illustrates this pivoting concept.

**Example 4.** Consider the Lemke’s method tableau

<table>
<thead>
<tr>
<th>Nonbasic variables</th>
<th>1</th>
<th>$w_1$</th>
<th>$w_3$</th>
<th>$z_2$</th>
<th>$z_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_0$</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>$w_2$</td>
<td>13</td>
<td>3</td>
<td>-2</td>
<td>5</td>
<td>-8</td>
</tr>
<tr>
<td>$z_1$</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>-2</td>
</tr>
</tbody>
</table>

From this tableau, it can be deduced that either $w_3$ or $z_3$ was the previous blocking variable because both of these complementary elements are nonbasic. For this example, the previous pivot involved $w_3$ exiting the basis so the current driving variable of Lemke’s method is $z_3$. By the minimum ratio test, it can be seen that increasing $z_3$ decreases all three basic variables. Therefore, the appropriate minimum ratio test is

$$\min \left\{ \frac{3}{2}, \frac{13}{8}, \frac{2}{2} \right\} = 1$$

which gives that $z_1$ must be the blocking variable. Hence, the pivot $(z_1, z_3)$ gives
as the post-pivot tableau.

In the absence of pivot cycling induced by degeneracy, Lemke’s method will finitely terminate either at a solution of the LCP(\(q, M\)) (corresponding to a pivot resulting in \(z_0 = 0\)) or on a secondary ray without yielding a solution. The latter ray termination corresponds to the absence of an exiting variable during the simple pivots of Lemke’s method for (3.4) and is examined in further detail in Section 3.3.1. Naturally, a considerable amount of research has been conducted to identify sufficient conditions under which Lemke’s method will terminate at a solution. Widely used results of this nature are provided in Theorem 3.1.

**Definition 3.1.** A matrix \(M\) is

(a) a \(P\) matrix if all its principal minors are positive;

(b) an \(E\) [strictly semimonotone] matrix if for every \(0 \neq z \geq 0\), there exists an index \(k\) such that \(z_k > 0\) and \((Mz)_k > 0\);

(c) an \(E_0\) [semimonotone] matrix if for every \(0 \neq z \geq 0\), there exists an index \(k\) such that \(z_k > 0\) and \((Mz)_k \geq 0\);

(d) an \(R_0\) [pseudo-regular] matrix if SOL(0, \(M\)) = \{0\}.

These matrix classes follow the inclusions \(P \subset E \subset E_0\) and \(E \subset R_0\).
Theorem 3.1 (Section 4.4, [47]). Lemke’s method terminates at a solution of LCP\((q, M)\) if \(M\) is a \(P\) matrix, an \(E\) matrix, or an \(E_0 \cap R_0\) matrix.

Unfortunately, the conditions on \(M\) required by Theorem 3.1 cannot be satisfied for LCPs arising from AGNEPs, thus the need for a rigorous analysis of Lemke’s method when applied to LCP\((q, M)\) (3.1).

As previously mentioned, there are three possible outcomes of Lemke’s method when applied to a general LCP\((q, M)\): (a) pivot cycling, (b) ray termination, or (c) solution identification. By assuming throughout that cycling will not occur, the last two cases will be studied. This blanket no-cycling assumption is made without proposing a cycling prevention scheme such as the lexicographic degeneracy resolution method of [47, Section 4.9]. This assumption is consistent with the practical implementation of Lemke’s method which typically does not include anti-cycling rules.

3.3.1 Ray termination

Because of the importance of ray termination in the analysis of Lemke’s method, a necessary condition for ray termination and a sufficient condition for the non-occurrence of ray termination for an LCP\((q, M)\) with \(M \in E_0\) are presented. Drawn from [180, Lemma 2], the former condition is stated in part (a) of Proposition 3.1 and the latter condition is stated in part (b) of the proposition.

Proposition 3.1. Consider Lemke’s method applied to LCP\((q, M)\) with the covering vector \(d > 0\). The following two statements hold.

(a) If the method terminates on a secondary ray, then there exists a tuple \((w^*, \bar{w}, z^*, \bar{z}, z_0^*, \bar{z}_0)\)
with \( z_0^* > 0 \) and \( \tilde{z} \neq 0 \) such that for all \( \tau \geq 0 \),

\[
0 \leq z^* + \tilde{z}\tau \perp w^* + \tilde{w}\tau \triangleq q + d(z_0^* + \tilde{z}_0\tau) + M(z^* + \tilde{z}\tau) \geq 0. \tag{3.6}
\]

(b) If \( M \in E_0 \), then the scalar \( \tilde{z}_0 \) satisfying (3.6) must equal zero; hence, if for every scalar \( z_0 > 0 \), SOL\((q + dz_0, M)\) is bounded, then LCP\((q, M)\) has a solution that can be computed by Lemke’s method with \( d \) as the covering vector.

As may be expected from Proposition 3.1(b), a sufficient condition for the semimonotonicity of the matrix \( M \) corresponding to the AGNEP \( G \) can be derived. For this result, it is assumed that the leading principal submatrix \( J_0 \) of \( M \) is semimonotone, where

\[
J_0 \triangleq \begin{bmatrix}
H_{11} & \cdots & H_{1F} & -B_1^T \\
\vdots & \ddots & \vdots & \vdots \\
H_{F1} & \cdots & H_{FF} & -B_F^T \\
B_1 & & & \\
& \ddots & & \\
& & B_F & \\
\end{bmatrix} \tag{3.7}
\]

and is obtained from the KKT conditions of the AVI corresponding to AGNEP \( G \) without the shared constraints. This property easily holds if the diagonal block

\[
\begin{bmatrix}
H_{11} & \cdots & H_{1F} \\
\vdots & \ddots & \vdots \\
H_{F1} & \cdots & H_{FF}
\end{bmatrix}
\]
is copositive \( \text{on } \mathbb{R}_{+}^{L_0} \), where \( L_0 \triangleq \sum_{i=1}^{F} n_i \); indeed, in this case, \( J_0 \) is copositive and is therefore semimonotone. Recall that an \( n \times n \) matrix \( M \) is copositive if \( x^T M x \geq 0 \) for all \( x \in \mathbb{R}_{+}^{n} \) [16].

Under the semimonotonicity assumption on the matrix \( J_0 \), the following result identifies a sign condition on the rows of the aggregated matrix

\[
A \triangleq \left[ \begin{array}{ccc} A_1 & \cdots & A_N \end{array} \right]
\]

that will ensure the semimonotonicity of \( M \) (3.3) from which a sufficient condition can be obtained for the existence of a solution to LCP\((q, M)\) (3.1).

**Proposition 3.2.** Let \( J_0 \) and \( M \) be given by (3.7) and (3.3), respectively. Assume that \( J_0 \in E_0 \). If the nonzero entries in each row of the matrix \( A \) are of a single sign, then \( M \in E_0 \).

**Proof.** It suffices to show that for every index set \( \alpha \subseteq \{1, \cdots, L\} \) and every positive vector \( z_\alpha \), there exists an index \( k \in \alpha \) such that \((Mz)_k \geq 0\), where \( z \triangleq (z_\alpha, 0) \) is the vector obtained from \( z_\alpha \) by assigning zeros to the \( \bar{\alpha} \)-components where \( \bar{\alpha} \) is the complement of \( \alpha \) in \( \{1, \cdots, L\} \). The desired claim is clearly true if \( z_\alpha \) does not contain an element of \( x \). Let \( \hat{z} \) be the subvector of \( z \) with \( \lambda^s \) removed. By taking into account the semimonotonicity of the matrix \( J_0 \) and the structure of the matrix \( M \), it suffices to consider the case that there is an index \( k \in \alpha \) such that \( z_k \) corresponds to some \( x_{ik'} \) variable and the corresponding component of \( J_0 \hat{z} \), which is here denoted by \((J_0 \hat{z})_{ik'}\), is nonnegative.

\[
(Mz)_k = (J_0 \hat{z})_{ik'} - (A_i^T)_{j} \lambda_i^s = (J_0 \hat{z})_{ik'} - \sum_{\ell=1}^{m_s} (A_i)_{\ell j} \lambda_{i\ell}^s = (J_0 \hat{z})_{ik'} - \sum_{\ell | \lambda_{i\ell}^s > 0} (A_i)_{\ell j} \lambda_{i\ell}^s.
\]
If \((A_i)_{\ell j} > 0\) for some \(\ell \in \{1, \cdots, m_s\}\) with \(\lambda_{i\ell}^s > 0\), then by the sign assumption, it follows that the entire \(\ell\)-th row of \(A\) is nonnegative. Thus, the row of \(Mz\) corresponding to \(\lambda_{i\ell}^s\) is nonnegative. Otherwise, \((A_i)_{\ell j} \leq 0\) for all \(\ell \in \{1, \cdots, m_s\}\) with \(\lambda_{i\ell}^s > 0\) so 
\[(Mz)_k \geq (J_0\tilde{z})_{ik'} \geq 0.\]

By Proposition 3.1(b), the solvability of the LCP\((q, M)\) by Lemke’s method under the assumptions of Proposition 3.2 hinges on the solution rays of LCP\((q + dz_0, M)\), if any. By definition, these solution rays are determined by the solutions of the homogeneous LCP\((0, M)\). In what follows, a technical result that relates the (nonzero) solutions of the latter LCP to those of the LCP\((0, J_0)\) is presented. This problem relationship is crucial for upcoming termination proofs in that it restricts the possible directions along which ray termination can occur.

**Proposition 3.3.** Let \(J_0\) and \(M\) be given by (3.7) and (3.3), respectively. Assume that the following two conditions hold:

(a) for every shared constraint \(\ell = 1, \cdots, m_s\), the implication holds:

\[
\begin{aligned}
&B_i x_i \geq 0, x_i \geq 0, \text{ for all } i = 1, \cdots, \mathcal{F} \\
&\left(\sum_{i=1}^{\mathcal{F}} A_i x_i\right)_{\ell} = 0 \\
&\Rightarrow x_i = 0 \text{ for all } i = 1, \cdots, \mathcal{F}; \tag{3.8}
\end{aligned}
\]

(b) for every player \(i = 1, \cdots, \mathcal{F}\), the implication holds:

\[
\begin{aligned}
&B_i^T \lambda_p^i + A_i^T \lambda_s^i \leq 0 \\
&\lambda_p^i, \lambda_s^i \geq 0 \\
&\Rightarrow A_i^T \lambda_s^i = 0. \tag{3.9}
\end{aligned}
\]

If \((\tilde{x}, \tilde{\lambda}^p, \tilde{\lambda}^s)\) is a solution of the LCP\((0, M)\), then \((\tilde{x}, \tilde{\lambda}^p)\) is a solution of the LCP\((0, J_0)\).
Proof. Let \((\bar{x}, \bar{\lambda}^p, \bar{\lambda}^s)\) be a solution of the LCP(0, M); it suffices to show that \(A_i^T \bar{\lambda}^s_i = 0\) for all \(i = 1, \cdots, F\). This is clearly true if \(\bar{\lambda}^s_i = 0\) for all \(i\). Assume that \(\bar{\lambda}^s_{i', \ell'} > 0\) for some pair \((i', \ell')\). The vector \(\bar{x}\) then satisfies the left-hand side of the implication (3.8).

Thus, \(\bar{x} = 0\). It follows that

\[
0 \leq \sum_{j=1}^{F} H_{ij} \bar{x}_i - B_i^T \bar{\lambda}^p_i - A_i^T \bar{\lambda}^s_i = -B_i^T \bar{\lambda}^p_i - A_i^T \bar{\lambda}^s_i,
\]

implying that \((\bar{\lambda}^s_i, \bar{\lambda}^p_i)\) satisfies the left-hand side of (3.9) for every \(i = 1, \cdots, F\). By assumption (b), \(A_i^T \bar{\lambda}^s_i = 0\) for all \(i = 1, \cdots, F\) and the proposition follows. \(\square\)

The implication (3.9) is difficult to satisfy when the matrix \(A\) is entrywise nonpositive. It turns out that this case requires an alternative condition that imposes a sign restriction on the vectors \(r_i\) and \(b\) (see condition (B) in Theorem 3.2).

To build intuition about the functioning of Lemke’s method when applied to AGNEPs with shared constraints, a simple 2-player example from [110] is provided next. In this example, condition (3.8) holds but \(A\) is entrywise negative. Furthermore, the example illustrates a special feature of Lemke’s method when applied to this class of AGNEPs, one which is formalized in the subsequent Proposition 3.4.

**Example 5.** Consider a 2-player game in which the individual optimization problems are

\[
\begin{align*}
\text{minimize} & \quad -x_1(1 - 0.5x_1 - 0.5x_2) \\
\text{subject to} & \quad 1 - x_1 - x_2 \geq 0 \quad (\lambda^s_i) \\
& \quad x_1 \geq 0
\end{align*}
\]
minimize $-x_2(2 - 0.5x_1 - 0.5x_2)$
subject to $1 - x_1 - x_2 \geq 0$ ($\lambda_2^{s}$)

$x_2 \geq 0$

The resulting equilibrium conditions are given by LCP($q,M$) with

$q = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$ and $M = \begin{bmatrix} 1 & 0.5 & 1 & 0 \\ 0.5 & 1 & 0 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}$.

This game has an infinite number of Nash equilibria given by

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $\alpha \geq 0$; $\begin{pmatrix} 0 \\ 1 \\ 0.5 + \alpha \\ 1 \end{pmatrix}$, $\alpha \geq 0$; and $\begin{pmatrix} \alpha \\ \frac{1 - \alpha}{0.5 - 0.5\alpha} \\ \frac{0.5 - 0.5\alpha}{1 + 0.5\alpha} \end{pmatrix}$, $\alpha \in (0,1)$.

However, Lemke’s method is not guaranteed to terminate at one of these solutions with the typical covering vector of ones. In actuality, the only solution that Lemke’s method can possibly find is $\begin{pmatrix} 1 & 0 & 0 & 1.5 \end{pmatrix}$ (a general result related to this observation is given by Proposition 3.4). The following 2 tableaux detail the first pivot in Lemke’s algorithm with the blocking variables in bold italics.
Note the identical shared constraint rows in these 2 tableaux. Due to these repeating rows, Lemke’s method does not have a unique blocking variable in the second tableau. If \( s_2 \) is made nonbasic, pivots of \( \langle s_2, x_2 \rangle, \langle w_1, \lambda s_2 \rangle, \langle z_0, x_1 \rangle \) terminate at the specified solution. If instead Lemke’s method makes \( s_1 \) nonbasic, pivots of \( \langle s_1, x_2 \rangle, \langle Unblocked, \lambda s_1 \rangle \) lead to ray termination and the method fails. This blocking variable issue is not unique to this example and may occur whenever Lemke’s method is employed to solve an AGNEP without taking special care of the Lagrange multipliers of the shared constraints.

If the variational equilibrium formulation of this game is considered (i.e., the case where \( \lambda^s_1 = \lambda^s_2 \) is enforced), the LCP\((q,M)\) is defined by

\[
q = \begin{pmatrix}
-1 \\
-2 \\
1
\end{pmatrix}
\text{ and } M = \begin{bmatrix}
1 & 0.5 & 1 \\
0.5 & 1 & 1 \\
1 & -1 & 0
\end{bmatrix}.
\]

Lemke’s method will successfully solve the latter problem with any positive covering vector and obtain the solution \( \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \), or equivalently \( \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \).

In what follows, an important property of the solutions of LCP\((q,M)\) obtained by Lemke’s method for \( M \) given by (3.3) is established. This result shows that Lemke’s method computes generalized Nash equilibria only of a special kind characterized by a distinguished
property of the multipliers of the shared constraints. The semimonotonicity of $J_0$ is not needed for the result; the only requirement is the structure of $M$.

**Proposition 3.4.** Let the vector $q$ and matrix $M$ be given by (3.2) and (3.3), respectively. If Lemke’s method finds a solution of the LCP($q, M$), then for each shared constraint $\ell = 1, \cdots, m_s$, there exists in that solution at most one $i \in \{1, \cdots, F\}$ such that $\lambda_{i\ell}^s > 0$.

**Proof.** The covering vector $d$ in partitioned form can be expressed as

$$d \triangleq \begin{pmatrix} d^x \triangleq \\ \vdots \\ d^F \end{pmatrix} \begin{pmatrix} d^1_x \\ \vdots \\ d^F_x \end{pmatrix} \begin{pmatrix} d^p \triangleq \\ \vdots \\ d^p_F \end{pmatrix} \begin{pmatrix} d^1_p \\ \vdots \\ d^p_F \end{pmatrix} \begin{pmatrix} d^s \triangleq \\ \vdots \\ d^s \end{pmatrix} \begin{pmatrix} d^s \\ \vdots \\ d^s \end{pmatrix}$$

in the same way that the variable $z$ is partitioned into $(x, \lambda^p, \lambda^s)$. Note that the subvector $d^s$ is taken to be

$$d^s \triangleq \begin{pmatrix} d^s \\ \vdots \\ d^s \end{pmatrix}$$

for some positive vector $d^s \in \mathbb{R}^{m_s}$. More generally, each $d^s$ can be replaced by a player-specific positive vector $d_{i}^s \in \mathbb{R}^{m_s}$ and the result still holds. With the covering vector as
specified, consider the operation of Lemke’s method and focus on the iteration where \( \lambda^s_{i\ell} \) is the first \( \lambda^s \)-variable to become basic. Define the set \( I' = \{1, 2, \ldots, i - 1, i + 1, \ldots, F\} \).

After the pivot of \( \lambda^s_{i\ell} \) into the basis (recall that the complement \( s^i_{i\ell} \) of \( \lambda^s_{i\ell} \) is nonbasic), \( s^i'_{i\ell} = s^i_{i\ell} = 0 \) for all \( i' \in I' \). In order for the variable \( \lambda^s_{i'\ell} \) with \( i' \in I' \) to become basic, its complement \( s^i'_{i'\ell} \) has to first leave the basis. This cannot happen unless \( s^i_{i\ell} \) is brought back into the basis, which implies that \( \lambda^s_{i\ell} \) would have to become nonbasic before \( \lambda^s_{i'\ell} \) becomes basic. Therefore, never in the pivot process will there be two multipliers \( \lambda^s_{i\ell} \) and \( \lambda^s_{i'\ell} \) with \( i \neq i' \) corresponding to the same shared constraint \( \left( \sum_{j=1}^{F} A_j x_j \geq b \right) \) both being basic variables.

With the multiplier result of Proposition 3.4, the first Lemke’s method termination result for an AGNEP can be developed. First, note that the ray termination condition (3.6) for Lemke’s method applied to the game \( G \) is defined by the following:

- for all \( \tau \geq 0 \) and \( i = 1, \ldots, F \),

\[
0 \leq x^*_i + \bar{x}_i \tau \perp w^p_i \tau + \bar{w}^p_i \tau \triangleq h_i + \sum_{j=1}^{F} H_{ij}(x^*_j + \bar{x}_j \tau) + d^p_i (z^*_0 + \bar{z}_0 \tau) - B^T_i (\lambda^s_p + \bar{\lambda}^s_p \tau) - A^T_i (\lambda^s_i + \bar{\lambda}^s_i \tau) \geq 0 
\]

(3.10)

\[
0 \leq \lambda^p_i \tau \perp w^p_i \tau + \bar{w}^p_i \tau \triangleq -r_i + d^p_i (z^*_0 + \bar{z}_0 \tau) + B_i (x^*_i + \bar{x}_i \tau) \geq 0 
\]

\[
0 \leq \lambda^s_i \tau \perp s^i + \bar{s}_i \tau \triangleq -b + d^s_i (z^*_0 + \bar{z}_0 \tau) + \sum_{j=1}^{F} A_j (x^*_j + \bar{x}_j \tau) \geq 0. 
\]

When ray termination occurs, the scalar \( \bar{z}_0 = 0 \) when \( M \) is semimonotone by Proposition 3.1(b). In this case, one of the three vectors \( \bar{x}, \bar{\lambda}^p \), and \( \bar{\lambda}^s \) must not be zero. Thus, for some player \( i \), one of 3 possibilities must occur after a suitable normalization:
(a) \( \tilde{x}_{ik} = 1 \) for some \( k \in \{1, \ldots, n_i\} \);

(b) \( \tilde{\lambda}_{ip}^p = 1 \) for some \( \ell \in \{1, \ldots, m_i\} \);

(c) \( \tilde{\lambda}_{is}^s = 1 \) for some \( \ell \in \{1, \ldots, m_s\} \).

We refer to these 3 cases, respectively, as ray termination: on a primal \( x \)-variable, on a private \( \lambda^p \)-variable, or on a shared \( \lambda^s \)-variable. The proof of Proposition 3.4 applies also to the case when Lemke’s method terminates in the third case. More precisely, if ray termination occurs when the (shared) multiplier \( \lambda_{is}^s \) is brought into the basis, then all the other \( \lambda_{i'\ell}^s \) for \( i' \neq i \) corresponding to the same \( \ell \)th shared constraint are nonbasic variables.

This observation motivates a modification of Lemke’s method that may be considered a degeneracy resolution technique for ties in the minimum ratio test arising from a shared constraint among the players. It turns out that under suitable conditions, this modification will compute a kind of NE introduced in the next section.

The following result identifies a sufficient condition under which ray termination on a primal \( x \)-variable can easily be ruled out.

**Proposition 3.5.** Let the vector \( q \) and matrix \( M \) be given by (3.2) and (3.3), respectively. Assume that \( M \in E_0 \), the principal submatrix \( J_0 \) given by (3.7) is \( R_0 \), and condition (a) of Proposition 3.3 holds. Applied to the LCP(\( q, M \)), Lemke’s method cannot ray terminate on a primal \( x \)-variable.

**Proof.** The semimonotonicity of \( M \) implies that \( \tilde{z}_0 = 0 \). If the algorithm ray terminates on a primal \( x \)-variable, then \( \tilde{x} \neq 0 \). By condition (a) of Proposition 3.3 it follows that \( \tilde{\lambda}^s = 0 \). Thus, the LCP(0, \( J_0 \)) has a nonzero solution, a contradiction of \( J_0 \in R_0 \). \( \square \)
3.4 Partial variational equilibria

By definition, a normalized NE in the sense of Rosen \cite{197} is a solution to the LCP\((q, M)\) for which there exists a positive vector \(e \in \mathbb{R}^F\) and a nonnegative vector \(\eta \in \mathbb{R}^{m_s}\) such that \(\lambda_i^\ell = \frac{\eta_\ell}{e_i}\) for all \(i = 1, \cdots, F\) and \(\ell = 1, \cdots, m_s\). In terms of the multipliers of the shared constraints, the NE computable by Lemke’s method and Rosen’s normalized NE are at two extremes in the following sense: for the former equilibrium, corresponding to every shared constraint \(\ell\), if a player has a positive multiplier, then all other players must have zero multipliers (see Proposition 3.4); in contrast, for the latter equilibrium, if a player has a positive multiplier, then all players must have positive multipliers. It turns out this spectrum of equilibria with distinct properties of the multipliers of the shared constraints can be partially filled with the introduction of coalitional multipliers. Specifically, let \(\alpha \subseteq \{1, \cdots, m_s\}\) be a nonempty index subset of the shared constraints and let \(\bar{\alpha}\) be its complement. Now consider the LCP\((\hat{q}, \hat{M})\) where \(\hat{M}\) is given by

\[
\begin{bmatrix}
H_{11} & \cdots & H_{1F} & -B_1^T & -(A_1^T)_{\alpha} & -(A_1^T)_{\bar{\alpha}} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
H_{F1} & \cdots & H_{FF} & -B_F^T & -(A_F^T)_{\alpha} & -(A_F^T)_{\bar{\alpha}} \\
\end{bmatrix}
\text{where}
\begin{bmatrix}
B_1 \\
\vdots \\
B_F \\
(A_1)_{\alpha} & \cdots & (A_F)_{\alpha} \\
(A_1)_{\bar{\alpha}} & \cdots & (A_F)_{\bar{\alpha}} \\
\vdots & \cdots & \vdots \\
(A_1)_{\bar{\alpha}} & \cdots & (A_F)_{\bar{\alpha}} \\
\end{bmatrix}
\]
and

\[ \tilde{q} \triangleq \begin{pmatrix} h \\ r \\ -b_\alpha \\ -b_\bar{\alpha} \\ \vdots \\ -b_\bar{\alpha} \end{pmatrix}. \]

This LCP, whose solutions are hereafter called \textit{partial variational equilibria}, is obtained by setting, for each \( \ell \in \alpha \), \( \lambda^s_{i\ell} = \lambda^s_i \) for all \( i = 1, \cdots, F \). As an abbreviation, a partial variational equilibrium is also called a partial VE. An economic interpretation of the multipliers of a partial VE can be given in terms of a price mechanism for an affine generalized Nash game with affine side constraints (refer to Section 2.6). Specifically, consider a modified game wherein player \( i \), taking the rival players’ strategy tuple \( x_{-i} \) and a price vector \( p_\alpha \) as parameters, solves the following quadratic program:

\[
\begin{align*}
\text{minimize} \quad & f_i(x_i, x_{-i}) - p_\alpha^T \left( \sum_{j=1}^{F} A_j x_j - b \right)_\alpha \\
\text{subject to} \quad & x_i \in K_i^\alpha(x_{-i}) \triangleq \left\{ x_i \in \mathbb{R}^{n_i}_{+} \mid B_i x_i \geq r_i, \left( \sum_{j=1}^{F} A_j x_j \geq b \right)_{\bar{\alpha}} \right\};
\end{align*}
\]

the description of this game is completed by imposing the \textit{market clearing condition} that is stated here in terms of a complementarity relation between the price vector \( p_\alpha \) and the shared constraints indexed by \( \alpha \):

\[
0 \leq p_\alpha \perp \left( \sum_{j=1}^{F} A_j x_j - b \right)_\alpha \geq 0,
\]

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or equivalently as a trivial price-minimization problem:

\[
\begin{align*}
\text{minimize} & \quad p^T \left( \sum_{j=1}^{\mathcal{F}} A_j x_j - b \right)_{\alpha} \\
\text{subject to} & \quad p_{\alpha} \geq 0
\end{align*}
\]

The Lemke-type NE and the VE can be obtained by letting \( \alpha \) to be the empty set and the full set \( \{1, \cdots, m_s\} \), respectively.

Appearing as early as the renowned general economic equilibrium model of [7], non-cooperative Nash games with side conditions and associated price mechanisms are quite pervasive in applications; [132] presents a general theory of such a game, albeit without the shared constraints but allowing the players’ optimization problems to be nonconvex. The lack of explicit bounds on the price vector \( p_{\alpha} \) is a source of difficulty for treating this problem as was discussed in Chapter 2. Supported by the theory herein, the LCP \((\hat{\mathbf{q}}, \hat{M})\) formulation provides a constructive approach for dealing with the affine case.

The partial VE concept divides the shared constraints into 2 groups: each constraint in one group (indexed by \( \alpha \)) has common multipliers across all players, while the constraints in the other group (indexed by \( \bar{\alpha} \)) have no such restriction. A refinement of this concept, which is here called a coalitional equilibrium, is possible in which each shared constraint \( \ell = 1, \cdots, m_s \) is associated with a partition of the player set \( \{1, \cdots, \mathcal{F}\} \) into \( \gamma_{\ell} \) coalitions (i.e., subsets), say \( C_k^{\ell} \) for \( k = 1, \cdots, \gamma_{\ell} \), such that \( C_k^{\ell} \cap C_{k'}^{\ell} = \emptyset \) for \( k \neq k' \), \( \bigcup_{k=1}^{\gamma_{\ell}} C_k^{\ell} = \{1, \cdots, \mathcal{F}\} \), and \( \lambda_{i_{\ell}}^s = \lambda_{j_{\ell}}^s \) for all \( i, j \in C_k^{\ell} \). Hence, a player is allowed to belong to different coalitions with respect to individual shared constraints. The partial VE is a special kind of coalitional equilibrium with the specifications: (a) for all \( \ell \in \alpha \), \( \gamma_{\ell} = 1 \) and \( C_1^{\ell} = \{1, \cdots, \mathcal{F}\} \), and (b) for all \( \ell \in \bar{\alpha} \), \( \gamma_{\ell} = \mathcal{F} \) and \( C_i^{\ell} = \{i\} \) for all \( i = 1, \cdots, \gamma_{\ell} \). An LCP formulation for a coalitional equilibrium can be derived to which the modified Lemke method and the
regularization technique can be applied. The details are omitted.

3.5 A modified Lemke’s method for partial VE

In this section, a modification of Lemke’s method is developed for solving the AGNEP $\mathcal{G}$. This modified algorithm, if successful, will compute a partial VE of the game. Sufficient conditions for successful termination will be presented subsequently. Throughout the method, it is assumed that degeneracy (i.e., ties in blocking variables) only occurs with respect to $s$-variables.
**Modified Lemke’s method.**

**Step 0:** Implement Lemke’s method until either (a) termination occurs, or (b) \( s_\ell \) becomes blocking for some \( \ell \).

- For (a), stop and declare solution or ray termination as appropriate;
- For (b), proceed to Step 1.

**Step 1:** Call \( \ell \) a distinguished shared constraint. Set Tableau\( _\ell \leftarrow \) Current Lemke tableau, \( i_\ell \leftarrow 1 \), and \( I_\ell \leftarrow \{1\} \). Pivot with \( s_{1\ell} \) as the blocking variable.

**Step 2:** Continue Lemke’s method until either (a) termination occurs, or (b) \( s_{\ell'} \) becomes blocking for some \( \ell' \neq \ell \).

- For (a), proceed to Step 3;
- For (b), return to Step 1 with index \( \ell' \).

**Step 3:** If a solution was obtained, stop. Otherwise,

- If ray termination occurred after aggregation to \( s_\ell \) and \( \lambda^s_\ell \) (see third bullet), return to Step 3 for the previous distinguished shared constraint.
- If ray termination occurred and \( I_\ell \neq \{1, \cdots, F\} \), set \( i_\ell \leftarrow i_\ell + 1 \) and \( I_\ell \leftarrow I_\ell \cup \{i_\ell\} \). Pivot with \( s_{i_\ell} \) as the blocking variable in Tableau\( _\ell \). Return to Step 2.
- If ray termination occurred and \( I_\ell = \{1, \cdots, F\} \), perform the following operations on Tableau\( _\ell \):
  1. Delete all but one row corresponding to shared constraint \( \ell \) and relabel the slack of that constraint \( s_\ell \);
  2. Combine all \( \lambda^s_{i_\ell} \)-variables into a single multiplier labeled \( \lambda^s_\ell \), called an aggregated shared constraint multiplier;
  3. Recalculate the \( \lambda^s_\ell \) column of the new tableau and call \( s_\ell \) aggregated;
  4. Pivot with \( s_\ell \) as the blocking variable;
  5. Return to Step 2.

Several remarks on the modified Lemke’s method are in order.
First, as a result of the aggregation in Step 3, ray termination will not occur on a non-aggregated shared constraint slack variable \( s \) or on a non-aggregated multiplier. Indeed, for a secondary ray to occur on a variable leading to termination, that variable must be nonbasic. Therefore, if a secondary ray occurs on a non-aggregated shared constraint slack variable \( s_{i\ell'} \), then \( s_{i\ell'} \) is nonbasic; moreover, either \( \ell' \) itself is the distinguished shared constraint in Step 1 or \( \ell' \) occurs in the sequence of pivots stemming from some current distinguished Tableau\( _\ell \) with \( \ell \neq \ell' \). Either way, the method will not terminate on \( s_{i\ell'} \) because either Step 3 will pick an alternate slack variable of the distinguished shared constraint as the blocking variable or an aggregation of the distinguished shared constraint will take place. A similar argument applies if a secondary ray is encountered on a non-aggregated shared constraint multiplier \( \lambda_{s_i\ell'} \). Second, if a solution is found only after Step 3 aggregation is employed for some shared constraint \( \ell \), the solution must be a partial VE of the AGNEP \( G \) as defined in Section 3.4. Third, it is possible for multiple pivots to take place prior to a return to Tableau\( _\ell \). While this is a waste of effort, there is no mechanism to take advantage of the work done in these pivots. However, one benefit of seeking an alternate blocking variable is that it provides a way for the method to take an alternate pivot path, thus avoiding immediate ray termination. Fourth, when a return to Tableau\( _\ell \) takes place, say with the slack \( s_{i\ell} \) replaced by \( s_{i'\ell} \) as the blocking variable, it suffices to exchange the two indices \( i \) and \( i' \) in the post-pivot tableau after \( s_{i\ell} \) was the blocking variable. Finally, the choice of a suitable alternative multiplier \( \lambda_{s_i\ell} \) in Step 3 has not been specified; depending on further problem structure, judicious choices of such driving variable could be made that involve several “look-ahead calculations”. Details of refinements like this are omitted.

The ray termination condition of the modified method after the shared constraints \( \alpha \subseteq \{1, \cdots, m_s\} \) are aggregated can be expressed as follows:
for all $\tau \geq 0$ and $i = 1, \ldots, F$,

$$
0 \leq x_i^* + \bar{x}_i \tau \perp w_i^{r,x} + \bar{w}_i^{r,x} \triangleq h_i + \sum_{j=1}^{F} H_{ij} (x_j^* + \bar{x}_j \tau) + d_i^r (z_0^* + \bar{z}_0 \tau)
$$

$$
-B_i^T (\lambda_i^{p,x} + \bar{\lambda}_i^p \tau) - (A_i^T)_{\bullet} (\lambda_i^{s,x} + \bar{\lambda}_i^s \tau) - (A_i^T)_{\bar{\alpha}} (\lambda_i^{s,x} + \bar{\lambda}_i^s \tau) \geq 0
$$

$$
0 \leq \lambda_i^{p,x} + \bar{\lambda}_i^p \tau \perp w_i^{p,x} + \bar{w}_i^{p,x} \triangleq -r_i + d_i^p (z_0^* + \bar{z}_0 \tau) + B_i (x_i^* + \bar{x}_i \tau) \geq 0
$$

$$
0 \leq \lambda_i^{s,x} + \bar{\lambda}_i^s \tau \perp s_i^x + \bar{s}_i^x \tau \triangleq \left[ -b + d^s (z_0^* + \bar{z}_0 \tau) + \sum_{j=1}^{F} A_j (x_j^* + \bar{x}_j \tau) \right] \geq 0
$$

$$
\left[ 0 \leq \lambda_i^{s,x} + \bar{\lambda}_i^s \tau \perp s_i^x + \bar{s}_i^x \tau \triangleq -b + d^s (z_0^* + \bar{z}_0 \tau) + \sum_{j=1}^{F} A_j (x_j^* + \bar{x}_j \tau) \geq 0 \right] \geq 0.
$$

When ray termination occurs, the scalar $\bar{z}_0 = 0$ when $M$ is semimonotone; moreover, one of the three vectors $\bar{x}$, $\bar{X}^p$, and $\bar{\lambda}_\alpha$ must not be zero. Furthermore, under the assumptions therein, the proof of Proposition 3.5 yields $\bar{x} = 0$. In what follows, sufficient conditions are provided for ruling out the two remaining ray termination cases $\left( \bar{X}^p \neq 0 \text{ or } \bar{\lambda}_\alpha \neq 0 \right)$, thus ensuring that the modified Lemke’s method will successfully compute a solution of the AGNEP $G$.

**Theorem 3.2.** Let the vector $q$ and matrix $M$ be given by (3.2) and (3.3), respectively. Assume that $M \in E_0$, the principal submatrix $J_0$ given by (3.7) is $R_0$, and condition (a) of Proposition 3.3 holds. The modified Lemke algorithm as described above will compute a solution of the AGNEP $G$ under either one of the following two conditions:

(A) condition (b) of Proposition 3.3 holds;

(B) the matrix $A$, the vector $b$, and the vectors $\{r_i\}_{i=1}^{F}$ are all nonpositive.
Proof. Assume that ray termination occurs on a private $\lambda_{p}^{i\ell}$-multiplier in (3.11). Without loss of generality, assume that $\tilde{\lambda}_{i'}^{p} = 1$ for some pair $(i', \ell')$. As noted above, $\tilde{x} = 0$. Therefore,

$$0 \leq x_{i}' \perp -B_{i'}^{T}\tilde{\lambda}_{i'}^{p} - (A_{i'}^{T})_{*\alpha} \tilde{\lambda}_{\alpha}^{s} - (A_{i'}^{T})_{\star\tilde{\alpha}} \left(\tilde{\lambda}_{i'}^{s}\right)_{\tilde{\alpha}} \geq 0.$$ 

If condition (A) holds, then $(A_{i'}^{T})_{\star\alpha} \tilde{\lambda}_{\alpha}^{s} + (A_{i'}^{T})_{\star\tilde{\alpha}} \left(\tilde{\lambda}_{i'}^{s}\right)_{\tilde{\alpha}} = 0$, implying that the LCP$(0, J_{0})$ has a nonzero solution, which is a contradiction. If (B) holds, it follows that

$$\left(\tilde{\lambda}_{i'}^{p}\right)^{T}B_{i'}x_{i}^{*} = -\left(\tilde{\lambda}_{\alpha}^{s}\right)^{T}(A_{i'}\cdot x_{i}^{*})_{\alpha} - \left(\tilde{\lambda}_{i'}^{s}\right)^{T}(A_{i'}\cdot x_{i}^{*})_{\tilde{\alpha}} \geq 0.$$ 

Hence,

$$0 = \left(\tilde{\lambda}_{i'}^{p}\right)^{T}\left[-r_{i'} + d_{i'}^{p}z_{0}^{*} + B_{i'}x_{i}^{*}\right] \geq \sum_{\ell = 1}^{m_{i'}} \left(\tilde{\lambda}_{i'}^{p}\right)^{T}\left[-r_{i'} + d_{i'}^{p}z_{0}^{*}\right]_{\ell} \geq -r_{i'} + d_{i'}^{p}z_{0}^{*} > 0,$$

which is a contradiction. Thus $\tilde{\lambda}_{i'}^{p} = 0$.

Assume that ray termination occurs on an aggregated shared constraint multiplier $\lambda_{\ell}^{s}$, for some $\ell' \in \alpha$. Without loss of generality, assume that $\tilde{\lambda}_{\ell}^{s} = 1$. By complementarity,

$$\left(-b + d^{s}z_{0}^{*} + \sum_{j = 1}^{F} A_{j}x_{j}^{*}\right)_{\ell'} = 0.$$ 

Since $-b_{\ell'} + d^{s}z_{0}^{*}$ is necessarily positive, there must exist a pair $(i_{0}, k_{0})$ such that $x_{i_{0}k_{0}}^{*} > 0$ and $(A_{i_{0}})_{\ell'k_{0}} < 0$ by the nonpositivity of $A$. Taking into account $\tilde{\lambda}_{i'}^{p} = 0$, $\tilde{x} = 0$, and $\tilde{z}_{0} = 0$,

$$\sum_{j = 1}^{F} H_{i_{0}j}\tilde{x}_{j} + d_{i_{0}}^{x}\tilde{z}_{0} - B_{i_{0}i}^{T}\tilde{\lambda}_{i_{0}}^{s} - (A_{i_{0}}^{T})_{\star\alpha} \tilde{\lambda}_{\alpha}^{s} - (A_{i_{0}}^{T})_{\star\tilde{\alpha}} \left(\tilde{\lambda}_{i_{0}}^{s}\right)_{\tilde{\alpha}} \geq -\left(A_{i_{0}}\right)_{\ell'k_{0}} > 0,$$

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contradicting $x_{i0k0}^* > 0$. □

**Remark 3.2.** There is no restriction on the matrices $B_i, i = 1, \cdots, F$, in the players’ private constraints for condition (B) of the above theorem. Notice that the requirement that the vector $b$ and the vectors $\{r_i\}_{i=1}^F$ are all nonpositive imply that $x = 0$ is feasible to all players’ problems. This requirement is consistent with the theory of generalized Nash equilibrium problems which invariably requires the existence of a reference vector satisfying certain conditions that is denoted $x^{ref}$ in [76, Theorem 1] and served by the zero vector in the present context. □

**Remark 3.3.** The assumption that $J_0 \in R_0$ is somewhat restrictive when players have private constraints. However, it can be relaxed at the expense of more notation; indeed, the only purpose of the assumption is to guarantee that the LCP($q', J_0$) can be solved by Lemke’s method where $q' \triangleq \begin{pmatrix} h & r \end{pmatrix}$. In this light, the utilization of the $R_0$ property guarantees that the augmented LCP($q' + d'z_0, J_0$) has no solution rays where $d' \triangleq \begin{pmatrix} d^x & d^p \end{pmatrix}$. This assumption makes the Lemke’s method termination statement of Proposition 3.1(b) applicable. Any other condition that guarantees the successful termination of Lemke’s method at a solution for LCP($q', J_0$) may be employed in Theorem 3.2 for the same result. □

**Remark 3.4.** The above modification of Lemke’s method admits several variations. For instance, two or more multipliers of a shared constraint $\ell$ can be grouped if they all lead to ray termination; such grouping does not need to wait until all the multipliers of shared constraint $\ell$ are tested as described above. Such a refinement of the method, if successful, allows for the computation of more general coalitional equilibria. □
3.6 Alternate generalized Nash equilibria via reformulations

Proposition 3.4 and the partial variational equilibria that can be identified by the modified Lemke’s method provided in Section 3.5 obviously do not cover all potential types of equilibria that may arise for AGNEP $\mathcal{G}$. Therefore, a different LCP is provided here with solutions that are distinct from those obtained by the original or modified Lemke’s method. This reformulation is based on the observation that instead of repeating the shared constraint $\sum_{j=1}^{\mathcal{F}} A_j x_j \geq b$ a total of $\mathcal{F}$ times in the LCP($q, M$) formulation, one copy of the constraint can be kept (say player 1’s copy without loss of generality) and the complementarity conditions

\begin{equation}
0 \leq \lambda_i^s \perp \sum_{j=1}^{\mathcal{F}} A_j x_j - b \geq 0, \quad i = 2, \ldots, \mathcal{F}
\end{equation}

(3.12)

can be replaced by

\begin{equation}
0 \leq \lambda_i^s \perp 1_{m_s}\beta - D_i^s \lambda_1^s \geq 0, \quad i = 2, \ldots, \mathcal{F},
\end{equation}

(3.13)

where $1_{m_s}$ is the $m_s$-vector of ones, $\beta \geq 0$ is a nonnegative parameter, and $D_i^s$ is a positive diagonal matrix of order $m_s$ with diagonal entries denoted $D_{i\ell}^s$ for $\ell = 1, \ldots, m_s$. Under the condition that

\begin{equation}
0 \leq \lambda_1^s \perp \sum_{j=1}^{\mathcal{F}} A_j x_j - b \geq 0,
\end{equation}

(3.14)

it can be seen that if the right-hand side of (3.13) holds with equality for index $\ell$ and some $\beta > 0$ at a solution, then the right-hand side of (3.12) holds with equality for index $\ell$. Hence, conditions (3.13) for $\beta > 0$ and (3.14) together imply (3.12) for AGNEP solutions. The case $\beta = 0$ is addressed below.
Note that the constraint (3.13) upper bounds the multipliers $\lambda_i$ during Lemke’s method pivoting when $z_0$ is bounded. In the parametric algorithm described below, such a constraint helps prevent ray termination on the $\lambda_i$-variable. Moreover, when a component inequality becomes binding (i.e., when $\lambda_i$ reaches the upper bound $(\beta + d_i z_0)/D_i$ for some $j \in \{1, \cdots, F\}$), player $j$’s multiplier $\lambda_j$ could become basic, thereby resulting in more than one multiplier corresponding to the same shared constraint being basic. Recall from Proposition 3.4 that this is impossible for Lemke’s method with the original problem formulation.

To describe the parametric procedure, consider applying Lemke’s method to the LCP($q_{\mathbf{f}}, M_{\mathbf{f}}$), which is expressed in tableau form with a covering vector added as

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>$z_0$</th>
<th>$x_1$</th>
<th>(x_F)</th>
<th>$\lambda_1$</th>
<th>(\lambda_F)</th>
<th>$\lambda_s$</th>
<th>(\lambda_F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>$h_1$</td>
<td>$d_1^c$</td>
<td>$H_{11}$</td>
<td>(H_{1F})</td>
<td>$-B_1^T$</td>
<td>(-A_1^T)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>(\vdots)</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$w_F$</td>
<td>$h_F$</td>
<td>$d_F^c$</td>
<td>$H_{F1}$</td>
<td>(H_{FF})</td>
<td>$-B_F^T$</td>
<td>(-A_F^T)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$w_1^p$</td>
<td>$-r_1$</td>
<td>$d_1^p$</td>
<td>$B_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$w_F^p$</td>
<td>$-r_F$</td>
<td>$d_F^p$</td>
<td>$B_F$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s_1$</td>
<td>$-b$</td>
<td>$d_s$</td>
<td>$A_1$</td>
<td>(A_F)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s_2$</td>
<td>$1_{m_s\beta}$</td>
<td>$d_s$</td>
<td></td>
<td></td>
<td>$-D_2^s$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td></td>
<td></td>
<td>$\vdots$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s_F$</td>
<td>$1_{m_s\beta}$</td>
<td>$d_s$</td>
<td></td>
<td></td>
<td>$-D_F^s$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In what follows, $\beta$ is allowed to be zero. Note that the first shared multiplier $\lambda^s$ that would become basic is one of the $\lambda^s_i$-variables. Furthermore, a $\lambda^s_i$ variable is basic for some $i \geq 2$
only if $\lambda_{it}^s$ is already basic. Thus, when $z_0$ reaches 0, the right-hand side of (3.14) must hold with equality whenever a $\lambda_{it}^s$ is basic for all $\beta \geq 0$. Therefore, it is evident that all solutions to the LCP($q_{it}, M_{it}$) obtained by Lemke’s method are solutions to the original LCP($q, M$).

In executing Lemke’s method, treat $\beta$ as a parameter. Additionally, define

\[
q \triangleq \begin{pmatrix}
  h \\
  r \\
  -b \\
  0 \\
  \vdots \\
  0
\end{pmatrix}
\quad \text{and} \quad
p \triangleq \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  \mathbf{1}_{m_s} \\
  \vdots \\
  \mathbf{1}_{m_s}
\end{pmatrix}.
\]

For extending Lemke’s method to this $\beta$-parametrized problem, consider the parametric LCP with primary variable $z$, artificial variable $z_0$, covering vector $d > 0$, and the parameter $\beta \geq 0$,

\[
0 \leq z \perp w \triangleq q + p\beta + dz_0 + Mz \geq 0.
\]

While serving as a parameter, the role of $\beta$ here is to guide the Lemke pivots toward a solution of the problem by driving the artificial variable $z_0$ to zero; thus, termination of Lemke’s method at a solution for LCP($q + p\beta, M$) for any value of $\beta \geq 0$ is satisfactory.

When the method terminates successfully, it will typically end with a solution of the LCP($q + p\beta, M$) for an interval of $\beta$ values. Therefore, multiple GNE of the AGNEP $\mathcal{G}$ can be obtained by evaluating the solution form for specific $\beta$ values in the given interval.

Unlike the Lemke’s method scheme in parametric form described in [47, Algorithm 4.5.4] or
the parametric principle pivoting method described in [47, Algorithm 4.5.2], the parametric version of Lemke’s method described herein works as follows. Consider the initial tableau and that at the beginning of a general pivot:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(w)</td>
<td>(q + p\bar{\beta})</td>
<td>(d)</td>
<td>(M)</td>
<td>(\rightarrow)</td>
</tr>
<tr>
<td>Basic variables</td>
<td>1</td>
<td>Nonbasic variables ((\bar{z}))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\bar{w})</td>
<td>(\bar{q} + \bar{p}\beta)</td>
<td>(\bar{M})</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Initial tableau: \(\beta \in [0, \infty)\)

General tableau: \(\beta \in [\beta, \bar{\beta}]\)

\(\bar{w}\) contains \(z_0\); \(\bar{q} + \bar{p}\beta \geq 0\)

Assume that \(\bar{z}_k\) is the driving nonbasic variable in the current iteration. The following ratio test is performed:

\[
\theta(\beta) \triangleq \min_j \left\{ \frac{\bar{q}_j + \bar{p}_j\beta}{-\bar{M}_{jk}} \mid \bar{M}_{jk} < 0 \right\}, \quad \beta \in [\beta, \bar{\beta}].
\]

If \(\bar{M}_{*k} \geq 0\), then ray termination occurs. Otherwise, there is a partition

\[
[\beta, \bar{\beta}] = \bigcup_{\ell=1}^{\ell} I_\ell
\]

into non-overlapping [except at endpoints] subintervals within each of which, say the \(\ell\)th one, an index \(j_\ell\) exists with \(\bar{M}_{j_\ell k} < 0\) such that for all \(\beta \in I_\ell\),

\[
\theta(\beta) = \frac{\bar{q}_{j_\ell} + \bar{p}_{j_\ell}\beta}{-\bar{M}_{j_\ell k}}.
\]

Such an index \(j_\ell\) determines the blocking basic variable. The pivots continue until \(z_0\) leaves the basis or ray termination occurs. In the former case, a range of values of \(\beta\) is obtained for each of which a solution to the LCP\((q + p\beta, M)\) is readily obtained. In order to derive alternative pivot sequences, each of the subintervals \(I_\ell\) could be examined, which could generate further subdivision of the \(\beta\) range. This parametric method is best illustrated...
through an application to the simple 2-player Example 5.

Example 6. (Continuation of Example 5) The initial tableau is given by

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>$z_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\lambda^s_1$</th>
<th>$\lambda^s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$w_2$</td>
<td>-2</td>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$s_1$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$\beta$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

After the initial pivot of $\langle w_2, z_0 \rangle$, $x_2$ is driving while $s_1$ is blocking. Recall that both $s_1$ and $s_2$ were blocking at this point before the reformulation. After pivoting $\langle s_1, x_2 \rangle$, $\lambda^s_1$ is driving in the following tableau and $s_2$ is the only possible blocking variable with an upper bound for $\beta$ equal to $\infty$.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>$w_2$</th>
<th>$x_1$</th>
<th>$s_1$</th>
<th>$\lambda^s_1$</th>
<th>$\lambda^s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>0.25</td>
<td>0.75</td>
<td>0.875</td>
<td>0.25</td>
<td>1</td>
<td>-0.75</td>
</tr>
<tr>
<td>$z_0$</td>
<td>0.5</td>
<td>0.5</td>
<td>0.25</td>
<td>0.5</td>
<td>0</td>
<td>-0.5</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1.5</td>
<td>0.5</td>
<td>-0.75</td>
<td>-0.5</td>
<td>0</td>
<td>-0.5</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$0.5 + \beta$</td>
<td>0.5</td>
<td>0.25</td>
<td>0.5</td>
<td>-1</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

If not for the constraint reformulation, Lemke’s method would have ray terminated at this point. After pivoting $\langle s_2, \lambda^s_1 \rangle$, the driving variable is $\lambda^s_2$ in the next tableau.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>$w_2$</th>
<th>$x_1$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$\lambda^s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>$0.75 + \beta$</td>
<td>1.25</td>
<td>1.125</td>
<td>0.75</td>
<td>-1</td>
<td>-1.25</td>
</tr>
<tr>
<td>$z_0$</td>
<td>0.5</td>
<td>0.5</td>
<td>0.25</td>
<td>0.5</td>
<td>0</td>
<td>-0.5</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1.5</td>
<td>0.5</td>
<td>-0.75</td>
<td>-0.5</td>
<td>0</td>
<td>-0.5</td>
</tr>
<tr>
<td>$\lambda^s_1$</td>
<td>$0.5 + \beta$</td>
<td>0.5</td>
<td>0.25</td>
<td>0.5</td>
<td>-1</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

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In this tableau, the $\beta$ value affects the blocking variable. There are 4 ratios to be considered by the minimum ratio test:

$$\min\{0.6 + 0.8\beta, 1, 3, 1 + 2\beta\} = \begin{cases} 
0.6 + 0.8\beta & \text{if } \beta \in [0, 0.5], \\
1 & \text{if } \beta \in [0.5, \infty).
\end{cases}$$

Therefore, two $\beta$ subintervals are created ($[0, 0.5]$ and $[0.5, \infty)$) whose common end point leads to a tie minimum ratio. For $\beta$ in the subinterval $[0.5, \infty)$, $z_0$ is the next blocking variable which leads to a solution of the form $\left(\begin{array}{c}0 \\
1 \\
\beta \\
1\end{array}\right)$. For $\beta$ in the subinterval $[0, 0.5]$, the next tableau is

<table>
<thead>
<tr>
<th></th>
<th>$1$</th>
<th>$w_2$</th>
<th>$x_1$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$w_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_2^4$</td>
<td>$0.6 + 0.8\beta$</td>
<td>1</td>
<td>0.9</td>
<td>0.6</td>
<td>-0.8</td>
<td>-0.8</td>
</tr>
<tr>
<td>$z_0$</td>
<td>$0.2 - 0.4\beta$</td>
<td>0</td>
<td>-0.2</td>
<td>0.2</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$1.2 - 0.4\beta$</td>
<td>0</td>
<td>-1.2</td>
<td>-0.8</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>$\lambda_1^5$</td>
<td>$0.2 + 0.6\beta$</td>
<td>0</td>
<td>-0.2</td>
<td>0.2</td>
<td>-0.6</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Because $x_1$ is the driving variable, the minimum ratio test is $\min\{1 - 2\beta, 1 - \frac{\beta}{3}, 1 + 3\beta\}$. It should be obvious that every $\beta \in [0, 0.5]$ results in the same minimum ratio of $1 - 2\beta$.

Therefore, no new $\beta$ subintervals are generated and $z_0$ must be the blocking variable. The solution obtained is of the form $\left(\begin{array}{c}1 - 2\beta \\
2\beta \\
\beta \\
1.5 - \beta\end{array}\right)$ for $\beta \in [0, 0.5]$. This solution has both multipliers $\lambda_1$ and $\lambda_2$ basic but not equal in value; in particular, at $\beta = 0$, $\lambda_1 = 0$ so the solution is degenerate. At $\beta = 0.5$, this solution coincides with the one from $\beta \in (0.5, \infty)$ as expected. $\square$
### 3.7 Regularization and generalized Nash equilibria

While it may be possible to design alternative ways to resolve ray termination of Lemke’s method when solving the AGNEP $\mathcal{G}$, this topic will not be discussed further. Instead, a brief study of how Lemke’s method can be used to obtain normalized Nash equilibrium in the sense of Rosen [197] will be presented. Recall that a normalized Nash equilibrium is a solution to the LCP($q, M$) for which there exist a positive vector $e \in \mathbb{R}^F$ and a nonnegative vector $\eta \in \mathbb{R}^{m_s}$ such that $\lambda_{i\ell}^e = \frac{\eta_{\ell}}{e_i}$ for all $i = 1, \ldots, F$ and $\ell = 1, \ldots, m_s$. In principle, if the vector $e$ is known, it is not difficult to show that such a normalized equilibrium corresponds to a solution of the LCP($\tilde{q}^e, \tilde{M}^e$) with

\[
\tilde{q}^e \triangleq \begin{pmatrix} e_1 h_1 \\ \vdots \\ e_F h_F \\ -r_1 \\ \vdots \\ -r_F \\ -b \end{pmatrix}
\quad \text{and} \quad
\tilde{M}^e \triangleq \begin{pmatrix}
    e_1 H_{11} & \cdots & e_1 H_{1F} & -B_1^T & -A_1^T \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    e_F H_{F1} & \cdots & e_F H_{FF} & -B_F^T & -A_F^T \\
    B_1 & \cdots & B_F \\
    A_1 & \cdots & A_F
\end{pmatrix}.
\]

It is worth emphasizing that when $e$ is a vector of ones, then the reduced problem can be solved to obtain a variational equilibrium. However, when $e$ does not satisfy this property, then a drawback of the matrix $\tilde{M}^e$ is that certain desirable properties of the leading principal submatrix $H_0$ (e.g., copositivity or positive semidefiniteness) could be destroyed by the scalings. [To be fair, it should also be noted that an appropriate scaling could uncover missing desirable properties of $H_0$, but exploring this possibility is outside the
Thus, a natural question arises as to whether normalized NE can be computed by Lemke’s method without relying on the conversion to the scaled LCP($\tilde{q}^e, \tilde{M}^e$).

Via a limiting argument, the answer is affirmative. This conclusion is based on the discovery of a basic connection between a normalized Nash equilibrium and the well-known regularization idea applied to the LCP($q, M$). More generally, such regularization produces GNE that are characterized by a sign-consistency property across the multipliers of the shared constraints; namely, if a player has a positive multiplier corresponding to a certain shared constraint, then all the rival players also have positive multipliers corresponding to the same (shared) constraint. Such tuples of multipliers $\lambda^s$ can be formally defined as follows. For any $m_s$ positive matrices $E_s^\ell \triangleq \left[ e_{ij}^\ell \right]_{i,j=1}^{F} \in \mathbb{R}^{F \times F}$ satisfying $e_{ij}^\ell e_{ji}^\ell = 1$ for all $i$ and $j$, let $\Lambda(E^s)$ be defined as follows:

$$\Lambda(E^s) \triangleq \left\{ \lambda^s \mid \lambda^s_{i\ell} = e_{ij}^\ell \lambda^s_{j\ell} \text{ for all } \ell = 1, \cdots, m_s \text{ and } i, j = 1, \cdots, F \right\}.$$ 

Rosen’s normalized Nash equilibria correspond to the case where each matrix $E_s^\ell$ is the same with entries given by $e_{ij}^\ell = e_j/e_i$ for a positive vector $e \in \mathbb{R}^F$. A variational equilibrium is a special kind of normalized NE where $e$ is the vector of ones. From this perspective, we see that Rosen’s normalized NE and the VE are indeed quite special in that $E_s^\ell$ is identical for every $\ell = 1, \cdots, m_s$.

To see how GNE with multipliers of the shared constraints belonging to the set $\Lambda(E^s)$ can be computed as solutions of LCPs, define the following matrix obtained by adding
diminishingly small positive scalars to the diagonal entries of $M$ given by (3.3):

$$
M_\nu \triangleq \begin{bmatrix}
H_{11} + E_1^{\nu;x} & \cdots & H_{1F} \\
\vdots & \ddots & \vdots \\
H_{F1} & \cdots & H_{FF} + E_F^{\nu;x} \\
B_1 & \cdots & E_1^{\nu;p} \\
\vdots & \ddots & \vdots \\
B_F & \cdots & E_F^{\nu;p} \\
A_1 & \cdots & A_F \\
\vdots & \cdots & \vdots \\
A_1 & \cdots & A_F
\end{bmatrix} - B_1^T - A_1^T - B_F^T - A_F^T,
$$

where each $E_i^{\nu;x,p,s}$ is a positive diagonal matrix with $\lim_{\nu \downarrow 0} E_i^{\nu;x,p,s} = 0$. The following proposition describes several properties of the LCP($q,M_\nu$).

**Proposition 3.6.** Let $M_\nu$ be given by (3.15) where each $E_i^{\nu;x,p,s}$ is as given above. The following statements hold.

(a) If $M \in E_0$, then $M_\nu$ is strictly semimonotone; thus, SOL($q,M_\nu$) $\neq \emptyset$ for every $q$.

(b) If $M \in R_0$, then

$$
\limsup_{\nu \downarrow 0} \{ \|z_\nu\| \mid z_\nu \in \text{SOL}(q,M_\nu) \} < \infty.
$$

(c) Assume that positive matrices $E_\ell^s \triangleq [e_{ij}^s]_{i,j=1}^F$ with entries satisfying $e_{ij}^\ell e_{ji}^\ell = 1$ for all $(i,j)$ exist such that for all $\ell = 1, \cdots, m_s$ and $i, j = 1, \cdots, F$,

$$
\lim_{\nu(\varepsilon) \downarrow 0} \frac{(E_i^{\nu;s})_{\ell \ell}}{(E_i^{\nu;s})_{ij}} = e_{ij}^\ell.
$$

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If \( \hat{\tau} = \lim_{\nu(\zeta(\kappa)) \downarrow 0} \nu(\in \kappa) \), then \( \hat{\tau} \triangleq (\hat{x}, \hat{\lambda}^p, \hat{\lambda}^s) \) is a solution of the LCP \((q, M)\) with \( \hat{\lambda}^s \in \Lambda(\mathcal{E}^s) \).

**Proof.** Part (a) is obvious. Part (b) follows from a standard normalization/limiting argument. To prove part (c), let \( \nu(\in \kappa) \). Assume that \( \lambda^{\nu,s}_{i',\ell'} > 0 \) for some pair \((i', \ell')\). It then holds, for any \( j = 1, \ldots, F \),

\[
\left( \sum_{i=1}^F A_i x^\nu_i \right)_{e'} + (E^{\nu,s}_{i',e'})_{e'\ell'} \lambda^{\nu,s}_{i',\ell'} = (E^{\nu,s}_{j,e'})_{e'\ell'} \lambda^{\nu,s}_{j,\ell'},
\]

which implies \( (E^{\nu,s}_{j,e'})_{e'\ell'} \lambda^{\nu,s}_{j,\ell'} \geq (E^{\nu,s}_{i',e'})_{e'\ell'} \lambda^{\nu,s}_{i',\ell'} > 0 \). Therefore, equality holds in (3.16) and, as a result, \( (E^{\nu,s}_{j,e'})_{e'\ell'} \lambda^{\nu,s}_{j,\ell'} = (E^{\nu,s}_{i',e'})_{e'\ell'} \lambda^{\nu,s}_{i',\ell'} \). Passing to the limit easily yields the desired conclusion that

\( \hat{\lambda}^s \in \Lambda(\mathcal{E}^s) \).

**3.8 Some applications**

In Section 3.5 a modification of Lemke’s method was presented with sufficient conditions for successful method termination; it is natural to question how restrictive these requirements are in practice. In Sections 3.8.1, 3.8.2 and 3.8.3 models of strategic games in environmental pollution, communication networks, and power markets are presented to demonstrate that these assumptions are indeed satisfied in realistic applications.

**3.8.1 Environmental pollution games**

Traditionally, environmental and resource economics focuses on the utilization of renewable resources (e.g., fisheries, forests) and exhaustible resources (e.g., minerals, coal, oil, natural gas) as well as public environmental goods (e.g., water, soil, air). Both extraction and utilization of natural resources, particularly exhaustible ones, lead to pollution
while abatement comes at a cost; in economics, pollution is often categorized as a negative *externality* and represents a cost imposed on society that remains uncompensated by the firms responsible. Naturally, several ideas have been proposed to economically value the societal cost of pollution so that the externality can be internalized by the market. Here, the design of “Pigouvian” taxes is considered. Such taxes, in theory, represent a penalty imposed on the firm that is identical in value to the resulting externality.

One instance of such a game considers a river with contaminated effluent inputs from diverse sources that subsequently mix and lead to increased pollution levels downstream. For example, these sources could be a set of pulp mills competing in a spatially distributed oligopoly. Assume that there are \( \mathcal{F} \) players, each indexed by \( i \) where \( i \in \{1, \cdots, \mathcal{F}\} \), and that there are \( \Theta \) production technologies, each indexed by \( \theta \in \{1, \cdots, \Theta\} \). Given a technology \( \theta \), player \( i \)'s production decisions are denoted by \( (x_{i\theta})_{\theta=1}^{\Theta} \). The player is charged a unit cost of \( c_{i\theta} \geq 0 \) for production decision \( x_{i\theta} \) that is bounded from above by a capacity \( \bar{x}_{i\theta} \). Additionally, processing requirements are imposed on the production decisions and are modeled by the constraint \( B_i x_i \geq r_i \) for all \( i \), where the vector \( r_i \) is nonpositive. Letting \( \rho_{\theta} \) denote the unit contribution of output of technology \( \theta \), the resulting output of firm \( i \) can be expressed as \( X_i \triangleq \sum_{\theta=1}^{\Theta} \rho_{\theta} x_{i\theta} \). The aggregated emission impacts are given by the expression \( \sum_{\theta=1}^{\Theta} U_{\theta} x_{i\theta} \), where \( U_{\theta} \) denotes the per unit emission rate associated with technology \( \theta \). A scalar conversion factor \( \beta \) is employed to represent the total emission level as a cost in the player’s objective functions (i.e., the cost of purchasing the required emission permits).

The commodity price is prescribed by the following equation:

\[
p(X) = a - bX, \quad \text{where} \quad X \triangleq \sum_{i=1}^{\mathcal{F}} X_i
\]
and scalars \( a, b > 0 \). Firm \( i \) solves the optimization problem

\[
\max_{x_i} \left( p(X)X_i - \sum_{\theta=1}^{\Theta} c_{i\theta}x_{i\theta} \right) - \beta \sum_{\theta=1}^{\Theta} U_{i\theta}x_{i\theta}
\]

subject to \( 0 \leq x_{i\theta} \leq \bar{x}_{i\theta}, \quad \theta = 1, \cdots, \Theta \)

\[
B_i x_i \geq r_i,
\]

\[
\alpha_0q_0 + \sum_{j=1}^{F} \alpha_{j\ell} \sum_{\theta=1}^{\Theta} c_{j\ell}x_{j\theta} \leq \bar{q}_{\text{max},\ell}, \quad \ell = 1, \cdots, L,
\]

where the final shared constraints stipulate the regulatory authority’s restrictions on pollution levels at \( L \) locations: \( \alpha_0, \cdots, \alpha_F \) are exponential (and thus positive) pollution decays, \( q_0 \) denotes the background pollution quantities, and \( \bar{q}_{\text{max},\ell} \) denotes the maximum concentration of pollutants allowable at locations \( \ell = 1, \cdots, L \). Assuming that the shared constraints are consistent, it can be deduced that the constants \( \{\bar{q}_{\text{max},\ell} - \alpha_0q_0\}_{\ell=1}^L \) are all nonnegative. It is easy to see that the conditions of (B) in Theorem 3.2 are satisfied. Condition (a) of Proposition 3.3 also holds. Therefore, if \( M \) is semimonotone and the LCP without the shared constraints is solved by Lemke’s method (the alternative to \( J_0 \in R_0 \)), the modified Lemke’s method will successfully compute a generalized Nash equilibrium of this game.

Notably, in [100], an open-loop dynamic game where players minimize their individual costs over a finite horizon is considered with price-taking farmers. In the interest of brevity, only a simpler static game was analyzed here so that the structure of the coupled constraints could be studied. A related question at a macro-level is examined in [218] where strategic behavior of countries was analyzed when international environmental regulations such as the Kyoto Protocol are in effect. A more detailed review of environmental games, both
3.8.2 Rate allocation in communication networks

Non-cooperative game-theoretic models have been utilized widely in analyzing strategic behavior in wired and wireless communication networks, specifically with respect to questions of routing [174], bandwidth allocation [4], and optical networks [177, 186]. The setting presented here is aligned with recent work [4, 240] where flow control, in the presence of congestion, is considered in communication networks.

Consider a network of $N$ nodes with $K$ links connecting these nodes. Assume that there are $F$ players sharing this set of resources and that player $i$ has an associated route $R_i$ that relates a unique source-destination pair and specifies a set of links. The routing matrix $A$ defines the set of routes and links and is defined as

$$A_{k,i} = \begin{cases} 1 & \text{if route } R_i \text{ uses link } k, \\ 0 & \text{otherwise}. \end{cases}$$

The [shared] link capacity constraints are given by $\sum_{i=1}^{F} A_{k,i} x_i \leq b_k$ for all $k \in \{1, \cdots, K\}$ where $b_k \in \mathbb{R}_+$. Player $i$’s payoff function is given by his utility of transmission flow $x_i$, denoted by the function $U_i(x_i)$, less the cost arising from congestion given by $f_i(x)$. In [4], the congestion cost is specified by $\sum_{k=1}^{K} P_i \left( \sum_{j \mid k \in R_j} x_j \right)$ where $P_i(\bullet)$ is an increasing convex function in its argument. If the utility functions are quadratic and the congestion cost metric is the same quadratic function for all players (denoted by $f(x)$), then the associated mapping in the variational formulation of the game has a constant Jacobian matrix defined
as

\[ H_0 = \begin{bmatrix} -\nabla^2_{x_1} U_1(x_1) & \cdots & -\nabla^2_{x_F} U_F(x_F) \\ \vdots & \ddots & \vdots \\ -\nabla^2_{x_F} U_F(x_F) & & -\nabla^2_{x_F} f_F(x_F) \end{bmatrix} + \nabla^2 f(x). \]

A shortcoming of this metric is that every player is charged the same amount, regardless of utilization. A natural extension lies in charging users in accordance with their utilization; in [240], a scaled congestion metric is considered which leads to a Jacobian matrix given by

\[ H_0 = \begin{bmatrix} -\nabla^2_{x_1} U_1(x_1) & \cdots & -\nabla^2_{x_1} f_1(x_1) \\ \vdots & \ddots & \vdots \\ -\nabla^2_{x_F} U_F(x_F) & & -\nabla^2_{x_F} f_F(x_F) \end{bmatrix} + \begin{bmatrix} \nabla^2_{x_1} f_1(x) & \cdots & \nabla^2_{x_1} f_1(x) \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_F} f_F(x) & \cdots & \nabla^2_{x_F} f_F(x) \end{bmatrix}, \]

a \( P \) matrix. In the resulting Nash equilibrium problem, player \( i \) solves the following optimization problem:

\[
\begin{align*}
\text{minimize} \quad & -U_i(x_i) + f_i(x) \\
\text{subject to} \quad & \sum_{i=1}^F A_{k,i} x_i \leq b_k \quad \text{for all } k \in \mathcal{K} \\
& x_i \geq 0.
\end{align*}
\]

It is relatively easy to show that the resulting game, when employing a quadratic utility and congestion metric, leads to an LCP that satisfies the requirements for applying the modified Lemke’s method. The details are omitted.
3.8.3 Strategic behavior in power markets

A key challenge facing deregulated electricity markets is strategic and possibly manipulative behavior on the part of electricity generators and traders. Arguably, equilibrium models represent a tractable avenue for accommodating the oligopolistic structure prevalent in these markets and allow for the characterization and computation of equilibria. In this subsection, a setting in which generating firms and the independent system operator (ISO) compete in a Cournot game is examined. Refer to Chapter 4 for a much more complete description and analysis of power market models.

Consider a collection of \( F \) generation firms where firm \( i \) may generate and sell power at node \( \ell \) (denoted by \( g_{i\ell} \) and \( s_{i\ell} \), respectively), where \( \ell \) is in the set of network nodes \( \{1, \cdots, N\} \). Furthermore, the cost of generation associated with firm \( i \) and node \( \ell \) is given by a linear cost \( c_{i\ell}g_{i\ell} \). The price of power at node \( \ell \) is assumed to be an affine function of aggregate sales, denoted by \( S_{\ell} \), and is defined as

\[
p_{\ell}(S_{\ell}) \triangleq P^0_{\ell} - \frac{P^0_{\ell}}{Q^0_{\ell}} S_{\ell} \quad \text{for all } \ell = 1, \cdots, N
\]

with \( P^0_{\ell} \) and \( Q^0_{\ell} \) being positive constants. The cost of power transmission from an arbitrary node (referred to as the hub) to node \( \ell \) is given by \( w_{\ell} \). [From its usage, there should not be any confusion between this \( w_{\ell} \) and the \( w \) of LCP(\( q, M \)) \([3.1]\).] While generators see transmission prices as exogenous parameters, these prices are an outcome of the equilibrium in the market as will be discussed shortly.
The resulting problem faced by generator $i$ may be stated as

$$\begin{align*}
\text{maximize} \quad & \sum_{\ell=1}^{N} \left[ p_{\ell}(s_{i\ell}; S_{-i\ell}^{*}) s_{i\ell} - c_{i\ell} g_{i\ell} - (s_{i\ell} - g_{i\ell}) w_{1}^{*} \right] \\
\text{subject to} \quad & 0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \quad \text{for all } \ell = 1, \cdots, N \\
& \sum_{j=1}^{F} s_{j\ell} \leq S_{\ell} \quad \text{for all } \ell = 1, \cdots, N \\
& \sum_{\ell=1}^{N} (s_{i\ell} - g_{i\ell}) = 0 \\
& s_{i\ell} \geq 0 \quad \text{for all } \ell = 1, \cdots, N, 
\end{align*}$$

where $S_{-i\ell}^{*} \triangleq \sum_{i' \neq i}^{N} s_{i'\ell}^{*}$, cap$_{i\ell}$ is the generation capacity of firm $i$ at node $\ell$ and $S_{\ell}$ is the maximum allowable sales to node $\ell$. By noting that $g_{i1} = \sum_{\ell=1}^{N} s_{i\ell} - \sum_{\ell=2}^{N} g_{i\ell}$, the firm problem reduces to

$$\begin{align*}
\text{maximize} \quad & \sum_{\ell=1}^{N} \left[ p_{\ell}(s_{i\ell}; S_{-i\ell}^{*}) s_{i\ell} - (w_{1}^{*} - w_{1}^{*}) s_{i\ell} \right] + \sum_{\ell=2}^{N} \left[ c_{i\ell} g_{i\ell} - (w_{\ell}^{*} - w_{1}^{*}) g_{i\ell} \right] \\
\text{subject to} \quad & 0 \leq \sum_{\ell=1}^{N} s_{i\ell} - \sum_{\ell=2}^{N} g_{i\ell} \leq \text{cap}_{i1} \\
& 0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \quad \text{for all } \ell = 2, \cdots, N \\
& \sum_{j=1}^{F} s_{j\ell} \leq S_{\ell} \quad \text{for all } \ell = 1, \cdots, N \\
& s_{i\ell} \geq 0 \quad \text{for all } \ell = 1, \cdots, N.
\end{align*}$$

The independent system operator (ISO) decides the flows on the transmission lines in
accordance with the solution of the optimization problem given by

$$\begin{align*}
\text{maximize} & \quad \sum_{\ell=1}^{N} y_\ell w_\ell^* \\
\text{subject to} & \quad T_k^- \leq \sum_{\ell=1}^{N} \text{PTDF}_{\ell k} y_\ell \leq T_k^+ \quad \text{for all } k = 1, \ldots, K,
\end{align*}$$

where $y_\ell$ represents the outflow at node $\ell$, $T_k^\pm$ denote the directional capacity associated with link $k \in \{1, \ldots, K\}$ (the set of nodal links), and PTDF$_{\ell k}$ denotes the power transfer distribution factor for the pair $(\ell, k)$ (the DC flow through link $k$ as a consequence of a unit MW injection at an arbitrary hub node and withdrawal at node $\ell$).

Finally, the market clearing requirement requires that the nodal outflows at node $\ell$ are given by the aggregate excess of sales over generation over the entire set of firms:

$$y_\ell^* = \sum_{i=1}^{F} (s_{i\ell}^* - g_{i\ell}^*) \quad \text{for all } \ell = 1, \ldots, N.$$

While the above market clearing equations are not in the form of an optimization problem, they can trivially be stated as a single linear program with the variable vector $w$ and $(y, s, g)$ as parameters:

$$\begin{align*}
\text{minimize} & \quad \sum_{\ell=1}^{N} w_\ell \left[ y_\ell^* - \sum_{i=1}^{F} (s_{i\ell}^* - g_{i\ell}^*) \right].
\end{align*}$$

In summary, this AGNEP has $F + 2$ players: $F$ generating firms, the ISO, and the market clearing mechanism. The shared constraints of this game are the regional sales caps: $S_\ell \leq \bar{S}_\ell$ for all $\ell = 1, \ldots, N$. Note that the variables $w_\ell$ and $y_\ell$ are not restricted in sign; nevertheless, each can be expressed as the difference of two nonnegative variables as in elementary linear programming. Omitting the details, it can be deduced that Theorem 3.2.
holds for this game if Lemke’s method successfully terminates for the game without the shared constraints via condition (B).

A host of related questions has been considered in the literature. In 1997, [230] presented a similar model where both investment and generation were considered, again leading to a generalized Nash equilibrium problem. Extensions to allow for the presence of arbitrageurs were provided in [151] while bilateral and pool-based models were investigated in [105]. More recently, two-period stochastic generalizations have been examined in [122] under a risk-aversion setting and in [123] under a risk-neutral setting. Notably, in all of these settings, variational equilibria (VE) were sought, the sole exception being [110] where the challenge of using Lemke’s method for obtaining more general generalized Nash equilibria was touched upon.

3.9 Computational results

In this section, the behavior of the modified Lemke’s method of Section 3.5 is investigated for a range of test problems. In Subsection 3.9.1 a short description of the test problems is provided. The performance and scalability (with respect to the number of players) of the modified Lemke’s method is examined relative to the original Lemke’s method in Subsection 3.9.2 while Subsection 3.9.3 focuses on the simplified river basin pollution game of Subsection 3.8.1 and provides a description of how the alternate generalized Nash equilibria may be computed via the problem reformulation method of Section 3.6. The possible failure of Lemke’s method is also illustrated.
3.9.1 Description of test problems

Since a standard set of affine generalized Nash equilibrium problems does not currently exist, four sets of problems were constructed to compare the behavior of the presented computation schemes:

(1) **4-player, 2 shared constraint game:** The first group of problems entails a simple 4-player, 2 shared constraint game without private constraints. The corresponding LCP has the form

\[
q \triangleq \begin{pmatrix}
  h_1 \\
  \vdots \\
  h_4 \\
  -b \\
  \vdots \\
  -b
\end{pmatrix} \quad \text{and} \quad M \triangleq \begin{pmatrix}
  H_{11} & \cdots & H_{14} & -A_1^T \\
  \vdots & \ddots & \vdots & \vdots \\
  H_{41} & \cdots & H_{44} & -A_4^T \\
  A_1 & \cdots & A_4 \\
  \vdots & \vdots & \vdots \\
  A_1 & \cdots & A_4
\end{pmatrix}
\]

where \( J_0 \) as defined by (3.7) is a symmetric positive semidefinite matrix in \( \mathbb{R}^{4 \times 4} \), \(-A_i \in \mathbb{R}^2_+\) for all \( i \in \{1, \cdots, 4\} \), \( h_i \in \mathbb{R} \) for all \( i \in \{1, \cdots, 4\} \), and \(-b \in \mathbb{R}^2_+\).

(2) **Networked power markets with piecewise linear price functions:** The second set of test problems is from [110] and models an electricity market with piecewise linear demand functions as described in Section 3.8.1. Each player’s [multivariate] optimization problem consists of six decision variables corresponding to generation and sales at each of three regions. Additionally, each player is subject to shared constraints representing regional sales caps. Parameters include regional sales caps (nonnegative), firm generation costs (nonnegative), firm generation capacities (nonnegative), transmission capacities...
(nonnegative), and price function parameters (breakpoints nonnegative, slopes negative). All three regions were assumed to have one breakpoint in the price function and the game consisted of 60 KKT conditions.

(3) Rate allocation in communication networks: The third set of problems is a simplification of the network congestion game presented in Subsection 3.8.2 and adapted from [240]. In the tested version of the game, 6 players compete for transmission on a linear network consisting of 6 nodes. Player \( i \) chooses \( x_i \geq 0 \) to maximize \( a_i x_i^2 - b_i x_i \) where \( a_i, b_i \geq 0 \). The shared constraints take the form of shared transmission line capacity on the linear network. Additionally, a congestion cost metric is imposed on every agent based on the aggregate residual capacity in a link.

(4) River basin pollution game: The last set of problems is the river basin pollution game from [160] composed of three players and two shared constraints, a simplified version of the game described in Subsection 3.8.1. Player \( i \) faces the optimization problem

\[
\begin{align*}
\text{minimize} & \quad [\alpha_i x_i + 0.01 (x_1 + x_2 + x_3) - \chi_i] x_i \\
\text{subject to} & \quad 3.25 x_1 + 1.25 x_2 + 4.125 x_3 \leq 100 \\
& \quad 2.2915 x_1 + 1.5625 x_2 + 2.8125 x_3 \leq 100 \\
& \quad x_i \geq 0
\end{align*}
\]

with parameters \( \alpha_1 = 0.01, \alpha_2 = 0.05, \alpha_3 = 0.01, \chi_1 = 2.9, \chi_2 = 2.88, \) and \( \chi_3 = 2.85 \).

The tests was carried out via a Matlab implementation on an Intel Core(TM)2 Quad CPU Q9550 (2.83GHz) with a Linux 10.04 operating system.
3.9.2 Comparison of performance

Each set of test problems consisted of 25 randomly generated problems. The problems examined were required to (a) have a solution, and (b) have ray termination (if any) only arising from a $\lambda^\ell$-multiplier being made basic. Each problem was solved 50 times due to the potentially random nature of Lemke’s method pivots when faced with tied blocking variables.

First, a comparison of the performance of the original Lemke’s method in terms of number of ray termination occurrences per 50 solution attempts versus the proposed modified Lemke’s method was made. The average number of complete pivots and partial pivots are presented in Table 3.1. Partial pivots represent the single column pivots used when a shared constraint is blocking to determine which $s_\ell$ variable may be made nonbasic without immediate ray termination when the associated $\lambda^\ell$-variable is entering the basis. It can be seen that the modified Lemke’s method accomplishes its goal of avoiding unnecessary ray termination for this simple problem. Next, the performance of the modified Lemke’s method for the more complex piecewise linear network demand function model from [110] is examined. Table 3.2 presents the results, and it is again observed that unnecessary ray termination is avoided by the modified method. It is worth emphasizing that this modification is achieved at relatively minor cost in terms of partial pivots.

Table 3.3 examines the modified Lemke’s method performance for the linear network problem from [240]. As the results show, the modified Lemke’s method eliminates all ray termination occurrences for relatively little cost in terms of additional partial and complete pivots.

Next, the performance of modified Lemke’s method is examined with respect to problem
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Table 3.3: Comparison of performance. Test problem 3
Figure 3.1: Problem scaling. Test problem 1

(a) Ray termination comparison for scaling

(b) Pivot comparison for scaling

scaling. These randomly generated problems are identical to the 4-player, 2 shared constraint problem above except that there are now \( F \) players solving univariate optimization problems. For each \( F \in \{2, \ldots, 20\} \), 25 problems were generated and solved 50 times each. The ray termination results are plotted in Figure 3.1a with pivot results in Figure 3.1b.

Confirming the results of Table 3.1, modified Lemke’s method does not ray terminate at all when the problem is scaled. However, Figure 3.1b shows that the number of partial pivots required by the modified Lemke’s method increases with problem size as should be expected. Despite this, each partial pivot evaluates at most one column of the tableau and is therefore not computationally intensive unless the problem is very large.

3.9.3 Constructing manifolds of equilibria

Last, the partial variational equilibria and the reformulation methods are examined for obtaining alternate equilibria for the river pollution problem of [160] as presented in Test problem (4) of Subsection 3.9.1. The partial VE problem formulations yielded two equilibria, denoted by X’s in Figure 3.2 and problem reformulations yielded 5 different manifolds.
of equilibria, shown as lines in Figure 3.2. One reformulation utilized the method previously described in Section 3.6 by keeping the first occurrence of the shared constraint KKT conditions and replacing the latter two with identical conditions. Two more complex reformulations involved a “stairstep” reformulation in which the replaced KKT conditions do not refer to the same $\lambda_{it}^s$-variable. As an example, one of the two “stairstep” reformulations used for this problem took the form:

$$q = \begin{pmatrix} -2.9 \\ -2.88 \\ -2.85 \\ 100 \\ \beta \\ \beta \\ 100 \\ \beta \\ \beta \end{pmatrix}$$

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The term “stairstep” refers to the placement of the new \(-1\) terms in the reformulated KKT conditions. There are obviously many different ways to reformulate problems in this manner with each such reformulation potentially providing additional equilibria. To generate the shown manifolds of equilibria, each reformulated problem was solved with Lemke’s method and successive division of the \(\beta\) interval as described in Section 3.6; each manifold corresponds to the solution of a reformulated problem for a specific \(\beta\) subinterval. Several \(\beta\) subintervals may yield solution manifolds for a given reformulation.

As a supplement to the above partial variational equilibrium results where the groupings of shared constraints are preset, selected pivots of the modified Lemke’s method are presented where the groupings of the shared constraints are carried out only if needed during the pivots. For this particular AGNEP, successful termination requires the presence of a shared constraint grouping rule such as that specified in Step 2 of the modification. The numbers

\[
M = \begin{bmatrix}
  0.04 & 0.01 & 0.01 & 3.25 & 0 & 0 & 2.2915 & 0 & 0 \\
  0.01 & 0.12 & 0.01 & 0 & 1.25 & 0 & 0 & 1.5625 & 0 \\
  0.01 & 0.01 & 0.04 & 0 & 0 & 4.125 & 0 & 0 & 2.8125 \\
  -3.25 & -1.25 & -4.125 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
  -2.2915 & -1.5125 & -2.8125 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 
\end{bmatrix}
\]
in Tables 3.5 through 3.11 below are rounded to 2 decimal digits to fit the page margins. After a few pivots from the original Table 3.4 of this game, Table 3.5 is obtained where the driving variable is the variable \( x_3 \); there is a tie in the blocking \( s_{s1} \)-variables. Initially, let \( s_{11} \) be the blocking variable; this pivot leads to Table 3.6 in which \( \lambda_{s1}^3 \) is the next driving variable. After pivoting on \( \lambda_{s1}^3 \), ray termination occurs on the next driving variable \( w_1 \). Return to Table 3.5 and choose \( s_{31} \) as the blocking variable; this pivot leads to Table 3.8 in which \( \lambda_{s3}^3 \) is the next driving variable. After pivoting on \( \lambda_{s3}^3 \), ray termination occurs on the next driving variable \( w_3 \). Finally, the same delayed ray termination occurs with \( s_{21} \) brought into the basis instead of \( s_{31} \); the details are omitted. Notice that Tables 3.6 and 3.8 are identical except for the exchange of \( s_{11} \) and \( s_{31} \). This property was mentioned in Section 3.5 and can be exploited to save computation time. At this point, group the first shared constraints and the corresponding multipliers (Table 3.10). After pivoting \( x_3 \) into the basis, \( \lambda_{s1}^3 \) becomes driving with the blocking variable \( z_0 \) as shown in Table 3.11. Thus, the solution \[
\begin{pmatrix}
21.1448 & 16.0279 & 2.7260 & 0.5744 & 0 & 0 & 0
\end{pmatrix}
\] is identified.

### 3.10 Conclusion

In this chapter, the solution of AGNEPs via Lemke’s method was analyzed with an emphasis on solutions that are not variational equilibria. An important source of Lemke’s method failure was identified and illustrated through a simple example, and a crucial limitation of Lemke’s method was proven. In order to remedy the problem of unnecessary ray termination of Lemke’s method, a modification to the method was developed and sufficient conditions for its successful termination were provided. This method was shown to help eliminate ray termination in problems that only ray terminated when shared constraint multipliers became basic. It is therefore a significant improvement to the current solution methodology. Since problem regularization is a commonly proposed solution method, a reg-
ularization framework was discussed and convergence properties were established. More specifically, it was shown that under different conditions, regularized solutions can converge to variational equilibria, normalized equilibria, and potentially other GNEs. Coalitional equilibria and partial variational equilibria were introduced to demonstrate the applicability of the modified Lemke’s method to a broad range of problems. Most importantly, this research developed a problem reformulation and associated solution methodology that has the ability to find solutions that were previously impossible to identify with Lemke’s method. The reformulation solution method also has the ability to find points on solution manifolds that could not previously be determined. Future work in this area could examine extensions of the problem reformulation method to find restricted Nash equilibria and its potential applicability to other parametric LCP problems. The application of this methodology to nonlinear equilibrium problems through a sequence of linearized problems may also be a fruitful area of research. Finally, while significant strides have been made in this research to improve the robustness of Lemke’s method for computing generalized Nash equilibria of various kinds, there are still realistic AGNEPs that escape treatment; further work is needed in this direction.
| \( w_1 \) | -2.9 | 1 | 0.04 | 0.01 | 0.01 | 3.25 | 0 | 0 | 2.29 | 0 | 0 |
| \( w_2 \) | -2.88 | 1 | 0.01 | 0.12 | 0.01 | 0 | 1.25 | 0 | 0 | 1.56 | 0 |
| \( w_3 \) | -2.85 | 1 | 0.01 | 0.01 | 0.04 | 0 | 0 | 4.13 | 0 | 0 | 2.81 |
| \( s_{11} \) | 100 | 1 | -3.25 | -1.25 | -4.13 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( s_{21} \) | 100 | 1 | -3.25 | -1.25 | -4.13 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( s_{31} \) | 100 | 1 | -3.25 | -1.25 | -4.13 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( s_{12} \) | 100 | 1 | -2.29 | -1.51 | -2.81 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( s_{12} \) | 100 | 1 | -2.29 | -1.51 | -2.81 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 3.4: The river basin game. Original formulation
Table 3.5: The river basin game. After 3 pivots from original, $x_3$ is the driving variable.

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Table 3.6: The river basin game. The pivot choice is on the distinguished shared multiplier $\lambda_{11}^s$.

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Table 3.7: The river basin game. Ray termination on \( w_1 \) after \( \lambda^s_{11} \) enters basis.
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Table 3.8: The river basin game. The pivot choice is on the distinguished shared multiplier $\lambda_{31}^s$. 
|   | 1 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
|   |   | $w_1$ | $w_2$ | $w_3$ | $s_{31}$ | $\lambda_{11}^s$ | $\lambda_{21}^s$ | $x_3$ | $\lambda_{12}^s$ | $\lambda_{22}^s$ | $\lambda_{32}^s$ |
| $z_0$ | 1.69 | 0.94 | 0.04 | 0 | 0.01 | -3.07 | -0.05 | 0.04 | -2.16 | -0.07 | 0 |
| $x_1$ | 28.38 | 3.43 | -3.15 | 0 | -0.28 | -11.14 | 3.94 | -1.14 | -7.85 | 4.93 | 0 |
| $x_2$ | 7.56 | -8.16 | 8.23 | 0 | -0.08 | 26.51 | -10.29 | -0.31 | 18.69 | -12.86 | 0 |
| $s_{11}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $s_{11}$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\lambda_{31}^s$ | 0.19 | -0.22 | -0.02 | 0.24 | -0.002 | 0.71 | 0.03 | -0.02 | 0.50 | 0.04 | -0.68 |
| $s_{12}$ | 25.22 | 5.43 | -5.18 | 0 | 0.76 | -17.64 | 6.48 | 0.30 | -12.43 | 8.10 | 0 |
| $s_{12}$ | 25.22 | 5.43 | -5.18 | 0 | 0.76 | -17.64 | 6.48 | 0.30 | -12.43 | 8.10 | 0 |

Table 3.9: The river basin game. Ray termination on $w_3$ after $\lambda_{31}^s$ enters basis
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Table 3.10: The river basin game. Table 3.5 after collapsing shared constraint 1 and its multipliers.
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<td>32.05</td>
<td>2.30</td>
<td>-43.85</td>
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<tr>
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<td>-5.63</td>
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<td>-16.32</td>
<td>-2.72</td>
<td>8.80</td>
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<tr>
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<td>1.19</td>
<td>-5.63</td>
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<td>-16.32</td>
<td>-2.72</td>
<td>8.80</td>
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Table 3.11: The river basin game. Solution after pivot on $\lambda_1^s$. 
Chapter 4

A survey of contemporary electricity market models

4.1 Introduction

The first power sector deregulation measures took place in Chile, England, and New Zealand in the 1990s with similar changes arising in California, New England, Pennsylvania-Jersey-Maryland (PJM), and New York in the late 1990s. Whereas the traditional power market structure is basically a government-regulated monopoly, deregulated markets are distinguished by the presence of an independent transmission system operator (ISO), competitive bidding for electricity generation, and purely financial markets for a variety of electricity-related services. These competitive market elements are often viewed as an intricate coupling between two systems, a physical electricity grid and a more abstract financial framework. The combination of these elements presents many interesting modeling and mathematical questions that will be assembled and discussed in this chapter.

\footnote{This chapter is an excerpt of a longer manuscript being prepared for submission (Co-authors: Uday Shanbhag and Jong-Shi Pang).}
To understand how the physical and financial market elements interact, it is important to understand [and then simplify] the systems involved. The physical electricity transmission system is best viewed as a network of generation and demand nodes connected by a fixed set of transmission lines. Instead of fully modeling the alternating current (AC) electricity flow through the network resulting from a certain combination of generation and demand quantities (e.g., [15] [136] [187] [188] [228] [229]), electricity flow is commonly assumed to be direct current (DC) so that Kirchhoff’s laws hold. The linearity of these laws makes them easy to model mathematically, and this simplification provides reasonable transmission estimates for most AC power flow situations [175]. Transmission limits are also usually imposed on certain transmission lines to capture the limitations of the electricity grid.

The financial layer of electricity markets can be seen as functioning on a higher level than the physical transmission network in the sense that decisions in the financial market determine how the transmission network will function. Namely, the clearing of the financial market [approximately] specifies the generation levels of each generator over a given time horizon, typically 24 hours; the approximation arises from possible transmission line failures, incorrect demand forecasts, and instantaneous supply-demand balancing. Closely associated with these financial market clearings is the concept of locational marginal price (LMP). In general, LMPs are determined for aggregated geographic areas of demand via a sensitivity analysis. Finally, financial markets for reserves, capacity, and financial transmission rights play important roles in electricity markets.

A notable power market restructuring effort took place in the United Kingdom [69] [80] and is popularly referred to as the “British Electricity Experiment.” Later developments in the United States showed two distinct deregulation trajectories for the power industry [40] [234]. One type of deregulated market design relies on centralized dispatch of generation capacity.
by an ISO based on submitted generator bid curves (a *POOLCO* market \[27, 87\]) while the other type of market involves bilateral transactions between generators and suppliers where the ISO only manages electricity grid imbalances (a *bilateral* market \[113\]). Prominent *POOLCO* markets in the United States include New York and the PJM Interchange; California and Texas follow the bilateral market framework.

The establishment of power markets after deregulation presents a challenge for both market designers and operators: how should the market be structured, cleared, and managed? The presence of strategic and selfish agents, possibly endowed with market power, makes this analysis more difficult than the original central planning-type problem. Furthermore, the concept of an “optimal” solution is somewhat ambiguous; steady-state Nash equilibria present an obvious starting point for examining this complex market structure. Because of their theoretical properties and practical importance, equilibrium models of power markets have been the subject of much research in the past two decades and have grown immensely in scope and intricacy. For instance, early models focused on strategic bidding in single-period deterministic power markets under a variety of assumptions, a relatively simplistic view of how the market functions that has since been extended in several directions. As the emphasis on reliability and environmental sustainability emerged, energy-based models have given way to expansive formulations that incorporate ancillary reserves, capacity, and emissions. In these models, a sequence of market clearings is necessary, leading to the study of multi-period models that incorporate both financial and physical transactions. Because of the importance of risk and uncertainty in power market decision-making as discussed in Chapter 2, research has more recently focused on the stochastic analogs of power market equilibrium problems (e.g., \[13, 30, 156, 189, 242\]).

In addition to the *POOLO*/bilateral market distinction, the nature of interactions between
generators and demanders raises another important modeling question. In what appears to be the earliest attempt at examining these markets, Green and Newbery [90] considered a supply-function equilibrium (SFE) framework for capturing the strategic interactions in the United Kingdom in a newly deregulated regime. Bolle [21] examined similar approaches and subsequently extended these directions to incorporate demand side bidding [20]. In the late 1990s, as deregulatory efforts proceeded in the U.S., SFE models were applied in the American context [10, 199]. An alternate approach for modeling strategic interactions is to assume a Cournot model in which firms bid in quantity, where Cournot prices are upper bounds on prices resulting from SFE models [126]. The resulting equilibrium conditions are compactly captured by a variational inequality or complementarity problem (e.g., [54, 105, 106, 111, 230]) and have proven to be an attractive approach given the tractability of such problems from the standpoint of both analysis and computation.

Given the sheer breadth of the research on power markets, there have been several past literature review efforts. Of these, the earliest known review is that of Hobbs and Helman [106], which concentrated on deterministic perfectly competitive and Nash-Cournot models as well as hierarchical models. A subsequent review by Ventosa, Báñez, Ramos, and Rivier [222] had a similar focus but also examined supply function equilibrium problems and incorporated agent-based models. The recent book by Gabriel, Conejo, Fuller, Hobbs, and Ruiz [86] also presents a broad overview of this field; the authors focus on both static and hierarchical equilibrium models as well as settings with discrete decisions while capturing a broad class of energy and environmental markets. Despite these literature reviews, many research gaps persist. Namely, there is a lack of a comprehensive treatment of power market equilibrium models characterized by the following properties: (a) a range of competitive interactions (perfectly competitive, Cournot, and conjectured supply function), and (b) the ability to incorporate additional markets such as those for
capacity and emission permits. Motivated by these deficiencies, this chapter provides a reasonably expansive review of power market research through a unified model and details the relevant theoretical and algorithmic foundations for the presented formulations.

The early efforts in modeling strategic interactions in power markets were largely restricted to deterministic and static regimes. In Section 4.2, the initial efforts focused on power generators and independent system operators are reviewed. Each generator attempts to maximize profit by generating and selling electricity on a DC power grid managed by an independent system operator. The behavioral assumptions of perfect, Cournot, or conjectured supply function competition are unified into a single model. The market formulations are studied through finite-dimensional variational inequalities that correspond, either wholly or partially, to the Nash equilibria of the game. Finally, tractable conditions for equilibrium existence and uniqueness are provided along with relevant computational methods. In Section 4.3, three different extensions to the unified model of Section 4.2 are presented: consumer surplus maximization, capacity markets, and emission permit schemes. The theoretical results of Section 4.2 are extended to include these more complex models in a straightforward manner. The chapter is concluded in Section 4.4.

### 4.2 Basic market models

In Subsection 4.2.1, three power market models are introduced and unified where the market consists of generators, an ISO with an associated DC transmission grid, and market clearing conditions. In each model, a different assumption regarding generator behavior is specified. Drawn from microeconomics, these assumptions are:

- Perfect competition (firms take prices as given);
- Cournot competition (firms realize that generation and sales decisions affect market
prices and take this into account when optimizing decisions);

- Conjectured supply function competition (firms believe that prices change based on competitor decisions in a specified manner).

The perfect competition assumption is often difficult to justify for deregulated power markets [48, 102, 200], so the Cournot and conjectured supply function approaches may be seen as more attractive. The unified model of this section shows that each of these behavioral assumptions is a specialization of a more general game formulation.

While Subsection 4.2.1 formulates the three power market models and the associated unified model, theoretical and computational considerations are addressed in Subsection 4.2.2. Several new results answering existence and uniqueness questions are proven, and numerical schemes for identifying equilibria based on equivalent variational inequality problems are described. Although the examined problems are somewhat simple, the theory and algorithms are applicable to more realistic market representations with the same problem form.

4.2.1 Competitive market models

Consider \( F \) generators competing on a fixed set of \( N \) nodes connected by a DC electricity transmission grid. Let firm \( i \)'s generation and sales at node \( \ell \) be denoted by \( g_{i\ell} \) and \( s_{i\ell} \), respectively, where generation and sales are not required to be equal (i.e., shipments between nodes may occur to realize locational price differences). This separation of generation and sales may seem superfluous but is important in power markets where transmission grid constraints become binding. A generator’s revenue comes from sales at each node and possibly transmission grid revenue as will be described momentarily. Generators incur costs from physical generation and possibly transmission grid payments. Assume that each generator
can own assets at multiple nodes with the cost of generation for firm $i$’s capacity at node $\ell$ given by an increasing convex cost function $c_{i\ell}(g_{i\ell})$.

The ISO/transmission grid is represented by a set of linear power transfer distribution factor (PTDF) constraints based on Kirchoff’s laws (for calculation methodology, refer to [150 Chapter 3]). The PTDF values arise by specifying an arbitrary node as the “hub” and calculating induced electricity flows for injection at the hub and withdrawal at a given node. Each node $\ell$ has an associated per-unit wheeling fee $w_{\ell}$ that is charged for generator shipment of electricity from the hub to node $\ell$. Given their directional property, wheeling fees are unrestricted in sign and a shipment in the opposite direction of the wheeling fee-specified direction incurs a negative wheeling fee (i.e., shipments in the opposite direction of a positive/negative wheeling fee incur a negative/positive wheeling fee). It follows that wheeling fees can be either revenue or costs for generators shipping electricity through the transmission network. The ISO chooses net injection/withdrawal quantities $y$ so that the PTDF constraints are not violated. However, it should be apparent that this ISO-chosen quantity is actually a result of the generators’ decisions. This requirement is specified in the market clearing conditions described next.

Market clearing in power market models guarantees the fulfillment of consumer demand and agreement of the ISO and generator net electricity transmission values. Namely, supply must at least meet demand and the ISO net injection/withdrawal quantities $y$ must equal the net result of generator decisions.

The perfect competition, Cournot competition, and conjectured supply function competition specifications of this model are developed as follows. For notational simplicity, define $g \triangleq \left( g_i \triangleq (g_{i\ell})_{\ell=1}^{N_i} \right)_{i=1}^{F}$, $s \triangleq \left( s_i \triangleq (s_{i\ell})_{\ell=1}^{N_i} \right)_{i=1}^{F}$, $y \triangleq (y_{\ell})_{\ell=1}^{N}$, $w \triangleq (w_{\ell})_{\ell=1}^{N}$, and mar-
ket prices $p \triangleq (p_\ell)_{\ell=1}^N$.

**Perfectly competitive equilibrium models**

In a perfectly competitive market model, each player treats prices as parameters rather than variables. Therefore, each generator treats the nodal electricity prices and wheeling fees exogenously in its profit optimization problem. Let $p^*_\ell$ and $w^*_\ell$ represent the electricity price and wheeling fee for node $\ell$, respectively. The generator is constrained by a capacity limit and a conservation requirement between sales to generation quantities. Mathematically, generator $i$ solves the problem

$$\begin{align*}
\text{maximize} & \quad \sum_{\ell=1}^N \left[ p^*_\ell s_{i\ell} - c_{i\ell}(g_{i\ell}) - (s_{i\ell} - g_{i\ell})w^*_\ell \right] \\
\text{subject to} & \quad 0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \quad \text{for all } \ell = 1, \cdots, N \\
& \quad s_{i\ell} \geq 0 \quad \text{for all } \ell = 1, \cdots, N \\
& \quad \sum_{\ell=1}^N (g_{i\ell} - s_{i\ell}) = 0.
\end{align*}$$

(4.1)

Although both $p^*_\ell$ and $w^*_\ell$ are treated exogenously in each generator’s problem, the values are endogenous to the game as a whole and factor into the ISO problem and market clearing conditions.

As described, the ISO manages the net flow of electricity along the transmission lines. In
this light, the ISO solves the following problem while treating wheeling fees $w^*$ exogenously:

$$\max_y \sum_{\ell=1}^{N} y_{\ell}w^*_{\ell}$$

subject to $T^-_k \leq \sum_{\ell=1}^{N} \text{PTDF}_{\ell k}y_{\ell} \leq T^+_k$ for all $k = 1, \cdots, K$,

where $y_{\ell}$ represents the nodal inflow to node $\ell$, $\{1, \cdots, K\}$ is the set of all transmission lines, $\text{PTDF}_{\ell k}$ denotes the power transfer distribution factor for node $\ell$ through line $k$, and $T^\pm_k$ denote the directional capacities associated with transmission line $k$. This objective function implies that the ISO makes decisions to maximize wheeling fee revenue, a standard assumption in power market literature even though it is not the governing principle of ISOs.

The first market clearing condition mandates that the nodal electricity outflow is equal to the aggregate nodal sales less generation:

$$y^*_{\ell} = \sum_{i=1}^{F} (s^*_{i\ell} - g^*_{i\ell}) \text{ for all } \ell = 1, \cdots, N,$$

or equivalently,

$$\min_w \sum_{\ell=1}^{N} w_{\ell} \left[ y^*_{\ell} - \sum_{i=1}^{F} (s^*_{i\ell} - g^*_{i\ell}) \right].$$

(4.3)

It is important to note that (4.3) has $w$ unrestricted in sign just as was originally specified in the game description. Furthermore, $g^*$, $s^*$, and $y^*$ are treated exogenously.

The second market clearing condition requires that consumer demand is satisfied. In this case, it is assumed that consumer demand at node $\ell$ is fixed at a specified value $D_{\ell} > 0$. In a slight adaptation of the approach taken in Chapter 2, the associated uniform price
method is given by the complementarity condition

\[ p_\ell \text{ free }, \quad \sum_{i=1}^{F} s_{i\ell}^* - D_\ell = 0 \quad \text{for all } \ell = 1, \cdots, N, \]

or equivalently,

\[ \max_p \sum_{\ell=1}^{N} p_\ell \left( \sum_{i=1}^{F} s_{i\ell}^* - D_\ell \right). \quad (4.4) \]

Note that neither (4.3) nor (4.4) takes prices and wheeling fees as given. Instead, both of these values are variables, thereby allowing their endogenous determination in the model.

Together, the concatenated optimization problems (4.1), (4.2), (4.3), and (4.4) define the most fundamental, albeit simplified, model of electricity market equilibrium under the perfect competition (PC) assumption. There are several extensions and variants of this model that are briefly summarized after a formal PC equilibrium definition is provided.
Perfectly Competitive Nash Equilibrium. A tuple \((g^*, s^*, y^*, w^*, p^*)\) is a perfectly competitive Nash equilibrium if and only if

\((PCE_1)\) for each \(i = 1, \ldots, F\),

\[
(g^*_i, s^*_i) \in \argmax_{g_i, s_i} \sum_{\ell=1}^{N} \left[ p^*_\ell s_{i\ell} - c_{i\ell}(g_{i\ell}) - (s_{i\ell} - g_{i\ell})w^*_\ell \right]
\]

subject to \(0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \) for all \(\ell = 1, \ldots, N\)

\(s_{i\ell} \geq 0 \) for all \(\ell = 1, \ldots, N\)

\[
\sum_{\ell=1}^{N} (g_{i\ell} - s_{i\ell}) = 0;
\]

\((PCE_2)\) \(y^* \in \argmax_{y} \sum_{\ell=1}^{N} y_{\ell}w^*_\ell\)

subject to \(T^-_k \leq \sum_{\ell=1}^{N} \text{PTDF}_{k\ell}y_{\ell} \leq T^+_k \) for all \(k = 1, \ldots, K\);

\((PCE_3)\) \(w^* \in \argmin_{w} \sum_{\ell=1}^{N} w_{\ell} \left[ y^*_{\ell} - \sum_{i=1}^{F} (s^*_{i\ell} - g^*_{i\ell}) \right] \), or equivalently,

\[
y^*_{\ell} = \sum_{i=1}^{F} (s^*_{i\ell} - g^*_{i\ell}) \quad \text{for all } \ell = 1, \ldots, N;
\]

\((PCE_4)\) \(p^* \in \argmax_{p} \sum_{\ell=1}^{N} p_{\ell} \left( \sum_{i=1}^{F} s^*_{i\ell} - D_{\ell} \right) \), or equivalently, \(\sum_{i=1}^{F} s^*_{i\ell} - D_{\ell} = 0. \square\)

This simple model has served as the basis of several more complex market formulations that address issues such as arbitrage, different pricing methods, and emission allowances.

In [106], perfectly competitive markets that include transporters/arbitragers are studied.
As opposed to generators, these players attempt to maximize profit by buying and selling electricity at different nodes to realize locational price differences with nodal demand being determined by the solution of consumer optimization problems. In accordance with the perfectly competitive assumption, consumers assume that they cannot affect market prices. In the related work [54], power markets in which the ISO imposes zonal as opposed to nodal pricing and charges both congestion charges and “postage stamp” transmission fees are modeled.

Competition in a market with emissions was examined in [41] with an allowance requirement imposed at three different points within the electricity supply chain. In this study, players were assumed to be perfectly competitive with respect to the prices of both electricity and emission allowances. The enforcement of emission allowance constraints for different point-of-regulation scenarios was not found to lead to significant decreases in total emissions due to either “leakage” or “contract reshuffling.” In [101], the effects of the Public Utility Regulatory Policies Act (PURPA) were analyzed with a focus on the interaction of large utilities, small power producers, and cogeneration plants. As stated by PURPA, utilities must purchase small supplier production at their avoided cost (assumed to be marginal cost) but may sell electricity on the grid for their average cost. It was found for a simulation of the New England power system that PURPA increased the cost for utilities, decreased the costs for small power producers, and caused changes in generation investment strategies.

**Nash-Cournot equilibrium models**

Through a modification of the price-taking market model, a Cournot game can be constructed in which generators compete in quantity and know that their production quantities affect prices. Unlike the previously formulated perfectly competitive model, nodal
demand is not an exogenous parameter in a Cournot model; instead, demand quantities are determined by inverse demand functions assumed to be known by generators. The nodal electricity price at node \( \ell \) is assumed to be a decreasing function of aggregate sales and is denoted by \( p_\ell(S_\ell) \), where \( S_\ell \triangleq \sum_{i=1}^{F} s_{i\ell} \). Because each generator treats the decisions of all other generators as given, the nodal price function is more correctly denoted by \( p_\ell(s_{i\ell}; S_{-i\ell}^*) \) for generator \( i \), where \( S_{-i\ell}^* \triangleq \sum_{i' \neq i} s_{i'\ell}^* \). As before, the cost of power transmission from an arbitrary hub node to node \( \ell \) is given by the wheeling fee \( w_\ell^* \). While generators treat transmission prices as exogenous parameters, these prices are an outcome of the ISO optimization problem and market clearing conditions as discussed previously. In summary, generator \( i \) solves the problem

\[
\max_{g_{i\ell}, s_{i\ell}} \sum_{\ell=1}^{N} \left[ p_\ell(s_{i\ell}; S_{-i\ell}^*) s_{i\ell} - c_{i\ell}(g_{i\ell}) - (s_{i\ell} - g_{i\ell}) w_\ell^* \right]
\]

subject to
\[
0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \quad \text{for all } \ell = 1, \cdots, N
\]
\[
s_{i\ell} \geq 0 \quad \text{for all } \ell = 1, \cdots, N
\]
\[
\sum_{\ell=1}^{N} (g_{i\ell} - s_{i\ell}) = 0.
\]

This Nash-Cournot power market model is completed with the ISO optimization problem (4.2) and the market clearing optimization problem (4.3). In this model, there is no need for the price clearing conditions (4.4) because the supply-demand relationship is already accounted for in the specified nodal pricing function. In practice, this electricity pricing function is often taken to be linear:

\[
p_\ell(S_\ell) \triangleq \frac{P^0_\ell}{Q^0_\ell} S_\ell \quad \text{for all } \ell = 1, \cdots, N
\]
with $P^0_\ell$ and $Q^0_\ell$ being positive constants.

Nash-Cournot models have been the focus of much power market research. As opposed to the lack of market power implied by the perfectly competitive model, a Cournot framework allows for the formulation and solution of models that can include generators with market power. It can also be argued that actual deregulated electricity markets are similar to oligopolies due to high barriers to entry and a small number of sizeable generation companies. Similar to the perfectly competitive case, there have been a variety of proposed extensions to this simple game.

A simple extension to this Nash-Cournot model arises when arbitragers are introduced. In [105], arbitragers were modeled as eliminating non-cost-based price differentials between nodes in the transmission network (i.e., buying and selling electricity at different nodes until wheeling fee costs equal nodal price differences). With their presence, it was proven that a bilateral market framework similar to that presented in this section is equivalent to a POOLCO model. Additionally, uniqueness of the equilibrium outcome was proven under certain conditions. A further generalization was presented in [151] where arbitrager actions were either anticipated by the generators or modeled as a separate Cournot optimization problem. The former model takes the form of a Stackelberg game while the latter case is similar to the game presented here. It was proven that both of these arbitrager assumptions lead to the same equilibrium prices, profit, and electricity supply.

An important analysis of market power with respect to the electricity industry was presented in [32] where a simplified version of the proposed model was used to demonstrate that transmission constraints can result in counterintuitive displays of market power. Namely, generators may be able to increase profits by producing excess electricity to force decreased
production of competing firms due to Kirchoff’s laws and transmission grid limitations. A similar idea was studied in [22] where the possibility of Bertrand competition in the electricity market was also discussed. The consequences of different transmission pricing methods was discussed in [230] with transmission prices determined by exogenous functions based on requested generator power flows rather than by an ISO optimization problem. Pricing variants involving both average-cost and marginal-cost pricing were also explored. In [60], a different ISO optimization problem form was considered. Instead of modeling the ISO as a profit-maximizing player, the authors model the ISO as an economic dispatch problem. In this framework, the goal of the ISO is to minimize the cost of meeting consumer demand while servicing nodes on a radial distribution network.

Lastly, the Nash-Cournot game has been expanded to an axiomatic bargaining setting where generators aim to maximize a collusive payoff function. Such a collusive game without an ISO was formulated in [97] as a nonconvex optimization problem and upper and lower bounding procedures on the global Nash bargaining solution were derived. An extension of this collusive framework was presented in [139] and subsequently published in [140] to include the transmission optimization problem of the ISO.

**Conjectured supply function equilibrium problems**

While perfectly competitive and Cournot models allow for agents to make quantity bids, the majority of market designs in the U.S. and elsewhere require that firms make price-quantity bids. In an effort to capture this intricacy, there has been much interest in equilibrium problems where players bid in supply functions. With the work of [126] providing the initial foundations, there has been significant research in this area over the last two decades (e.g., [5, 6, 11, 12, 21, 89, 90, 92, 114, 126, 137, 172, 199, 226, 227, 229, 241]). However, this problem form is generally nonconvex and infinite-dimensional, making complementarity-
based approaches less useful. It can be noted that Cournot quantity bids can be viewed as
supply functions with infinite slopes, but solution methods and theoretical results cannot
be readily generalized from the Cournot framework to supply function games.

Complementarity-based approaches have been useful in constructing a related conjectured
supply function (CSF) model \[55, 111\] even though this formulation does not apply to
supply function equilibrium problems in general. In a CSF model, generators utilize a [not
necessarily correct] belief regarding how competitors will change their decisions based on
market price. In the following CSF model description, \(p_\ell\) denotes the price at node \(\ell\) while
\(p_{i\ell}\) denotes generator \(i\)'s belief about the price at node \(\ell\). These values are related with a
market clearing condition as will be specified later.

- Each node \(\ell\) is characterized by a continuous nodal demand function \(q_\ell(p_\ell)\) of price
  (e.g., a linear demand function \(q_\ell(p_\ell) = Q_0^\ell - \frac{Q^0_0}{p_\ell} p_\ell\));

- Generator \(i\) conjectures that net competitor sales correspond to the function
  \(\sigma_{i\ell}(S_{-i\ell}^*, p_{i\ell})\), where \(S_{-i\ell}^* \triangleq \sum_{i' \neq i} s_{i'\ell}\) is the cumulative supply of firm \(i\)'s competitors.

With these two functions, generator \(i\)'s optimization problem is

\[
\begin{align*}
\text{maximize} & \quad \sum_{\ell = 1}^{N} \left[ p_{i\ell} s_{i\ell} - c_{i\ell}(g_{i\ell}) - (s_{i\ell} - g_{i\ell}) w_{i\ell}^* \right] \\
\text{subject to} & \quad 0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \quad \text{for all } \ell = 1, \cdots, N \\
& \quad s_{i\ell} \geq 0 \quad \text{for all } \ell = 1, \cdots, N \\
& \quad \sum_{\ell = 1}^{N} (g_{i\ell} - s_{i\ell}) = 0 \\
& \quad s_{i\ell} + \sigma_{i\ell}(S_{-i\ell}^*, p_{i\ell}) = q_\ell(p_{i\ell}).
\end{align*}
\]

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The CSF equilibrium model is completed with the addition of the transmission optimization problem of the ISO (4.2), the nodal clearing optimization problems (4.3), and a market clearing condition stipulating that, with \( p_i \equiv (p_{i\ell})_{\ell=1}^N \), there is a set of optimal solutions \( (g^*_i, s^*_i, p^*_i)_{\ell=1}^F \) such that \( p^*_i = p^* \) for all \( i = 1, \cdots, F \). It is worth emphasizing that in the CSF framework, generator \( i \) optimizes over its generation, sales, and belief regarding prices (as denoted by \( p_{i\ell} \)). At equilibrium, the believed prices must match the actual market clearing prices.

A noteworthy feature of generator \( i \)'s optimization problem (4.6) is that the cumulative competitor sales variable \( S^*_{-i\ell} \) is present in the constraints; moreover, this constraint could be firm-dependent in a slightly different model formulation. As was discussed in the Chapter 1, the term generalized is used to describe Nash equilibria of games with this type of constraint.

For a CSF model, an important problem specialization arises when the conjectured supply function \( \sigma_{i\ell}(S^*_{-i\ell}, p_{\ell}) \) has a “slope-intercept” form:

\[
\sigma_{i\ell}(S^*_{-i\ell}, p_{\ell}) = S^*_{-i\ell} + \beta_{i\ell}(S^*_{-i\ell}, p^*_{\ell})(p_{\ell} - p^*_{\ell}),
\]

where \( p^*_{\ell} \) is the equilibrium market clearing price and \( \beta_{i\ell}(S^*_{-i\ell}, p^*_{\ell}) \) is a conjectured correction term to approximate how much deviation from the equilibrium market price will alter net competitor sales. In this situation, the optimization problem (4.6) can be converted into one where competitor decisions appear only in the objective function, thereby removing difficulties inherent to generalized Nash equilibrium problems. Indeed, substituting this form of \( \sigma_{i\ell}(S^*_{-i\ell}, p_{\ell}) \) into the constraint equation

\[
s_{i\ell} + \sigma_{i\ell}(S^*_{-i\ell}, p_{i\ell}) = q_{\ell}(p_{i\ell})
\]
gives
\[ s_{it} + S^*_{-i\ell} = \beta_{i\ell}(S^*_{-i\ell}, p_{i\ell}^s) + q_{i\ell}(p_{i\ell}) - \beta_{i\ell}(S^*_{-i\ell}, p_{i\ell}^s) p_{i\ell}^s. \] (4.7)

If the mapping from \( p_{i\ell} \) to \( q_{i\ell}(p_{i\ell}) - \beta_{i\ell}(S^*_{-i\ell}, p_{i\ell}^s) p_{i\ell}^s \) is invertible, \( p_{i\ell} \) can be explicitly determined from (4.7). In general,
\[ p_{i\ell} = \rho_{i\ell}(s_{i\ell}; S^*_{-i\ell}, p_{i\ell}^s) \]
for some function \( \rho_{i\ell}(s_{i\ell}; S^*_{-i\ell}, p_{i\ell}^s) \). With this expression, (4.6) becomes

\[
\begin{align*}
\text{maximize} & \quad \sum_{\ell=1}^{N} \left[ \rho_{i\ell}(s_{i\ell}; S^*_{-i\ell}, p_{i\ell}^s) s_{i\ell} - c_{i\ell}(g_{i\ell}) - (s_{i\ell} - g_{i\ell}) w_{i\ell}^s \right] \\
\text{subject to} & \quad 0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \quad \text{for all } \ell = 1, \cdots, N \\
& \quad s_{i\ell} \geq 0 \quad \text{for all } \ell = 1, \cdots, N \\
& \quad \sum_{\ell=1}^{N} (g_{i\ell} - s_{i\ell}) = 0. \\
\end{align*}
\]
(4.8)

Upon close examination, (4.8) resembles the perfectly competitive and Cournot models when the function \( \rho_{i\ell} \) is specified appropriately. It is precisely this observation that allows for the following unified power market model.

A unified equilibrium formulation

Formulating a CSF model entails specifying three different functions \( \beta_{i\ell}(S^*_{-i\ell}, p_{i\ell}^s) \), \( q_{i\ell}(p_{i\ell}) \), and \( \rho_{i\ell}(s_{i\ell}; S^*_{-i\ell}, p_{i\ell}^s) \), and, as has already been indicated, perfectly competitive and Cournot models can be obtained by appropriately defining these functions. The specialization of the nodal demand function \( q_{i\ell}(p_{i\ell}) \) is relatively straightforward according to which model is chosen. The “correction term” \( \beta_{i\ell}(S^*_{-i\ell}, p_{i\ell}^s) \) is more difficult to specify but can be seen as the change in net competitor sales resulting from marginal price deviations from equilib-
rium. Using this marginal change concept, the perfectly competitive and Cournot model specializations of \( \beta_{i\ell}(S^*_{-i\ell}, p^*_\ell) \) are more apparent. Namely, it should be obvious that \( \beta_{i\ell}(S^*_{-i\ell}, p^*_\ell) \geq 0 \) because a higher-than-equilibrium price should result in higher-than-equilibrium net competitor sales. Therefore, two relevant limits for \( \beta_{i\ell}(S^*_{-i\ell}, p^*_\ell) \) are 0 and \( \infty \). In typical CSF models, \( \beta_{i\ell}(S^*_{-i\ell}, p^*_\ell) \) is taken to be a rational function instead of a constant so that the intensity of the net competitor sales correction can vary. The interested reader is referred to [111] for additional details on CSF model specifications from a similar but not identical market formulation.

Before delving into the stated problem specializations, it is important to note that some form of supply-demand market clearing condition is required for each of these three market models. Luckily, the relevant market clearing condition can be captured by the simple equation
\[
S^*_\ell = q_\ell(p^*_\ell),
\]
where \( S^*_\ell \) is the nodal supply and \( q_\ell(p^*_\ell) \) is the nodal demand at the equilibrium market price \( p^*_\ell \). Recalling that this condition could be enforced through the optimization problem (4.4) for the perfectly competitive market model, a more general optimization problem can easily be formulated to accommodate different \( q_\ell(p) \) functions:

\[
p^*_\ell \in \arg\max_{p_\ell} \left[ \int_0^{p_\ell} q_\ell(p'_\ell)dp'_\ell - S^*_\ell p_\ell \right],
\]
which becomes \( S^*_\ell = q_\ell(p^*_\ell) \) by the second fundamental theorem of calculus.

**Perfect competition:** Because the demand for node \( \ell \) in a perfectly competitive game is fixed at \( D_\ell \), it is obvious that \( q_\ell(p_\ell) = D_\ell \) for all \( i = 1, \ldots, F \) and \( \ell = 1, \ldots, N \). In essence, this specification indicates that the nodal demand is independent of market price. Let \( \beta_{i\ell}(S^*_{-i\ell}, p^*_\ell) = \infty \) for all \( i = 1, \ldots, F \) and \( \ell = 1, \ldots, N \), basically implying that any deviation from the equilibrium price is believed to be infeasible by each player. After
normalization by $\beta_{it}(S_{-it}^*, p_{it}^*)$ and noting that $s_{it}$ is bounded by $\sum_{\ell=1}^N c_{it\ell}$, (4.7) gives that $p_{it} = p_{it}^*$ (i.e., generator $i$'s belief about the market price at node $\ell$ agrees with the actual market price). Thus, $\rho_{it}(s_{it}; S_{-it}^*, p_{it}^*) = p_{it}^*$ for all $i = 1, \cdots, F$ and $\ell = 1, \cdots, N$. In summary, for all $i = 1, \cdots, F$ and $\ell = 1, \cdots, N$,

- $q_{it}(p_{it}) = D_t$;
- $\beta_{it}(S_{-it}^*, p_{it}^*) = \infty$;
- $\rho_{it}(s_{it}; S_{-it}^*, p_{it}^*) = p_{it}^*$.

**Cournot:** For a Cournot problem, nodal demand is not fixed. Instead, it is given by the inverse demand function $q_{it}(p_{it}) = Q_0^\ell - \frac{Q_0^\ell}{P_0^\ell} p_{it}$. Consequently, each generator assumes

$q_{it}(p_{it}) = Q_0^\ell - \frac{Q_0^\ell}{P_0^\ell} p_{it}$, an inverse demand function based on their believed market price. Let $\beta_{it}(S_{-it}^*, p_{it}^*) = 0$ for all $i = 1, \cdots, F$ and $\ell = 1, \cdots, N$, indicating that no player believes that net competitor production will change because of deviation from the equilibrium price. It follows from (4.7) that $p_{it} = P_0^\ell - \frac{P_0^\ell}{Q_0^\ell} (s_{it} + S_{-it}^*)$. Therefore, the Cournot model is obtained by specifying, for all $i = 1, \cdots, F$ and $\ell = 1, \cdots, N$,

- $q_{it}(p_{it}) = Q_0^\ell - \frac{Q_0^\ell}{P_0^\ell} p_{it}$;
- $\beta_{it}(S_{-it}^*, p_{it}^*) = 0$;
- $\rho_{it}(s_{it}; S_{-it}^*, p_{it}^*) = P_0^\ell - \frac{P_0^\ell}{Q_0^\ell} (s_{it} + S_{-it}^*)$.

**Conjectured supply function:** No special treatment is needed to model conjectured supply function games in this unified framework. Therefore, instead of specifying required function forms, a specialization resulting in a model identical to that of [111] is presented. As in the Cournot model, let $q_{it}(p_{it})$ be an affine function $q_{it}(p_{it}) = Q_0^\ell - \frac{Q_0^\ell}{P_0^\ell} p_{it}$. Consider
the rational function \( \beta_{i\ell}(S^*_{-i\ell}, p^*_\ell) = \frac{S^*_{-i\ell}}{p^*_\ell - A_{-i\ell}} \) where \( A_{-i\ell} \) is a positive player-specific constant for node \( \ell \) that serves as an equilibrium price comparison. After substitution and simplification, (4.7) gives

\[
p_{i\ell} = \frac{Q^0_{\ell} - s_{i\ell} - S^*_{-i\ell} + \frac{S^*_{-i\ell}}{p^*_\ell - A_{-i\ell}} p^*_\ell}{Q^0_{\ell} + \frac{S^*_{-i\ell}}{p^*_\ell - A_{-i\ell}}}.
\]

For this particular CSF model, for all \( i = 1, \cdots, F \) and \( \ell = 1, \cdots, N \),

- \( q_{i\ell}(p_{i\ell}) = Q^0_{\ell} - \frac{Q^0_{\ell}}{p^*_\ell} p_{i\ell} \);
- \( \beta_{i\ell}(S^*_{-i\ell}, p^*_\ell) = \frac{S^*_{-i\ell}}{p^*_\ell - A_{-i\ell}} \);
- \( \rho_{i\ell}(s_{i\ell}; S^*_{-i\ell}, p^*_\ell) = \frac{Q^0_{\ell} - s_{i\ell} - S^*_{-i\ell} + \frac{S^*_{-i\ell}}{p^*_\ell - A_{-i\ell}} p^*_\ell}{Q^0_{\ell} + \frac{S^*_{-i\ell}}{p^*_\ell - A_{-i\ell}}} \).

In summary, the proper specification of \( q_{i\ell}(p_{i\ell}), \beta_{i\ell}(S^*_{-i\ell}, p^*_\ell), \) and \( \rho_{i\ell}(s_{i\ell}; S^*_{-i\ell}, p^*_\ell) \) allows for the recovery of the three models presented thus far. Therefore, a unified equilibrium problem can be formulated to obtain a host of perfectly and imperfectly competitive electricity market equilibrium problems studied in the literature. Similar to the perfectly competitive, Cournot, and conjectured supply function game formulations, other constraints can be imposed within the respective optimization problems; in particular, bounds on prices can be easily included if needed. This flexibility is a distinct advantage of the optimization-based formulation of the equilibrium problem.
Unified Nash Equilibrium. Given an appropriate specification of $\rho_i(\ell; s_i, S_i^\ast)$ and $q_\ell(p_i)$, a tuple $(g^*, s^*, y^*, w^*, p^*)$ is a Nash equilibrium if and only if the following conditions hold, where $S_{i\ell}^* = \sum_{i' \neq i} s_{i'\ell}^*$ and $S_i^* = \sum_{i=1}^F s_i^*$,

\(\text{(UE}_1\text{)}\) for each $i = 1, \cdots, F$,

\[
(g_i^*, s_i^*) \in \arg\max_{g_i, s_i} \sum_{\ell=1}^N \left[ \rho_i(\ell; S_{i\ell}^*, p_{i\ell}^*) s_{i\ell} - c_i(\ell) - (s_{i\ell} - g_{i\ell}) w_i^* \right]
\]

subject to $0 \leq g_{i\ell} \leq \text{cap}_{i\ell}$ for all $\ell = 1, \cdots, N$

$s_{i\ell} \geq 0$ for all $\ell = 1, \cdots, N$

$\sum_{\ell=1}^N (g_{i\ell} - s_{i\ell}) = 0$;

\(\text{(UE}_2\text{)} y^* \in \arg\max_{y} \sum_{\ell=1}^N y_{\ell} w_i^*$

subject to $T_{k}^- \leq \sum_{\ell=1}^N \text{PTDF}_{\ell k} y_{\ell} \leq T_{k}^+$ for all $k = 1, \cdots, K$;

\(\text{(UE}_3\text{)} w^* \in \arg\min_w \sum_{\ell=1}^N w_{\ell} \left[ y_{\ell}^* - \sum_{i=1}^F (s_{i\ell}^* - g_{i\ell}^*) \right]$, or equivalently,

\[
y_{\ell}^* = \sum_{i=1}^F (s_{i\ell}^* - g_{i\ell}^*) \text{ for all } \ell = 1, \cdots, N;
\]

\(\text{(UE}_4\text{)} \) for each $\ell = 1, \cdots, N$,

\[
p_{\ell}^* \in \arg\max_{p_{\ell}} \left[ \int_{0}^{p_{\ell}} q_\ell(p_{\ell}') dp_{\ell}' - S_{\ell}^* p_{\ell} \right], \text{ or equivalently, } S_{\ell}^* = q_\ell(p_{\ell}^*). \quad \square
\]

For theoretical reasons that will become evident in Section 4.2.2 it is worth noting that
the ISO problem (UE2) and the flow-balancing problem (UE3) can be absorbed into the generator problems by changing (UE1) to (UE1'):

\[
(\text{UE1}') \text{ for each } i = 1, \cdots, F,
\]

\[
(g_i^*, s_i^*) \in \text{argmax} \sum_{\ell=1}^{N} \left[ \rho_{i\ell}(s_i; S_{-i\ell}^*, p_{i\ell}^*)s_i - c_{i\ell}(g_{i\ell}) \right]
\]

subject to

\[
0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \quad \text{for all } \ell = 1, \cdots, N
\]

\[
s_{i\ell} \geq 0 \quad \text{for all } \ell = 1, \cdots, N
\]

\[
\sum_{\ell=1}^{N} (g_{i\ell} - s_{i\ell}) = 0
\]

(4.9)

where the last two constraints are shared by all players. By enforcing common multipliers for these constraints (i.e., finding the variational equilibrium as was discussed in Chapter 3), the wheeling fees are given by

\[
w_{\ell} = \sum_{k=1}^{K} \text{PTDF}_{\ell k} (\lambda_k^+ - \lambda_k^-) \quad \text{for all } \ell = 1, \cdots, N
\]

with \(\lambda_k^+\) and \(\lambda_k^-\) being the common Lagrange multipliers associated with the upper and lower bounds on transmission, respectively.
Other shared constraints

In the development of the CSF market model, it was noted that the CSF concept is naturally related to a generalized Nash game where the [exogenous] amount of cumulative competitor nodal sales $S_{\cdot,i\ell}^*$ is present in each generator’s constraint set. Although the final CSF (and thus unified) model did not contain this type of coupling constraint as the result of the given assumptions/manipulations, generalized Nash games are still important in other areas of power market research. For instance, there are several realistic constraints that may couple generator feasible regions together, including regional sales caps [110], piecewise linear price functions [184], shared limited resources [110, 206], and minimum generation requirements [28]. In addition to the problem simplifications already specified, it is assumed here that regional sales caps and minimum generation requirements are imposed on generators even though these constraints are typically imposed on consumers/load-serving entities in reality. In the following development, an extended optimization problem for the generators containing nodal sales caps $S_{\ell}$ and minimum generation requirements $Q_{\ell}$ is formulated. Although a special subset of nodes could be defined at which these shared constraints are enforced, it is notationally simpler to enforce these constraints at every node and then specify $S_{\ell'} \triangleq 1 + \sum_{i=1}^{F} \text{cap}_{i\ell}$ and $Q_{\ell'} \triangleq -1$ for all nodes $\ell'$ not requiring these constraints (the $\pm 1$ ensures that degeneracy is not accidentally introduced through these extraneous constraints).
(UE\textsubscript{1} with shared constraints) for each $i = 1, \cdots, F$,

\[
(g^*_i, s^*_i) \in \arg\max_{g_i, s_i} \sum_{\ell=1}^{N} \left[ \rho_{i\ell}(s_{i\ell}; S^*_{-i\ell}, p^*_\ell) s_{i\ell} - c_{i\ell}(g_{i\ell}) - (s_{i\ell} - g_{i\ell}) w^*_\ell \right]
\]

subject to

\[
\sum_{\ell=1}^{N} (g_{i\ell} - s_{i\ell}) = 0
\]

\[
\begin{cases}
0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \\
s_{i\ell} \geq 0 \\
s_{i\ell} + \sum_{j \neq i} s_{j\ell}^* \leq S_\ell \\
g_{i\ell} + \sum_{j \neq i} g_{j\ell}^* \geq Q_\ell
\end{cases}
\]

for all $\ell = 1, \cdots, N$.

An optimization problem analogous to (4.9) can be obtained by absorbing (UE\textsubscript{2}) and (UE\textsubscript{3}) into (UE\textsubscript{1} with shared constraints). Although omitted here because of its similarity to (4.9), this combined problem is referenced later as (UE\textsubscript{1}' with shared constraints).

Clearly, a necessary condition for feasibility is $\sum_{\ell=1}^{N} S_\ell \geq \sum_{\ell=1}^{N} Q_\ell$. This feasibility condition, however, is not sufficient to yield the existence of a Nash equilibrium.

4.2.2 Analysis

In the same vein as the earlier chapters of this dissertation, the power market games of Section 4.2.1 will be analyzed via finite-dimensional variational inequalities [75] and linear complementarity problems [47]. This approach is natural (see Chapter 1) and allows for the rich theoretical results of these fields to be leveraged. In addition to the previously cited papers dealing exclusively with power markets, the recent survey [76] presented a comprehensive review on the application of this mathematical programming methodology.
to non-cooperative games. While the survey provided some broad existence results, the applicability of these results to the unified problem is not immediate. Therefore, the following developments prove existence and uniqueness for the given power market models with shared constraints.

**Existence of equilibria**

For the power market models formulated in Section 4.2.1, unbounded variables include the net transmission quantities $y$, the wheeling fees $w$, and the market prices $p$. Therefore, Corollary 1.1, Theorem 1.2, and Theorem 1.3 cannot be applied and special care must be taken when proving equilibrium existence.

To demonstrate different approaches for proving Nash equilibrium existence, two existence results are presented. The first result applies to the perfectly competitive market model and relies on the observation that the mapping $F : K \rightarrow K$ is integrable on $K$; see the discussion of gradient maps in Section 1.3.1. With this integrability property, it is simple to establish that a Nash equilibrium corresponds to an optimal solution of a single optimization problem through an examination of necessary and sufficient optimality conditions. The second existence result requires slightly more analysis but is basically a fixed-point proof that requires an assumption on the form of $q_{\ell}(p_{\ell})$ to guarantee feasible region compactness.

To gain intuition about the first game equivalence, note that the necessary and sufficient optimality conditions of the perfectly competitive game, given the convexity (but not necessarily differentiability) of objective function in $(PCE_1)$, are
• for all \( i \) and \( \ell \),

\[
0 \in \partial g_{i\ell}(g^*_{i\ell}) - \sum_{k=1}^{K} \text{PTDF}_{\ell k}(\lambda^+_k - \lambda^-_k) - \mu_{i\ell}^1 + \mu_{i\ell}^2 - \mu_{i\ell}^3
\]

\[
0 \leq s_{i\ell}^* \perp -p^*_\ell + \sum_{k=1}^{K} \text{PTDF}_{\ell k}(\lambda^+_k - \lambda^-_k) + \mu_{i\ell}^3 \geq 0
\]

\[
0 \leq \mu_{i\ell}^1 \perp g_{i\ell}^* \geq 0
0 \leq \mu_{i\ell}^2 \perp \text{cap}_{i\ell} - g_{i\ell}^* \geq 0;
\]

• for all \( i \),

\[\mu_{i}^3 \text{ free }, \sum_{\ell=1}^{N} (g_{i\ell}^* - s_{i\ell}^*) = 0;\]

• for all \( \ell \),

\[p^*_\ell \text{ free }, \sum_{i=1}^{F} s_{i\ell}^* - D_\ell = 0;\]

• for all \( k \),

\[
0 \leq \lambda^+_k \perp T^+_k - \sum_{\ell=1}^{N} \text{PTDF}_{\ell k} \sum_{i=1}^{F} (s_{i\ell}^* - g_{i\ell}^*) \geq 0
\]

\[
0 \leq \lambda^-_k \perp \sum_{\ell=1}^{N} \text{PTDF}_{\ell k} \sum_{i=1}^{F} (s_{i\ell}^* - g_{i\ell}^*) - T^-_k \geq 0.
\]

To establish the stated integrability property, it suffices to note that these optimality
conditions are equivalent to those of the optimization problem

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{\mathcal{F}} \sum_{\ell=1}^{\mathcal{N}} c_{i\ell}(g_{i\ell}) \\
\text{subject to} & \quad \sum_{i=1}^{\mathcal{F}} s_{i\ell} - D_\ell = 0 \quad \text{for all } \ell = 1, \ldots, \mathcal{N} \\
& \quad \sum_{\ell=1}^{\mathcal{N}} (g_{i\ell} - s_{i\ell}) = 0 \quad \text{for all } i = 1, \ldots, \mathcal{F} \\
& \quad \{ 0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \} \quad \text{for all } i = 1, \ldots, \mathcal{F} \\
& \quad s_{i\ell} \geq 0 \quad \text{and } \ell = 1, \ldots, \mathcal{N} \\
& \quad T^-_k \leq \sum_{\ell=1}^{\mathcal{N}} \sum_{i=1}^{\mathcal{F}} PTDF_{t_\ell k} (s_{i\ell} - g_{i\ell}) \quad \text{for all } k = 1, \ldots, \mathcal{K} \\
& \quad T^+_k \geq \sum_{\ell=1}^{\mathcal{N}} \sum_{i=1}^{\mathcal{F}} PTDF_{t_\ell k} (s_{i\ell} - g_{i\ell}) \quad \text{for all } k = 1, \ldots, \mathcal{K}.
\end{align*}
\] (4.10)

Therefore, if \((g^*, s^*, y^*, w^*, p^*)\) is a Nash equilibrium, then \((g^*, s^*)\) is an optimal solution of (4.10). Conversely, if \((g^*, s^*)\) is an optimal solution of (4.10), then \((g^*, s^*, y^*, w^*, p^*)\) is a Nash equilibrium by defining, for all \(\ell = 1, \ldots, \mathcal{N}\),

- \(y^*_\ell = \sum_{i=1}^{\mathcal{F}} (s^*_{i\ell} - g^*_{i\ell})\);
- \(w^*_\ell = \sum_{k=1}^{\mathcal{K}} PTDF_{t_k} (\lambda^+ - \lambda^-)\);
- \(p^*_\ell\) as the Lagrange multiplier of the constraint \(\sum_{i=1}^{\mathcal{F}} s^*_{i\ell} - D_\ell = 0\).

**Proposition 4.1** (Existence of a perfectly competitive Nash equilibrium). Assume that the cost function \(c_{i\ell}(g_{i\ell})\) is convex for all \(i\) and \(\ell\). For the perfectly competitive power market game defined by (PCE_1)–(PCE_4), a necessary and sufficient condition for Nash
equilibrium existence is that there exists a sales and generation pair \((\hat{g}, \hat{s})\) such that

\[
\sum_{\ell=1}^{N} (\hat{g}_{i\ell} - \hat{s}_{i\ell}) = 0 \quad \text{for all } i = 1, \cdots, F
\]

\[
\sum_{i=1}^{F} \hat{s}_{i\ell} - D_{\ell} = 0 \quad \text{for all } \ell = 1, \cdots, N
\]

\[
\left\{ \begin{array}{l}
0 \leq \hat{g}_{i\ell} \leq \text{cap}_{i\ell} \\
\hat{s}_{i\ell} \geq 0
\end{array} \right. \quad \text{for all } i = 1, \cdots, F
\]

\[
\text{and } \ell = 1, \cdots, N
\]

\[
\left\{ \begin{array}{l}
T_{k}^{-} \leq \sum_{\ell=1}^{N} \text{PTDF}_{tk} \sum_{i=1}^{F} (\hat{s}_{i\ell} - \hat{g}_{i\ell}) \\
T_{k}^{+} \geq \sum_{\ell=1}^{N} \text{PTDF}_{tk} \sum_{i=1}^{F} (\hat{s}_{i\ell} - \hat{g}_{i\ell})
\end{array} \right. \quad \text{for all } k = 1, \cdots, K.
\]

**Proof.** Because (PCE$_1$)–(PCE$_4$) is equivalent to (4.10), the latter optimization problem can be analyzed. Namely, the feasible set of (4.10) is convex, compact, and nonempty (by the stated \((\hat{g}, \hat{s})\) assumption). Therefore, the convex optimization problem (4.10) has an optimal solution. \(\square\)

**Remark 4.1.** The existence of the point \((\hat{g}, \hat{s})\) simply assumes that there is a combination of player generation and sales decisions that satisfies transmission constraints and meets demand requirements. Thus, this assumption is very weak and can be taken as given for all functioning markets. \(\square\)

It is important to note that the integrability property of the perfectly competitive game allows for a compactness-based existence proof even though \(y, w,\) and \(p\) are not necessarily bounded in the formulation (PCE$_1$)–(PCE$_4$). This result is possible because the equivalent optimization problem (4.10) is solely in terms of \(g\) and \(s\), thus requiring compactness only
in these variables. Unfortunately, this type of problem equivalence does not hold for the other presented power market models so a different existence proof approach must be used.

To prove Nash equilibrium existence for the unified model with shared constraints, consider the problems (UE\textsuperscript{1} with shared constraints) and (UE\textsubscript{4}). Because a subdifferential version of Corollary 1.1 will be leveraged in the next proof, feasible region compactness must be established for the relevant VI formulation. Define the polyhedron

\[
K \triangleq \left\{ \begin{array}{l}
(g,s) | \quad \text{for all } i = 1, \ldots, \mathcal{F} \\
\sum_{\ell=1}^{N} (g_{i\ell} - s_{i\ell}) = 0 \\
\left\{ \begin{array}{l}
0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \\
0 \leq s_{i\ell} \leq 0
\end{array} \right. \\
\text{for all } \ell = 1, \ldots, N \\
\sum_{i=1}^{\mathcal{F}} s_{i\ell} \leq 3s_{\ell} \\
\sum_{i=1}^{\mathcal{F}} g_{i\ell} \geq Q_{\ell} \\
\text{for all } \ell = 1, \ldots, N \\
T_{k}^{-} \leq \sum_{\ell=1}^{N} \text{PTDF}_{tk} \sum_{i=1}^{\mathcal{F}} (s_{i\ell} - g_{i\ell}) \\
T_{k}^{+} \geq \sum_{\ell=1}^{N} \text{PTDF}_{tk} \sum_{i=1}^{\mathcal{F}} (s_{i\ell} - g_{i\ell})
\end{array} \right. \right\}, \quad (4.11)
\]

which is clearly bounded. It remains to be shown that the feasible region of \( p \) is bounded, a property that is not immediate for the unified model problem. Assume that \( q_{\ell}(p_{\ell}) \) is a continuous, strictly decreasing function of \( p_{\ell} \) and therefore has an inverse. Let constants
\( p_\ell \) and \( \bar{p}_\ell \) satisfy the inequality

\[-\infty < p_\ell < q_\ell^{-1}(\bar{S}_\ell) < q_\ell^{-1}(0) < \bar{p}_\ell < \infty.\]

With these price bounds in mind, consider a constrained version of the optimization problem in (UE4):

\[
\max_{\underline{p}_\ell \leq p_\ell \leq \bar{p}_\ell} \left[ \int_0^{\underline{p}_\ell} q_\ell(p_\ell') dp_\ell' - S_\ell^* p_\ell \right], \tag{4.12}
\]

It holds that (4.12) is equivalent to the optimization problem in (UE4) for all \( S_\ell \in [0, \bar{S}_\ell] \) (i.e., any optimal solution \( p_\ell^* \) of must lie strictly between the bounds \( \underline{p}_\ell \) and \( \bar{p}_\ell \)). Indeed, any optimal solution \( p_\ell^* \) of (4.12) must satisfy the following complementarity conditions for some multiplier \( \nu_\ell \):

\[
0 \leq p_\ell^* - \underline{p}_\ell \perp S_\ell^* - q_\ell(p_\ell^*) + \nu_\ell \geq 0
\]

\[
0 \leq \nu_\ell \perp \bar{p}_\ell - p_\ell^* \geq 0.
\]

- Assume that \( p_\ell^* = \underline{p}_\ell \). It holds that \( \nu_\ell = 0 \) and \( \bar{S}_\ell \geq S_\ell^* \geq q_\ell(p_\ell^*) \). This implies that \( p_\ell^* \geq q_\ell^{-1}(\bar{S}_\ell) \), a contradiction.

- Assume that \( p_\ell^* = \bar{p}_\ell \). It holds that \( S_\ell^* - q_\ell(p_\ell^*) + \nu_\ell = 0 \), implying that \( q_\ell(p_\ell^*) = S_\ell^* + \nu_\ell \geq S_\ell^* \geq 0 \). Therefore, \( p_\ell^* \leq q_\ell^{-1}(0) \), a contradiction.

Thus, a continuous and strictly decreasing \( q_\ell(p_\ell) \) allows for the restriction of the price feasible region to the bounded interval \([\underline{p}_\ell, \bar{p}_\ell]\) without loss of generality. As a consequence, the desired existence proof is straightforward.

**Proposition 4.2** (Existence of a unified Nash equilibrium). Assume that the cost function \( c_{i\ell}(g_{i\ell}) \) is convex for all \( i \) and \( \ell \). Furthermore, assume that the nodal demand function \( q_\ell(p_\ell) \) is a continuous and strictly decreasing function of \( p_\ell \) for all \( \ell \). If \( K \) as defined by (4.11) is nonempty and the function \( \rho_{i\ell}(s_{i\ell}; S_{-i\ell}, p_\ell)s_{i\ell} \) is concave in \( s_{i\ell} \) for each fixed \((S_{-i\ell}, p_\ell)\)
and all \(i\) and \(\ell\), then a Nash equilibrium exists for the power market game defined by (UE\(_1\) with shared constraints) and (UE\(_2\)–(UE\(_4\)).

**Proof.** To formulate a generalized VI \([75, \text{Section 1.6}]\) that is equivalent to the unified market model with shared constraints, define the set

\[
\hat{\mathcal{K}} \triangleq K \times \prod_{\ell=1}^{\mathcal{N}} \left[ \bar{p}_\ell, \overline{p}_\ell \right]
\]

and the set-valued map

\[
F(g, s, p) \triangleq \left( \left[ \partial_{g_{i\ell}} c_{i\ell}(g_{i\ell}) \right]_{i, \ell=1}^{\mathcal{F}, \mathcal{N}} \right. \left[- \partial_{s_{i\ell}} \left( \rho_{i\ell}(s_{i\ell}; S_{-i\ell}, p_\ell) s_{i\ell} \right) \right]_{i, \ell=1}^{\mathcal{F}, \mathcal{N}} \right)\]

\[
\left. \left[ S_\ell - q_\ell(p_\ell) \right]_{\ell=1}^{\mathcal{N}} \right).
\]

Consider the generalized variational inequality defined by the pair \((\hat{\mathcal{K}}, F)\); this problem attempts to identify a tuple \((g^*, s^*, p^*) \in \hat{\mathcal{K}\)} and subgradients \(a_{i\ell} \in \partial c_{i\ell}(g_{i\ell}^*)\) and \(b_{i\ell} \in -\partial s_{i\ell} (\rho_{i\ell}(s_{i\ell}^*; S_{-i\ell}^*, p_\ell^*) s_{i\ell}^*)\) such that for every \((g, s, p) \in \hat{\mathcal{K}}\),

\[
\sum_{i=1}^{\mathcal{F}} \sum_{\ell=1}^{\mathcal{N}} \left[ a_{i\ell}(g_{i\ell} - g_{i\ell}^*) + b_{i\ell}(s_{i\ell} - s_{i\ell}^*) \right] + \sum_{\ell=1}^{\mathcal{N}} (S_\ell^* - q_\ell(p_\ell^*)) (p_\ell - p_\ell^*) \geq 0.
\]

Because \(\hat{\mathcal{K}}\) is nonempty (by assumption), convex, and compact, existence follows from an adaptation of Corollary \([1.1]\).

**Remark 4.2.** It is simple to verify that the Cournot specification of the unified model fulfills the required assumptions of Proposition \([4.2]\). Depending on the specification of \(A_{-i\ell}\) for each \(i = 1, \cdots, \mathcal{F}\) and \(\ell = 1, \cdots, \mathcal{N}\), existence for the given CSF example from \([111]\) holds as well.
Uniqueness of equilibria

Equilibrium uniqueness is a desirable theoretical property for Nash games and typically requires stronger assumptions than those for solution existence. For instance, establishing solution existence for an optimization problem may only require objective function convexity while uniqueness commonly needs strong convexity (or solution existence and strict convexity). As was discussed in Section 1.3.2, monotone and $P$ functions play a similar role for VIs; thus, uniqueness for VIs still hinges on verifying certain matrix-theoretic properties. Two different uniqueness proofs are provided, one for the perfectly competitive market model and one for the unified model. This approach mirrors that used to establish equilibrium existence for the respective problems.

Based on the equivalence between (PCE$_1$)–(PCE$_4$) and (4.10), a unique solution $(g^*, s^*)$ of (4.10) corresponds to a unique solution $(g^*, s^*, y^*, w^*, p^*)$ of (PCE$_1$)–(PCE$_4$) if LICQ holds for $(g^*, s^*)$. However, it can easily be seen that the objective function of (4.10) cannot be made strictly convex in both $g$ and $s$. Therefore, solution uniqueness cannot be proven but a weaker statement based on $F$-uniqueness still holds.

**Proposition 4.3** (Partial uniqueness of the perfectly competitive Nash equilibrium). If the cost function $c_{i\ell}(g_{i\ell}) \in C^2$ has $\frac{d^2c_{i\ell}(g_{i\ell})}{dg_{i\ell}^2} > 0$ for all $i$ and $\ell$ and the feasibility condition of Proposition 4.1 is satisfied, then $g^*$ and the aggregate nodal sales values $\sum_{\ell=1}^{N} s_{i\ell}^*$ for each $i$ are unique for the perfectly competitive power market game defined by (PCE$_1$)–(PCE$_4$).

**Proof.** Formulating (4.10) as a primal VI,

$$F(g, s) \triangleq \begin{pmatrix} \frac{dc_{i\ell}(g_{i\ell})}{dg_{i\ell}} \bigg|_{i, \ell=1}^{F,N} \\ 0 \end{pmatrix}$$
\[ JF(g, s) \triangleq \begin{bmatrix} \text{diag} \left( \frac{d^2 c_{i\ell}(g_{i\ell})}{dg_{i\ell}^2} \right)_{i, \ell = 1}^{\mathcal{F}, \mathcal{N}} \left[ 0 \right] \end{bmatrix}. \]

By assumption, \( F \) fulfills the monotonicity requirement of Corollary 1.2, implying that \( \text{SOL}(K, F) \) is \( F \)-unique and therefore that \( g^* \) is unique. The uniqueness of \( \sum_{\ell=1}^{\mathcal{N}} s_{i\ell}^* \) follows immediately. \( \square \)

Because integrability does not hold for the unified market model, equilibrium uniqueness for \((\text{UE}_1 \text{ with shared constraints}) \) and \((\text{UE}_2)-(\text{UE}_4) \) must be proven via a different argument. Namely, strengthening the assumptions of Proposition 4.2 allows for the application of Propositions 1.5 and 1.6 to the primal VI formulation of the game. This result requires that the conjectured price function is simplified to depend on \((S_{\ell}, p_{\ell})\), meaning that it only depends on sales through the total nodal sales \( S_{\ell} \triangleq \sum_{i=1}^{\mathcal{F}} s_{i\ell} \); a uniqueness result for the more general conjectured price function of \((\text{UE}_1 \text{ with shared constraints}) \) and \((\text{UE}_2)-(\text{UE}_4) \) remains illusive.

**Proposition 4.4** (Uniqueness of the unified Nash equilibrium). Assume that

(A) \( \hat{K} \) as defined in the proof of Proposition 4.2 is nonempty;

(B) LICQ holds for \( \hat{K} \) at the Nash equilibrium point;

(C) for each pair \((i, \ell)\), the function \( c_{i\ell}(g_{i\ell}) \in C^2 \) with \( \frac{d^2 c_{i\ell}(g_{i\ell})}{dg_{i\ell}^2} > 0 \);

(D) for each \( \ell \), the function \( q_{\ell}(p_{\ell}) \in C^1 \) with \( \frac{dq_{\ell}(p_{\ell})}{dp_{\ell}} < 0 \);

(E) for each pair \((i, \ell)\), the conjectured price function depends on \((S_{\ell}, p_{\ell})\)

\[ (\text{i.e., } \rho_{i\ell}(s_{i\ell}, S_{-i\ell}, p_{\ell}) \triangleq \rho_{i\ell}(S_{\ell}, p_{\ell})) \]

with the following properties for all \((S_{\ell}, p_{\ell}) \in \left[ 0, \sum_{i=1}^{\mathcal{F}} \sum_{\ell=1}^{\mathcal{N}} \text{cap}_{i\ell} \right] \times \left[ \underline{P}_{\ell}, \overline{P}_{\ell} \right] \):
1. \( \frac{\partial p_{it}(S_t, p_t)}{\partial S_t} < 0; \)
2. \( \frac{\partial^2 p_{it}(S_t, p_t)}{\partial S_t^2} \leq 0; \)
3. \( \frac{\partial p_{it}(S_t, p_t)}{\partial p_t} \geq 0; \)
4. \( \frac{\partial^2 p_{it}(S_t, p_t)}{\partial p_t \partial S_t} \geq 0. \)

The Nash equilibrium point of the unified power market game defined by (UE\(_1\) with shared constraints) and (UE\(_2\))–(UE\(_4\)) is unique.

**Proof.** Given the differentiability assumptions here, the \( F \) mapping of the primal VI is defined as

\[
F(g, s, p) \triangleq \left( \begin{array}{c}
\left[ \frac{dc_{it}(g_{it})}{dg_{it}} \right]_{i, t = 1}^{\mathcal{F}, \mathcal{N}} \\
\left[ -\rho_{it}(S_t, p_t) - s_{it} \frac{\partial \rho_{it}(S_t, p_t)}{\partial S_t} \right]_{i, t = 1}^{\mathcal{F}, \mathcal{N}} \\
\left[ S_t - q_t(p_t) \right]_{t = 1}^{\mathcal{N}}
\end{array} \right),
\]

although the more general Clarke subdifferentials of Proposition 4.2 are also technically correct. Therefore, the game of (UE\(_1\) with shared constraints) and (UE\(_2\))–(UE\(_4\)) corresponds to \( \text{VI}(\hat{K}, F) \), where \( \hat{K} \) is defined in the proof of Proposition 4.2. It is claimed that \( F \) is a \( P \) function by Proposition 1.5. For notational simplicity, define the following values (suppressing dependence on \((s_{it}; S_{-it}, p_t)):\n
- \( i_{it} \triangleq -\frac{\partial p_{it}(S_t, p_t)}{\partial S_t} - s_{it} \frac{\partial^2 \rho_{it}(S_t, p_t)}{\partial S_t^2}; \)
- \( \xi_{it} \triangleq -\frac{\partial p_{it}(S_t, p_t)}{\partial S_t}; \)
- \( \upsilon_{it} \triangleq -\frac{\partial p_{it}(S_t, p_t)}{\partial p_t}; \)
- \( \eta_{it} \triangleq -s_{it} \frac{\partial^2 \rho_{it}(S_t, p_t)}{\partial p_t \partial S_t}; \)
\[ \theta_\ell \triangleq -1 / \left( \frac{dq_\ell(p_\ell)}{dp_\ell} \right). \]

By assumption, \( \nu_{i\ell}, \xi_{i\ell}, \theta_\ell > 0 \) and \( \nu_{i\ell}, \eta_{i\ell} \leq 0 \) for all \( i \) and \( \ell \).

Because \( JF(g,s,p) \) can be arranged to have a block diagonal structure according to the nodal index \( \ell \), it suffices to examine

\[
J_{z\ell}F_\ell(z_\ell) \triangleq \begin{bmatrix}
\text{diag} \left( \left( \frac{d^2 c_{i\ell}(g_{i\ell})}{dg_{i\ell}^2} \right)_{i=1}^F \right) \\
\tilde{P}_\ell & a_\ell \\
1^T & 1/\theta_\ell
\end{bmatrix},
\]

where \( z_\ell \triangleq (g_\ell,s_\ell,p_\ell) \), \( 1^T \) is a unit row vector,

\[
\tilde{P}_\ell \triangleq \begin{bmatrix}
\nu_{1\ell} + \xi_{1\ell} & \nu_{1\ell} & \nu_{1\ell} & \cdots \\
\nu_{2\ell} & \nu_{2\ell} + \xi_{2\ell} & \nu_{2\ell} & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\nu_{F\ell} & \nu_{F\ell} & \cdots & \nu_{F\ell} + \xi_{F\ell}
\end{bmatrix}, \quad \text{and} \quad a_\ell \triangleq \begin{bmatrix}
\nu_{1\ell} + \eta_{1\ell} \\
\nu_{2\ell} + \eta_{2\ell} \\
\vdots \\
\nu_{F\ell} + \eta_{F\ell}
\end{bmatrix}.
\]

To apply Proposition 1.5(a) to \( J_{z\ell}F_\ell(z_\ell) \), a rectangular feasible region is required. Unfortunately, this formulation has a polyhedral but not necessarily rectangular \( \tilde{K} \). Luckily, the rectangular region

\[
\tilde{K} \triangleq \prod_{i=1}^F \prod_{\ell=1}^N \left( [0, \text{cap}_{i\ell}] \times \left[ 0, \sum_{i=1}^F \sum_{\ell=1}^N \text{cap}_{i\ell} \right] \right) \times \sum_{\ell=1}^N \left[ \mathbf{p}_\ell, \mathbf{p}_\ell \right] \supset \tilde{K}
\]

and its \( \ell \)th partition \( \tilde{K}_\ell \) can be used to prove that \( F \) is a \( P \) function on \( \tilde{K} \); this statement follows from Definition 1.5(a) in that \( P \) functions on a Cartesian space are \( P \) functions on a Cartesian subset of that space.
By assumption, the upper diagonal block of $J_{z\ell}F_{\ell}(z\ell)$ is a $P$ matrix. Therefore, it suffices to verify that all principal submatrices of the lower diagonal block are also $P$ matrices. By assumption, $1/\theta_{\ell} > 0$ so, without loss of generality, only $\tilde{P}_\ell$ and the entire lower diagonal block need to be evaluated.

Consider $\tilde{P}_\ell$ first. It holds that

$$\det(\tilde{P}_\ell) = \det\begin{bmatrix} \sum_{i=1}^{F} \nu_{i\ell} + \xi_{1\ell} & \xi_{2\ell} - \xi_{1\ell} & \xi_{3\ell} - \xi_{1\ell} & \cdots \\ \nu_{2\ell} & \xi_{2\ell} & & \\ \vdots & \ddots & & \\ \nu_{F\ell} & & \cdots & \xi_{F\ell} \end{bmatrix}$$

$$= \det\begin{bmatrix} \nu_{1\ell} + \xi_{1\ell} & -\xi_{1\ell} & \cdots & -\xi_{1\ell} \\ \nu_{2\ell} & \xi_{2\ell} & & \\ \vdots & \ddots & & \\ \nu_{F\ell} & & \cdots & \xi_{F\ell} \end{bmatrix}$$

$$= \prod_{i=2}^{F} \xi_{i\ell} \det \left( \nu_{1\ell} + \xi_{1\ell} \left( 1 + \frac{\nu_{2\ell}}{\xi_{2\ell}} + \frac{\nu_{3\ell}}{\xi_{3\ell}} + \cdots + \frac{\nu_{F\ell}}{\xi_{F\ell}} \right) \right),$$

where the first equality follows from adding rows $2$–$F$ to the first row and subsequently subtracting the first column from the other columns, the second equality follows from subtracting rows $2$–$F$ from row 1, and the final equality follows from the block determinant formula

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(D) \det(A - BD^{-1}C)$$

when $D$ is invertible. From the stated assumptions, $\det(\tilde{P}_\ell) > 0$ as desired.
Using a similar approach, it can be concluded that

\[
\det \begin{bmatrix}
\hat{P}_\ell & a_\ell \\
1^T & 1/\theta_\ell
\end{bmatrix} = \det (1/\theta_\ell) \det \left( \hat{P}_\ell - a_\ell \theta_\ell 1^T \right) > 0
\]

because

\[
\det \left( \hat{P}_\ell - a_\ell \theta_\ell 1^T \right) = \det \begin{bmatrix}
\nu_1 + \xi_1 - \theta(v_1 + \eta_1) & -\xi_1 & \cdots & -\xi_1 \\
\nu_2 - \theta(v_2 + \eta_2) & \xi_2 \\
\vdots & \ddots & \ddots \\
\nu_F - \theta(v_F + \eta_F) & \xi_F
\end{bmatrix}
\]

\[
= \left( \prod_{i=2}^F \xi_i \right) \det \left( \nu_1 - \theta(v_1 + \eta_1) + \xi_1 \left( 1 + \frac{\nu_2 - \theta(v_2 + \eta_2)}{\xi_2} + \cdots + \frac{\nu_F - \theta(v_F + \eta_F)}{\xi_F} \right) \right) > 0.
\]

Therefore, \( J_{z}\ell F_\ell (z_\ell) \) is a \( P \) matrix on \( \tilde{K}_\ell \) so \( F \) is a \( P \) function on \( \tilde{K} \). The existence of a unique solution \( (g^*, s^*, p^*) \) follows from Proposition 4.6 and Proposition 4.2. Uniqueness of the solution \( (g^*, s^*, y^*, w^*, p^*) \) of (UE1 with shared constraints) and (UE2)–(UE4) follows from LICQ holding and setting \( y^* = \sum_{i=1}^{\mathcal{F}} (s_{i\ell}^* - g_{i\ell}^*) \) and \( w_{\ell}^* = \sum_{k=1}^{K} \mathrm{PTDF}_{\ell k}(\lambda_k^+ - \lambda_k^-) \).

\[ \square \]

**Remark 4.3.** The assumptions needed for Proposition 4.4 are rather general. Similar to the perfectly competitive case of Proposition 4.3, cost function convexity is strengthened to strict convexity by Assumption \( (C) \). Furthermore, Assumptions \( (E)1–3 \) require that conjectured prices decrease as total sales increase, are concave in total sales, and increase with equilibrium price, respectively. These properties should not be unexpected for conjectured prices. Assumption \( (E)4 \) is required but unfortunately does not have a simple interpreta-

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tion; in some sense, it can be seen as conveying a relative difference between conjectured price function changes caused by total sales versus equilibrium price.

Remark 4.4. It is simple to see that, given the appropriate cost and nodal demand function properties, a wide variety of power market games have unique solutions by Proposition 4.4. Namely, any Cournot game formulated as (UE$_1$ with shared constraints) and (UE$_2$)–(UE$_4$) with pricing functions depending solely on total sales has a unique solution if LICQ holds and the pricing functions are strictly decreasing and concave. The traditional linear inverse demand function model obviously fits this framework. Depending on how prices in CSF models are specified, these models may also have unique solutions by Proposition 4.4.

4.2.3 Computation

Given the existence and uniqueness results proven in Section 4.2.2, the next obvious question for this unified power market model becomes computational: how can equilibria be found when they are known to exist? Much like solution methods for optimization problems, VI solution algorithms are iterative and perform selected computations until a solution is found with a desired level of accuracy. For the problems examined here, projection- and Newton-type methods are proven to converge under mild assumptions. To simplify the following analysis, it is assumed that shared constraints are not present and

- for (PCE$_1$)–(PCE$_4$), the assumptions of Proposition 4.1 hold with $c_{i\ell}(g_{i\ell}) \in C^1$;
- for (UE$_1$)–(UE$_4$) specialized to the Cournot setting, the assumptions of Proposition 4.2 hold with $c_{i\ell}(g_{i\ell}) \in C^1$;
• for (UE1)–(UE4), the assumptions of Proposition 4.4 hold.

Therefore, the perfectly competitive and the Cournot market formulations are guaranteed to have solution existence and differentiability of the defining map $F$. The assumptions required for existence can obviously be strengthened to give uniqueness, but this simplification is avoided so that algorithms capable of identifying nonunique VI solutions can be discussed. For the more general unified model, solution uniqueness is assumed so that potential convergence difficulties arising from multiple equilibria are avoided.

For both the projection- and Newton-type solution algorithms, the following necessary and sufficient conditions for a solution to $VI(K, F)$ are important. The notation $\Pi_K(x)$ denotes the projection of point $x$ onto the set $K$; with a polyhedral $K$, this operation can be formulated as the strictly convex optimization problem

$$\Pi_K(x) \equiv \arg\min_{y \in K} \frac{1}{2} (y - x)^T (y - x).$$

**Proposition 4.5** (Propositions 1.5.8 and 1.5.9 of [75]). Let $K \subseteq \mathbb{R}^n$ be a closed convex set and the map $F : K \to \mathbb{R}^n$ be arbitrary. A vector $x \in SOL(K, F)$ if and only if

(a) $F^\text{nat}_K(x) = 0$,

where $F^\text{nat}_K(x) \triangleq x - \Pi_K(x - F(x))$ (the natural map of $VI(K, F)$);

(b) there exists a vector $z$ such that $x = \Pi_K(z)$ and $F^\text{nor}_K(z) = 0$,

where $F^\text{nor}_K(z) \triangleq F(\Pi_K(z)) + z - \Pi_K(z)$ (the normal map of $VI(K, F)$). □

From this proposition, it can be seen that solving $VI(K, F)$ is equivalent to solving a set of nonsmooth equations. This relationship is fundamental to many VI solution methods because it (a) provides a tractable scheme for solution verification, and (b) suggests a
variety of different algorithms based on nonlinear equations.

Note that, by definition, using the normal map to solve \( VI(K, F) \) is equivalent to using the natural map except for the fact that the maps deal with different [but related] variables. From a computational standpoint, the normal map is preferable because its domain of definition is \( \mathbb{R}^n \); the domain of definition for the natural map is the potentially more complicated region \( K \subseteq \mathbb{R}^n \).

**Projection-based algorithms**

It has been noted that \( (PCE_1)–(PCE_4) \) is equivalent to the convex optimization problem (4.10) by integrability. Similarly, because \( (UE_4) \) is not technically needed in the Cournot version of the unified model [it is redundant because of the \( \rho_{i\ell} \) specification], the Cournot version of \( (UE_1)–(UE_3) \) is seen to be equivalent to the convex optimization problem

\[
\begin{align*}
\text{maximize}_{g, s} & \quad \sum_{i=1}^{\mathcal{F}} \sum_{\ell=1}^{\mathcal{N}} \left[ \left( \sum_{i=1}^{\mathcal{F}} \frac{P^0_i}{Q^0_\ell} \sum_{i=1}^{\mathcal{F}} s_{i\ell} \right) s_{i\ell} - c_{i\ell}(g_{i\ell}) \right] \\
\text{subject to} & \quad \sum_{\ell=1}^{\mathcal{N}} (g_{i\ell} - s_{i\ell}) = 0 \quad \text{for all } i = 1, \cdots, \mathcal{F} \\
 & \quad \begin{cases} 
 0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \\
 0 \leq s_{i\ell} \leq c_{i\ell}(g_{i\ell}) \\
 0 \leq s_{i\ell} \leq 0 
\end{cases} \quad \text{for all } i = 1, \cdots, \mathcal{F} \quad \text{and } \ell = 1, \cdots, \mathcal{N} \\
\end{align*}
\]

\[
\begin{align*}
\left\{ T^-_k \leq \sum_{\ell=1}^{\mathcal{N}} \text{PDTDF}_{\ell k} \sum_{i=1}^{\mathcal{F}} (s_{i\ell} - g_{i\ell}) \right\} & \quad \text{for all } k = 1, \cdots, \mathcal{K}. \\
\left\{ T^+_k \geq \sum_{\ell=1}^{\mathcal{N}} \text{PDTDF}_{\ell k} \sum_{i=1}^{\mathcal{F}} (s_{i\ell} - g_{i\ell}) \right\} & \quad \text{for all } k = 1, \cdots, \mathcal{K}. 
\end{align*}
\]
A wide variety of optimization solution methods are applicable for finding equilibria of these two models but will not be discussed here. Instead, VI-specific solution methods for these problems are provided. The interested reader is referred to [173] for a detailed presentation of state-of-the-art optimization algorithms.

**Basic projection method.** Much as the name suggests, the basic projection algorithm attempts to find a zero of $F_{nat}^K$ via a simple iterative approach.

<table>
<thead>
<tr>
<th>Basic Projection Algorithm [75, Algorithm 12.1.1].</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $\tau &gt; 0$, $x^0 \in K$, and $k = 0$.</td>
</tr>
<tr>
<td><strong>Step 1:</strong> If $x^k = \Pi_K(x^k - \tau^{-1}F(x^k))$, stop.</td>
</tr>
<tr>
<td><strong>Step 2:</strong> Set $x^{k+1} = \Pi_K(x^k - \tau^{-1}F(x^k))$, $k \leftarrow k + 1$, and return to Step 1.</td>
</tr>
</tbody>
</table>

This algorithm cannot be expected to converge for every mapping $F$ and all $\tau > 0$. As such, convergence is typically guaranteed by assuming, among other things, strong monotonicity and Lipschitz continuity of $F$. Unfortunately, it can be proven that neither the perfectly competitive model nor the Cournot model satisfy strong monotonicity. Luckily, a much weaker condition is required for convergence if $F$ is a gradient map of a convex function.

**Theorem 4.1** (Exercise 12.8.1, [75]). Let $K \subseteq \mathbb{R}^n$ be a closed convex set, $F$ be a Lipschitz continuous map on $K$ with Lipschitz constant $L$, and $\text{SOL}(K, F) \neq \emptyset$. If $F$ is the gradient map of a convex function and $\tau > L/2$, then every sequence generated by the basic projection algorithm converges to some $x^* \in \text{SOL}(K, F)$. 

By the representation of the perfectly competitive model as (4.10), it is obvious that

$$F(g, s) \triangleq \begin{pmatrix} \frac{dc_{it}(g_{it})}{dg_{it}} & \frac{F, N}{i, t = 1} \\ 0 \end{pmatrix}$$
is a gradient map. Let $K$ be the feasible set of (4.10). It only remains to be shown that $F$ is Lipschitz continuous. If $c_{i\ell}(g_{i\ell}) \in C^1$ on $\mathbb{R}$, $F$ is obviously continuous and Lipschitz continuity follows from the compactness of $K$. A similar development can be performed for the Cournot model of (UE$_1$)–(UE$_3$) represented as (4.13).

**Proposition 4.6.** The following statements hold.

(a) Given the assumptions of Proposition 4.1 and $c_{i\ell}(g_{i\ell}) \in C^1$, the basic projection algorithm with a sufficiently large $\tau > 0$ converges to a solution of (PCE$_1$)–(PCE$_4$).

(b) Given $c_{i\ell}(g_{i\ell}) \in C^1$, the basic projection algorithm with a sufficiently large $\tau > 0$ converges to a solution for the Cournot version of (UE$_1$)–(UE$_3$). □

As mentioned previously, there are several extensions of the basic projection method that lead to better theoretical properties. The extragradient method [75, Algorithm 12.1.9] utilizes two projections per iteration to solve variational inequalities that only satisfy pseudo-monotonicity, the weakest form of monotonicity. The hyperplane projection algorithm [75, Algorithm 12.1.12] combines projection iterations with a line search scheme to apply the method to variational inequalities lacking a Lipschitz continuous map $F$. Instead of discussing these extensions of projection-based methods, an algorithm that identifies a specific type of solution for problems lacking solution uniqueness is presented next.

**Tikhonov regularization.** The concept of regularization was briefly introduced in Section 3.7 and solves $\text{VI}(K, F)$ through a sequence of well-behaved, perturbed problems. For Tikhonov regularization, the perturbation $\text{VI}(K, F + \epsilon I)$ is analyzed where $\epsilon > 0$ and, with slight abuse of notation, $\epsilon I$ is an identity matrix with diagonal entries $\epsilon$. Originally developed for variational inequalities with monotone $F$, the Tikhonov regularization method has been proven to converge for $P_0$ mappings $F$ under certain conditions. In these situations,
solution existence guarantees that SOL\((K, F + \epsilon I)\) is unique because Tikhonov-regularized monotone \((P_0)\) maps are strictly monotone \((P)\) (see Theorem 1.4 and Proposition 1.6).

As \(\epsilon \to 0\), it is obvious that VI\((K, F + \epsilon I)\) converges to VI\((K, F)\). Letting \(x(\epsilon)\) be the unique solution of VI\((K, F + \epsilon I)\), the question of method convergence becomes proving the existence of \(\lim_{\epsilon \to 0} x(\epsilon)\). For monotone \(F\), the following result provides a straightforward condition for this existence.

**Theorem 4.2 (Theorem 12.2.3, [75])**. Let \(K \subseteq \mathbb{R}^n\) be a closed convex set, \(F : K \to \mathbb{R}^n\) be continuous and monotone on \(K\), and \(x(\epsilon) \in \text{SOL}(K, F + \epsilon I)\). If \(\text{SOL}(K, F) \neq \emptyset\), then \(\lim_{\epsilon \to 0} x(\epsilon) = x^\infty\) where \(x^\infty\) is the unique least Euclidean-norm solution of VI\((K, F)\).

This theorem proves a very special convergence property for the Tikhonov solution trajectory for monotone variational inequalities: it will identify the least-norm solution of the original VI\((K, F)\).

**Tikhonov Regularization Algorithm [75, Algorithm 12.2.9]**.

Let \(x^0 \in K\), \(k = 0\), and the sequence \(\{\epsilon_k\} \to 0\) where \(\epsilon_k > 0\) for all \(k\).

**Step 1**: If \(x^k \in \text{SOL}(K, F)\), stop.

**Step 2**: Let \(x^{k+1}\) be the solution of VI\((K, F + \epsilon_k I)\).

**Step 3**: Set \(k \leftarrow k + 1\) and return to Step 1.

To solve for the solution of VI\((K, F + \epsilon_k I)\), the basic projection algorithm or its improvements can be utilized when \(c_{i\ell}(g_{i\ell}) \in C^1\).

It is simple to see that the primal variational inequalities arising from (4.10) and (4.13) have continuous and monotone \(F\) maps when \(c_{i\ell}(g_{i\ell}) \in C^2\). Therefore, the Tikhonov regularization algorithm can be applied as stated in the following proposition.
Proposition 4.7. The following statements hold.

(a) Given the assumptions of Proposition 4.1 and \( c_i \ell (g_i \ell) \in C^2 \), the Tikhonov regulariza-
tion algorithm converges to the least-norm \((g, s)\) solution of \((\text{PCE}_1)–(\text{PCE}_4)\).

(b) Given the convex function \( c_i \ell (g_i \ell) \in C^2 \), the Tikhonov regularization algorithm con-
verges to the least-norm \((g, s)\) solution of the Cournot version of \((\text{UE}_1)–(\text{UE}_3)\).

Although the Tikhonov algorithm applied to the primal VI formulations of (4.10) and (4.13) is guaranteed to converge, the size of these variational inequalities for power market models can be prohibitive. Therefore, a distributed regularization approach may be used in lieu of a single Tikhonov application. In such a distributed computation scheme, the VI is separated into smaller, more manageable subproblems that are each solved with a Tikhonov regularization approach. Under certain conditions, the distributed scheme is guaranteed to converge to the least-norm solution that would have been obtained by implementing the non-distributed Tikhonov method. For Nash games, individual players are an obvious means of problem separation.

Drawn from [121], the presented distributed Tikhonov method allows each subproblem to be solved with independently chosen regularization parameters. Therefore, given a starting point, the next iterate can be computed in a parallel manner; the only information that must be shared between subproblems is the previous iterate value. To make the method precise, define the whole variational inequality as the Cartesian product of \( N \) subproblems:

\[
\text{VI}(K, F) \text{ where } K \triangleq \prod_{\nu = 1}^N K_{\nu} \text{ and } F(x) \triangleq \left(F_{\nu}(x)\right)_{\nu = 1}^N (4.14)
\]

with \( K_{\nu} \subseteq \mathbb{R}^n \) being closed and convex.
Iterative Tikhonov Regularization Algorithm [121].

For each $\nu = 1, \cdots, N$, let $x_0^\nu \in K_\nu$, $\gamma^k_\nu > 0$ for all $k$, $\epsilon^k_\nu > 0$ for all $k$, and the sequence $\{\epsilon^k_\nu\} \to 0$. Let $k = 0$.

Step 1: If $x_0^\nu \in \text{SOL}(K_\nu, F_\nu)$ for all $\nu$, stop.

Step 2: Set $x_{k+1}^\nu = \prod_{K_\nu}(x_k^\nu - \gamma^k_\nu(F_\nu(x_k) + \epsilon^k_\nu x_k))$ for each $\nu$.

Step 3: Set $k \leftarrow k + 1$ and return to Step 1.

It can be seen that this algorithm falls somewhere between a pure Tikhonov method and a basic projection method. Instead of solving VI($K_\nu, F_\nu + \epsilon^k_\nu I_\nu$) (where $I_\nu$ is an appropriately dimensioned identity matrix) in Step 2 as in the Tikhonov regularization algorithm, a projection step is taken. Thus, Step 2 cannot be claimed to solve the regularized subproblem. However, Step 2 does not correspond to the basic projection algorithm either because the true $F_\nu$ is regularized by $\epsilon^k_\nu I_\nu$.

Theorem 4.3 (Theorem 2.4, [121]). Let VI($K, F$) be defined as (4.14). Assume that the following conditions hold:

(a) $F$ is monotone and Lipschitz continuous with Lipschitz constant $L$;

(b) SOL($K, F$) is nonempty and bounded;

(c) The sequences $\{\gamma^k_\nu\}$ and $\{\epsilon^k_\nu\}$ satisfy, for all $\nu = 1, \cdots, N$,

1. $\sum_{k=1}^{\infty} \gamma^k_\nu \epsilon^k_\nu = \infty$;

2. $\lim_{k \to \infty} \frac{\max_{\nu} \gamma^k_\nu}{\min_{\nu} \gamma^k_\nu (\min_{\nu} \epsilon^k_\nu)} = 0$;

3. $\sum_{k=1}^{\infty} (\gamma^k_\nu)^2 < \infty$;

4. $\sum_{k=1}^{\infty} (\gamma^k_\nu \epsilon^k_\nu)^2 < \infty$;

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5. \[ \lim_{k \to \infty} \frac{\max_{\nu} \epsilon^k_{\nu} - \min_{\nu} \epsilon^k_{\nu}}{(\min_{\nu} \gamma^k_{\nu})^2(\min_{\nu} \epsilon^k_{\nu})} = 0; \]

6. \[ \lim_{k \to \infty} \frac{\max_{\nu} \gamma^k_{\nu} \max_{\nu} \epsilon^k_{\nu} - \min_{\nu} \gamma^k_{\nu} \min_{\nu} \epsilon^k_{\nu}}{(\min_{\nu} \gamma^k_{\nu})(\min_{\nu} \epsilon^k_{\nu})} = 0; \]

7. \[ \lim_{k \to \infty} \epsilon^k_{\nu} = 0 \text{ for all } \nu = 1, \ldots, N. \]

The iterative Tikhonov regularization sequence \( \{x^k\} \triangleq \{(x^k_{\nu})_{\nu=1}^N\} \) converges to \( x^* \), the least-norm solution of \( \text{VI}(K,F) \).

To illustrate how this method can be applied, its implementation for the Cournot version of \((\text{UE}_1)–(\text{UE}_3)\) is presented next. It can also be applied to \((\text{PCE}_1)–(\text{PCE}_4)\), but solution set boundedness is required for convergence. In general, this boundedness is difficult to prove through results such as Proposition 1.3 assuming solution uniqueness is a much easier approach. With this in mind, the final convergence result is presented given the assumptions required for solution uniqueness although only solution set boundedness is technically required.

Because of the need for readily available subproblems, the iterative Tikhonov regularization algorithm is naturally applied to the primal VI formulation of the Cournot \((\text{UE}_1)–(\text{UE}_3)\) (the equivalent \((4.13)\) does not allow for this type of problem division). Define the generator, ISO, and market clearing feasible regions as

\[
K^\text{Gen}_i \triangleq \left\{ (g_i, s_i) \mid \sum_{\ell=1}^N (g_{i\ell} - s_{i\ell}) = 0, \right. \\
0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \quad \text{for all } \ell = 1, \ldots, N, \\
\left. s_{i\ell} \geq 0 \quad \text{for all } \ell = 1, \ldots, N \right\},
\]

\[
K^\text{ISO} \triangleq \left\{ y \mid T_k^- \leq \sum_{\ell=1}^N \text{PTDF}_{k\ell} y_{\ell} \leq T_k^+ \quad \text{for all } \ell = 1, \ldots, N \right\}, \text{ and } K^\text{MC} \triangleq \mathbb{R}^N,
\]

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respectively. Let the generator, ISO, and market clearing mappings be

\[
F_{\text{Gen}}^i(g, s, y, w) \triangleq \left( \begin{array}{c}
\left( \frac{dc_{i\ell} - w_{\ell}}{g_{i\ell}} \right)_{\ell=1}^N \\
\left( -P_0^\ell + \frac{p_0^\ell}{Q_0^\ell} \sum_{i=1}^F s_{i\ell} + \frac{p_0^\ell}{Q_0^\ell} s_{i\ell} + w_{\ell} \right)_{\ell=1}^N 
\end{array} \right)
\]

\[
F_{\text{ISO}}(g, s, y, w) \triangleq (-w_{\ell})_{\ell=1}^N , \quad \text{and} \quad F_{\text{MC}}^i(g, s, y, w) \triangleq \left( y_{\ell} - \sum_{i=1}^F (s_{i\ell} - g_{i\ell}) \right)_{\ell=1}^N .
\]

Define \(z_i = (g_i, s_i)\), \(\{\gamma_{i, k}^{\text{Gen}}\}\) and \(\{\epsilon_{i, k}^{\text{Gen}}\}\) as generator \(i\)'s parameter sequences, \(\{\gamma_{i, k}^{\text{ISO}}\}\) and \(\{\epsilon_{i, k}^{\text{ISO}}\}\) as the ISO's parameter sequences, and \(\{\gamma_{i, k}^{\text{MC}}\}\) and \(\{\epsilon_{i, k}^{\text{MC}}\}\) as the market clearing player’s parameter sequences. A step of the iterative Tikhonov regularization algorithm performs \(F + 2\) independent calculations:

\[
\begin{aligned}
z_{i+1}^k &= \Pi_{K_{\text{Gen}}}^i \left( z_i^k - \gamma_{i, k}^{\text{Gen}} \left( F_{i}^\text{Gen} \left( z_i^k, y_i^k, w_i^k \right) + \epsilon_{i, k}^{\text{Gen}} z_i^k \right) \right) \quad \text{for all } i = 1, \cdots, F \\
y_{i+1}^k &= \Pi_{K_{\text{ISO}}} y_i^k - \gamma_{i, k}^{\text{ISO}} \left( F_{i}^\text{ISO} \left( z_i^k, y_i^k, w_i^k \right) + \epsilon_{i, k}^{\text{ISO}} y_i^k \right) \\
w_{i+1}^k &= \Pi_{K_{\text{MC}}} w_i^k - \gamma_{i, k}^{\text{MC}} \left( F_{i}^\text{MC} \left( z_i^k, y_i^k, w_i^k \right) + \epsilon_{i, k}^{\text{MC}} w_i^k \right).
\end{aligned}
\]  

Define

\[
K \triangleq \prod_{i=1}^F K_{\text{Gen}}^i \times K_{\text{ISO}} \times K_{\text{MC}}
\]

and

\[
F(g, s, y, w) \triangleq \left( \begin{array}{c}
\left( \frac{dc_{i\ell}(g_{i\ell}) - w_{\ell}}{d g_{i\ell}} \right)_{i=1}^F \\
\left( -P_0^\ell + \frac{p_0^\ell}{Q_0^\ell} \sum_{i=1}^F s_{i\ell} + \frac{p_0^\ell}{Q_0^\ell} s_{i\ell} + w_{\ell} \right)_{i=1}^F \\
-w_{\ell} \\
y_{\ell} - \sum_{i=1}^F (s_{i\ell} - g_{i\ell}) 
\end{array} \right)_{\ell=1}^N .
\]
Proposition 4.8. Assume that the following conditions hold:

1. The assumptions of Proposition 4.4.
2. For all \( i \) and \( \ell \), the function \( c_{i\ell}(g_{i\ell}) \in C^2 \) has a bounded, positive second derivative \( \frac{d^2 c_{i\ell}(g_{i\ell})}{dg_{i\ell}^2} \);
3. Assumption (c) of Theorem 4.3 holds.

The iterative Tikhonov regularization algorithm described by (4.15) converges to the unique solution of the Cournot version of (UE\(_1\))–(UE\(_3\)).

Proof. Given that properties of \( c_{i\ell}(g_{i\ell}) \), it can easily be verified that \( F \) is Lipschitz continuous and monotone on \( K \). Thus, assumption (a) of Theorem 4.3 holds. From the convex, compact, and nonempty feasible region of (4.13), a solution is known to exist. The equilibrium is unique [and therefore bounded] by Proposition 4.4. Theorem 4.3 provides the desired result. 

Remark 4.5. The identification of sequences \( \{\gamma_{i,\text{Gen},k}\}, \{\gamma_{i,\text{ISO},k}\}, \{\gamma_{i,\text{MC},k}\}, \{\epsilon_{i,\text{Gen},k}\}, \{\epsilon_{i,\text{ISO},k}\}, \{\epsilon_{i,\text{MC},k}\} \) that satisfy assumption (c) of Theorem 4.3 is straightforward. See Lemma 2.5 of [121] for the specification of a valid set of these sequences generated from random, player-specific positive integers.

As was seen in this development, the shared constraint version of (UE\(_1\)) was not included in this analysis. This simplification originated in the different model specifications required when solving for generalized versus variational Nash equilibria (see Chapter 3). In its presented formulation, the unified model with shared constraints in this chapter is a GNEP because no requirements are imposed on the Lagrange multipliers of the shared
constraints. Therefore, generator problems with shared constraints cannot be easily separated, implying that the iterative Tikhonov algorithm would likely not realize the benefits of parallel computation because the Cournot model could only be decomposed into 3 (as opposed to $F + 2$) subproblems. If the game is slightly adapted to solve for a variational equilibrium by requiring shared constraint multipliers to be identical between players, the game can successfully be divided into $F + 4$ subproblems after exogenously imposing the shared constraints

$$\sum_{i=1}^{F} s_{i\ell} \leq S_{\ell} \quad \text{and} \quad \sum_{i=1}^{F} g_{i\ell} \geq Q_{\ell} \quad \text{for all } \ell = 1, \cdots, N$$

as the pricing optimization problems

$$\min_{p^S \geq 0} \sum_{\ell=1}^{N} p_{\ell}^S \left( S_{\ell} - \sum_{i=1}^{F} s_{i\ell} \right) \quad \text{and} \quad \min_{p^Q \geq 0} \sum_{\ell=1}^{N} p_{\ell}^Q \left( \sum_{i=1}^{F} g_{i\ell} - Q_{\ell} \right)$$

and slightly altering the generator objective functions. It can be verified that this variational equilibrium version of the game satisfies the convergence requirements of the iterative Tikhonov algorithm.

**Newton-based algorithms**

For $(UE_1$ with shared constraints) and $(UE_2)$–$(UE_4)$, Newton-type solution methods can be utilized for identifying solutions. In essence, these methods leverage the first-order Taylor expansion of the mapping $F$ to solve a set of nonlinear equations. Similar to the traditional Newton’s method implementation for nonlinear equations (see [173, Algorithm 11.1]), the $JF(x)$ (or the generalized Jacobian $\partial F(x)$) needs to be somewhat well-behaved for the method to function properly. For instance, singular Jacobian matrices can cause the method to stall. The power market model defined by $(UE_1$ with shared constraints)
and (UE$_2$)–(UE$_4$) satisfies several of these requirements.

Some definitions are in order before discussing Newton-type solution methods. Solution stability is a desirable property for Newton-type convergence in that it guarantees that solutions of slightly perturbed variational inequalities [with respect to map $F$] are near the unperturbed solution. With this in mind, the concepts of tangent cones, recession cones, and critical cones play important roles in identifying directions in which the slightly perturbed variational inequalities may be poorly behaved. For VI problems in which the feasible region $K$ can be expressed as the Cartesian product of lower-dimensional sets, the matrix concepts of semicopositivity and $R_0$ pairs can be leveraged. These properties generalize the definitions of semimonotone and pseudo-regular matrices (see Definition 3.1) to VI($K, F$) from the LCP setting.

**Definition 4.1.** Let map $F : U \to \mathbb{R}^n$ be continuous where $U$ is an open subset of $\mathbb{R}^n$ containing $K$. A solution $x^*$ of VI($K, F$) is stable if for every open neighborhood $N$ of $x^*$ satisfying $\text{cl}N \subset U$ and $\text{SOL}(K, F) \cap \text{cl}N = \{x^*\}$, there exist two positive scalars $c$ and $\epsilon$ such that, for every $G \in B(F; \epsilon, K \cap \text{cl}N)$ and every $x \in \text{SOL}(K, G) \cap N \neq \emptyset$,
\[
\|x - x^*\| \leq c \|F(x) - G(x)\|. 
\]

**Definition 4.2.** Let $K \subseteq \mathbb{R}^n$.

- The tangent cone of $K$ at point $x \in K$, denoted by $\mathcal{T}(x; K)$, is the set of all vectors $d \in \mathbb{R}^n$ for which there exist a sequence of vectors $\{y^\nu\} \subset K$ and a sequence of positive scalars $\{\tau^\nu\}$ such that
\[
\lim_{\nu \to \infty} y^\nu = x, \quad \lim_{\nu \to \infty} \tau^\nu = 0, \quad \text{and} \quad \lim_{\nu \to \infty} \frac{y^\nu - x}{\tau^\nu} = d.
\]

- The recession cone of $K$, denoted by $K_\infty$, is the set of all directions $d \in \mathbb{R}^n$ such that for
some vector $y \in K$, the ray $\{y + \tau d \mid \tau \geq 0\}$ is contained in $K$.

- The critical cone of the pair $(K, F)$ at point $x \in K$, denoted by $\mathcal{C}(x; K, F)$, is given by $\mathcal{T}(x; K) \cap F(x)^\perp$ (i.e., the intersection of the tangent cone and the orthogonal complement of $F$ at $x$).

\[\mathcal{T}(x; K) \cap F(x)^\perp\] \[\square\]

**Definition 4.3.** Let $K \subseteq \mathbb{R}^n$ be given by the Cartesian product $\prod_{\nu=1}^{N} K_\nu$, $K_\nu \subseteq \mathbb{R}^{n_\nu}$, and $\sum_{\nu=1}^{N} n_\nu = n$. Let $M$ be an $n \times n$ matrix.

- If each $K_\nu$ is a cone in $\mathbb{R}^{n_\nu}$, $M$ is
  
  (a) **semipositive on** $K$ if for every nonzero vector $x \in K$, there exists an index $\nu \in \{1, \cdots, N\}$ such that $x_\nu \neq 0$ and $x_\nu^T (Mx)_\nu \geq 0$;

  (b) **strictly semipositive on** $K$ if

  $$\max_{1 \leq \nu \leq N} x_\nu^T (Mx)_\nu > 0 \text{ for all } x \in K \setminus \{0\}.$$ 

- The pair $(K, M)$ is an $R_0$ pair if $\text{SOL}(K_\infty, M) = \{0\}$. \[\square\]

As with proving theoretical properties such as monotonicity, it can be difficult to prove that a solution $x^* \in \text{SOL}(K, F)$ is stable directly. Therefore, two existing VI results will be combined in the upcoming proof of stability. The first result establishes a special $R_0$ pair given strict semipositivity while the second result proves stability under certain conditions.

**Corollary 4.1** (Corollary 3.5.3, [75]). Let $K \subseteq \mathbb{R}^n$ be given by the Cartesian product $\prod_{\nu=1}^{N} K_\nu$, $K_\nu \subseteq \mathbb{R}^{n_\nu}$ be a closed convex cone, $\sum_{\nu=1}^{N} n_\nu = n$, and $M \in \mathbb{R}^{n \times n}$. If $M$ is strictly semipositive on $K$, then $M$ is semipositive on $K$ and $(K, M)$ is an $R_0$ pair. \[\square\]
Proposition 4.9 (Proposition 5.3.15, [75]). Let $K$ be a Cartesian product of $N$ polyhedra $K_\nu \subseteq \mathbb{R}^{n_\nu}$, $x^* \in \text{SOL}(K, F)$, and mapping $F$ be continuously differentiable in a neighborhood of $x^*$. If $JF(x^*)$ is semicopositive on $C(x^*; K, F)$, then $x^*$ is a stable solution of $\text{VI}(K, F)$ if and only if $(C(x^*; K, F), JF(x^*))$ is an $R_0$ pair. \hfill \Box

With the state results and the assumptions of Proposition 4.4, the following proposition proves that the unique solution $x^*$ of $(\text{UE}_1 \text{ with shared constraints})$ and $(\text{UE}_2)$–$(\text{UE}_4)$ is stable.

**Proposition 4.10** (Stability of the unified Nash equilibrium). If the assumptions of Proposition 4.4 hold, the unique solution of $(\text{UE}_1 \text{ with shared constraints})$ and $(\text{UE}_2)$–$(\text{UE}_4)$ is stable.

**Proof.** For Proposition 4.9 to be applicable, certain properties must be proven on the unknown critical cone $C\left(x^*; \hat{K}, F\right)$ where the pair $\left(\hat{K}, F\right)$ is as defined in the proof of Proposition 4.4. Fortunately, these properties can be proven on the whole space $\mathbb{R}^n$ and, by implication, must hold for the specified critical cone.

Let each $K_\nu$ be the trivial closed, convex cone $\mathbb{R}$ and $K = \mathbb{R}^n$, an open, unbounded rectangle. In the definition of semicopositivity, $M = JF(x^*)$ which corresponds to the affine map $F'(x) = JF(x^*)x$. Because $JF(x^*)$ is a $P$ matrix by the proof of Proposition 4.4, $F'$ is a $P$ function on $K$. It can be concluded from the definition of a $P$ function that $JF(x^*)$ is strictly semicopositive on $K$ because each $K_\nu$ is a cone. By Corollary 4.1, $JF(x^*)$ is semicopositive on $K$ [and therefore on $C\left(x^*; \hat{K}, F\right)$] and $(K, JF(x^*))$ is an $R_0$ pair. Because $(C\left(x^*; \hat{K}, F\right), JF(x^*))$ is also an $R_0$ pair. Proposition 4.9 gives the desired conclusion. \hfill \Box
Josephy-Newton method. A well-known iterative solution method for VI($K,F$) when stability holds relies on the normal map $F_{nor}^K$ as defined in Proposition 4.5. Similar to Newton’s method for solving nonlinear equations, the Josephy-Newton method utilizes a first-order approximation of $F$ to iteratively identify a zero of the normal map. After a series of manipulations, there are two equivalent formulations of this method: the first scheme solves a nonsmooth equation arising from the normal map, while the second scheme expresses this equation as the solution of an equivalent VI. Convergence of the Josephy-Newton method is proven via an appeal to solution stability.

Given an iterate $z^k$, define the subsequent iterate $z^{k+1}$ as a solution of the nonsmooth equation

$$F\left(\Pi_K(z^k)\right) + JF(\Pi_K(z^k))\left(\Pi_K(z^{k+1}) - \Pi_K(z^k)\right) + z^{k+1} - \Pi_K(z^{k+1}) = 0. \quad (4.18)$$

Under conditions that guarantee convergence of these iterates to $z^\infty$, the solution of VI($K,F$) is given by $x^* = \Pi_K(z^\infty)$. From the definition of $F_{nor}^K$ and letting $x^k = \Pi_K(z^k)$, (4.18) is seen to be equivalent to solving VI($K,F^k$) where

$$F^k(x) \triangleq F(x^k) + JF(x^k)(x - x^k).$$

Therefore, the Josephy-Newton method can be implemented in either the variable $z$ or the variable $x$; the latter algorithm is provided here.
Josephy-Newton Method for the VI [75, Algorithm 7.3.1].

Let $x^0 \in K$, $\epsilon > 0$, and $k = 0$.

**Step 1:** If $x^k \in \text{SOL}(K,F)$, stop.

**Step 2:** Let $x^{k+1}$ be any solution of the $\text{VI}(K,F^k)$ such that $x^{k+1} \in B(x^k,\epsilon)$.

**Step 3:** Set $k \leftarrow k + 1$ and return to Step 1.

The only remaining question for the Josephy-Newton method is under what conditions converge to the solution $x^*$ of $\text{VI}(K,F)$ is guaranteed. As with many Newton-type results, convergence can only be claimed for initial points sufficiently close to the optimal solution.

**Theorem 4.4** (Theorem 7.3.5, [75]). Let $F$ be continuously differentiable. Assume that $x^*$ is a stable solution of $\text{VI}(K,F^*)$, where $F^*(x) \triangleq F(x^*) + JF(x^*)(x - x^*)$ is the strong first-order approximation of $F$ at $x^*$. For every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x^0 \in K \cap B(x^*,\delta)$, the Josephy-Newton method generates a well-defined sequence $\{x^k\}$ in $B(x^*,\delta)$, and every such sequence converges to $x^*$.

**Proposition 4.11.** Given the assumptions of Proposition 4.4, the Josephy-Newton iterates starting from any point $x^0$ sufficiently close to the solution $x^*$ of (UE$_1$ with shared constraints) and (UE$_2$)–(UE$_4$) will converge to $x^*$.

**Proof.** Let $\text{VI}(\hat{K},F)$ be as defined in the proof of Proposition 4.4 and $F^*(x)$ be as defined in Theorem 4.4. The proof of Proposition 4.4 guarantees that $\text{VI}(\hat{K},F^*)$ has a unique solution. Furthermore, $F^*(x^*) = F(x^*)$ and, therefore, $\text{SOL}(\hat{K},F^*)$ is equivalent to $\text{SOL}(\hat{K},F)$ by (1.3). It is also simple to see that Proposition 4.10 guarantees that $x^*$ is a stable solution of $\text{VI}(\hat{K},F^*)$. Theorem 4.4 gives the desired result.

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Semismooth Newton method. A different solution approach to (UE$_1$ with shared constraints) and (UE$_2$)--(UE$_4$) relies on an implementation of Newton’s method for semismooth equations.

**Definition 4.4.** Let $U \subseteq \mathbb{R}^n$ be an open set and mapping $F : U \rightarrow \mathbb{R}^n$ be a locally Lipschitz continuous function on $U$. $F$ is **semismooth** at a point $x \in U$ if $F$ is directionally differentiable near $x$ and there exists a neighborhood $U' \subseteq U$ of $x$ and a function $\Delta : (0, \infty) \rightarrow [0, \infty)$ with $\lim_{t \to 0} \Delta(t) = 0$ such that for any $y \in U'$ distinct from $x$,

$$\frac{\|F'(y; y - x) - F'(x; y - x)\|}{\|y - x\|} \leq \Delta(\|y - x\|).$$

For VI($K, F$) where $K$ is polyhedral and $F$ is continuously differentiable, as in (UE$_1$ with shared constraints) and (UE$_2$)--(UE$_4$), $F^{\text{nor}}_K$ and $F^{\text{nat}}_K$ are both semismooth; the normal (or natural) map representation of SOL($K, F$) from Proposition 4.5 can be solved via algorithms for semismooth equations. For semismooth equations, the Jacobian $JF(x)$ usually required by Newton’s method is replaced by the Clarke generalized Jacobian $\partial F(x)$.

---

**Semismooth Newton Method** [75, Algorithm 7.5.1].

Let $G$ be the normal map of VI($K, F$) (i.e., $G(z) \triangleq F(\Pi_K(z)) + z - \Pi_K(z)$), $x^*$ be a solution of VI($K, F$), $z^0 \in \mathbb{R}^n$, and $k = 0$.

**Step 1:** If $G(z^k) = 0$, set $x^* = \Pi_K(z^k)$ and stop.

**Step 2:** Select an element $H^k \in \partial G(z^k)$. Find a direction $d^k \in \mathbb{R}^n$ such that

$$G(z^k) + H^k d^k = 0.$$ 

**Step 3:** Set $z^{k+1} = z^k + d^k$, $k \leftarrow k + 1$, and return to Step 1.
If $H^k$ is nonsingular in this algorithm, $d^k$ is unique and can be easily found. Given that $\partial G(z^*)$ is nonsingular (i.e., all $H \in \partial G(z^*)$ are nonsingular), the semismooth Newton method is guaranteed to converge when the starting point $z^0$ is sufficiently close to the optimal solution $z^*$ similar to Theorem 4.4.

**Theorem 4.5** (Theorem 7.5.3, [75]). Let $U \subseteq \mathbb{R}^n$ be an open set and mapping $G : U \rightarrow \mathbb{R}^n$ be semismooth at $z^* \in U$ with $G(z^*) = 0$. If $\partial G(z^*)$ is nonsingular, then there exists a $\delta > 0$ such that, if $z^0 \in B(z^*, \delta)$, the sequence $\{z^k\}$ generated by the semismooth Newton method is well-defined and converges to $z^*$.

**Proposition 4.12.** Let $G$ be the normal map associated with VI($\hat{K}, F$) with ($\hat{K}, F$) defined as in the proof of Proposition 4.4. Given the assumptions of Proposition 4.4 and nonsingularity of $\partial G(z^*)$, the semismooth Newton method iterates starting from any point $z^0$ sufficiently close to the solution $z^*$ will converge to $z^*$. The solution of (UE$_1$ with shared constraints) and (UE$_2$)–(UE$_4$) is given by $\Pi_K(z^*)$.

A noticeable deficiency in these Newton-based methods is that convergence is only guaranteed when the starting point is sufficiently close to a solution. Using a merit function approach, this requirement can be partially relaxed under certain conditions. A merit function $\theta : K \rightarrow \mathbb{R}_+$, by definition, has the property that $\theta(x) = 0$ if and only if $x \in \text{SOL}(K, F)$. Using a given merit function $\theta$, the optimization problem

$$\minimize_{x \in K} \theta(x)$$

(4.19)

can be solved with globally convergent computational approaches such as iterative line search or trust region algorithms. Unfortunately, merit functions are typically nonconvex.
and may not be differentiable; consequently, a solution \( x^* \) of (4.19) is a stationary point but may have \( \theta(x^*) > 0 \) (i.e., it is possible that \( x^* \notin \text{SOL}(K,F) \)). The conditions under which a stationary point of (4.19) is indeed a solution of VI\((K,F)\) are related to regularity properties. Thus, although implementing a merit function-based approach within the specified Newton methods gives global as opposed to local convergence, the limiting point is only guaranteed to solve the desired problem under additional assumptions. For a much more comprehensive discussion of merit function methods, the interested reader is referred to [75, Chapters 8 and 10].

4.3 Model extensions

A variety of model extensions have been proposed in the power market literature, each of which attempts to better represent current and emerging issues. While some of these proposed extensions have been rigorously studied, other additions have been addressed in only a vague and simplistic manner. Therefore, this section describes and analyzes the theoretical consequences of three important power market extensions: consumer surplus maximization, capacity markets, and emission permit auctions. Each of these extensions is currently the subject of considerable debate between regulators, ISOs, and industry, thus making them relevant additions to the unified framework.

4.3.1 Consumer surplus maximization

Due to how the electricity industry has developed, consumers commonly do not know the real-time price of electricity consumption and are instead charged at an average rate for their use. This lack of a price signal is undesirable from an economic perspective because it limits the natural balancing of supply and demand. As such, electricity price spikes have been observed several times in the recent past. In two days of June 1998, electricity
prices in the Midwest varied from $25 to $7500 per MWh. After an analysis by the Federal Energy Regulatory Commission (FERC) [1], it was determined that the conditions causing this price spike were unlikely to occur again. In July 1999, prices spiked to $10,000 per MWh under very different conditions. According to [37], the primary culprit behind these price spikes was demand inelasticity and not the complex interaction of factors identified by FERC in [1]. Furthermore, it was shown that a small percentage of consumers paying the real-time electricity price would be very helpful in avoiding these extreme prices.

In spite of the recognition that price signals of some form should be incorporated into consumer electricity sales, technological limitations have made this effort impractical until recently. Now that real-time electricity pricing and other price-responsive demand reduction methods have become more viable, much research has been devoted to the effects, implementation, and success of these efforts. Generally, these efforts fall under the heading of demand response, the field of inducing changes in customer electricity use as a response to electricity prices [3]. The interested reader is referred to [35, 91, 125, 154, 214, 223] for current research on demand response.

Consumer surplus maximization is a simple way to account for demand response in power markets, and, under certain specifications of the function \( q_\ell(p_\ell) \), the game defined by (UE\(_1\) with shared constraints) and (UE\(_2\))–(UE\(_4\)) already includes this extension. In this model, the nodal consumer is assumed to choose demand such that consumer surplus is maximized; this concept was briefly discussed in Section 2.7. Defining \( p^*_\ell \) as the equilibrium market price at node \( \ell \) and \( P^C_\ell - \frac{P^C_\ell}{2Q^C_\ell} q_\ell \) as an affine consumer demand curve at node \( \ell \) where \( q_\ell \) is the quantity demanded and \( P^C_\ell, Q^C_\ell > 0 \), the consumer surplus maximization problems are
(Affine demand UE₅) for each \( \ell = 1, \ldots, N \),

\[
\text{maximize} \quad PC_\ell q_\ell - \frac{PC_\ell}{2QC_\ell} q_\ell^2 - p_\ell^* q_\ell,
\]

or equivalently,

\[
q_\ell(p_\ell^*) \triangleq QC_\ell - \frac{QC_\ell}{PC_\ell} p_\ell^* \quad \text{for all } \ell = 1, \ldots, N.
\]

In Section 2.7, \( q_\ell \) was constrained to be nonnegative. Although valid, the resulting piecewise linear function \( q_\ell(p_\ell) \) is not strictly decreasing as required for existence and uniqueness in Propositions 4.2 and 4.4. It is simple to see that the Cournot demand specification of \( q_\ell(p_\ell) \) can be achieved by setting \( PC_\ell = P_\ell^0 \) and \( QC_\ell = Q_\ell^0 \).

**Remark 4.6.** It should be obvious that more general strictly decreasing demand curves can be used to calculate consumer surplus as long as the function \( q_\ell(p_\ell^*) \) can be explicitly determined; the affine demand specification here is motivated by its mathematical simplicity, its applicability to Cournot models, and the desire to avoid unnecessary notation. The more general consumer surplus maximization problem will be referred to as (UE₅).

For a unified model (UE₁ with shared constraints) and (UE₂)–(UE₄), this maximization has been implicitly included if there exist nodal demand functions such that the optimal demand associated with consumer surplus maximization is given by \( q_\ell(p_\ell) \) for all \( \ell \). With this in mind, solution existence and uniqueness trivially hold by previous results.

**Proposition 4.13.** If the strictly decreasing functions \( (q_\ell(p_\ell))_{\ell=1}^N \) arise from consumer surplus maximization problems (UE₅), then a solution exists for (UE₁ with shared constraints) and (UE₂)–(UE₅) under the assumptions of Proposition 4.2. The solution is unique under the assumptions of Proposition 4.4.
Although this extension is simple, it is crucial in the following model additions.

4.3.2 Capacity markets

Because electricity demand is projected to continue increasing for the foreseeable future [71], generation capacity must likewise increase over time. The current difficulty is that while an ISO may determine how much additional capacity will be needed in the future, profit-maximizing generators must individually decide to make these capacity investments in a deregulated market. When considering the significant capital commitment and delayed profit realization associated with generation investments along with future market uncertainty, it is understandable that investment in new generation capacity may not be adequate given current power market structures as will be discussed momentarily. Peaking capacity (i.e., capacity with high marginal generation cost that runs to help meet peak demands) also plays an important role in future capacity expansion because of the inelastic nature of electricity demand.

Risk aversion alone can decrease generator capacity investments because of considerable lead times and uncertain electricity prices in the future. Unfortunately, deregulated power markets also commonly contain regulatory features that further discourage companies from investing in new capacity [79 83 119]. For instance, price caps meant to mitigate price spikes can decrease investment in peaking capacity by making this capacity less profitable on the rare occasions when it is utilized. This regulation-induced revenue deficiency is commonly referred to as the “missing money” problem [49 50]. To address this investment issue, markets for generation capacity have been proposed and implemented by many ISOs to make the installation of capacity more attractive (e.g., [23 52 79 83 107 108 116 119 170 210]).
Capacity markets typically have one of three forms based on the microeconomic mechanism utilized by the ISO: price-based, quantity-based, or hybrid. As discussed in [116], the capacity market should ideally achieve an equilibrium between marginal cost and marginal benefit of additional capacity. In a locational capacity model, let the equilibrium price and quantity for node $\ell$ be denoted by the pair $(p_{\ell}^{\text{cap,}*}, q_{\ell}^{\text{cap,}*})$. The capacity market developed by the ISO should lead to this equilibrium point where the additional capacity $q_{\ell}^{\text{cap,}*}$ at each node $\ell$ is judged by the ISO to be sufficient to satisfy future reliability standards and demand.

In a price-based market, the ISO establishes a fixed per-unit (e.g., installed MW) subsidy $p_{\ell}^{\text{cap,}*}$ for each node $\ell$ to induce the desired capacity levels $(q_{\ell}^{\text{cap,}*})_{\ell=1}^{N}$. A quantity-based market takes the opposite approach in that the ISO establishes a total required capacity $q_{\ell}^{\text{cap,}*}$ for each node $\ell$ to induce the equilibrium prices $(p_{\ell}^{\text{cap,}*})_{\ell=1}^{N}$. A hybrid capacity market model [107, 170, 210] combines elements of these frameworks and has become prevalent in the electricity industry. For a hybrid model, the ISO develops a capacity demand curve based on its target capacity level and pays a per-unit amount to generators according to the total capacity supplied; this payment is provided by consumers (i.e., load serving entities) through nodal uplift charges. Unlike the quantity-based approach in which the demand curve is vertical (i.e., inelastic), the hybrid capacity demand curve is usually a nonincreasing piecewise linear function, a difference that leads to less abrupt price changes in the neighborhood of the target capacity level. Unlike the price-based market, capacity in a hybrid market receives a variable per-unit payment depending on the total capacity supply. It should be noted that the hybrid approach is more easily implemented than the price- and quantity-based approaches because the marginal cost and marginal benefit curves do not need to be known to a high degree of certainty.
Although many publications have discussed the advantages and disadvantages of each of these market forms, a vast majority (if not all) of this research has dealt with these markets in relative isolation. The theoretical consequences of implementing different types of capacity market have yet to be studied in the framework of a comprehensive power market model. Therefore, the unified model results are now extended to include quantity-based and hybrid capacity market forms.

**Quantity-based capacity markets**

Assume that the ISO decides that $\text{CAP}_\ell$ is the capacity required at node $\ell$ to satisfy future demand and reliability standards. The ISO subsequently requires that at least this much capacity is provided by consumers (i.e., load serving entities). Because consumers typically do not own substantial generation capacity, it can be assumed that all of this required capacity is purchased from generators. To reflect these market changes, modify $(\text{UE}_1$ with shared constraints) to
\((\text{UE}_i^{\text{cap}})\) with shared constraints\) for each \(i = 1, \ldots, \mathcal{F},\)

\[\begin{array}{l}
(g_i^*, s_i^*, s_{i\text{cap}}^*) \in \arg\max_{g_i, s_i, s_{i\text{cap}}} \sum_{\ell=1}^{N} \left[ p_{i\ell}(s_{i\ell}; S_{-i\ell}^*, p_{i\ell}^*) s_{i\ell} - (s_{i\ell} - g_{i\ell})w_{i\ell}^* + p_{i\ell}^\text{cap,*} s_{i\ell}^\text{cap,*} \right] \\
\text{subject to } \sum_{\ell=1}^{N} (g_{i\ell} - s_{i\ell}) = 0 \\
0 \leq \sum_{\ell=1}^{N} s_{i\ell}^\text{cap,*} \leq \sum_{\ell=1}^{N} s_{i\ell}^\text{cap,*} \\
\begin{cases}
0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \\
s_{i\ell} \geq 0 \\
s_{i\ell} + \sum_{j \neq i} s_{j\ell}^* \leq \mathcal{S}_{\ell} \\
g_{i\ell} + \sum_{j \neq i} g_{j\ell}^* \geq \mathcal{Q}_{\ell}
\end{cases}
\end{array}\]

for all \(\ell = 1, \ldots, N,\)

where \(p_{i\ell}^\text{cap,*}\) is the [endogenous] capacity subsidy for node \(\ell\) and \(s_{i\ell}^\text{cap,*}\) is the capacity sold to the consumer at node \(\ell\) by generator \(i\). To reflect the capacity payments from the consumer at node \(\ell\) to the generators, modify (Affine demand UE_5) to

\((\text{Affine demand UE}_5^{\text{cap}})\) for each \(\ell = 1, \ldots, N,\)

\[\begin{array}{l}
\max_{q_{\ell}} P_{\ell}^C q_{\ell} - \frac{P_{\ell}^C}{2Q_{\ell}^C} q_{\ell}^2 - p_{\ell}^* q_{\ell} - p_{\ell}^\text{cap,*} \sum_{i=1}^{\mathcal{F}} s_{i\ell}^\text{cap,*}. \\
\end{array}\]

The term \(p_{\ell}^\text{cap,*} \sum_{i=1}^{\mathcal{F}} s_{i\ell}^\text{cap,*}\) is called the nodal uplift charge. Note that this charge is unrelated to the consumer demand at node \(\ell\) and is therefore treated as constant. In reality, the ISO can distribute these nodal uplift charges among the consumers in any manner as long
as the total payments to generators is the still \( \sum_{\ell=1}^{N} p_{\ell}^{\text{cap},*} \sum_{i=1}^{F} s_{i\ell}^{\text{cap},*} \). A common payment
distribution metric is based on historical nodal peak demand; the assumption that each
consumer pays for its own capacity is used here for simplicity. Similar to before, the more
general version of this consumer maximization problem will be denoted by \( \text{UE}_{5}^{\text{cap},q} \).

To complete this quantity-based capacity market, a market clearing condition must be
enforced to determine the prices \( (p_{\ell}^{\text{cap}})_{\ell=1}^{N} \):

\[
0 \leq p_{\ell}^{\text{cap}} \perp \sum_{i=1}^{F} s_{i\ell}^{\text{cap}} - \text{CAP}_{\ell} \geq 0 \quad \text{for all } \ell = 1, \cdots, N. \tag{4.20}
\]

The price \( p_{\ell}^{\text{cap}} \) cannot be bounded above in this market formulation, leading to diffi-
culty when proving equilibrium existence. Therefore, \( 4.20 \) will be incorporated into
\( \text{UE}_{1}^{\text{cap},q} \) with shared constraints as shared constraints with a common multiplier in a
manner similar to the perfectly competitive equivalent formulation \( 4.10 \):

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\[ \left( \text{UE}_{i}^{\text{cap}, \ell} \right) \text{ with shared constraints} \] for each \( i = 1, \cdots, \mathcal{F}, \)

\[
\left( g_{i}^{*}, s_{i}^{*}, s_{i}^{\text{cap}, *} \right) \in \arg\max_{g_{i}, s_{i}, s_{i}^{\text{cap}}} \sum_{\ell=1}^{N} \left[ \rho_{i\ell}(s_{i\ell}; S_{i\ell}^{*}, p_{i\ell}^{*}) s_{i\ell} - c_{i\ell}(g_{i\ell}) - (s_{i\ell} - g_{i\ell}) w_{i\ell} \right]
\]

subject to

\[
\sum_{\ell=1}^{N} (g_{i\ell} - s_{i\ell}) = 0
\]

\[
0 \leq \sum_{\ell=1}^{N} s_{i\ell}^{\text{cap}} \leq \sum_{\ell=1}^{N} \text{cap}_{i\ell}
\]

\[
\left\{ \begin{array}{l}
0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \\
0 \leq s_{i\ell} \leq \text{cap}_{i\ell} \\
 s_{i\ell} + \sum_{j \neq i} s_{j\ell} \leq S_{\ell} \\
g_{i\ell} + \sum_{j \neq i} g_{j\ell} \geq Q_{\ell} \\
 \sum_{i=1}^{\mathcal{F}} s_{i\ell}^{\text{cap}} - \text{CAP}_{\ell} \geq 0
\end{array} \right. \text{ for all } \ell = 1, \cdots, N,
\]

where the Lagrange multiplier of the first constraint (i.e., capacity market clearing for each \( \ell \)) is \( p_{i\ell}^{\text{cap}} \) for all \( i = 1, \cdots, \mathcal{F}. \)
Before proving existence, define

\[
\begin{align*}
K & \triangleq \\
& \left\{ (g,s,s^{\text{cap}}) \mid \begin{array}{l}
\text{for all } i = 1, \ldots, F \\
\sum_{\ell = 1}^{N} (g_{i\ell} - s_{i\ell}) = 0 \\
0 \leq \sum_{\ell = 1}^{N} s_{i\ell}^{\text{cap}} \leq \sum_{\ell = 1}^{N} \text{cap}_{i\ell} \\
\{ 0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \} \quad \text{for all } \ell = 1, \ldots, N \\
s_{i\ell} \geq 0 \\
\end{array} \\
\text{for all } \ell = 1, \ldots, N \\
\sum_{i \in F} s_{i\ell} \leq S_{\ell} \\
\sum_{i \in F} g_{i\ell} \geq Q_{\ell} \\
\sum_{i = 1}^{N} s_{i\ell}^{\text{cap}} - \text{CAP}_{\ell} \geq 0 \\
\text{for all } k = 1, \ldots, K \\
T_{k}^{-} \leq \sum_{\ell = 1}^{N} \sum_{i \in F} \text{PTDF}_{\ell k} \sum_{i = 1}^{F} (s_{i\ell} - g_{i\ell}) \\
T_{k}^{+} \geq \sum_{\ell = 1}^{N} \sum_{i \in F} \text{PTDF}_{\ell k} \sum_{i = 1}^{F} (s_{i\ell} - g_{i\ell})
\right\}.
\end{align*}
\]

(4.21)

**Proposition 4.14.** Assume that the cost function \(c_{i\ell}(g_{i\ell})\) is convex for all \(i\) and \(\ell\). Furthermore, assume that the nodal demand function \(q_{\ell}(p_{\ell})\) is a continuous and strictly decreasing function of \(p_{\ell}\) for all \(\ell\). If \(K\) as defined by (4.21) is nonempty and the function \(\rho_{i\ell}(s_{i\ell}; S_{-i\ell}, p_{\ell})\) is concave in \(s_{i\ell}\) for each fixed \((S_{-i\ell}, p_{\ell})\) and all \(i\) and \(\ell\), then a Nash equilibrium exists for the power market game defined by \((\text{UE}_{1}^{\text{cap}})\text{ with shared constraints})\), \((\text{UE}_{2})-(\text{UE}_{4})\), \((\text{UE}_{5}^{\text{cap}})\), and (4.20).
Proof. The optimality conditions of \((UE_1^{capq} \text{ with shared constraints})\) and \((4.20)\) correspond to \((UE_1^{capq},' \text{ with shared constraints})\). Because \((UE_5^{capq})\) is already accounted for in the specification of \(q_e(p_e)\) in \((UE_4)\), it suffices to examine the primal VI formulation of \((UE_1^{capq},' \text{ with shared constraints})\) and \((UE_2)-(UE_4)\). After incorporating \((UE_2)\) and \((UE_3)\) into \((UE_1^{capq},' \text{ with shared constraints})\), this VI is defined by

\[
\hat{K} \triangleq K \times \prod_{\ell=1}^{N} \left[ \overline{p}_\ell, \underline{p}_\ell \right] \quad \text{and} \quad F \triangleq \begin{pmatrix}
\partial_{g_i\ell} c_{i\ell}(g_{i\ell}) |_{i,\ell=1}^{\mathcal{F},N} \\
-\partial_{s_{i\ell}} (\rho_{i\ell}(s_{i\ell}; S_{-i\ell}, p_i) s_{i\ell}) |_{i,\ell=1}^{\mathcal{F},N} \\
[0] \\
[S_{\ell} - q_{\ell}(p_{\ell})]_{\ell=1}^{N}
\end{pmatrix}.
\]

Corollary 1.1 gives existence because \(\hat{K}\) is compact and nonempty. □

Hybrid capacity markets

In a hybrid capacity market, the ISO does not establish a required capacity level to be met. Instead, the ISO formulates a capacity demand curve that will [hopefully] result in the desired capacity levels. Let this demand function be continuous, nonincreasing, and denoted by \(p_{\ell}^{cap} \left( \sum_{i=1}^{\mathcal{F}} \hat{s}_{i\ell}^{cap} \right)\). Because these demand functions are known, it is natural to model generators as Cournot players in the hybrid capacity market.
$$\left( g_i^*, s_i^*, s_{i}^{\text{cap,*}} \right) \in \arg\max_{g_i, s_i, s_{i}^{\text{cap}}} \sum_{\ell = 1}^{N} \left[ \rho_{i\ell}(s_{i\ell}; S_{-i\ell}^*, p^*_\ell) s_{i\ell} - c_{i\ell}(g_{i\ell}) - (s_{i\ell} - g_{i\ell})w^*_\ell \\
+ p^\text{cap}_{\ell}(s_{i\ell}; S_{-i\ell}^{\text{cap,*}}) s_{i\ell} \right]$$

subject to

$$\sum_{\ell = 1}^{N} (g_{i\ell} - s_{i\ell}) = 0$$

$$0 \leq \sum_{\ell = 1}^{N} s_{i\ell}^{\text{cap}} \leq \sum_{\ell = 1}^{N} \text{cap}_{i\ell}$$

$$\left\{ \begin{array}{l}
0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \\
 s_{i\ell} \geq 0 \\
 s_{i\ell} + \sum_{j \neq i}^{j} s_{j\ell}^* \leq S_{\ell} \\
g_{i\ell} + \sum_{j \neq i}^{j} g_{j\ell}^* \geq Q_{\ell} 
\end{array} \right\} \text{ for all } \ell = 1, \ldots, N,$$

Because the capacity requirement is not fixed in the hybrid market, it follows that consumption decisions affect capacity payments. Acting strategically with respect to the capacity market, consumers can be modeled as solving the following optimization problems:

**Affine demand UE$_5^{\text{cap,h}}$** for each $\ell = 1, \ldots, N$,

$$\text{maximize } P^\text{C}_\ell q_\ell - \frac{P^\text{C}_\ell}{2Q^\text{C}_\ell} q^2_\ell - p^\text{cap}_\ell (r_\ell q_\ell) r_\ell q_\ell,$$

where $r_\ell$ is the nodal capacity requirement for each unit of demand as specified by the ISO and $p^\text{cap}_\ell (r_\ell q_\ell) r_\ell q_\ell$ represents the payments for capacity. Upon simplification, if the mapping from $q_\ell$ to

$$\frac{P^\text{C}_\ell}{Q^\text{C}_\ell} q_\ell - r_\ell p^\text{cap}_\ell (r_\ell q_\ell) - r^2_\ell q_\ell p^\text{cap}_\ell (r_\ell q_\ell)$$
is invertible, then \( q_\ell \) can be expressed as a function of \( p_\ell \) (i.e., \( q_\ell(p_\ell) \)).

Notice that the capacity price function is parametrized by \( r_\ell q_\ell \) instead of \( \sum_{i=1}^{\mathcal{F}} s_{i\ell}^{\text{cap}} \) in the consumer surplus optimization problems. This substitution reflects that price is determined by demand (as well as supply) at equilibrium, leading to the obvious market clearing condition

\[
\sum_{i=1}^{\mathcal{F}} s_{i\ell}^{\text{cap}} = r_\ell q_\ell(p_\ell) \quad \text{for all } \ell = 1, \ldots, \mathcal{N}. \tag{4.22}
\]

Because the market clearing condition (4.22) incorporates both \( s_{i\ell}^{\text{cap}} \) and \( p_\ell \) variables, define
the set $K$ over all variables simultaneously.

$$
K \triangleq \left\{ \left( g, s, s^{\text{cap}}, p \right) \mid \begin{array}{l}
\bullet \text{ for all } i = 1, \cdots, F \\
N \sum_{\ell=1}^N \left( g_{i\ell} - s_{i\ell} \right) = 0 \\
0 \leq \sum_{\ell=1}^N s_{i\ell}^{\text{cap}} \leq \sum_{\ell=1}^N \text{cap}_{i\ell} \\
\begin{cases}
0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \\
\text{s}_{i\ell} \geq 0
\end{cases} \text{ for all } \ell = 1, \cdots, N \\
\bullet \text{ for all } \ell = 1, \cdots, N \\
\sum_{i=1}^F s_{i\ell} \leq \bar{S}_\ell \\
\sum_{i=1}^F g_{i\ell} \geq Q_\ell \\
\sum_{i=1}^N s_{i\ell}^{\text{cap}} = r_q \ell q_\ell(p_\ell) \\
\bullet \text{ for all } k = 1, \cdots, K \\
T_k^- \leq \sum_{\ell=1}^N \text{PTDF}_{\ell k} \sum_{i=1}^F \left( s_{i\ell} - g_{i\ell} \right) \\
T_k^+ \geq \sum_{\ell=1}^N \text{PTDF}_{\ell k} \sum_{i=1}^F \left( s_{i\ell} - g_{i\ell} \right)
\end{array} \right. \right\} \quad (4.23)
$$

Although $p$ is not explicitly bounded in $K$, it is implicitly bounded if the function $q_\ell(p_\ell)$ is strictly decreasing as was proven in Section 4.2.2.

**Proposition 4.15.** Assume that the cost function $c_{i\ell}(g_{i\ell})$ is convex for all $i$ and $\ell$. Furthermore, assume that the nodal demand functions $q_\ell(p_\ell)$ derived from (Affine demand UE$^{\text{cap}_h}$) are continuous and strictly decreasing functions of $p_\ell$ for all $\ell$. If $K$ as defined by (4.23) is nonempty and the function $\rho_{i\ell}(s_{i\ell}; S_{-i\ell}, p_\ell) s_{i\ell}$ is concave in $s_{i\ell}$ for each fixed $(S_{-i\ell}, p_\ell)$.
and all \( i \) and \( \ell \), then a Nash equilibrium exists for the power market game defined by 
\((\text{UE}_1^{\text{cap}_h} \text{ with shared constraints}), (\text{UE}_2)–(\text{UE}_4), (\text{Affine demand UE}_5^{\text{cap}_h})\), and (4.22). \( \square \)

**Proposition 4.16.** Assume that \( K \) as defined by (4.23) is nonempty and LICQ holds for 
\( K \) at the Nash equilibrium point. If Assumptions (C)–(E) of Proposition 4.4 hold and

\[
\begin{align*}
(a) & \quad \frac{dp^{\text{cap}}_\ell(S^{\text{cap}}_\ell)}{dS^{\text{cap}}_\ell} < 0 \\
(b) & \quad \frac{d^2p^{\text{cap}}_\ell(S^{\text{cap}}_\ell)}{(dS^{\text{cap}}_\ell)^2} \leq 0
\end{align*}
\]

for all \( \ell \) and all \( S^{\text{cap}}_\ell \in \left[ 0, \sum_{i=1}^{\mathcal{F}} \sum_{\ell=1}^{\mathcal{N}} \text{cap}_{i\ell} \right] \), then the power market game defined by 
\((\text{UE}_1^{\text{cap}_h} \text{ with shared constraints}), (\text{UE}_2)–(\text{UE}_4), (\text{Affine demand UE}_5^{\text{cap}_h})\), and (4.22) has a unique solution.

**Proof.** It suffices to note that the primal VI associated with this game is defined by \( K \) from (4.23) and

\[
F(g, s, S^{\text{cap}}, p) \triangleq \begin{pmatrix}
\left[ \frac{dc_{i\ell}(g_{i\ell})}{dg_{i\ell}} \right]_{i, \ell=1}^{\mathcal{F}, \mathcal{N}} \\
\left[ -\rho_{i\ell}(S_\ell, p_\ell) - s_{i\ell} \frac{dp^{\text{cap}}_{i\ell}(S^{\text{cap}}_\ell)}{dS^{\text{cap}}_\ell} \right]_{i, \ell=1}^{\mathcal{F}, \mathcal{N}} \\
\left[ -p^{\text{cap}}_\ell(S^{\text{cap}}_\ell) - s^{\text{cap}}_{i\ell} \frac{dp^{\text{cap}}_\ell(S^{\text{cap}}_\ell)}{dS^{\text{cap}}_\ell} \right]_{i, \ell=1}^{\mathcal{F}, \mathcal{N}} \\
\left[ S_\ell - q_\ell(p_\ell) \right]_{\ell=1}^{\mathcal{N}}
\end{pmatrix}
\]

Uniqueness follows from an adaptation of the proof of Proposition 4.4. \( \square \)
4.3.3 Emission permit markets

Various frameworks have been proposed to help markets internalize the costs of environmental externalities of electricity production. Arguably the most successful example of such a market is the SO$_2$ emission allowance system established by Title IV of the 1990 Clean Air Act. The implementation of this scheme is credited with helping reduce acid rain deposition in the eastern United States for a significantly lower cost than a command-and-control approach [29]. However, most power market models do not account for the effects of these emission markets.

Following the SO$_2$ system, tradeable permit markets are commonly proposed for addressing pollution concerns. Basically, each permit allows its holder to emit a certain amount of pollutant and can be traded on the open market. Each polluter faces a non-compliance fine if its permit holdings do not cover its emission levels. Because the regulator fixes the number of permits that are issued, the aggregate pollution level can be controlled and is usually decreased over time. When the supply of emission permits is tight, polluters can either purchase permits or invest in pollution mitigation technologies. Given the polluters are profit maximizers, this market mechanism should result in the least-cost pollution abatement strategy because polluters should trade and invest until the marginal permit cost equals the marginal cost of pollution abatement.

There are two primary forms of permit markets: ambient permits and emissions permits [8, 9, 155]. In the ambient permit framework, each permit is location-specific and grants the ability to pollute such that the ambient pollutant concentration at the location increases by a certain amount. Thus, based on how diffusion of the pollution is modeled, a single generation plant may need to purchase ambient permits for several different locations. Although the monitoring and administration of this system is obviously complex, it should
not result in geographic regions with unacceptably high ambient pollutant concentrations. Emission permits take a different approach and are not location-specific. Instead, an emission permit grants the ability to emit a certain amount of pollution regardless of location, making a single [or sometimes zonal] market possible. The possible emergence of pollutant hot-spots for emission permit markets has been an area of substantial research, and several market adaptations have been proposed to help eliminate this possibility [219]. A popular market change involves a hybrid-type market in which free trading of emission permits is allowed as long as the trade does not result in the violation of pre-specified pollution standards at any monitoring location [131, 147, 148].

Because of the relatively small number of generators in the electricity market, market power is always a concern. When emission permit markets are implemented, the potential for market power may be unintentionally introduced (e.g., [93, 153]). Examples of this behavior have been seen in the California electricity market where generators have bid production without accounting for emission permit cost in order to inflate electricity prices [127]. In [152], the economic welfare implications of this strategic bidding were studied. These potential market power issues provide further motivation for expanding the scope of the unified power market model of Section 4.2.

From a theoretical perspective, the implementation of ambient and/or emission permit markets is very similar to that for quantity-based capacity markets. This relationship arises because the number of permits is fixed and price is determined by demand. For simplicity, only the ambient permit setting will be presented here; the emission permit model trivially follows. In the following development, note the similarities between this model and that studied in Chapter 2.
For ambient permit markets, let \( \{1, \cdots, R\} \) be the set of air quality monitoring locations and \( U_{rt}^r \) be generator \( i \)'s permit \( r \) requirement for a unit of generation at node \( \ell \). The constant \( U_{rt}^i \) takes into account both the amount of pollution-per-unit emitted from generator \( i \)'s production at node \( \ell \) and the regulator-determined dispersion characteristics of this pollution. In the case of air pollution, these dispersion characteristics are related to wind direction, atmospheric mixing, and pollutant decay. Let \( p^r \) be the [endogenous] price of a single permit for monitoring location \( r \). With \( R^r \) being the total number of permits issued for monitoring location \( r \) such that air quality standards are not violated, the obvious auction-based market clearing conditions are

\[
0 \leq p^r \perp R^r - \sum_{i=1}^{F} \sum_{\ell=1}^{N} U_{rt}^i g_{i\ell} \geq 0 \quad \text{for all } r = 1, \cdots, R. \tag{4.24}
\]

Because generators must pay for these permits, this cost must be included in their optimization problems:
\((\text{UE}_{1}^{\text{emit}} \text{ with shared constraints})\) for each \(i = 1, \ldots, \mathcal{F},\)

\[
(g_{i}^{*}, s_{i}^{*}) \in \arg\max_{g_{i}, s_{i}} \sum_{\ell = 1}^{N} \left[ \rho_{i\ell}(s_{i\ell}; S_{-i\ell}, p_{i\ell}) s_{i\ell} - c_{i\ell}(g_{i\ell}) - (s_{i\ell} - g_{i\ell}) w_{i\ell}^{*} - \sum_{r = 1}^{R} U_{i\ell}^{r} P_{r}^{*} g_{i\ell} \right]
\]

subject to \(\sum_{\ell = 1}^{N} (g_{i\ell} - s_{i\ell}) = 0\)

\[
\begin{align*}
0 & \leq g_{i\ell} \leq \text{cap}_{i\ell} \\
s_{i\ell} & \geq 0 \\
s_{i\ell} + \sum_{j \neq i} s_{j\ell}^{*} & \leq \bar{S}_{\ell} \\
g_{i\ell} + \sum_{j \neq i} g_{j\ell}^{*} & \geq Q_{\ell}
\end{align*}
\]

for all \(\ell = 1, \ldots, N.\)

The consumer surplus maximization problem given by \((\text{UE}_{5})\) remains unchanged because consumers are not required to pay for emission-related permits directly. After incorporating \(4.24\) as a shared constraint with common multipliers in the generator problems to
obtain \((\text{UE}_1^{\text{emit},r} \text{ with shared constraints})\) [omitted], define

\[
K \triangleq \begin{cases}
(g, s) | & \text{for all } i = 1, \ldots, F \\
\sum_{\ell=1}^{N} (g_{i\ell} - s_{i\ell}) = 0 \\
\left\{ 
\begin{array}{l}
0 \leq g_{i\ell} \leq \text{cap}_{i\ell} \\
s_{i\ell} \geq 0
\end{array}
\right. & \text{for all } \ell = 1, \ldots, N \\
\sum_{i=1}^{F} s_{i\ell} \leq S_{\ell} \\
\sum_{i=1}^{F} g_{i\ell} \geq Q_{\ell}
\end{cases}
\]

\[
\text{for all } \ell = 1, \ldots, N
\]

\[
\sum_{r=1}^{R} \sum_{i=1}^{N} \sum_{\ell=1}^{F} U_{i\ell}^{r} g_{i\ell} \geq 0
\]

\[
\text{for all } r = 1, \ldots, R
\]

\[
T_{k}^{-} \leq \sum_{\ell=1}^{N} \sum_{i=1}^{F} (s_{i\ell} - g_{i\ell})
\]

\[
T_{k}^{+} \geq \sum_{\ell=1}^{N} \sum_{i=1}^{F} (s_{i\ell} - g_{i\ell})
\]

Proposition 4.17. Assume that the cost function \(c_{i\ell}(g_{i\ell})\) is convex for all \(i\) and \(\ell\). Furthermore, assume that the nodal demand function \(q_{\ell}(p_{\ell})\) derived from \((\text{UE}_5)\) is a continuous and strictly decreasing function of \(p_{\ell}\) for all \(\ell\). If \(K\) as defined by \((4.25)\) is nonempty and the function \(\rho_{i\ell}(s_{i\ell}; S_{-i\ell}, p_{\ell})s_{i\ell}\) is concave in \(s_{i\ell}\) for each fixed \((S_{-i\ell}, p_{\ell})\) and all \(i\) and \(\ell\), then a Nash equilibrium exists for the power market game defined by \((\text{UE}_1^{\text{emit}} \text{ with shared constraints}), (\text{UE}_2)-(\text{UE}_4), (\text{UE}_5)\), and \((4.24)\). The equilibrium is
unique if LICQ holds for \( K \) at the Nash equilibrium and Assumptions (C)–(E) of Proposition 4.4 hold.

As may be expected, this ambient permit framework can easily be combined with capacity markets for additional theoretical results along the lines of those proved here.

### 4.4 Conclusion

This chapter has combined and expanded on traditional power market models in several ways. After describing and formulating perfectly competitive, Cournot, and conjectured supply function models for electricity markets, a new unified formulation was presented. This unified model encompasses each of these traditional market forms along with more exotic behavioral assumptions. For the perfectly competitive model, Nash equilibrium existence and uniqueness was proven through an appeal to game integrability. The existence and uniqueness results for the unified model were established using variational inequality theory. From the presented theory and proof methods, developing results for related market models should not be difficult for the reader.

Although knowing that an equilibrium exists for a game is useful, tractible computation schemes are needed if these solutions are to be identified. For the unified model, projection- and Newton-based algorithms were presented and proven to be applicable under suitable conditions. The basic projection algorithm and the Tikhonov regularization algorithm can be directly applied to the model and are guaranteed to terminate at an equilibrium given appropriate parameters. The distributed Tikhonov regularization scheme from [121] shows promise but requires that the solution set is bounded, a property that cannot be easily established for the unified problem. For the Newton-based methods, solution stability
allows for the application of the Josephy-Newton and the Semismooth Newton methods.

Finally, three different model extensions were presented. Consumer surplus maximization was a useful addition from a microeconomic perspective because it made the specification of nodal demand functions less abstract. The consumer surplus maximization problem is also fundamental to the more complex capacity and emission market additions. Both quantity-based and hybrid capacity markets were studied with equilibrium existence and uniqueness results proven when possible. For managing pollution, both ambient permits and emission permits were discussed. Mathematically, these markets are implemented in a very similar manner and solution existence was proven. With these additions, it may be useful to compare these extended unified models to those of Chapter 2.

There are certainly many elements that have not been included in this unified power market model. Namely, Stackelberg games, uncertainty/risk, and continuous-time dynamics have not been addressed. Each of these areas would be a fruitful direction for future research. It is hoped that this chapter can serve as a basis for the comprehensive development of further model extensions and that it will obviate the need for repetitive research on closely related power market models.
Chapter 5

Differential Nash equilibria: Equivalence and distributed computation\(^6\)

5.1 Introduction

Unlike the static (i.e., discrete-time) non-cooperative Nash equilibrium problems \([73, 76]\) that have been studied from an optimization viewpoint in the preceding chapters, continuous-time non-cooperative Nash games have been minimally analyzed from a constrained optimization perspective. Namely, past research into numerical methods for differential Nash games has focused on optimal control problems without algebraic constraints; in realistic applications, unconstrained player problems are unlikely. From the theoretical viewpoint, slightly more is known with past research proving equilibrium existence for differential Nash games with pure control constraints. However, results for games with mixed state-control

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control constraints are lacking. The primary goal of this chapter is to at least partially address these computational and theoretical deficiencies for differential Nash games with mixed state-control constraints. Under certain conditions,

(a) it is proven that an optimal control problem solution methodology can be applied to identify equilibria;

(b) the convergence of a new Jacobi-type solution algorithm is established.

The starting point of this work is the recent paper [95] that analyzes the convergence of a time-stepping method for solving a linear-quadratic (LQ) optimal control problem with mixed state-control constraints. Such an LQ optimal control problem is the building block of an open-loop differential linear-quadratic Nash game in which each of a finite number of non-cooperative players solves an LQ optimal control problem with mixed state-control constraints and an objective function that depends on rivals’ variables and constrained by private, mixed state-control constraints. The main contribution of this work is to provide a constructive analysis of this differential LQ game by considering two cases:

- **The Symmetric case.** A “symmetry” condition is provided under which a differential LQ game with mixed state-control constraints is equivalent to a “concatenated” LQ optimal control problem; the implication of this equivalence is that all such differential LQ games can be solved by the convergent time-stepping algorithm described in [95].

- **The Asymmetric case.** A “spectral radius” condition is provided under which the differential LQ game can be solved by a distributed Jacobi-type algorithm. Such a distributed algorithm has been used extensively for solving static Nash games arising from signal processing problems for communication (e.g., [143, 181, 182, 183, 208]). The general convergence theory of the Jacobi algorithm for solving static Nash games
was developed in [70] and is extended to the specified differential LQ game in this chapter.

In addition to the equivalences and computational methods noted above, examples of both symmetric and asymmetric differential Nash games that satisfy the assumptions of their respective results are provided. The symmetric example is a continuous-time Nash-Cournot equilibrium problem while the asymmetric case is illustrated by a conjectured supply function equilibrium problem. Thus, the assumptions required by the proven theorems are shown to be not entirely unreasonable.

The remainder of the chapter is organized in five sections. Section 5.2 formulates the LQ optimal control problem being solved by each player, develops a variational principle regarding solution optimality, and summarizes the assumptions and existence theorem presented in [95]. Section 5.3 proves the equivalence of a symmetric differential LQ game with a concatenated LQ optimal control problem, and Section 5.4 proves the convergence of a Jacobi-type iterative scheme for solving an asymmetric differential LQ game. Illustrative examples of both symmetric and asymmetric differential LQ games are presented in Section 5.5. The chapter concludes in Section 5.6.

5.2 The differential LQ game with mixed-state control constraints

Consider the linear-quadratic $F$-player non-cooperative game on the time interval $[0, T]$ where $T < \infty$ is the finite horizon. Each player, indexed by $i \in \{1, \cdots, F\}$, chooses an absolutely continuous state function $x_i : [0, T] \rightarrow \mathbb{R}^{n_i}$ and a bounded measurable (thus integrable) control function $u_i : [0, T] \rightarrow \mathbb{R}^{m_i}$ to solve an LQ optimal control problem; these state and control variables are constrained by a player-specific linear inequality sys-
Define $x \triangleq (x_i)_{i=1}^F$ and $u \triangleq (u_i)_{i=1}^F$ for all players’ state and control variables, respectively. Furthermore, define $x_{-i} \triangleq (x_j)_{j \neq i}$ and $u_{-i} \triangleq (u_j)_{j \neq i}$ for the state and control tuples of player $i$’s rivals. Taking $(x_{-i}, u_{-i})$ as constant, player $i$ solves the following continuous-time optimal control problem:

$$\min_{x_i, u_i} f_i(x_i, x_{-i}, u_i, u_{-i}) \triangleq x_i(T)^T \left( c_i + \sum_{j=1}^F W_{ij} x_j(T) \right) +$$

$$\int_0^T \left( \begin{array}{c} x_i(t) \\ u_i(t) \end{array} \right)^T \left( \begin{array}{c} p_i(t) \\ q_i(t) \end{array} \right) + \sum_{j=1}^F \left[ \begin{array}{cc} P_{ij} & Q_{ij} \\ R_{ij} & S_{ij} \end{array} \right] \left( \begin{array}{c} x_j(t) \\ u_j(t) \end{array} \right) \right) dt$$

subject to $x_i(0) = \xi_i$

$$\begin{cases} \dot{x}_i(t) = a_i + A_i x_i(t) + B_i u_i(t) \\ b_i + C_i x_i(t) + D_i u_i(t) \geq 0 \end{cases}$$

for almost all $t \in [0, T]$, where $\xi_i \in \mathbb{R}^{n_i}$ is a given initial state, $c_i \in \mathbb{R}^{n_i}$, $W_{ij} \in \mathbb{R}^{n_i \times n_j}$, $P_{ij} \in \mathbb{R}^{n_i \times n_j}$, $Q_{ij} \in \mathbb{R}^{n_i \times m_j}$, $R_{ij} \in \mathbb{R}^{m_i \times n_j}$, $S_{ij} \in \mathbb{R}^{m_i \times m_j}$, $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, $a_i \in \mathbb{R}^{n_i}$, $b_i \in \mathbb{R}^{\ell_i}$, $C_i \in \mathbb{R}^{\ell_i \times n_i}$, $D_i \in \mathbb{R}^{\ell_i \times m_i}$, $p_i : [0, T] \to \mathbb{R}^{n_i}$, and $q_i : [0, T] \to \mathbb{R}^{m_i}$. The matrices $W_{ii}$, $P_{ii}$, and $S_{ii}$ are symmetric for all players $i$. A feasible solution of (5.1) is a pair $(x_i, u_i)$ such that $x_i$ is absolutely continuous, $u_i$ is integrable, and the constraints of (5.1) are satisfied as stated. A noteworthy feature of the problem (5.1) is the mixed state-control constraint $b_i + C_i x_i(t) + D_i u_i(t) \geq 0$. Throughout this work, the matrix $C_i$ is permitted to be zero (i.e., pure control constraints), but a key condition for problem solvability yields a special relation among the matrices $(A_i, B_i, C_i, D_i)$ (see Assumption $(E_i)$ in Section 5.2.2). An aggregated pair $(x^*, u^*)$ is a Nash equilibrium (NE) of the above game if and only if for
each $i = 1, \cdots, \mathcal{F}$,

$$
(x_i^*, u_i^*) \in \arg\min_{x_i, u_i} f_i(x_i, x_{i-1}^*, u_i, u_{i-1}^*)
$$
subject to $(x_i, u_i)$ feasible to (5.1).

(5.2)

5.2.1 A review of past research

The importance of many-player, nonzero-sum differential games has led to a considerable
body of academic research in the field (e.g., [36, 70, 115, 142, 204, 216, 220]). Unfortunately,
many of the examined problems are simplistic and the previously derived results are not
applicable to the current setting due to the absence of mixed state-control constraints.
Despite this deficiency in the literature, it is important to recall the traditional formulation
and solution methodologies for nonzero-sum differential games so that the advantages of
the current methodology can be fully appreciated.

If players do not face any constraints on state or control, the game can be expressed as

$$
\begin{align*}
\begin{cases}
\text{minimize} & f_1(x, u) \\
\vdots \\
\text{minimize} & f_{\mathcal{F}}(x, u)
\end{cases}
\end{align*}
$$
subject to $\dot{x}(t) = g(t, x(t), u(t))$ and $x(0) = \xi$.

(5.3)

One of the first theoretical developments for many-player, nonzero-sum differential games
was presented in [36] where it was proven that the derived partial differential equation
system for (5.3) was a generalization of Isaac’s equations [115]. The proof technique uti-
lized value functions and the necessity of the partial differential equations at a solution
was demonstrated. For a special form of the game, it was also proven that the derived
partial differential equations were sufficient for optimality. When specialized to a simple
linear-quadratic differential game without constraints, the game was shown to be normal and a system of ordinary differential equations was found to be sufficient for a solution to the game’s partial differential equations. Despite this simplification, the ordinary differential equations could only be explicitly solved for a subinterval of time. This solution shortcoming was echoed in several later papers.

In 1971, [142] discussed the idea of playability for linear-quadratic nonzero-sum differential Nash games without algebraic constraints. The framework followed the formulation of (5.3), and each player minimized a quadratic objective function based on control decisions and a target state vector. Assuming open-loop controls, it was stated that the invertibility of a given linear operator is sufficient for game playability (i.e., the presence of a unique Nash equilibrium for the game) and several different sufficient conditions for the stated invertibility were derived. If the given operator was not invertible, the game was said to be not playable. However, this terminology is misleading as invertibility is sufficient for solution existence but is not necessary. This incorrect interpretation was addressed in [70] where a generalized [but still unconstrained] version of the problem from [142] was studied; the main result established that either every starting point has a unique equilibrium or the space of starting points is separated into a proper hyperplane possessing nonunique equilibria and its complement possessing no equilibria. In a different [unconstrained] generalization of the problem from [142], [220] proved the existence of an upper bound $T_0$ for the terminal time such that a Nash equilibrium exists if $T < T_0$ for general convex player objective functions.

Different theoretical tools must be employed when constraints are present in player optimal control problems. In an extension of [142] and [220], [216] examined a linear-quadratic differential game with constraints restricting player $i$’s controls to lie within a unit ball
in Euclidean space $\mathbb{R}^{m_i}$ for all $i$. A penalty function method was utilized to establish the existence of a unique open-loop Nash equilibrium for the penalized game, and convergence to an open-loop Nash equilibrium of the original game was established for a subsequence of penalization parameters. However, this existence result, like that in [220], only applies for a sufficiently small time interval $(0, T)$.

A significant drawback of the results of [36, Chapter 4], [220], and [216] is that existence was only proven for sufficiently small time intervals $(0, T)$. This problem was resolved in [204] for linear-quadratic differential games with player $i$’s controls constrained to lie in a compact, convex subset of $\mathbb{R}^{m_i}$ for all $i$ using weak topology properties and a fixed-point theorem of Tikhonov.

None of this previous research has dealt with the difficulties that arise from mixed state-control constraints (other than [95] which only examines a “one-player” game). Furthermore, specific equilibrium computation methods were not developed in these papers. Both of these issues are addressed in this chapter.

5.2.2 Properties of (5.1): A synopsis

For player $i$, suppose (5.2) holds for an arbitrary pair $(x_{-i}, u_{-i})$. Let $(x_i, u_i)$ be a feasible pair to (5.1) and let $(x_i^\tau, u_i^\tau) \triangleq (x_i^\tau, u_i^\tau) + \tau(x_i - x_i^*, u_i - u_i^*)$ for $\tau \in [0, 1]$. Since $(x_i^\tau, u_i^\tau)$
Consequently, a necessary condition for (5.2) to hold with an arbitrary pair \((x_{-i}, u_{-i})\) is that

\[
0 \leq f_i(x_i^*, x_{-i}, u_i^*, u_{-i}) - f_i(x_i^*, x_{-i}, u_i^*, u_{-i})
\]

\[
= \tau (x_i(T) - x_i^*(T)) \left( c_i + 2W_{ii}x_i^*(T) + \sum_{j \neq i} W_{ij}x_j(T) \right)
\]

\[
+ \tau^2 (x_i(T) - x_i^*(T))^T W_{ii} (x_i(T) - x_i^*(T)) + \tau \int_0^T \begin{pmatrix} x_i(t) - x_i^*(t) \\ u_i(t) - u_i^*(t) \end{pmatrix}^T \begin{pmatrix} p_i(t) \\ q_i(t) \end{pmatrix} dt
\]

\[
+ 2 \begin{pmatrix} P_{ii} & \frac{1}{2} (Q_{ii} + R_{ii}^T) \\ \frac{1}{2} (Q_{ii}^T + R_{ii}) & S_{ii} \end{pmatrix} \begin{pmatrix} x_i^*(t) \\ u_i^*(t) \end{pmatrix} + \sum_{j \neq i} \begin{pmatrix} P_{ij} & Q_{ij} \\ R_{ij} & S_{ij} \end{pmatrix} \begin{pmatrix} x_j(t) \\ u_j(t) \end{pmatrix} dt
\]

\[
+ \tau^2 \int_0^T \begin{pmatrix} x_i(t) - x_i^*(t) \\ u_i(t) - u_i^*(t) \end{pmatrix}^T \begin{pmatrix} P_{ii} & \frac{1}{2} (Q_{ii} + R_{ii}^T) \\ \frac{1}{2} (Q_{ii}^T + R_{ii}) & S_{ii} \end{pmatrix} \begin{pmatrix} x_i(t) - x_i^*(t) \\ u_i(t) - u_i^*(t) \end{pmatrix} dt.
\]

Consequently, a necessary condition for (5.2) to hold with an arbitrary pair \((x_{-i}, u_{-i})\) is that

\[
(x_i(T) - x_i^*(T))^T \left( c_i + 2W_{ii}x_i^*(T) + \sum_{j \neq i} W_{ij}x_j(T) \right)
\]

\[
+ \int_0^T \begin{pmatrix} x_i(t) - x_i^*(t) \\ u_i(t) - u_i^*(t) \end{pmatrix}^T \begin{pmatrix} p_i(t) \\ q_i(t) \end{pmatrix} + \sum_{j \neq i} \begin{pmatrix} P_{ij} & Q_{ij} \\ R_{ij} & S_{ij} \end{pmatrix} \begin{pmatrix} x_j(t) \\ u_j(t) \end{pmatrix} dt + 2 \begin{pmatrix} P_{ii} & \frac{1}{2} (Q_{ii} + R_{ii}^T) \\ \frac{1}{2} (Q_{ii}^T + R_{ii}) & S_{ii} \end{pmatrix} \begin{pmatrix} x_i^*(t) \\ u_i^*(t) \end{pmatrix} dt \geq 0
\]

for all pairs \((x_i, u_i)\) feasible to (5.1), which is a variational principle for optimality to this optimal control problem.
For notational convenience, let

\[
\Xi_{ij} \triangleq \begin{cases}
2 \begin{bmatrix}
P_{ii} & \frac{1}{2} (Q_{ii} + R_{ii}^T) \\
\frac{1}{2} (Q_{ii}^T + R_{ii}) & S_{ii}
\end{bmatrix} & \text{if } i = j \\
\begin{bmatrix}
P_{ij} & Q_{ij} \\
R_{ij} & S_{ij}
\end{bmatrix} & \text{if } i \neq j.
\end{cases}
\]

In general, \(\Xi_{ij} \neq \Xi_{ji}^T\) for \(i \neq j\), reflecting the asymmetric impact of the strategy of player \(j\) on player \(i\)'s objective function and vice versa. The case where symmetry between these matrices holds (i.e., \(\Xi_{ij} = \Xi_{ji}^T\) for \(i \neq j\)) is addressed in Section 5.3.

In [95], a set of postulates was introduced under which an LQ optimal control problem such as (5.1) is shown to have a solution; this solution is obtained as the limit of a sequence of piecewise trajectories formed by interpolating the discrete-time iterates computed from a time-stepping procedure. Except for Assumption (E_i), the other assumptions provided in the reference are fairly mild. In what follows, these results are summarized for the problem (5.1) by starting with the assumptions of [95]:

(A_i) the matrices \(W_{ii}\) and \(\Xi_{ii}\) are symmetric positive semidefinite;

(B_i) the functions \(p_i(t)\) and \(q_i(t)\) are Lipschitz continuous on \([0, T]\);

(C_i) a continuously differentiable function \(\hat{x}_{i}^{fs}\) with \(\hat{x}_{i}^{fs}(0) = \xi_{i}\) and a continuous function \(\hat{u}_{i}^{fs}\) exist such that for all \(t \in [0, T]\): \(d\hat{x}_{i}^{fs}(t)/dt = a_i + A_i\hat{x}_{i}^{fs}(t) + B_i\hat{u}_{i}^{fs}(t)\) and \(\hat{u}_{i}^{fs}(t) \in U_i(\hat{x}_{i}^{fs}(t))\), where

\[
U_i(x_i) \triangleq \{ u \in \mathbb{R}^{m_i} \mid b_i + C_i x_i + D_i u_i \geq 0 \};
\]
(D_i) \[ S_{ii}u_i = 0, D_iu_i \geq 0 \] implies \( u_i = 0 \) (a primal condition);

(E_i) \[ D_i^T \mu_i = 0, \mu_i \geq 0 \] implies \( (C_iA_i^kB_i) \mu_i = 0 \) for all integers \( k = 0, \cdots, n_i - 1 \), or equivalently, for all nonnegative integers \( k \) (a dual condition).

**Remark 5.1.** Assumptions (A_i) and (B_i) can be easily verified. Assumption (C_i) claims the existence of a continuously differentiable state trajectory and a continuous control trajectory that are feasible to (5.1), a slightly more abstract but demonstrable property. The algebraic implications of Assumptions (D_i) and (E_i) can be proven directly or can be claimed based on matrix properties. For instance, if the symmetric matrix \( S_{ii} \) is positive definite, Assumption (D_i) holds; Assumption (E_i) holds if \( D_i \) has full row rank or is entry-wise positive. A more detailed discussion of the meaning and consequences of Assumption (E_i) can be found in [95, Section 3.2].

To derive the necessary and sufficient optimality conditions for the LQ optimal control problem (5.1), start with the definition of the Hamiltonian function \( H_i(x, u, \lambda_i) \):

\[
\begin{pmatrix} x_i \\ u_i \end{pmatrix}^T \begin{pmatrix} p_i \\ q_i \end{pmatrix} + \sum_{j=1}^F \begin{bmatrix} P_{ij} & Q_{ij} \\ R_{ij} & S_{ij} \end{bmatrix} \begin{pmatrix} x_j \\ u_j \end{pmatrix} + \lambda_i^T (a_i + A_i x_i + B_i u_i),
\]

where \( \lambda_i \) is the vector of costate (also called adjoint) variables of the ordinary differential equation \( \dot{x}_i = a_i + A_i x_i + B_i u \). The Lagrangian function for the optimal control problem is therefore

\[
L_i(x, u, \lambda_i, \mu_i) \triangleq H_i(x, u, \lambda_i) - \mu_i^T (b_i + C_i x_i + D_i u_i),
\]

where \( \mu_i \) is the vector of Lagrange multipliers for \( b_i + C_i x_i + D_i u_i \geq 0 \). The [boundary-value] differential affine variational inequality (DAVI) associated with the LQ problem (5.1)
\[
\begin{align*}
\dot{\lambda}_i(t) &= \left( -p_i(t) - \sum_{j \neq i} (P_{ij}x_j(t) + Q_{ij}u_j(t)) \right) + a_i \\
\dot{x}_i(t) &= \left( \begin{bmatrix} -A_i^T & -2P_{ii} \\ 0 & A_i \end{bmatrix} \begin{bmatrix} \lambda_i(t) \\ x_i(t) \end{bmatrix} + \begin{bmatrix} -[Q_{ii} + R_{ii}^T] \\ 0 \end{bmatrix} \right) \begin{bmatrix} u_i(t) \\ \mu_i(t) \end{bmatrix} \\
0 &= q_i(t) + \sum_{j \neq i} (R_{ij}x_j(t) + S_{ij}u_j(t)) + \left[ Q_{ii} + R_{ii} \right] x_i(t) \\
&+ 2S_{ii}u_i(t) + B_{ii}^T \lambda_i(t) - D_{ii}^T \mu_i(t) \\
0 &\leq \mu_i(t) \perp b_i + C_i x_i(t) + D_i u_i(t) \geq 0
\end{align*}
\] 

(5.5)

Remark 5.2. The formulation of (5.5) follows from Pontryagin’s minimum principle. The differential equations are given by \( \dot{\lambda}_i = -\frac{\partial L_i}{\partial x_i} \) and \( \dot{x}_i = \frac{\partial L_i}{\partial \lambda_i} \), respectively, the bracketed equation arises from \( \frac{\partial L_i}{\partial u_i} = 0 \), the bracketed complementarity condition enforces complementary slackness, and \( \lambda_i(T) \) equals the partial derivative of the objective function cost associated with terminal time \( T \) with respect to \( x_i(T) \). \( \square \)

As a parameterized DAVI with \((x_i, \lambda_i)\) as the pair of differential variables, \((u_i, \mu_i)\) as the pair of algebraic variables, and \((x_{-i}, u_{-i})\) as parameters, the tuple \((x_i, u_i, \lambda_i, \mu_i)\) is a weak solution of (5.5) in the sense of Carathéodory if (a) \((x_i, \lambda_i)\) is absolutely continuous and \((u_i, \mu_i)\) is square-integrable on \([0, T]\), (b) the differential equations and the two algebraic
conditions hold for almost all \( t \in (0, T) \), and (c) the initial and boundary conditions are satisfied.

With detailed proofs given in [95], Theorem 5.1 below summarizes the main properties of the optimal control problem (5.1), the DAVI (5.5), and their relationship. The theorem starts with the Assumptions (A)–(E), under which part (I) asserts the existence of a weak solution of the DAVI (5.5) via a constructive argument. Part (II) of Theorem 5.1 states that any weak solution of the DAVI (5.5) is an optimal solution of (5.1), thus proving sufficiency of the Pontryagin optimality principle (see, e.g., [26, 43, 88, 141, 190]). Part (III) asserts several properties of an optimal solution of (5.1); this part is analogous to well-known results for a finite-dimensional convex quadratic programming. From these properties, part (IV) proves the reverse implication between solutions of optimal control problem (5.1) and DAVI (5.5), namely that any optimal solution of (5.1) must be a weak solution of the DAVI (5.5). An alternative characterization of optimality in terms of the variational inequality (5.4) is stated in part (V). Since this property was not derived in [95], a proof is provided. Finally, part (VI) asserts solution uniqueness from part (I) under the positive definiteness of the matrix \( S_{ii} \).

**Theorem 5.1.** Under Assumptions (A)–(E), the following statements (I)–(VI) hold for arbitrary integrable pair \((x_{-i}, u_{-i})\).

**I: Solvability of the DAVI** The DAVI (5.5) has a weak solution \((x^*_i, u^*_i, \lambda^*_i, \mu^*_i)\), provided that the pair \((x_{-i}, u_{-i})\) is Lipschitz continuous.

**II: Sufficiency of Pontryagin** If \((x^*_i, u^*_i, \lambda^*_i, \mu^*_i)\) is any weak solution of (5.5), then the pair \((x^*_i, u^*_i)\) is an optimal solution of the problem (5.1).

**III: Gradient characterization of optimal solutions** If \((\tilde{x}_i, \tilde{u}_i)\) and \((\bar{x}_i, \bar{u}_i)\) are any two optimal solutions of (5.1), then the following three properties hold:
(a) for almost all $t \in [0, T]$,
\[
\begin{bmatrix}
P_{ii} & \frac{1}{2} \left( Q_{ii} + R_{ii}^T \right) \\
\frac{1}{2} \left( Q_{ii}^T + R_{ii} \right) & S_{ii}
\end{bmatrix}
\begin{bmatrix}
\hat{x}_i(t) - \bar{x}_i(t) \\
\hat{u}_i(t) - \bar{u}_i(t)
\end{bmatrix} = 0;
\]

(b) $W_{ii} \hat{x}_i(T) = W_{ii} \bar{x}_i(T)$;

(c) $0 = \left( \hat{x}_i(T) - \bar{x}_i(T) \right)^T \left( c_i + \sum_{j \neq i} W_{ij} x_j(T) \right) + \int_0^T \left( \hat{x}_i(t) - \bar{x}_i(t) \right)^T \left( \begin{bmatrix} p_i(t) \\ q_i(t) \end{bmatrix} + \sum_{j \neq i} \begin{bmatrix} P_{ij} & Q_{ij} \\ R_{ij} & S_{ij} \end{bmatrix} \begin{bmatrix} x_j(t) \\ u_j(t) \end{bmatrix} \right) dt.$

Given any optimal solution $(\hat{x}, \hat{u})$ of (5.1), a feasible tuple $(\bar{x}, \bar{u})$ of (5.1) is optimal if and only if conditions (a), (b), and (c) hold.

(IV: Necessity of Pontryagin) Let $(x^*_i, u^*_i, \lambda^*_i, \mu^*_i)$ be the tuple obtained from part (I). A feasible tuple $(\bar{x}_i, \bar{u}_i)$ of (5.1) is optimal if and only if $(\bar{x}_i, \bar{u}_i, \lambda^*_i, \mu^*_i)$ is a weak solution of (5.5).

(V: Variational characterization of optimality) The satisfaction of the variational principle (5.4) for all pairs $(x, u)$ feasible to (5.1) is necessary and sufficient for an optimal solution $(x^*_i, u^*_i)$ of (5.1).

(VI: Uniqueness under positive definiteness) If $S_{ii}$ is positive definite and the pair $(x_{-i}, u_{-i})$ is Lipschitz continuous, then (5.1) has a unique optimal solution $(\hat{x}_i, \hat{u}_i)$ such that $\hat{x}_i$ is continuously differentiable and $\hat{u}_i$ is Lipschitz continuous on $[0, T]$; moreover, for any optimal $\hat{\lambda}_i$ and all $t \in [0, T]$,
\[
\hat{u}_i(t) \in \arg\min_{u_i \in U_i(\hat{x}_i)} H_i(\hat{x}_i(t), x_{-i}(t), u_i, u_{-i}(t), \hat{\lambda}_i(t)).
\]
Proof. For (V), the derivation of (5.4) has proven necessity. The sufficiency follows easily from the positive semidefiniteness Assumption (A_i). Incidentally, the validity of this part only requires Assumption (A_i). □

5.3 The symmetric case

Theorem 5.1 pertains only to the individual players’ optimal control problems and says nothing about the game as a whole. In what follows, under an additional symmetry condition, it is shown that this inequality constrained differential LQ game is equivalent to a single concatenated optimal control problem that is under the scope of applicability of Theorem 5.1. Define aggregate vector \( \mathbf{c} \equiv (c_i)_{i=1}^F \) with similar definitions holding for \( p \) and \( q \) and aggregate matrix \( \mathbf{W} \equiv \begin{bmatrix} (W_{ij})_{i,j=1}^F + \text{diag}(W_{ii})_{i=1}^F \end{bmatrix} \) with similar definitions for

\[
\begin{bmatrix}
P & Q \\
R & S
\end{bmatrix} \equiv \begin{bmatrix}
(P_{ij})_{i,j=1}^F + \text{diag}(P_{ii})_{i=1}^F & (Q_{ij})_{i,j=1}^F + \text{diag}(Q_{ii})_{i=1}^F \\
(R_{ij})_{i,j=1}^F + \text{diag}(R_{ii})_{i=1}^F & (S_{ij})_{i,j=1}^F + \text{diag}(S_{ii})_{i=1}^F
\end{bmatrix}.
\]
Furthermore, define $\Xi \triangleq [\Xi_{ij}]_{i,j=1}^F$. Using this notation, define the aggregated LQ optimal control problem in the variables $(x, u)$:

$$\begin{align*}
\text{minimize} \quad & x(T)^T(c + \frac{1}{2} W x(T)) + \\
& \int_0^T \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} + \frac{1}{2} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \ dt \\
\text{subject to, for all } i \in \{1, \ldots, F\}, \\
& x_i(0) = \xi_i \\
& \begin{cases} \dot{x}_i(t) = a_i + A_i x_i(t) + B_i u_i(t) \\
& b_i + C_i x_i(t) + D_i u_i(t) \geq 0 \end{cases} \text{ for almost all } t \in [0, T].
\end{align*}$$

(5.6)

Note that the matrix $\begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ in the objective function is a principal rearrangement of $\Xi$, thereby implying that properties of $\Xi$ such as symmetry and positive semidefiniteness will hold for $\begin{bmatrix} P & Q \\ R & S \end{bmatrix}$.

Aggregate the Assumptions $(A_i)–(E_i)$ to obtain a set of conditions pertaining to the above single LQ optimal control problem.

(A) the matrices $W$ and $\Xi$ are symmetric positive semidefinite;

(B) Assumption $(B_i)$ holds for all $i = 1, \ldots, F$;

(C) Assumption $(C_i)$ holds for all $i = 1, \ldots, F$;

(D) $[S u_i = 0 \text{ and } D_i u_i \geq 0 \text{ for all } i = 1, \ldots, F]$ implies $[u_i = 0 \text{ for all } i = 1, \ldots, F]$;

(E) Assumption $(E_i)$ holds for all $i = 1, \ldots, F$.

Under the above conditions, the following equivalence for the “symmetric” differential LQ
game can be derived.

**Theorem 5.2.** Under Assumption (A), the following statements hold.

(I: Equivalence) A pair \((x^*, u^*)\) is a Nash equilibrium if and only if \((x^*, u^*)\) is an optimal solution of (5.6).

(II: Existence) Under the additional Assumptions (B)–(E), a Nash equilibrium exists such that \(x^*\) is absolutely continuous and \(u^*\) is square-integrable on \([0, T]\).

(III: Uniqueness) If \(S\) is positive definite in addition to (A)–(E), then \((x^*, u^*)\) is the unique Nash equilibrium such that \(x^*\) is continuously differentiable and \(u^*\) is Lipschitz continuous on \([0, T]\).

**Proof.** This result follows from two observations under Assumption (A): (a) the pair \((x^*, u^*)\) is a Nash equilibrium if and only if the following variational property (an aggregation of (5.4)) holds for all pairs \((x, u)\) feasible to (5.6):

\[
\sum_{i=1}^{F} (x_i(T) - x_i^*(T))^T \left( c_i + 2W_{ii}x_i^*(T) + \sum_{j \neq i} W_{ij}x_j^*(T) \right) + \\
\sum_{i=1}^{F} \int_0^T \left( x_i(t) - x_i^*(t) \right)^T \left( \begin{array}{c} p_i(t) \\ q_i(t) \end{array} \right) \left( \begin{array}{cc} P_{ij} & Q_{ij} \\ R_{ij} & S_{ij} \end{array} \right) \left( \begin{array}{c} x_j^*(t) \\ u_j^*(t) \end{array} \right) dt \geq 0, \\
+ 2 \left[ P_{ii} \quad \frac{1}{2} (Q_{ii} + R_{ii}^T) \quad \frac{1}{2} (Q_{ii}^T + R_{ii}) \quad S_{ii} \right] \left( \begin{array}{c} x_i^*(t) \\ u_i^*(t) \end{array} \right) \right] dt \geq 0, \\
(5.7)
\]

and (b) the variational condition (5.7) is necessary and sufficient for the pair \((x^*, u^*)\) to be optimal for the LQ optimal control problem (5.6). Assertions (II) and (III) follow from the theory in [95] applied to (5.6) and the equivalence in (I). \(\square\)
A consequence of the equivalence between the symmetric differential LQ game and the single LQ optimal control problem is that the numerical time-stepping method presented in [95] can be used to solve for Nash equilibria. This method consists of solving convex quadratic subprograms obtained by discretizing (5.6) in time; the convergence of the method provides a constructive proof for the existence result in part (I) of Theorem 5.2. Details are omitted and can be found in the cited reference.

5.4 The asymmetric case

Next consider the asymmetric case where the matrix $\Xi$ is not necessarily symmetric. In this situation, the equivalence proven in Theorem 5.2 does not hold and the computation method of [95] cannot be applied to (5.6) for equilibrium identification. Instead, the symmetry assumption required in Section 5.3 is replaced with a spectral condition on a certain condensed matrix obtained from $\Xi$. Under this matrix-theoretic spectral condition, it is shown that a distributed algorithm will converge to the unique NE of the differential LQ game. To simplify analysis, Assumption (W) (a diagonal condition, see below) is made, thereby removing the dependence of the rivals’ terminal states in each player’s objective function (i.e., the sum $\sum_{j \neq i} W_{ij}x_j(T)$ is eliminated). The analysis can be extended to handle the case where some $W_{ij}$ are nonzero; in this case, the corresponding diagonal block $W_{ii}$ is required to be positive definite with its smallest eigenvalue “dominating” the sum of the nonzero off-diagonal blocks, roughly speaking.

The treatment in this section is an extension of that for the static Nash game whose details can be found in [76]. Unlike the algorithm for the latter game that solves finite-dimensional optimization problems, the algorithm below solves a sequence of LQ optimal control problems, which in turn need to be discretized for numerical computation by, for
example, the time-stepping method described in [95].

In addition to Assumption (W), two more assumptions are needed in this section, (Â) and (̂D), which are stated below. Assumption (Â) assumes that the diagonal blocks Ξ

_{ii} are positive definite matrices, albeit not necessarily symmetric. This assumption implies (C_i) for all \( i = 1, \cdots, F \). Assumption (̂D) yields the uniform boundedness Lemma 5.1 through a proof similar to that of Gronwall’s lemma.

(Â) For all \( i = 1, \cdots, F \), the matrices Ξ

_{ii} are positive definite with minimum eigenvalues \( \sigma_i > 0 \); the matrices \( W_{ii} \) remain (symmetric) positive semidefinite;

(̂D) For all \( i = 1, \cdots, F \), the following implication holds: \( D_i u_i \geq 0 \Rightarrow u_i = 0 \).

(W) For all \( i = 1, \cdots, F \), the matrices \( W_{ij} = 0 \) for all \( j \neq i \).

Lemma 5.1. Under Assumption (̂D), there exists a constant \( \zeta > 0 \) such that for all \( i = 1, \cdots, F \),

\[
\sup_{t \in [0, T]} \| x_i(t) \| \leq \zeta \quad \text{and} \quad \text{ess sup}_{t \in [0, T]} \| u_i(t) \| \leq \zeta
\]

for any trajectory \((x_i, u_i)\) feasible to (5.1).

Proof. Given (̂D) and the constraint \( b_i + C_i x_i(t) + D_i u_i(t) \geq 0 \) holding for almost all \( t \in [0, T] \), there exist positive \( \alpha_i \) and \( \widehat{\alpha}_i \) such that for all feasible \((x_i, u_i)\),

\[
\| u_i(t) \| \leq \alpha_i \| b_i + C_i x_i(t) \| \leq \widehat{\alpha}_i (1 + \| x_i(t) \|)
\] (5.8)
holds when \( u_i(t) \) is well-defined. Now consider that for all \( t \in [0, T] \) with \( \dot{x}_i(t) \) well-defined,

\[
\frac{d}{dt} \|x_i(t)\| \leq \|\dot{x}_i(t)\| = \|a_i + A_i x_i(t) + B_i u_i(t)\| \\
\leq \|a_i\| + \|A_i\| \|x_i(t)\| + \|B_i\| \tilde{a}_i(1 + \|x_i(t)\|) \\
\leq \beta_i (1 + \|x_i(t)\|)
\]

where \( \beta_i > 0 \). With manipulations similar to the proof of Gronwall’s lemma,

\[-\beta_i \|x_i(t)\| + \frac{d}{dt} \|x_i(t)\| \leq \beta_i, \text{ implying} \]

\[
\frac{d}{dt} \left( e^{-\beta_i t} \|x_i(t)\| \right) \leq \beta_i e^{-\beta_i t}; \\
\text{thus, } \|x_i(t)\| \leq e^{\beta_i t} \left( \|x_i(0)\| + \int_0^t \beta_i e^{-\beta_i s} ds \right).
\]

It follows that a constant \( \zeta' > 0 \) exists such that \( \|x_i(t)\| \leq \zeta' \) for all \( t \in [0, T] \). The existence of the desired constant \( \zeta \) follows readily from (5.8). \( \square \)

Consider the following Jacobi-type iterative solution method for the asymmetric differential LQ game. The analysis of a related Gauss-Seidel version of the algorithm is similar and omitted.

\begin{boxedtext}{Jacobi method for asymmetric differential LQ games.}
For each \( i = 1, \cdots, \mathcal{F} \), let \( x_i^0 \) be a continuously differentiable Lipschitz function and \( u_i^0 \) be a Lipschitz continuous function with \((x_i^0, u_i^0)\) feasible to (5.1). Let \( k = 0 \).

**Step 1:** If \((x_i^k, u_i^k)\) solves (5.1) given \((x_{-i}^k, u_{-i}^k)\) for each \( i \), stop.

**Step 2:** Let \((x_i^{k+1}, u_i^{k+1})\) solve (5.1) given \((x_{-i}^k, u_{-i}^k)\) for each \( i \).

**Step 3:** Set \( k \leftarrow k + 1 \) and return to Step 1.
\end{boxedtext}
Remark 5.3. The presented Jacobi method is a distributed computation scheme for asymmetric differential LQ games that operates in a manner similar to its namesake, the Jacobi method for solving a linear system of equations. Like the iterative Tikhonov method provided in Section 4.2.3, this algorithm can be implemented in parallel with players sharing only their optimal trajectories at the end of each iteration (i.e., each player solves (5.1) parametrized by its rivals' optimal trajectories from the previous iteration). Finding the [exact] solution of (5.1) in Step 2 requires the use of a separate [inner] algorithm such as the iterative discretization method of [95]. As a consequence of the [upcoming] convergence of this computational scheme, every differential Nash game satisfying the stated assumptions has a unique equilibrium.

Remark 5.4. The Lipschitz continuity of \( (x^k_j, u^k_j) \) for \( j \neq i \) is needed so that Assumption (B\(_i\)) can be satisfied for the computation of the next pair \( (x^{k+1}_i, u^{k+1}_i) \). In turn, the positive definiteness of \( \Xi_{ii} \) ensures the preservation of the Lipschitz property of \( (x^{k+1}_i, u^{k+1}_i) \) and thus the Lipschitz continuity of the entire sequence of iterates. Under Assumptions (\( \hat{A} \)), (B), (W), (\( \hat{D} \)), and (E), the sequence of iterates \( \{ (x^k, u^k) \} \) is well defined with \( x^k \) being continuously differentiable and \( u^k \) being Lipschitz continuous on \( [0, T] \) for all \( k \).

The convergence of this method in the differential context has not been analyzed before and is the focus of the rest of the chapter.
From the variational condition (5.4),

\[
\left( x_i^k(T) - x_i^{k+1}(T) \right)^T \left( c_i + 2W_{ii}x_i^{k+1}(T) \right) + \\
\int_0^T \left( x_i^k(t) - x_i^{k+1}(t) \right)^T \left( \begin{array}{c} p_i(t) \\ q_i(t) \end{array} \right) + \sum_{j \neq i} \left[ \begin{array}{cc} P_{ij} & Q_{ij} \\ R_{ij} & S_{ij} \end{array} \right] \left( \begin{array}{c} x_j^{k+1}(t) \\ u_j^{k+1}(t) \end{array} \right) dt \geq 0
\]

and

\[
\left( x_i^{k+1}(T) - x_i^k(T) \right)^T \left( c_i + 2W_{ii}x_i^k(T) \right) + \\
\int_0^T \left( x_i^{k+1}(t) - x_i^k(t) \right)^T \left( \begin{array}{c} p_i(t) \\ q_i(t) \end{array} \right) + \sum_{j \neq i} \left[ \begin{array}{cc} P_{ij} & Q_{ij} \\ R_{ij} & S_{ij} \end{array} \right] \left( \begin{array}{c} x_j^{k-1}(t) \\ u_j^{k-1}(t) \end{array} \right) dt \geq 0.
\]
Adding,

\[
0 \leq 2 \left( x_i^k(T) - x_i^{k+1}(T) \right)^T W_{ii} \left( x_i^{k+1}(T) - x_i^k(T) \right) + \\
\int_0^T \left( x_i^k(t) - x_i^{k+1}(t) \right)^T \left( \sum_{j \neq i} P_{ij} Q_{ij} R_{ij} S_{ij} \right) \left( x_j^k(t) - x_j^{k-1}(t) \right) \left( u_j^k(t) - u_j^{k-1}(t) \right) dt \\
+ 2 \left[ P_{ii} \frac{1}{2} \left( Q_{ii} + R_{ii} \right) \right] \left( x_i^{k+1}(t) - x_i^k(t) \right) \left( u_i^{k+1}(t) - u_i^k(t) \right) dt
\]

by leveraging the positive definiteness of \( \Xi_{ii} \) and the Cauchy-Schwarz inequality. By the Cauchy-Schwarz inequality for integrals,

\[
\sigma_i^\Xi \left( \int_0^T \left\| \begin{bmatrix} x_i^k(t) - x_i^{k+1}(t) \\ u_i^k(t) - u_i^{k+1}(t) \end{bmatrix} \right\|^2 dt \right)^{1/2} \leq \sum_{j \neq i} \left\| P_{ij} Q_{ij} \right\| \left( \int_0^T \left\| \begin{bmatrix} x_j^k(t) - x_j^{k-1}(t) \\ u_j^k(t) - u_j^{k-1}(t) \end{bmatrix} \right\|^2 dt \right)^{1/2}.
\]

(5.9)

Defining the difference

\[
e_i^k \triangleq \sqrt{\sigma_i^\Xi} \left( \int_0^T \left\| \begin{bmatrix} x_i^k(t) - x_i^{k+1}(t) \\ u_i^k(t) - u_i^{k+1}(t) \end{bmatrix} \right\|^2 dt ,
\]

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the vector \( e^k \triangleq \left( e^k_i \right)_{i=1}^F \), and the matrix \( \Gamma \triangleq \left[ \Gamma_{ij} \right]_{i,j=1}^F \), where

\[
\Gamma_{ij} \triangleq \begin{cases} 
0 & \text{if } i = j \\
\frac{1}{\sqrt{\sigma^2_i \sigma^2_j}} \| \Xi_{ij} \| & \text{if } i \neq j,
\end{cases}
\]

it can be deduced by aggregating the inequalities (5.9) for \( i = 1, \ldots, F \) that

\[ e^k \leq \Gamma e^{k-1}. \]

(5.10)

Thus, if the spectral radius \( \rho(\Gamma) < 1 \), then the sequence \( \{e^k\} \) contracts and converges to zero. In particular, the sequence \( \{(x^k_i(t), u^k_i(t))\} \) converges strongly to a square-integrable, thus integrable, limit \( (x^\infty_i(t), u^\infty_i(t)) \) in the space \( L^2[0, T] \):

\[
\lim_{k \to \infty} \int_0^T \left\| \begin{pmatrix} x^k_i(t) - x^\infty_i(t) \\
u^k_i(t) - u^\infty_i(t) \end{pmatrix} \right\| dt = 0.
\]

Thus, for an infinite index set \( \kappa \subset \{1, 2, \cdots, \infty\} \), the subsequence of functions \( \{(x^k_i(t), u^k_i(t))\}_{k \in \kappa} \) converges pointwise to \( (x^\infty_i(t), u^\infty_i(t)) \) for almost all \( t \in [0, T] \) by the Riesz-Fischer Theorem. Hence, \( b_i + C_i x^\infty_i(t) + D_i u^\infty_i(t) \geq 0 \) for almost all \( t \). Moreover,

\[
\lim_{k \to \infty} \int_0^t \left\| \begin{pmatrix} x^k_i(\tau) - x^\infty_i(\tau) \\
u^k_i(\tau) - u^\infty_i(\tau) \end{pmatrix} \right\| d\tau = 0 \quad \text{for all } t \in (0, T].
\]

Therefore, for all \( t \in (0, T] \),

\[
\lim_{k \to \infty} \int_0^t \left( a_i + A_i x^k_i(\tau) + B_i u^k_i(\tau) \right) d\tau = \int_0^t \left( a_i + A_i x^\infty_i(\tau) + B_i u^\infty_i(\tau) \right) d\tau.
\]

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At times $t$ where $\{x^k_i(t)\}_{k \in \kappa}$ converges to $x^\infty_i(t)$, expressing $x^k_i(t)$ as the integration of $\dot{x}^k_i(\tau)$ from $(0, t]$ with $x^k_i(0) = \xi_i$ gives

$$x^\infty_i(t) = \lim_{k(\in\kappa) \to \infty} x^k_i(t) = \xi_i + \lim_{k(\in\kappa) \to \infty} \int_0^t \left( a_i + A_i x^k_i(\tau) + B_i u^k_i(\tau) \right) d\tau$$

$$= \xi_i + \int_0^t \left( a_i + A_i x^\infty_i(\tau) + B_i u^\infty_i(\tau) \right) d\tau.$$

From Lemma [5.1] the sequence $\{(x^k_i(t), u^k_i(t))\}_{k \in \kappa}$ is uniformly bounded for $t \in [0, T]$; thus, the limit pair $(x^\infty_i(t), u^\infty_i(t))$ is bounded for almost all $t \in [0, T]$. For any two times $t < s$ in $[0, T]$,

$$x^k_i(t) - x^k_i(s)$$

$$= \int_t^s \left( a_i + A_i x^k_i(\tau) + B_i u^k_i(\tau) \right) d\tau$$

$$= \int_t^s \left( a_i + A_i x^\infty_i(\tau) + B_i u^\infty_i(\tau) \right) d\tau$$

$$+ \int_t^s \left( A_i \left( x^k_i(\tau) - x^\infty_i(\tau) \right) + B_i \left( u^k_i(\tau) - u^\infty_i(\tau) \right) \right) d\tau.$$

Therefore, the family $\{x^k_i(t)\}_{k \in \kappa}$ is equicontinuous; by the well-known Arzelà-Ascoli theorem, this sequence has a subsequence that converges uniformly to a continuous function on $t \in [0, T]$. Without loss of generality, assume that $\{x^k_i(t)\}_{k \in \kappa}$ converges uniformly to $x^\infty_i(t)$ which is continuous. It follows that for all $t \in [0, T]$,

$$x^\infty_i(t) = \xi_i + \int_0^t \left( a_i + A_i x^\infty_i(\tau) + B_i u^\infty_i(\tau) \right) d\tau.$$

The pair $(x^\infty_i, u^\infty_i)$ is feasible to (5.1). Moreover, since

$$\lim_{k \to \infty} \left( x^k_i(T) - x^{k+1}_i(T) \right)^T W_{ii} \left( x^{k+1}_i(T) - x^k_i(T) \right) = 0$$
and $W_{ii}$ is symmetric positive semidefinite, it follows that

$$\lim_{k(\varepsilon_k) \to \infty} W_{ii} \left( x_i^{k+1}(T) - x_i^\infty(T) \right) = 0$$

and

$$\lim_{k(\varepsilon_k) \to \infty} \left( x_i^{k+1}(T) - x_i^\infty(T) \right)^T W_{ii} \left( x_i^{k+1}(T) - x_i^\infty(T) \right) = 0.$$

To show that the tuple $(x_i^\infty, u_i^\infty)_{i=1}^F$ is a Nash equilibrium of the game, it suffices to show that

$$( x_i(T) - x_i^\infty(T) )^T \left( c_i + 2W_{ii}x_i^\infty(T) \right) +
\int_0^T \left( \begin{array}{c}
x_i(t) - x_i^\infty(t) \\
u_i(t) - u_i^\infty(t)
\end{array} \right)^T \left( \begin{array}{c}
p_i(t) \\
q_i(t)
\end{array} \right) + \sum_{j \neq i} \left[ \begin{array}{cc}
P_{ij} & Q_{ij} \\
R_{ij} & S_{ij}
\end{array} \right] \left( \begin{array}{c}
x_j^\infty(t) \\
u_j^\infty(t)
\end{array} \right) dt \geq 0$$

for every feasible pair $(x_i, u_i)$ of (5.1) and for all $i = 1, \cdots, F$. For all $k$,

$$( x_i(T) - x_i^{k+1}(T) )^T \left( c_i + 2W_{ii}x_i^{k+1}(T) \right) +
\int_0^T \left( \begin{array}{c}
x_i(t) - x_i^{k+1}(t) \\
u_i(t) - u_i^{k+1}(t)
\end{array} \right)^T \left( \begin{array}{c}
p_i(t) \\
q_i(t)
\end{array} \right) + \sum_{j \neq i} \left[ \begin{array}{cc}
P_{ij} & Q_{ij} \\
R_{ij} & S_{ij}
\end{array} \right] \left( \begin{array}{c}
x_j^{k+1}(t) \\
u_j^{k+1}(t)
\end{array} \right) dt \geq 0.$$
It holds that

\[
(x_i(T) - x_i^{k+1}(T))^T \left( c_i + 2W_{ii}x_i^{k+1}(T) \right) \\
= \left( x_i(\mathcal{T}) - x_i^{k+1}(\mathcal{T}) \right)^T \left( c_i + 2W_{ii}x_i^{k+1}(\mathcal{T}) \right) \\
+ (x_i(T) - x_i^{\infty}(T))^T \left( c_i + 2W_{ii}x_i^{k+1}(\mathcal{T}) \right) \\
= 2 \left( x_i(\mathcal{T}) - x_i^{k+1}(\mathcal{T}) \right)^T W_{ii} \left( x_i^{k+1}(\mathcal{T}) - x_i^{\infty}(\mathcal{T}) \right) + \\
(x_i(T) - x_i^{\infty}(\mathcal{T}))^T \left( c_i + 2W_{ii}x_i^{k+1}(\mathcal{T}) \right) + \left( x_i(\mathcal{T}) - x_i^{k+1}(\mathcal{T}) \right)^T \left( c_i + 2W_{ii}x_i^{\infty}(\mathcal{T}) \right).
\]

Therefore,

\[
\lim_{k(\in\kappa)\to\infty} \left( x_i(\mathcal{T}) - x_i^{k+1}(\mathcal{T}) \right)^T \left( c_i + 2W_{ii}x_i^{k+1}(\mathcal{T}) \right) = \left( x_i(\mathcal{T}) - x_i^{\infty}(\mathcal{T}) \right)^T \left( c_i + 2W_{ii}x_i^{\infty}(\mathcal{T}) \right).
\]

Similarly,

\[
\lim_{k\to\infty} \int_0^T \begin{pmatrix}
(x_i(t) - x_i^{k+1}(t)) & (p_i(t)) \\
(u_i(t) - u_i^{k+1}(t)) & (q_i(t))
\end{pmatrix}

+ 2 \begin{bmatrix}
P_{ii} & \frac{1}{2} (Q_{ii} + R_{ii}) \\
\frac{1}{2} (Q_{ii}^T + R_{ii}) & S_{ii}
\end{bmatrix}
\begin{pmatrix}
x_i^{k+1}(t) \\
u_i^{k+1}(t)
\end{pmatrix}
dt
\]

\[
= \int_0^T \begin{pmatrix}
(x_i(t) - x_i^{\infty}(t)) & (p_i(t)) \\
(u_i(t) - u_i^{\infty}(t)) & (q_i(t))
\end{pmatrix}

+ 2 \begin{bmatrix}
P_{ii} & \frac{1}{2} (Q_{ii} + R_{ii}) \\
\frac{1}{2} (Q_{ii}^T + R_{ii}) & S_{ii}
\end{bmatrix}
\begin{pmatrix}
x_i^{\infty}(t) \\
u_i^{\infty}(t)
\end{pmatrix}
dt.
\]
Finally,

$$\int_0^T \begin{pmatrix} x_i(t) - x_i^{k+1}(t) \\ u_i(t) - u_i^{k+1}(t) \end{pmatrix}^T \begin{bmatrix} P_{ij} & Q_{ij} \\ R_{ij} & S_{ij} \end{bmatrix} \begin{pmatrix} x_j^k(t) \\ u_j^k(t) \end{pmatrix} \, dt$$

$$= \int_0^T \begin{pmatrix} x_i(t) - x_i^\infty(t) \\ u_i(t) - u_i^\infty(t) \end{pmatrix}^T \begin{bmatrix} P_{ij} & Q_{ij} \\ R_{ij} & S_{ij} \end{bmatrix} \begin{pmatrix} x_j^\infty(t) \\ u_j^\infty(t) \end{pmatrix} \, dt$$

$$+ \int_0^T \begin{pmatrix} x_i(t) - x_i^\infty(t) \\ u_i(t) - u_i^\infty(t) \end{pmatrix}^T \begin{bmatrix} P_{ij} & Q_{ij} \\ R_{ij} & S_{ij} \end{bmatrix} \begin{pmatrix} x_j^k(t) \\ u_j^k(t) \end{pmatrix} \, dt$$

$$+ \int_0^T \begin{pmatrix} x_i^\infty(t) - x_i^{k+1}(t) \\ u_i^\infty(t) - u_i^{k+1}(t) \end{pmatrix}^T \begin{bmatrix} P_{ij} & Q_{ij} \\ R_{ij} & S_{ij} \end{bmatrix} \begin{pmatrix} x_j^k(t) \\ u_j^k(t) \end{pmatrix} \, dt,$$

from which it can be deduced that

$$\lim_{k \to \infty} \int_0^T \begin{pmatrix} x_i(t) - x_i^{k+1}(t) \\ u_i(t) - u_i^{k+1}(t) \end{pmatrix}^T \begin{bmatrix} P_{ij} & Q_{ij} \\ R_{ij} & S_{ij} \end{bmatrix} \begin{pmatrix} x_j^k(t) \\ u_j^k(t) \end{pmatrix} \, dt$$

$$= \int_0^T \begin{pmatrix} x_i(t) - x_i^\infty(t) \\ u_i(t) - u_i^\infty(t) \end{pmatrix}^T \begin{bmatrix} P_{ij} & Q_{ij} \\ R_{ij} & S_{ij} \end{bmatrix} \begin{pmatrix} x_j^\infty(t) \\ u_j^\infty(t) \end{pmatrix} \, dt.$$

Summarizing the above analysis, the following convergence, existence, and uniqueness result for the asymmetric differential LQ Nash game that is the counterpart of Theorem 5.2 has been obtained. The condition $\rho(\Gamma) < 1$ is a familiar condition for the contraction of fixed-point iterations for computing static Nash equilibria (e.g., [76, Section 6]).

**Theorem 5.3.** Under Assumptions ($\hat{A}$), (B), (W), ($\hat{D}$), (E), the following statements hold for the sequence $\{(x_i^k, u_i^k)\}$ generated by the specified Jacobi method.
(I: Well-definedness) The sequence \( \{ (x^k_i, u^k_i) \} \) is well-defined with \( x^k_i \) being continuously differentiable and \( u^k_i \) Lipschitz continuous on \([0, T]\) for all \( k \).

(II: Contraction and strong convergence) If \( \rho(\Gamma) < 1 \), the sequence \( \{ (x^k, u^k) \} \) converges strongly to a pair \((x^\infty, u^\infty)\) that is the unique Nash equilibrium of the differential LQ game.

**Proof.** It remains to be shown that the differential game has a unique Nash equilibrium. Let \((\hat{x}, \hat{u})\) and \((\tilde{x}, \tilde{u})\) denote two Nash equilibria. After defining the error vector \( e \triangleq (e_i)_{i=1}^F \), where

\[
e_i \triangleq \sigma_i \sqrt{\int_0^T \left\| \begin{pmatrix} \hat{x}_i(t) - \tilde{x}_i(t) \\ \hat{u}_i(t) - \tilde{u}_i(t) \end{pmatrix} \right\|^2 dt},
\]

\( e \leq \Gamma e \) holds. Since \( \rho(\Gamma) < 1 \), \( e = 0 \) and the Nash equilibria must be identical. \( \square \)

5.5 Example problems

Given the abstract framing of the symmetric and asymmetric problems in the previous two sections, two concrete examples are presented to illustrate that these types of problems arise naturally in microeconomic games. The first example model is an adaptation of the well-known Nash-Cournot equilibrium problem while the second is a conjectured supply function equilibrium problem. Although Nash-Cournot problems are typically studied in a static setting, some research has been conducted on oligopolistic markets formulated as differential games (e.g., \[38, 44, 62, 166, 212\]). There has been little research on differential conjectured supply function games, but the example formulation represents a natural extension to the static problem form. Although game generalizations may seem straightforward, the presented examples should be treated with caution; the conditions that must be satisfied should not be haphazardly assumed to hold for similar problems.
5.5.1 A Nash-Cournot game

In non-cooperative microeconomic equilibrium problems, \( \mathcal{F} \) players produce and sell a commodity to maximize profit subject to constraints such as manufacturing capacity and available budget. For the Nash-Cournot version of this problem, each player believes that their output affects the commodity price that is represented as a function of total output.

For a two-player, two-node problem, let the production of player \( i \) at node \( \ell \) and time \( t \) be denoted by \( g_{i\ell}(t) \), the sales of player \( i \) to node \( \ell \) at time \( t \) be denoted by \( s_{i\ell}(t) \), and the ramp rate (i.e., instantaneous change in production) of player \( i \) at node \( \ell \) and time \( t \) be denoted by \( r_{i\ell}(t) \). Let the linear Cournot pricing function at node \( \ell \) at time \( t \) be given by

\[
P_{\ell}^0 - \frac{P_{\ell}^0}{Q_{\ell}^0} \left( \sum_{i=1}^{\mathcal{F}} s_{i\ell}(t) \right) \quad (P_{\ell}^0 \text{ and } Q_{\ell}^0 \text{ positive for all } \ell)
\]

and the quadratic production cost of player \( i \) at node \( \ell \) and time \( t \) be given by

\[
c_{i\ell}^1 g_{i\ell}(t) + c_{i\ell}^2 g_{i\ell}(t)^2 \quad (c_{i\ell}^1 \text{ and } c_{i\ell}^2 \text{ positive for all } i \text{ and } \ell).
\]

If each player pays a marginal transportation cost of \( w(t) \) at time \( t \) for product movement from node 2 to node 1, the transportation cost for firm \( i \) is \( (s_{i1}(t) - g_{i1}(t))w(t) \). Assume that \( w(t) \) is a given Lipschitz continuous function on \( t \in [0, T] \).
With this problem description, player 1’s optimal control problem is

\[
\minimize_{g_1, s_1, r_1} \int_0^T \begin{pmatrix}
g_{11}(t) \\
g_{12}(t) \\
s_{11}(t) \\
s_{12}(t) \\
r_{11}(t) \\
r_{12}(t)
\end{pmatrix}^T \begin{pmatrix}
c_{11}^1 - w(t) \\
c_{12}^1 \\
-P_1^0 + w(t) \\
-P_2^0 \\
0 \\
0
\end{pmatrix} \begin{pmatrix}
g_{11}(t) \\
g_{12}(t) \\
s_{11}(t) \\
s_{12}(t) \\
r_{11}(t) \\
r_{12}(t)
\end{pmatrix} dt
\]

subject to (next page)
\[ g_{11}(0) = g_{11}^0, \ g_{12}(0) = g_{12}^0, \ \text{and for almost all } t \in [0, T], \]
\[
\begin{align*}
\dot{g}_{11}(t) &= r_{11}(t) \\
\dot{g}_{12}(t) &= r_{12}(t) \\
\end{align*}
\]
\[
-\Xi_{1\ell} + r_{1\ell}(t) \geq 0 \quad \text{for } \ell = 1, 2 \\
r_{1\ell} - r_{1\ell}(t) \geq 0 \quad \text{for } \ell = 1, 2 \\
g_{11}(t) + g_{12}(t) - s_{11}(t) - s_{12}(t) \geq 0 \\
-g_{11}(t) - g_{12}(t) + s_{11}(t) + s_{12}(t) \geq 0.
\]

The state and control variables of this problem are
\[
\{g_{11}, g_{12}\} \quad \text{and} \quad \{s_{11}, s_{12}, r_{11}, r_{12}\}, \ \text{respectively.}
\]

The first group of constraints describes ramp rates limits with \( r_{i\ell} \) and \( r_{i\ell} \) being the limiting ramp rates for player \( i \)'s production at node \( \ell \). The last two constraints equate total production and total sales.

Player 2's objective function is easily shown to be identical to that given above except with 1 and 2 interchanged in the player index \( i \). Therefore, it is apparent that \( \Xi \triangleq \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{bmatrix} \)
is the symmetric matrix

\[
\begin{pmatrix}
2c_{11}^2 & 2c_{12}^2 & 2P_1^0 & 2P_1^0 & P_1^0 \\
2c_{12}^2 & 2P_1^0 & 2Q_1^0 & P_1^0 & P_2^0 \\
2P_1^0 & 2Q_1^0 & 2P_2^0 & P_2^0 & P_2^0 \\
2Q_1^0 & 2P_2^0 & 2Q_2^0 & 2P_2^0 & P_2^0 \\
P_1^0 & P_2^0 & P_2^0 & 2P_2^0 & 0 \\
Q_1^0 & Q_2^0 & Q_2^0 & 2Q_2^0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Furthermore, \( \Xi \) must be positive semidefinite by row diagonal dominance, thereby fulfilling Assumption (A). Assumption (B) is obvious by the assumed Lipschitz continuity of \( w(t) \). Assumption (C) follows from setting \( r_{ij}(t) = 0 \) and \( s_{ij}(t) = g_{ij}(t) \) for \( i, j = 1, 2 \) and
\( t \in [0, \mathcal{T}] \) to obtain a feasible trajectory with \( g_{ij}(t) = g_{ij}^0 \) for \( t \in [0, \mathcal{T}] \). Since

\[
S \triangleq \begin{bmatrix}
2S_{11} & S_{12} \\
S_{21} & 2S_{22}
\end{bmatrix} =
\begin{bmatrix}
\frac{2P_1^0}{Q_1^0} & \frac{P_1^0}{Q_1^0} & \frac{P_2^0}{Q_2^0} & \frac{2P_2^0}{Q_2^0} \\
\frac{2P_2^0}{Q_2^0} & \frac{P_1^0}{Q_1^0} & \frac{2P_1^0}{Q_1^0} & \frac{2P_2^0}{Q_2^0} \\
0 & 0 & \frac{P_1^0}{Q_1^0} & \frac{P_1^0}{Q_1^0} \\
0 & 0 & \frac{2P_2^0}{Q_2^0} & \frac{2P_2^0}{Q_2^0} \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

and

\[
D_i \triangleq \begin{bmatrix}
1 & 1 \\
1 & -1 \\
-1 & -1 \\
-1 & 1 \\
1 & 1
\end{bmatrix}
\]

for players \( i = 1, 2 \), \( Su = 0 \) implies that \( s_{11} = s_{12} = s_{21} = s_{22} = 0 \) and \( D_iu_i \geq 0 \) for \( i = 1, 2 \) gives \( r_{11} = r_{12} = r_{21} = r_{22} = 0 \). Hence, (D) holds. It can be seen that \( [D^T \mu_i = 0, \mu_i \geq 0] \) can only hold when \( \mu_{i1} = \mu_{i3}, \mu_{i2} = \mu_{i4}, \) and \( \mu_{i5} = \mu_{i6} \). Therefore,
\[(C_iA_i^0B_i)^T \mu_i \text{ equals}
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 1 \\
-1 & -1
\end{bmatrix}
\begin{bmatrix}
I \\
0 0 1 0 \\
0 0 0 1 \\
1 & -1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
\mu_{i1} \\
\mu_{i2} \\
\mu_{i3} \\
\mu_{i4} \\
\mu_{i5} \\
\mu_{i6}
\end{bmatrix}^T
\begin{bmatrix}
0 \\
0 \\
\mu_{i5} - \mu_{i6} \\
\mu_{i5} - \mu_{i6} \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
\mu_{i5} - \mu_{i6} \\
\mu_{i5} - \mu_{i6} \\
0 \\
0
\end{bmatrix}.
\]

Since \(A_i\) is a zero matrix, \(A_i^k\) for every positive integer \(k\) is 0. Hence, (E) holds. It has been established that this differential Nash-Cournot equilibrium problem can be formulated as an equivalent symmetric optimal control problem satisfying Assumptions (A)–(E). As a consequence, Theorem 5.1 guarantees the existence of an equilibrium and [95] provides a convergent iterative method for finding an Nash equilibrium.

### 5.5.2 A conjectured supply function game

To identify an asymmetric differential game arising from microeconomic Nash equilibrium problems, consider a conjectured supply function (CSF) problem [55, 111, 112, 213] and Section 4.2. In the Nash-Cournot problem, symmetry arises from the assumption that each player uses the same commodity pricing function and that no player anticipates competitor production/sales changes with respect to price. In a conjectured supply function equilibrium problem, players instead use a function to predict how total competitor production will change based on price. For this example, the presented model will be simplified to include only one node so that production and sales quantities are equivalent and transmission is not needed.

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For player \(i\), let the function \(\sigma_i(G_{-i}(t), p_i(t), t)\) represent the relationship between price and total competitor production in time \(t\):

\[
\sigma_i(G_{-i}(t), p_i(t), t) \triangleq G_{-i}(t) + \beta_i(G_{-i}(t), p^*_i(t), t)(p_i(t) - p^*_i(t)),
\]

where \(G_{-i}(t)\) is the total amount of competitor production expected at the specified equilibrium price \(p^*_i(t)\) at time \(t\). Notice that players may expect different equilibrium price trajectories here; this setting generalizes the case in which players use the same equilibrium price trajectory of \(p^*_i(t) = p^*(t)\) for \(i = 1, 2\). It follows that, depending on the specification of \(\beta_i(G_{-i}(t), p^*_i(t), t)\), the conjectured total production from other players will rise or fall if the realized price \(p_i(t)\) does not equal the equilibrium price \(p^*_i(t)\). Upon substitution into the production-pricing relationship

\[
g_i(t) + \sigma_i(G_{-i}(t), p_i(t), t) = Q^0 - \frac{Q^0}{P^0} p_i(t),
\]

invertibility of \(\frac{Q^0}{P^0} + \beta_i(G_{-i}(t), p^*_i(t), t)\) provides an explicit equation for player \(i\)'s conjectured price \(p_i(t)\). This invertibility should hold in market settings: \(\beta_i(G_{-i}(t), p^*_i(t), t)\) is expected to be nonnegative so that total competitor production is believed to change in the same direction as price differences (i.e., higher prices than expected at equilibrium should not decrease conjectured production). In the special case assumed here where \(\beta_i(G_{-i}(t), p^*_i(t), t) \triangleq B_{-i}\) for some positive constant \(B_{-i}\),

\[
p_i(t) = \frac{Q^0 - G_i(t) + B_{-i}p^*_i(t)}{\frac{Q^0}{P^0} + B_{-i}}.
\]

Using this conjectured price, formulate player 1’s optimal control problem as a cost minimization problem in which the conjectured supply function price is used for determining
revenue and costs are comprised of a quadratic production cost and a quadratic ramp rate cost:

\[
\minimize_{g_1, r_1} \int_0^T \begin{pmatrix} g_1(t) \\ r_1(t) \end{pmatrix}^T \begin{pmatrix} c_{11}^1 - \frac{Q^0 + B_{-1}p_1^*(t)}{Q^0 + B_{-1}} \\ 0 \end{pmatrix} \begin{pmatrix} g_1(t) \\ r_1(t) \end{pmatrix} \, dt \\
+ \begin{pmatrix} c_{11}^2 + \frac{1}{Q^0 + B_{-1}} & 0 \\ 0 & c_{12}^1 \end{pmatrix} \begin{pmatrix} g_1(t) \\ r_1(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{Q^0 + B_{-1}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_2(t) \\ r_2(t) \end{pmatrix} \, dt
\]

subject to \( g_1(0) = g_1^0, \) and for almost all \( t \in [0, T], \)

\[
\begin{cases}
\dot{g}_1(t) = r_1(t) \\
-x_1 + r_1(t) \geq 0 \\
x_1 - r_1(t) \geq 0
\end{cases} \Rightarrow x_1 \leq \dot{g}_1(t) \leq x_1.
\]

Similarly, player 2's optimal control problem just interchanges 1 and 2 for the player index \( i. \) If the player supply conjectures are not identical (i.e., \( B_{-1} \neq B_{-2} \)),

\[
\Xi_{12} = \begin{bmatrix}
\frac{1}{Q^0 + B_{-1}} & 0 \\
0 & 0
\end{bmatrix} \neq \begin{bmatrix}
\frac{1}{Q^0 + B_{-2}} & 0 \\
0 & 0
\end{bmatrix} = \Xi_{21}^T.
\]

It follows that a conjectured supply function game in which players have different conjectures is not a symmetric game. With \( p_1^*(t) \) and \( p_2^*(t) \) being Lipschitz continuous functions, it is simple to see that \((\hat{A}), (B), (W), (\hat{D}),\) and \((E)\) hold. To prove that \( \rho(\Gamma) < 1, \) con-
sider the fact that \( \rho(\Gamma) \leq \|\Gamma^k\|^{\frac{1}{k}} \) for all natural numbers \( k \). Examining \( k = 1 \) with the Euclidean norm, \( \|\Gamma\| \) is the largest eigenvalue of

\[
(\Gamma^T \Gamma)^{\frac{1}{2}} = \begin{pmatrix}
0 & \frac{1}{\sqrt{\sigma_2^\Xi \sigma_1^\Xi}} \|\Xi_{21}\| \\
\frac{1}{\sqrt{\sigma_1^\Xi \sigma_2^\Xi}} \|\Xi_{12}\| & 0
\end{pmatrix} \begin{pmatrix}
0 & \frac{1}{\sqrt{\sigma_1^\Xi \sigma_2^\Xi}} \|\Xi_{12}\| \\
\frac{1}{\sqrt{\sigma_1^\Xi \sigma_2^\Xi}} \|\Xi_{12}\| & 0
\end{pmatrix}^{\frac{1}{2}}
\]

\[
= \frac{1}{\sqrt{\sigma_1^\Xi \sigma_2^\Xi}} \begin{pmatrix}
\|\Xi_{12}\| & 0 \\
0 & \|\Xi_{21}\|
\end{pmatrix}
\]

where, as defined in (\( \hat{A} \)), \( \sigma_i^\Xi \) is the minimum eigenvalue of \( \Xi_{ii} \). For this problem,

\[
\sigma_1^\Xi \triangleq \min\left( c_{11}^2 + \frac{1}{Q_0^0 + B_{-1}}, c_{12}^1 \right) \quad \text{and} \quad \sigma_2^\Xi \triangleq \min\left( c_{21}^2 + \frac{1}{Q_0^0 + B_{-2}}, c_{22}^1 \right).
\]
Hence, if

\[ \| \Xi_{12} \| \]

\[ \| \quad \frac{1}{Q_0^0 + B_{-1}} < \sqrt{\min \left( c_{11}^2 + \frac{1}{Q_0^0 + B_{-1}}, c_{12}^1 \right)} \min \left( c_{21}^2 + \frac{1}{Q_0^0 + B_{-2}}, c_{22}^1 \right) \]

\[ \| \quad \sqrt{\sigma_1^\Xi \sigma_2^\Xi} \]

\[ \| \quad \frac{1}{Q_0^0 + B_{-2}} < \sqrt{\min \left( c_{11}^2 + \frac{1}{Q_0^0 + B_{-1}}, c_{12}^1 \right)} \min \left( c_{21}^2 + \frac{1}{Q_0^0 + B_{-2}}, c_{22}^1 \right) \]

\[ \| \quad \| \Xi_{21} \| \],

then \( \rho(\Gamma) < 1 \). The above condition can clearly be satisfied for a wide variety of parameter values. Thus, it has been proven that Theorem 5.3 holds for the above CSF problem specification and the presented Jacobi iterative algorithm will converge to the unique differential Nash equilibrium.

### 5.6 Conclusion

In this paper, solution existence of differential LQ Nash games was studied via a constructive approach. Under certain assumptions, the equivalence of a symmetric differential LQ
game to a single concatenated linear-quadratic optimal control problem was established. Using this equivalence, the convergent numerical method developed in [95] for solving LQ optimal control problems can be leveraged to identify differential Nash equilibria. For asymmetric differential LQ games, a Jacobi-type iterative solution scheme was proven to converge under certain conditions to a unique differential Nash equilibrium. Examples of common microeconomic games fulfilling the assumptions for each differential game form were presented. It is hoped that the results derived in this chapter will pave the way for further research on differential LQ Nash games with mixed state-control constraints. Obvious future research directions include extending this work to optimal control problems with pure state constraints and analyzing whether inexact solutions to (5.1) can be used in the Jacobi-type solution method. Unfortunately, the theoretical foundation for problems with pure state constraints is not firmly established, thereby decreasing the promise of that extension.
Chapter 6

Conclusion

This dissertation has advanced knowledge in the field of Nash equilibrium problems in several ways.

In Chapter 2, a general capacity expansion model for multiple markets and risk averse players was analyzed. Equilibrium existence was proven for two price determination methods (inverse supply/demand function, uniform price auctions), and consumer surplus maximization was included to change the traditional fixed-demand auction to an auction in which demand is price elastic. Unlike previous results, the model in this chapter did not assume that event probabilities were identical for all players, included exponential risk aversion, and is not limited to a proof-defined number of markets. A potential direction for further research in this area involves determining whether other pricing mechanisms can be incorporated into the model without sacrificing desirable theoretical properties such as solution existence while retaining the faithfulness of the models with regard to applications.

Chapter 3 provided a detailed analysis of Lemke’s method as applied to linear complement-
tarity problems arising from games that include shared constraints. One of the first, and perhaps most important, results of this work was the very specialized multiplier property that solutions identified by Lemke’s method must possess. In essence, Lemke’s method is only capable of identifying a subclass of all potential equilibria. This deficiency was the focus of the rest of the chapter. New equilibrium concepts such as partial variational equilibria, coalitional equilibria, and regularization-based equilibria were introduced, and a simple looping-type modification of Lemke’s method was proposed to identify partial variational equilibria “on-the-fly.” Sufficient conditions were provided for successful termination of the modified Lemke’s method. Finally, a unique problem reformulation technique and an associated solution method were described that allow for an additional array of equilibria to be identified. Further work in this area may focus on the application of this methodology to linearized nonlinear equilibrium problems and additional adaptations for solving an even wider variety of LCPs.

Chapter 4 developed and studied a unified power market framework that encompasses several traditional models. At its most basic level, the unified market model combines the optimization problems of generators (perfectly competitive, Cournot, or conjectured supply function), an ISO, and market clearing conditions. For the unified model, sufficient conditions for equilibrium existence, uniqueness, and stability were proven. From a computational perspective, it was shown that several projection- and Newton-based methods for variational inequalities can be successfully applied to identify a solution of the unified model. Lastly, three important model extensions somewhat related to the model of Chapter 2 were introduced: consumer surplus maximization, capacity markets, and emission permit auctions. Equilibrium existence and uniqueness results for these extended models were proven when possible. Additional research directions arising from this chapter are plentiful. For instance, unified models that study hierarchical behavior, uncertainty/risk,
and continuous-time dynamics would all be relevant when analyzing the properties of actual deregulated electricity markets.

The final chapter of the dissertation moved to continuous-time domain to study differential Nash equilibrium problems. Starting from a recently developed computational method for solving a linear-quadratic optimal control problem with mixed state-control constraints, a differential Nash game comprised of these problems was analyzed. When a symmetry property was satisfied, it was proven that the differential Nash game was equivalent to a single concatenated linear-quadratic optimal control problem and the previously mentioned computational method could be applied. For asymmetric games that satisfy a spectral radius condition, it was shown that a Jacobi-type iterative method converges to the unique game equilibrium. Cournot and conjectured supply function games were presented to illustrate that the required problem assumptions are not overly restrictive. Future research in this direction could attempt to develop a similar solution method for games in which pure state constraints are present.
Bibliography


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