STUDIES OF SOME ISSUES IN SOLVING SURFACE INTEGRAL EQUATIONS
AND THE EQUIVALENCE PRINCIPLE ALGORITHM

BY

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THESIS

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ABSTRACT

This thesis describes the progressive development analysis of the equivalence principle algorithm (EPA) beginning with the development of various operators that constitute it. We begin with the formulation for the electric field integral equation (EFIE), and visit the necessary treatment of the magnetic field integral equation (MFIE), the combined field integral equation (CFIE), and the Poggio-Miller-Chu-Harrington-Wu-Tsai (PMCHWT) along with various singularity extraction schemes before we develop the EPA relations. The explicit expressions of the translation operators are also derived. The EFIE, the MFIE, and the CFIE formulations are used to verify the accuracy of such operators as the $L$, the $K$, the $\hat{n} \times L$, and the $\hat{n} \times K$ operators that are at the heart of the EPA formulation. Very detailed derivations and analyses of these operators with proper scaling factors are included to avoid the inaccuracy due to iterative solvers. Later we provide pertinent results for all of them for verification and comparison.
To Baba and Ma, the serene oasis in the great desert of life!

“When the heart is hard and parched up, come upon me with a shower of mercy. When grace is lost from life, come with a burst of song.”

(Rabindranath Tagore, Gitanjali, Verse 39)
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<td>CFIE</td>
<td>Combined Field Integral Equation</td>
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<td>CS</td>
<td>Current Solver</td>
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<tr>
<td>dB</td>
<td>deciBel</td>
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<tr>
<td>dBsm</td>
<td>deciBel per Squared Meter</td>
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<td>EFIE</td>
<td>Electric Field Integral Equation</td>
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<td>EM</td>
<td>Electromagnetic</td>
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<td>Equivalence Principle Algorithm</td>
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<td>Equivalence Surface</td>
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<td>FDTD</td>
<td>Finite Difference Time Domain</td>
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<td>Finite Element Method</td>
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<td>HH</td>
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<td>HV</td>
<td>Horizontal-Vertical</td>
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<td>IO</td>
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<tr>
<td>MFIE</td>
<td>Magnetic Field Integral Equation</td>
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<tr>
<td>MoM</td>
<td>Method of Moments</td>
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<tr>
<td>OI</td>
<td>Outside-In</td>
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<tr>
<td>PEC</td>
<td>Perfect Electric Conductor</td>
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<td>PMCHWT</td>
<td>Poggio-Miller-Chang-Harrington-Wu-Tsai</td>
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<td>--------------</td>
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</tr>
<tr>
<td>RCS</td>
<td>Radar Cross Section</td>
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<td>Rao-Wilton-Glisson</td>
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<td>Surface Integral Equation</td>
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<td>VH</td>
<td>Vertical-Horizontal</td>
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CHAPTER 1

INTRODUCTION

The problem of computing the scattered radiation from a scatterer or a system of scatterers can be dealt with by various techniques. Broadly, these techniques can be categorized in three classes: the finite difference time domain (FDTD) method, the finite element method (FEM), and the method of moments (MoM). While the first two involve the meshing of the scatterer over the volume, the third one, more often than not, is meshed over the surface, giving rise to what is known as the surface integral equations (SIE), resulting in the reduction in the number of unknowns. These surface integral equations have been at the heart of some of the ground-breaking algorithms, albeit with the use of highly sophisticated techniques that can deal with large radiation problems unknowns of the orders of millions to billions quite efficiently. The ultimate focus of this thesis is the step-by-step development of the equivalence principle algorithm (EPA), which is a domain decomposition method (DDM) for integral equations that essentially reduces the computational cost in large, intricate systems with periodicity or large array structure, reducing the computational cost and, hence, decreasing the run time.

This thesis begins with the derivation of the integral equation form of the equivalence principle and extinction theorem, the foundation of the SIEs and their implications in Chapter 2. Then in Chapter 3, we discuss the derivation of the electric field integral equation (EFIE) (and hence, the \( \mathcal{L} \) operator) for the perfect electric conductor (PEC) objects and the formulation for the system matrix elements for the EFIE, using the well-known Rao-Wilton-Glisson (RWG) basis functions and an elaborate treatment of singular terms by means of explicit expressions. We also include the scattered field expression. In Chapter 4, we explain the concept of the radar cross section (RCS) and its classification and briefly mention the numerical integration by means of Gaussian quadrature. Chapter 5 deals with the magnetic field integral equation (MFIE) for PEC objects (and hence, the \( \hat{n} \times \mathcal{K} \) and the \( \mathcal{K} \) operators).
with elaborate formulations and singularity treatment. In Chapter 6, we develop a concrete but effective singularity threshold analysis to reduce the computational effort and time. Chapter 7 briefly describes various combined field integral equations (CFIE) for PEC scatterers to avoid the resonance problem. Chapter 8 derives the Poggio-Miller-Chang-Harrington-Wu-Tsai (PMCHWT) and Müller formulations for dielectric scatterers. Chapter 9 very elaborately describes the EPA matrix formulation along with the translation operator. We conclude this Chapter with a discussion of the $\mathcal{K}$, the $\mathbf{n} \times \mathcal{L}$, and the $\mathbf{n} \times \mathcal{K}$ operators. The properly scaled matrix versions of the PMCHWT and the EPA are derived in Chapter 10. Chapter 11 discusses the tap basis formulation of the EPA for both the PEC and dielectric scatterers. In Chapter 12, we discuss the simulation results from the various formulations developed, discussed, and analyzed so far.
The equivalence principle and the extinction theorem are the cornerstones in deriving the EPA. Another great feature of them is that they provide an easier and faster way to verify the accuracy of the various cases (i.e., EFIE, MFIE, CFIE, OI, and IO problems, the EPA etc). To state the theorem, we consider a region, $V_2$, enclosed by a surface, $S$. Let us define two regions, 1 and 2, which refer to the regions outside and inside the surface, $S$, respectively. Region 1 is indicated by $V_1$. We further assume that a radiation field impinges upon surface, $S$, from a current source, $J_{inc}$, in region 1, while there is no source in region 2. The geometry appears in Figure 2.1. Using the vector wave equation in region 1, we can write

$$\nabla \times \nabla \times E(r) - k^2 E(r) = i\omega \mu J_{inc}(r)$$

(2.1)
where, $\mu$ indicates the permeability in $V_1$ and $\omega$ is the angular frequency. For a homogeneous medium, the wave vector is given by $k = \omega \sqrt{\mu \epsilon}$, where $\epsilon$ is the permittivity of the medium. Now if we define

$$E(r) = i\omega \mu \int\int\int_{V_1} dV J_{\text{inc}}(r') \cdot \mathbf{G}^t(r, r')$$  \hspace{1cm} (2.2)$$

where, $t$ indicates the transpose operation. Here, primed quantities refer to the source while the nonprimed quantities refer to the observation points. Now plugging (2.2) in (2.1), we have

$$i\omega \mu \nabla \times \nabla \times \int\int\int_{V_1} dV J_{\text{inc}}(r') \cdot \mathbf{G}^t(r, r') - i\omega \mu k^2 \int\int\int_{V_1} dV J_{\text{inc}}(r') \cdot \mathbf{G}^t(r, r')$$

$$= i\omega \mu \int\int\int_{V_1} dV J_{\text{inc}}(r') \delta(r - r')$$

$$= i\omega \mu J_{\text{inc}}(r)$$  \hspace{1cm} (2.3)$$

In writing the above, we used the sifting property of Dirac delta function, $\delta$. From the second line in (2.3), we obtain

$$\nabla \times \nabla \times \mathbf{G}^t(r, r') - k^2 \mathbf{G}^t(r, r') = \mathbf{I} \delta(r - r')$$  \hspace{1cm} (2.4)$$

Thus, $\mathbf{G}(r, r')$, which is called the dyadic Green’s function, is the impulse response to the vector wave equation [1]. It is given by

$$\mathbf{G}(r, r') = \left( \mathbf{I} + \frac{\nabla \nabla}{k^2} \right) \frac{e^{ik|r-r'|}}{4\pi|r-r'|}$$  \hspace{1cm} (2.5)$$

Now, premultiplying (2.4) by $E(r)$ and postmultiplying (2.1) by $\mathbf{G}(r, r')$, and subtracting the latter from the former, we have
\[ \mathbf{E}(\mathbf{r}) \cdot \nabla \times \nabla \times \mathbf{G}^t(\mathbf{r}, \mathbf{r}') - \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) \cdot \mathbf{G}^t(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \mathbf{E}(\mathbf{r}) - i\omega \mu \mathbf{J}_{\text{inc}}(\mathbf{r}) \cdot \mathbf{G}^t(\mathbf{r}, \mathbf{r}') \] (2.6)

Integrating both sides over \( V_1 \), we have

\[ \mathbf{E}(\mathbf{r}') = \mathbf{E}_{\text{inc}}(\mathbf{r}') + \iiint_{V_1} dV \left[ \mathbf{E}(\mathbf{r}) \cdot \nabla \times \nabla \times \mathbf{G}^t(\mathbf{r}, \mathbf{r}') - \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) \cdot \mathbf{G}^t(\mathbf{r}, \mathbf{r}') \right] \] (2.7)

where we used (2.2) and the sifting property of the Dirac delta function, \( \delta \).

Now, using the vector identity [1]

\[ -\nabla \cdot \left\{ \mathbf{E}(\mathbf{r}) \times [\nabla \times \mathbf{G}(\mathbf{r}, \mathbf{r}')] \right\}^t + \nabla \times \mathbf{E}(\mathbf{r}) \times \mathbf{G}^t(\mathbf{r}, \mathbf{r}') = \mathbf{E}(\mathbf{r}) \cdot \nabla \times \nabla \times \mathbf{G}^t(\mathbf{r}, \mathbf{r}') 
- \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) \cdot \mathbf{G}^t(\mathbf{r}, \mathbf{r}') \] (2.8)

and using Gauss’ divergence theorem, (2.7) can be written as

\[ \mathbf{E}(\mathbf{r}') = \mathbf{E}_{\text{inc}}(\mathbf{r}') + \oint_{S + S_{\text{inf}}} d\mathbf{r}' \hat{\mathbf{n}} \cdot \left\{ \mathbf{E}(\mathbf{r}) \times [\nabla \times \mathbf{G}(\mathbf{r}, \mathbf{r}')] \right\}^t + \nabla \times \mathbf{E}(\mathbf{r}) \times \mathbf{G}^t(\mathbf{r}, \mathbf{r}') \]  
- \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) \cdot \mathbf{G}^t(\mathbf{r}, \mathbf{r}') \]  
(2.9)

where \( \hat{\mathbf{n}} \) is the outward unit normal to surface, \( S \). In (2.9), we make use of the vector identity

\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c} \] (2.10)

and Maxwell’s first equation [2]
\[ \nabla \times \mathbf{E}(\mathbf{r}) = i\omega \mu \mathbf{H}(\mathbf{r}) \]  \hspace{1cm} (2.11)

Also \( S_{infr} \) is a surface that demarcates the outermost boundary of region 1. Using the reciprocity condition of the dyadic Green’s function, we get [2]

\[ \mathbf{G}^t(\mathbf{r}, \mathbf{r}') = \mathbf{G}(\mathbf{r}', \mathbf{r}) \]  \hspace{1cm} (2.12)

Then, we obtain

\[ \mathbf{E}(\mathbf{r}) = i\omega \mu \iiint_{V_1} dV \mathbf{G}^t(\mathbf{r}', \mathbf{r}) \cdot \mathbf{J}_{inc}(\mathbf{r}') \]  \hspace{1cm} (2.13)

and from (2.9), we have

\[ \mathbf{E}(\mathbf{r}') = \mathbf{E}_{inc}(\mathbf{r}') + \oint_{S+S_{infr}} d\mathbf{r}' \left[ i\omega \mu \mathbf{G}(\mathbf{r}', \mathbf{r}) \cdot \mathbf{n} \times \mathbf{H}(\mathbf{r}) - \left[ \nabla \times \mathbf{G}(\mathbf{r}', \mathbf{r}) \right] \cdot \mathbf{n} \times \mathbf{E}(\mathbf{r}) \right] \]  \hspace{1cm} (2.14)

In the above we used the reciprocity condition [2]

\[ \left[ \nabla \times \mathbf{G}(\mathbf{r}, \mathbf{r}') \right]^t = -\nabla \times \mathbf{G}(\mathbf{r}', \mathbf{r}) \]  \hspace{1cm} (2.15)

Now, from the Sommerfeld’s radiation condition [3,4], we know

\[ \lim_{r \to \infty} \left[ \nabla \times \mathbf{U}(\mathbf{r}) - ik \hat{\mathbf{r}} \times \mathbf{U}(\mathbf{r}) \right] = 0 \]  \hspace{1cm} (2.16)

where \( \mathbf{U}(\mathbf{r}) = \{ \mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r}) \} \). In our case, \( \mathbf{n} = \hat{\mathbf{r}} \). Thus the Sommerfeld’s radiation condition clearly states that in (2.14), \( \mathbf{n} \times \mathbf{E}(\mathbf{r}) \), and \( \mathbf{n} \times \mathbf{H}(\mathbf{r}) \) decay as \( 1/r \) as \( r \to \infty \). Moreover, \( \mathbf{G}(\mathbf{r}, \mathbf{r}') \) and \( \nabla \times \mathbf{G}(\mathbf{r}, \mathbf{r}') \) decay as \( 1/r \) as \( r \to \infty \) [1]. Therefore, the terms within the square brackets on the right-hand side of (2.14) decay as \( 1/r^2 \) as
$r \to \infty$. However, these terms cancel each other to the leading order, leaving terms of $O(1/r^3)$ and beyond. Thus, even though $dS \to r^2$, the second term on the right-hand side varies as $1/r$ and for $r \to \infty$, the integral over $S_{mf}$ vanishes. Another more intuitive yet simpler way to arrive at the overall $1/r$ decay is that the first term on the right-hand side is the incident electric field, and so is the left-hand side. Therefore, the second term on the right-hand side must also be an electric field and hence it must vary as $1/r$. We also note that if $r'$ lies outside of $V_1$, because of the sifting property of the delta function, $\delta$, the right-hand side of (2.14) is zero. Taking this into account and also swapping primed and non-primed quantities, (2.14) can be written as

$$E_{inc}(r) + \oint_S dS' \left[ i\omega \mu \mathbf{G}(r, r') \cdot \hat{n}' \times \mathbf{H}(r') + \nabla \times \mathbf{G}(r, r') \cdot \hat{n}' \times \mathbf{E}(r') \right] = \begin{cases} E(r), & r \in V_1 \\ 0, & r \in V_2 \end{cases} \quad (2.17)$$

Now defining the surface currents on $S$ as

$$\mathbf{J}_S = \hat{n} \times \mathbf{H}, \quad \mathbf{M}_S = -\hat{n} \times \mathbf{E} = \mathbf{E} \times \hat{n} \quad (2.18)$$

we can rewrite (2.17) as

$$E_{inc}(r) + \oint_S dS' \left[ i\omega \mu \mathbf{G}(r, r') \cdot \mathbf{J}_S(r') - \nabla \times \mathbf{G}(r, r') \cdot \mathbf{M}_S(r') \right] = \begin{cases} E(r), & r \in V_1 \\ 0, & r \in V_2 \end{cases} \quad (2.19)$$

This is the extinction theorem derived from the equivalence principle [2]. Physically it means that surface current sources placed on $S$ produce a total field equal to the original problem outside the fictitious equivalence surface (ES) and extinct field inside. Similarly, using the duality principle, we can derive the equation for the magnetic field. The original problem has effectively been converted into an equivalent problem where there is nonzero field outside and zero field inside, as is evident from the comparison with the expression for surface currents for an interface discontinuity.
boundary problem given as

\[
J_S = \mathbf{n} \times [\mathbf{H} - 0], \quad M_S = -\mathbf{n} \times [\mathbf{E} - 0] = [\mathbf{E} - 0] \times \mathbf{n}
\] (2.20)

where the unit normal, \( \mathbf{n} \), points from the region of zero field to the region of nonzero field. For the cases in the following chapters, we will use the extinction theorem as one of the tools to verify the accuracy of our routines. It should also be stressed that a formula akin to (2.19) can also be formulated in an identical fashion by noting that there is no current source in region 2 (which implies that there is no incident field for this case), and using the corresponding permeability, permittivity, and wave number in region 2, should the medium in region 2 be different from that in region 1. It is also to be emphasized that the the extinction principle can be applied equally well if both regions 1 and 2 contain the same medium. This feature can be especially helpful in solving the surface integral equations (SIE) in inside-out and outside-in manner. This point will be further elaborated when we consider scattering by dielectric bodies.
CHAPTER 3

THE ELECTRIC FIELD INTEGRAL EQUATION (EFIE) FOR PEC OBJECTS

The electric field integral equation (EFIE) for a PEC object relates the incident electric field to the scattered surface electric current of surface $S$. To derive the EFIE for a PEC scatterer, let us again consider the scenario depicted in Figure 2.1. We assume that $\mathbf{J}_{\text{inc}}$ is a current that generates a field $\mathbf{E}_{\text{inc}}$ that impinges on the PEC objects, which, thereupon, generates a scattered field. For the derivation, we consider a hypothetical surface, $S^+$, that is just large enough to contain the surface $S$. The equivalence principle and the extinction theorem will be applied to it. When the equivalence currents are placed on $S^+$, they produce the total field outside $S^+$ and extinguish the field inside [1]. The equivalence currents are still given by (2.18) on $S^+$. It should still be pointed out that $S$ and $S^+$ are so close together that we still indicate the currents as $\mathbf{J}_S$ and $\mathbf{M}_S$. Because $\hat{n} \times \mathbf{E} = 0$ on $S^+$, only $\mathbf{J}_S$ is required to represent the field outside. Thus the equivalence theorem and extinction principle from (2.19) becomes [1]

$$
\mathbf{E}_{\text{inc}}(\mathbf{r}) + i\omega \mu \int_{S} dS' \mathbf{G}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_S(\mathbf{r}') = 
\begin{cases} 
\mathbf{E}(\mathbf{r}), & \mathbf{r} \in V_1 \\
0, & \mathbf{r} \in V_2 
\end{cases}
$$

(3.1)

Using the extinction theorem part in (3.1), we finally obtain [1]

$$
\mathbf{E}_{\text{inc}}(\mathbf{r}) = -i\omega \mu \int_{S} dS' \mathbf{G}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_S(\mathbf{r}')
$$

(3.2)

It is, however, customary to write (3.2) as [1,5]
\[ \hat{t} \cdot \mathbf{E}_{\text{inc}}(\mathbf{r}) = -i\omega \mu \int_S dS' \hat{t} \cdot [\mathbf{G}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_S(\mathbf{r}')] \]  

(3.3)

where \( \hat{t} \) is the unit tangent to the surface \( S \). As the tangential electric field component is continuous across a current sheet, the above integral equation should only be applied on \( S \) \[^{[1]}\]. It is also important to note that the EFIE can be applied to both open (metallic sheets, for example) and closed (metallic cubes and spheres, for example) geometric structures. If we express the surface current in terms of the sum of weighted local bases as

\[ \mathbf{J}(\mathbf{r}) = \sum_{n=1}^{N} a_n \mathbf{f}_n(\mathbf{r}) \]  

(3.4)

then (3.3) can be written as

\[ \hat{t} \cdot \mathbf{E}_{\text{inc}}(\mathbf{r}) = -i\omega \mu \int_S dS' \hat{t} \cdot \left[ \mathbf{G}(\mathbf{r}, \mathbf{r}') \cdot \left( \sum_{n=1}^{N} a_n \mathbf{f}_n(\mathbf{r}') \right) \right] \]  

(3.5)

Let us assume that the Rao-Wilton-Glisson (RWG) triangular basis function is used, which is defined as \[^{[5,6]}\]

\[ f_n(\mathbf{r}) = \begin{cases} 
\frac{l_n}{2A_n^+} \rho_n^+(\mathbf{r}), & \mathbf{r} \in T_n^+ \\
\frac{l_n}{2A_n^-} \rho_n^- (\mathbf{r}), & \mathbf{r} \in T_n^- \\
0, & \text{otherwise}
\end{cases} \]  

(3.6)

where \( T_n^+ \) and \( T_n^- \) are the triangles that share edge \( n \). Here, \( l_n \) is the length of the edge, and \( A_n^+ \) and \( A_n^- \) indicate the areas of the triangles \( T_n^+ \) and \( T_n^- \), respectively. We also define \[^{[5]}\]
Figure 3.1: Definition of the RWG basis function (after [5]).

\[
\begin{align*}
\rho_n^+(r) &= v_n^+ - r, \quad r \text{ in } T_n^+ \\
\rho_n^-(r) &= r - v_n^-, \quad r \text{ in } T_n^-
\end{align*}
\]

(3.7)

Here, \(v_n^+\) and \(v_n^-\) indicate the vertices lying opposite to the shared edge on triangles \(T_n^+\) and \(T_n^-\), respectively. These specifics are included in Figure 3.1. It should be stressed that the definitions in (3.7) are slightly different from the actual definitions by Rao, Wilton, and Glisson [6]. There are a few specifics [5] which are important for the implementation, viz.

1. RWG basis functions are assigned only by the interior edges that are shared by two triangles, not by exterior edges that are not shared. For example, if we consider a flat PEC plate to be analyzed using EFIE, the current coefficients (i.e., \(a_n\)'s) are assigned only to any one of the two surfaces of this open structure (as opposed to closed structure, e.g., a PEC sphere) and no basis function is assigned to the edges on the the border of the flat plate.

2. The RWG basis function has no component normal to any edge other than the edge it shares, and the component of the function normal to this edge is unity. Surface divergence of \(f_n(r')\) yields
\[
\n\nabla_S \cdot f_n(r) = \begin{cases} 
-\frac{\ln \frac{r}{A_n}}{A_n}, & r \in T_n^+ \\
\frac{\ln \frac{r}{A_n}}{A_n}, & r \in T_n^- \\
0, & \text{otherwise}
\end{cases}
\quad \text{(3.8)}
\]

3. Since the surface divergence is finite for the RWG basis functions, they are \textit{divergence conforming} basis functions. Since the surface divergence is finite and (3.8) resembles the well-known continuity equation, the surface divergence of RWG basis functions indicates charge density, and (3.8) indicates that there is no accumulation of charges on an edge.

4. Since the RWG basis is defined on triangular facets, they represent only surface components. As both \( E_{\text{inc}}(r) \) and \( J_S(r) \) in (3.3) are expressed in terms of RWG basis functions, they will automatically be surface components; we do not need to consider the dot multiplication with \( \hat{t} \).

5. We clearly note that the RWG basis function is linear (1st order) in longitudinal direction while it is constant (0th order) in transverse direction. RWG basis function, therefore, is a mixed-order basis function.

Testing (3.5) with RWG (of course, we omit the \( \hat{t} \cdot \) from both sides so that the Galerkin testing can be applied), we obtain (we write only single term over one pair of basis functions for the source and testing, or observation, triangles) [5]

\[
\iint_{f_m} dS f_n(r) \cdot E_{\text{inc}}(r) = -i\omega \mu \int \int_{f_m} dS \int \int_{f_n} dS' f_m(r) \cdot \left[ G(r, r') \cdot \sum_{n=1}^{N} a_n f_n(r') \right]
\quad \text{(3.9)}
\]

Now, we note that in (3.9) there is a singularity in the dyadic Green’s function at \( r = r' \) and the \( \nabla \) operator, acting on Green’s function, makes it all the more singular. To avoid this problem, we move the gradient into the testing function from \( (I + \nabla \nabla) \) in \( \mathcal{G}(r, r') \) by using the integration by parts. We, thus, have [5]

\[
\bar{Z} \cdot a = b
\quad \text{(3.10)}
\]
where the matrix elements \( Z_{mn} \) are given by

\[
Z_{mn} = -i\omega \mu \iint_{T_m} dS \iint_{T_n} dS' \left( f_m(r) \cdot f_n(r') - \frac{1}{k^2} [\nabla \cdot f_m(r)][\nabla' \cdot f_n(r')] \right) \frac{e^{ik|r-r'|}}{4\pi|r-r'|}
\]  

(3.11)

and the excitation vector elements \( b_m \) are given by

\[
b_m = \iint_{T_m} dS f_m(r) \cdot E_{inc}(r)
\]

(3.12)

Once we have computed \( Z \) and \( b \), we can also obtain \( a \). Now we can define an \textit{integro-differential} surface operator called the \( \mathcal{L} \) operator. Although there are various definitions of this operator in the literature, we define it as [7]

\[
\mathcal{L}(r, r')X(r') = i\omega \mu \oint_S dS' \left[ I + \frac{1}{k^2} \nabla \nabla \cdot \left( X(r') \frac{e^{ik|r-r'|}}{4\pi|r-r'|} \right) \right]
\]

\[
= i\omega \mu \oint_S dS' \left[ X(r') + \frac{1}{k^2} \nabla \nabla' \cdot X(r') \right] \frac{e^{ik|r-r'|}}{4\pi|r-r'|}
\]

(3.13)

It is more efficient to compute the matrix elements triangle-by-triangle rather than in a basis-by-basis manner. We can express (3.11) for one pair of source and observation triangular facets (understandably in \( Z_{mn} \) there will be four such terms corresponding to the combinations of positive and negative triangles for each of the source and testing facets) as [5]

\[
I_{EFIE} = -\frac{i\omega \mu l_m l_n}{4\pi A_m^\pm A_n^\pm} \iint_{T_m^\pm} dS \iint_{T_n^\pm} dS' \left( \frac{1}{4} \rho_m^\pm(r) \cdot \rho_n^\pm(r') \pm \frac{1}{k^2} \right) \frac{e^{ik|r-r'|}}{4\pi|r-r'|}
\]

(3.14)

Obviously the variable sign depends on the triangles involved. Applying an M-point Gaussian quadrature rule (normalized by triangle area) over source and observation facets for a nonsingular case (where source and observation facets are far away), (3.14) is given by
\[ I_{EFIE} \approx -\frac{i\omega \mu l_m l_n}{4\pi} \sum_{p=1}^{M} \sum_{q=1}^{M} w_p w_q \left( \frac{1}{4} \mathbf{\rho}_m^\pm (\mathbf{r}_p) \cdot \mathbf{\rho}_n^\pm (\mathbf{r}_q') \pm \frac{1}{k^2} \right) \frac{e^{ik|\mathbf{r}_p - \mathbf{r}_q'|}}{4\pi|\mathbf{r}_p - \mathbf{r}_q'|} \]  

(3.15)

While there is no hard and fast rule to stipulate the separation between the source and testing facets, Gibson [5] proposes that a reasonable estimate would be that if the centroids of the facets are more than 0.1\(\lambda\) to 0.2\(\lambda\) away (\(\lambda\) is the wavelength), we can consider it a nonsingular case. A rather good result can be obtained for a 0.3\(\lambda\) threshold separation assumption. However, if this threshold is pushed beyond this, the computation time is increased significantly, since the computation of the singular part is more time consuming. But as it turns out that, if we use an electrically small object compared to \(\lambda\), we will end up using singularity for all terms in the impedance matrix! We, however, defer the discussion on singularity threshold to Chapter 6.

Now let us consider the singular case. The singularity basically comes from

\[ G_0 = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \]  

(3.16)

which can be rewritten as [5, 8]

\[ G_0 = \frac{1}{4\pi} \left( \frac{e^{ik|\mathbf{r} - \mathbf{r}'|} - 1}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \]  

(3.17)

In (3.17), the term in the square brackets can easily be numerically integrated using low-order Gaussian quadrature after expanding the exponential term in the numerator in the Taylor series. The second extracted term outside the square brackets requires special treatment. In the limit where \(|\mathbf{r} - \mathbf{r}'|\) approaches zero, we obtain

\[ \lim_{R \to 0} \frac{1}{4\pi} \left( \frac{e^{ikR} - 1}{R} \right) = \frac{ik}{4\pi} \]  

(3.18)

where \(R = |\mathbf{r} - \mathbf{r}'|\). Also in relation to \(R\), we define \(\mathbf{R} = \mathbf{r} - \mathbf{r}'\). From now onward, we will use these symbols, unless otherwise stated or implied. Upon insertion of the second term on the right-hand side of (3.17) (term outside the square brackets) into
(3.14), we obtain terms of the form [5, 8]

\[ I_{1EFIE} = \iint_{T_n^+} dS \rho_m^+(r) \cdot \iint_{T_n^-} dS' \rho_n^+(r') \frac{R}{R} \]  \hspace{1cm} (3.19)

and

\[ I_{2EFIE} = \iint_{T_n^+} dS \iint_{T_n^-} dS' \frac{R}{R} \]  \hspace{1cm} (3.20)

In (3.19) and (3.20), the inner integral is evaluated using an analytical method while the outer integral is computed using the numerical integration (Gaussian quadrature). The special singularity treatment for this part is done after [5, 8]. This method is applicable to any polygon since the calculation involves the edges of the polygon. In our case, the polygon is none other than a triangle (number of polygon edges, \(M_p = 3\)). To understand the case, let us refer to Figure 3.2. We consider a line segment, \(C\) (which is part of a polygon), with endpoints \(r^+\) and \(r^-\) located in the plane of a polygon (indicated as \(S\)). In the following analysis, projection of any point \(r_{\text{any}}\) onto the polygon surface, \(S\) is given by [5]

\[ \rho_{\text{any}} = r_{\text{any}} - \hat{n} (r_{\text{any}} \cdot \hat{n}) \]  \hspace{1cm} (3.21)

Here, \(\hat{n}\) is the unit normal to the polygon plane, \(S\). Similarly, the projections of the source and observation or testing vectors \(r'\) and \(r\) on \(S\) are given by \(\rho'\) and \(\rho\) respectively. The quantities in Figure 3.2 are given by [5, 8]

\[ \rho^\pm = r^\pm - \hat{n} (r^\pm \cdot \hat{n}) \]  \hspace{1cm} (3.22)

\[ \hat{i} = \frac{\rho^+ - \rho^-}{|\rho^+ - \rho^-|} \]  \hspace{1cm} (3.23)
Figure 3.2: Different quantities for the segment $C$, which is part of a polygon (after [5,8]).

\[ \hat{u} = \hat{l} \times \hat{n} \]  
\[ l^{\pm} = (\rho^{\pm} - \rho) \cdot \hat{l} \]  
\[ P^{0} = |(\rho^{\pm} - \rho) \cdot \hat{u}| \]
\[ P^\pm = |(\rho^\pm - \rho)| = \sqrt{(P^0)^2 + (l^\pm)^2} \]  

(3.27)

\[ \hat{P}^0 = \frac{(\rho^\pm - \rho) - l^\pm \hat{l}}{P^0} \]  

(3.28)

\[ R^0 = \sqrt{(P^0)^2 + d^2} \]  

(3.29)

\[ R^\pm = \sqrt{(P^\pm)^2 + d^2} \]  

(3.30)

Here, \( \hat{l} \) and \( \hat{u} \) are the unit vectors along and perpendicular to the line segment \( C \) lying in the polygon plane \( S \). The distance \( d \) of the point of observation from the surface \( S \) is given by [5, 8]

\[ d = \hat{n} \cdot (r - r^\pm) \]  

(3.31)

Now that we have defined the various quantities, we want to evaluate the singular integral. We mainly stick to the treatment in [8, 9]. In the plane, \( S \), we set local polar coordinates with the origin at \( \rho \), the polar vector is \( P \), and the polar angle is \( \zeta \). Thus, we write

\[ R = r - r' = r - \rho - (r' - \rho) = d - P \]  

(3.32)

and

\[ R = \sqrt{P^2 + d^2} \]  

(3.33)
In the following analysis, we also define

\[
\nabla_S f(P, \zeta) = \hat{P} \frac{\partial f}{\partial P} + \hat{\zeta} \frac{1}{P} \frac{\partial f}{\partial \zeta}
\]

(3.34)

and

\[
\nabla_S \cdot F(P, \zeta) = \frac{1}{P} \frac{\partial (PF_P)}{\partial P} + \frac{1}{P} \frac{\partial F_\zeta}{\partial \zeta}
\]

(3.35)

Now, we note that

\[
\nabla_S \cdot \left( \frac{R}{P} \hat{P} \right) = \frac{1}{P} \frac{\partial}{\partial P} \left( \frac{PR}{P} \right) = \frac{1}{P} \frac{\partial R}{\partial P} = \frac{1}{R}
\]

(3.36)

Hence,

\[
I_{\text{inner}_{\text{EFIE}}} = \iint_T \frac{dS'}{R} = \lim_{\epsilon \to 0} \iint_{T-\epsilon} \frac{dS'}{R} + \lim_{\epsilon \to 0} \iint_{T_{\epsilon}} \frac{dS'}{R} = \lim_{\epsilon \to 0} \iint_{T-\epsilon} dS' \nabla_S \cdot \left( \frac{R}{P} \hat{P} \right) + \lim_{\epsilon \to 0} \iint_{T_{\epsilon}} \frac{dS'}{R}
\]

(3.37)

Here \(T_\epsilon\) refers to an infinitesimally small disk of radius \(\epsilon\) centered at \(\rho\) that implies the residual. But, as \(\epsilon \to 0\), from (3.33), \(R = \sqrt{\epsilon^2 + d^2} \to |d|\), whereas, \(\iint_{T_{\epsilon}} dS' = S_\epsilon\) and \(S_\epsilon \to 0\). Therefore, the second integral on the third line of (3.37) vanishes leaving us with
\[ I_{\text{Inner2EFIE}} = \lim_{\epsilon \to 0} \int_T \int S' \nabla' S \cdot \left( \frac{R}{P} \hat{P} \right) \]

\[ = \lim_{\epsilon \to 0} \int_T \int S' \nabla' S \cdot \left( \frac{R}{P} \hat{P} \right) - \lim_{\epsilon \to 0} \int_T \int S' \nabla' S \cdot \left( \frac{R}{P} \hat{P} \right) \]

\[ = \lim_{\epsilon \to 0} \int_{\partial T} dl' \frac{R}{P'^2} P \cdot \hat{u} - \lim_{\epsilon \to 0} \int_{\partial T} R \frac{\epsilon}{\epsilon^2} (\epsilon \cdot \hat{u}) \epsilon d\zeta' \]

\[ = \sum_{i=1}^{M_p} \int_{\partial_i T} dl' \frac{R}{P'^2} \left( P^0_i + \hat{i}i' \right) \cdot \hat{u}_i - \lim_{\epsilon \to 0} \int_{\partial T} \frac{\sqrt{\epsilon^2 + d^2}}{\epsilon} (\epsilon \cdot \hat{u}) d\zeta' \]

\[ = \sum_{i=1}^{M_p} \int_{\partial_i T} dl' \frac{R}{P'^2} \left( P^0_i + \hat{i}i' \right) \cdot \hat{u}_i - |d| \int_{\partial T} (\hat{\epsilon} \cdot \hat{u}) d\zeta' \]

\[ = \sum_{i=1}^{M_p} \int_{\partial_i T} dl' \frac{R}{P'^2} \left( P^0_i + \hat{i}i' \right) \cdot \hat{u}_i - |d| \alpha (\rho) \quad (3.38) \]

In (3.38), we make use of the Gauss surface divergence theorem in the third line on the right-hand side. \( \partial T_\epsilon \) and \( \partial T \) refer to the boundary of \( T_\epsilon \) and \( T \), respectively. For the \( T_\epsilon \), we used \( P = \epsilon, \epsilon = \hat{\epsilon} \epsilon\), and the Jacobian, \( dl' = \epsilon d\zeta' \). We further note that \( \hat{\epsilon} \cdot \hat{u} = 1 \), since \( \hat{\epsilon} = \hat{u} \). The last line in (3.38) indicates that we have written it for a polygon which has \( M_p \) line segments as edges on plane, \( S \). We define

\[ \alpha (\rho) = \int_{\partial T_\epsilon} d\zeta' \quad (3.39) \]

In (3.39), \( \alpha (\rho) \) is the angular extent of the circular arc portion of the \( \partial T_\epsilon \) lying within \( T \) [8]. Therefore, if the point located by \( \rho \) lies outside \( T \), \( T_\epsilon \) is empty and \( \alpha (\rho) = 0 \); if it lies completely inside \( T \), \( \alpha (\rho) = 2\pi \); if it lies on \( \partial T_\epsilon \), \( \alpha (\rho) = \pi \); if it coincides with a vertex, \( \alpha (\rho) \) becomes the angle between two edges of \( T \) meeting at the vertex. All these scenarios are depicted in Figure 3.3.

Now, we note that on the line segment, \( C_i, \hat{L}_i \cdot \hat{u}_i = 0 \), and \( P^0_i \cdot \hat{u}_i \) is a constant, which can be pulled out of the integral. Further,

\[ \frac{R}{P^2} = \frac{R^2}{P^2 R} = \frac{P^2 + d^2}{P^2 R} = \frac{1}{R} + \frac{d^2}{P^2 R} \quad (3.40) \]
We, therefore, obtain from (3.38)

$$I_{\text{inner2EFIE}} = \sum_{i=1}^{M_{\rho}} \left( P^0_i \cdot \hat{u}_i \right) \int_{\partial_i T} d' \left( \frac{1}{R} + \frac{d'^2}{P^2 R} \right) - |d| \alpha(\rho) \quad (3.41)$$

Again, we find that on $C_i$, $P = P^0_i + \hat{l}_i l'$, and $P^0_i \cdot \hat{l}_i = 0$. Hence,

$$P^2 = \left( P^0_i + \hat{l}_i \right) \cdot \left( P^0_i + \hat{l}_i \right)$$
$$= (P^0_i)^2 + (l')^2 \quad (3.42)$$

Using (3.33) and (3.42), we obtain
\[ R^2 = (P_i^0)^2 + (l')^2 + d^2 \]  
\[ (3.43) \]

Hence,

\[ \int_{\partial_i} \frac{dl'}{R} = \int_{l_i^{-}}^{l_i^{+}} \frac{dl'}{\sqrt{(P_i^0)^2 + (l')^2 + d^2}} = \ln \left( \frac{R_i^+ + l_i^+}{R_i^- + l_i^-} \right) \]  
\[ (3.44) \]

\[ \int_{\partial_i} \frac{dl'}{P^2 R} = \int_{l_i^{-}}^{l_i^{+}} \frac{dl'}{P^2 R} = \int_{l_i^{-}}^{l_i^{+}} \frac{dl'}{P^2 R} \left[ \frac{\left( P_i^0 \right)^2 + (l')^2}{P_i^0 R_i^+} \right] = \frac{1}{|P_i^0 d|} \left[ \tan^{-1} \left( \frac{|d| l_i^+}{|P_i^0| R_i^+} \right) - \tan^{-1} \left( \frac{|d| l_i^-}{|P_i^0| R_i^-} \right) \right] \]  
\[ (3.45) \]

Here, \( P_i^0 = P_i^0 \cdot \hat{u}_i \), and, definitely, \( P_i^0 \) is signed. We further note that \( \alpha (\rho) \) is the sum of the angles between adjacent vertices around the polygon [8]. Therefore, \( \alpha (\rho) \) is the difference in angle between the angle made by \( r^+ \) with \( P_i^0 \) and that by \( r^- \) with \( P_i^0 \). Since \( \hat{P}_i^0 \cdot \hat{u}_i = 1 \), we obtain

\[ \alpha (\rho) = \sum_{i=1}^{M_i} \hat{P}_i^0 \cdot \hat{u}_i \left[ \tan^{-1} \left( \frac{l_i^+}{P_i^0} \right) - \tan^{-1} \left( \frac{l_i^-}{P_i^0} \right) \right] \]  
\[ (3.46) \]

From the trigonometric identity [9]

\[ \tan^{-1} Q_1 - \tan^{-1} Q_2 = \tan^{-1} \left( \frac{Q_1 - Q_2}{1 - Q_1 Q_2} \right) \]  
\[ (3.47) \]

we obtain [8]
\[
\tan^{-1}\left(\frac{l_i^\pm}{P_i^0}\right) - \tan^{-1}\left(\frac{|d| l_i^\pm}{|P_i^0| R_i^\pm}\right) = \tan^{-1}\left(\frac{P_i^0 l_i^\pm}{(R_i^0)^2 + |d| R_i^\pm}\right)
\]

(3.48)

Using (3.44), (3.45), (3.46), (3.48), and the fact that \(P_i^0 = \hat{P}_i^0 P_i^0\) in (3.41), we have

\[
I_{\text{Inner}_{EFIE}} = \int \int_{T_i^\pm} \frac{dS'}{R}
\]

\[= \sum_{i=1}^{M_p} \hat{P}_i^0 \cdot \hat{u}_i \left[ P_i^0 \ln \left( \frac{R_i^+ + l_i^+}{R_i^- + l_i^-} \right) \right]
\]

\[\quad - \sum_{i=1}^{M_p} \hat{P}_i^0 \cdot \hat{u}_i |d| \left[ \tan^{-1}\left(\frac{P_i^0 l_i^+}{(R_i^0)^2 + |d| R_i^+}\right) - \tan^{-1}\left(\frac{P_i^0 l_i^-}{(R_i^0)^2 + |d| R_i^-}\right) \right]
\]

(3.49)

Next, we note that

\[
\nabla_S R = \hat{P} \frac{\partial R}{\partial P} = \hat{P} \frac{\partial (\sqrt{P^2 + d^2})}{\partial P} = \frac{P}{R}
\]

(3.50)

Therefore, we write
\[ I_{\text{Partial Inner}_{\text{EFIE}}} = \iint_T dS' \frac{\rho' - \rho}{R} \]
\[ = \iint_T dS' \frac{P}{\hat{R}} \]
\[ = \lim_{\epsilon \to 0} \iint_{T - T' - T_\epsilon} dS' \frac{P}{\hat{R}} + \lim_{\epsilon \to 0} \iint_{T_\epsilon} dS' \frac{P}{\sqrt{\epsilon^2 + d^2}} \]
\[ = \lim_{\epsilon \to 0} \left[ \iint_{\partial T - \partial T_\epsilon} dl' \hat{R} \hat{u} \right] \]
\[ = \int_{\partial T} dl' \hat{R} \hat{u} - \lim_{\epsilon \to 0} \int_{\partial T_\epsilon} \sqrt{\epsilon^2 + d^2} \hat{u} \epsilon d\zeta' \]
\[ = \int_{\partial T} dl' \hat{R} \hat{u} \quad (3.51) \]

In the above, we used the substitution \( \rho' - \rho = P \) (please refer to Figure 3.2), (3.50), and the Gauss surface divergence theorem. Now, using (3.43), we obtain from (3.51)

\[ I_{\text{Partial Inner}_{\text{EFIE}}} = \int_{\partial T} dl' \hat{R} \hat{u} \]
\[ = \sum_{i=1}^{M_p} \hat{u}_i \int_{L_i^-}^{L_i^+} dl' \sqrt{(P_{0i})^2 + (l')^2 + d^2} \]
\[ = \sum_{i=1}^{M_p} \hat{u}_i \left[ \frac{Rl'}{2} + \frac{(R_{0i})^2}{2} \ln (R + l') \right]_{L_i^-}^{L_i^+} \]
\[ = \frac{1}{2} \sum_{i=1}^{M_p} \hat{u}_i \left[ (R_{0i})^2 \ln \left( \frac{R_{0i} + l_i^+}{R_{0i} - l_i^-} \right) + R_{0i}^+ l_i^+ - R_{0i}^- l_i^- \right] \quad (3.52) \]

At this point it is imperative that we point out a few specifics:

1. If the observation point \( r \) lies in \( S \) on the edge or its extension, from (3.26), \( P^0 = 0 \), and from (3.31), \( d = 0 \), and thus from (3.49), \( I_{\text{Inner}_{\text{EFIE}}} = 0 \) [8].

2. Since \( P^0 = d = 0, R^0 = 0 \), this causes the second term (in square brackets) in (3.49) to be of indeterminate form. Also, \( \hat{P}^0 \) blows up. Thus we encounter a
numerical problem. To circumvent the problem, we move the point of observation to a very small distance, $\epsilon_1$, away from the edge, by setting [5]

$$\mathbf{r} = \mathbf{r} + \epsilon_1 \hat{\mathbf{u}}$$

(3.53)

3. Also, in case of an object with cubic geometry or two surfaces meeting at the right angle, we might encounter the case where $P^0 = 0$. In this case, we treat it exactly the same way as in (3.53).

To evaluate the inner integral in (3.19), we write the basis function $\mathbf{\rho}_n^\pm(\mathbf{r}')$ as

$$\mathbf{\rho}_n^\pm(\mathbf{r}') = \mp (\mathbf{\rho}' - \mathbf{\rho}_v^\pm) = \mp [(\mathbf{\rho}' - \mathbf{\rho}) + (\mathbf{\rho} - \mathbf{\rho}_v^\pm)]$$

(3.54)

Here, $\mathbf{\rho}_v^\pm$ is the projection of the vertex, $\mathbf{v}_n^\pm$, on the plane, $S$. Plugging the above relation into the inner integral of (3.19) leads to [5,8]

$$\int \int_{T_n^\pm} dS' \frac{\mathbf{\rho}_n^\pm(\mathbf{r}')}{R} = \mp \left[ \int \int_{T_n^\pm} dS' \mathbf{\rho}' - \mathbf{\rho} + (\mathbf{\rho} - \mathbf{\rho}_v^\pm) \right] \int \int_{T_n^\pm} \frac{dS'}{R}$$

(3.55)

We can evaluate (3.55) using (3.49) and (3.52).

For an incident plane wave (all waves can be expressed in plane wave expansion), we can evaluate the excitation as below (of course there will be two terms from positive and negative triangles)

$$I_{inc_{EFIE}} = \frac{L_m}{2A_m^\pm} E_{0_{inc}} \cdot \int \int_{T_m^\pm} dS \mathbf{\rho}_m^\pm(\mathbf{r}) e^{-ikr \hat{\mathbf{r}}_{inc}}$$

$$= \frac{L_m}{2A_m^\pm} (\hat{\theta}_{inc}, \hat{\phi}_{inc}) \cdot \int \int_{T_m^\pm} dS \mathbf{\rho}_m^\pm(\mathbf{r}) e^{-ikr \hat{\mathbf{r}}_{inc}}$$

$$\approx \frac{L_m}{2} (\hat{\theta}_{inc}, \hat{\phi}_{inc}) \cdot \sum_{p=1}^M w_p \mathbf{\rho}_p^\pm(\mathbf{r}_p) e^{-ikr \hat{\mathbf{r}}_{inc}}$$

(3.56)
Here, \((\hat{\theta}_{inc}, \hat{\phi}_{inc})\) may be \(\hat{\theta}_{inc}, \hat{\phi}_{inc}\) or any linear combination thereof.

Once the matrix system (3.10) is solved for the coefficient vector \(a\), we move to find the scattered field expression. The scattered electric field is given by [4]

\[
E_{sca}(r) = i\omega A_S(r) - \frac{1}{i\omega \mu \epsilon} \nabla_S \left[ \nabla_S \cdot A_S(r) \right] - \frac{1}{\epsilon} \nabla \times F_S(r) \quad (3.57)
\]

The subscript \(S\) indicates surface quantity here. \(A_S(r)\) and \(F_S(r)\) are vector magnetic potential and vector electric potential, respectively. They are given by [4]

\[
A_S(r) = \frac{\mu}{4\pi} \int_S dS' \mathbf{J}_S(r') \frac{e^{ikR}}{R} \quad (3.58)
\]

and

\[
F_S(r) = \frac{\epsilon}{4\pi} \int_S dS' \mathbf{M}_S(r') \frac{e^{ikR}}{R} \quad (3.59)
\]

where \(R\) is defined as the way we pointed it out, following (3.18).

Since more often than not, we are interested in the far-field approximation, we consider the case when \(R \gg \lambda\) and \(r \gg r'\). Now to derive the far-field approximation, we consider [10]

\[
R = |r - r'| = \sqrt{(r - r') \cdot (r - r')}
= \sqrt{r \cdot r - 2r \cdot r' - r' \cdot r'} \\
\approx \sqrt{r^2 - 2r \cdot r'} \\
= r \sqrt{1 - 2\frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{r}} \\
= r \sqrt{1 - 2\hat{\mathbf{r}} \cdot \frac{\mathbf{r}'}{r}} \\
\approx r \left(1 - \frac{1}{2} 2\hat{\mathbf{r}} \cdot \frac{\mathbf{r}'}{r} \right) \\
= r - \hat{\mathbf{r}} \cdot \mathbf{r}' \quad (3.60)
\]
This more sophisticated far-field approximation will be used to account for the phase in the numerator of $e^{ikR}/R$, because even though the error is small in $R = |r - r'|$ approximation, the exponent is multiplied by $k$ as in $e^{ik|r-r'|}$. Hence, the total error is multiplied by $k$. Thus, for a large $k$, even a small error will result in a large total error. In order for the phase to be negligible, $k|error| \ll \pi$. Since this criterion does not occur in the denominator, we use the far-field approximation $R = |r - r'| \approx r$. Using these approximations in (3.58) and (3.59), we have

$$A_S(r) \approx \frac{\mu}{4\pi r} e^{ikr} N_S$$  \hspace{1cm} (3.61)

and

$$F_S(r) \approx \frac{\epsilon}{4\pi r} e^{ikr} L_S$$  \hspace{1cm} (3.62)

where

$$N_S = \iint_S dS' J_S (r') e^{-ik\hat{r} \cdot r'}$$  \hspace{1cm} (3.63)

$$L_S = \iint_S dS' M_S (r') e^{-ik\hat{r} \cdot r'}$$  \hspace{1cm} (3.64)

Substituting (3.61) and (3.62) into (3.57) and keeping only the dominant terms [10], we obtain

$$E_{sca}(r) = \frac{-ik}{4\pi r} e^{ikr} \left[ \hat{r} \times L_S - \eta N_{S_{\theta,\phi}} \right]$$  \hspace{1cm} (3.65)

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance of the medium, and $N_{S_{\theta,\phi}}$ is given by $N_{S_{\theta,\phi}} = N_S - \hat{r} N_{S_r}$. The inclusion of only $N_{S_{\theta,\phi}}$ in place $N_S$ ensures that the scattered far-field has no $\theta$- or $\phi$-component. Therefore, from (3.61), the contribution to the radiated electric far-field from each triangle and basis function can be written as [5]
\[
\mathbf{E}_{\text{sc}a}(\mathbf{r}) \approx \frac{i\omega \mu}{4\pi} \frac{e^{ikr}}{r} a_m L_m \int \int_{T_m^\pm} dS' \rho_m^\pm (\mathbf{r}') e^{-ikr'} \hat{r}_{\text{inc}}
\]

\[
\approx \frac{i\omega \mu}{4\pi} \frac{e^{ikr}}{r} a_m L_m \sum_{q=1}^{M_p} w_p \rho_m^\pm (\mathbf{r}'_q) e^{-ikr'_q} \hat{r}_{\text{inc}}
\]

(3.66)

It may be noted that the term inside the square brackets in (3.17) has a discontinuous derivative at \( R = 0 \), and its integration via standard Gaussian quadrature will be inaccurate [11]. It was suggested that extraction of more terms for singularity as below will result in better results [11].

\[
G_0 = \frac{1}{4\pi} e^{ikR} \\
= \left[ \frac{e^{ikR}}{R} - \frac{1}{R} + k^2 \frac{R}{2} \right] + \frac{1}{4\pi} \left[ \frac{1}{R} - k^2 \frac{R}{2} \right]
\]

(3.67)

In (3.67), the expression for the integration of the second term inside the second bracketed quantity is given in the Appendix of [11]. Actually, in (3.67), more terms from the Taylor series have been included. Because, although the term inside the square brackets on the right-hand side of (3.17) is finite and continuous, its first derivative, clearly, is not. In (3.67), the first derivative of the terms inside the square brackets is finite and continuous. In fact, by including more and more Taylor series terms, we can make the second, the third, the fourth, etc., finite and continuous because we are making the term(s) inside the square brackets smoother. Oijala and Taskinen [11] provide elaborate recursive formulas for a generalized scenario. To find the field at any point in space, we apply (2.19) which will produce the total field outside the object and zero field inside.
4.1 Radar Cross Section (RCS)

The radar cross section (RCS) is a very important parameter that defines the target characteristics in areas ranging from avionics to defense. The RCS is defined as the area intercepting the amount of power that, when scattered isotropically, produces at the receiver a density that is equal to the density scattered by the actual target [10]. For a three-dimensional target, the RCS is given by [10]

\[
\sigma_{3-D} = \lim_{r \to \infty} \left[ 4\pi r^2 \frac{|E_{sca}(r)|^2}{|E_{inc}(r)|^2} \right] = \lim_{r \to \infty} \left[ 4\pi r^2 \frac{|H_{sca}(r)|^2}{|H_{inc}(r)|^2} \right]
\]

(4.1)

Complete electromagnetic scattering information is contained in the polarization scattering matrix that relates the two possible transverse incident field vectors to two possible transverse scattered field vectors. Only two components, \( (\hat{\theta}, \hat{\phi}) \), of the field are used since Maxwell’s equations require that the electric field be transverse to the direction of propagation \( (\hat{r} = \hat{k}) \) [12]. The polarization scattering matrix (also called the S-matrix) is given by [12]

\[
E_{sca} = \overline{S} \cdot E_{inc}
\]

(4.2)

where \( \overline{S} \) is a 2 \( \times \) 2 dyadic (tensor) given by
\[
\begin{bmatrix}
E_{\text{sca},\theta} \\
E_{\text{sca},\phi}
\end{bmatrix} =
\begin{bmatrix}
S_{\theta_{\text{sca}}\theta_{\text{inc}}} & S_{\theta_{\text{sca}}\phi_{\text{inc}}} \\
S_{\phi_{\text{sca}}\theta_{\text{inc}}} & S_{\phi_{\text{sca}}\phi_{\text{inc}}}
\end{bmatrix}
\cdot
\begin{bmatrix}
E_{\text{inc},\theta} \\
E_{\text{inc},\phi}
\end{bmatrix}
\] (4.3)

The diagonal terms are the two independent co-polarized scattering elements while the off-diagonal terms are the two cross-polarized scattering terms. The scattering matrix completely represents the scattering properties of a target. The four terms above are complex-valued with amplitude and phase. The elements of \( \mathbf{S} \) are related to the RCS by [12]

\[
S_{ij} = \sqrt{\frac{\sigma_{ij}}{4\pi r^2}}
\] (4.4)

Here \( i = \{\theta_{\text{sca}}, \phi_{\text{sca}}\} \) and \( j = \{\theta_{\text{inc}}, \phi_{\text{inc}}\} \). Also \( S_{ij} \) can be expressed as [12]

\[
S_{ij} = S_{ij}(\theta_{\text{sca}}, \phi_{\text{sca}}, \theta_{\text{inc}}, \phi_{\text{inc}})
\] (4.5)

The monostatic (or backscattered) RCS is the case when the transmitting antenna and the receiving antenna are at the same location. Therefore, for monostatic case we have either of the following two cases:

1. \( S_{\theta_{\text{sca}}\theta_{\text{inc}}} \) with \( \theta_{\text{sca}} = \theta_{\text{inc}} \).
2. \( S_{\phi_{\text{sca}}\phi_{\text{inc}}} \) with \( \phi_{\text{sca}} = \phi_{\text{inc}} \).

The bistatic RCS is the case when the transmitting antenna and the receiving antenna are at different locations. Therefore for the bistatic case, any of \( S_{\theta_{\text{sca}}\theta_{\text{inc}}} \), \( S_{\theta_{\text{sca}}\phi_{\text{inc}}} \), \( S_{\phi_{\text{sca}}\theta_{\text{inc}}} \), and \( S_{\phi_{\text{sca}}\phi_{\text{inc}}} \) may be possible. However for the cases when \( S_{\theta_{\text{sca}}\theta_{\text{inc}}} \) and \( S_{\phi_{\text{sca}}\phi_{\text{inc}}} \) are considered, \( \theta_{\text{sca}} \neq \theta_{\text{inc}} \) and \( \phi_{\text{sca}} \neq \phi_{\text{inc}} \) must be true respectively. It may also be noted that observations made toward directions that satisfy Snell’s law of reflection are usually referred to as specular [10].

Now, we refer to a four-way classification of the RCS:

1. Vertical-vertical (VV) polarization: It is also called \( \theta\theta \) polarization. For this case, the RCS is given by
\[
\sigma_{VV} = \lim_{r \to \infty} \left[ 4\pi r^2 \left| \frac{E_{\theta_{sca}}}{E_{\theta_{inc}}} \right|^2 \right] \tag{4.6}
\]

2. Vertical-horizontal (VH) polarization: It is also called \( \theta \phi \) polarization. For this case, the RCS is given by

\[
\sigma_{VH} = \lim_{r \to \infty} \left[ 4\pi r^2 \left| \frac{E_{\phi_{sca}}}{E_{\phi_{inc}}} \right|^2 \right] \tag{4.7}
\]

3. Horizontal-vertical (HV) polarization: It is also called \( \phi \theta \) polarization. For this case, the RCS is given by

\[
\sigma_{HV} = \lim_{r \to \infty} \left[ 4\pi r^2 \left| \frac{E_{\phi_{sca}}}{E_{\phi_{inc}}} \right|^2 \right] \tag{4.8}
\]

4. Horizontal-horizontal (HH) polarization: It is also called \( \phi \phi \) polarization. For this case, the RCS is given by

\[
\sigma_{HH} = \lim_{r \to \infty} \left[ 4\pi r^2 \left| \frac{E_{\phi_{sca}}}{E_{\phi_{inc}}} \right|^2 \right] \tag{4.9}
\]

Here \( E_{\psi_j} \) is given by [9]

\[
E_{\psi_j} = E_j \cdot \hat{\psi}_j \tag{4.10}
\]

where \( \hat{\psi}_j \) may be either \( \hat{\theta}_j \) or \( \hat{\phi}_j \), and \( j = \{sca, \ inc\} \). This nomenclature stems from the fact that \( \theta \) is the zenith or inclination angle which is the angular tilt with reference to the vertical direction (z-axis), and \( \phi \) is the azimithal angle which is angular rotation with reference to the horizontal direction (x-axis). It may be mentioned that \( \sigma_{VH} = \sigma_{HV} \).
4.2 Numerical Integration

To numerically integrate, we use the Gaussian quadrature. The most commonly referenced Gauss-Legendre locations and weights for triangles are the symmetric quadrature rules of [5, 13]. The weights are given in Tables 9.1 and 9.2 of [5, pp. 267–268] for various quadrature rules. The weights in these tables are normalized by triangle area, $A$, as [5]

$$\int \int_S f(S)dS \approx A \sum_{i=1}^{M_p} w(\alpha_i, \beta_i, \gamma_i) f(r(\alpha_i, \beta_i, \gamma_i)) \quad (4.11)$$

and

$$r(\alpha, \beta, \gamma) = \gamma v_1 + \alpha v_2 + \beta v_3 \quad (4.12)$$

where $v_1$, $v_2$, and $v_3$ are the Cartesian coordinates of the three vertices of the triangle. We use the 4-point Gaussian quadrature rule for nonsingular integrals and the 7-point Gaussian quadrature for singular and near-singular integrals. Obviously, higher order Gaussian quadrature rules will considerably improve the accuracy, albeit at the expense of longer run times.
CHAPTER 5

THE MAGNETIC FIELD INTEGRAL EQUATION (MFIE) FOR PEC OBJECTS

The MFIE for a PEC object relates the incident magnetic field to the scattered surface electric current of surface $S$. For a PEC object, using the duality principle [2, 4, 10], we can write from (3.1)

$$
\mathbf{H}_{inc}(\mathbf{r}) + \nabla \times \oint_S dS' \mathbf{G}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_S(\mathbf{r}') = \begin{cases} 
\mathbf{H}(\mathbf{r}), \mathbf{r} \in V_1 \\
0, \mathbf{r} \in V_2 
\end{cases} \quad (5.1)
$$

Using the extinction part of (5.1) (exactly as we did to derive the EFIE equation for a PEC scatterer before), we obtain

$$
\mathbf{H}_{inc}(\mathbf{r}) = -\nabla \times \oint_S dS' \mathbf{G}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_S(\mathbf{r}') \quad (5.2)
$$

We obtain from (5.2)

$$
\hat{n} \times \mathbf{H}_{inc}(\mathbf{r}) = -\hat{n} \times \nabla \times \oint_S dS' \mathbf{G}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_S(\mathbf{r}') \quad (5.3)
$$

Now, we notice that the gradient of the Green’s function makes the kernel more singular than the EFIE kernel, and the principal value integral method can be used to evaluate this singular integral [1, 8]. Using the technique very similar to that described by Jin [4, pp. 419–421], we obtain

$$
\hat{n} \times \mathbf{H}_{inc}(\mathbf{r}) = \frac{1}{2} \mathbf{J}_S(\mathbf{r}) - \hat{n} \times P.V. \iint_S dS' \nabla G_0(\mathbf{r}, \mathbf{r}') \times \mathbf{J}_S(\mathbf{r}') \quad (5.4)
$$
The first term represents the residue term while the second term represents the principal value term. It should be emphasized that the MFIE can only be applied to closed surfaces, while the EFIE can be applied to both open and closed surfaces. Although both the EFIE and the MFIE were derived from the extinction principle, for a PEC object \( \hat{n} \times E_{\text{inc}}(r) = 0 \), while \( \hat{n} \times H_{\text{inc}}(r) = J_S(r) \). Thus, the electric field is continuous across the surface while the magnetic field has a jump discontinuity. Now, \( \nabla G_0(r, r') \) is given by [7]

\[
\nabla G_0(r, r') = -\nabla' G_0(r, r') = -(r - r') \frac{(1 - i k |r - r'|)}{4\pi |r - r'|^3} e^{ik|r-r'|} \quad (5.5)
\]

Plugging (5.5) into (5.4) and using RWG expansion for the electric current, we obtain [5]

\[
\hat{n} \times H_{\text{inc}}(r) = \frac{1}{2} \sum_{n=1}^{N} a_n f_n(r) + P.V. \iint_S dS' \hat{n}(r) \times \left[ (r - r') \times \sum_{n=1}^{N} a_n f_n(r') \right] \quad (5.6)
\]

Using Galerkin testing (5.6) with RWG functions, we obtain an equation similar to (2.14). The matrix elements \( Z_{mn} \) are given by [5]

\[
Z_{mn} = \frac{1}{2} \iint_{f_n=f_n} dS f_m(r) \cdot f_n(r)
+ \frac{1}{4\pi} \iint_{f_m} dS f_m(r) \cdot \hat{n}(r) \times \iint_{f_n} dS' [(r - r') \times f_n(r')] (1 - i k |r - r'|) \frac{e^{ik|r-r'|}}{|r - r'|^3}
= \frac{1}{2} \iint_{f_m=f_n} dS f_m(r) \cdot f_n(r)
- \frac{1}{4\pi} \iint_{f_m} dS \hat{n}(r) \times f_m(r) \cdot \iint_{f_n} dS' [(r - r') \times f_n(r')] (1 - i k |r - r'|) \frac{e^{ik|r-r'|}}{|r - r'|^3} \quad (5.7)
\]

The second line in (5.7) was obtained from the vector identity [4] in (2.10).

Either of the two forms given in (5.7) can be used. Since the first term in both forms in (3.39) is the residue term, this will contribute only if the triangular patches overlap, while the second term in both lines in (3.39) is accounted for only if the
triangular patches do not overlap. The excitation vector elements $b_m$ are given by [5]

$$b_m = \int_{T_m} dS f_m(r) \cdot [\hat{n}(r) \times H_{\text{inc}}(r)]$$

(5.8)

Now we can define another \textit{integro-differential} surface operator called the $\mathcal{K}$ operator. Although there are various definitions of this operator in the literature, we define it as [7]

$$\mathcal{K}(r, r')X(r') = -\frac{\hat{n}(r) \times X(r)}{2} + P.V. \int_{S} dS' X(r') \times \nabla G_0(r, r')$$

(5.9)

Using (2.17) and (3.42), the EFIE and the MFIE can be written in compact notation as

$$\hat{t}(r) \cdot E_{\text{inc}}(r) = -\hat{t}(r) \cdot [\mathcal{L}(r, r') J_S(r')]$$

(5.10)

and

$$\hat{n}(r) \times H_{\text{inc}}(r) = \hat{n}(r) \times [\mathcal{K}(r, r') J_S(r')]$$

(5.11)

respectively.

As in the case of EFIE, we can express (5.7) for one pair of source and observation patches (certainly in $Z_{mn}$, there will be four such terms corresponding to the combinations of positive and negative triangles for each of the source and testing patches) as [5]

$$I_{\text{MFIE}} = \frac{L_m L_n}{4\pi A_m^\pm A_n^\pm} \left[ \frac{\pi}{2} \int_{T_m^\pm} dS \rho_m^\pm(r) \cdot \rho_n^\pm(r) \right]$$

$$+ \frac{L_m L_n}{4\pi A_m^\pm A_n^\pm} \int_{T_m^\pm} dS \frac{1}{4} \rho_m^\pm(r) \cdot \hat{n}(r) \times \int_{T_n^\pm} dS' (r - r')$$

$$\times \rho_n^\pm(r') (1 - i\kappa |r - r'|) e^{i\kappa |r - r'|^3}$$

(5.12)
Applying an $M$-point Gaussian quadrature rule over source and observation patches for a completely nonsingular case, (5.12) yields

$$I_{\text{MFIE nonsingular}} \approx \frac{L_m L_n}{4\pi} \sum_{p=1}^{M} \sum_{q=1}^{M} w_p w_q \frac{1}{4} \rho_m^\pm(r_p) \cdot \hat{n}(r_p) \times (r_p - r'_q)$$

$$\times \rho_n^\pm(r'_q) (1 - i k |r_p - r'_q|) \frac{e^{ik|r_p-r'_q|}}{|r_p - r'_q|^3}$$

(5.13)

For partial or full overlap, we have the term

$$I_{\text{MFIE overlap}} = \frac{L_m L_n}{4\pi A_n^\pm} \left[ \pi \sum_{p=1}^{M} w_p \rho_m^\pm(r_p) \cdot \rho_n^\pm(r_p) \right]$$

(5.14)

When there is no overlap, only (5.13) is needed. For full overlap, only (5.14) is used. For a partial overlap, for overlapping patches, we compute contribution from (5.14), while for the rest, (5.13) is applied.

For the treatment of near-singularity (there is no complete singularity owing to the residue term), we consider Figure 5.1, in addition to the one we have considered for the EFIE case. Here, we follow the treatment by [5,14]. We define a vector

$$R_n = r - v_n^\pm = (r - r') + (r' - v_n^\pm) = (r - r') \mp \rho_n^\pm$$

(5.15)

From (5.15), we have

$$r - r' = R_n \pm \rho_n^\pm$$

(5.16)

Thus we write [5]

$$(r - r') \times \rho_n^\pm = R_n \times \rho_n^\pm = R_n \times \rho_n^\pm$$

(5.17)

From the innermost integral of the second term of (5.12), we write
Figure 5.1: Definition of quantities for the polygon segment $C$ in excess of Figure 3.2 for MFIE singular integrals (after [5, 14]).

\[
I_{MFIE_{inner}} = \int\int_{T_n^+} dS' \mathbf{R} \times \rho_n^+(\mathbf{r}') (1 - ikR) \frac{e^{ikR}}{R^3} \\
= \mathbf{R}_n \times \int\int_{T_n^+} dS' \rho_n^+(\mathbf{r}') (1 - ikR) \frac{e^{ikR}}{R^3} 
\] (5.18)

We have moved $\mathbf{R}_n$ out of the integral because it is a constant vector. We note that
\[ (1 - ikR) \frac{e^{ikR}}{R^3} = \frac{(1 - ikR) e^{ikR} - (1 + \frac{1}{2}k^2 R^2) + (1 + \frac{1}{2}k^2 R^2)}{R^3} \]
\[ = \frac{(1 - ikR) e^{ikR} - (1 + \frac{1}{2}k^2 R^2)}{R^3} + \frac{1}{R^3} + \frac{k^2}{2R} \]  
\[ (5.19) \]

Therefore, employing (5.19) in (5.18) we have [5]

\[ I_{MFE_{inner}} = R_n \times \int_{T_{n}^\pm} dS' \rho_n^+(r') \left( \frac{(1 - ikR) e^{ikR} - (1 + \frac{1}{2}k^2 R^2)}{R^3} \right) \]
\[ + R_n \times \left[ a_n(r) + \frac{k^2}{2} b_n(r) \right] \]  
\[ (5.20) \]

where \( a_n(r) \) and \( b_n(r) \) are given by [5,14]

\[ a_n(r) = \int_{T_{n}^\pm} dS' \frac{\rho_n^+(r')}{R^3} \]  
\[ (5.21) \]

and

\[ b_n(r) = \int_{T_{n}^\pm} dS' \frac{\rho_n^+(r')}{R} \]  
\[ (5.22) \]

Equation (5.22) can be evaluated by using (3.49), (3.52), and (3.55), as before. However, (5.21) can be written as

\[ \int_{T_{n}^\pm} dS' \frac{\rho_n^+(r')}{R^3} = \pm \left[ \int_{T_{n}^\pm} dS' \frac{\rho' - \rho}{R^3} + (\rho - \rho_{v_n^+}) \int_{T_{n}^\pm} dS' \frac{1}{R^3} \right] \]  
\[ (5.23) \]

Now we want to evaluate \( a_n(r) \). We note that [9,14]

\[ \nabla_S \cdot \left( \frac{P}{P^2 R} \right) = \nabla_S \cdot \left( \frac{1}{PR} \hat{P} \right) = \frac{1}{P} \frac{\partial}{\partial P} \left( P \frac{1}{P R} \right) = \frac{1}{P} \frac{\partial}{\partial P} \left( \frac{1}{R} \right) = -\frac{1}{R^3} \]  
\[ (5.24) \]

Hence,
\[ I_{\text{Inner scalarMFIE}} = \iiint_T \frac{dS'}{R^3} \]

\[ = - \lim_{\epsilon \to 0} \iiint_{T-T_c} \frac{dS'}{R} - \lim_{\epsilon \to 0} \iiint_{T_c} \frac{dS'}{R^3} \]

\[ = - \lim_{\epsilon \to 0} \iiint_{T-T_c} dS' \nabla_S \cdot \left( \frac{P}{P^2 R} \right) - \lim_{\epsilon \to 0} \iiint_{T_c} \frac{dS'}{R^3} \]

Here the second term in the third line of (5.25) is zero applying the same reasoning that we deduced right after (3.37). Now,

\[ I_{\text{Inner scalarMFIE}} = - \lim_{\epsilon \to 0} \iiint_{T-T_c} dS' \nabla_S' \cdot \left( \frac{1}{PR} \hat{P} \right) \]

\[ = - \lim_{\epsilon \to 0} \iiint_{T} dS' \nabla_S' \cdot \left( \frac{1}{PR} \hat{P} \right) + \lim_{\epsilon \to 0} \iiint_{T} dS' \nabla_S' \cdot \left( \frac{1}{PR} \hat{P} \right) \]

\[ = - \lim_{\epsilon \to 0} \int_{\partial T} dl' \frac{1}{P^2 R} \mathbf{P} \cdot \hat{u} + \lim_{\epsilon \to 0} \int_{\partial T} \epsilon \left( \frac{1}{P^2 R} \hat{u} \right) \epsilon \hat{u} \cdot \epsilon d\zeta' \]

\[ = \sum_{i=1}^{M_p} \int_{\partial T} dl' \frac{1}{P^2 R} \left( \mathbf{P}_i^0 + \hat{\mathbf{l}}_i \hat{\mathbf{r}}' \right) \cdot \hat{\mathbf{u}}_i + \lim_{\epsilon \to 0} \int_{T_c} \frac{1}{\epsilon^2 + \epsilon^2} \left( \epsilon \hat{u} \right) \cdot \epsilon \hat{u} \cdot \epsilon d\zeta' \]

\[ = \sum_{i=1}^{M_p} \int_{\partial T} dl' \frac{1}{P^2 R} \left( \mathbf{P}_i^0 + \hat{\mathbf{l}}_i \hat{\mathbf{r}} ' \right) \cdot \hat{\mathbf{u}}_i + \lim_{\epsilon \to 0} \int_{T_c} \frac{1}{d} \left( \epsilon \cdot \hat{u} \right) d\zeta' \]

\[ = \sum_{i=1}^{M_p} \int_{\partial T} dl' \frac{1}{P^2 R} \left( \mathbf{P}_i^0 + \hat{\mathbf{l}}_i \hat{\mathbf{r}} ' \right) \cdot \hat{\mathbf{u}}_i + \frac{1}{|d|} \alpha (\rho) \] (5.26)

In deriving (5.26), we took the same route as we did for (3.38). We note that \( \left( \mathbf{P}_i^0 + \hat{\mathbf{l}}_i \hat{\mathbf{r}} ' \right) \cdot \hat{\mathbf{u}}_i = \mathbf{P}_i^0 \cdot \hat{\mathbf{u}}_i \), for reasons already described in the discussion preceding (3.40) and it is only a constant that can be pulled out of the integral in the last line of (5.26). Thus the first term in the last line of (5.26) is nothing but the integral described by (3.45), while in the second term we can use (3.46). Now, with the aid of the trigonometric identity [14]
\[
\tan^{-1} \frac{Q_2}{Q_1} - \tan^{-1} \frac{Q_2Q_4}{Q_1Q_3} = \tan^{-1} \left( \frac{Q_1Q_2}{Q_1^2 + Q_4^2 + Q_3Q_4} \right) \tag{5.27}
\]

we finally arrive at [14]

\[
\int \int_{T^3_n} \frac{dS'}{R^3} = \sum_{i=1}^{M_p} \hat{P}_i \cdot \hat{u}_i \frac{1}{|d|} \tan^{-1} \left( \frac{P_{i}^{0}_{l_i}}{(R_{i}^{0})^2 + |d|R_{i}^{+}} \right) - \sum_{i=1}^{M_p} \hat{P}_i \cdot \hat{u}_i \frac{1}{|d|} \tan^{-1} \left( \frac{P_{i}^{0}_{l_i}}{(R_{i}^{0})^2 + |d|R_{i}^{-}} \right) \quad (d \neq 0) \tag{5.28}
\]

For \( |d| \to 0 \), (5.26) is unbounded for \( \rho \in S \), since the singularity in not integrable. Nevertheless, for \( \rho \notin S \) the limit as \( |d| \to 0 \) exists. Therefore, setting \( \alpha (\rho) = 0 \), and \( P_i^0 \neq 0 \) we evaluate for small argument case (i.e., \( |d| \to 0 \)) [14]

\[
\lim_{d \to 0} \frac{1}{|d|} \tan^{-1} \frac{|d| l_i^+}{P_i^0 R_i^+} = \frac{l_i^+}{P_i^0 R_i^+} \tag{5.29}
\]

This leads to

\[
\int \int_{T^3_n} \frac{dS'}{R^3} = - \sum_{i=1}^{M_p} \hat{P}_i \cdot \hat{u}_i \left[ \frac{l_i^+}{P_i^0 R_i^+} - \frac{l_i^-}{P_i^0 R_i^-} \right] \quad (d = 0) \tag{5.30}
\]

Combining (5.28) and (5.30), we write [5]

\[
\int \int_{T^3_n} \frac{dS'}{R^3} = \sum_{i=1}^{M_p} \hat{P}_i \cdot \hat{u}_i \frac{1}{|d|} \tan^{-1} \left( \frac{P_{i}^{0}_{l_i}}{(R_{i}^{0})^2 + |d|R_{i}^{+}} \right) - \sum_{i=1}^{M_p} \hat{P}_i \cdot \hat{u}_i \frac{1}{|d|} \tan^{-1} \left( \frac{P_{i}^{0}_{l_i}}{(R_{i}^{0})^2 + |d|R_{i}^{-}} \right) \quad (d \neq 0) \\
= - \sum_{i=1}^{M_p} \hat{P}_i \cdot \hat{u}_i \left[ \frac{l_i^+}{P_i^0 R_i^+} - \frac{l_i^-}{P_i^0 R_i^-} \right] \quad (d = 0) \tag{5.31}
\]
Now, we note that

\[
\nabla S \left( \frac{1}{R} \right) = \hat{P} \frac{\partial}{\partial P} \left( \frac{1}{R} \right) = -\frac{P}{R^3}
\]

(5.32)

Using this, we write \[9,14\]

\[
I_{\text{Inner vector MFIE}} = \iint_T dS' \rho' - \rho \frac{P}{R^3}
\]

\[
= \iint_T dS' \frac{P}{R^3}
\]

\[
= \lim_{\epsilon \to 0} \iint_{T-T_\epsilon} dS' \frac{P}{R^3} + \lim_{\epsilon \to 0} \iint_{T_\epsilon} dS' \frac{P}{R^3}
\]

\[
= \frac{P}{R^3} \nabla' \left( \frac{1}{R} \right) + \lim_{\epsilon \to 0} \iint_{T_\epsilon} dS' \frac{\hat{P} \epsilon}{(\epsilon^2 + d^2)^{\frac{3}{2}}}
\]

\[
\]

\[
= -\lim_{\epsilon \to 0} \int_{\partial T - \partial T_\epsilon} dl' \left( \frac{1}{R} \right) \hat{u}
\]

\[
= -\int_{\partial T} dl' \left( \frac{1}{R} \right) \hat{u} + \lim_{\epsilon \to 0} \int_{\partial T_\epsilon} \frac{\epsilon}{\sqrt{\epsilon^2 + d^2}} \hat{u} d\zeta'
\]

\[
= \int_{\partial T} dl' \left( \frac{1}{R} \right) \hat{u} - \sum_{i=1}^{M_p} \hat{u}_i \int_{L^-_i}^{L^+_i} dl' \frac{1}{\sqrt{(P'^0)_i^2 + (l')^2 + d^2}}
\]

\[
= -\sum_{i=1}^{M_p} \hat{u}_i \ln \left( \frac{R_i^+ + l_i^+}{R_i^- + l_i^-} \right)
\]

(5.33)

Therefore, we write \[5\]

\[
\iint_{T_n^\pm} dS' \rho' - \rho \frac{P}{R} = -\sum_{i=1}^{M_p} \hat{u}_i \ln \left( \frac{R_i^+ + l_i^+}{R_i^- + l_i^-} \right)
\]

(5.34)

It should be pointed out that although we have not presented them here, the more generalized recursive relations to compute the singular integrals presented in the Appendix of \[11\] can be evaluated fairly easily using the same techniques that we used
to evaluate the singular integrals for the EFIE and the MFIE. All the observations
that we have made for the EFIE (in the paragraph following (3.52)) hold equally well
for the MFIE.

The excitation for MFIE can be evaluated as [5]

\[ I_{\text{incMFIE}} = \frac{L_m}{2A_m^2} \left( \hat{n} \times H_{0\text{inc}} \right) \cdot \iint_{T_m^+} dS \rho_m^\pm(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} \]

\[ = \frac{L_m}{2A_m^2 \eta} \left[ \hat{n} \times (\hat{r}_{\text{inc}} \times \mathbf{E}_{0\text{inc}}) \right] \cdot \iint_{T_m^+} dS \rho_m^\pm(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} \]

\[ = \frac{L_m}{2A_m^2 \eta} \left[ \hat{n} \times (\hat{r}_{\text{inc}} \times \left( \hat{\theta}_{\text{inc}}, \hat{\phi}_{\text{inc}} \right)) \right] \cdot \iint_{T_m^+} dS \rho_m^\pm(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} \]

\[ \approx \frac{L_m}{2\eta} \left[ \hat{n} \times (\hat{r}_{\text{inc}}, \hat{\theta}_{\text{inc}}) \right] \cdot \sum_{p=1}^{M} w_p \rho_m^\pm(\mathbf{r}_p) e^{-i\mathbf{k} \cdot \mathbf{r}_p} \]  

(5.35)

It should be stressed that \( \hat{k}_{\text{inc}} = -\hat{r}_{\text{inc}} \), while \( \hat{k}_{\text{sca}} = \hat{r}_{\text{sca}} \).

For the radiated electric far-field, we use (3.66). To find the electric field at any
point in space, we apply (2.19) which will produce the total field outside the object
and zero field inside.

Now, let us verify the accuracy of the singular terms in both the EFIE and the
MFIE by reproducing the results in [14] and [5], respectively. First, we consider the
results in [14]. Let us write the scattered magnetic field of a triangular patch (it does
not matter if the triangle is positive or negative) as [14]

\[ H_{\text{scan}}^\pm = -\frac{1}{8\pi A_n^\pm} R_n \times \iint_{T_n^\pm} dS' \rho_n^\pm(\mathbf{r}') \left( 1 - ikR \right) e^{ikR} - \left( 1 + \frac{1}{2} k^2 R^2 \right) \right] \]

\[ - \frac{1}{8\pi A_n^\pm} R_n \left[ a_n(\mathbf{r}) + \frac{k^2}{2} b_n(\mathbf{r}) \right] \]  

(5.36)

We assume that the triangle we consider here is an equilateral triangle with sides
0.1λ. The plot appearing in Figure 5.2 is identical to the one given in [14]. Because
\( a_n \) and \( b_n \) are real, it is sufficient to examine the behavior of \( \Re \left[ H_{\text{scan}}^\pm \right] \) as \( R \to 0 \). The plot shows the magnitude of \( a_n \) and \( b_n \) versus perpendicular distance \( z \)
above the triangle centroid. As the observation point approaches the plane of the
Figure 5.2: Real part of the scattered magnetic field for a single equilateral triangular patch (reproduction of different terms adapted from [14, Figure 7.8 (a)]).

Triangle, the term $a_n$, with $1/R^3$ singularity, dominates, while the $1/R$ singularity term has a minuscule contribution. Also, we note that at about a distance of $0.3\lambda$ above the plane of the triangle, the $1/R^3$ singularity contribution converges with the integral contribution. Therefore, it apparently seems to be a candidate for singularity threshold consideration.

Next we consider the results in [5]. Let us compute the scalar quantity

$$I_{\text{singular, scalar}} = \left| a_n(r) + \frac{k^2}{2} b_n(r) \right|$$

(5.37)

For this case, we consider an equilateral triangle with sides $1\lambda$. In Figures 5.3 and 5.4, we compare the results obtained from the the third, the fourth, and the fifth order Gaussian quadrature rules with the analytical expression in (3.61). We find that the results reproduced here are identical to those in [5]. In Figure 5.3, the observation point is placed above the centroid of the triangle and the separation distance is varied, while in Figure 5.4, the observation point is placed at the mid-point of one of the sides of the triangle under consideration. In both cases, the comparison is good at larger distances but is increasingly poor at shorter distances, as expected. Again, as we find in the reproduction of the results in [14], here the singularity threshold seems to be $0.3\lambda$. 

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Figure 5.3: MFIE Singular integration (reproduction of different terms adapted from [5, Figure 7.8 (a)]).

Figure 5.4: Singular integration (reproduction of different terms adapted from [5, Figure 7.8 (b)]).

It may be mentioned here that the RWG functions, when used in the MFIE, are not as accurate as the EFIE for the same problem [15]. This inaccuracy stems from numerical integration [11, 16], choice of solid angle factor [17, 18], and the choice of basis and testing functions. Using a \textit{curl conforming} basis functions [19, 20], the accuracy can be considerably improved. This improvement happens because the \textit{curl}
conforming basis functions are in the dual space [21,22] of the RWG basis functions. In fact, this relative inaccuracy of the MFIE will be verified in Chapter 12.

Now let us reconsider (3.37). Here, $G_0(r - r')$ has a singularity at $r = r'$. Now, $\nabla G_0(r - r')$ is even more singular. We extracted only one singular term from $G_0(r - r')$ for EFIE. Extending this idea to $\nabla G_0(r - r')$, we have [11]

$$\nabla G_0(r - r') = \nabla \left[ G_0(r - r') - \frac{1}{4\pi R} \right] + \frac{1}{4\pi} \nabla \frac{1}{R} \tag{5.38}$$

Here the second term can be evaluated using the Appendix of [11]. The first term (the term with $\nabla$ operating on the expression in square brackets), however, is still discontinuous at $R = 0$. To remedy this problem, which might cause numerical problem, we extract two terms from $\nabla G_0(r - r')$ and write [11]

$$\nabla G_0(r - r') = \nabla \left[ G_0(r - r') - \frac{1}{4\pi R} + \frac{k^2}{8\pi} \frac{1}{R} \right] + \frac{1}{4\pi} \nabla \frac{1}{R} - \frac{k^2}{8\pi} \nabla R \tag{5.39}$$

The first term now has continuous higher-order derivative, and thus can be integrated numerically using its Taylor series expansion. The third term can also be analytically integrated using the formulas given in the Appendix of [11]. Let us pay close attention to the $\nabla \left(1/R\right)$ term. Although we can integrate it over the source patch (inner integral) in a closed form using again the Appendix of [11], it still has a mild logarithmic singularity at the common points of the testing patches for the outer integral for the testing patch. This singularity is particularly pronounced when the source patch shares a side with the testing patch and the source and testing triangles are not in the same plane [11]. Although this singularity is mild and can be eliminated using a higher order Gaussian quadrature for the outer integral, however, the treatment compromises the computational time. Ylä-Oijala and Taskinen [11] propose a unique way to mitigate this mild logarithmic singularity by exchanging the order of the integration after converting the inner surface integral into a line integral by using the Gauss divergence theorem. This treatment, however, is beyond the scope of this thesis.
CHAPTER 6

SINGULARITY TREATMENT: THE CORRECT YARDSTICK

An important question in the singularity treatment in the surface integral equations (SIE) is the threshold of the singularity treatment. The matrix equation for such a SIE (for example, the EFIE, or the MFIE) can be written as

\[ \mathbf{Z} \cdot \mathbf{a} = \mathbf{b} \] (6.1)

While the singularity treatment for all the terms in the system matrix, \( \mathbf{Z} \), yields the most accurate result, it comes at the expense of a huge computational cost. It is, therefore, imperative that a threshold for the singularity treatment be set up, so that only the singular terms are computed using the analytical method while the rest be treated as regular integrals, and as such, be treated using numerical integration techniques like the Gaussian quadrature. As a yardstick for the singularity treatment, [5] suggests that integrals involving source and observation triangle pairs in the EFIE separated by less than \( 0.1\lambda - 0.2\lambda \) be treated as singular, while in the MFIE the prescribed threshold be \( 0.2\lambda - 0.3\lambda \). While this definition works pretty well, we ask if we should take these figures as the threshold. Let us consider a PEC sphere of radius \( 0.1\lambda \) to be simulated using both the electric field integral equation (EFIE) and the magnetic field integral equation (MFIE). Clearly, we will end up treating all the terms in \( \mathbf{Z} \) as singular! The whole scheme is, thus, ludicrous in view of the different problem sizes appearing in the electromagnetic (EM) community.

In this chapter, we study a feasibility study for a sensible reference for the singularity treatment. We consider an equilateral triangle, which will be considered as the source triangle, and, also, a point of observation. There may be two cases here:

**Case I:** The point of observation lies above the triangle centroid. The distance between the triangle centroid and the point of observation is denoted as \( d_1 \).
Case II: The point of observation lies above the mid-point of one of the sides of the triangle. The distance between the mid-point of the triangle above which the point of observation lies and the point of observation is denoted as $d_2$.

First, we consider the feasibility of the sides of the equilateral triangle as the yardstick for the singularity treatment. To consider various cases, we take two sets for the length of each side of the triangle (denoted as $L$) as 0.05 m to 0.5 m, assuming a frequency of 300 MHz. To do the analysis we reconsider (5.37) as

$$I_{\text{singular}} = \left| a(r) + \frac{k^2}{2} b(r) \right|$$  \hspace{1cm} (6.2)

where

$$a(r) = \iint_T dS' \frac{\rho(r')}{R^3}$$  \hspace{1cm} (6.3)

and

$$b(r) = \iint_T dS' \frac{\rho(r')}{R}$$  \hspace{1cm} (6.4)

Here, $R = r - r'$ and $R = |r - r'|$. The primed quantity here indicates the source point while the nonprimed quantity refers to observation point. Also, $\rho$ is given by

$$\rho(r) = r - v$$  \hspace{1cm} (6.5)

where $v$ refers to any of the three vertices of the equilateral triangle, and $T$ indicates the triangle itself. We compare the analytical method to compute the singular integrals in (6.2) with the simple 7-point Gaussian quadrature, where, obviously, the analytical expressions give accurate results. Figure 6.1 indicates the percentage error plots for Case I for different values of the length of each edge of the triangles, $L$, when the triangle edge is treated as the yardstick. Similarly, Figure 6.2 indicates the percentage error plots for Case II for different values of $L$ when the triangle edge
Similar cases for the wavelength as the yardstick are illustrated in Figures 6.3–6.6. It might be noted here that the cases like $L = 1.0 \text{ m}$ or $L = 1.0\lambda$ are used only for the purpose of comparison. In practice, these cases are, understandably, almost never used.

When we use the sides of the triangle as the yardstick, we observe that, irrespective of the cases, approximately above $0.4L$ for $d_1$ or $d_2$, we get the same result from both the analytical and numerical integration, i.e., the error is negligible. In fact, the error plots do not go through drastic change in the scale of error. As a check, we note that the error at $d_1/L = 0.4$ changes from $2.49\%$ and $0.78\%$ as the edge length changes from $L = 0.05 \text{ m}$ to $L = 1.0 \text{ m}$ for Case I, and at $d_2/L = 0.4$, they change from $4.42\%$ to $0.34\%$ for Case II for the triangle edge as yardstick. When we use wavelength as the yardstick, we note that the error at $d_1/\lambda = 0.3$ changes from $8.11 \times 10^{-7}\%$ to $2.60\%$ as $L$ changes from $L = 0.05\lambda$ and $L = 1.0\lambda$ for Case I, and at $d_2/\lambda = 0.3$, they change from $1.81\%$ and $1.02\%$ for Case II when the triangle edge is the yardstick. Thus the criterion of $d_{1(2)} = 0.4L - 0.6L$ is quite a good threshold for singularity extraction for various object and patch sizes. Using the triangle edge to set the singularity threshold, we can avoid the strange requirement of treating all

Figure 6.1: The percentage error plot against normalized distance from the point of observation.
the system matrix elements as singular for electrically very small objects and can, therefore, save considerable time in simulation. Another, perhaps, hitherto not-so-evident feature can be appreciated by looking at the error at the leftmost point on each plot. The error at $d_1/L = 0.05$ changes from 635.6% to 480.6% for a change of $L$ from $L = 0.05$ m to $L = 1.0$ m for Case I, and at $d_2/L = 0.05$, the error changes from from 272.9% to 128.3% for Case II for the triangle edge as reference. These numbers seem to be almost at comparable levels. But the true surprise lies in the case when we assume wavelength as the reference. The errors at $d_1/\lambda = 0.05$ for $L = 0.05\lambda$ and $L = 1.0\lambda$ are 0.036% and 480.7%, respectively for Case I, and at $d_2/\lambda = 0.05$, they are 2.20% and 128.3%, respectively, for Case II. Thus the error values change over several decades. Furthermore, we find that for the case when we have $L/\lambda$ very small, i.e. for electrically very small objects, then even at very small value of $d_{1(2)}$, the agreement between the analytical and numerical results is very good, which implies that the numerical integration provides very good approximation even for strongly singular cases. This observation corroborates the fact that for very small objects, $\lambda$ becomes uncorrelated to the definition of the singularity threshold.
Thus, the problem becomes static, since the change of phase over the object surface is negligible ($\lambda$ is very large compared to the scatterer dimension, $\omega \approx 0$, and thus $k = \omega \sqrt{\epsilon \mu} \approx 0$). The source-free vector wave equation for a homogeneous medium ($\nabla \times \nabla \times E - k^2 E = 0$) turns into the Laplace equation given by $\nabla^2 E \approx 0$ (because $\nabla \times \nabla \times E = \nabla (\nabla \cdot E) - \nabla^2 E = \nabla (\rho / \epsilon) - \nabla^2 E = -\nabla^2 E$; charge density, $\rho = 0$, for source-free case). However, the same problem when viewed from the angle of triangle edge as reference looks quite different. This observation, therefore, points out that it is the triangle side that should be the correct reference rather than the wavelength. It might also be noted that the change in error in Case II for the leftmost point on the error plot is relatively smaller compared to that in Case I. This is because of the perception of the distance of the observation. In Case I, we define it from the triangle centroid (naturally a more appropriate reference), while in Case II, we take the mid-point of the edge above which the point of observation lies as the reference.

To verify our finding, we consider an electrically small scattering object, which will benefit the most from using $L$ as the singularity reference instead of $\lambda$ as the singularity yardstick. We consider a perfect electrically conducting (PEC) sphere of radius $0.1\lambda$. We assume that a plane wave impinges upon the PEC sphere and that
Figure 6.4: The percentage error plot against normalized distance from the point of observation. The inset shows the case for $L = 0.2\lambda$ only.

Figure 6.5: The percentage error plot against normalized distance from the point of observation.
the incident azimuth is $\phi_{\text{inc}} = 270^\circ$, and the incident inclination is $\theta_{\text{inc}} = 0^\circ$. The vertical-vertical (VV) bistatic radar cross section (RCS) obtained by using our newly acquired singularity reference appears in Figure 6.7. For comparison, the Mie series result is also superimposed. We find a good agreement between the two results, which substantiates the validity of our singularity yardstick.
Figure 6.7: The vertical-vertical (VV) bistatic RCS of a PEC sphere of radius 0.1\(\lambda\).
CHAPTER 7
THE COMBINED FIELD INTEGRAL EQUATION (CFIE) FOR PEC OBJECTS

The EFIE and the MFIE, when applied to a closed surface, cannot produce unique solutions for all frequencies because homogeneous solutions exist that satisfy the boundary conditions with zero incident field, \( i.e., \) they have \textit{null space} \([5]\). These spurious solutions correspond to interior resonant modes of the object itself when no fields are radiated outside, \( i.e., \) for the lossless case. The basic reason behind this is that the tangential components of a single incident field are not sufficient to uniquely determine the surface currents at these resonant frequencies \([23]\). Since the \textit{null spaces} of the EFIE and the MFIE are different, the most widely used method is to use a linear combination of the EFIE and the MFIE \([24]\), called the combined field integral equation (CFIE). However, there are different definitions of CFIE. Gibson \([5]\) defines the CFIE as

\[
CFIE = \alpha EFIE - \frac{i}{k} (1 - \alpha) MFIE
\]  

(7.1)

where \( \alpha \) is \( 0.2 \leq \alpha \leq 0.5 \) while Chew \textit{et al.} \([25]\) defines it as

\[
CFIE = \alpha EFIE + (1 - \alpha) MFIE
\]  

(7.2)

\( \alpha \) is \( 0 \leq \alpha \leq 1 \) and as yet another variant, Chew, Tong, and Hu \([1]\) define it as

\[
CFIE = \alpha EFIE + \eta (1 - \alpha) MFIE
\]  

(7.3)

where \( \alpha \) is about 0.5. Harrington \([24]\) uses a slightly modified version of (7.3), where the \( (1 - \alpha) \) factor does not appear in the second term on the right-hand side. In
Chapter 12, we include the results for the first two of them. However, since the electric and magnetic fields have a ratio of intrinsic impedance, $\eta = \sqrt{\mu/\epsilon}$, (7.3) makes good sense, while (7.1) attempts at introducing a complex resonance frequency for the CFIE by including a factor $i = \sqrt{-1}$. In fact, the result from (7.2) is not as satisfactory.
CHAPTER 8

SCATTERING BY DIELECTRIC OBJECTS

So far, we have only discussed the scattering by PEC or impenetrable objects. In this chapter, we will consider scattering by dielectric or penetrable objects. As we will find, the scattering by impenetrable objects presents just a special case for the scattering by penetrable objects. To derive the necessary formulas, we reconsider the scenario in Figure 2.1. However, here $S$ refers to the boundary of the dielectric body. Let $S^+$ be a surface just large enough to contain the surface $S$, while $S^-$ be a surface just small enough to be contained by $S$. Here, $S$, $S^+$, and $S^-$ are so close to each other, that the surface can still be described only by $S$ for all practical purposes. If we place surface equivalence currents on $S^+$, the extinction principle applies inside the surface. Hence, we can remove the dielectric object without perturbing the fields outside [1]. These surface currents are given by (2.18). Applying the extinction theorem part of (2.19) to $S^-$ (i.e., the point of observation lies on $S^-$, while the source points are on $S^+$), we obtain

\[ E_{inc}(r) = -\mathcal{L}_1(r, r')J_S(r') - \mathcal{K}_1(r, r')M_S(r') \]  

(8.1)

This is the EFIE for penetrable or dielectric bodies [1]. Using the operator notations that we developed earlier, we rewrite (8.1) as

\[ E_{inc}(r) = -\mathcal{L}_1(r, r')J_S(r') - \mathcal{K}_1(r, r')M_S(r') \]  

(8.2)

In (8.2), the subscript 1 refers to region 1. Since there is no incident field inside the surface (i.e., in region 2), when the point of observation lies on $S^+$, while the source points are on $S^-$, we write
Applying the duality principle to (8.2) and (8.3), we obtain

$$0 = -\mathcal{L}_2(r, r')J_S(r') - \mathcal{K}_2(r, r')M_S(r')$$

(8.3)

and

$$\mathbf{H}_{\text{inc}}(r) = \mathcal{K}_1(r, r')J_S(r') - \frac{1}{\eta_1^2}\mathcal{L}_1(r, r')M_S(r')$$

(8.4)

We clearly note that the EFIE equation (5.10) and the MFIE equation (5.11) are clearly the special case of (8.2) and (8.4), since $M_S(r) = 0$ for a PEC scatterer.

Referring to (8.2)–(8.5), we note that we have only two unknowns, but four equations to solve. We will use the weighted sum of these equations to avoid the internal resonance problem. Thus, the redundancy helps keep the spurious solutions from internal resonance at bay. Using the weighted sum, we have

$$a_1 \mathbf{E}_{\text{inc}}(r) = -\left[a_1\mathcal{L}_1(r, r') + a_2\mathcal{L}_2(r, r')\right]J_S(r') - \left[a_1\mathcal{K}_1(r, r') + a_2\mathcal{K}_2(r, r')\right]M_S(r')$$

(8.6)

and

$$b_1 \mathbf{H}_{\text{inc}}(r) = \left[b_1\mathcal{K}_1(r, r') + b_2\mathcal{K}_2(r, r')\right]J_S(r') - \left[b_1\eta_1\mathcal{L}_1(r, r') + b_2\eta_2\mathcal{L}_2(r, r')\right]M_S(r')$$

(8.7)

Now to solve for the surface currents, we need to enforce only the tangential components of (8.6) and (8.7). If we use $a_1 = a_2 = b_1 = b_2 = 1$, the resulting formulation is called the PMCHWT (with the initials from Poggio, Miller, Chang, Harrington, Wu, and Tsai) formulation [7, 24–26]. This method is especially suitable for dielectrics
with high contrast [1]. When \( a_1 = b_1 = 1, a_2 = -\epsilon_2/\epsilon_1, \) and \( b_2 = \mu_2/\mu_1, \) the method is called the Müller formulation [27]. This method is particularly useful for dielectrics with low contrast [1]. It should, nonetheless, be mentioned that the PMCHWT formulation that we derived could well be arrived at by using the non-extinct part of (2.19), and the continuity of fields across the interface between dielectrics given by [7]

\[
\hat{n}_1 \times \mathbf{E}_{\text{Region} 1} = \hat{n}_2 \times \mathbf{E}_{\text{Region} 2}, \quad \hat{n}_1 \times \mathbf{H}_{\text{Region} 1} = \hat{n}_2 \times \mathbf{H}_{\text{Region} 2} \tag{8.8}
\]

enforced on \( S. \)

We can, as well, find a CFIE formulation from (8.2)–(8.5) using [28]

\[
CFIE = \alpha EFIE + (1 - \alpha) \hat{n} \times MFIE \tag{8.9}
\]

or

\[
CFIE = \alpha \hat{n} \times EFIE + (1 - \alpha) MFIE \tag{8.10}
\]

Experience [29] shows that even a formula (not quite correct though) as crude as

\[
CFIE = \alpha EFIE + (1 - \alpha) MFIE \tag{8.11}
\]

works reasonably well, even if not pin-point accurate. In (8.11), \( 0 \leq \alpha \leq 1.0. \) Here, (8.2) and (8.3) are the EFIE equations while (8.4) and (8.5) are the MFIE equations. Thus, we finally get two equations to solve for two unknowns (i.e., the surface currents). It should be stressed that while the \( K_1(r, r') \) is given by (5.9), \( K_2(r, r') \) is given by [28]

\[
K_2(r, r')X(r') = \frac{\hat{n}(r) \times X(r)}{2} + \text{P.V.} \int_S dS' X(r') \times \nabla G_{0_{\text{Region} 2}}(r, r') \tag{8.12}
\]

To understand the origin of this sign reversal, we consider Figure 8.1. For the computation of the residue integral for the singular case (\( r = r' \)), we may deform the
Figure 8.1: Computation of the residue integral: deformation is such that the point of observation, \( \mathbf{r} \), resides in (a) region 2, (b) region 1.

surface \( S \) in any of the two ways [4] shown in Figure 8.1. To compute \( K_1(\mathbf{r}, \mathbf{r}′) \), we use the detouring technique in Figure 8.1(a), while to compute \( K_2(\mathbf{r}, \mathbf{r}′) \), we use the detouring technique in Figure 8.1(b). It is worthwhile to note that in the PMCHWT formulation, the residue integral parts from \( K_1(\mathbf{r}, \mathbf{r}′) \) and \( K_2(\mathbf{r}, \mathbf{r}′) \) precisely cancel each other. This, however, is not the case for the Müller formulation or the CFIE case. We can, therefore, avoid computing the self-terms or the diagonal terms in the system matrices of \( K_1(\mathbf{r}, \mathbf{r}′) \) and \( K_2(\mathbf{r}, \mathbf{r}′) \) for the PMCHWT formulation. Numerical experiments have shown that while the PMCHWT formulation always yields accurate results, the CFIE formulation is not as accurate [4].
CHAPTER 9

THE EQUIVALENCE PRINCIPLE ALGORITHM (EPA)

9.1 The Equivalence Principle Algorithm (EPA)

The equivalence principle algorithm (EPA) allows for the simplification of a system of scatterers in which the scatterers are replaced by some fictitious surfaces called the equivalence surfaces (ES) with surface currents [30–32]. The problem is thus changed from solving a matrix problem of the currents over the surfaces of the objects to the problem of solving currents over the ES, which is done by relating the radiated fields from the induced currents on the objects via scattering matrices called the equivalence principle operators (EPO), and the interaction between ES via translation matrices called the translation operators (TO) [30–32]. The advantages are [33]:

1. The unknowns on the surfaces of the scatterers are moved to the ES. The ES will generally have coarser mesh than that on the scatterers, which will reduce the number of unknowns. This is specifically true because the near-field radiation from the object decays to insignificant value before it reaches the ES. Hence, the fields across the ES are smoother and the currents on the ES can be approximated with fewer functions, as the finer features of the objects no longer need to be modeled. This smoothness of the ES accounts for a substantial reduction in the computational time.

2. If we have multiple objects with identical geometry and size, we need to compute the EPO of only one of them, since the rest must be identical. Thus the feature of reusability speeds up the code, while considerably simplifying the problem.

3. There could be several objects wrapped by the ES, that originally had larger total surface area than the ES itself, thus further reducing the number of unknowns.

4. If we know that the same object or a group of same objects is part of various scattering scenarios, we need not compute the EPO every time. Rather, we invoke
the saved EPO as and where needed.

The EPA can basically be decomposed into three problems: the outside-in (OI), the current solver, and the inside-out (IO) problems [30–32]. To construct the EPA, we refer to (2.19) and, using the \( \mathcal{L} \) and \( \mathcal{K} \) operators defined in (3.13) and (5.9), respectively, we write the scattered electric and magnetic fields outside a surface \( S \) in terms of surface currents as

\[
E_{\text{sca}}(r) = \mathcal{L}(r, r') J_S(r') + \mathcal{K}(r, r') M_S(r') \quad (9.1)
\]

\[
H_{\text{sca}}(r) = -\mathcal{K}(r, r') J_S(r') + \frac{1}{\eta^2} \mathcal{L}(r, r') M_S(r') \quad (9.2)
\]

where \( J_S \) and \( M_S \) are the surface currents defined by (2.18) and \( \hat{n} \) is the outward unit normal to \( S \). Now the EPO works by taking the incident currents along an ES (OI) and solving the current on the surface of the object (the current solver) and projecting them to get the scattered current on the ES (IO). The incident currents on the ES due to incident electromagnetic field is given by [33]

\[
J_{\text{Inc}} = \hat{n} \times H_{\text{Inc}}, \quad M_{\text{Inc}} = -\hat{n} \times E_{\text{Inc}} = E_{\text{Inc}} \times \hat{n} \quad (9.3)
\]

Using the equivalence principle with zero fields outside \( S \), with the surface currents in (9.3), the resulting nonzero fields inside \( S \) are given by [30–33]

\[
E_{\text{Inc}}^{\text{inside}}(r) = -\mathcal{L}^{OI}(r, r') J_{\text{Inc}}(r') - \mathcal{K}^{OI}(r, r') M_{\text{Inc}}(r') \quad (9.4)
\]

\[
H_{\text{Inc}}^{\text{inside}}(r) = \mathcal{K}^{OI}(r, r') J_{\text{Inc}}(r') - \frac{1}{\eta^2} \mathcal{L}^{OI}(r, r') M_{\text{Inc}}(r') \quad (9.5)
\]

We notice that we could also get the above equations directly from (2.19) and its
electromagnetic dual by putting \( \mathbf{E}(\mathbf{r}) = 0 \) and \( \mathbf{H}(\mathbf{r}) = 0 \). This is the OI problem.

Once we know the incident fields inside the ES, we can solve for scattered currents on the surface of the scatterer by using any suitable solver like EFIE for the PEC object, the PMCHWT or the Müller formulation for high and low contrast dielectric bodies, respectively, \textit{etc}. For example, for the PMCHWT we can write the current solver operator as [7,33]

\[
Z = \begin{bmatrix}
-L_{\text{out}} - L_{\text{in}} & -K_{\text{out}} - K_{\text{in}} \\
K_{\text{out}} + K_{\text{in}} & -\frac{1}{\eta_{\text{out}}} L_{\text{out}} - \frac{1}{\eta_{\text{in}}} L_{\text{in}}
\end{bmatrix}
\] (9.6)

Here the subscripts \textit{out} and \textit{in} indicate outside and inside the dielectric body, respectively. For EFIE, of course, \( Z = -L_{\text{PEC}} \). Using the current solver operator, we can write the scattered surface currents on the object as

\[
\begin{bmatrix}
\mathbf{J}_{\text{obj sca}} \\
\mathbf{M}_{\text{obj sca}}
\end{bmatrix} = Z^{-1} \cdot \begin{bmatrix}
\mathbf{E}_{\text{inside}}^{\text{inc}} \\
\mathbf{H}_{\text{inside}}^{\text{inc}}
\end{bmatrix}
\] (9.7)

Now we again use the equivalence principle to get the scattered currents on the ES that gives nonzero scattered fields outside and zero field inside. The scattered fields outside are given by [30–33]

\[
\mathbf{E}_{\text{S sca}}(\mathbf{r}) = \mathcal{L}^{\text{IO}}(\mathbf{r}, \mathbf{r}') \mathbf{J}_{\text{obj sca}}(\mathbf{r}') + \mathcal{K}^{\text{IO}}(\mathbf{r}, \mathbf{r}') \mathbf{M}_{\text{obj sca}}(\mathbf{r}')
\] (9.8)

\[
\mathbf{H}_{\text{S sca}}(\mathbf{r}) = -\mathcal{K}^{\text{IO}}(\mathbf{r}, \mathbf{r}') \mathbf{J}_{\text{obj sca}}(\mathbf{r}') + \frac{1}{\eta^2} \mathcal{L}^{\text{IO}}(\mathbf{r}, \mathbf{r}') \mathbf{M}_{\text{obj sca}}(\mathbf{r}')
\] (9.9)

We notice that we could also get the above equations directly from (2.5), and its electromagnetic dual by putting \( \mathbf{E}_{\text{inc}}(\mathbf{r}) = 0 \) and \( \mathbf{H}_{\text{inc}}(\mathbf{r}) = 0 \). The equivalent scattered currents on the ES are given by [30–33]

\[
\mathbf{J}_{\text{S sca}} = \hat{n} \times \mathbf{H}_{\text{S sca}}, \quad \mathbf{M}_{\text{S sca}} = -\hat{n} \times \mathbf{E}_{\text{S sca}} = \mathbf{E}_{\text{S sca}} \times \hat{n}
\] (9.10)
This is the IO problem. Thus the scattered currents along the ES can be related to
the incident current along the ES through the following expression [30–33]

\[
\begin{bmatrix}
J_{\text{sca}}
M_{\text{sca}}
\end{bmatrix}
= \begin{bmatrix}
-\hat{n} \times K_{\text{IO}} & \frac{1}{\eta} \hat{n} \times L_{\text{IO}} \\
-\hat{n} \times L_{\text{IO}} & -\hat{n} \times K_{\text{IO}}
\end{bmatrix} \cdot Z^{-1} \cdot \begin{bmatrix}
-\mathcal{L}^{\text{OI}} & -K^{\text{OI}} \\
K^{\text{OI}} & -\frac{1}{\eta^2} \mathcal{L}^{\text{OI}}
\end{bmatrix} \cdot \begin{bmatrix}
J_{\text{inc}}
M_{\text{inc}}
\end{bmatrix}
\]

(9.11)

Here, \( S \) is the EPO. The IO and the OI operators are defined as [30–33]

\[
T_{\text{IO}} = \begin{bmatrix}
-\hat{n} \times K_{\text{IO}} & \frac{1}{\eta} \hat{n} \times L_{\text{IO}} \\
-\hat{n} \times L_{\text{IO}} & -\hat{n} \times K_{\text{IO}}
\end{bmatrix}
\]

(9.12)

\[
T_{\text{OI}} = \begin{bmatrix}
-\mathcal{L}^{\text{OI}} & -K^{\text{OI}} \\
K^{\text{OI}} & -\frac{1}{\eta^2} \mathcal{L}^{\text{OI}}
\end{bmatrix}
\]

(9.13)

For PEC scatterers, the EPO becomes [30]

\[
S^{\text{PEC}} = \begin{bmatrix}
-\hat{n} \times K^{\text{PEC}} \\
-\hat{n} \times L^{\text{PEC}}
\end{bmatrix} \cdot [-L^{\text{PEC}}]^{-1} \cdot \begin{bmatrix}
-\mathcal{L}^{\text{OI}} & -K^{\text{OI}}
\end{bmatrix}
\]

(9.14)

Now, let us derive the TO. If we have one surface with known surface currents, \( J_{S_1} \)
and \( M_{S_1} \), the fields on another surface are given by [30–33]

\[
E_{S_2}(r_2) = L^{\text{TO}}(r_2, r'_1) J_{S_1}(r'_1) + K^{\text{TO}}(r_2, r'_1) M_{S_1}(r'_1)
\]

(9.15)

\[
H_{S_2}(r_2) = -K^{\text{TO}}(r_2, r'_1) J_{S_1}(r'_1) + \frac{1}{\eta^2} L^{\text{TO}}(r_2, r'_1) M_{S_1}(r'_1)
\]

(9.16)

and the surface currents are given by
Figure 9.1: The three steps in the EPO: (a) the outside-in (OI) propagation, (b) the current solver (CS), and (c) the inside-out (IO) propagation (after [34]).

\[
J_{S_2} = \hat{n}_2 \times H_{S_2}, \quad M_{S_2} = -\hat{n}_2 \times E_{S_2} = E_{S_2} \times \hat{n}_2 \quad (9.17)
\]

Henceforth, we can write

\[
\begin{bmatrix}
J_{S_2} \\
M_{S_2}
\end{bmatrix} = \begin{bmatrix}
-\hat{n}_2 \times \mathcal{K}^{TO} & \frac{1}{\eta^2} \hat{n}_2 \times \mathcal{L}^{TO} \\
-\hat{n}_2 \times \mathcal{L}^{TO} & -\hat{n}_2 \times \mathcal{K}^{TO}
\end{bmatrix} \cdot \begin{bmatrix}
J_{S_1} \\
M_{S_1}
\end{bmatrix} = \mathcal{T}_{21} \cdot \begin{bmatrix}
J_{S_1} \\
M_{S_1}
\end{bmatrix} \quad (9.18)
\]

where \(\mathcal{T}_{21}\) is the TO. It might be mentioned that the translation need not necessarily be between two ESs. It might be between one ES wrapping a scatterer and another scatterer without any ES, or even between two scatterers without any ES. This is because the formula in (9.18) is completely general and no \textit{a priori} assumptions have been used. This, in fact, is a very nice feature that lets us use the ES as needed. Therefore, the procedures to form the EPA equations can be summed up as shown in Figure 9.1.

Now, let us extend the ideas developed so far to a case where two arbitrary homogeneous scatterers are wrapped inside two separate ES [33]. Let us identify the ES as ES\(_1\) and ES\(_2\). The total scattered currents on ES\(_1\) are due to the incident currents
on $ES_1$ and the scattering contributions from $ES_2$. Hence, we can write [33]

$$
\begin{bmatrix}
  J_{S_{1\text{sc}}a} \\
  M_{S_{1\text{sc}}a}
\end{bmatrix}
= S_{11} \cdot \begin{bmatrix}
  J_{S_{1\text{inc}}} \\
  M_{S_{1\text{inc}}}
\end{bmatrix}
+ S_{11} \cdot T_{12} \cdot \begin{bmatrix}
  J_{S_{2\text{sc}}a} \\
  M_{S_{2\text{sc}}a}
\end{bmatrix}
$$

(9.19)

Here, the first term on the right-hand side is just from (9.11). The second term is the contribution from the scattered surface currents from $ES_2$. First, the scattered currents on $ES_2$ are translated via the TO, $T_{12}$, to get the incident currents on $ES_1$. This current then passes through the three steps, namely, the OI problem, the current solver, and the IO problem, much the same way as the first term on the right-hand side, and these steps are reflected in $S_{11}$. Similarly for $ES_2$, we can write

$$
\begin{bmatrix}
  J_{S_{2\text{sc}}a} \\
  M_{S_{2\text{sc}}a}
\end{bmatrix}
= S_{22} \cdot \begin{bmatrix}
  J_{S_{2\text{inc}}} \\
  M_{S_{2\text{inc}}}
\end{bmatrix}
+ S_{22} \cdot T_{21} \cdot \begin{bmatrix}
  J_{S_{1\text{sc}}a} \\
  M_{S_{1\text{sc}}a}
\end{bmatrix}
$$

(9.20)

We, therefore, can write a matrix system, which upon rearrangement gives the current coefficients, as [33]

$$
\begin{bmatrix}
  I_{11} & -S_{11} \cdot T_{12} \\
  -S_{22} \cdot T_{21} & I_{22}
\end{bmatrix}
\cdot
\begin{bmatrix}
  J_{S_{1\text{sc}}a} \\
  M_{S_{1\text{sc}}a}
\end{bmatrix}
= S_{11} \cdot \begin{bmatrix}
  J_{S_{1\text{inc}}} \\
  M_{S_{1\text{inc}}}
\end{bmatrix}
$$

(9.21)

Equation (9.21) can be extended to any number of scatterers with or without ES. For example, if for the second scatterer we did not have an ES, we would modify (9.21) to

$$
\begin{bmatrix}
  I_{11} & -S_{11} \cdot T_{12} \\
  -L_{22} \cdot T_{21} & I_{22}
\end{bmatrix}
\cdot
\begin{bmatrix}
  J_{S_{1\text{sc}}a} \\
  M_{S_{1\text{sc}}a} \\
  J_{\text{obj2sc}} \\
  M_{\text{obj2sc}}
\end{bmatrix}
= S_{11} \cdot \begin{bmatrix}
  J_{S_{1\text{inc}}} \\
  M_{S_{1\text{inc}}} \\
  J_{\text{obj2inc}} \\
  M_{\text{obj2inc}}
\end{bmatrix}
$$

(9.22)

Now we will derive the matrix representation of the EPA. To do that, we define
the inner product of two vector functions \( \mathbf{A} \) and \( \mathbf{B} \) as

\[
\langle \mathbf{A}, \mathbf{B} \rangle = \int \int_S dS \mathbf{A} \cdot \mathbf{B}
\]

(9.23)

This definition will be implicitly assumed in our analysis to follow. We can expand different quantities in (9.11) in terms of the basis function as follows [30,33]:

\[
J_{S,inc}(\mathbf{r}) = \sum_{n=1}^{N_S} j_{S,inc} f_{S,n}(\mathbf{r})
\]

(9.24)

\[
M_{S,inc}(\mathbf{r}) = \sum_{n=1}^{N_S} m_{S,inc} f_{S,n}(\mathbf{r})
\]

(9.25)

\[
J_{S,sca}(\mathbf{r}) = \sum_{n=1}^{N_S} j_{S,sca} f_{S,n}(\mathbf{r})
\]

(9.26)

\[
M_{S,sca}(\mathbf{r}) = \sum_{n=1}^{N_S} m_{S,sca} f_{S,n}(\mathbf{r})
\]

(9.27)

\[
J_{obj,sca}(\mathbf{r}) = \sum_{n=1}^{N_{obj}} j_{obj,sca} f_{obj,n}(\mathbf{r})
\]

(9.28)

\[
M_{obj,sca}(\mathbf{r}) = \sum_{n=1}^{N_{obj}} m_{S,sca} f_{obj,n}(\mathbf{r})
\]

(9.29)

Using these definitions, (9.4) and (9.5) can be written as [33]

65
\[ E_{inc}^{inside}(r) = - \sum_{n=1}^{N_S} j_{S,n_{inc}} L^{OI}(r, r') f_{S,n}(r') - \sum_{n=1}^{N_S} m_{S,n_{inc}} K^{OI}(r, r') f_{S,n}(r') \] (9.30)

\[ H_{inc}^{inside}(r) = \sum_{n=1}^{N_S} j_{S,n_{inc}} K^{OI}(r, r') f_{S,n}(r') - \frac{1}{\eta^2} \sum_{n=1}^{N_S} m_{S,n_{inc}} L^{OI}(r, r') f_{S,n}(r') \] (9.31)

Thus, (9.30) and (9.31) can be converted into the following matrix equation

\[
\begin{bmatrix}
E_{inc}^{inside} \\
H_{inc}^{inside}
\end{bmatrix} = \begin{bmatrix}
-L^{OI} F^t_S & -K^{OI} F^t_S \\
K^{OI} F^t_S & -\frac{1}{\eta^2} L^{OI} F^t_S
\end{bmatrix} \cdot \begin{bmatrix}
j_{S,inc} \\
m_{S,inc}
\end{bmatrix} \] (9.32)

where

\[ F_S = \begin{bmatrix}
f_{S,1} \\
f_{S,2} \\
\vdots \\
f_{S,N_S}
\end{bmatrix} \] (9.33)

\[ j_{S,inc} = \begin{bmatrix}
j_{S,1_{inc}} \\
j_{S,2_{inc}} \\
\vdots \\
j_{S,N_{inc}}
\end{bmatrix} \] (9.34)

\[ m_{S,inc} = \begin{bmatrix}
m_{S,1_{inc}} \\
m_{S,2_{inc}} \\
\vdots \\
m_{S,N_{inc}}
\end{bmatrix} \] (9.35)

and \( t \) indicates transpose operation. Equation (9.7) can be rewritten as
\[ Z \cdot \begin{bmatrix} J_{\text{obj, sca}} \\ M_{\text{obj, sca}} \end{bmatrix} = \begin{bmatrix} E_{\text{inc}}^{\text{inside}} \\ H_{\text{inc}}^{\text{inside}} \end{bmatrix} \] (9.36)

Testing the above with basis functions over the surface of the object, we get

\[
\begin{bmatrix} \dot{j}_{\text{obj, sca}} \\ \dot{m}_{\text{obj, sca}} \end{bmatrix} = Z^{-1} \cdot \begin{bmatrix} b_{\text{inc}}^{E, \text{inside}} \\ b_{\text{inc}}^{H, \text{inside}} \end{bmatrix} \] (9.37)

where

\[
\dot{j}_{\text{obj, sca}} = \begin{bmatrix} \dot{j}_{\text{obj, 1, sca}} \\ \dot{j}_{\text{obj, 2, sca}} \\ \vdots \\ \dot{j}_{\text{obj, N_{obj, sca}}} \end{bmatrix} \] (9.38)

\[
\dot{m}_{\text{obj, sca}} = \begin{bmatrix} m_{\text{obj, 1, sca}} \\ m_{\text{obj, 2, sca}} \\ \vdots \\ m_{\text{obj, N_{obj, sca}}} \end{bmatrix} \] (9.39)

\[
[Z]_{m,n} = \left\langle f_{\text{obj, m}}(r), Z(r, r'), f_{\text{obj, n}}^\dagger(r') \right\rangle \] (9.40)

\[
[b_{\text{inc}}^{E, \text{inside}}]_m = \left\langle f_{\text{obj, m}}(r), E_{\text{inc}}^{\text{inside}}(r') \right\rangle \] (9.41)

\[
[b_{\text{inc}}^{H, \text{inside}}]_m = \left\langle f_{\text{obj, m}}(r), H_{\text{inc}}^{\text{inside}}(r') \right\rangle \] (9.42)

Now testing (9.30) with the basis functions over the object surface, the \(m\)-th element
is given by [33]

\[
\begin{bmatrix}
  b_{E,\text{inside}}^E \\
  b_{H,\text{inside}}^H
\end{bmatrix}_{inc}
= \langle f_{obj,m}(r), E_{inc}^{\text{inside}}(r') \rangle
\]
\[
= -\sum_{n=1}^{N_S} \langle f_{obj,m}(r), \mathcal{L}^{\text{OI}}(r,r'), f_{S,n}(r') \rangle \hat{j}_{S,n_{\text{inc}}}
\]
\[
- \sum_{n=1}^{N_S} \langle f_{obj,m}(r), \mathcal{K}^{\text{OI}}(r,r'), f_{S,n}(r') \rangle m_{S,n_{\text{inc}}}
\]
\[
= -\sum_{n=1}^{N_S} \left[ \mathcal{L}^{\text{OI}} \right]_{m,n} \hat{j}_{S,n_{\text{inc}}} - \sum_{n=1}^{N_S} \left[ \mathcal{K}^{\text{OI}} \right]_{m,n} m_{S,n_{\text{inc}}}
\] (9.43)

Henceforward, we can write

\[
\begin{bmatrix}
  b_{E,\text{inside}}^E \\
  b_{H,\text{inside}}^H
\end{bmatrix}_{inc}
= - \left[ \mathcal{L}^{\text{OI}} \right] \cdot \hat{j}_{S,n_{\text{inc}}} - \left[ \mathcal{K}^{\text{OI}} \right] \cdot m_{S,n_{\text{inc}}}
\] (9.44)

Likewise, from (9.31) using the same procedure as above, we have

\[
\begin{bmatrix}
  b_{E,\text{inside}}^H \\
  b_{H,\text{inside}}^H
\end{bmatrix}_{inc}
= \left[ \mathcal{K}^{\text{OI}} \right] \cdot \hat{j}_{S,n_{\text{inc}}} - \frac{1}{\eta^2} \left[ \mathcal{L}^{\text{OI}} \right] \cdot m_{S,n_{\text{inc}}}
\] (9.45)

The resulting matrix equation, therefore, becomes [30,33]

\[
\begin{bmatrix}
  b_{E,\text{inside}}^E \\
  b_{H,\text{inside}}^H
\end{bmatrix}_{inc}
= \begin{bmatrix}
  -\mathcal{L}^{\text{OI}} & -\mathcal{K}^{\text{OI}} \\
  \mathcal{K}^{\text{OI}} & -\frac{1}{\eta^2} \mathcal{L}^{\text{OI}}
\end{bmatrix} \cdot \begin{bmatrix}
  \hat{j}_{S,n_{\text{inc}}} \\
  m_{S,n_{\text{inc}}}
\end{bmatrix}
\] (9.46)

Testing (9.37) with basis functions on the object and using (9.46), we have

\[
\begin{bmatrix}
  J_{\text{obj, sca}} \\
  M_{\text{obj, sca}}
\end{bmatrix}
= \begin{bmatrix}
  F_{\text{obj}}^i & 0_{\text{obj}}^i \\
  0_{\text{obj}}^i & F_{\text{obj}}^i
\end{bmatrix} \cdot \mathcal{Z}^{-1} \cdot \begin{bmatrix}
  -\mathcal{L}^{\text{OI}} & -\mathcal{K}^{\text{OI}} \\
  \mathcal{K}^{\text{OI}} & -\frac{1}{\eta^2} \mathcal{L}^{\text{OI}}
\end{bmatrix} \cdot \begin{bmatrix}
  \hat{j}_{S,n_{\text{inc}}} \\
  m_{S,n_{\text{inc}}}
\end{bmatrix}
\] (9.47)

where
\[
F_{obj} = \begin{bmatrix}
    f_{obj,1} \\
    f_{obj,2} \\
    \vdots \\
    f_{obj,N_{obj}}
\end{bmatrix}
\]  
(9.48)

Here, \(0_{obj}\) denotes a zero vector of the same dimension as \(F_{obj}\).

Now, using (9.8)–(9.10) and (9.47), we have

\[
\begin{bmatrix}
    J_{sca} \\
    M_{sca}
\end{bmatrix} = \begin{bmatrix}
    F^t_S & 0^t_S \\
    0^t_S & F^t_S
\end{bmatrix} \cdot \begin{bmatrix}
    j_{sca} \\
    m_{sca}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    -\hat{n} \times K^{IO} & \frac{1}{\eta^2} \hat{n} \times L^{IO} \\
    \hat{n} \times L^{IO} & -\hat{n} \times K^{IO}
\end{bmatrix} \cdot \begin{bmatrix}
    J_{objca} \\
    M_{objca}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    -\hat{n} \times K^{IO} & \frac{1}{\eta^2} \hat{n} \times L^{IO} \\
    \hat{n} \times L^{IO} & -\hat{n} \times K^{IO}
\end{bmatrix} \cdot \begin{bmatrix}
    F^t_{obj} & 0^t_{obj} \\
    0^t_{obj} & F^t_{obj}
\end{bmatrix} \cdot \bar{Z}^{-1} \cdot \begin{bmatrix}
    -L^{OI} & -K^{OI} \\
    K^{OI} & -\frac{1}{\eta^2} L^{OI}
\end{bmatrix} \cdot \begin{bmatrix}
    j_{inc} \\
    m_{inc}
\end{bmatrix}
\]  
(9.49)

where

\[
j_{sca} = \begin{bmatrix}
    \hat{j}_{sca,1} \\
    \hat{j}_{sca,2} \\
    \vdots \\
    \hat{j}_{sca,N_{sca}}
\end{bmatrix}
\]  
(9.50)

\[
m_{sca} = \begin{bmatrix}
    m_{sca,1} \\
    m_{sca,2} \\
    \vdots \\
    m_{sca,N_{sca}}
\end{bmatrix}
\]  
(9.51)

Employing the last line of (9.49), we have
From (9.52), we directly find

\[
\begin{bmatrix}
\mathbf{j}_{sca}
\end{bmatrix} = U^{-1} \cdot \begin{bmatrix}
\mathbf{j}_{inc}
\end{bmatrix} = \mathbf{S} \cdot \begin{bmatrix}
\mathbf{j}_{inc}
\end{bmatrix} \tag{9.53}
\]

where

\[
U = \begin{bmatrix}
\bar{u} & 0 \\
0 & \bar{u}
\end{bmatrix} \tag{9.54}
\]

\[
[u]_{m,n} = \langle \mathbf{f}_{S,m}(\mathbf{r}), \mathbf{f}^t_{S,n}(\mathbf{r}) \rangle \tag{9.55}
\]
\[
\begin{align*}
[L^{IO,\hat{n} \times}]_{m,n} &= \langle f_{S,m}(r), \hat{n} \times L^{IO}(r,r'), f_{obj,n}^t(r') \rangle \tag{9.56}
\end{align*}
\]

\[
\begin{align*}
[K^{IO,\hat{n} \times}]_{m,n} &= \langle f_{S,m}(r), \hat{n} \times K^{IO}(r,r'), f_{obj,n}^t(r') \rangle \tag{9.57}
\end{align*}
\]

where \( \mathbf{U} \), which is a sparse matrix, is also called a Grammian [1, 35]. If we use the RWG basis function for a closed geometry, there will be only five nonzero entries in each row and column, and \( \mathbf{U} \) is a block diagonal sparse matrix with \( \mathbf{u} \) blocks [33]. We further write the EPO operator in a compact format as [33]

\[
\mathbf{S} = \mathbf{U}^{-1} \cdot \mathbf{T}^{IO} \cdot \mathbf{Z}^{-1} \cdot \mathbf{T}^{OI} \tag{9.58}
\]

where

\[
\mathbf{T}^{IO} = \begin{bmatrix}
-K^{IO,\hat{n} \times} & \frac{1}{\eta^2} L^{IO,\hat{n} \times} \\
-L^{IO,\hat{n} \times} & -K^{IO,\hat{n} \times}
\end{bmatrix} \tag{9.59}
\]

and

\[
\mathbf{T}^{OI} = \begin{bmatrix}
-L^{OI} & -K^{OI} \\
-K^{OI} & -\frac{1}{\eta^2} L^{OI}
\end{bmatrix} \tag{9.60}
\]

Now that we have derived the matrix formulation for the EPO, we will derive the matrix formulation for the TO. From (9.18), we can write

\[
\begin{bmatrix}
F^t_{S_2} & 0^t_{S_2} & j_{S_2 sca} \\
0^t_{S_2} & F^t_{S_2} & m_{S_2 sca}
\end{bmatrix} \cdot \begin{bmatrix}
\hat{n}_2 \times \mathbf{K}^{TO} \\
\hat{n}_2 \times \mathbf{L}^{TO}
\end{bmatrix} \cdot \begin{bmatrix}
F^t_{S_1} & 0^t_{S_1} & j_{S_1 sca} \\
0^t_{S_1} & F^t_{S_1} & m_{S_1 sca}
\end{bmatrix} = \begin{bmatrix}
\hat{n}_2 \times \mathbf{K}^{TO} \\
\hat{n}_2 \times \mathbf{L}^{TO}
\end{bmatrix} \cdot \begin{bmatrix}
F^t_{S_1} & 0^t_{S_1} & j_{S_1 sca} \\
0^t_{S_1} & F^t_{S_1} & m_{S_1 sca}
\end{bmatrix} \tag{9.61}
\]

From (9.61), we obtain [33]
\[
\begin{bmatrix}
F_{S_2} & 0_{S_2} \\
0_{S_2} & F_{S_2}
\end{bmatrix}
\begin{bmatrix}
F_{S_2}' & 0_{S_2}' \\
0_{S_2}' & F_{S_2}'
\end{bmatrix}
\begin{bmatrix}
j_{S_{2\text{aca}}} \\
m_{S_{2\text{aca}}}
\end{bmatrix}
= 
\mathbf{U}_2 \cdot 
\begin{bmatrix}
j_{S_{2\text{aca}}} \\
m_{S_{2\text{aca}}}
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
F_{S_2} & 0_{S_2} \\
0_{S_2} & F_{S_2}
\end{bmatrix}
\begin{bmatrix}
\hat{n}_2 \times \mathbf{K}^{TO} & \frac{1}{\eta} \hat{n}_2 \times \mathbf{L}^{TO} \\
\hat{n}_2 \times \mathbf{L}^{TO} & -\hat{n}_2 \times \mathbf{K}^{TO}
\end{bmatrix}
\begin{bmatrix}
j_{S_{1\text{aca}}} \\
m_{S_{1\text{aca}}}
\end{bmatrix}
\]

\[
\begin{bmatrix}
F_{S_1}' & 0_{S_1}' \\
0_{S_1}' & F_{S_1}'
\end{bmatrix}
\begin{bmatrix}
j_{S_{1\text{aca}}} \\
m_{S_{1\text{aca}}}
\end{bmatrix}
= 
\begin{bmatrix}
-\mathbf{K}^{TO,\hat{n} \times} & \frac{1}{\eta} \mathbf{L}^{TO,\hat{n} \times} \\
-\mathbf{L}^{TO,\hat{n} \times} & -\mathbf{K}^{TO,\hat{n} \times}
\end{bmatrix}
\begin{bmatrix}
j_{S_{2\text{aca}}} \\
m_{S_{2\text{aca}}}
\end{bmatrix}
\]

(9.62)

Therefore, from (9.62), we have

\[
\begin{bmatrix}
j_{S_{2\text{aca}}} \\
m_{S_{2\text{aca}}}
\end{bmatrix}
= 
\mathbf{U}_2^{-1} \cdot
\begin{bmatrix}
-\mathbf{K}^{TO,\hat{n} \times} & \frac{1}{\eta} \mathbf{L}^{TO,\hat{n} \times} \\
-\mathbf{L}^{TO,\hat{n} \times} & -\mathbf{K}^{TO,\hat{n} \times}
\end{bmatrix}
\begin{bmatrix}
j_{S_{1\text{aca}}} \\
m_{S_{1\text{aca}}}
\end{bmatrix}
= 
\mathbf{T}_{21} \cdot 
\begin{bmatrix}
j_{S_{2\text{aca}}} \\
m_{S_{2\text{aca}}}
\end{bmatrix}
\]

(9.63)

where

\[
\left[\mathbf{L}^{TO,\hat{n} \times}\right]_{m,n} = \langle \mathbf{f}_{S_{2,m}} (\mathbf{r}), \hat{n}_2 \times \mathbf{L}^{TO} (\mathbf{r}, \mathbf{r}'), \mathbf{f}_{S_{1,n}} (\mathbf{r}') \rangle
\]

(9.64)

and

\[
\left[\mathbf{K}^{TO,\hat{n} \times}\right]_{m,n} = \langle \mathbf{f}_{S_{2,m}} (\mathbf{r}), \hat{n}_2 \times \mathbf{K}^{TO} (\mathbf{r}, \mathbf{r}'), \mathbf{f}_{S_{1,n}} (\mathbf{r}') \rangle
\]

(9.65)

Thus we can define the TO as

\[
\mathbf{T}_{21} = \mathbf{U}_2^{-1} \cdot 
\begin{bmatrix}
-\mathbf{K}^{TO,\hat{n} \times} & \frac{1}{\eta} \mathbf{L}^{TO,\hat{n} \times} \\
-\mathbf{L}^{TO,\hat{n} \times} & -\mathbf{K}^{TO,\hat{n} \times}
\end{bmatrix}
\]

(9.66)
where $\mathbf{U}_2$ is a Grammian which can be obtained in a fashion similar to $\mathbf{U}$.

Now let us consider the matrix equation using the above relations for the translation of scattered current from one surface to another. Equation (9.21) can be modified to

$$
\begin{bmatrix}
I_{11} & -\mathbf{S}_{11} \cdot \mathbf{T}_{12} \\
-\mathbf{S}_{22} \cdot \mathbf{T}_{21} & I_{22}
\end{bmatrix}
\begin{bmatrix}
j_{s_{1\text{aca}}} \\
m_{s_{1\text{aca}}}
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{S}_{11} \cdot j_{s_{1\text{inc}}} \\
m_{s_{1\text{inc}}}
\end{bmatrix}
\begin{bmatrix}
j_{s_{2\text{aca}}} \\
m_{s_{2\text{aca}}}
\end{bmatrix}
\begin{bmatrix}
\mathbf{S}_{22} \cdot j_{s_{2\text{inc}}} \\
m_{s_{2\text{inc}}}
\end{bmatrix}
$$

(9.67)

To simplify this relation, we can write

$$
\begin{bmatrix}
\mathbf{S}_{11}^{-1} & -\mathbf{T}_{12} \\
-\mathbf{T}_{21} & \mathbf{S}_{22}^{-1}
\end{bmatrix}
\begin{bmatrix}
j_{s_{1\text{aca}}} \\
m_{s_{1\text{aca}}}
\end{bmatrix}
= 
\begin{bmatrix}
j_{s_{1\text{inc}}} \\
m_{s_{1\text{inc}}}
\end{bmatrix}
\begin{bmatrix}
j_{s_{2\text{aca}}} \\
m_{s_{2\text{aca}}}
\end{bmatrix}
\begin{bmatrix}
j_{s_{2\text{inc}}} \\
m_{s_{2\text{inc}}}
\end{bmatrix}
$$

(9.68)

The final step is to find the incident current coefficients on the right-hand side of (9.53) in terms of the incident fields, which are known. To determine them, we expand (9.3) in terms of the basis functions using (9.24) and (9.25) as

$$
\mathbf{J}_{s_{\text{inc}}} (\mathbf{r}) = \sum_{n=1}^{N_S} j_{s_{n\text{inc}}} \mathbf{f}_{s,n} (\mathbf{r}) = \mathbf{\hat{n}} \times \mathbf{H}_{s_{\text{inc}}} (\mathbf{r})
$$

(9.69)

$$
\mathbf{M}_{s_{\text{inc}}} (\mathbf{r}) = \sum_{n=1}^{N_S} m_{s_{n\text{inc}}} \mathbf{f}_{s,n} (\mathbf{r}) = -\mathbf{\hat{n}} \times \mathbf{E}_{s_{\text{inc}}} (\mathbf{r})
$$

(9.70)

Now Galerkin testing (9.69) and (9.70), we have

$$
\langle \mathbf{f}_{s,m} (\mathbf{r}) , \mathbf{J}_{s_{\text{inc}}} (\mathbf{r}) \rangle = \sum_{n=1}^{N_S} j_{s_{n\text{inc}}} \langle \mathbf{f}_{s,m} (\mathbf{r}) , \mathbf{f}_{s,n}^t (\mathbf{r}) \rangle = \langle \mathbf{f}_{s,m} (\mathbf{r}) , \mathbf{\hat{n}} \times \mathbf{H}_{s_{\text{inc}}} (\mathbf{r}) \rangle
$$

(9.71)
\[
\langle f_{S,m}(r), M_{S,\text{inc}}(r) \rangle = \sum_{n=1}^{N_S} m_{S,n,\text{inc}} \langle f_{S,m}(r), f_{S,n}^*(r) \rangle = -\langle f_{S,m}(r), \hat{n} \times E_{S,\text{inc}}(r) \rangle
\]

(9.72)

Using these two relations, we can directly write

\[
\begin{bmatrix}
  j_{S,\text{inc}} \\
  m_{S,\text{inc}}
\end{bmatrix} = \mathbf{U}^{-1} \begin{bmatrix}
  b_{\text{inc}}^{\hat{n} \times H} \\
  b_{\text{inc}}^{\hat{n} \times E}
\end{bmatrix}
\]

(9.73)

where

\[
\left[ b_{\text{inc}}^{\hat{n} \times H} \right]_m = \langle f_{S,m}(r), \hat{n} \times H_{S,\text{inc}}(r) \rangle
\]

(9.74)

and

\[
\left[ b_{\text{inc}}^{\hat{n} \times E} \right]_m = -\langle f_{S,m}(r), \hat{n} \times E_{S,\text{inc}}(r) \rangle
\]

(9.75)

It should be pointed that while the general MoM, when applied to the EFIE, the MFIE, the CFIE, and the PMCHWT or the Müller formulation, gives a dense system matrix, the system of matrix equation given by (9.67) or (9.68) returns a sparse matrix. This result is because of the presence of the identity matrices. Thus, these equations can be conveniently solved by any suitable iterative sparse matrix solver.

Before we end this chapter, let us consider the computation of different elements in the EPO and the TO. The \( K \) and \( \hat{n} \times K \) terms can be directly computed from the MFIE section discussed earlier. It should, nevertheless, be mentioned that the \( K \) and \( \hat{n} \times K \) only have near-singular terms, but no singular terms. In fact, the diagonal or self-terms can be calculated using the first term on the right-hand side of (5.9) and its corresponding version for \( K \) and \( \hat{n} \times K \), respectively. To further appreciate this, let us refer to Figure 9.2. Also we refer to the second term in the last line of the right-hand side of (5.7). Clearly,
Figure 9.2: Computation of the diagonal or self-terms in $\mathcal{K}$ and $\hat{\mathbf{n}} \times \mathcal{K}$.

\[
(r - r') \times \rho_n(r') = |(r - r') \times \rho_n(r')| \hat{\mathbf{n}}
\]  

(9.76)

Evidently, for the diagonal or self-terms,

\[
\rho_m(r) \cdot (r - r') \times \rho_n(r') = |(r - r') \times \rho_n(r')| \rho_m(r) \cdot \hat{\mathbf{n}} = 0
\]  

(9.77)

The operator $\hat{\mathbf{n}} \times \mathcal{K}$ acting upon the basis function and when Galerkin tested, can be written in the form [4]

\[
\langle f_m(r), \hat{\mathbf{n}} \times \mathcal{K}(r, r'), f_n(r') \rangle = -\langle \hat{\mathbf{n}}(r) \times f_m(r), \mathcal{K}(r, r'), f_n(r') \rangle
\]  

(9.78)

Thus, for $\hat{\mathbf{n}} \times \mathcal{K}$ we have

\[
[\hat{\mathbf{n}} \times \rho_m(r)] \cdot [(r - r') \times \rho_n(r')] = |(r - r') \times \rho_n(r')| \hat{\mathbf{n}} \times \rho_m(r) \cdot \hat{\mathbf{n}} = 0
\]  

(9.79)

$\mathcal{K}$ and $\hat{\mathbf{n}} \times \mathcal{K}$, therefore, have no singular terms, only off-diagonal near singular terms. It should also be recalled that this term is obtained using the residue theorem [4]. The application of the residue theorem enables us to get quite accurate results for $\mathcal{K}$ and $\hat{\mathbf{n}} \times \mathcal{K}$ without doing any singularity treatment. Of course, more precise results
can be obtained by the singularity treatment described in the MFIE in Chapter 5. The operator \( \hat{n} \times \mathcal{L} \), however, demands more care. This operator, acting upon the basis function and when Galerkin tested, can be written in the form [7]

\[
\langle f_m (r), \hat{n} (r) \times \mathcal{L} (r, r'), f_n (r') \rangle = - \langle \hat{n} (r) \times f_m (r), \mathcal{L} (r, r'), f_n (r') \rangle
\]

In writing (9.78) and (9.80), we invoked (2.10). In the vein of (3.11), we can write (9.80) as

\[
I_{\hat{n} \times \mathcal{L}} = - \langle \hat{n} (r) \times f_m (r), \mathcal{L} (r, r'), f_n (r') \rangle \\
= i\omega \mu \int \int_{T_m} dS \int \int_{T_n} dS' (\hat{n} \times f_m (r)) \cdot f_n (r') + \\
\frac{i\omega \mu}{k^2} \int \int_{T_m} dS \int \int_{T_n} dS' (\hat{n} \times f_m (r)) \cdot \left[ \nabla G_0 (r, r') \right] \left[ \nabla' \cdot f_n (r') \right]
\]

In the above, we have avoided the derivative of \( \hat{n} \times f_m (r) \) by applying the integration by parts. The first term in the second line on the right-hand side of (9.81) can be treated in a way very similar to the one shown for the EFIE in Chapter 3 for both the singular and the regular surface integrals. For the second term, it can be shown that [7]

\[
\int \int_{T_m} dS \hat{n} \times \rho_m^\pm (r) \nabla G_0 (r, r') = \int \int_{\partial T_m^\pm} dt \hat{t} \cdot \rho_m^\pm (r) G_0 (r, r')
\]

Now the singular integral can be evaluated in the same way as in the EFIE section. However, we need to use the Gaussian quadrature routine for the line integral, which is more complicated than the Gaussian quadrature routine for the surface integral and requires a higher order for accuracy. We can develop an alternative formulation that allows for the reuse of the Gaussian quadrature routine for the surface integral, and hence simplifies the problem. In (9.81), plugging in the expression from (5.5), we can write
\[ I_{\hat{n} \times \mathbb{L}} = -\langle \mathbf{n}(r) \times f_m(r), \mathbb{L}(r, r') \cdot f_n(r') \rangle \]
\[
= i\omega\mu \int_S \int_{T_{n}^{+}} dS' \int_{T_{n}^{-}} dS \hat{n} \times f_m(r) \cdot f_n(r') - \\
\frac{i\omega\mu}{k^2} \int_S \int_{T_{n}^{+}} dS \hat{n} \times f_m(r) \cdot \int_{T_{n}^{-}} dS' \left[ \nabla' \cdot f_n(r') \right] R \frac{1 - ikR}{4\pi R^3} e^{ikR} \]  
(9.83)

The first term on the right-hand side can easily be dealt with using the techniques developed in the EFIE, while the second term can be handled by making use of (5.16) and (5.19) for singularity treatment. However, understandably, this time, the \( \rho^\pm_n \) term will not vanish since there is no cross product. Once these substitutions are made, the rest is similar to the MFIE case. With that end in view, we write the inner integral of (9.83)

\[ I_{\hat{n} \times \mathbb{L}_{inner}} = I_{IP_{Non singular}} + I_{IP_{Singular}} \]
\[
= \left[ \int_{T_{n}^{+}} dS' \left( R_n \pm \rho_n(r') \right) \frac{(1 - ikR) e^{ikR} - \left( 1 + \frac{1}{2}k^2R^2 - 1 \right)}{R^3} \right] + \\
\left[ R_n \left( \int_{T_{n}^{+}} \frac{dS'}{R^3} + \frac{k^2}{2} \int_{T_{n}^{+}} \frac{dS'}{R} \right) \pm \left( a_n(r) + \frac{k^2}{2} b_n(r) \right) \right] \]  
(9.84)

In going from (9.83) to (9.84), we used (5.15), (5.16), (5.19), (5.21), and (5.22), as just hinted earlier. Here, the + sign indicates the positive source triangle, while the – sign indicates the negative source triangle. Using (9.84) into (9.83), gives the required form.

9.2 Taylor Series Expansion

Here, we briefly mention the Taylor series expansion of the nonsingular parts that arose en route to the singularity extraction for the EFIE, the MFIE, the CFIE, the PMCHWT, and the EPA.

\[ \lim_{R \to 0} \frac{e^{ikR} - 1}{R} \approx \lim_{R \to 0} \left[ ik - \frac{k^2}{2} R - i\frac{k^3}{6} R^2 + \frac{k^4}{24} R^3 \right] \]  
(9.85)
\[
\lim_{R \to 0} \frac{(1 - ikR)e^{ikR} - (1 + \frac{k^2}{2}R^2)}{R^3} \approx \lim_{R \to 0} \left[ \frac{k^3}{3} - \frac{k^4}{8}R - i\frac{k^5}{30}R^2 + \frac{k^6}{144}R^3 \right] + \\
\lim_{R \to 0} \left[ i\frac{k^7}{840}R^4 - \frac{k^8}{5760}R^5 - i\frac{k^9}{45360}R^6 \right]
\] (9.86)
CHAPTER 10

SCALING THE MATRIX EQUATIONS

If we look at the first two Maxwell’s equations for EM waves in a homogeneous medium given by [1]

\[ \nabla \times \mathbf{E} = i\omega \mu \mathbf{H} \] (10.1)

\[ \nabla \times \mathbf{H} = -i\omega \varepsilon \mathbf{E} + \mathbf{J} \] (10.2)

for a plane wave illumination, we obtain

\[ \frac{|\mathbf{E}|}{|\mathbf{H}|} = \eta \] (10.3)

Looking at (2.18), we expect a similar relation between surface magnetic and electric currents, namely,

\[ \frac{|\mathbf{J}_s|}{|\mathbf{M}_s|} = \frac{1}{\eta} \] (10.4)

Since \( \eta \) can be quite high depending on the medium, the matrix equations we usually solve (e.g., the EFIE, the EFIE, the CFIE, the PMCHWT, and the EPA) might give relatively inaccurate results dependent upon the choice of the matrix solver. This inaccuracy is more pronounced especially for the iterative solvers. To circumvent this problem, we need to scale the matrix equation appropriately. To understand the methodology, we first consider a typical PMCHWT matrix equation (after testing
with the appropriate basis function) given by

\[
\begin{pmatrix}
-(\mathcal{L}_1 + \mathcal{L}_2) & -(\mathcal{K}_1 + \mathcal{K}_2) \\
(\mathcal{K}_1 + \mathcal{K}_2) & -\left(\frac{1}{\eta_1^2} \mathcal{L}_1 + \frac{1}{\eta_2^2} \mathcal{L}_2\right)
\end{pmatrix}
\cdot
\begin{pmatrix}
j_{\text{sca}} \\
m_{\text{sca}}
\end{pmatrix}
= \begin{pmatrix}
b_{\text{inc}}^E \\
b_{\text{inc}}^H
\end{pmatrix}
\tag{10.5}
\]

In (10.5), \( j_{\text{sca}} \) and \( m_{\text{sca}} \) are the current coefficients to be solved and \( b_{\text{inc}}^E \) and \( b_{\text{inc}}^H \) are the Galerkin tested incident electric and magnetic fields (known), respectively. Now, (10.5) can be written as

\[
\begin{pmatrix}
j_{\text{sca}} \\
m_{\text{sca}}
\end{pmatrix}
= \begin{pmatrix}
-(\mathcal{L}_1 + \mathcal{L}_2) & -(\mathcal{K}_1 + \mathcal{K}_2) \\
(\mathcal{K}_1 + \mathcal{K}_2) & -\left(\frac{1}{\eta_1^2} \mathcal{L}_1 + \frac{1}{\eta_2^2} \mathcal{L}_2\right)
\end{pmatrix}^{-1}
\cdot
\begin{pmatrix}
b_{\text{inc}}^E \\
b_{\text{inc}}^H
\end{pmatrix}
\tag{10.6}
\]

Looking at (10.3) and (10.4), we want the following transformation

\[
\begin{pmatrix}
b_{\text{inc}}^E \\
b_{\text{inc}}^H
\end{pmatrix} \Rightarrow \begin{pmatrix}\frac{1}{\eta} b_{\text{inc}}^E \\
\frac{1}{\eta} b_{\text{inc}}^H
\end{pmatrix}
\quad \quad \quad \quad
\begin{pmatrix}j_{\text{sca}} \quad m_{\text{sca}}\end{pmatrix} \Rightarrow \begin{pmatrix}j_{\text{sca}} \quad \frac{1}{\eta} m_{\text{sca}}\end{pmatrix}
\tag{10.7}
\]

Hence we write an equation as

\[
\begin{pmatrix}
j_{\text{sca}} \\
\frac{1}{\eta} m_{\text{sca}}
\end{pmatrix}
= \begin{pmatrix}a & b \\
c & d
\end{pmatrix}
\cdot
\begin{pmatrix}\frac{1}{\eta} b_{\text{inc}}^E \\
\frac{1}{\eta} b_{\text{inc}}^H
\end{pmatrix}
\tag{10.8}
\]

where we define the transformation matrix as

\[
\begin{pmatrix}a & b \\
c & d
\end{pmatrix}
= \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\tag{10.9}
\]

and \( a, b, c, \) and \( d \) are constants to be determined. From (10.8), we obtain

\[
j_{\text{sca}} = \frac{a}{\eta} A \cdot b_{\text{inc}}^E + b B \cdot b_{\text{inc}}^H
\tag{10.10}
\]
and

\[ \frac{1}{\eta} m_{sca} = \frac{c}{\eta} \mathbf{C} \cdot \mathbf{b}_{inc}^E + d \mathbf{D} \cdot \mathbf{b}_{inc}^H \quad (10.11) \]

Using (10.9) in (250), and from the resulting matrix equation, we have

\[ j_{sca} = \mathbf{A} \cdot \mathbf{b}_{inc}^E + \mathbf{B} \cdot \mathbf{b}_{inc}^H \quad (10.12) \]

and

\[ m_{sca} = \mathbf{C} \cdot \mathbf{b}_{inc}^E + \mathbf{D} \cdot \mathbf{b}_{inc}^H \quad (10.13) \]

Comparing (10.10) to (10.12), and (10.11) to (10.13), we find

\[ a = \frac{1}{\eta}; \quad b = 1; \quad c = 1; \quad d = \frac{1}{\eta} \quad (10.14) \]

Since \( \mathbf{A} \) corresponds to \( - (\mathbf{L}_1 + \mathbf{L}_2)^{-1} \), \( \mathbf{B} \) corresponds to \( - (\mathbf{K}_1 + \mathbf{K}_2)^{-1} \), \( \mathbf{C} \) corresponds to \( (\mathbf{K}_1 + \mathbf{K}_2)^{-1} \), and \( \mathbf{D} \) corresponds to \( - \left( \frac{1}{\eta_1} \mathbf{L}_1 + \frac{1}{\eta_2} \mathbf{L}_2 \right)^{-1} \), we finally deduce the properly scaled PMCHWT matrix equation to be

\[ \begin{bmatrix} j_{sca} \\ \frac{1}{\eta} m_{sca} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\eta} (\mathbf{L}_1 + \mathbf{L}_2) & - (\mathbf{K}_1 + \mathbf{K}_2) \\ (\mathbf{K}_1 + \mathbf{K}_2) & -\eta \left( \frac{1}{\eta_1} \mathbf{L}_1 + \frac{1}{\eta_2} \mathbf{L}_2 \right) \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{\eta} \mathbf{b}_{inc}^E \\ \frac{1}{\eta} \mathbf{b}_{inc}^H \end{bmatrix} \quad (10.15) \]

Now, we want to write the properly scaled matrix equation for the EPA. To derive the properly scaled version, we first refer to the unscaled version given in (9.53) and (9.73). After proper scaling, (9.73) can be written as

\[ \begin{bmatrix} j_{sinc} \\ \frac{1}{\eta} m_{sinc} \end{bmatrix} = \mathbf{U}^{-1} \cdot \begin{bmatrix} \frac{1}{\eta} \mathbf{b}_{inc}^{\tilde{h} \times E} \\ \frac{1}{\eta} \mathbf{b}_{inc}^{\tilde{h} \times H} \end{bmatrix} \quad (10.16) \]
The current-solver (CS) part in (9.53) can be derived directly from the system matrix in (9.53), namely

\[
\begin{bmatrix}
- (L_1 + L_2) & - (K_1 + K_2) \\
(K_1 + K_2) & - \left( \frac{1}{\eta_1} L_1 + \frac{1}{\eta_2} L_2 \right)
\end{bmatrix} \Rightarrow \begin{bmatrix}
- \frac{1}{\eta} (L_1 + L_2) & - (K_1 + K_2) \\
(K_1 + K_2) & - \frac{1}{\eta} \left( \frac{1}{\eta_1} L_1 + \frac{1}{\eta_2} L_2 \right)
\end{bmatrix}
\]

\[\text{(10.17)}\]

Since the CS problem must be premultiplied to the incident field vector consisting of incident electric and magnetic fields (cf. the right-hand side of (10.15)), we clearly deduce that the incident current vector premultiplied by the OI vector (on the right-hand side of (9.53)) must be scaled very much like the incident field vector on the right-hand side of 10.15) (i.e., the electric field must be divided by \(\eta\), while magnetic field is left as it is). This procedure is fulfilled by the transformation

\[
\begin{bmatrix}
\frac{1}{\eta} j_{\text{obj,ска}} \\
m_{\text{obj,ска}}
\end{bmatrix} = \begin{bmatrix}
- \frac{1}{\eta} (L_1 + L_2) & - (K_1 + K_2) \\
(K_1 + K_2) & - \frac{1}{\eta} \left( \frac{1}{\eta_1} L_1 + \frac{1}{\eta_2} L_2 \right)
\end{bmatrix}^{-1} \cdot \begin{bmatrix}
- \frac{1}{\eta} L^{OI} & - K^{OI} \\
K^{OI} & - \frac{1}{\eta} \frac{1}{\eta_1} L^{OI}
\end{bmatrix} \cdot \begin{bmatrix}
\frac{1}{\eta} j_{\text{inc}} \\
m_{\text{inc}}
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
\frac{1}{\eta} j_{\text{obj,ска}} \\
m_{\text{obj,ска}}
\end{bmatrix} = \begin{bmatrix}
- \frac{1}{\eta} (L_1 + L_2) & - (K_1 + K_2) \\
(K_1 + K_2) & - \eta \left( \frac{1}{\eta_1} L_1 + \frac{1}{\eta_2} L_2 \right)
\end{bmatrix}^{-1} \cdot \begin{bmatrix}
- \frac{1}{\eta} L^{OI} & - K^{OI} \\
K^{OI} & - \eta \frac{1}{\eta_1} L^{OI}
\end{bmatrix} \cdot \begin{bmatrix}
j_{\text{inc}} \\
m_{\text{inc}}
\end{bmatrix}
\]

\[\text{(10.18)}\]

Therefore, we write the required form of (9.53) as

\[
\begin{bmatrix}
\frac{1}{\eta} \dot{j}_{\text{ска}} \\
m_{\text{ска}}
\end{bmatrix} = \bar{U}^-1 \cdot \begin{bmatrix}
aA & bB \\
cC & dD
\end{bmatrix} \cdot \begin{bmatrix}
\frac{1}{\eta} j_{\text{obj,ска}} \\
m_{\text{obj,ска}}
\end{bmatrix}
\]

\[\text{(10.19)}\]

and simultaneously we must fulfill

\[
\begin{bmatrix}
\dot{j}_{\text{ска}} \\
m_{\text{ска}}
\end{bmatrix} = \bar{U}^-1 \cdot \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \cdot \begin{bmatrix}
\frac{1}{\eta} j_{\text{obj,ска}} \\
m_{\text{obj,ска}}
\end{bmatrix}
\]

\[\text{(10.20)}\]

Comparing (10.19) and (10.20), we have
\[ a = 1; \ b = \eta; \ c = \frac{1}{\eta}; \ d = 1 \] (10.21)

Therefore, the final properly scaled EPA matrix equation is

\[
\begin{bmatrix}
\frac{\mathbf{j}_{\text{sca}}}{\nu \mathbf{m}_{\text{sca}}}
\end{bmatrix}
= \mathbf{U}^{-1} \cdot
\begin{bmatrix}
-\frac{\mathbf{K}^{\text{IO},\hat{\nu}\times}}{\nu^{\hat{\nu}}} \frac{\nu}{\eta_1} \mathbf{L}^{\text{IO},\hat{\nu}\times} & -\frac{1}{\nu} \left( \mathbf{L}_1 + \mathbf{L}_2 \right) - (\mathbf{K}_1 + \mathbf{K}_2) \\
-\frac{1}{\nu} \mathbf{L}^{\text{IO},\hat{\nu}\times} - \frac{\nu}{\eta_1} \mathbf{K}^{\text{IO},\hat{\nu}\times} & \frac{\nu}{\eta_1} \left( \mathbf{L}_1 + \mathbf{K}_2 \right) - \eta \left( \frac{\nu}{\eta_1} \mathbf{L}_1 + \frac{\nu}{\eta_2} \mathbf{L}_2 \right)
\end{bmatrix}^{-1} \cdot
\begin{bmatrix}
-\frac{1}{\nu} \mathbf{L}^{\text{IO},\hat{\nu}\times} & -\mathbf{K}^{\text{IO},\hat{\nu}\times} \\
\mathbf{K}^{\text{IO},\hat{\nu}\times} & -\frac{\nu}{\eta_1} \mathbf{L}^{\text{IO},\hat{\nu}\times}
\end{bmatrix}
\cdot
\begin{bmatrix}
\frac{\mathbf{j}_{\text{inc}}}{\nu \mathbf{m}_{\text{inc}}}
\end{bmatrix}
\] (10.22)

The properly scaled translation operator can be derived in precisely the same way we derived the scaled IO matrix operator. It can be written as

\[
\begin{bmatrix}
\frac{\mathbf{j}_{\text{sca}}}{\nu \mathbf{m}_{\text{sca}}}
\end{bmatrix}
= \mathbf{U}^{-1} \cdot
\begin{bmatrix}
-\frac{\mathbf{K}^{\text{TO},\hat{\nu}\times}}{\nu^{\hat{\nu}}} \frac{\nu}{\eta_1} \mathbf{L}^{\text{TO},\hat{\nu}\times} & -\frac{1}{\nu} \mathbf{L}^{\text{TO},\hat{\nu}\times} - \frac{\nu}{\eta_1} \mathbf{K}^{\text{TO},\hat{\nu}\times}
\end{bmatrix}
\cdot
\begin{bmatrix}
\frac{\mathbf{j}_{\text{sca}}}{\nu \mathbf{m}_{\text{sca}}}
\end{bmatrix}
\] (10.23)

It should be pointed out that in our derivation, we assumed that the whole space is filled by a material of intrinsic impedance \( \eta_1 \), and the scatterers are dielectric bodies of intrinsic impedance \( \eta_2 \). We use a standard intrinsic impedance \( \eta \) (in vacuum, \( \eta = \eta_0 \)) to scale all the matrices. If the material pervading the space is vacuum, we can use \( \eta_1 = \eta = \eta_0 \).
11.1 The EPA Using the Tap Basis for the PEC Objects and Its Matrix Representation

So far, we have derived the EPA formulation for scatterers which are physically apart. Problems arise when the ES intersects a scatterer, because the current discontinuity ensues. To overcome this unexpected scenario, a new formulation called the tap basis formulation of the EPA has been proposed [31, 32, 34, 36]. The tap basis provides an efficient means to subdivide one scatterer by one or more ESs. To understand the concept, let us consider a thin PEC strip depicted in Figure 11.1. Two ESs denoted as $ES_1$ and $ES_2$ enclose the thin PEC strip. We consider three parts of the thin PEC strip: the part on the left, which is fully enclosed by $ES_1$, is denoted by $obj_1$; that on the right, which is fully enclosed by $ES_2$, is denoted by $obj_2$; and the third part, which straddles the ES, is denoted by $obj_3$. This third part will be treated as if it were external to the ES, and its interaction with other parts will be computed directly. Since the potential in the PEC is the same everywhere, there can be no field inside it. From this we can write

![Figure 11.1: A PEC strip split into two by two ESs sharing a common face (after [33]).](image)
\[ \mathbf{E}_{\text{sca}} + \mathbf{E}_{\text{inc}} = \mathbf{E}_{\text{tot}} = 0 \] (11.1)

From (11.1), we can write the following system of equations:

\[ \mathbf{L}_{11} \mathbf{J}_{\text{obj}_1,\text{sca}} + \mathbf{L}_{12} \mathbf{J}_{\text{obj}_2,\text{sca}} + \mathbf{L}_{13} \mathbf{J}_{\text{obj}_3,\text{sca}} = -\mathbf{E}_{\text{obj}_1,\text{inc}} \] (11.2)

\[ \mathbf{L}_{21} \mathbf{J}_{\text{obj}_1,\text{sca}} + \mathbf{L}_{22} \mathbf{J}_{\text{obj}_2,\text{sca}} + \mathbf{L}_{23} \mathbf{J}_{\text{obj}_3,\text{sca}} = -\mathbf{E}_{\text{obj}_2,\text{inc}} \] (11.3)

\[ \mathbf{L}_{31} \mathbf{J}_{\text{obj}_1,\text{sca}} + \mathbf{L}_{32} \mathbf{J}_{\text{obj}_2,\text{sca}} + \mathbf{L}_{33} \mathbf{J}_{\text{obj}_3,\text{sca}} = -\mathbf{E}_{\text{obj}_3,\text{inc}} \] (11.4)

where the operator \( \mathbf{L}_{ij} \) operating on current \( \mathbf{J}_{\text{obj}_j,\text{sca}} \) gives the electric field scattered from \( \text{obj}_i \), i.e., it gives the electric field scattered by \( \text{obj}_i \) contributed by the scattered current on \( \text{obj}_j \). Evidently, these are simple EFIE equations. Now, the \( \text{obj}_3 \) surface, which has been intersected by the ES, will be designated the tap basis. The EPA operators can be introduced at this point to give

\[ \mathbf{E}_{\text{obj}_1,\text{inc}} = \mathbf{T}_{11}^{QI} \cdot \begin{bmatrix} \mathbf{J}_{S_1,\text{inc}} \\ \mathbf{M}_{S_1,\text{inc}} \end{bmatrix} \] (11.5)

\[ \mathbf{E}_{\text{obj}_2,\text{inc}} = \mathbf{T}_{22}^{QI} \cdot \begin{bmatrix} \mathbf{J}_{S_2,\text{inc}} \\ \mathbf{M}_{S_2,\text{inc}} \end{bmatrix} \] (11.6)

\[ \mathbf{L}_{12} \mathbf{J}_{\text{obj}_2,\text{sca}} = \mathbf{T}_{11}^{QI} \cdot \mathbf{T}_{12} \cdot \mathbf{T}_{22}^{IO} \cdot \mathbf{J}_{\text{obj}_2,\text{sca}} \] (11.7)

\[ \mathbf{L}_{21} \mathbf{J}_{\text{obj}_1,\text{sca}} = \mathbf{T}_{22}^{QI} \cdot \mathbf{T}_{21} \cdot \mathbf{T}_{11}^{IO} \cdot \mathbf{J}_{\text{obj}_1,\text{sca}} \] (11.8)
Here, the tap contribution $t_{31(2)}$ becomes a new unknown in the tap basis formulation. Using (11.5)–(11.10), (11.2)–(11.4) can be recast as

\begin{equation}
\mathcal{L}_{11} J_{obj_{1, sca}} + \mathcal{T}^{O1}_{11} \cdot \mathcal{T}^{I1}_{12} \cdot \mathcal{T}^{I2}_{22} \cdot J_{obj_{2, sca}} + \mathcal{L}_{13} J_{obj_{3, sca}} = -\mathcal{T}^{O1}_{11} \cdot \begin{bmatrix} J_{S1, inc} \\ M_{S1, inc} \end{bmatrix} \tag{11.11}
\end{equation}

\begin{equation}
\mathcal{T}^{O1}_{22} \cdot \mathcal{T}^{I1}_{21} \cdot \mathcal{T}^{I2}_{11} \cdot J_{obj_{1, sca}} + \mathcal{L}_{22} J_{obj_{2, sca}} + \mathcal{L}_{23} J_{obj_{3, sca}} = -\mathcal{T}^{O1}_{22} \cdot \begin{bmatrix} J_{S2, inc} \\ M_{S2, inc} \end{bmatrix} \tag{11.12}
\end{equation}

\begin{equation}
t_{31} + t_{32} + \mathcal{L}_{33} J_{obj_{3, sca}} = -E_{obj_{3, inc}} \tag{11.13}
\end{equation}

From (11.13), we obtain

\begin{equation}
J_{obj_{3, sca}} = -\mathcal{L}^{-1}_{33} (t_{31} + t_{32} + E_{obj_{3, inc}}) \tag{11.14}
\end{equation}

Now, we note that

\begin{equation}
\mathcal{T}^{I2}_{22} \cdot J_{obj_{2, sca}} = \begin{bmatrix} J_{S2, sca} \\ M_{S2, sca} \end{bmatrix} \tag{11.15}
\end{equation}

Using (11.15) in (11.11), and doing a slight reshuffle, we have
\[ J_{\text{obj},sca} = L_{11}^{-1} \left( -\mathcal{T}_{11}^{OI} \cdot \begin{bmatrix} J_{S_1,\text{inc}} \\ M_{S_1,\text{inc}} \end{bmatrix} - \mathcal{T}_{11}^{OI} \cdot \mathcal{T}_{12} \cdot \begin{bmatrix} J_{S_2,sca} \\ M_{S_2,sca} \end{bmatrix} - L_{13} J_{\text{obj},sca} \right) \] (11.16)

Furthermore,

\[ \mathcal{T}_{11}^{IO} \cdot J_{\text{obj},sca} = \begin{bmatrix} J_{S_1,sca} \\ M_{S_1,sca} \end{bmatrix} \] (11.17)

Substituting (11.16) into (11.17), we have

\[ \begin{bmatrix} J_{S_1,sca} \\ M_{S_1,sca} \end{bmatrix} = \mathcal{T}_{11}^{IO} \cdot L_{11}^{-1} \left[ -\mathcal{T}_{11}^{OI} \cdot \left( \begin{bmatrix} J_{S_1,\text{inc}} \\ M_{S_1,\text{inc}} \end{bmatrix} + \mathcal{T}_{12} \cdot \begin{bmatrix} J_{S_2,sca} \\ M_{S_2,sca} \end{bmatrix} \right) - L_{13} J_{\text{obj},sca} \right] \] (11.18)

Similarly, using (11.16) into (11.9), we have

\[ t_{31} = L_{31} L_{11}^{-1} \left[ -\mathcal{T}_{11}^{OI} \cdot \left( \begin{bmatrix} J_{S_1,\text{inc}} \\ M_{S_1,\text{inc}} \end{bmatrix} + \mathcal{T}_{12} \cdot \begin{bmatrix} J_{S_2,sca} \\ M_{S_2,sca} \end{bmatrix} \right) - L_{13} J_{\text{obj},sca} \right] \] (11.19)

Now, we can define a scattering matrix operator for the tap basis formulation as

\[ S_{\text{tap},11} = \begin{bmatrix} \mathcal{T}_{11}^{IO} \\ L_{31} \end{bmatrix} \cdot \left( -L_{11}^{-1} \right) \begin{bmatrix} \mathcal{T}_{11}^{OI} \\ L_{13} \end{bmatrix} \] (11.20)

Using this definition, we can combine (11.18) and (11.19) to write

\[ \begin{bmatrix} J_{S_1,sca} \\ M_{S_1,sca} \\ t_{31} \end{bmatrix} - S_{\text{tap},11} \cdot \begin{bmatrix} \mathcal{T}_{12} \cdot \begin{bmatrix} J_{S_2,sca} \\ M_{S_2,sca} \end{bmatrix} \end{bmatrix} = S_{\text{tap},11} \cdot \begin{bmatrix} J_{S_1,\text{inc}} \\ M_{S_1,\text{inc}} \\ J_{\text{obj},sca} \end{bmatrix} \] (11.21)

Thus, we have recast (11.11) into a new form in (11.21). Likewise, we can recast (11.12) as
\[
\begin{bmatrix}
J_{S_2,\text{sca}} \\
M_{S_2,\text{sca}} \\
t_{32}
\end{bmatrix} - S_{\text{tap},22} \cdot \begin{bmatrix}
\mathcal{T}_{21} \cdot [J_{S_1,\text{sca}}] \\
M_{S_1,\text{sca}} \\
J_{\text{obj},\text{sca}}
\end{bmatrix} = S_{\text{tap},22} \cdot \begin{bmatrix}
J_{S_2,\text{inc}} \\
M_{S_2,\text{inc}} \\
0
\end{bmatrix}
\]  
(11.22)

It should, however, be noted that (11.21) and (11.22) are yet to be EPA formulations for the tap basis, since we have \(J_{\text{obj},\text{sca}}\) instead of taps. We can, nevertheless, convert them into the tap formulation by substituting (11.14) into them, and rearranging the resulting equations.

\[
\begin{bmatrix}
J_{S_1,\text{sca}} \\
M_{S_1,\text{sca}} \\
t_{31}
\end{bmatrix} - S_{\text{tap},11} \cdot \begin{bmatrix}
\mathcal{T}_{12} \cdot [J_{S_2,\text{sca}}] \\
M_{S_2,\text{sca}} \\
- \mathcal{L}_{33}^{-1} (t_{31} + t_{32})
\end{bmatrix} = S_{\text{tap},11} \cdot \begin{bmatrix}
J_{S_1,\text{inc}} \\
M_{S_1,\text{inc}} \\
- \mathcal{L}_{33}^{-1} E_{\text{obj},\text{inc}}
\end{bmatrix}
\]  
(11.23)

\[
\begin{bmatrix}
J_{S_2,\text{sca}} \\
M_{S_2,\text{sca}} \\
t_{32}
\end{bmatrix} - S_{\text{tap},22} \cdot \begin{bmatrix}
\mathcal{T}_{21} \cdot [J_{S_1,\text{sca}}] \\
M_{S_1,\text{sca}} \\
- \mathcal{L}_{33}^{-1} (t_{31} + t_{32})
\end{bmatrix} = S_{\text{tap},22} \cdot \begin{bmatrix}
J_{S_2,\text{inc}} \\
M_{S_2,\text{inc}} \\
- \mathcal{L}_{33}^{-1} E_{\text{obj},\text{inc}}
\end{bmatrix}
\]  
(11.24)

Now, from (11.9) using the Galerkin testing we can write,

\[
t_{31,\text{coeff}} = \mathbf{\overline{u}}_3^{-1} \cdot \mathbf{L}_{31} \cdot j_{\text{obj},\text{sca}}
\]  
(11.25)

where \(t_{31,\text{coeff}}\) and \(j_{\text{obj},\text{sca}}\) are coefficient matrices for the tap and \(\text{obj}_1\), respectively, and

\[
[\mathbf{\overline{u}}_3]_{m,n} = \langle f_{\text{obj},m} (r), f_{\text{obj},n}^* (r) \rangle
\]  
(11.26)

\[
[\mathbf{L}_{31}]_{m,n} = \langle f_{\text{obj},m} (r), \mathcal{L}_{31} (r,r'), f_{\text{obj},n}^* (r') \rangle
\]  
(11.27)

The scattering matrix can, thus, be written as
\[
\mathbf{S}_{tap,11} = \mathbf{U}_{13}^{-1} \cdot \left[ \frac{\mathbf{T}^{IO}}{\mathbf{L}_{31}} \right] \cdot (-\mathbf{L}^{-1}_{11}) \cdot \left[ \frac{\mathbf{T}^{OJ}}{\mathbf{L}_{11}} \right] \mathbf{L}_{13} \tag{11.28}
\]

where

\[
\mathbf{U}_{13} = \begin{bmatrix}
\bar{u}_1 & 0 \\
0 & \bar{u}_3
\end{bmatrix} \tag{11.29}
\]

\[
[\mathbf{L}_{13}]_{m,n} = \langle f_{obj,m}(r), \mathcal{L}_{31}(r,r'), f'_{obj,n}(r') \rangle \tag{11.30}
\]

Henceforward, (11.23) can be converted into a matrix equation

\[
\begin{bmatrix}
j_{S_1,sca} \\
m_{S_1,sca} \\
t_{31,coeff}
\end{bmatrix}
- \mathbf{S}_{tap,11} \cdot \begin{bmatrix}
j_{S_2,sca} \\
m_{S_2,sca} \\
t_{31,coeff} + t_{32,coeff}
\end{bmatrix}
= \mathbf{S}_{tap,11} \cdot \begin{bmatrix}
j_{S_1,inc} \\
m_{S_1,inc} \\
b_E^{obj,inc}
\end{bmatrix} \tag{11.31}
\]

where \(t_{32,coeff}\) can be given by an equation similar to (11.25). Likewise, (11.24) can be converted into the matrix equation

\[
\begin{bmatrix}
j_{S_2,sca} \\
m_{S_2,sca} \\
t_{32,coeff}
\end{bmatrix}
- \mathbf{S}_{tap,22} \cdot \begin{bmatrix}
j_{S_1,sca} \\
m_{S_1,sca} \\
t_{31,coeff} + t_{32,coeff}
\end{bmatrix}
= \mathbf{S}_{tap,22} \cdot \begin{bmatrix}
j_{S_2,inc} \\
m_{S_2,inc} \\
b_E^{obj,inc}
\end{bmatrix} \tag{11.32}
\]

The procedure used to derive these equations is identical to the one we used for the conventional EPA. Equations (11.31)–(11.32) can easily be solved using a suitable iterative solver.
11.2 The EPA Using the Tap Basis for Dielectric Objects

Since the PEC scatterers are the special case of the dielectric objects, here we derive the formulation for the dielectric scatterers. Once we obtain the tap basis formulations of the EPA for the PEC objects, we can modify them to obtain the tap formulations of the EPA for dielectric scatterers toward the more generalized treatment. Equations (11.2)–(11.4) can be modified for a case of \( M_d \) dielectric objects. For example, (11.2) is written as [33]

\[
Z_{11} \cdot \begin{bmatrix} J_{obj_1, sca} \\ M_{obj_1, sca} \end{bmatrix} + Z_{12} \cdot \begin{bmatrix} J_{obj_2, sca} \\ M_{obj_2, sca} \end{bmatrix} + Z_{13} \cdot \begin{bmatrix} J_{obj_3, sca} \\ M_{obj_3, sca} \end{bmatrix} + \ldots \\
+ Z_{1M_d} \cdot \begin{bmatrix} J_{obj_{M_d}, sca} \\ M_{obj_{M_d}, sca} \end{bmatrix} = \begin{bmatrix} E_{obj_1, inc} \\ H_{obj_1, inc} \end{bmatrix}
\] (11.33)

As a rule of thumb, the following substitutions into (11.2)–(11.4) result in the formulations for the dielectric objects.

\[
-\mathcal{L}_{ij} \Rightarrow Z_{ij} \quad E_{obj_1, inc} \Rightarrow \begin{bmatrix} E_{obj_1, inc} \\ H_{obj_1, inc} \end{bmatrix} \quad J_{obj_1, sca} \Rightarrow \begin{bmatrix} J_{obj_1, sca} \\ M_{obj_1, sca} \end{bmatrix} \quad t \Rightarrow \begin{bmatrix} t_{ij}^E \\ t_{ij}^H \end{bmatrix}
\] (11.34)

Now, let us consider a generalized formulation for the EPA using the tap basis for dielectric objects. Let us assume that there are \( N_t \) tap basis functions that connect across \( N_t \) boundaries and let there be \( N_{ES} \) number of ESs. The tap basis currents are indicated by the subscript \( t_i \), where \( i \) can be any number between 1 and \( N_t \). The tap bases themselves are indicated by \( t_{t_{ES}} \). Thus for the \( i \)-th ES, we have [33]
\[
\begin{bmatrix}
J_{S_i,sca} \\
M_{S_i,sca} \\
t_{E_{t_1}} \\
t_{H_{t_1}} \\
t_{E_{t_2}} \\
t_{H_{t_2}} \\
\vdots \\
t_{E_{t_{Nt_1}}} \\
t_{H_{t_{Nt_1}}} \\
\end{bmatrix}
- S_{ii} \cdot \sum_{j \neq i} N_{ES} T_{ij} \cdot 
\begin{bmatrix}
J_{S_j,sca} \\
M_{S_j,sca} \\
t_{E_{t_1}} \\
t_{H_{t_1}} \\
t_{E_{t_2}} \\
t_{H_{t_2}} \\
\vdots \\
t_{E_{t_{Nt_j}}} \\
t_{H_{t_{Nt_j}}} \\
\end{bmatrix} 
= S_{ii} \cdot 
\begin{bmatrix}
J_{S_i,inc} \\
M_{S_i,inc} \\
E_{t_1,inc} \\
H_{t_1,inc} \\
E_{t_2,inc} \\
H_{t_2,inc} \\
\vdots \\
E_{t_{Nt_i},inc} \\
H_{t_{Nt_i},inc} \\
\end{bmatrix}
\]
(11.35)

where

\[
Z_t = 
\begin{bmatrix}
Z_{t_1t_1} & Z_{t_1t_2} & \cdots & Z_{t_1t_{Nt}} \\
Z_{t_2t_1} & Z_{t_2t_2} & \cdots & Z_{t_2t_{Nt}} \\
\vdots & \vdots & \ddots & \vdots \\
Z_{t_{Nt_1}t_1} & Z_{t_{Nt_1}t_2} & \cdots & Z_{t_{Nt_1}t_{Nt}} \\
\end{bmatrix}
\]
(11.36)

and the EPO is defined as

\[
S_{ii} = 
\begin{bmatrix}
T_{ii}^{IO} \\
Z_{t_1i} \\
Z_{t_2i} \\
\vdots \\
Z_{t_{Nt_i}i} \\
\end{bmatrix}
\cdot (-Z_{ii}^{-1}) \cdot 
\begin{bmatrix}
T_{ii}^{IO} & Z_{it_1} & Z_{it_1} & \cdots & Z_{it_{Nt}} \\
\end{bmatrix}
\]
(11.37)

Now that we have derived the generalized formula for the tap basis case of the EPA, we want to derive the matrix formulation, which will be quite along the line of the conventional EPA. With that end in view, we assume the following expansions.
\[ J_{S_i,\text{inc}}(r) = \sum_{m=1}^{N_{S_i}} j_{S_i,m,\text{inc}} f_{S_i,m}(r) \]  
\[ M_{S_i,\text{inc}}(r) = \sum_{m=1}^{N_{S_i}} m_{S_i,m,\text{inc}} f_{S_i,m}(r) \]  
\[ J_{S_i,\text{sca}}(r) = \sum_{m=1}^{N_{S_i}} j_{S_i,m,\text{sca}} f_{S_i,m}(r) \]  
\[ M_{S_i,\text{sca}}(r) = \sum_{m=1}^{N_{S_i}} m_{S_i,m,\text{sca}} f_{S_i,m}(r) \]  
\[ J_{\text{obj},\text{sca}}(r) = \sum_{m=1}^{N_{\text{obj},i}} j_{\text{obj},i,m,\text{sca}} f_{\text{obj},i,m}(r) \]  
\[ M_{\text{obj},\text{sca}}(r) = \sum_{m=1}^{N_{\text{obj},i}} m_{\text{obj},i,m,\text{sca}} f_{\text{obj},i,m}(r) \]  
\[ J_{t_n,\text{sca}}(r) = \sum_{m=1}^{N_t} j_{t_n,m,\text{sca}} f_{t_n,m}^{J,M}(r) \]  
\[ M_{t_n,\text{sca}}(r) = \sum_{m=1}^{N_t} m_{t_n,m,\text{sca}} f_{t_n,m}^{J,M}(r) \]
\[ t^E_{tni}(r) = \sum_{m=1}^{N_t} t^E_{tni,m} f_{tn,m}(r) \] (11.46)

\[ t^H_{tni}(r) = \sum_{m=1}^{N_t} m^H_{tni,m} f_{tn,m}(r) \] (11.47)

It is worthwhile to mention that we can assign the same basis functions for both the tap and tap currents, i.e., \( f_{tn,m}^*, f_{tn,m} \). The tap can be related to the scattered currents on the object by [33]

\[
\begin{bmatrix}
    t^E_{tni} \\
    t^H_{tni}
\end{bmatrix} = Z_{tni} \cdot \begin{bmatrix}
    J_{obj, sca} \\
    M_{obj, sca}
\end{bmatrix} \] (11.48)

Using the basis function expansions (11.42), (11.43), (11.45), and (11.46) into (11.48), and then Galerkin testing it with the basis functions corresponding to the tap, we obtain

\[
\begin{bmatrix}
    t^E_{tni,coeff} \\
    t^H_{tni,coeff}
\end{bmatrix} = U_{tn}^{-1} \cdot Z_{tni} \cdot \begin{bmatrix}
    j_{obj, sca} \\
    m_{obj, sca}
\end{bmatrix} \] (11.49)

In (11.49), \( j_{obj, sca} \), and \( m_{obj, sca} \) are the vectors consisting of the current coefficients in the same manner as in (9.38), and (9.39), for example. The various quantities in (11.49) are defined as

\[
\begin{bmatrix}
    \bar{u}_{tn} \\
    0
\end{bmatrix}
\] (11.50)

\[
[\bar{u}_{tn}]_{m,n} = \langle f_{tn,m}(r), f_{tn,n}(r) \rangle \] (11.51)
\[
\begin{bmatrix}
Z_{t,i}
\end{bmatrix}_{m,n} = \left\langle f_{t,j,m}(r), Z(r, r'), f_{obj,i,n}(r') \right\rangle
\] (11.52)

The matrix form of the EPO can be obtained in much the same way as we obtained in the conventional EPA case from (11.37) as

\[
S_{ii} = U_{i,t}^{-1} \cdot \begin{bmatrix}
T_{ii}^{IO} \\
Z_{t_{1i}} \\
Z_{t_{2i}} \\
\vdots \\
Z_{t_{Nt}}
\end{bmatrix} \cdot \left(-Z_{ii}^{-1}\right) \cdot \begin{bmatrix}
T_{ii}^{OJ} \\
Z_{i_{t1}} \\
Z_{i_{t2}} \\
\vdots \\
Z_{i_{tNt}}
\end{bmatrix}
\] (11.53)

where

\[
U_{i,t} = \begin{bmatrix}
U_i & 0 & 0 & \cdots & 0 \\
0 & U_{t_1} & 0 & \cdots & 0 \\
0 & 0 & U_{t_2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & U_{t_n}
\end{bmatrix}
\] (11.54)

\[
\begin{bmatrix}
Z_{i_{tj}}
\end{bmatrix}_{m,n} = \left\langle f_{obj,i,m}(r), Z(r, r'), f_{J,M}^{(J,M)}(r') \right\rangle
\] (11.55)

The element \( U_i \) is given by the \( i \)-th component of (9.55), and \( \overline{T}_{ii}^{IO} \) and \( \overline{T}_{ii}^{OJ} \) are given by the \( i \)-th components of (9.59) and (9.60), respectively.

Now, we extend (11.13) for the generalized dielectric case to [33]
Galerkin testing the above equation by the basis functions on the tap, we obtain

\[
\begin{bmatrix}
  \mathbf{J}_{t_1,sca} \\
  \mathbf{M}_{t_1,sca} \\
  \vdots \\
  \mathbf{J}_{t_Nt,sca} \\
  \mathbf{M}_{t_Nt,sca}
\end{bmatrix}
= \mathbf{Z}_t^{-1} \cdot 
\begin{bmatrix}
  \mathbf{E}_{t_1,inc} \\
  \mathbf{H}_{t_1,inc} \\
  \vdots \\
  \mathbf{E}_{t_Nt,inc} \\
  \mathbf{H}_{t_Nt,inc}
\end{bmatrix} - \sum_{i=1}^{N_{ES}} \mathbf{U}_{t_i} \cdot 
\begin{bmatrix}
  \mathbf{t}_E^{t_{t_1}} \\
  \mathbf{t}_H^{H_{t_1}} \\
  \vdots \\
  \mathbf{t}_E^{t_{t_Nt}} \\
  \mathbf{t}_H^{H_{t_Nt}}
\end{bmatrix}
\]

(11.57)

where

\[
\mathbf{Z}_t = 
\begin{bmatrix}
  \mathbf{Z}_{t_1 t_1} & \mathbf{Z}_{t_1 t_2} & \cdots & \mathbf{Z}_{t_1 t_Nt} \\
  \mathbf{Z}_{t_2 t_1} & \mathbf{Z}_{t_2 t_2} & \cdots & \mathbf{Z}_{t_2 t_Nt} \\
  \vdots & \vdots & \ddots & \vdots \\
  \mathbf{Z}_{t_Nt t_1} & \mathbf{Z}_{t_Nt t_2} & \cdots & \mathbf{Z}_{t_Nt t_Nt}
\end{bmatrix}
\]

(11.58)

\[
[\mathbf{Z}_{t_i,t_j}]_{m,n} = \langle \mathbf{f}_{t_i,m}(\mathbf{r}), \mathbf{Z}(\mathbf{r}, \mathbf{r}'), \mathbf{f}_{t_j,n}(\mathbf{r}') \rangle
\]

(11.59)

\[
[\mathbf{b}_{t_i,inc}^E]_m = \langle \mathbf{f}_{t_i,m}(\mathbf{r}), \mathbf{E}_{t_i,inc}(\mathbf{r}) \rangle
\]

(11.60)
\[ \begin{bmatrix} \mathbf{b}^H_{t_i,\text{inc}} \end{bmatrix}_m = \langle \mathbf{f}_{t_i,m}(\mathbf{r}), \mathbf{H}_{t_i,\text{inc}}(\mathbf{r}) \rangle \] 

(11.61)

In the same vein, the final EPA matrix equation to be solved is given by [33]

\[
\begin{bmatrix}
\mathbf{j}_{s_i,\text{acq}} \\
\mathbf{m}_{s_i,\text{acq}} \\
\mathbf{t}_{t_1,i} \\
\mathbf{H}_{t_1,i} \\
\mathbf{t}_{t_2,i} \\
\mathbf{H}_{t_2,i} \\
\vdots \\
\mathbf{t}_{t_{N_t},i} \\
\mathbf{H}_{t_{N_t},i}
\end{bmatrix}
- \mathbf{S}_{ii} \cdot 
\begin{bmatrix}
\sum_{j=1}^{N_{ES}} \mathbf{U}_{t_{i,j}} \\
\sum_{j=1}^{N_{ES}} \mathbf{U}_{t_{1,j}} \\
\sum_{j=1}^{N_{ES}} \mathbf{U}_{t_{2,j}} \\
\vdots \\
\sum_{j=1}^{N_{ES}} \mathbf{U}_{t_{N_t,j}}
\end{bmatrix}
- \mathbf{Z}_t^{-1} \cdot 
\begin{bmatrix}
\mathbf{j}_{s_i,\text{inc}} \\
\mathbf{m}_{s_i,\text{inc}} \\
\mathbf{b}_{t_1,\text{inc}} \\
\mathbf{b}_{H_{t_1,\text{inc}}} \\
\mathbf{b}_{t_2,\text{inc}} \\
\mathbf{b}_{H_{t_2,\text{inc}}} \\
\vdots \\
\mathbf{b}_{t_{N_t},\text{inc}} \\
\mathbf{b}_{H_{t_{N_t},\text{inc}}}
\end{bmatrix}
= \mathbf{S}_{ii} \cdot 
\begin{bmatrix}
\mathbf{j}_{s_i,\text{inc}} \\
\mathbf{m}_{s_i,\text{inc}} \\
\mathbf{b}_{t_1,\text{inc}} \\
\mathbf{b}_{H_{t_1,\text{inc}}} \\
\mathbf{b}_{t_2,\text{inc}} \\
\mathbf{b}_{H_{t_2,\text{inc}}} \\
\vdots \\
\mathbf{b}_{t_{N_t},\text{inc}} \\
\mathbf{b}_{H_{t_{N_t},\text{inc}}}
\end{bmatrix}
\] 

(11.62)

The properly scaled matrices for the tap formulation of the EPA can be derived in a similar fashion as we did for the PMCHWT and the conventional EPA.
CHAPTER 12
RESULTS

The results for the various cases described so far will be discussed here. Unless explicitly mentioned otherwise, we assume a PEC sphere of radius $0.1\lambda$ with an incident electric field of frequency $300$ MHz, inclination or zenith angle of $0^\circ$, and azimuth of $270^\circ$ as our test case. In Figure 12.1, the total electric field (dB) plot for the EFIE is given. Since we used the surface currents to compute the electric field, the field is extinct inside the sphere, verifying the extinction theorem and equivalence principle, given by (2.19). The electric field is not very sharp only because of the lower number of field points chosen for the plot to reduce the run time. As expected, the total electric field tends to decay to almost unity (which is the magnitude of the incident electric field) as we move farther away from the surface. The VV bistatic RCS appears in Figure 12.2, and this agrees with the analytical result obtained from the Mie series. Now we consider the total electric field (dB) plot for the MFIE in Figure 12.3. We have shown earlier that although the MFIE operator has near-singularity, it has no singularity in the diagonal elements. Therefore, exclusion of any singularity treatment produces fairly accurate results. We want to demonstrate this point further. The VV bistatic RCS plots for cases without and with singularity treatment appear in Figure 12.4 and Figure 12.5, respectively. Evidently, the difference is barely perceptible to naked eyes. Thus, where moderate accuracy is acceptable, omission of the singularity treatment for the MFIE is a means to save time and effort.

For the CFIE, we consider all three cases given by (7.1)–(7.3). We will designate the CFIE equations given by (7.1)–(7.3) as CFIE-1, CFIE-2, and CFIE-3, respectively. The electric field plots of these CFIE equations are given in Figures 12.6–12.8, respectively. As we might expect, the extinction is much better (cf. the extinction in Figure 12.3) than in the MFIE. The VV bistatic RCS for all three cases appears in Figures 12.9–12.11. Again, there is a very good correspondence of the CFIE results with that obtained from the Mie series calculation. We note from the computational
point of view, the CFIE is not really advantageous if we are not concerned with the pin-point accuracy of the results. The difference in performance is hardly conspicuous and the EFIE gives good enough and quicker results with less effort.
Now we consider the PMCHWT case. We refer to (10.15), i.e., the properly scaled version. Although, we use singularity treatment for the $\mathcal{L}$ operator, but we neglect
Figure 12.5: The vertical-vertical (VV) bistatic RCS of a PEC sphere of radius $0.1\lambda$ computed using the MFIE with singularity treatment and compared against the Mie series.

Figure 12.6: Total electric field (dB) plot from the CFIE-1.

that for the $\mathcal{K}$ operator, since the latter has only near-singularity. Moreover, in the PMCHWT formulation, the diagonal terms for the $\mathcal{K}$ operator cancel and we do
not compute these terms. The diagonal terms, however, do not cancel in the other
dielectric formulations, and hence, must be retained. The relative dielectric constant
is given by $\epsilon_r = 4.0$. The principle of extinction is shown to be enforced inside a
Figure 12.9: The vertical-vertical (VV) bistatic RCS of a PEC sphere of radius $0.1\lambda$ computed using the CFIE-1 and compared against the Mie series.

Figure 12.10: The vertical-vertical (VV) bistatic RCS of a PEC sphere of radius $0.1\lambda$ computed using the CFIE-2 and compared against the Mie series.

dielectric sphere of radius $0.25\lambda$ as shown in Figure 12.12. One comment is in order here. If we compare this field plot in Figure 12.12 with the that from PEC sphere, we see that the scattering in this case is lower. This is actually expected, because in
Figure 12.11: The vertical-vertical (VV) bistatic RCS of a PEC sphere of radius $0.1\lambda$ computed using the CFIE-3 and compared against the Mie series.

Figure 12.12: Total electric field (dB) plot from the PMCHWT.

the dielectric sphere the fields penetrate (although not evident here, since it shows the extinction of field inside) inside the dielectric body and hence intensity of the
Figure 12.13: The vertical-vertical (VV) bistatic RCS of a dielectric sphere of radius $0.15\lambda$ ($\epsilon_r = 4.0$) computed using the PMCHWT and compared against the Mie series.

Figure 12.14: The vertical-vertical (VV) bistatic RCS of a dielectric sphere of radius $0.25\lambda$ ($\epsilon_r = 4.0$) computed using the PMCHWT and compared against the Mie series.

scattered field is weaker. Although the comparison is asymmetric because of different radiiuses of the dielectric sphere, it does verify the the field pattern. The plots of VV bistatic RCS for spheres of radiiuses $0.15\lambda$ and $0.25$ appear in Figures 12.13 and 12.14,
respectively. In both cases, the results from the PMCHWT agrees well with the Mie scattering results.

Now that we have dealt with the PMCHWT, we are ready to analyze the EPA. We will use a step-by-step approach to verify the different components of the EPA. The electric field plots of the OI problem without and with singularity treatment are included in Figures 12.15 and 12.16, respectively. Similarly, the electric field plots of the IO problem without and with singularity treatment are included in Figures 12.17 and 12.18, respectively. In all the plots the extinction principle has been maintained verifying the accuracy of the properly scaled formulation. The VV bistatic RCS plot of the EPA compared against the EFIE is included in Figure 12.19. We find that the RCS plots of the two cases are almost indistinguishable, which implies the accuracy of the OI and IO operators. Therefore, it is the core current solver (CS) portion (e.g., the EFIE, the PMCHWT etc solvers) that will determine the overall accuracy of the EPA algorithm. The bistatic RCS plots from EPA scheme using spherical and cubical ES compared with that from the EFIE are given in Figure 12.20. Again, the EPA using both the spherical and cubical ESs produces almost identical results and agrees very well except at $\theta_{sca} = 90^\circ$. This result, however, is predictable. The ideal bistatic RCS at $\theta_{sca} = 90^\circ$ should be infinitesimally small for a PEC strip of zero thickness.

Figure 12.15: Total electric field (dB) plot from the outside-in (OI) problem with no singularity treatment.
Figure 12.16: Total electric field (dB) plot from the outside-in (OI) problem with singularity treatment.

Figure 12.17: Total electric field (dB) plot from the inside-out (IO) problem with no singularity treatment.

Now, let us consider the EPA with the translation operator \( i.e., \) the multiple
Figure 12.18: Total electric field (dB) plot from the inside-out (IO) problem with singularity treatment.

Figure 12.19: The vertical-vertical (VV) bistatic RCS of a PEC sphere of radius 0.1λ computed using the EPA and compared against the EFIE.

scatterer case) with proper scaling. We consider two tiny PEC strips (each of the dimensions 0.2λ × 0.1λ) in two cases. In both the cases, we use spherical ES. In
Figure 12.20: The vertical-vertical (VV) bistatic RCS of a thin PEC strip \((0.2\lambda \times 0.1\lambda)\) computed using the EPA (spherical and cubical ES) and compared against the EFIE.

Figure 12.21: Spherical ES and object mesh of two thin PEC strips \((0.2\lambda \times 0.1\lambda)\) with the ESs kept apart.

The first case, the ESs are apart, while in the second case, they are touching each other. The mesh configurations are shown in Figures 12.21 and 12.22, respectively.
Figure 12.22: Spherical ES and object mesh of two thin PEC strips \((0.2\lambda \times 0.1\lambda)\) with the ESs touching each other.

Figure 12.23: The vertical-vertical (VV) bistatic RCS of two thin PEC strips \((0.2\lambda \times 0.1\lambda)\) enclosed by two spherical ESs (kept apart) computed using the EPA and compared against the EFIE.

The VV bistatic RCS plots are given in Figures 12.23 and 12.24, respectively. Again, there is a very good agreement (despite using quite a coarse meshing) with the EFIE.
Figure 12.24: The vertical-vertical (VV) bistatic RCS of two thin PEC strips (0.2\(\lambda\) \(\times\) 0.1\(\lambda\)) enclosed by two spherical ESs (touching each other) computed using the EPA and compared against the EFIE.

result, underscoring the accuracy of the scaled translation operators derived. Then, we consider two PEC spheres of radius 0.15\(\lambda\), each enclosed in a spherical ES of radius 0.2\(\lambda\). The meshing configuration is shown in Figure 12.25. Each sphere (ES or object) has 209 nodes and 414 faces. The VV bistatic RCS plot appears in Figure 12.26. Again, despite using a coarse meshing configuration, the results from the EPA and EFIE agree very well.

As the final example, we consider a thin PEC strip with the dimensions 0.6\(\lambda\) \(\times\) 0.1\(\lambda\). For this case we apply the EPA with the tap basis. It has been observed that if the tap bases are assigned so that the \(obj_1\) and \(obj_2\) (please refer to Figure 11.1) are very close to that portion of the encasing ES straddling the object, numerical accuracy drops and that gives rise to significant error in computation. Keeping this very important nuance in mind, we assign the ESs accordingly. The meshing of the ESs appears in Figure 12.27. In this figure, the mesh in red indicates the tap. The VV bistatic RCS of the object from the EPA using the tap basis appears in Figure 12.28. As a means to compare the performance of the EPA with the EFIE, the VV bistatic RCS from the EFIE is also included and, as expected, a very good agreement between the results from the EPA with the tap basis and the EFIE was observed. This substantiates the accuracy of the tap formulation developed in Chapter 11.
Figure 12.25: Spherical ES of radius $0.2\lambda$ and object mesh of two PEC spheres (each of radius $0.15\lambda$) with the ESs kept apart.

Figure 12.26: The vertical-vertical (VV) bistatic RCS of two PEC spheres (each of radius $0.15\lambda$) enclosed by two spherical ESs of radius $0.2\lambda$ (kept apart) computed using the EPA and compared against the EFIE.
Figure 12.27: Meshing of the cubical ESs and the thin PEC object of dimensions $0.6\lambda \times 0.1\lambda$ partitioned into two strips and the tap (the transparent mesh in red connecting the strips with blue mesh and green background).

Figure 12.28: The vertical-vertical (VV) bistatic RCS of the thin PEC object of dimensions $0.6\lambda \times 0.1\lambda$ computed using the EPA with the tap basis and compared against the EFIE.
REFERENCES


[29] T. Xia, Center for Computational Electromagnetics Lab (CCEML), Univ. of Illinois at Urbana-Champaign, Urbana, IL, private communication, Jul. 2013.


