MODAL INTERACTIONS AND TARGETED ENERGY TRANSFERS IN LAMINAR VORTEX-INDUCED VIBRATIONS OF A RIGID CYLINDER WITH STRONGLY NONLINEAR INTERNAL ATTACHMENTS

BY

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DISsertation

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We study the effect of coupling an essentially nonlinear element to a sprung rigid circular cylinder undergoing “vortex-induced vibration” (VIV) in an incompressible flow. The essentially nonlinear device is termed a “nonlinear energy sink” (NES); in this work we use two configurations of NES: translational NES and rotational NES. For the translational NES, consisting of a mass, a linear damper, and an essentially nonlinear spring, the NES mass is constrained to move perpendicular to the mean flow. For the rotational case, the NES mass is constrained to rotate at a fixed radius about the oscillating center of the cylinder. Using a variational multiscale residual-based stabilized finite-element method, we consider the intermediate Reynolds number ($Re$) regime $20 \leq Re \leq 120$, with the cylinder motion constrained to be perpendicular to the mean flow. The nonlinear interaction of the NES and flow via the rigid body motion of the cylinder leads to several new response regimes of the coupled system of flow-cylinder-NES.

The translational NES promotes nearly one-way transfer of energy to itself from the primary structure (the cylinder) and the flow, resulting in reduction of the amplitude of the limit-cycle oscillation (LCO) by as much as 75%, depending on the parameters characterizing the NES. Various mechanisms of VIV suppression by the NES are discussed. A reduced-order model (ROM) based on a single-degree-of-freedom (DOF) self-excited oscillator is developed to approximate the limit-cycle oscillation of the cylinder undergoing VIV. This self-excited oscillator models the interaction of the flow and the cylinder and, in principle, is similar to other phenomenological ROMs considered in the literature. Then, a coupled two-DOF reduced-order model for the system with the internal NES is constructed by coupling the single-DOF NES to the single-DOF self-excited oscillator. Hence, the infinite-dimensional system of flow-cylinder-NES is reduced to a two-DOF model. We examine carefully the range of system parameters where the coupled ROM is valid. The two targeted energy transfer mechanisms responsible for passive VIV suppression that are observed in the finite-element computations are fully reproduced using the two-DOF reduced-order model within the range of validity of the ROM. This reduction of the dynamics to a tractable low-dimensional reduced-order model facilitates the approximate
analysis of the underlying dynamics and provides the basis for predictive design of the NES for VIV suppression. Two other approaches of model reduction to obtain a more advanced ROM that can be predictive were also considered, the first based on “proper orthogonal decomposition” (POD), and the second based on dynamics of the pressure around the cylinder.

We show that, besides passive VIV suppression, the rotational NES can also lead to a flow state that is qualitatively different from the usual Kármán vortex street, with the length and width of the attached vortices significantly altered, and the amplitude and frequency of the lift force and cylinder displacement significantly modified. In fact, our finding is that the nonlinear action of the rotational NES can drastically alter the wake structure downstream, which indicates that the internal NES can affect the external flow, even though no direct contact between the NES and the flow exists. We also explore the dependence of the critical $Re$ for the Hopf bifurcation (from steady, symmetric flow past a motionless cylinder to oscillatory, asymmetric flow past a moving cylinder) on the stiffness of the sprung cylinder, and discuss the effect of a rotational NES on the bifurcation diagram. In addition, we demonstrate the existence of multiple long-time solutions of the Navier-Stokes equation in the presence of a rotational NES. Finally, we discuss an approximate analytical approach to explain some of the numerical findings, which provides good agreement with our computational results.

This work provides a first study of flow-structure interaction of a bluff body possessing a strongly nonlinear internal attachment, and examines the interesting nonlinear dynamic phenomena that exist in such a system. The underlying nonlinear dynamics that govern these phenomena is passive targeted energy transfer from the flow and the bluff body to the NES, which introduces new interesting dynamics with no counterparts in analogous linear settings. As such, this work can be regarded as a contribution in the new field of constructive use of intentional strong nonlinearity in mechanics and design.
To my beloved parents and Dipti, for their love and support.
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CHAPTER 1
INTRODUCTION

A bluff-body flow is one in which pressure drag is much larger than viscous drag. Examples include flows past chimneys, bridges, tube banks, drill strings, as well as flow past a stalled airfoil under high angle of attack. Flow separation aft of a bluff body can result in alternate shedding of vortices from opposite sides of the body, thus generating an oscillating lift force on the body. An elastically supported bluff body can be excited to undergo large amplitude “vortex-induced vibration” (VIV) when the natural frequency of the body is close to the oscillating frequency of the lift force. VIV also increases the drag force on the body as compared to a motionless body. The large amplitude and low frequency of VIV can lead to fatigue failure. Commonly encountered bluff bodies in engineering applications can be idealized as cylindrical structures with circular or rectangular cross-sections. The geometrically simplest case of bluff-body vortex-induced vibration involves “cross-flow” past a circular cylinder. The circular cylinder is an iconic representative problem of the flow past a bluff body and has been the subject of investigations for several decades. VIV of a rigid circular cylinder and its passive suppression is of practical significance for many applications involving bluff bodies. A brief summary of the relevant literature is presented in the following section, after which we describe the motivation and scope of the current work.

1.1 Vortex shedding past a motionless cylinder

Though relatively simple, the flow past a motionless cylinder provides valuable information about the physical phenomenon that is relevant to VIV. At sufficiently small values of the Reynolds number $Re = U_0 D/\nu$, where $U_0$ is the
free-stream speed, \( D \) is the cylinder diameter, and \( \nu \) is the kinematic viscosity, the flow past a fixed rigid cylinder is steady, two-dimensional, and symmetric about the diametral plane parallel to the mean flow, with no lift force perpendicular to the direction of the mean flow. Beyond about \( Re = 47 \), the flow becomes time-periodic and a series of alternating vortices is observed in the wake, leading to a time-periodic lift force and formation of the Kármán vortex street. The flow remains two-dimensional up to about \( Re = 175 \), beyond which a spanwise periodicity appears. At still higher \( Re \), the flow becomes temporally chaotic and ultimately turbulent. In spite of several complex transitions in the flow regimes, the flow past a motionless cylinder is governed by a single robust parameter known as the Strouhal number \( St = f_s D/U_0 \), where \( f_s \) is the characteristic shedding frequency.

1.2 Forced and free vortex-induced vibration

The study of a cylinder moving with prescribed motion in a flow field is a step towards understanding VIV. The motion of the cylinder introduces a forcing frequency to the system in addition to the natural Strouhal frequency of the flow. The early experimental study by Bishop and Hassan (1964) at around \( Re \approx 6000 \) showed that the interaction of these frequencies leads to several complex responses near the forcing frequency and its harmonics; they also showed the possibility of hysteretic lock-in behavior. It is found that in the case of lock-in response the vortex shedding is controlled by the structure, in the sense that the shedding frequency deviates from the Strouhal frequency and approaches the forcing frequency. The lower \( Re \) behavior of lock-in in forced vibration was studied experimentally by Koopmann (1967) for \( Re < 300 \) and in recent computational work by Placzek et al. (2009) at \( Re = 100 \).

Self-excited VIV is a complicated problem compared to the motionless cylinder or a cylinder undergoing prescribed oscillation, due to intrinsic nonlinear coupling of the fluid and structure. When the cylinder is not held fixed, the time-dependent lift force associated with the shedding of alternating vortices
leads to VIV, as reviewed by Bearman (1984), and more recently by Sarpkaya (2004) and Williamson and Govardhan (2004).

In the fully-coupled problem of flow past an elastically supported cylinder, there are three interacting frequencies. The Strouhal frequency \( St \) is the characteristic vortex shedding frequency of a fixed cylinder, depending only on \( Re \). The second frequency is the natural frequency \( f_n \) of the structure defined by its mass and stiffness. The third frequency \( f_{ex} \) or \( f_{CL} \) is the excitation frequency in forced oscillation or the lift force frequency in the case of VIV. When \( f_n \) or \( f_{ex} \) is close to \( St \), the flow is slaved to the structural vibration, and the response frequency is either at \( f_n \) or \( f_{ex} \) or close to them; such a scenario is called lock-in or synchronization.

1.3 Low-dimensional phenomenological models

When detailed experimental data and detailed computational results are not available, for instance in quick design iterations and for systems with several design parameters, “reduced-order models” (ROM) can be very useful. The quest for an accurate phenomenological ROM for VIV dates back to 1970s; the majority of the initial approaches to development of a ROM involved modeling the phenomenon as a wake-body coupled oscillator. Such a phenomenological reduced-order model consists of a wake oscillator representing the dynamics of an appropriate flow variable, namely the lift coefficient or the velocity in the near wake, and a structural oscillator. The central idea for the wake oscillator models is that the dynamics of the wake can be described using a Van der Pol-type of oscillator that models the self-sustained oscillations. The major challenge in developing wake-oscillator models is arriving at an appropriate form of coupling between the wake oscillator and the structural oscillator. One of the earliest and best-known wake-oscillator models was proposed by Hartlen and Currie (1970), based on the Rayleigh oscillator for the wake with the lift coefficient as the flow variable; coupling was assumed proportional to the cylinder velocity. The model predicted lock-in behavior quite accurately, but was not
able to predict the multiple solution branches observed experimentally. Several refinements to the model of Hartlen and Currie (1970) were proposed by Skop and Griffin (1973), Skop and Balasubramanian (1997), and Krenk and Nielsen (1999), with use of a Van der Pol oscillator for the wake and coupling terms based on the observed physics.

A relatively recent study by Facchinetti et al. (2004) focused exclusively on various possible approaches to the coupling between the wake and body oscillators. Facchinetti et al. (2004) noted that the inertial coupling between the wake and body oscillators represents the phenomenon more precisely as compared to either velocity or displacement coupling. Several variants of such wake-body oscillator models have been summarized in the review articles by Parkinson (1989) and Gabbai and Benaroya (2005). Most of these ROMs were developed based on estimating the parameters of the wake oscillator using experimental data in a higher Re range, on the order of \( Re = 10000 \). As discussed by Williamson and Govardhan (2004), lower Re VIV is qualitatively different than higher Re VIV, thus limiting the applicability of the previously discussed wake-body oscillator models for the case of \( Re = 100 \). Another limitation in the existing ROMs is that they are based on an assumed oscillator for the flow variable (Rayleigh, Van der Pol, Stuart-Landau, etc). Experimental or computational data is used to determine the parameters and coupling between the wake and structural oscillators, but the form of the flow-structure interaction nonlinearity may not conform to the nonlinear characteristic inherent in these assumed wake oscillators. A detailed study of some of the relevant phenomenological models and their comparison with our lower Re VIV computational results has been documented by Christopher (2011).

1.4 Suppression of VIV

In those cases where VIV is undesirable, both active (Skaugset and Larsen (2003)) and passive (Bernitsas and Raghavan (2008); Assi et al. (2010); Owen et al. (2001); Dong et al. (2008)) approaches to its suppression have been con-
sidered. Passive control of the wake is one of the means by which VIV can be suppressed, since shedding of vortices is the primary driving force responsible for VIV. Tokumaru and Dimotakis (1991) demonstrated passive control of the wake past a stationary cylinder by imposing a harmonic rotary motion to the cylinder. The possibility of controlling vortex shedding from an oscillating cylinder by imposing rotary motion to the cylinder has been discussed by Chen et al. (1993). Another mechanism that has been widely explored for control of the wake past a stationary cylinder is based on placing a small secondary control cylinder in the wake of the primary cylinder. In an experimental study, Strykowski and Sreenivasan (1990) demonstrated that the presence of a secondary cylinder alters the local stability of the flow by smearing and diffusing concentrated shear layers behind the body.

One of the primary mechanisms for vortex shedding and formation of the Kármán vortex street is the interaction of the shear layers formed on either side of the cylinder. Presumably, one would expect to mitigate the vortex shedding and hence the VIV by reducing the interactions of the shear layers. Use of a splitter plate is a well-known approach to reduce the interaction of the shear layers. For a stationary cylinder, an experimental study by Roshko (1954b) showed that, with appropriate choice of the splitter plate dimensions and its placement aft of the cylinder, the periodic vortex shedding of vortices can be completely eliminated. However, the passive wake control approaches listed above and many more discussed in the literature face additional challenges and are rendered ineffective when the cylinder is allowed to move freely in response to the oscillating lift force generated by vortex shedding. Passive suppression of “Wake Induced Vibration” (WIV) of a flexibly supported cylinder located in the wake of a stationary cylinder has been discussed by Assi et al. (2010), who used a splitter plate with a rotational DOF to reduce the interactions between the shear layers. In a related approach to flow control to suppress VIV, Galvao et al. (2008) used several hydrofoils to force the flow pattern around the cylinder to resemble that of potential flow, with the aim of reducing the drag on the cylinder, as well as VIV amplitude.
1.5 Motivation for nonlinear VIV suppression using a high-fidelity computational model

To date, active approaches to suppression of VIV either suffer from what, in applications, would be excessive power requirements, or require suction and blowing involving internal “plumbing” within the cylinder, and all passive design approaches involve addition of appendages (e.g., shrouds, helical appendages, freely rotating plates) to the cylindrical geometric boundary between the fluid and structure. These appendages can increase or decrease the drag, and can also lead to structural issues of their own. In this work, we discuss use of a different approach to passive suppression of VIV for a rigid cylinder. This approach, known as “targeted energy transfer” (TET), (Vakakis et al. (2008)), involves attaching an essentially nonlinear element (a “nonlinear energy sink,” or NES) to a primary structure in order to promote nearly one-way transfer of energy from the latter to the NES. By “essential nonlinearity” we mean a nonlinear force-response characteristic that lacks a linear part, so the resulting dynamics cannot be linearized. The previous applications of TET to suppress limit-cycle oscillations are discussed in the works of Lee et al. (2007a), Lee et al. (2007b), and Gendelman et al. (2010). For a 2-DOF model of a generic transport wing in a simple model of subsonic flow, Gendelman et al. (2010) showed that aeroelastic flutter, at speeds significantly in excess of the nominal critical flutter speed, could be greatly suppressed or completely eliminated by a single-DOF NES. Numerous applications of TET in other fields have been considered, including vibration and shock mitigation of mechanical components (Georgiadis et al. (2005); Wierschem et al. (2012); AL-Shudeifat et al. (2013)), blast and seismic mitigation (Nucera et al. (2007), Nucera et al. (2010)), acoustic isolation (Bellet et al. (2010)), and drill-string instability suppression (Vigué et al. (2009)).

In conformance with what would seem to be the simplest implementation for marine risers, tension legs, and other hollow-cylinder applications, we consider the case in which the NES is located within the interior of the cylindrical domain whose outer boundary is in contact with the flow. In this case, the only way in
which the VIV suppression system can affect the flow is through its alteration of the rigid-body motion of the cylindrical boundary. Two configurations of NES that will be used in this study are the translational NES and rotational NES. The translational NES configuration (Lee et al. (2007a)) is comprised of a small mass relative to the cylinder, a damping element to dissipate energy and an essentially nonlinear spring. A more recent configuration of NES based on a rotating mass has been proposed by Gendelman et al. (2012), wherein TET is realized based on a strongly nonlinear inertial coupling between the NES and the primary structure. The efficacy of both these configurations in suppressing LCO of the cylinder and the effect of TET on the flow will be studied.

While there are key differences (Anagnostopoulos and Bearman (1992)) between VIV in the laminar regime and the turbulent regime of greater interest in applications, the flow that we consider, at $Re = 100$, is of interest for three main reasons. First, the $Re = 100$ flow past a rigid circular cylinder restrained by a linear spring and constrained to move perpendicularly to the mean flow has become a very popular testbed for investigating free response in VIV. Beyond the reviews cited above, recent work at $Re = 100$ includes the discovery by Singh and Mittal (2005) of a hysteretic jump at the high-$Re$ end of the lock-in region, a study of blockage effects by Prasanth and Mittal (2008), and additional studies by Prasanth and Mittal (2009), Placzek et al. (2009), and Bahmani and Akbari (2010). Second, the flow at this $Re$ is laminar and two-dimensional, and hence can be readily simulated in order to perform detailed parametric studies without concern for issues such as unresolved scales and the attendant unmodeled dynamics associated with simulations of three-dimensional turbulent flow. Third, as shown by (Roshko (1954a)), one of the notable features of the flow past a fixed cylinder is that the Strouhal number $St = f_s D/U_0$ (a dimensionless shedding frequency where $f_s$ is a properly defined dimensional shedding frequency) maintains a nearly constant value ($St = 0.19 \pm 0.02$) over $10^2 \leq Re \leq 10^4$; i.e., from the laminar two-dimensional regime well into the turbulent regime. As discussed by Roshko (1954a) and Williamson (1996), significant aspects of vortex shedding by a cylinder are qualitatively similar over wide ranges of $Re$. 

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including at least part of the laminar regime. Thus, there is some reason to believe that methods that suppress VIV at \( Re = 100 \) will be useful, or can be adapted for use at much higher \( Re \).

### 1.6 Outline of the dissertation

The main objective of this work is to study the effect of introducing an essentially nonlinear device into a rigid circular cylinder undergoing VIV in the laminar regime. We use a high-fidelity variational multiscale residual-based stabilized finite-element method to solve the coupled flow-structure problem. We perform a coarse parametric survey to identify desirable combinations of the NES parameters, and analyze the response to understand the effect of the NES.

In chapter 2, we provide a detailed description of the model. A brief description of the variational multiscale residual-based stabilized finite-element formulation is provided Appendix A. The computational model is validated by comparing the current model results with standard results available in the literature. Flow past a stationary cylinder, forced vibration, VIV at \( Re = 100 \) and the laminar lock-in regime are confirmed before adding the NES to the cylinder.

The effect of an ungrounded translational NES in passive suppression of VIV is studied in chapter 3; the essential nonlinearity in this case is due to cubic stiffness of the NES. The two suppression mechanisms that were found for several combinations of NES parameters are discussed. The effect of the NES on the lock-in regime around \( Re = 100 \) is studied. The effectiveness of the NES is compared with an equivalent linear absorber.

In chapter 4, the effect of a rotational NES with essential nonlinearity due to inertia coupling is discussed. Apart from passive suppression, the rotational NES is found to alter the wake structure significantly. An approximate analytical model is developed to understand the effect of the internal device on the wake structure. The effect of the NES on the flow field is studied by performing
a proper orthogonal decomposition (POD) on the solution.

Chapter 5 focuses on developing a low-dimensional model for the system with translational NES. The lowest-order model based on computational results obtained in chapter 3 is developed. The analysis provided proves the low-dimensionality of the problem, and also elicits the underlying complicated nonlinear interaction which makes it very hard to generalize the lowest-order model to be a predictive tool. In chapter 6, two more attempts at model reduction are presented; however, these models need further investigation and development.

The effect of spring stiffness on VIV in the sub-critical $Re$ is studied in chapter 7 to obtain the bifurcation diagram in the frequency-Reynolds number plane. The possibility of lock-in response, and modes of VIV in sub-critical $Re$ are also discussed. The effect of introducing a rotational NES on the onset of instability of a sprung cylinder is also examined in the same chapter. Introducing the NES results in new response regimes which are absent otherwise; these are discussed in chapter 8.

Conclusions and suggestions for future work are presented in chapter 9.
CHAPTER 2

DESCRIPTION AND VALIDATION OF THE COMPUTATIONAL MODEL

In this chapter we discuss the computational model without including the NES, in order to validate our computational model before embarking on the study of passive VIV suppression using the NES. We discuss the governing equations of the problem and state the scope of physical parameters used in the following chapters. We perform several benchmark computations to validate our model against standard results available in the literature. Starting with the simplest case of flow past a stationary cylinder, we move towards self-excited VIV, verifying at each stage that our computations are accurate.

2.1 Physical model and governing equation

We consider two-dimensional flow past a circular cylinder of diameter $D$, with steady and uniform free-stream velocity $U_0 e_x$. We take the flow to be governed by the Navier-Stokes equations for a constant-density, constant-viscosity fluid in an inertial reference frame, which we write in dimensionless form as

\begin{align}
\frac{\partial \mathbf{v}^*}{\partial \tau} + \mathbf{v}^* \cdot \nabla^* \mathbf{v}^* &= -\nabla^* p^* + \frac{1}{Re} \nabla^* \mathbf{v}^* \\
\nabla^* \cdot \mathbf{v}^* &= 0
\end{align}

where the dimensionless time is defined by $\tau = tU_0/D$, and we have scaled the length, velocity, and pressure by $D$, $U_0$, and $\rho f U_0^2/2$ respectively. Here, $(\cdot)^*$ denotes a dimensionless quantity.

The cylinder is constrained to move in the $y$–direction, perpendicular to the direction of the mean flow. The dimensional equation of motion of the cylinder
is
\[ \dot{M}_{cyl} \frac{d^2 y_1}{dt^2} + \dot{K}_{cyl} y_1 = \dot{F}_L \]  

(2.2)

where \( \dot{M}_{cyl} \) and \( \dot{K}_{cyl} \) are the mass per unit length and stiffness per unit length of the cylinder, respectively, and \( \dot{F}_L \) is the dimensional lift force per unit length. Using the same scaling for time and length as discussed above, we obtain the dimensionless equation

\[ \frac{d^2 Y_1}{d\tau^2} + \omega_0^2 Y_1 = \frac{2C_L}{\pi m^*} \]  

(2.3)

for the cylinder motion, where \( \omega_0^* = \dot{K}_{cyl} D^2/(\dot{M}_{cyl} U_0^2) \) is the dimensionless natural frequency of the spring-mounted cylinder, the parameter \( m^* = \rho_b/\rho_f \) is the ratio of cylinder density to fluid density, and \( C_L \) is a time-dependent lift coefficient defined by \( C_L = 2\dot{F}_L/(\rho_f U_0^2 D) \), which drives the cylinder motion based on the asymmetry of vortex shedding from alternate sides of the cylinder.

In terms of dimensionless flow quantities, we write \( C_L \) and the drag coefficient \( C_D \) as

\[ C_L = \int_{\Gamma_{cyl}(t)} (\sigma^* \cdot n) \cdot e_y d\Gamma \]  

(2.4a)

\[ C_D = \int_{\Gamma_{cyl}(t)} (\sigma^* \cdot n) \cdot e_x d\Gamma \]  

(2.4b)

where \( \sigma^* \) is the dimensionless stress tensor, \( n \) is the unit outward normal to the surface of the cylinder, and \( \Gamma_{cyl}(t) \) is the (moving) cylinder boundary.

The stiffness of the oscillator is chosen so that the dimensionless natural frequency \( f_n^* = \frac{\omega_0^*}{2\pi} = \frac{1}{2\pi} \sqrt{\dot{K}_{cyl}/\dot{M}_{cyl} D/U_0} \) is close to the natural shedding frequency of a fixed stationary cylinder at \( Re = 100 \), which is \( St = 0.168 \) from both our computation of flow past the stationary cylinder and previous results.

In all of our computational studies, we restrict the cylinder motion to be single-degree of freedom, oscillating in a direction perpendicular to the mean flow direction. Although, we have not studied the effect of other degrees of freedom on the results, we expect that the effects in our system would be small, based on the results available in the literature. Jauvtis and Williamson (2004) carried out a two-degree of freedom VIV study and showed that the effect of the in-line
degree of freedom is insignificant for \( m^* > 6 \); however, for \( m^* < 6 \) dramatic changes in the response have been reported. Bearing this in mind, we fix the mass ratio \( m^* \) to be 10 for all our studies, a value chosen to validate the results by comparison to previous work by Prasanth and Mittal (2009). We note that this density ratio is comparable to the steel/water density ratio.

The Navier-Stokes equations are discretized on a time-dependent domain, and their solution, coupled to the rigid-body dynamics of the cylinder, is obtained using the variational multiscale finite-element formulation developed by Calderer and Masud (2010), relevant details of which can be found in Appendix A and the references cited therein. The computational domain is the doubly-connected area shown schematically in Fig. 2.1, with the circular boundary representing the cylinder. The discretization of the computational domain is described in Appendix B. We assume the flow to be two-dimensional and laminar; hence, the computations performed here are restricted to \( Re < 180 \).

The temporal discretization is based on the generalized-\( \alpha \) method for first-order equations (Jansen et al. (2000)), in which the parameter \( \rho_\infty \) has been set equal to 0. The generalized-\( \alpha \) method is an implicit second-order accurate scheme. The coupled structural equations are solved using the generalized-\( \alpha \) method for second-order systems described by Chung and Hulbert (1993), with the same zero value of \( \rho_\infty \) used for solving the fluid equations. All the computational
results discussed in this dissertation were obtained with a time step size of $0.025U_0/D$, unless stated otherwise.

The coupled system of equations (2.3) and the discretized equations for the fluid flow, used to evaluate the lift force, are solved in a staggered fashion. In the solution of the fluid-structure interaction problem, it was assumed that there would be no need for equilibrating iterations in each time step, since the structure considered here is rigid. This assumption was verified by computations in which the fluid and structural responses at each time step were computed iteratively, with no significant effect on the response as discussed in appendix E. In all the simulations discussed in this chapter, we assume a uniform initial inlet velocity condition for the fluid and a quiescent condition for the cylinder, except as discussed in §2.2.4.

2.2 Validation of computational model

The formulation and code are validated, and the adequacy of the spatial and temporal resolutions are assessed, by computing the flow past a motionless cylinder, the forced vibration of the cylinder at $Re = 100$, the vortex-induced vibration at $Re = 100$, and the synchronization or lock-in range $Re$ in the laminar regime.

2.2.1 Vortex street past a stationary cylinder at $Re = 100$

We first computed the flow past a fixed circular cylinder at $Re = 100$. The $2S$ alternating-vortex mode characteristic of the flow at this $Re$ (Williamson and Govardhan (2004)) is excited by round-off error asymmetric about the mid-plane.

The time histories of the drag and lift coefficients are shown in Figs. 2.2a and 2.2b, respectively. The maximum amplitude of the oscillating lift coefficient is found to be 0.328, and the time-averaged drag coefficient is $<C_D> = 1.36$. The frequency of oscillation of the lift coefficient is equal to the shedding frequency,
Table 2.1: Comparison of results for stationary cylinder. (Values and citations for previous work taken from Shiels, Leonard & Roshko (2001).)

<table>
<thead>
<tr>
<th>Source</th>
<th>St</th>
<th>$&lt;C_D&gt;$</th>
<th>max($C_L$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stansby &amp; Slaouti (1993)</td>
<td>0.166</td>
<td>1.32</td>
<td>0.35</td>
</tr>
<tr>
<td>Anagnostopoulos (1994)</td>
<td>0.167</td>
<td>1.20</td>
<td>0.27</td>
</tr>
<tr>
<td>Henderson (1995)</td>
<td>0.166 *</td>
<td>1.35</td>
<td>0.33 *</td>
</tr>
<tr>
<td>Zhou, So &amp; Lam (1999)</td>
<td>0.162</td>
<td>1.48</td>
<td>0.31</td>
</tr>
<tr>
<td>Shiels, Leonard &amp; Roshko (2001)</td>
<td>0.167</td>
<td>1.33</td>
<td>0.30</td>
</tr>
<tr>
<td>present work</td>
<td>0.168</td>
<td>1.36</td>
<td>0.328</td>
</tr>
</tbody>
</table>

* Unpublished values quoted by Shiels, Leonard & Roshko (2001)

and the frequency of the drag coefficient is twice the shedding frequency. The Strouhal number, maximum amplitude of the lift coefficient, and time-averaged drag coefficient are shown in Table 2.1, along with computational results of previous authors, for which the comparison is excellent. Adequacy of the spatial resolution was verified by computing the flow past the fixed cylinder with much higher spatial resolution (with 196 elements representing the cylindrical boundary and with a total of 40272 nodes for the same size of computational domain). For this much finer mesh, the maximum lift coefficient was found to be 2.75% lower, and the mean drag coefficient was found to be 1.15% lower.

The spanwise vorticity ($\Omega_z$) plot is shown in Fig. 2.3: the Kármán vortex street in the wake is shown in 2.3a. The wake structure close to the cylinder shown in Figs. 2.3b through 2.3e corresponds to four equally separated instants of time in one shedding cycle. The pressure on the surface of the cylinder over a period of the shedding cycle is shown in Fig. 2.4 as the coefficient of pressure $C_p = 2(p - p_0)/(\rho_f U_0^2)$. The wake structure and pressure surface plots serve as useful tools to study the effect of cylinder motion and NES on the flow field in the following chapters.
(a) Drag coefficient

(b) Lift coefficient

Figure 2.2: Force coefficients for a stationary cylinder at $Re = 100$.

Figure 2.3: (a) Wake structure past fixed cylinder at $Re = 100$ corresponding to $t = 0$; Spanwise vorticity close to cylinder, (b) $t = 0$, (c) $t = T/4$, (d) $t = T/2$, and (d) $t = 3T/4$. 

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Figure 2.4: Pressure around the cylinder for one oscillation cycle of lift coefficient.

2.2.2 Laminar regime $St$-$Re$ relation

The Strouhal number, $St = f_s D/U_0$ is the characteristic frequency for vortex shedding by a motionless cylinder, and depends solely on $Re$. The value of $St$ is found to remain close to 0.2 over a large range, $10^2 \leq Re \leq 10^4$. However, at lower $Re$, $St$ varies considerably. Starting with a value of around 0.11 at the first Hopf bifurcation, approximately at $Re \approx 47$, it gradually increases to 0.2. By compiling various experimental and computational data, Roshko (1954a) provided best-fit equations for the variation of $St$ with $Re$ for different flow regimes. For the two-dimensional laminar vortex-shedding regime ($50 \leq Re \leq 150$) the relation is given by equation (2.5).

$$St = 0.212 - \frac{4.5}{Re}$$ (2.5)

Though there is general agreement on the trend of $St$ vs $Re$ reported by several researchers, the data available also show considerable scatter. Typically, $St$ is computed based on the oscillating frequency of the lift force or by measuring the velocity in the near wake region, both of which exhibit significant sensitivity to the experimental parameters. In the case of computational studies, the domain size and spatial resolution play a critical role in the computed lift force and hence have significant influence on the value of $St$. In our computational work,
we wish to study the VIV and its passive suppression by NES with a cylinder that is in resonant interaction with the flow. For this purpose, we will tune the natural frequency of our spring cylinder to be close or coincident with the $St$ corresponding to the $Re$ of interest. In order to make sure we have the correct resonant interaction, we have computed $St$ over the range $47 \leq Re \leq 120$. At each $Re$, we have computed the flow past a stationary cylinder for a sufficiently long time to get a fully-developed LCO in the lift coefficient, and to perform spectral analysis to compute its frequency $f_{CL}$, which is used to compute the Strouhal number, $St = \frac{f_{CL} D}{U_0}$. The variation of $St$ with $Re$ based on our computational model is shown in Fig. 2.5a. Comparing with the summary of $St$-$Re$ available in the literature, shown in Fig. 2.5b (taken from Williamson (1996)), we observe good agreement of the current results with the results in literature. The best fit curve of the form $St = a + \frac{b}{Re}$ to our computational results gives us the relation

$$St = 0.212 - \frac{4.351}{Re}$$

Comparison of equations (2.5) and (2.6) reinforces the good agreement in frequency predicted by our computational model with the data available in the literature.
2.2.3 Vortex shedding from cylinder undergoing forced vibration

In this section, we discuss the effect of motion of the cylinder on the vortex shedding and fluid forces by prescribing sinusoidal motion of the cylinder. Since in this case the cylinder motion is known a priori, this is a relatively simple computation, compared to the fully coupled problem of VIV, and serves as a good validation step.

The forced oscillation of the cylinder in flow has been discussed extensively by Bishop and Hassan (1964) in their experimental study at \( Re \approx 6000 \), where they identified the interactions of forcing frequency with lift force frequency, synchronization, and hysteresis. Though their work was for higher \( Re \), the basic characteristics are found to be relevant at lower \( Re \). Due to forced oscillation of the cylinder, there are two frequencies in the system: the characteristic shedding frequency \( f_s \) and frequency of forcing \( f_{ex} \). The response of the system is dependent on the relation between these two frequencies.

At low \( Re \), the experimental study by Koopmann (1967) for \( Re < 300 \) revealed several important aspects of synchronization in the laminar flow regime. Synchronization was found to be dependent on the frequency ratio \( f^* = f_s / f_{ex} \) and amplitude of the cylinder motion. There is a threshold amplitude of the cylinder motion below which synchronization is not possible for any value of \( f^* \). For small amplitudes above the threshold, synchronization occurs over a narrow range of \( f^* \) on either side of \( f^* = 1 \). As the amplitude of the cylinder increases, the synchronization is found to occur over a wider range of \( f^* \). These observations in the laminar regime were confirmed by Placzek et al. (2009) in their computational study at \( Re = 100 \). In order to validate our computational model for the case of forced oscillation of the cylinder, we perform forced vibration computations with parameters identical to those of Placzek et al. (2009) and compare the results.
Figure 2.6: Lift coefficient and its frequency content: (a,b) $f^* = 0.5$, (c,d) $f^* = 0.9$, (e,f) $f^* = 1.1$, and (g,h) $f^* = 1.5$. 

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The cylinder motion is prescribed as

\[ y_p(t) = A \sin (f^* \omega_s t). \]  (2.7)

where \( \omega_s = 2\pi f_s \) is the natural shedding frequency of the stationary cylinder in a flow with \( Re = 100 \). The dimensionless amplitude of the cylinder motion is chosen as \( A = 0.25 \), which is above the threshold amplitude suggested by Koopmann (1967). In the study by Koopmann (1967), for \( Re = 100 \) and \( A = 0.25 \) the frequency ratio of synchronization is in the range of \( 0.85 \leq f^* \leq 1.15 \); we perform a forced vibration study for \( f^* = 0.5, 0.9, 1.1, \) and 1.5, which were the frequency ratios used by Placzek et al. (2009). These frequency ratios represent both the cases for synchronization on either side of \( f^* = 1.0 \) and away from synchronization.

The time history of the lift coefficient and its frequency content are shown in Fig. 2.6. For the cases \( f^* = 0.9 \) and \( f^* = 1.1 \), synchronized response can be observed in Figs. 2.6c and 2.6e, respectively; i.e., there is a single dominant frequency corresponding to the forcing frequency, and vortex shedding deviates from the Strouhal frequency and coincides with the forcing frequency. Hence the frequency content for the synchronized cases shown in Figs. 2.6d and 2.6f show a single dominant frequency corresponding to that of the forcing frequency. For \( f^* = 0.5 \) and \( f^* = 1.5 \), the forcing frequency ratio is out of the synchronization range and, hence, effectively there are two components of frequency interacting. The lift time histories shown in Figs. 2.6a and 2.6g are not sinusoidal and exhibit quasi-periodicity. The dominant frequency for \( f^* = 0.5 \) and \( f^* = 1.5 \) shown in Figs. 2.6b and 2.6h, respectively, corresponds to the Strouhal frequency at \( Re = 100 \). The phase plots for synchronized response shown in Figs. 2.7b and 2.7c are closed curves indicating the periodic nature of the response. For the nonsynchronized responses shown in Figs. 2.7a and 2.7d, the orbits fill the phase space densely, indicating the quasi-periodic nature of the response.

In the experimental study by Bishop and Hassan (1964) the phase difference
Figure 2.7: Phase plot of the lift and prescribed displacement (a) $f^* = 0.5$, (b) $f^* = 0.9$, (c) $f^* = 1.1$, and (d) $f^* = 1.5$
between lift and displacement of the cylinder was found to exhibit a discontinuous jump when the forcing frequency crossed $f^* = 1$, and the jump was also reported to be hysteretic in nature. In our forced oscillation computations at lower $Re$, the phase difference between lift and displacement shown in Figs. 2.8a and 2.8b does not show such a jump. This is consistent with the results reported by Placzek et al. (2009). The spanwise vorticity showing the wake structure near the cylinder for all four cases of excitation frequency ratio $f^*$ are shown in Fig. 2.9, corresponding to the time instant at which the cylinder is at its positive extreme position of excursion. For the case of synchronized response with wake structure shown in Figs. 2.9b and 2.9c, the vorticity distributions are approximately the same and similar to that of the stationary cylinder wake shown in Fig. 2.3. For the case with no synchronization, the wake structures in Figs. 2.9a and 2.9d show markedly different distributions of vorticity close to the cylinder.

2.2.4 Vortex-induced vibration of a spring-mounted cylinder

We simulated VIV by mounting the cylinder on a linear spring, with the motion of the cylinder constrained to be perpendicular to that of the mean flow, and with its equilibrium position on the midline of the domain. The motion of
Figure 2.9: Spanwise vorticity plot corresponding to the time instant at which the cylinder is at its maximum amplitude of \( y_1(t)/D = 0.25 \); (a) \( f^* = 0.5 \), (b) \( f^* = 0.9 \), (c) \( f^* = 1.1 \), and (d) \( f^* = 1.5 \)
Figure 2.10: VIV of the spring-mounted cylinder at $Re = 100$. 
the cylinder under the action of the oscillating lift $C_L$ is governed by equation (2.3). As noted earlier, the sprung cylinder resonates at the vortex shedding frequency of the stationary cylinder. The response of the cylinder, both transient and steady-state, is shown in Fig. 2.10a, with maximum dimensionless oscillation amplitude of 0.49. The time-periodic lift coefficient driving the cylinder is shown in Fig. 2.10b, where it can be observed that at lock-in the lift coefficient amplitude is 0.212, which is lower than its value (0.328) for the stationary cylinder. The drag coefficient in Fig. 2.10c shows the averaged drag to be around 1.85 which is much higher than the stationary cylinder value of 1.36. The wake structure with the cylinder undergoing VIV is shown in Fig. 2.11; the spanwise vorticity is stretched in the cross-stream direction near the cylinder. The wake is quite similar to the Kármán vortex street past a stationary cylinder. Close to the cylinder, the vorticity distribution is shown in Fig. 2.11b through 2.11e, starting with the time instant when the cylinder is at its maximum positive amplitude, followed by three equally spaced time instants spanning one period of oscillation. Some difference in vorticity distribution close to the cylinder can be observed when compared to the stationary cylinder shown in Figs. 2.3b through 2.3e. The formation of opposite signed vorticity is minimized in the case of VIV, due to the cylinder carrying the attached vorticity with it as it moves. As mentioned in §2.1, a staggered approach was employed to couple the fluid and structure in computing the VIV solution shown in Fig. 2.10 with no equilibrating iterations. Performing the fluid-structure equilibrating iterations to compute VIV at $Re = 100$ was found to be insignificant, as discussed in Appendix E.1.

For a fixed value of the cylinder natural frequency $f_n^* = f_n D/U_0 = 0.167$, which is close to the natural shedding frequency at $Re = 100$, lock-in response was observed over a range of $Re$, as shown in Fig. 2.12, with jumps in response at the ends of the lock-in regime. The jump at the high-$Re$ end of the lock-in range is hysteretic. As pointed out by Prasanth and Mittal (2009), hysteresis of VIV in the laminar regime is quite different than that experimentally investigated by Khalak and Williamson (1999) in the range $5000 \leq Re \leq 16000$. We
Figure 2.11: (a) Wake structure past cylinder undergoing VIV at $Re = 100$ corresponding to $t = 0$; Spanwise vorticity close to cylinder, (b) $t = 0$, (c) $t = T/4$, (d) $t = T/2$, and (d) $t = 3T/4$

Figure 2.12: Lock-in response regime of laminar VIV: ■, $Re$ increasing; ×××, $Re$ decreasing.
have verified accurate prediction of the lock-in regime in our model by sweeping $Re$ from 70 at the low end to 140 at the high end. Each simulation at successively higher $Re$ was restarted using initial conditions from the solution at the previous $Re$. The computed lock-in regime and the predicted response therein are shown in Fig. 2.12. The maximum amplitude response ($\max \{y_1/D\} = 0.55$) occurs at $Re = 82.4$, associated with a jump at the low-$Re$ limit of the lock-in range, which extends from $Re = 82.4$ up to 134. The hysteretic jumps near the high end of the $Re$ range occur at $Re = 134$ for a forward $Re$ sweep and at $Re = 123$ for a reverse $Re$ sweep. These results compare well with those of Singh and Mittal (2005), who considered only the case in which motion of the cylinder in both the $x$- and $y$-directions is allowed, and who noted that allowing for “in-line” motion in the $x$-direction has little effect, compared to the case in which vibration is solely in the transverse direction.

### 2.3 Conclusions

We have summarized the computational framework for the system without NES. We examined flow past the motionless cylinder and found that the forces and their frequency are in good agreement with results available in the literature. The forced vibration response at $Re = 100$ is computed to confirm that the proper synchronization behavior is replicated by our computational model. The VIV at $Re = 100$ exhibits large amplitude vibration of the cylinder at $\approx 0.49D$. The synchronization in the laminar regime is computed and found to be in agreement with a similar computational study by Prasanth and Mittal (2009). Through the computations discussed in this chapter, we have confirmed that our computational model is capable of simulating all of the relevant scenarios of laminar flow past both motionless and flexibly supported cylinders accurately. Hence, the computational domain size and spatial and temporal resolutions should be adequate to accurately represent the physical phenomenon around $Re = 100$. Thus, in the following chapters we use this model and incorporate the NES into this system to study its effect on VIV in the laminar flow regime.
In this chapter, we discuss the application of TET to suppress the VIV of a rigid circular cylinder. The single-DOF translational NES with cubic nonlinearity in restoring force is employed to achieve TET. The oscillating lift force that drives the cylinder to LCO is computed using a high fidelity finite-element method as discussed in chapter 2. We first describe the coupled system, including the translational NES, and then summarize the main results of the computational study. We also perform a comparison study with an optimized linear “tuned mass damper” (TMD) and compare the effectiveness of the NES to that of the TMD.

### 3.1 Coupled system with translational NES

The coupled system, with the NES attached to the cylinder, is depicted in Fig. 3.1. The mass of the NES per unit length is taken to be a specified fraction of the cylinder mass per unit length, thereby keeping the sum of the two constant. Hence, the value of the density ratio, $m^* = 10$, is retained, and the frequency of the periodic solution of the coupled system remains close to the natural shedding frequency of the stationary cylinder at $Re = 100$, as it is without the NES. The motion of the coupled system is governed by equations (3.1a) and (3.1b)

\[
(\hat{M}_{cyl} - \hat{M}_{nes}) \frac{d^2 y_1}{dt^2} + \hat{K}_{cyl} y_1 + \hat{C}_{nes} \left( \frac{dy_1}{dt} - \frac{dy_2}{dt} \right) + \hat{K}_{nes} (y_1 - y_2)^3 = \hat{F}_L
\]

\[
(\hat{M}_{nes}) \frac{d^2 y_2}{dt^2} + \hat{C}_{nes} \left( \frac{dy_2}{dt} - \frac{dy_1}{dt} \right) + \hat{K}_{nes} (y_2 - y_1)^3 = 0
\]
wherein the displacement of the NES (as measured by the NES DOF $y_2(t)$) is also constrained to be perpendicular to the mean flow. The resonance interaction of the NES with the cylinder results in transfer of energy from the cylinder to the NES, thereby reducing the amplitude of the cylinder vibration.

Dividing equations (3.1a) and (3.1b) by $M^*=\dot{M}_{cyl} - \dot{M}_{nes}$, nondimensionalizing time and length as for the dimensionless Navier-Stokes equations (2.1a) and (2.1b), and identifying $\epsilon = \dot{M}_{nes}/M^*$, we obtain the dimensionless equations of motion

$$\ddot{Y}_1 + \omega_0^2 Y_1 + \lambda (\dot{Y}_1 - \dot{Y}_2) + \kappa (Y_1 - Y_2)^3 = \frac{\dot{F}_L D}{U_0^2 M^*} \frac{2C_L(1 + \epsilon)}{\pi m^*} \tag{3.2a}$$
$$\epsilon \ddot{Y}_2 + \lambda (\dot{Y}_2 - \dot{Y}_1) + \kappa (Y_2 - Y_1)^3 = 0 \tag{3.2b}$$

where $\dot{\cdot} = d(\cdot)/d\tau$ and $\ddot{\cdot} = d^2(\cdot)/d\tau^2$. The dimensionless parameters characterizing the NES are the mass ratio $\epsilon$, dimensionless damping coefficient $\lambda = D\dot{C}_{nes}/U_0 M^*$, and dimensionless stiffness $\kappa = D^4\dot{K}_{nes}/U_0^2 M^*$. 

Figure 3.1: Schematic of spring-mounted cylinder in flow with NES.
3.2 Passive VIV suppression results

Here, we computationally demonstrate how attachment of a lightweight NES can suppress VIV. As discussed previously, placing the NES in the interior of the cylinder isolates it from direct excitation by the flow, and protects it from the external environment, while avoiding geometric complexity. As shown below, addition of the NES to the fluid-structure interaction problem leads to global changes in the dynamics of the system.

The effectiveness of the NES was studied by varying the three dimensionless NES parameters $\epsilon$, $\lambda$, and $\kappa$. All simulation results are presented for $Re = 100$ and $m^* = 10$. The results for the three sets of NES parameters considered here are representative of many other sets for which computations were performed.

For two of the sets of NES parameters, our computations demonstrate significant reduction of cylinder displacement. For another set, the NES does not appreciably reduce LCO amplitude. These results are discussed and analyzed in this section, according to the type of suppression achieved. The effective suppression mechanisms we find (“mechanism I” and “mechanism II”) are related to those Gendelman et al. (2010) previously discovered for a finite-dimensional aeroelastic oscillator, and the same nomenclature for them is adopted here. The results discussed in this chapter were obtained without fluid-structure equilibrating iterations; the effect of fluid-structure equilibration on both suppression mechanisms are discussed in Appendix E.2.

3.2.1 Suppression Mechanism I

For NES parameters $\epsilon = 0.11$, $\lambda = 0.0283$, and $\kappa = 1.59$, the response of the cylinder and the NES relative displacement are shown in Fig. 3.2a and 3.2b, respectively. This passive nonlinear suppression of cylinder VIV (mechanism I) corresponds to quasiperiodic responses of the cylinder, the NES, and the flow. As shown by Gendelman et al. (2010), suppression mechanism I in their aeroelastic problem gives rise to relaxation oscillations along a slow invariant man-
Figure 3.2: Suppression mechanism I: Time history of response with 78% reduction in RMS value of displacement.
Figure 3.3: Wake structure past the cylinder undergoing VIV with suppression mechanism I at $Re = 100$; Small amplitude (a) $t = 0$, (b) $t = 0$, (c) $t = T/4$, (d) $t = T/2$, and (d) $t = 3T/4$; Large amplitude (a) $t = 0$, (b) $t = 0$, (c) $t = T/4$, (d) $t = T/2$, and (d) $t = 3T/4$

ifold of a reduced-order model of the augmented cylinder-NES system. Their analysis was based on a slow-fast partition of the dynamics; we will perform a similar analysis in chapter 5 and show that mechanism I of VIV suppression is a “strongly modulated response” (SMR). Fig. 3.2a shows that the maximum amplitude of oscillation is approximately $0.18D$, compared to a steady periodic amplitude of $0.49D$ for the cylinder without the NES. Thus, a 78% reduction in the RMS displacement can be attributed to the NES. Such SMR is known to be a very efficient vibration absorption mechanism (Starosvetsky and Gendelman (2008)).

In our parametric search, SMRs were observed for several NES parameters. Several time series are shown in Fig. 3.4, which summarizes realization of SMR in the cylinder for various NES parameters. Based on the computations reported above, and on computations for other sets of NES parameters, it appears that the result shown in Fig. 3.2a is representative of the SMR regime, with the best
Figure 3.4: Time history of systems with SMR for various NES parameters; (a) $\epsilon = 0.03093$, $\lambda = 0.00525$, and $\kappa = 0.05674$; (b) $\epsilon = 0.075269$, $\lambda = 0.010953$, and $\kappa = 0.03551$; (c) $\epsilon = 0.11$, $\lambda = 0.01415$, and $\kappa = 1.22316$; (d) $\epsilon = 0.11$, $\lambda = 0.03395$, and $\kappa = 1.83473$; (e) $\epsilon = 0.25$, $\lambda = 0.01591$, and $\kappa = 2.7521$. 
possible suppression occurring when the mass of the NES is about 10% of that of the primary cylinder and the NES support is relatively stiff (\(\kappa > 1\)). For the smallest value of \(\kappa\) shown in Fig. 3.4a, the mass of the NES also needs to be small in order for the NES to engage in resonance interaction, but the reduction in the cylinder displacement is not appreciable. At the other extreme of very large \(\kappa\) and \(\epsilon\) shown in Fig. 3.4c, the cylinder and NES are not in clear 1:1 resonance interaction with the lift force.

### 3.2.2 Suppression Mechanism II

The response of the cylinder and the NES relative displacement are shown in Fig. 3.5a and 3.5b, respectively, for \(\epsilon = 0.066, \lambda = 0.0217,\) and \(\kappa = 0.704.\) This second mechanism (“mechanism II”) corresponds to periodic responses of the flow, cylinder, and NES, resulting in partial suppression of VIV. In the finite-dimensional aeroelastic context of Gendelman et al. (2010), it was found that this mechanism is realized when the transient dynamics of the augmented system is attracted by a nontrivial stable point on a slow invariant manifold of an appropriate reduced-order model. Hence, it differs qualitatively from mechanism I. Fig. 3.5a shows that after an initial transient, in which oscillations grow to an amplitude of \(0.21D\), the response slowly settles into a suppressed LCO of amplitude \(0.125D\), with weak modulation. The reduction in the RMS value of cylinder displacement is about 74%, compared to the system without the NES.

Results for several sets of parameters that give partially suppressed LCOs are shown in Fig. 3.7. For relatively small mass of the NES, the envelope of the response is modulated as shown in Fig. 3.7a: with increase in mass, the modulation is absent. Results shown in Fig. 3.5a are representative of the “partial suppression” regime, with the best suppression occurring when the mass of the NES is about 6% that of the primary cylinder, and the NES support is softer (\(\kappa < 1\)). For very large mass and stiffness of NES, the response shown in Fig. 3.7e is realized with relatively larger damping.

As mentioned earlier, both of these suppression mechanisms were observed in
Figure 3.5: Suppression mechanism II: Time history of response with 74% reduction in RMS value of displacement.
Figure 3.6: Wake structure past the cylinder with NES under suppression mechanism II; (a) $t = 0$, (b) $t = 0$, (c) $t = T/4$, (d) $t = T/2$, and (d) $t = 3T/4$

Figure 3.7: Time history of systems with suppressed LCO for various NES parameters; (a) $\epsilon = 0.05263$, $\lambda = 0.0268$, and $\kappa = 0.2897$; (b) $\epsilon = 0.04167$, $\lambda = 0.03183$, and $\kappa = 0.344$; (c) $\epsilon = 0.06383$, $\lambda = 0.02709$, and $\kappa = 0.70266$; (d) $\epsilon = 0.06601$, $\lambda = 0.02172$, and $\kappa = 0.7511$; (e) $\epsilon = 0.25$, $\lambda = 0.1591$, and $\kappa = 2.7521$. 
the prior application of an NES to suppress aeroelastic instability (Gendelman et al. (2010)). In that work, a third mode of suppression, namely complete LCO elimination, was also observed. Whether that mode is possible in this infinite-dimensional system is not known. A detailed search for a set of NES parameters giving complete suppression, as well as a detailed study of the dependence of suppression on the three dimensionless NES parameters, is a very resource-intensive task, and was not performed. A total of around 300 sets of parameters were simulated in a coarse survey of the parameters space.

3.2.3 System with less effective NES

To make clear that the effects we observe under suppression mechanisms I and II are not solely attributable to the additional damping introduced into the structural system by the NES (characterized by $\lambda$), we consider here a set of NES parameters for which the reduction in LCO displacement is not significant.

For $\epsilon = 0.02$, $\lambda = 0.0259$, and $\kappa = 1.010$, the displacement history of the cylinder and the NES relative displacement are shown in Fig. 3.8a and 3.8b, respectively, and the corresponding lift coefficient is presented in Fig. 3.9a. We see that the reduction in LCO amplitude is approximately 10%, much less.

Figure 3.8: Cylinder displacement (a) and NES relative displacement (b) for a system with less effective NES.
than the 78% reduction achieved for $\epsilon = 0.11$, $\lambda = 0.0283$, and $\kappa = 1.59$, discussed in §3.2.1, and that the phase angle between the displacement and the lift force also settles to a constant value after the initial transient, as shown in Fig. 3.9b. Thus, we infer that the suppression mechanisms identified in §3.2.1 and §3.2.2 are due to dynamic interaction of the NES with the flow field through the cylinder. These results clearly show that the suppression achieved for the sets of parameters considered in the two previous sections is not due to the damping introduced by the NES, but instead results from resonance captures that facilitate targeted energy transfer from the flow to the NES, where energy is ultimately dissipated.

3.2.4 Dynamics of the interaction of the NES with the flow

In this section, we discuss the dynamical interaction of an NES with the cylinder in laminar flow, mediated by the cylinder, with no direct contact between the NES and the flow. We first investigate the effect of the NES on the phase relation between the driving lift force and the resulting cylinder displacement.

Fig. 3.10 shows the time series of cylinder displacement and lift coefficient plotted on the same axes, for a system without the NES. Note that, after the
Figure 3.10: Phase relation between cylinder displacement and lift force without NES: \(- - -, C_L; \_\_\_\_, y_1/D\).

Figure 3.11: Phase relation between cylinder displacement and lift force with suppression mechanism I: \(- - -, C_L; \_\_\_, y_1/D\).
initial transient response, the phase difference between the lift force and the cylinder displacement approaches zero. The phase relations for systems with effective NES suppression via mechanisms I and II are shown in Figs. 3.11 and 3.12, respectively. In contrast, Fig. 3.11 shows that for NES parameters $\epsilon = 0.11$, $\lambda = 0.0283$, and $\kappa = 1.59$ (corresponding to suppression mechanism I), there is a time-varying phase difference between the displacement and lift force. Figure 3.12 shows that for $\epsilon = 0.066$, $\lambda = 0.0217$, and $\kappa = 0.704$ (corresponding to mechanism II), the modulation is considerably weaker than for mechanism I.

These qualitative observations of the phase difference between lift force and cylinder displacement are further confirmed by taking the Hilbert transform of these time series and computing the phase difference as a function of time. With no NES, the result is an asymptotic approach to zero following the initial transient, as shown in Fig. 3.13. For mechanism I, Fig. 3.14 shows that the phase difference varies in time, with large fluctuations (about 90°, peak-to-peak) corresponding to a strongly modulated response. Fig. 3.15 shows that the variation in phase difference for the system exhibiting suppression mechanism II is much smaller (less than 25°, peak-to-peak), and the effect of the NES is to shift the phase difference from zero to a mean value on the order of -90°. Thus, for both mechanisms I and II, the addition of the NES results in a non-stationary phase relation between the driving lift force and the resulting displacement, leading to
3.2.5 Strongly nonlinear interaction of NES

As stated above, the NES is an essentially nonlinear element lacking an inherent linearized natural frequency. Depending upon the magnitude and frequency content of the driving force, the NES can engage in resonance capture with the primary structure, resulting in one or more strong resonant interactions with the primary system as a result of dissipation of energy within the NES. In the present application, the driving force is provided by the unsteady lift on the cylinder generated by vortex shedding. From a dynamical systems perspective, the lift force can be regarded as the response of an intrinsic fluid oscillator with variable frequency and amplitude based on the motion of the cylinder and imposed flow conditions. It is well known that the frequency of shedding is slaved to the structure during lock-in, in the sense that the shedding frequency departs
from that for a stationary cylinder at the same $Re$, and assumes a value dictated by, and generally close to, the sprung cylinder’s natural frequency (Anagnostopoulos (1994)). Thus, the NES-augmented system can be viewed as a system of three coupled oscillators: i) a cylinder mounted on a linear spring, with dimensionless frequency $f^*_n = (1/2\pi)\sqrt{K_{cyl}/M_{cyl}D/U_0}$, ii) an NES with no inherent frequency, and iii) an intrinsic oscillator characterizing the oscillatory lift, with dimensionless frequency $f^*_L = f_L D/U_0$, representing the vortex shedding, where $f_L$ is the dimensional frequency associated with the fluctuating lift.

Frequency data for the three NES-augmented oscillators considered above is summarized in Table 3.1. In all of our simulation results, the natural frequency of the sprung cylinder is $f^*_n = f_n D/U_0 = 0.167$, which is chosen to be close to the natural shedding frequency of the stationary cylinder, $St = f_s D/U_0 = 0.168$.

Table 3.1 shows that for the very effective NES mechanisms I and II, the coupled...
system undergoes 1 : 1 : 1 resonance interaction, in which all three oscillators oscillate with same frequency. The strong interaction of the NES with the flow field, mediated by the cylinder, can be inferred by comparing the time series of the lift coefficient for suppression mechanisms I (Fig 3.11) or II (Fig. 3.12), to the case with no NES (Fig. 3.10) or with a much less effective NES (Fig. 3.9a). For 1 : 1 : 1 resonance interaction, the intrinsic fluid oscillator receives strong feedback from the cylinder and responds with a larger amplitude lift coefficient and additional frequency content. When the NES is relatively less effective, the frequency of the lift coefficient approaches a value close to that found for free response.

3.2.6 Energy Balance with NES

For the case with an NES, there are no previous experimental or computational results to which we can compare, so an “internal” validation was undertaken. Specifically, we performed an energy balance (shown in Appendix C), obtained
Table 3.1: Dominant frequencies of lift coefficient

\[ f_{C_L}^* = f_{L,D/U_0}, \text{ cylinder oscillation} \]
\[ f_{y_1}^* = f_{y_1,D/U_0} \text{ and NES oscillation} \]
\[ f_{y_2}^* = f_{y_2,D/U_0} \]

<table>
<thead>
<tr>
<th></th>
<th>( f_{C_L,D/U_0} )</th>
<th>( f_{y_1,D/U_0} )</th>
<th>( f_{y_2,D/U_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stationary cylinder</td>
<td>0.1682</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Free response</td>
<td>0.1644</td>
<td>0.1644</td>
<td>-</td>
</tr>
<tr>
<td>NES mechanism I</td>
<td>0.1668</td>
<td>0.1668</td>
<td>0.1668</td>
</tr>
<tr>
<td>NES mechanism II</td>
<td>0.1696</td>
<td>0.1696</td>
<td>0.1696</td>
</tr>
<tr>
<td>Less effective NES</td>
<td>0.1627</td>
<td>0.1661</td>
<td>0.1661</td>
</tr>
</tbody>
</table>

by taking the scalar product of the velocity vector and the momentum equation for the fluid (2.1a), integrating over the annular domain (using integration by parts to reduce the contributions from the pressure gradient and the nonlinear inertial term to surface integrals), and accounting for the kinetic energy of the cylinder and NES, the potential energy in the linear and nonlinear springs, and dissipation in the fluid (by viscosity) and by the NES damper. The results show that the mean value (averaged over 50 time points spanning one cycle of oscillation) of the absolute value of the relative error (signed residual in (C.1), divided by the magnitude of the largest term on the left-hand side) was less than 0.3%. The magnitude of the largest relative error over the same time interval was 0.7%. These results show that our “full” formulation (i.e., with the NES) is energetically consistent. (We believe that a significant fraction of the very small residual is associated with the fact that the time derivatives on the left-hand side of (C.1) were approximated using dynamical variables separated by time increments (1/50th of a cycle) much larger than the time step (≈ 1/250 of a cycle) used to compute the dynamical variables.)

3.3 Tuned mass damper for VIV suppression

The effectiveness of the NES is compared with that of a linear absorber attached to the cylinder and having the same mass as the NES that it replaces.
3.3.1 Formulation

The equations of motion of the “tuned mass damper” (TMD) coupled to those of the cylinder (and by virtue of the lift, coupled to the flow) are

\[
(\hat{M}_{cyl} - \hat{M}_{tmd}) \frac{d^2 y_1}{dt^2} + \hat{K}_{cyl} y_1 + \hat{C}_{tmd} \left( \frac{dy_1}{dt} - \frac{dy_{tmd}}{dt} \right) + \hat{K}_{tmd} (y_1 - y_{tmd}) = \hat{F}_L
\]

(3.3a)

\[
\hat{M}_{tmd} \frac{d^2 y_{tmd}}{dt^2} + \hat{C}_{tmd} \left( \frac{dy_{tmd}}{dt} - \frac{dy_1}{dt} \right) + \hat{K}_{tmd} (y_{tmd} - y_1) = 0
\]

(3.3b)

The dimensionless equations of motion for the TMD system are obtained using a scaling completely analogous to that used for the NES case, giving

\[
\ddot{Y}_1 + \omega^*_{tmd}^2 Y_1 + \lambda_{tmd} (\ddot{Y}_1 - \dot{Y}_{tmd}) + \kappa_{tmd} (Y_1 - Y_{tmd}) = \frac{\hat{F}_L D}{U_0^2 \hat{M}^*_{tmd}} = \frac{2\hat{C}_L (1 + \epsilon)}{\pi m^*}
\]

(3.4a)

\[
\epsilon_{tmd} \ddot{Y}_{tmd} + \lambda_{tmd} (\ddot{Y}_{tmd} - \dot{Y}_1) + \kappa_{tmd} (Y_{tmd} - Y_1) = 0
\]

(3.4b)

where \( M^*_{tmd} = \hat{M}_{cyl} - \hat{M}_{tmd} \) and \( \epsilon_{tmd} = \hat{M}_{tmd}/M^*_{tmd} \). The dimensionless parameters for the TMD are the mass ratio \( \epsilon_{tmd} \), dimensionless damping coefficient \( \lambda_{tmd} = \hat{C}_{tmd} D/\hat{M}^*_{tmd} U_0 \), and dimensionless stiffness \( \kappa_{tmd} = \hat{K}_{tmd} D^2/\hat{M}^*_{tmd} U_0^2 \).

The TMD is tuned for flow with \( Re = 100 \) and optimized damping is used. The tuning and optimization of damping for the TMD follows the work by Krenk (2005), with the following dimensionless damping coefficient and frequency ratio.

\[
\zeta_{tmd} = \frac{\epsilon_{tmd}}{2(1 + \epsilon_{tmd})}
\]

(3.5a)

\[
\frac{\omega^*_{tmd}}{\omega_0^*} = \frac{1}{1 + \epsilon_{tmd}}
\]

(3.5b)

where \( \omega^*_{tmd} = \hat{K}_{tmd} D^2/\hat{M}_{tmd} U_0^2 \) and \( \hat{C}_{tmd} = 2\zeta_{tmd} \sqrt{\hat{K}_{tmd} \hat{C}_{tmd}} \). Since the mass of the TMD is the same as that of the NES, equations (3.5a) and (3.5b) are used to compute the stiffness and damping of the TMD.
3.3.2 Comparison of TMD and NES in laminar VIV suppression

In this section, we discuss VIV suppression using a TMD. The effectiveness of the TMD is compared to the NES, for NES parameters ($\epsilon = 0.11$, $\lambda = 0.0283$, and $\kappa = 1.59$) corresponding to suppression mechanism I, and with the TMD tuned to perform optimally at $Re = 100$. The TMD mass is taken equal to the NES mass, so that $\epsilon_{tmd} = 0.11$. The cylinder displacement and the TMD relative displacement are shown in Fig. 3.16a and 3.16b, respectively.
As expected, the TMD with damping optimized for $Re = 100$, almost completely suppressed VIV at $Re = 100$. However, it is well known that the TMD’s effectiveness is sensitive to the frequency of excitation, or in the present case to the $Re$-dependent shedding frequency. We thus compare the TMD to the NES over a range of $Re$. The maximum displacement amplitude at each $Re$ determines the lock-in curve shown in Fig. 3.17. For response of the system without NES and with the Suppression Mechanism I NES parameters, each simulation at successively higher $Re$ was started using initial conditions from a time point at the previous $Re$, whereas simulations with the TMD at each $Re$ began with zero initial conditions for both the flow and the structure.

As indicated above, the NES has no preferred frequency, and responds to a broad range of excitation frequency. The TMD very effectively suppresses VIV close to the frequency at which it was tuned, but loses effectiveness as the forcing frequency varies with $Re$ away from $Re = 100$. We note that, unlike the TMD damping coefficient and frequency, the NES parameters used were not optimized. The advantage of the NES over the traditional TMD is clear from Fig. 3.17: the unoptimized NES provides superior suppression over the entire range, except for $87 \leq Re \leq 103$, a range that includes the $Re$ at which the TMD was optimized. We note that for the TMD system the lock-in range is approximately $80 \leq Re \leq 125$, in comparison to the system without NES, for which the lock-in range is $82.4 \leq Re \leq 134$. The NES parameters ($\epsilon = 0.11$, $\lambda = 0.0283$, and $\kappa = 1.59$) corresponding to suppression mechanism I narrow down this lock-in range to $87.7 \leq Re \leq 120$.

The basic difference in the mechanism of VIV suppression by NES and TMD can be understood by looking at the interplay of the three frequency components in the system. As stated earlier, the three frequencies in VIV are the characteristic fixed-cylinder Strouhal frequency $St$ which depend solely on $Re$, the dimensionless natural frequency of the cylinder $f_n^* = f_n D/U_0$ which depends on $Re$ as well, and the dimensionless frequency of vortex shedding or lift coefficient $f_{C_L}^* = f_{C_L} D/U_0$. The lift frequency is equal to $St$ when the system in not in lock-in vibration, but once in the lock-in regime, the frequency of vibration
Figure 3.18: Frequency synchronization over the lock-in regime: ●, Stationary cylinder Strouhal frequency $St$; ▲, dimensionless natural frequency of cylinder $f_n^* = f_n D/U_0$; ★, dimensionless frequency of lift coefficient $f_{CL}^* = f_{CL} D/U_0$
$f_{yi}^*$ and $f_{C_L}^*$ will coincide, and both of them can be different from, but close to, $f_n^*$. Hence, the lock-in regime is described as the response of the coupled system when vortex shedding is controlled by the structure, with the shedding frequency deviating from the fixed-cylinder Strouhal frequency.

We have computed the frequency of lift from the time history of the response, and its variation with $Re$ is shown in Fig. 3.18. The dimensionless natural frequency of the cylinder can be computed at various $Re$. The Strouhal frequency is computed from the best-fit equation (2.6). For the system without the NES, the interaction of the frequencies shown in Fig. 3.18a clearly shows the lock-in behavior. The lift frequency $f_{C_L}^*$ jumps from being equal to $St$ to a value very close to $f_n^*$ at $Re \approx 84$, and remains close to $f_n^*$ until $Re \approx 134$. Thus, the system without a NES remains in the lock-in state in the range $84 \leq Re \leq 134$.

For the system with NES under mechanism I, the frequency variation is shown in Fig. 3.18b. The range of lock-in, where $f_{C_L}^*$ is close to $f_n^*$, is reduced to the narrower range of $88 \leq Re \leq 101$. Beyond $Re = 101$, the presence of the NES drives the lift frequency $f_{C_L}^*$ a close to $St$, thereby reducing the amplitude of the response. For the system with a TMD the variation of lift frequency is shown in Fig. 3.18c. The TMD is very effective close to $Re = 100$ since its damping is optimized to that frequency; thus suppressing the vibration of the cylinder in a part of the lock-in range, $94 \leq Re \leq 104$. However, the TMD fails to drive the lift frequency $f_{C_L}^*$ towards $St$ as does the NES. Hence, the lock-in regime extends from $80 \leq Re \leq 125$ for the system with a TMD, and we clearly see that the lack of preferential frequency in the NES reduces the lock-in range significantly in the case of the system with NES.

3.4 Conclusions

The numerical investigation described herein shows that VIV of a sprung circular cylinder in two-dimensional laminar flow can be passively suppressed through TET introduced by an NES attached internally to the cylinder. The interaction
of the NES and fluid, mediated by the sprung cylinder, results in alteration of the phase relation between the lift force and the cylinder displacement, thereby reducing cylinder displacement. A strong 1 : 1 : 1 resonance interaction is required to achieve effective TET, and hence passive suppression of VIV. The computational results reveal the existence of two suppression mechanisms very similar to those found in finite-dimensional systems, with reductions in cylinder displacement of over 75%. Furthermore, for a given set of NES parameters, suppression of VIV is achievable over the entire range of lock-in \( Re \), which is itself narrowed by the NES. Since the NES is a small, lightweight attachment to the primary structure, not directly exposed to the flow field, requiring no power supply, internal “plumbing”, or electronics, and inherently broadband, the potential of this passive control concept at higher \( Re \) is worthy of consideration.

As noted previously (Vakakis et al. (2008); Gendelman et al. (2010)), addition of a local, essentially nonlinear attachment can introduce global changes in the dynamics, some of which may have unwanted effects. It follows that strongly nonlinear design approaches of the type proposed herein should be studied carefully, to avoid introducing phenomena that run counter to design objectives. It is thus important to carefully establish the limits of the proposed nonlinear designs, as well as the permissible ranges of parameters and initial/forcing conditions.
In this chapter, we study the effect of coupling an essentially nonlinear rotating element to a sprung rigid circular cylinder undergoing VIV. The essentially nonlinear device, termed the rotational NES, discussed by Gendelman et al. (2012), consists of a small mass and a viscous damper. The NES mass is constrained to rotate at a fixed radius about the oscillating center of the cylinder, with the cylinder motion constrained to be perpendicular to the mean flow. The use of a rotational NES to modify VIV leads to several interesting results, which are of interest beyond the suppression of VIV. These include the appearance of temporal chaos in the cylinder motion and the flow at Reynolds numbers where flow past a fixed circular cylinder is time-periodic, and considerable elongation of the region of attached vorticity during each shedding cycle.

Chaotic response in the wake of an oscillating cylinder has potential application to the enhancement of mixing and chemical reaction rates (cf. Deshmukh and Vlachos (2005)). This has been considered in the context of flow past a periodically excited cylinder at higher $Re$ by Karniadakis and Triantafyllou (1989a) and Batcho and Karniadakis (1991). Chaotic flow at lower $Re$, and without the need for periodic excitation, potentially has significant practical advantages in several mixing applications.

4.1 Formulation of coupled system with rotational NES

A schematic of the rotating NES attached to the cylinder is shown in Fig. 4.1. The damping of the NES necessary for dissipation is assumed to be a linear
viscous damper. The equations of motion for the coupled system shown in Fig. 4.1 are written as

\[
(\hat{M}_{\text{cyl}} + \hat{M}_{\text{nes}}) \frac{d^2 y_1}{dt^2} + \hat{K}_{\text{cyl}} y_1 = F_L + \hat{M}_{\text{nes}} r_0 \frac{d}{dt} \left( \frac{d\theta}{dt} \sin \theta \right) \tag{4.1a}
\]

\[
\hat{M}_{\text{nes}} r_0^2 \frac{\dot{\theta}^2}{dt^2} + r_0^2 \hat{C}_{\text{nes}} \frac{d\theta}{dt} = \hat{M}_{\text{nes}} r_0 \frac{d^2 y_1}{dt^2} \sin \theta \tag{4.1b}
\]

where \( \hat{M}_{\text{cyl}} \) and \( \hat{K}_{\text{cyl}} \) are the mass and the stiffness per unit length of cylinder, respectively. The NES mass and damping per unit length of the cylinder are denoted by \( \hat{M}_{\text{nes}} \) and \( \hat{C}_{\text{nes}} \), respectively. The generalized-\( \alpha \) predictor-multicorrector numerical scheme used for forward time integration of the coupled system (4.1a)-(4.1b) is discussed in Appendix D.

Nondimensionalizing the time and the length by scaling with \( U_0/D \) and \( D \), respectively, dimensionless variables are introduced as

\[
Y_1 = \frac{y_1}{D} \quad \bar{r}_0 = \frac{r_0}{D} \quad \tau = \frac{t U_0}{D}
\]
Dividing equations (4.1a) and (4.1b) by \((\hat{M}_{cyl} + \hat{M}_{rnes})\), and identifying the dimensionless mass ratio of the NES \(\epsilon_r = \hat{M}_{rnes}/(\hat{M}_{cyl} + \hat{M}_{rnes})\), the dimensionless equations of motion are obtained as

\[
\frac{d^2 Y_1}{d\tau^2} + \frac{D^2}{U_0^2} \frac{\hat{K}_{cyl}}{M_{cyl} + M_{rnes}} Y_1 = \frac{D}{U_0^2} \frac{F_L}{M_{cyl} + M_{rnes}} + \epsilon_r \hat{r}_0 \frac{d}{d\tau} \left( \frac{d\theta}{d\tau} \sin \theta \right) \tag{4.2a}
\]

\[
\epsilon_r \hat{r}_0 U_0^2 \frac{d^2 \theta}{d\tau^2} + \frac{D^2 \hat{r}_0^2}{M_{cyl} + M_{rnes}} \frac{\hat{C}_{rnes}}{U_0} \frac{d\theta}{d\tau} = \epsilon_r \hat{r}_0 U_0 \frac{d^2 Y_1}{d\tau^2} \sin \theta \tag{4.2b}
\]

The term involving the lift force \(F_L\) is nondimensionalized using the dimensionless density \(m^* = \rho_b/\rho_f\) and dimensionless lift coefficient

\[
C_L = \frac{F_L}{\frac{1}{2} \rho_f U_0^2 D} \tag{4.3}
\]

giving

\[
\ddot{Y}_1 + \omega_{r}^* Y_1 = \frac{2C_L}{\pi m^*} + \epsilon_r \hat{r}_0 \frac{d}{d\tau} \left( \dot{\theta} \sin \theta \right) \tag{4.4a}
\]

\[
\ddot{\theta} + \lambda_r \dot{\theta} = \frac{\ddot{Y}_1}{\hat{r}_0} \sin \theta \tag{4.4b}
\]

where we again use the notation \(\dot{\cdot} = \frac{d\cdot}{d\tau}\) and \(\dddot{\cdot} = \frac{d^2\cdot}{d\tau^2}\). The dimensionless natural frequency of the cylinder supported on the linear spring is

\[
\omega_{r}^* = \frac{D^2}{U_0^2} \frac{\hat{K}_{cyl}}{M_{cyl} + M_{rnes}}
\]

The dimensionless parameters characterizing the NES are the mass ratio \(\epsilon_r\), radius ratio \(\hat{r}_0\), and damping ratio \(\lambda_r = \frac{D \hat{C}_{rnes}}{\hat{M}_{rnes} U_0}\). In deriving the equations of motion, \(\theta\) is taken to be positive counterclockwise, and \(\theta = 0\) and \(\theta = \pi\) correspond to fixed points of the system. In all the simulation results discussed in this paper, the initial conditions for the \(\theta\) DOF are taken as \(\theta(0) = \pi/2\) and \(\dot{\theta}(0) = 0\). In the following section, we discuss the computational results obtained by varying the NES parameters \(\hat{r}_0\), \(\lambda_r\), and \(\epsilon_r\). The strong nonlinearity arises due to the inertial coupling of the angular motion of the NES.
and the rectilinear motion of the cylinder. The strong nonlinear interaction of
the NES and the cylinder results in several interesting response regimes, not
previously observed.

4.2 Passive suppression of VIV using rotational NES

In this section, we discuss the computational results of VIV with a rotating
NES, obtained by varying the NES parameters $\bar{r}_0$, $\lambda_r$, and $\epsilon_r$. The interac-
tion of the NES with the flow, mediated by the cylinder, leads to TET to the
NES, resulting in suppression of vibration of the cylinder. Two types of passive
suppression mechanisms were observed: “strongly modulated response” (SMR
mechanism I) and suppressed LCO (LCO mechanism II). The best passive sup-
pression found for each mechanism is reported in this section.

Post-processing of the numerically computed time series was performed by nu-
merical wavelet transforms (WT) employing a Matlab-based algorithm. Al-
though the algorithm provides the opportunity to use two kinds of mother
wavelets, namely Morlet and Cauchy, in the applications considered herein we
have used only the Morlet wavelet, which is a Gaussian-windowed complex si-
nusoid. The frequency of the mother wavelet is the user parameter that enables
one to tune the frequency and time resolution of the results. In the WT contour
plots (WT spectra) we depict the amplitude of the WT of the signal as function
of frequency (vertical axis) and time (horizontal axis). Heavily shaded areas
correspond to regions where the amplitude of the WT is high, whereas lightly
shaded ones imply regions where the WT amplitude is low. Such plots enable
us to deduce the temporal evolution of the dominant frequency components of
the computed time series, and, in addition, to identify transitions between dif-
ferent modes of the response as the nonlinear dynamical process evolves in time
(Vakakis et al. (2008)).
Figure 4.2: Mechanism I: Response with 71% reduction in RMS value of cylinder.
4.2.1 SMR - Mechanism I

For NES parameters $\bar{r}_0 = 0.2$, $\epsilon_r = 0.3$, and $\lambda_r = 0.212$, the cylinder exhibits a strongly modulated response. The time history of the cylinder displacement, NES angle, and lift coefficient are shown in Figs. 4.2a - 4.2d. The reduction in the RMS value of the displacement of the cylinder is about 71%, compared to the system without the NES. The time history of the NES angular position shown in Fig. 4.2b also indicates that the direction of rotation of the NES changes without any particular trend, while the cylinder undergoes relaxation oscillations.

The frequency content of all the time series shown in Fig. 4.2 is shown in Figs. 4.3a - 4.3d. The NES frequency content is extracted by taking the wavelet transform of the cosine of the NES angle. The cylinder displacement and the lift coefficient have a dominant frequency close to the natural shedding frequency, whereas the NES has no dominant frequency, indicating that the NES is not in a clear 1:1 resonance with either the cylinder displacement or the lift coefficient. The presence of higher harmonics can also be observed from frequency content; the multiple harmonics are specific to the system with a rotational NES.

4.2.2 Suppressed LCO - Mechanism II

A second mechanism of passive VIV suppression was observed with NES parameters $\bar{r}_0 = 0.2$, $\epsilon_r = 0.25$, and $\lambda_r = 0.849$. The cylinder displacement, the NES angle, and the lift coefficient are shown in Figs. 4.4a - 4.4d. The suppressed LCO of the displacement of the cylinder is shown in Fig. 4.4a, with the maximum amplitude of the cylinder displacement being about $0.25D$. Thus, in this case a reduction in the RMS value of cylinder displacement of about 50% can be attributed to the rotating NES.

From Fig. 4.4b, we can observe that the NES remains nearly motionless while the amplitude of the cylinder displacement is very small. Once the cylinder is excited to a sufficiently large amplitude oscillation, the NES starts interacting with the cylinder, and after some initial transient the system settles into a 1:1:1
Figure 4.3: Mechanism I: Frequency content of the cylinder, the NES angle, and the lift coefficient.
Figure 4.4: Time history of cylinder, NES angle, and lift coefficient for system under suppression Mechanism II.
Figure 4.5: Mechanism II: Frequency content of cylinder, NES and lift coefficient time history.
resonance, as can also be seen from the frequency analysis of the time signals shown in Figs. 4.5a - 4.5d. We note that the direction of rotation of the NES reverses with a frequency one-half that of the cylinder displacement and lift coefficient.

4.3 Effect of rotational NES on the wake structure

For NES parameters $\bar{r}_0 = 0.5$, $\epsilon_r = 0.3$, and $\lambda_r = 0.0034$, the response is characterized by a very slow decay of the amplitude of the cylinder displacement and lift coefficient followed by a burst into a chaotic mode as shown in Fig. 4.6. Fig. 4.6a shows that the amplitude of the cylinder displacement undergoes considerable decay before bursting into chaotic oscillation, the smallest amplitude of the cylinder being $0.1D$. A similar observation can be made for the lift coefficient from Fig. 4.6d; the smallest amplitude of the lift coefficient amplitude is about $0.045$. Time integration of this system is performed for a very long duration (for 4000 dimensionless time units) to ascertain the long-term nature of the solution. As can be seen from Fig. 4.6, the alternate slow decay to a small amplitude and burst into chaotic motion recurs, thus confirming that the observed response is not an initial transient but a stable attractor of the system for these sets of values for the NES parameters.

The angular motion of the NES in this response is characterized by a nearly unidirectional rotation as the cylinder amplitude slowly decays. The direction of rotation after chaotic transition varies, but remains unchanged in any particular slow-decaying portion of the solution. This long-duration nearly unidirectional rotation of the NES differs from the NES rotation predicted for mechanisms I and II described in §4.2.

The frequency content for the slowly modulated response is shown in Fig. 4.7. Unlike the results discussed above, the frequency content deviates from the Strouhal frequency on a slow time scale. In the regime of chaotic motion, there is no clear dominant frequency, indicating that the response in these intermit-
Figure 4.6: Time history of cylinder, NES angle, drag coefficient and lift coefficient with slow modulation cycles.
Figure 4.7: Frequency content for the cylinder displacement, NES angle, drag coefficient, and lift coefficient in the slow modulation solution.
tent chaotic regimes is essentially broadband in nature. But once the system locks on to the slowly decaying response, there is clearly a dominant frequency close to the Strouhal frequency in both the cylinder displacement (Fig. 4.7a) and NES angle (Fig. 4.7b), but not for the lift coefficient (Fig. 4.7d).

A distinct feature of this response is that the frequency of the response is no longer constant in time, but continuously decreases within the slowly varying portion of the response, reaching a value lower than \( St \) before entering the chaotic regime. The frequency detuning from the Strouhal frequency is about 15\% at the beginning of the slowly modulated response, and reaches 33\% towards the end of the slow decay. We also note that Figs. 4.6d and 4.7d show that the lift coefficient does not have a dominant frequency close to the Strouhal frequency. During the slow decay portion of the response, the dominant frequency in the lift coefficient is about twice the Strouhal frequency. Such a transition in lift frequency content has been reported at higher \( Re \) and has been attributed to switching in the timing of shedding. This motivates us to study the wake structure carefully for this response.

More interesting features were found in the wake structure for this response with slow decay. The wake structure for a time instant close to the end of slow decay is shown in Fig. 4.8a. It is well known that in two-dimensional VIV, the wake structure remains quite similar to that of the Kármán vortex street, with small differences that can be attributed to the motion of the cylinder. With a rotational NES, however, the attached vorticity extends to about 10 diameters before destabilizing (the solid line above the cylinder represents 10\( D \)).

Elongation of the attached vortices has been observed only when the cylinder amplitude envelope is slowly modulated, and was not observed with a translational NES (Tumkur et al. (2013)) or for mechanisms I and II with a rotational NES. Fig 4.8a shows the wake structure at \( tU_0/D = 910.125 \), when the solution is very close to transition into the chaotic regime. The wake structure shown in Fig. 4.8b is for \( tU_0/D = 1181.375 \), which is well into the chaotic regime. The
Figure 4.8: Modified wake structure due to rotational NES.

significant differences in both the near and far fields of the wake can be seen by comparing to the classical Kármán vortex street past a stationary cylinder shown in Fig. 4.8c for the same Re, and to the case of free VIV without an NES shown in Fig. 4.8d. This implies that VIV with a rotating NES leads to modification in the near-field vorticity distribution, and that the wake is modified in the far field by the upstream vorticity distribution. This suggests, as discussed in §4.6.2, that in some sense the flow partially approaches the steady symmetric solution. Since Re = 100 is well above the critical Re ≈ 47, the unsteady solution dominates towards the end of the slow decay motion, resulting in destabilization of the steady solution, and the system locks-out of the slow decaying envelope.
The time history of the drag coefficient during the slowly modulated response is shown in Fig. 4.6c. The average drag coefficient continuously decreases during the slowly decaying motion of the system. The mean drag coefficient just before the chaotic transition is about 1.2, compared to the mean value of 1.85 for a system undergoing VIV without the NES. Hence, there is a significant reduction in drag attributed to the rotational NES. The reduction in drag also supports our conjecture that the solution is approaching the steady symmetric state.

From the time series of the response and wake structure, one can associate the dynamics of the system with relaxation oscillations, with the important difference of intermittent chaotic transitions during relaxation. The plausibility of such relaxation oscillations can be attributed to TET due to the NES. Indeed, the NES continuously extracts energy from the fluid (dissipating it internally through its viscous damper), thus diminishing the capacity of the flow to force the cylinder motion. Such TET directed towards the NES reduces the amplitude of the cylinder motion, make the flow around the cylinder approach that of steady symmetric flow. However, a stage is reached when cylinder motion increases more rapidly than NES angular velocity or NES dissipation, resulting in dominance of the unsteady flow solution, which leads to sudden relaxation and transition of the regular motion to a chaotic regime. The relaxation cycle then repeats itself after the NES angular velocity regains its capacity to extract energy from the flow as described above.

4.4 Persistence of elongated wake solution at $Re = 60$

We show that the solution with an elongated region of attached vorticity is not an isolated case, by computing flows at $Re = 60$ for NES parameters different than those used above. Since the Strouhal frequency varies with $Re$ for $Re < 200$, we first determine $St$ at $Re = 60$, so that the parameters of the cylinder without the NES can be tuned to the resonance condition in order to observe large-amplitude oscillation of the cylinder. The time series of the lift coefficient at $Re = 60$ (for which $St = 0.14$), computed by holding the cylinder
stationary, is shown in Fig. 4.9a, and its frequency content is shown in Fig. 4.9b. Next, while retaining the mass ratio $m^* = 10$, we tune the natural frequency of the cylinder to be in resonance with the lift, and simulate VIV of the cylinder to obtain the large-amplitude motion of the cylinder at $Re = 60$. The cylinder displacement and lift coefficient at $Re = 60$ are shown in Figs. 4.10a and 4.10b, respectively.

We then introduce a rotating NES into the system and find that the elongated region of attached vorticity is not observed at $Re = 60$ with the same
parameters of the NES as at $Re = 100$. However, by modifying the NES parameters in a coarse parametric search, we were able to find the elongated vortex solution for $r_0 = 0.3$, $e_r = 0.3$, and $\lambda_r = 0.00943$. The time series of the cylinder displacement, NES angle, and lift coefficient are shown in Figs. 4.11a to 4.11d, respectively. The spanwise vorticity distribution (Fig. 4.12) shows the elongated region of attached vorticity at $Re = 60$, for $tU_0/D = 570$. The time series shown in Fig. 4.11 resemble those in Fig. 4.6. The typical scenario of chaotic oscillation of the cylinder and lift force leading to slowly decaying motion of the cylinder is also observed in this case. During the slow decaying motion of the cylinder, the lift coefficient decays to a very small value, and the NES engages in nearly unidirectional rotational motion. From the time series in Fig. 4.11, it can be observed that the time dependence of the frequency and its detuning as discussed for $Re = 100$ are similar at $Re = 60$. The maximum frequency detuning and maximum elongation in the attached vortex correspond to the same time instant. However, there are clear differences in the response at $Re = 60$ and $Re = 100$. Overall, the magnitude of the lift coefficient is much smaller during the entire response at $Re = 60$, which is to be expected since $Re = 60$ is closer to the Hopf bifurcation that occurs near $Re \approx 47$. The time periods of the response during which the cylinder motion is chaotic and in the slowly decaying mechanism are much shorter at $Re = 60$. As will be discussed in §4.6.1, based on an approximate analysis we conjecture that during the slowly modulated response the effect of the flow is largely to provide damping. As we move from $Re = 100$ to $Re = 60$, we expect that the damping provided by the flow increases due to increased viscous dissipation, and that this reduces the time durations of the chaotic and slow decay regimes. Hence, these results indicate that the elongated attached vortex is not an isolated case for some specific values of $Re$ and the NES parameters, but rather can be observed over a range of $Re$ when the NES parameters are appropriately chosen.
Figure 4.11: Time history of cylinder, NES, and lift coefficient with slow modulation cycles at $Re = 60$. 

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Figure 4.12: Elongated attached vortex solution with rotational NES at \( Re = 60 \).

Figure 4.13: Persistence of elongated wake solution for small change in \( Re \) with \( \epsilon_r = 0.3, \bar{r}_0 = 0.5, \) and \( \lambda_r = 0.0034 \).

4.5 Persistence of elongated modes for small changes in NES parameters

We next discuss the persistence of the elongated vortex solution for small changes in the NES parameters and the Reynolds number. The elongated wake solution observed at \( Re = 100 \) with NES parameters \( \bar{r}_0 = 0.5, \epsilon_r = 0.3, \) and \( \lambda_r = 0.0034 \) is varied by perturbing the four parameters around their original values, one at a time. This study is aimed at gaining a preliminary understanding of the region in the parameter space for which the elongated vortex solution exists.

For the nominal values of the NES parameters, we did not find the elongated wake solution for \( Re < 100 \). However, on the higher end, the elongated vortex solution persists up to at least \( Re = 105 \). The slowly decaying cylinder displacement and elongated vortex are shown in Figs. 4.13a and 4.13c, respectively, for
Figure 4.14: Persistence of elongated wake solution for small change in $\epsilon_r$ with $Re = 100$, $\bar{r}_0 = 0.5$, and $\lambda_r = 0.0034$.

$Re = 101$. The cylinder displacement and spanwise vorticity distribution at $Re = 105$ are shown in Figs. 4.13b and 4.13d, respectively. For all intermediate values of $Re$ in the range $100 \leq Re \leq 105$, an elongated vortex solution was observed. As we increase $Re$ above 100, the portion of the time series showing chaotic characteristics increases, and the slowly decaying envelope is shortened and modulated.

The effect of changing the mass of the NES on the elongated vortex structure is shown in Fig. 4.13. When $\epsilon_r$ is reduced to 0.27, the elongated vortex solution is still found, and the corresponding cylinder displacement and spanwise vorticity are shown in Figs. 4.14a and 4.14c, respectively. The cylinder displacement and spanwise vorticity distribution for larger NES mass ($\epsilon_r = 0.35$) are shown in Figs. 4.14b and 4.14d, respectively. For the smaller NES mass, the slowly decaying envelope extends over a longer time. In addition, the decay does not have the typical linear envelope observed in other cases.

The persistence of the elongated vortex with variation in the NES damping is shown in Fig. 4.15. The range of NES damping over which the elongated wake persists is found to be $0.00255 \leq \lambda_r \leq 0.01698$. For low NES damping, the wake elongation persists over a longer time. Since the NES damping is responsible for dissipating energy transferred from the flow to the cylinder, the slow decay in the cylinder displacement is consistent with the fact that the energy dissipation
Figure 4.15: Persistence of elongated wake solution for small change in $\lambda_r$ with $Re = 100$, $\epsilon_r = 0.3$, and $\bar{r}_0 = 0.5$.

Figure 4.16: Persistence of elongated wake solution for small change in $\bar{r}$ with $Re = 100$, $\epsilon_r = 0.3$, and $\lambda_r = 0.0034$.

is reduced for small damping, as can be observed in the cylinder displacement time history shown in Fig. 4.15a, and in the corresponding spanwise vorticity distribution shown in Fig. 4.15c. The stronger influence of NES damping at the upper end of the range ($\lambda_r = 0.01698$) is consistent with the more rapid dissipation of energy that occurs in this case, as can be seen in the cylinder displacement time series shown in Fig. 4.15b, and the corresponding spanwise vorticity shown in 4.15d.

The effect of radius of the rotating mass on the elongated wake solution is shown in Fig. 4.16. Figure 4.16b shows that the elongated vortex persists down to $\bar{r}_0 = 0.45$. (For $\bar{r} > 0.5$, the mass would rotate outside the cylinder, a case we do not consider.)
We note that the slowly decaying envelope of the cylinder displacement shown in Fig. 4.16a with $\bar{r}_0 = 0.45$ is very similar to that for the lower-mass case ($\bar{r}_0 = 0.27$) shown in Fig. 4.14a. This supports the hypothesis that the angular momentum of the rotating NES plays a critical role in the elongated vortex solution. The angular momentum is a function of the moment of inertia of the rotating mass, which in turn is directly proportional to the radius and mass of the NES. Hence, a decrease in mass or radius of the NES affects the angular momentum of the NES in similar ways, leading to similar modifications in the elongated vortex response in both cases.

In addition to the NES parameters and $Re$, we also show that the elongated vortex solution persists for changes in initial conditions. For this case, we initially “lock” the NES, so that there is no NES motion during an initial transient. Since the NES mass is a fixed fraction of the cylinder mass, the combined mass of the system remains the same as for the system without the NES. Thus, normal VIV at $Re = 100$ is expected with the NES locked. After the VIV solution is fully developed, the NES is set free to interact with the cylinder. The results are shown in Fig. 4.17, where the NES is locked until $tU_0/D = 288$, at which time VIV is fully developed, with cylinder amplitude $y_1/D = 0.49$ and maximum lift coefficient of $\max \{C_L\} = 0.212$, as for the system without NES. Beyond $tU_0/D = 288$, the simulation is continued by allowing the NES to interact with the motion of the cylinder. As seen in Fig. 4.17a, the system enters into exactly the same response as shown in Fig. 4.6. The spanwise vorticity (Fig. 4.18) shows that the elongated vortex develops from the standard VIV solution just as it does from the usual initial condition.

All computational results presented in this chapter thus far were obtained without fluid-structure equilibrating iterations. The influence of performing fluid-structure equilibrating iterations on this slowly decaying and elongated wake solution is discussed in Appendix E.3; additionally, the effect of higher temporal resolution is examined.
Figure 4.17: Time history of cylinder, NES angle, and lift coefficient with slow modulation cycles at $Re = 100$ starting from a fully developed VIV solution without NES.
4.6 Discussion and interpretation

4.6.1 Approximate NES-induced mass model

The cylinder motion is driven by the oscillating lift force and the essentially nonlinear force due to the NES, as shown on the right-hand side of equation (4.1a). Consider the latter term

\[ N_l = \hat{M}_{rnes} r_0 \frac{d}{dt} \left( \frac{d\theta}{dt} \sin \theta \right) \]

The dynamical interaction of the rotating NES with the flow, mediated by the cylinder, can be approximated by adding an effective time varying NES-induced mass \( M_{eff}(t) \) to the cylinder. This effective NES-induced mass can be obtained by taking the ratio of the nonlinear restoring force to the acceleration of the cylinder

\[ M_{eff}(t) = -\frac{N_l(t)}{(d^2 y/dt^2)} \quad (4.5) \]

which can be rewritten in dimensionless form as

\[ m_{eff} = \frac{c r_0}{Y_1} \frac{d}{dr} \left( \dot{\theta} \sin \theta \right) \quad (4.6) \]

For the case where the NES locks into a slowly-decaying motion, the dimensionless effective NES-induced mass is calculated over each slowly-decaying portion of the time series, and is shown in Fig. 4.19a. Since the effective NES-induced mass is a ratio of two quantities oscillating about zero, the plot will involve

Figure 4.18: Elongated wake at \( Re = 100 \) starting from a fully developed VIV solution without NES.
large peaks that have been filtered by Gaussian smoothing. In each portion of
the time series shown separately in Fig. 4.19a, the dimensionless effective NES-
induced mass approaches unity toward the end of the decaying motion; i.e., the
effective NES-induced mass is as large as the mass of the cylinder and NES.
This dynamic mass added to the cylinder also explains the large detuning of the
frequency that has been observed in the response toward the end of the slowly
decaying motion. The natural frequency of the structural system including the
effective NES-induced mass is

$$\omega_{\text{detuned}} = \frac{\omega^*}{\sqrt{2}} \quad (4.7)$$

Hence, due to doubling of the mass, the frequency of the system is reduced to
$1/\sqrt{2} \approx 70\%$ of the original value, which also closely corresponds to the 33%
frequency detuning observed in the cylinder response towards the end of the
slow modulation cycle.

The complementary cumulative distribution function (CCDF) of the effective
NES-induced mass is shown in Fig. 4.19b. The CCDF plots indicate that the
probability of mass being additive in this response is high, since \( CCDF(m_{eff} = 0) \approx 0.95 \) in all the time portions with slowly decaying motion. The effective NES-induced mass for systems with the rotating NES under passive suppression mechanisms I and mechanism II is shown in Figs. 4.20b and 4.20a, respectively, indicating that there is no trend for added mass to the cylinder in these cases; hence, the lock-in response observed in these cases are with stationary frequency, and that frequency remains close to the Strouhal frequency. It is interesting to compare the effective added mass to the cylinder in the cases of passive suppression mechanisms I and II discussed earlier, where no elongated vortex structures were observed. The \( CCDF \) of the effective NES-induced mass for these two mechanisms are shown in Figs. 4.21a and 4.21b, from where we deduce that \( m_{eff} \) fluctuates about the average value of zero, which is the value corresponding to no effective added mass \( (m_{eff} = 0) \), so no elongation of the vortex structure occurs in these cases. This result indicates (although it does not rigorously prove) that the increase in NES induced mass is directly connected to the change in the vortex structure during the described relaxation cycles. The value of \( CCDF(m_{eff} = 0) \approx 0.35 \) and \( CCDF(m_{eff} = 0) \approx 0.4 \) for mechanisms I and II, respectively, show that, on average, there is no net added mass.

Figure 4.20: Effective NES-induced mass when system is under mechanisms I and II.
4.6.2 Approximate analysis of the slowly modulated solution

In this section, we perform an approximate analysis of the slowly-decaying motion discussed in §4.3 in order to prove that the effect of the flow during the slowly-decaying motion giving rise to the elongated vortex structure is, in effect, similar to linear viscous dissipation. Through this analysis, we correlate the observed effective NES-induced mass and the frequency detuning observed in the response. To this end, we consider the unforced system of equations governing the coupled motion of the cylinder and NES

\[
\ddot{Y}_1 + \omega_r^2 Y_1 = \epsilon_r \frac{d}{d\tau} \left( \dot{\theta} \sin \theta \right) \quad (4.8a)
\]

\[
\ddot{\theta} + \lambda_0 \dot{\theta} = \frac{\dot{Y}_1}{\dot{r}_0} \sin \theta \quad (4.8b)
\]

The ordinary differential equations (ODEs) (4.8a) and (4.8b) are solved by prescribing the initial conditions taken from the finite-element solution (referred to as the PDE solution) of the fully coupled system at the beginning of the slowly decaying motion, which are \(Y_1(550) = -0.309623, \dot{Y}_1(550) = -0.028324, \theta(550) = 90.935038\), and \(\dot{\theta}(550) = 1.167909\). The solution computed from the unforced ODE system (4.8a) and (4.8b) is compared to the fully coupled finite-
Based on the solution of the unforced ODE system, we conjecture that the slow modulation response is a perturbation of the solution of the underlying Hamiltonian system. We approximate the solution of the Hamiltonian system analytically by neglecting the damping of the NES from equation (4.8b), and
thus consider the undamped and unforced system of equations

\[ \ddot{Y}_1 + \omega^2 Y_1 = \epsilon \bar{r}_0 \dot{\theta} \sin \theta + \epsilon \bar{r}_0 \dot{\theta} \cos \theta \]  

(4.9a)

\[ \ddot{\theta} = \frac{\dot{Y}_1}{\bar{r}_0} \sin \theta \]  

(4.9b)

Assuming a harmonic oscillation for the cylinder and unidirectional rotation of the NES, we approximate the response of system (4.9) according to the ansatz

\[ \tilde{Y}_1 = \bar{r}_0 \alpha \cos \omega \tau \]

\[ \theta = \omega \tau \]

where the amplitude \( \alpha \) and frequency \( \omega \) are assumed to be varying on a time scale slower than \( \tau \). Substituting this ansatz into (4.9a) and (4.9b) provides us with the frequency-amplitude relation

\[ \alpha = \frac{\epsilon \omega^2}{\omega_f^2 - \omega^2} \]  

(4.10)

Using the relation (4.10), we estimate the frequency detuning as the ratio of frequencies at two instants of time in the slow decaying portion of the time series, according to

\[ \frac{\omega_f}{\omega_i} = \sqrt{\frac{\alpha_f (\alpha_i + \epsilon)}{\alpha_i (\alpha_f + \epsilon)}} \]  

(4.11)

Using the amplitudes of the cylinder from the PDE solution in (4.11), the analytical estimate of frequency ratio is \( \omega_f/\omega_i = 0.777 \) for the portion of the time series shown in Fig. 4.23a. The computational frequency content is extracted by taking a wavelet transform of the cylinder displacement time series in the range \( 450 \leq tU_0/D \leq 950 \), shown in Fig. 4.23b. The computational frequency ratio is found to be \( \omega_f/\omega_i = 0.789 \). The good agreement between the frequency ratio obtained by computation and by the analytically approximated value reinforces our conjecture that the observed slow modulation cycle is a perturbation of the underlying undamped system.

Motivated by the previous analysis, we model the effect of the flow on the
Figure 4.23: Frequency detuning in one slow modulation cycle.
unforced system of oscillators by introducing an additional damping term into
the unforced system (4.8a) and (4.8b), leading to the following modified system
of equations
\begin{align}
\ddot{Y}_1 + \beta \dot{Y}_1 + \omega^* Y_1 &= \epsilon_r r_0 \frac{d}{d\tau} \left( \dot{\theta} \sin \theta \right) \\
\ddot{\theta} + \lambda_r \dot{\theta} &= \frac{\dot{Y}_1}{r_0} \sin \theta
\end{align}

where \( \beta \) is an “ad-hoc” viscous damping term used to model the effect of the
flow on the transverse oscillation of the cylinder during the relaxation cycle. We
estimate \( \beta \) by minimizing the RMS error between the response of the system
(4.12a) and (4.12b) and the PDE response during a slowly decaying portion
of the time series, and use the PDE solution to prescribe initial conditions for
(4.12a) and (4.12b). For the optimized damping value of \( \beta = 0.009 \) we obtain
a good match between the PDE and ODE solutions, for which the cylinder dis-
placement is shown in Figs. 4.24a and 4.24b for the PDE and ODE systems,
respectively, and NES angular displacement is shown in Figs. 4.24c and 4.24d
for the PDE and ODE systems, respectively. We note that the modulations in
the envelope of the cylinder displacement are now reduced when compared to
the Hamiltonian system. Thus, the “ad-hoc” linear damping models the slowly
decaying response quite accurately.

Further, if we initiate the solution of system (4.12a) and (4.12b) with a smaller
amplitude of the cylinder as the initial condition, we obtain very good agree-
ment for the response for \( \beta = 0.059 \), as can be observed in Figs. 4.25a and
4.25b for cylinder displacement, and in Figs. 4.25c and 4.25d for the NES an-
gular displacement, respectively. We see that at this larger damping value, we
can match the system response more accurately if the solution of (4.12a) and
(4.12b) is started later in the slow modulation cycle. To see the effect of damp-
ing as a function of initial conditions, and to determine the damping values that
most accurately models the PDE solution, we show in Fig. 4.26 the solutions
of the ODE system for several initial conditions.
Figure 4.24: Unforced ODE with linear damping $\beta = 0.009$ and initial conditions form PDE solution, comparison of PDE and ODE solutions.

The optimal damping value increases monotonically as the integration starts from progressively later stages in the slowly-decaying portion of the relaxation cycle. We see that at the beginning of the slowly-decaying response, the effect of the flow is quite small and nearly negligible. In a sense, the cylinder motion reaches, via a chaotic transition, a state where the fluid coupling is minimized and flow effects are similar to the action of a linear viscous damper. In essence, the inertia effects of the flow past the moving cylinder are nearly negligible and viscous effects dominate the fluid-structure interaction in the slow portion of the relaxation cycle. As the cylinder continues its motion as a nearly unforced oscillator, the flow interacts with the cylinder, and coupled motion resumes towards the end of the slowly-decaying response, at which time the cylinder dynamics undergoes a fast transition to chaotic dynamics. At that stage the fast portion of the relaxation cycle is reached. As mentioned previously the slow part of the cycle (by ‘slow’ and ‘fast’, we refer to the temporal evolution of the envelope of the cylinder motion) corresponds to effective energy transfer of energy from the flow to the internal rotating NES, whereas the fast part corresponds
Figure 4.25: Unforced ODE with linear damping $\beta = 0.059$ and initial conditions form PDE solution, comparison of PDE and ODE solutions.

Figure 4.26: Optimal linear viscous damping required to approximate the slow modulation cycle.
to the point where the NES is incapable of further dissipating energy from the flow, and transition to chaotic motion results. When the fast part of the cycle is reached, the vortex elongation observed in the slow part is eliminated. It is interesting that after some chaotic transients the relaxation cycle repeats itself following the same process. Our numerical simulations show that, once initiated, the relaxation cycle seems to be robust.

4.7 Proper orthogonal decomposition (POD) of elongated attached vortex solution

In this section, we consider the velocity field in a noninertial reference frame moving with the cylinder, and use it to generate a set of orthogonal solenoidal velocity modes onto which the computed velocity field (in the same noninertial frame) can be projected. (We have also projected the Navier-Stokes equations, in the same noninertial frame, onto these modes, which will be discussed in section 6.1.)

We begin with the computed velocity field for a single simulation of the flow (i.e., a numerical approximation to the solution of the initial-boundary value problem) in the inertial reference frame in which it is computed, which we denote by $v_{\text{fluid}}^i(x, y, t)$. We refer that velocity field to a noninertial frame $y' = y - y_1(t)$ moving with the cylinder velocity $dy_1/dt$. At each point on the computational domain discussed in §2.1, the result is $v_{\text{fluid}}^n(x, y', t) = v_{\text{fluid}}^i(x, y, t) - dy_1/dt$.

We then restrict the domain of $v_{\text{fluid}}^n(x, y', t)$ to the set of points $(x, y)$ that always remain within the original domain on which the finite-element computation was performed ($x_{\text{min}} \leq x \leq x_{\text{max}}$ and $y'_{\text{min}} \leq y' \leq y'_{\text{max}}$), and use the “method of snapshots” at $N_s$ time points to compute a set of $N_m$ spatial modes (the “POD modes”), which we denote by $u_{ji}^n(x, y', t)$, $1 \leq j \leq N_m$. (The projection onto $N_m$ modes is performed in such a way that there is no set of $N_m$ solenoidal modes capturing more of the kinetic energy of the computed field $v_{\text{fluid}}^n(x, y', t)$.) At each time $t_k$ ($1 \leq k \leq N_s$), we can then project the computed velocity field $v_{\text{fluid}}^n(x, y', t)$ onto these POD modes over the range $y'_{\text{min}} \leq y' \leq y'_{\text{max}}$. One result of this projection is a set of coefficients at each
We first perform POD for one slowly-decaying portion of the relaxation cycle shown in Fig. 4.6, composed of 1057 equally spaced snapshots taken from \( tU_0/D = 550 \) to \( tU_0/D = 950 \) of the time series. We first compute the cumulative energy content of the retained modes \((k)\) based on the eigenvalues of the velocity correlation matrix

\[
E_k = \sum_{i=1}^{N_k} \frac{\lambda_i}{\sum_{i=1}^{N_s} \lambda_i}
\]

where \( A \) is the annular domain whose inner boundary is the surface of the cylinder, and whose outer boundary is defined by \( x_{\text{min}}, x_{\text{max}}, y'_{\text{min}}, \) and \( y'_{\text{max}} \).

Figure 4.27: Energy distribution of POD modes at \( Re = 100 \)

For this case, where the elongated vortex attributed to the rotating NES exists, the cumulative energy content in the first \( N_k \) retained modes is shown in Fig. 4.27a. For comparison, we also perform POD for the flow with cylinder undergoing VIV absent the NES, which is periodic at \( Re = 100 \); in this case we use 50 snapshots spanning one period of oscillation of the cylinder. The cumulative energy content of the first \( N_k \) POD modes for the periodic VIV solution without NES is plotted in Fig. 4.27b. We note that, as compared to the cylinder undergoing VIV without the NES, more modes are required when the NES is
Figure 4.28: Vorticity of POD modes computed using one slowly-decaying portion of the solution (NES parameters: \( \bar{r}_0 = 0.5, \epsilon_r = 0.3, \) and \( \lambda_r = 0.0034 \)).

Present, in fact, nine modes are required to capture approximately 97% of the energy, whereas in the absence of the NES, the two most energetic modes capture the same fraction of energy. This initial result indicates that, even though the rotating NES is not in direct contact with the fluid, it still has important dynamical effects on the flow.

We next proceed to analyze the spatial structure of the modes and temporal coefficients of the POD modes. We compute the vorticity field of each POD mode using that mode’s vector velocity field. The vorticity distribution corresponding to the twelve most energetic POD modes for the system with a rotating NES with parameters \( \bar{r}_0 = 0.5, \epsilon_r = 0.3, \) and \( \lambda_r = 0.0034 \), corresponding to an elongated vortex solution, are shown in Fig. 4.28, while those for the same system without the NES are shown in Fig. 4.29. Comparing the first four modes between Figs. 4.28 and 4.29, we see the influence of the elongated vortex on the POD modes. Mode 5 in Fig. 4.28 for the system with the NES shows a spatial structure which is not typical, as it is neither symmetric nor antisymmetric. There is no analogous mode resembling this mode for the NES-less case in Fig. 4.29. Due to symmetry of the boundaries and boundary conditions, one would expect the modes to be either symmetric or anti-symmetric. However, the pres-
Figure 4.29: Vorticity of POD modes for system without NES undergoing VIV $Re = 100$.

The temporal coefficients computed by projecting the solution onto the POD modes are shown in Fig. 4.30. As for the spatial structures shown in Fig. 4.28, the time coefficients exhibit the influence of the NES. The temporal coefficients corresponding to modes 4, 6 and 7 oscillate about a nonzero mean, indicating that there is a unidirectional flow of energy to/out of these modes.

To confirm that the observed differences in POD mode structure are due solely to the presence of the rotating NES, we perform another POD for a portion of the time series where the NES motion differs considerably from the first time period. For this case, we choose the second slowly-decaying portion of the solution from the time series shown in Fig. 4.6, in the range $1625 \leq tU_0/D \leq 2025$. During this portion of the solution, the NES continues to rotate after the chaotic transition, but the direction of rotation is opposite to that of the first portion of the solution. Since the direction of rotation of the NES is reversed compared to the earlier time period, the angular momentum of the NES between the two portions will be different, which should influence the flow in a different sense. The angular momentum of the NES in the two portions of the time series in

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Figure 4.30: Time coefficients corresponding to POD modes computed using one slowly-decaying portion of the solution (NES: $\bar{\nu}_0 = 0.5$, $\epsilon_r = 0.3$, and $\lambda_r = 0.0034$).

Figure 4.31: Vorticity of POD modes computed using second portion of the slowly-decaying solution with rotational NES.
question is shown in Fig. 4.32. The twelve most energetic modes for the second slow-decay cycle are shown in Fig. 4.31. Comparing the mode shapes to those in Fig. 4.28, we see that mode 5 is now a mirror image between these two cases. This indicates that the angular momentum of the NES, which is extracted from the vorticity in the flow, influences the POD modes in a significant way.

Without the NES, equal and oppositely signed vortices are shed from either side of cylinder, which are separated by a phase of 180 degrees. So, the total angular momentum in the system is conserved over a shedding period. However, when we introduce the rotational NES into system, this symmetry is broken as the rotation of the NES extracts angular momentum from the flow, effectively creating a deficit of angular momentum.

4.7.1 POD of a set of realizations in one slowly-decaying portion of the solution

In this section, we discuss the evolution of the POD modes during the first slowly-decaying portion of the time series, in order to understand the influence of the rotating NES on the POD modes. We sample 65 sets of snapshots in the range $500 \leq tU_0/D \leq 1000$; each set contains around 50 consecutive snapshots of flow data, and corresponds to one quasi-period of the cylinder oscillation. Hence, we perform 65 proper orthogonal decompositions and analyze the POD modes and their energy evolution as the solution progresses through the slowly decaying envelope.
The energy content of the POD modes at three different portions of the slow decay is shown in Fig. 4.33. The energy content shown in Fig. 4.33a corresponds to a time period where the response is at the beginning of the slowly-decaying mechanism; we observe that the energy distribution is quite similar to Fig. 4.27b. As we proceed along the slow decay, there is a continuous redistribution of energy among modes, as shown by Figs. 4.33b and 4.33c. The continuous evolution of energy for the first twelve modes during the slowly-decaying part of the solution is shown in Fig. 4.34.
Figure 4.34: Evolution of POD modes energy distribution during one slowly-decaying solution regime.

4.8 Characterization of the chaotic response

Ever since Van Atta and Gharib (1987) showed that Sreenivasan’s observation of temporal chaos in flow past a tensioned piano wire (Sreenivasan (1985)) could be attributed to vibration of the wire, there has been interest in chaotic VIV, particularly because VIV apparently offers a platform for the study of temporal chaos in an unforced shear flow at a Reynolds number sufficiently small that the flow is laminar. From an applications standpoint, one of the key issues is the fact that, for a flexible structure, chaotic vibration and periodic vibration have quite different implications for fatigue life (cf. Modarres-Sadeghi et al. (2011)). From a fundamental fluid mechanics standpoint, chaotic response in the range $60 \leq Re \leq 120$ allows for the detailed study of a chaotic flow in a regime where a two-dimensional flow is presumably linearly stable, and where spatial resolution requirements are quite modest. We note that, unlike the computations of others in which temporal complexity is found in flow past a cylinder in this range of $Re$ (Karniadakis and Triantafyllou (1989b); Li et al. (1992); Nakano and Rockwell (1994); Cossu and Morino (2000); Baek and Sung (2000)), we consider a rigid cylinder with no end effects, apply no periodic excitation, and introduce no “structural damping coefficient”.

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Here, we focus on several measures of the temporal chaos, including the attractor dimension of both the cylinder velocity and the cross-stream and stream-wise components of the fluid velocity at several locations in the flow, as well as wavelet transformations of the same quantities. We also present results of the cross-correlation of the cylinder velocity with each velocity component at several points in the flow.

The solution with chaotic response shown in Fig. 4.6 is analyzed further to ascertain the chaotic behavior of the flow at $Re = 100$, which is known to be in the laminar VIV regime for a sprung cylinder without NES. The velocity of the cylinder during the chaotic regime of the solution is shown in Fig. 4.35a, and its frequency content analyzed by Wavelet Transform (WT) and “fast-Fourier transform” (FFT) are shown in Figs. 4.35c and 4.35e respectively. The broadband frequency content indicates the chaotic nature of the cylinder motion in this time range of the solution. For comparison, the cylinder velocity for the case of NES-less periodic VIV at $Re = 100$ is shown in Fig. 4.35a, and its WT and FFT are shown in Figs. 4.35d and 4.35f, respectively. The broadband frequency content for the system with NES indicates the chaotic nature of the cylinder motion in this time range, whereas the periodic solution shows the single dominant frequency close to the Strouhal frequency as expected.

The fluid motion in the same time range during which chaotic solution was observed is analyzed at two fixed locations in the wake close to the cylinder. The first point is located at one diameter downstream of the rear of the cylinder when its center lies on the mid-line, and one diameter above the mid-line. For the NES-less periodic solution at $Re = 100$, the velocity components in an inertial frame, and their frequency content obtained by WT and FFT are shown in Figs. Figs. 4.36a, 4.36c, and 4.36e, respectively for the streamwise component, and in Figs. 4.36b, 4.36d, and 4.36f, respectively for the cross-stream component. The WT and FFT clearly show a dominant frequency, indicating the periodic nature of the flow in the wake.
Figure 4.35: Cylinder velocity exhibiting chaotic nature with NES (a,c,e) and periodic solution at $Re = 100$ for system without NES (b,d,f).
The cross-correlations of the cylinder velocity with the fluid velocity for the streamwise and cross-stream components are computed as

\[ a_u(\tau_c) = \frac{1}{U_0^2(T_2 - T_1)} \int_{T_1}^{T_2} V_{cyl}(t)u(x_p, y_p, t + \tau_c)dt \]  
(4.14a)

\[ a_v(\tau_c) = \frac{1}{U_0^2(T_2 - T_1)} \int_{T_1}^{T_2} V_{cyl}(t)v(x_p, y_p, t + \tau_c)dt \]  
(4.14b)

The cross-correlations for the case of periodic solution without NES are shown in Figs. 4.36g and 4.36h for the streamwise and cross-stream velocity, respectively, with \( T_1 = 150 \) and \( T_2 = 275 \). Both the cross-correlations remain periodic, as one would expect for a periodic solution.

For the case of the system with rotational NES exhibiting chaotic cylinder motion, the fluid velocity one diameter from the rear of the cylinder (when it is on the mid-line) and one diameter above the mid-line are shown in Figs. 4.37a and 4.37b for the streamwise and cross-stream directions, respectively. The frequency content for the streamwise velocity obtained via WT and FFT is shown in Figs. 4.37c and 4.37e, respectively, and in Figs. 4.37d and 4.37f, respectively, for the cross-stream direction. Similar to the motion of the cylinder, the flow in the near wake is also chaotic as can be observed from the frequency spectrum and comparison to the periodic case shown in Fig. 4.36. The cross-correlations for the case of chaotic solution are shown in Figs. 4.37g and 4.37h for the streamwise and cross-stream velocity components, respectively, with \( T_1 = 125 \) and \( T_2 = 700 \). The correlation decays with time for both streamwise and cross-stream velocity; this strongly suggests the chaotic nature of the flow at \( Re = 100 \) due to presence of the rotational NES.

For the same solution with a rotational NES, a second spatial location is also analyzed, which is three diameters downstream of the rear of the cylinder (when it is on mid-line) and one diameter above the mid-line. The streamwise and cross-stream velocity components are shown in Fig. 4.38a and 4.38b, respectively. The frequency content analyzed via WT and FFT is shown in Figs. 4.37c and 4.37e, respectively for the streamwise component, and in Figs. 4.37d
Figure 4.36: Fluid velocity exhibiting periodic nature at a spatial point located 1D downstream of rear stagnation point of stationary cylinder and 1D offset from the cylinder mid-plane.
Figure 4.37: Fluid velocity exhibiting chaotic nature at a spatial point located $1D$ downstream of rear stagnation point of stationary cylinder and $1D$ offset from the cylinder mid-plane.
and 4.37f, respectively, for the cross-stream component. The cross-correlations for both the velocity components with the cylinder velocity are shown in Fig. 4.38g and 4.38h with $T_1 = 125$ and $T_2 = 700$, indicating decay with time, thus again suggesting chaotic behavior of the flow at this location also. Thus, based on the results shown in Figs. 4.37 and 4.38, the presence of an internal dissipative rotational NES can lead to temporally chaotic flow in the laminar regime.

To complement the frequency spectra and autocorrelation functions, we estimate the correlation dimension as a measure of the fractional dimension of the chaotic attractor. We use the algorithm of Grassberger and Procaccia (1983), and apply the code developed by Hegger et al. (1999) to the cylinder displacement time series shown in figure 4.6a, with logarithmic plots of the correlation dimension $C_m(\sigma)$ vs the “distance” $\sigma$ shown in figure 4.39 for six values of the embedding dimension $m$, using about 300 points per quasi-period, a time delay corresponding to 60 sampled points, and a Theiler window parameter corresponding to 800 sampled points.

The results, along with least-squares lines fitted over the indicated range of $\sigma$, show that the slope (corresponding to the correlation dimension) approaches a limiting value of about 3.2. This value is consistent with a low-dimensional attractor expected for a low-$Re$ flow with two additional degrees of freedom (the cylinder oscillation and NES rotation), and is comparable to to the fractal dimensions of 2.48, 3.10, and 4.65 found for chaotic two-dimensional flow past a NACA 0012 airfoil ($Ma = 0.2, 20^\circ$ angle of attack) at $Re = 1600, 2000, and 3000$, respectively (Pulliam and Vastano, 1993).

It is thus clear that a rotational NES can lead to temporal chaos of the cylinder motion and the flow, at Reynolds numbers where standard VIV is strictly time-periodic.
Figure 4.38: Fluid velocity exhibiting chaotic nature at a spatial point located 3D downstream of rear stagnation point of stationary cylinder and 1D offset from the cylinder mid-plane.
Figure 4.39: Convergence of correlation dimension (slope of least-squares fitted line to correlation sum data) with increase in embedding dimension $m$: $C_m(\sigma)$; $-$, least-squares fit for $C_m(\sigma)$ computed a range of $\sigma$ shown. (a) $m = 1$, $D_{corr} = 0.9743$. (b) $m = 2$, $D_{corr} = 1.8626$. (c) $m = 3$, $D_{corr} = 2.5524$. (d) $m = 4$, $D_{corr} = 2.8813$. (e) $m = 5$, $D_{corr} = 3.1818$. (f) $m = 6$, $D_{corr} = 3.1219$. 
4.9 Conclusions

Apart from passive VIV suppression, the rotational NES is found to strongly influence the wake structure close to the cylinder. The transient wake modification is also associated with motion of the coupled system, which is very close to its corresponding Hamiltonian system. The solution with an elongated wake was found to have an increased effective NES-induced mass, leading to a shift in the response frequency away from the Strouhal frequency. The slow decay in response, and the frequency shift approximated by our analytical model, match that of the full PDE solution. The POD decomposition of the elongated wake flow indicates continuous redistribution of the energy content among modes as the system responds with slow-decaying envelope motion. The elongated wake solution is found to exist at different combinations of NES parameters at different $Re$. The elongated wake solution for one set of NES parameters is also found to be robust for small changes to the NES parameter values.

Unforced (i.e., autonomous) chaotic motion of the cylinder at lower Reynolds number is found for a system with a rotational NES. The chaotic flow is characterized by analyzing the frequency spectrum of the fluid velocity in the wake and cross-correlation of the fluid velocity with the cylinder velocity. Based on the limited computational results discussed in this chapter, the system with rotational NES has very rich dynamics with far more interesting effects than just passive VIV suppression.
In this chapter, we discuss an approach to develop a ROM for the laminar VIV of a rigid circular cylinder. We utilize the high fidelity computational results available from the finite-element solution and develop a data-driven approach to formulate an ODE for the coupled flow-cylinder system. Our goal is to utilize the available data and construct a corresponding low-dimensional model. This approach may be rigorously extended to develop a predictive ROM for laminar VIV once we characterize the low-dimensional behavior of the coupled system. We will not assume a standard LCO oscillator \textit{a priori}; instead, we assume a generic second-order ODE to model the combined flow-cylinder interaction. The generic ODE is assumed to have nonlinear damping and restoring force coefficients that are functions of the state variables with several unknown parameters. We estimate the unknown parameters of the ODE such that the solution approximately matches the high-fidelity computational results. Coupling a NES to the ODE yields a two-DOF ROM, approximating the complete flow-cylinder-NES coupled system.

5.1 Description of the model\textsuperscript{1}

We assume the single-DOF self-excited oscillator to model the LCO of the cylinder in laminar flow as

\[
\ddot{y} + c_N \frac{dy}{dt} + q_{NL} y = 0 \quad \text{(5.1)}
\]

\textsuperscript{1}This work was performed in collaboration with Professor Oleg Gendelman and graduate student Elad Domany, Technion Israel Institute of Technology, Technion, Israel
where $\tilde{M}_{cyl}$ is the mass per unit length of the cylinder. The nonlinear damping coefficient $c_{NL}$ and restoring force coefficient $q_{NL}$ in equation (5.1) model the inherently unstable nature of the coupled system of flow and cylinder. An essential characteristic of a self-excited oscillator model is that it should have negative damping while the amplitude of the cylinder is lower than the limit-cycle amplitude, thereby driving the cylinder to larger amplitude oscillations. Once the limit-cycle amplitude is reached, the damping force should nullify in order to support the self-sustained oscillation of the cylinder. Hence, we assume the nonlinear damping of the self-excited oscillator model to be a function of the state variables, namely the displacement and the velocity of the cylinder. In order to keep the analysis general, we assume the frequency of oscillation of the self-excited oscillator to also be a function of the state variables. The explicit form of the nonlinear functions $c_{NL}$ and $q_{NL}$ will be discussed in Section 5.2.

Figure 5.1: Coupled system of cylinder and NES with nonlinear damping and nonlinear restoring force modeling the flow.

The self-excited oscillator model (5.1) is coupled to a single-DOF NES with cubic nonlinear stiffness that models the internal nonlinear dynamic absorber with essential stiffness nonlinearity, or NES. We note that the strong nonlinearizable stiffness nonlinearity is essential for our targeted energy transfer design (Vakakis et al. (2008)). The schematic of the coupled system of the cylinder and NES is shown in Fig. 5.1. The governing second-order ODEs for the coupled system in
dimensional form are written as

\[ M \frac{d^2 y_1}{dt^2} + c_{NL} \frac{dy_1}{dt} + q_{NL} y_1 + C_{nes} \left( \frac{dy_1}{dt} - \frac{dy_2}{dt} \right) + K_{nes} (y_1 - y_2)^3 = 0 \]  

\[ \dot{M}_{nes} \frac{d^2 y_2}{dt^2} + \dot{C}_{nes} \left( \frac{dy_2}{dt} - \frac{dy_1}{dt} \right) + \dot{K}_{nes} (y_2 - y_1)^3 = 0 \]

where, \( \dot{M}_{nes}, \dot{C}_{nes}, \) and \( \dot{K}_{nes} \) are the mass, damping and stiffness per unit length of the NES. The reduced mass per unit length of the cylinder is taken as \( M^* = \dot{M}_{cyl} - \dot{M}_{nes} \), ensuring no mass addition due to the internal NES. Dimensionless equations of motion are obtained by dividing equations (5.2a) and (5.2b) by \( M^* \), scaling the length using the diameter of the cylinder to define the dimensionless DOFs \( Y_1 = y_1/D \) and \( Y_2 = y_2/D \), and defining the dimensionless time as \( \tau = \omega_0 t \), where \( \omega_0 \) is the dominant frequency of the response. The dominant frequency in a laminar lock-in VIV response will be close to the Strouhal frequency (the natural shedding frequency at a particular \( Re \)). Using the notations \( \dot{\cdot}(\cdot) = \frac{d(\cdot)}{d\tau} \) and \( \ddot{\cdot}(\cdot) = \frac{d^2(\cdot)}{d\tau^2} \), the simplified dimensionless equations of motion are written as

\[ \ddot{Y}_1 + \gamma \dot{Y}_1 + Y_1 + \omega_n Y_1 + \epsilon \lambda \left( \dot{Y}_1 - \dot{Y}_2 \right) + \frac{4}{3} \epsilon \kappa (Y_1 - Y_2)^3 = 0 \]  

\[ \ddot{Y}_2 + \lambda \left( \dot{Y}_2 - \dot{Y}_1 \right) + \frac{4}{3} \kappa (Y_2 - Y_1)^3 = 0 \]

with

\[ \gamma = \frac{c_{NL}}{\omega_0 M^*} \quad \epsilon = \frac{\dot{M}_{nes}}{M^*} \quad \lambda = \frac{\dot{C}_{nes}}{M_{nes} \omega_0} \quad \kappa = \frac{3}{4} \frac{D^2 \dot{K}_{nes}}{M_{nes} \omega_0^2} \]

In writing equation (5.3a), the ratio of nonlinear force to the cylinder mass is expressed using the dominant frequency \( \omega_0 \) as:

\[ \frac{q_{NL}}{M^*} = \omega_0^2 (1 + \omega_n) \]

The yet unknown functional forms of the parameters \( \gamma \) and \( \omega_n \) of the self-excited oscillator model (5.3a) are assumed to be polynomial functions of the
displacement and the velocity of the cylinder. The justification to assume the polynomial form is based on the premise that a sufficiently higher degree of the polynomial provides a good approximation to any complicated function.

The second degree polynomials in terms of the state variable \( s \) for the damping \( \gamma(Y_1, \dot{Y}_1) \) and the frequency \( \omega_n(Y_1) \) are written as

\[
\gamma(Y_1, \dot{Y}_1) = \gamma_0 + \gamma_{10} Y_1 + \gamma_{01} \dot{Y}_1 + \gamma_{20} Y_1^2 + \gamma_{02} \dot{Y}_1^2 + \gamma_{11} Y_1 \dot{Y}_1 \\
\omega_n(Y_1) = \omega_1 Y_1 + \omega_2 Y_1^2 + \omega_3 Y_1^3 + \omega_4 Y_1^4
\] (5.4a, 5.4b)

For brevity of the analysis, we transform the coordinates of the NES DOF to the cylinder-fixed frame and use the NES relative displacement DOF defined as \( W = Y_1 - Y_2 \). Subtracting equation (5.3b) from (5.3a), we obtain the equations of motion of the cylinder in terms of \( Y_1 \) and the relative displacement of the NES in terms of \( W \) as

\[
\ddot{Y}_1 + Y_1 + \gamma(Y_1, \dot{Y}_1) \dot{Y}_1 + \omega_n(Y_1) Y_1 + \epsilon \lambda \dot{W} + \frac{4}{3} \epsilon \kappa W^3 = 0 \\
\ddot{W} + Y_1 + \gamma(Y_1, \dot{Y}_1) \dot{Y}_1 + \omega_n(Y_1) Y_1 + \lambda(1 + \epsilon) \dot{W} + \frac{4}{3}(1 + \epsilon) \kappa W^3 = 0
\] (5.5a, 5.5b)

The set of dimensionless coupled ODEs (5.5a) and (5.5b) model the complicated interaction of the flow with the NES-augmented cylinder. All the information related to the flow is encapsulated in the nonlinear damping \( \gamma(Y_1, \dot{Y}_1) \) and the nonlinear frequency \( \omega_n(Y_1) \) functions.

### 5.2 Estimation of the parameters of the self-excited oscillator model

In this section, we discuss the method to estimate the unknown coefficients in the assumed functions \( \gamma \) and \( \omega_n \), using the high-fidelity computational results obtained from the finite-element solution. The amplitude envelope of the cylinder displacement during the initial transient growth of the LCO is computed by taking the Hilbert transform of the time series of the finite-element solution. We approximate this envelope using our self-excited oscillator model by determining the unknown coefficients in the parameters \( \gamma \) and \( \omega_n \). To this end, the envelope
of the cylinder displacement predicted by the assumed self-excited oscillator is obtained by averaging out the fast dynamics. Consider the dimensionless equation of motion (5.5a) which, upon excluding the NES coupling terms, reduces to

$$\dddot{Y}_1 + \gamma(Y_1, \dot{Y}_1) \dot{Y}_1 + Y_1 + \omega_n(Y_1) Y_1 = 0$$  \hspace{1cm} (5.6)

The slow-fast partition of dynamics is performed by applying the complexification-averaging technique (Vakakis et al. (2008)), using the complex transformation $$\dot{Y}_1 + iY_1 = \varphi e^{i\tau}$$. With this representation, we introduce a slow/fast decomposition of the dynamics by expressing the complex measure $$(\dot{Y}_1 + iY_1)$$ in terms of a slow-varying complex amplitude $$\varphi$$ and a fast oscillation $$e^{i\tau}$$, with $$i = \sqrt{-1}$$. In considering this representation we assume that there exists a single (normalized) fast-frequency equal to unity in the dynamics. Extensions, however, to cases where more fast frequencies exist have been considered (Vakakis et al. (2008)). Averaging out the fast dynamics with single dominant frequency, the averaged equation of motion of the cylinder is given by

$$\dot{\varphi} = - \oint \gamma(Y_1, \dot{Y}_1) \dot{Y}_1 d\tau - \oint \omega_n(Y_1) Y_1 d\tau$$  \hspace{1cm} (5.7)

This complex differential equation governs the slow evolution of the modulation (envelope) of the response of the reduced order model (5.6). This is also referred to as the slow flow of (5.6). Each contour integral in (5.7) denotes integration with respect to a single normalized period of the fast oscillation (equal to $$2\pi$$). Substituting the polynomial functions for $$\gamma$$ and $$\omega_n$$ from equations (5.4a) and (5.4b) respectively into the slow flow (5.7), we express the evolution equation for the complex variable $$\varphi$$ governing the slow-flow of the cylinder motion as

$$\dot{\varphi} = \frac{3}{8} i\omega_2|\varphi|^2\varphi + \frac{5}{16} i\omega_4|\varphi|^4\varphi - \frac{1}{2} \gamma_0 \varphi - \frac{1}{8} \gamma_0 \varphi - \frac{3}{8} \gamma_2 \varphi^2 - \frac{1}{8} \gamma_1 \varphi^2 + \frac{1}{8} \gamma_1 \varphi^2$$  \hspace{1cm} (5.8)

As can be observed from equation (5.8), the coefficients of the terms of odd degree in the assumed polynomials are nullified by the process of averaging.
Hence, the simplified expressions for the polynomial functions, sufficient for further analysis, are given by

\[
\begin{align*}
\gamma(Y_1, \dot{Y}_1) &= \gamma_0 + \gamma_{20} Y_1^2 + \gamma_{02} Y_1^2 + \gamma_{11} Y_1 \dot{Y}_1 \\
\omega_n(Y_1) &= \omega_2 Y_1^2 + \omega_4 Y_1^4
\end{align*}
\]  

(5.9a)  

(5.9b)

We define the residue \( \Theta \), as the difference between the envelope of the displacement time series generated by the finite-element solution \(|\tilde{Y}_1|\) and the envelope of the slow-flow \(|\varphi|\) generated by integrating equation (5.8) as

\[
\Theta(p) = W_1 \int_{\tau_1}^{\tau_2} \left( |\varphi| - |\tilde{Y}_1| \right)^2 d\tau + W_2 \int_{\tau_1}^{\tau_2} \left( \angle \varphi - \angle \tilde{Y}_1 \right)^2 d\tau
\]  

(5.10)

In (5.10) \( \tau_1 \leq \tau \leq \tau_2 \) is the time interval of the initial transient growth of the LCO, \( \angle \varphi \) is the phase of the slow-flow \( \varphi \), and \( \angle \tilde{Y}_1 \) is the slow phase of \( \tilde{Y}_1 \). \( W_1 \) and \( W_2 \) are weighting factors for the slow amplitude and the slow-phase, respectively. The unknown coefficients of the damping function and frequency function in equation (5.8) are expressed as a parameter vector defined as \( p = [\gamma_0 \ \gamma_{20} \ \gamma_{02} \ \gamma_{11} \ \omega_2 \ \omega_4] \). We determine \( p \) such that the residue \( \Theta \) is minimized by solving the unconstrained minimization problem

\[
\text{Minimize } \Theta(p) \quad \text{ for } p \in \mathbb{R}
\]  

(5.11)

The minimization (5.11) is performed using the standard MATLAB subroutine "fminunc". The minimization routine requires an initial guess \( p_0 \) as input to start the minimization procedure by solving for the gradient of the objective function with respect to the parameters.

5.2.1 Global analysis of the self-excited oscillator model coupled to the NES

Assuming that the parameters of the self-excited oscillator model are determined to a sufficient degree of accuracy, we proceed to analyze the slow-flow of the coupled system to determine the fixed points and the corresponding slow invariant manifold of the system.
5.2.2 Slow-flow of the coupled system

The coupled system of equations (5.5a) and (5.5b) are averaged using the complexification-averaging technique with the following complex transformation for the cylinder and the NES DOF,

\[ \dot{Y}_1 + iY_1 = \varphi e^{i\tau} \]
\[ \dot{W} + iW = \xi e^{i\tau} \]

where again we have assumed that there is a single normalized frequency equal to unity in the dynamics. Then, averaging with respect to this fast frequency, the resulting modulation equations are expressed as:

\[ \dot{\varphi}(\tau) = -\frac{\epsilon \lambda}{2} \xi + \frac{i \epsilon \kappa}{2} |\xi|^2 \xi - \int \gamma(Y_1, \dot{Y}_1) \dot{Y}_1 d\tau - \int \omega_n(Y_1) Y_1 d\tau \] (5.12a)
\[ \dot{\xi}(\tau) = \frac{i}{2}(\varphi - \xi) - \left(1 + \epsilon\right) \frac{\lambda}{2} \xi + \frac{i \kappa (1 + \epsilon)}{2} |\xi|^2 \xi - \int \gamma(Y_1, \dot{Y}_1) \dot{Y}_1 d\tau - \int \omega_n(Y_1) Y_1 d\tau \] (5.12b)

For brevity, a new variable \( \Phi \) is introduced to represent the terms of the averaged damping and frequency of the self-excited oscillator,

\[ \Phi = -\int \gamma(Y_1, \dot{Y}_1) \dot{Y}_1 d\tau - \int \omega_n(Y_1) Y_1 d\tau \] (5.13)

which, when substituting the polynomial functions for \( \gamma(Y_1, \dot{Y}_1) \), \( \omega_n(Y_1) \), and averaging with respect to the fast frequency, yields the averaged expression

\[ \Phi = \frac{3i}{8} \omega_2 |\varphi|^2 \varphi + \frac{5i}{16} \omega_4 |\varphi|^4 \varphi - \frac{1}{2} \gamma_0 \varphi - \frac{1}{8} \gamma_20 |\varphi|^2 \varphi - \frac{3}{8} \gamma_{02} |\varphi|^2 \varphi + \frac{i}{8} \gamma_{11} |\varphi|^2 \varphi \] (5.14)

Hence, the equations of motion are rewritten in terms of \( \Phi \) as

\[ \dot{\varphi}(\tau) = \Phi - \frac{\epsilon \lambda}{2} \xi + \frac{i \epsilon \kappa}{2} |\xi|^2 \xi \] (5.15a)
\[ \dot{\xi}(\tau) = \Phi + \frac{i}{2} (\varphi - \xi) - \left(1 + \epsilon\right) \frac{\lambda}{2} \xi + \frac{i \kappa (1 + \epsilon)}{2} |\xi|^2 \xi \] (5.15b)
Introducing the polar transformations,

\[ \varphi(\tau) = R(\tau) e^{i\delta_1(\tau)}, \quad \xi(\tau) = P(\tau) e^{i\delta_2(\tau)}, \quad \delta(\tau) = \delta_1(\tau) - \delta_2(\tau) \]

leads to

\[
\begin{align*}
\dot{R} e^{i\delta} + i \delta_1 R e^{i\delta} &= -\frac{\epsilon}{2} \lambda P + i \frac{\epsilon}{2} \kappa P^3 + \tilde{\Phi} \quad (5.16a) \\
\dot{P} + i \delta_2 P &= \frac{i}{2} R e^{i\delta} - \frac{i}{2} P - \frac{(1 + \epsilon)}{2} \lambda P + i \frac{\epsilon}{2} (1 + \epsilon) P^3 + \tilde{\Phi} \quad (5.16b)
\end{align*}
\]

where \( \tilde{\Phi} \) is given by

\[
\tilde{\Phi} = \Phi \left( \frac{3}{8} i \omega_2 R^3 + \frac{5}{16} i \omega_4 R^5 + \frac{1}{8} i \gamma_{11} R^3 - \frac{1}{2} \gamma_9 R - \frac{1}{8} \gamma_{20} R^3 - \frac{3}{8} \gamma_{02} R^3 \right) e^{i\delta}
\]

\[
\begin{align*}
Re \left\{ \tilde{\Phi} \right\} &= \cos \delta \left( -\frac{1}{2} \gamma_9 R - \frac{1}{8} \gamma_{20} R^3 - \frac{3}{8} \gamma_{02} R^3 \right) \\
&\quad + \sin \delta \left( -\frac{3}{8} \omega_2 R^3 - \frac{5}{16} \omega_4 R^5 - \frac{\gamma_{11}}{8} R^3 \right) \quad (5.17a) \\
Im \left\{ \tilde{\Phi} \right\} &= \cos \delta \left( \frac{3}{8} \omega_2 R^3 + \frac{5}{16} \omega_4 R^5 + \frac{\gamma_{11}}{8} R^3 \right) \\
&\quad + \sin \delta \left( -\frac{1}{2} \gamma_9 R - \frac{1}{8} \gamma_{20} R^3 - \frac{3}{8} \gamma_{02} R^3 \right) \quad (5.17b)
\end{align*}
\]

Separating the equations of motion (5.16a) and (5.16b) into real and imaginary equations gives

\[
\begin{align*}
\dot{R} \cos \delta - \dot{\delta}_1 R \sin \delta &= -\frac{\epsilon}{2} \lambda P + Re \left\{ \tilde{\Phi} \right\} \quad (5.19a) \\
\dot{R} \sin \delta + \dot{\delta}_1 R \cos \delta &= \frac{\epsilon}{2} \kappa P^3 + Im \left\{ \tilde{\Phi} \right\} \quad (5.19b) \\
\dot{P} &= -\frac{(1 + \epsilon)}{2} \lambda P - \frac{1}{2} R \sin \delta + Re \left\{ \tilde{\Phi} \right\} \quad (5.19c) \\
\dot{\delta}_2 P &= \frac{1}{2} R \cos \delta - \frac{1}{2} P + \frac{\kappa}{2} (1 + \epsilon) P^3 + Im \left\{ \tilde{\Phi} \right\} \quad (5.19d)
\end{align*}
\]
Therefore, the evolution equations for the coupled cylinder-NES system in real variables are written as

\[
\begin{align*}
\dot{R} &= -\frac{\epsilon}{2} \lambda P \cos \delta + \frac{\epsilon}{2} \kappa P^3 \sin \delta + \left( -\frac{1}{2} \gamma_0 R - \frac{1}{8} \gamma_{20} R^3 - \frac{3}{8} \gamma_{02} R^3 \right) \\
\dot{P} &= -\frac{1}{2} \left( 1 + \epsilon \right) \lambda P - \frac{1}{2} R \sin \delta + \cos \delta \left( -\frac{1}{2} \gamma_0 R - \frac{1}{8} \gamma_{20} R^3 - \frac{3}{8} \gamma_{02} R^3 \right) \\
&\quad + \sin \delta \left( \frac{3}{8} \omega_1 R^3 - \frac{5}{16} \omega_4 R^3 - \frac{711}{2} R^3 \right) \\
\dot{\delta} &= \frac{\epsilon}{2} \lambda \frac{P}{R} \sin \delta - \frac{1}{2} \frac{R}{P} \cos \delta + \frac{\epsilon}{2} \kappa \frac{P^3}{R} \cos \delta - \left( 1 + \epsilon \right) \gamma_0 + \frac{1}{2} \\
&\quad + \frac{1}{R} \left[ \cos \delta \left( \frac{3}{8} \omega_2 R^3 + \frac{5}{16} \omega_4 R^3 + \frac{711}{2} R^3 \right) + \sin \delta \left( -\frac{1}{2} \gamma_0 R - \frac{1}{8} \gamma_{20} R^3 - \frac{3}{8} \gamma_{02} R^3 \right) \right]
\end{align*}
\]

The averaged equations (5.20a)-(5.20c) form the basis for further analysis. The approximations that have been made in deriving the equations (5.20a)-(5.20c) are related to the complexification-averaging operations performed previously and to the existence of a single normalized fast frequency equal to unity in the dynamics. In addition, the unknown damping and frequency of the cylinder excited by the flow are approximated.

### 5.3 Slow invariant manifold

The conservation of energy for the coupled problem is obtained by finding the first integral of motion

\[
\frac{dE}{d\tau} = \frac{d}{d\tau} \left( \frac{1}{2} \dot{Y}_1^2 + \frac{1}{2} \dot{Y}_2^2 + \frac{1}{2} \int (1 + \omega_n(Y_1)) Y_1 d\tau + \frac{1}{4} \kappa W^4 \right) = -\epsilon \lambda \dot{W}^2 - \gamma (Y_1, \dot{Y}_1) Y_1^2
\]

Applying complexification-averaging to the dissipative terms on the right side of the equation (5.21), conservation of energy is found to be

\[
\epsilon \lambda Z + Y \left( \gamma_0 + \frac{1}{4} Y \left( 3\gamma_{02} + \gamma_{20} \right) \right) = 0
\]
Equation (5.22), obtained after substitution of the variables \( Y = R_0^2 \) and \( Z = P_0^2 \), represents a parabola in the \( Y - Z \) plane. The subscript 0 denotes that the variables correspond to fixed points of the system (5.20a)-(5.20c); i.e., to the steady state solutions of the averaged coupled problem.

The evolution equations (5.20a)-(5.20c) are simplified using the previous energy conservation as given by equation (5.22), resulting in the set of equations

\[
0 = \lambda \frac{P_0^2}{R_0} \left( \frac{R_0}{P_0} \cos \delta_0 - 1 \right) - \kappa P_0^4 \sin \delta_0 (5.23a)
\]

\[
0 = (1 + \epsilon) \lambda P - 0 - \epsilon \lambda \frac{P_0^2}{R_0} \cos \delta_0 + R_0 \sin \delta_0 
+ \frac{1}{4} \left( \gamma_{11} + 3 \left( \omega_2 + \frac{5}{6} \omega_4 R_0^2 \right) \right) R_0^3 \sin \delta_0 \quad (5.23b)
\]

\[
0 = 1 - \left( \frac{R_0}{P_0} - \frac{\kappa P_0^2}{R_0} \right) \cos \delta_0 - (1 + \epsilon) \kappa P_0^2 
+ \frac{1}{4} \left( \gamma_{11} + 3 \left( \omega_2 + \frac{5}{6} \omega_4 R_0^2 \right) \right) R_0^2 \left( 1 - \frac{R_0}{P_0} \cos \delta_0 \right) \quad (5.23c)
\]

Solving for \( \sin \delta_0 \) and \( \cos \delta_0 \) from the first two equations and using the trigonometric relation \( \sin^2 \delta_0 + \cos^2 \delta_0 = 1 \), we obtain the equation of the slow invariant manifold (SIM)

\[
\epsilon ZG \left[ \epsilon (Z - Y) - 2Y \right] + Y \left[ Y f_1^2 (Z - Y) + Z^2 \left( \lambda^2 - 2\kappa Y f_1 \right) - ZG \right] = 0 \quad (5.24)
\]

where

\[
f_1(Y) = 1 + \frac{Y}{4} \left( \gamma_{11} + 3 \left( \omega_2 + \frac{5}{6} \omega_4 Y \right) \right)
\]

\[
G(Y, Z) = \lambda^2 (Z - Y) - YZ^2 \kappa^2
\]

The SIM represents the fixed points of the coupled averaged problem. The intersection of the parabola defined by equation (5.22) and the SIM defined by the equation (5.24) provides the possible solutions of the coupled system (Gendelman et al. (2010); Gendelman and Bar (2010)). The analytical solution for the intersection is not attempted here due to the complexity of the equations; instead, graphical solutions will be presented in this paper.
In previous work on similar dynamical systems (Gendelman et al. (2010); Gendelman and Bar (2010); Gendelman (2011)), the SIM was derived based on asymptotic analysis of the averaged flow. Based on the asymptotic analysis, solution to the leading order equations with time scale $\tau_0$ resulted in a simplified cubic expression for the SIM. The cubic equation for the SIM reveals several important characteristics regarding the global dynamics of the strongly nonlinear system. In the current system of the self-excited oscillator with the NES, we refrain from asymptotic analysis, as we cannot make any assumption on the order of magnitude of the terms that are estimated from the finite-element solution. However, the conjectures from the previous work are restated here, as they are also relevant for the current system, but are not apparent due to the complicated nature of the SIM equation (5.24). For reference the SIM based on asymptotic analysis by Gendelman and Bar (2010) is repeated below, where the same notation as in the current analysis has been used.

$$Y(\tau_1) = Z(\tau_1) \left( \lambda^2 + (1 - \kappa Z(\tau_1))^2 \right)$$  \hspace{1cm} (5.25)

The equation (5.25) is a function of super-slow time $\tau_1 = \epsilon \tau_0$, representing the

![Diagram](image)

Figure 5.2: Structure of the slow invariant manifold for $\lambda \leq 1/ \sqrt{3}$ with two stable and one unstable branches.
slow drift of the averaged flow. The SIM (5.25) will have one stable branch for sufficiently strong damping ($\lambda > 1/\sqrt{3}$) or one unstable and two stable branches for relatively small damping ($\lambda \leq 1/\sqrt{3}$). A typical (super-slow) flow on the SIM with $\lambda \leq 1/\sqrt{3}$ is depicted in Fig. 5.2. In the case when the super-slow flow can bring the system to the fold points $Z_1$ or $Z_2$, where the stable branches cease to exist, the flow jumps at slow time scale to the landing points $Z_u$ or $Z_d$ on the respective stable branch. This is a typical relaxation oscillation cycle and has been referred to as a strongly modulated response (SMR). Realization of a SMR depends on whether the slow drift brings the system to the fold points. This implies that one should study the higher order system in the limit $\tau_0 \to \infty$.

Analysis of such super-slow time scale dynamics resulted in the equation of a parabola (5.22), which also represents the conservation of energy. Noting these previous results, we can analyze the global dynamics of the current ROM by studying the intersections of the parabola (5.22) and SIM (5.24).

In the following Section reduced order models are constructed for different regimes of the dynamics. Specifically, we consider limit cycle oscillation - LCO and strongly modulated response - SMR regimes detected by finite-element simulations, and compare to the responses predicted by the reduced-order models and the numerical simulations in order to validate the previous analytical reduction methodology.

5.4 Results

In this Section, we present the reconstruction of the approximate dynamics of the coupled system using our self-excited oscillator-based ROM. First, the LCO of the cylinder-flow system without the NES is presented, to ascertain that our self-excited oscillator model is capturing the cylinder and flow interaction accurately. Then, the ROM for the NES-coupled system is presented for two types of passive suppression mechanisms that were observed in the finite-element simulations. We also discuss the range of validity of the current approach by highlighting the limitations of the proposed approach.
Throughout this section the finite-element computational results are referred to as “PDE” (partial differential equation simulations), the ROM solution as “ODE” (ordinary differential equation reduced order models), and “HT” stands for Hilbert transform. All the results discussed in this Section correspond to a flow with $Re = 100$ and dimensionless density ratio of 10, which is defined as the ratio of the density of the cylinder to the fluid. The initial guess vector for parameter estimation is $p_0 = [-0.1 \ 0.1 \ 0.1 \ 0.1 \ 0.01 \ 0.01]$, and is retained for all the cases studied here; moreover, the weighting factors are set to $W_1 = 100$ and $W_2 = 1$.

5.4.1 LCO of the cylinder in flow without the NES

![Figure 5.3: LCO of the cylinder in flow with $Re = 100$, and the optimized fit of the transient envelope](image)

The LCO due to VIV of the sprung cylinder in flow with $Re = 100$ is as shown in Fig. 5.3a. The LCO amplitude of the cylinder displacement is equal to $0.49D$, where $D$ denotes the diameter of the cylinder. The initial transient growth of the LCO in the normalized time interval $84 \leq \tau \leq 168$ is used to estimate the nonlinear damping and frequency. The parameters that produce the best fit to the transient envelope shown in Fig. 5.3b are given by

$$
\gamma(Y_1, \dot{Y}_1) = -0.1872 + 0.4581Y_1^2 + 1.1740\dot{Y}_1^2 - 1.2585Y_1\dot{Y}_1
$$

$$
\omega_n(Y_1) = -4.0654Y_1^2 + 28.5953Y_1^4
$$
Knowing $\gamma(Y_1, \dot{Y}_1)$ and $\omega_n(Y_1)$, we integrate equation (5.6) to recover the LCO

![Graph](image)

Figure 5.4: ROM solution for cylinder in flow at $Re = 100$.

using the self-excited oscillator as shown in Fig. 5.4. The amplitude $Y_1$ of the cylinder increases under the action of the negative damping force until zero net damping force is reached, after which the response settles to a self-sustained oscillation. The initial condition for forward integration is taken from the finite-element solution at $\tau = 84$. The LCO predicted by the self-excited oscillator as shown in Fig. 5.4 exhibits fully developed amplitude of $0.434D$. Thus, the self-excited oscillator predicts the LCO of the cylinder displacement with a marginal error of $11.43\%$. In addition, the long-time solution remains at constant amplitude without any divergence. Hence, the relatively simple system of the cylinder in flow without the NES is approximated quite accurately by the ROM. In the following section this modeling approach is employed for two systems with internal NESs.

### 5.4.2 Suppressed LCO

The displacement of the cylinder and the NES relative displacement obtained by finite-element simulation are as shown in the Figs. 5.5a and 5.5b, respectively; this solution is identified as “suppressed LCO”. The suppressed LCO is observed for the NES with parameter values $\kappa = 5.991$, $\lambda = 0.485$, and $\epsilon = 0.052$. The envelope of the cylinder displacement grows and saturates in the time interval $84 \leq \tau \leq 120$. The coefficients of the assumed nonlinear damping and frequency functions are evaluated by fitting an approximation to this transient cylinder displacement envelope. The optimized fit is as shown in Fig. 5.6; the envelope

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of the PDE solution is shown in the same plot for comparison.

The damping and frequency as functions of the displacement and velocity

\( \gamma(Y_1, \dot{Y}_1) = -0.2586 + 1.7467Y_1^2 + 5.0400\dot{Y}_1^2 - 0.7871Y_1\dot{Y}_1 \)

\( \omega_n(Y_1) = -2.6512Y_1^2 - 0.0271Y_1^4 \)

Knowing \( \gamma(Y_1, \dot{Y}_1) \) and \( \omega_n(Y_1) \), we forward integrate the coupled nonlinear ODEs (5.15a) and (5.15b). The initial conditions for forward integration are taken from the finite-element solution at \( \tau = 84 \). The slow-flow solution obtained by integration of the ROM is shown in Figs. 5.7a and 5.7b, respectively. For comparison, the PDE solution of the cylinder displacement and the NES

Figure 5.5: PDE solution of suppressed LCO mechanism.

Figure 5.6: Optimized transient growth envelope fit of the averaged ODE to match the PDE envelope for the suppressed LCO mechanism.
Table 5.1: Quantitative comparison of PDE and ROM solution for SMR I. (Error is computed between PDE and ROM)

<table>
<thead>
<tr>
<th></th>
<th>PDE</th>
<th>ROM</th>
<th>SIM</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cylinder displacement</td>
<td>0.1989</td>
<td>0.2052</td>
<td>0.2052</td>
<td>3.167</td>
</tr>
<tr>
<td>NES relative displacement</td>
<td>0.4003</td>
<td>0.3657</td>
<td>0.3657</td>
<td>-8.643</td>
</tr>
</tbody>
</table>

are shown in the same figures. The error in the amplitude of the cylinder displacement at the fully developed LCO is approximately 3.167% and that in the NES relative displacement approximately -8.643%.

![Figure 5.7](image)

Figure 5.7: ODE solution of the suppressed LCO mechanism.

![Figure 5.8](image)

Figure 5.8: Intersection of the SIM and parabola for suppressed LCO.

The topology of the intersection of the parabola (5.22) and SIM (5.24) are as shown in Fig. 5.8. The only intersection is on the stable upper branch of the SIM; thus, the only possible attractor for the dynamics is a stable LCO (Gendelman and Bar (2010)). The intersection point of the SIM and the parabola also gives the amplitude of the slow-flow envelope of both the cylinder displace-
ment and the NES relative displacement as predicted by the ROM. The LCO responses obtained by PDE and ROM are compared with the intersection of the parabola and the SIM in Table 5.1, and good agreement between the ROM and PDE is observed. Also, the intersection of the SIM and the parabola agrees with the solution obtained by forward integration. However, it should be noted that the intersection of the parabola and SIM occur close to the fold of the SIM. Hence, a small error in estimating the system parameters can result in a large deviation in the solution predicted by the ROM.

5.4.3 Strongly Modulated Response

The second type of response is a strongly modulated response (SMR) obtained for the NES with parameters $\kappa = 9.735$, $\lambda = 0.242$ and $\epsilon = 0.111$. The PDE solution of the cylinder and NES relative displacement are shown in Figs. 5.5a and 5.5b, respectively; this solution is identified as SMR. The envelope of the cylinder displacement grows from zero amplitude to approximately $0.18D$ and then starts modulating. However, the “fast-frequency” of both the cylinder and NES remain close to the natural shedding frequency. Hence, we expect our averaged system to exhibit such a response. The cylinder displacement in the time interval $84 \leq \tau \leq 105$ is used for parameter estimation. An approximation to this transient cylinder displacement envelope determines the unknown coefficients of the damping and frequency functions; the accuracy of the fit is shown in Fig. 5.10. The damping and frequency as a function of displacement
and velocity of the cylinder are found to be

\[
\gamma(Y_1, \dot{Y}_1) = -0.2922 + 3.8255Y_1^2 + 11.2766\dot{Y}_1^2 - 2.9919Y_1\dot{Y}_1
\]

\[
\omega_n(Y_1) = -9.2658Y_1^2 + 0.058Y_1^4
\]

The forward integration of the ROM given by equations (5.15a) and (5.15b) and comparison with the envelope of the PDE solution are shown in Figs. 5.11a and 5.11b, respectively. As can be observed from Figs. 5.11a and 5.11b, the maximum and minimum amplitudes of the modulated envelopes are predicted accurately by the ROM. The modulation frequencies of the cylinder displacement and NES relative displacement also agree quite well with those predicted by the PDE solution. The reduction of dynamics on the SIM is shown in Fig. 5.12; the only intersection of the SIM and parabola is on the unstable middle branch of the SIM. Thus, the SMR is the only stable attractor for this system, which is consistent with the solution predicted by solving the PDE and with the solution reproduced by the ROM.

5.4.4 Scope of validity of the ROM

We have demonstrated a framework of model reduction for a complicated coupled system utilizing data obtained from a finite-element solution. Such a low-dimensional model has the obvious utility of providing computationally inexpensive long time forward integration; in addition, the low-dimensional model also reveals the existence of the SIM. In this section, we discuss the limitations
(a) Cylinder displacement  
(b) NES Relative displacement

Figure 5.11: ODE solution for SMR mechanism.

Figure 5.12: Intersection of SIM and parabola for SMR.
of the current approach to model reduction. As noted in the introduction, our
goal is not to develop a predictive model for laminar VIV, and in fact, this is not
the most general approach to obtain a ROM for VIV. In this section, we illus-
trate the inability of the current approach to model a high dimensional response.

**Single dominant frequency:**
The current approach to model reduction is based on the assumption that there
is a single dominant fast frequency, so that the averaging procedure of Section
5.2.1 remains valid. This is not a very restrictive assumption; as mentioned ear-
lier, the laminar VIV is characterized by the response being at a single dominant
frequency close to the Strouhal frequency. However, the NES-coupled system
with SMR response will have some additional frequency content, but at gener-
ally lower amplitude when compared to the dominant frequency. Hence even in
the case of SMRs our reduced modeling technique can be applied provided that
the additional fast frequency harmonics are small compared to the dominant
one.

**Small mass ratio:**

Figure 5.13: PDE solution of a high-dimensional SMR mechani-

The NES mass ratio $\epsilon$ is assumed to be small in this analysis, since the com-
putation of the SIM relies on an asymptotic process using epsilon as the small
parameter. In the development of the 2-DOF model, the flow and cylinder are
modeled as a single-DOF self-excited oscillator. If the NES mass is large, the
Figure 5.14: Optimized transient growth envelope fit of averaged ODE to match the PDE envelope for SMR mechanism.

Figure 5.15: ODE solution for a high-dimensional SMR mechanism.

Figure 5.16: Intersection of SIM and parabola for a high-dimensional SMR.
assumed single-DOF model is not appropriate to represent the combined flow-cylinder system.

**High-dimensionality of the response:**
The governing equations of the coupled problem are the Navier-Stokes equations for the flow and second-order ODEs for the cylinder and NES. The original governing equations can admit solutions that may not be possible to replicate using our assumed 2-DOF model; we illustrate one such solution in this section. We consider a SMR from the PDE solution as shown in Figs 5.13a and 5.13b, respectively, which corresponds to system parameters $\kappa = 1.294$, $\lambda = 0.105$ and $\epsilon = 0.0638$. The optimized fit is shown in Fig. 5.14, which yields the coefficients of the assumed nonlinear damping and frequency functions as

$$
\gamma(Y_1, \dot{Y}_1) = -0.2283 + 0.7182Y_1^2 + 1.9545\dot{Y}_1^2 - 0.2169Y_1\dot{Y}_1
$$

$$
\omega_n(Y_1) = -0.9408Y_1^2 - 0.0391Y_1^4
$$

Although, the system has a small mass ratio and response is at a single dominant fast frequency, the ODE forward integration fails to reproduce the PDE solution as shown in Figs. 5.15a and 5.15b, respectively. We can observe the complex nature of the solution by inspecting the intersection of the SIM and parabola, as shown in Fig. 5.16. There are multiple intersections, thus making it impossible to determine which solution the ROM is attracted to. Thus, the SIM provides us with a tool to ascertain if the model reduction is appropriate to a given response. In summary, we require a simple intersection either on a stable branch or the unstable branch. Qualitatively, when the intersection is not simple, the original problem which is of higher dimensionality is not amenable to the current approach to model reduction.

### 5.5 Conclusions

A data-driven approach is adopted to develop a ROM for laminar VIV of a sprung rigid circular cylinder with attached NES. The self-excited oscillator model for the cylinder-flow interaction is assumed in generic form; the final form...
of the model is determined using the high-fidelity computational data. The self-excited oscillator with state variables-dependent damping and frequency accurately models the LCO of the system without the NES. The two-DOF model for the coupled system with NES is developed from this framework. The fully coupled system ROM is tested exhaustively against high-fidelity finite-element solutions; two representative cases are presented in detail in this paper. The simple two-DOF ROM qualitatively reproduces the dynamics of the coupled fluid-cylinder-NES system dynamics. The accuracy of the ROM in reproducing the dynamics with just two-DOF depends on satisfying three criteria. First, the PDE solution should have only one dominant fast frequency with quasi-periodic modulation of amplitude, thus being in conformance with the assumptions of the averaging principle. Second, the dimensionless mass ratio $\epsilon$, which is used as a small parameter, should indeed be small. Third, the reduction of the dynamics on the SIM should be represented by a simple intersection of the SIM and the parabola; namely, the intersection should be either on the upper or lower branch of the SIM, or the intersection should be on the unstable middle branch only, away from the fold and the jump points.
CHAPTER 6

ALTERNATIVE REDUCED-ORDER MODELING OF FLUID-STRUCTURE INTERACTION

In this chapter, we discuss two additional methodologies that were considered in order to formulate a ROM for the VIV of the cylinder. The objective of these efforts was to construct such a ROM for fluid-structure interaction in order to couple it with the NES and explore the coupled system dynamics. Having such a ROM will facilitate detailed parametric study of VIV with the aim to identify the optimal parameters of the NES for optimal VIV suppression. Moreover, it can lead to predictive design for passive VIV suppression and to better understanding of the nonlinear dynamics induced by the internal NES on the fluid-structure interaction.

6.1 Proper orthogonal decomposition

The finite-element computations provide us with the pressure and the vector velocity fields. These are performed for the case of, (i) a motionless cylinder, (ii) flow past a cylinder undergoing a prescribed motion, and (iii) free VIV of an elastically supported cylinder. The computed fields of the velocity and pressure are used to extract the energetic spatial coherent structures and a set of orthogonal basis are then constructed by “proper orthogonal decomposition” (POD) (Sirovich (1987); Holmes et al. (1996); Noack et al. (2003)). Galerkin projection of the governing Navier-Stokes equation onto this POD basis gives us a set of coupled “ordinary differential equations” (ODE) governing the temporal coefficients of the POD modes. This set of coupled nonlinear ODEs modeling the fluid flow can be coupled to the equations of motion of the cylinder augmented with a NES to obtain a reduced-order model (ROM) for our complete system.

1This work was performed in collaboration with Dr. Guy Ben-Dov, currently Research Scientist at KLA-Tencor, Migdal Haemek, Israel.
integrating the flow, the cylinder, and the NES.

6.1.1 Formulation

The method of snapshots as proposed by Sirovich (1987) is used to construct the POD modes. We consider \( N_s \) snapshots of the pressure \( p \) and the vector velocity \( \mathbf{v} \) over one period of shedding cycle derived directly from the computational data during which the lift force has been fully developed and the cylinder undergoes a limit-cycle oscillation (LCO). Only the time-varying component of the velocity and the pressure snapshots are retained by subtracting the corresponding mean values over one period of the shedding cycle. We restrict the domain of \( p \) and \( \mathbf{v} \) to the subdomain \( \Omega \) that always lies within the original domain on which the finite-element computation was performed. In addition, the Laplacian of the pressure field is computed for each of the snapshots. The temporal correlation matrix \( C_{\mathbf{v}ij} \) for vector velocity and \( C_{ij}^{\nabla^2 p} \) for Laplacian of pressure are computed as below.

\[
C_{\mathbf{v}ij} = \int_{\Omega} \mathbf{v}(x,y,t_i) \mathbf{v}(x,y,t_j) d\Omega \quad (6.1a)
\]

\[
C_{ij}^{\nabla^2 p} = \int_{\Omega} \nabla^2 p(x,y,t_i) \nabla^2 p(x,y,t_j) d\Omega \quad (6.1b)
\]

where \( i, j = 1, 2, ..., N_s \)

where \( \Omega \) is the spatial domain considered for POD analysis. The eigenvalue decomposition of \( C_{\mathbf{v}ij} \) yields a set of eigenvalues and eigenvectors \( \{\lambda_k, \psi_k\} \) and the eigenvalue decomposition of \( C_{ij}^{\nabla^2 p} \) yields a set of eigenvalues and eigenvectors \( \{\sigma_k, \phi_k\} \), where \( k = 1, 2, ..., N_s \), with \( N_s \) being the number of snapshots used to construct the correlation matrix. The eigenvectors and corresponding field quantities are used to construct the POD modes for the velocity and pressure.
as follows:

\[ U_i(x, y) = \sum_{k=1}^{N_s} \psi_{ik} u(x, y, t_k) \quad (6.2a) \]

\[ V_i(x, y) = \sum_{k=1}^{N_s} \psi_{ik} v(x, y, t_k) \quad (6.2b) \]

\[ P_i(x, y) = \sum_{k=1}^{N_s} \phi_{ik} p(x, y, t_k) \quad (6.2c) \]

\[ Q_i(x, y) = \sum_{k=1}^{N_s} \phi_{ik} \nabla^2 p(x, y, t_k) \quad (6.2d) \]

The approximation for the vector velocity (\( \tilde{V} \)) and the pressure (\( \tilde{p} \)) fields are computed as below using the first few energetic velocity modes \( N_v \), and pressure modes \( N_p \), where \( N_v, N_p \leq N_s \).

\[ \tilde{V}(x, y, t) = v_0(x, y) + \sum_{i=1}^{N_v} a_i(t) V_i(x, y) \quad (6.3a) \]

\[ \tilde{p}(x, y, t) = p_0(x, y) + \sum_{i=1}^{N_p} b_i(t) P_i(x, y) \quad (6.3b) \]

where \( v_0(x, y) \) and \( p_0(x, y) \) are the mean velocity and the mean pressure, respectively. The unknown temporal coefficients of the velocity and the pressure modes are \( a_i(t) \) and \( b_i(t) \), respectively. The Galerkin projection of Navier-Stokes equations onto the approximated velocity and pressure fields results in a set of coupled ODEs for evolution of the temporal coefficients \( a_i(t) \) as

\[ \frac{da_i(t)}{dt} = - \sum_{k=0}^{N_v} \sum_{j=0}^{N_s} a_k(t) a_j(t) \int \left[ \left( U_k \frac{\partial U_j}{\partial x} + V_k \frac{\partial U_j}{\partial y} \right) U_i + \left( U_k \frac{\partial V_j}{\partial x} + V_k \frac{\partial V_j}{\partial y} \right) V_i \right] d\Omega \]

\[ - \int \left[ \sum_{j=0}^{N_s} b_j(t) \left( \frac{\partial P_j}{\partial x} U_i + \frac{\partial P_j}{\partial y} V_i \right) \right] d\Omega \]

\[ + \sum_{j=0}^{N_p} a_j(t) \frac{1}{Re} \int \left[ \left( \frac{\partial^2 U_j}{\partial x^2} + \frac{\partial^2 U_j}{\partial y^2} \right) U_i + \left( \frac{\partial^2 V_j}{\partial x^2} + \frac{\partial^2 V_j}{\partial y^2} \right) V_i \right] d\Omega \quad (6.4) \]

with \( a_0(t) = 1, b_0(t) = 1, \) and \( 1 \leq i \leq N_v \).

The ODEs governing the evolution of the coefficients \( a_i(t) \) involve the unknown time coefficients of pressure \( b_i(t) \), and we use the pressure Poisson equation to
express the coefficients $b_i(t)$ in terms of $a_i(t)$.

$$\nabla^2 p = -2 \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} - \left( \frac{\partial u}{\partial x} \right)^2 - \left( \frac{\partial v}{\partial y} \right)^2 \quad (6.5)$$

The mean Laplacian of the pressure $\nabla^2 p_0$ and the POD modes $Q_i$ are used to compute an approximation to the pressure Laplacian as shown below.

$$\nabla^2 \tilde{p}(x,y,t) = \nabla^2 p_0(x,y) + \sum_{i=1}^{N_v} b_i(t) Q_i(x,y) \quad (6.6)$$

The projected pressure Poisson equation (6.5) onto the approximated velocity and approximated Laplacian of the pressure field give the required relation between the coefficients $b_i(t)$ and $a_i(t)$ as below:

$$b_i(t) = - \sum_{j=0}^{N_v} \sum_{k=0}^{N_v} a_j(t) a_k(t) \int \left( 2 \frac{\partial U_j}{\partial y} \frac{\partial V_k}{\partial x} + \frac{\partial U_j}{\partial x} \frac{\partial U_k}{\partial x} + \frac{\partial V_j}{\partial y} \frac{\partial V_k}{\partial y} \right) Q_i d\Omega \quad (6.7)$$

Substituting $b_i(t)$ into equation (6.4) provides us with an ODE system only in terms of only velocity modes temporal coefficients $a_i(t)$ as

$$\frac{da_i(t)}{dt} = \sum_{j=0}^{N_v} \beta_{ij} a_j(t) + \sum_{j=0}^{N_v} \sum_{k=0}^{N_v} \xi_{ijk} a_j(t) a_k(t) \quad (6.8)$$

with $a_0(t) = 1$ and $1 \leq i \leq N_v$.

**Shift mode:** The stability of the solution of this set of ODEs can be enhanced by using a shift mode as discussed in detail by Noack et al. (2003). The shift mode is computed using the time-averaged velocity field $v_0$ and steady symmetric solution $v_s$ as

$$V_\Delta = \frac{1}{T} \int v(x,y,t) dt - v_s = v_0 - v_s \quad (6.9)$$

The steady symmetric solution $v_s$ is obtained from the finite-element solution past a motionless cylinder at a time instant where the vortex shedding has not yet started. The Gram-Schmidt orthogonalization procedure is applied to orthonormalize the shift mode representing the mean flow direction with respect
to the velocity POD modes as shown below:

$$\mathbf{V}_{\Delta}^{GS} = \mathbf{V}_{\Delta} - \sum_{i=1}^{N_v} \mathbf{V}_i \int \mathbf{V}_{\Delta} \cdot \mathbf{V}_i d\Omega$$  \hspace{1cm} (6.10)

$$\mathbf{V}_{(N_v+1)} = \frac{\mathbf{V}_{\Delta}^{GS}}{|\mathbf{V}_{\Delta}^{GS}|}$$  \hspace{1cm} (6.11)

The shift mode is introduced as the \((N_v + 1)\)th mode of the POD system, given by

$$\mathbf{V}_{(N_v+1)} = \frac{\mathbf{V}_{\Delta}^{GS}}{|\mathbf{V}_{\Delta}^{GS}|}$$  \hspace{1cm} (6.12)

6.1.2 Noninertial frame of reference

In order to model the moving cylinder problem, the governing Navier-Stokes equations are transformed to a noninertial frame of reference \((X,Y)\) that is moving with the cylinder. The coordinate transformation from the current inertial coordinate system \((x,y,t)\) to the new non-inertial coordinate system \((X,Y,t)\) is given by

\[
\begin{align*}
    X &= x \\
    Y &= y - y_{cyl}(t) \\
    U(X,Y,t) &= u(x,y,t) \\
    V(X,Y,t) &= v(x,y,t) - \frac{dy_{cyl}(t)}{dt}
\end{align*}
\]

Applying this transformation, the momentum equations are transformed as

$$\begin{cases}
    \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = -\frac{1}{\rho} \frac{\partial p}{\partial X} + \frac{1}{Re} \left( \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right) \\
    \frac{\partial^2 y_{cyl}(t)}{dt^2} + U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} = -\frac{1}{\rho} \frac{\partial p}{\partial Y} + \frac{1}{Re} \left( \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right)
\end{cases}$$  \hspace{1cm} (6.13, 6.14)

but the continuity equation remains unchanged as

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$  \hspace{1cm} (6.15)
Moreover, the projection of the Navier-Stokes equations in the noninertial frame onto the POD basis (derived in terms of the new non-inertial frame) gives an additive forcing term that couples the flow and cylinder motions as

\[
\frac{da_i(t)}{dt} = f_{\text{noninertial},i}(t) + \sum_{j=0}^{N_v} \beta_{ij} a_j(t) + \sum_{j=0}^{N_v} \sum_{k=0}^{j} \xi_{ijk} a_j(t)a_k(t)
\]  

(6.16)

with \(a_0(t) = 1\) and \(1 \leq i \leq N_v\).

The complete procedure is outlined below:

- Compute the pressure and the velocity fields in the inertial frame.
- Transform the computational data to the noninertial frame moving with cylinder.
- Obtain POD modes in the noninertial frame.
- Project the Navier-Stokes equation, with cylinder motion prescribed, onto modal basis.
- Integrate the resulting nonlinear ODEs to compute the time coefficients of the POD modes.
- Numerically reconstruct the velocity and the pressure fields using the integrated time coefficients and compare these reconstructed time series to the exact time series derived by direct numerical integration of the FE model.

6.1.3 Results

**Motionless cylinder:**

The flow past a motionless cylinder is computed at \(Re = 100\) and the corresponding POD modes are constructed. The cumulative modal energy content is measured based on the eigenvalues of the velocity and the pressure POD modes and are presented in Figs. 6.1a and 6.1b, respectively. As can be observed, the first 10 modes contribute to the most of the energy content, so it is concluded that including 10 POD modes is sufficient for accurate reconstruction of the FE model.
results. The temporal coefficients obtained by the forward integration of the

ODEs of the reduced order model (ROM) are used to reconstruct the pressure and the velocity fields at a point in the wake (located one and a half diameter downstream of the cylinder center and one diameter above the cylinder mid-plane) are as shown in Figs. 6.2a, 6.2b, and 6.2c for the streamwise velocity, the cross-stream velocity, and the pressure, respectively. As can be observed, for the case of the motionless cylinder the ODEs solution of the ROM reconstructs the pressure and the velocity time series at a point in the wake with sufficient accuracy.

The reconstruction of the complete field quantity over the domain at an instant of time \( tU_0/D = 192.375 \) (marked with a heavy dot in Figs. 6.2a-6.2c) are shown Figs. 6.3a-6.3c for the streamwise velocity, Figs. 6.3d-6.3f for the cross-stream velocity, and Figs. 6.3g-6.3i for the pressure, respectively.

**Moving Cylinder:**

As discussed in §6.1.2, the issue of the moving cylinder boundary in the finite-element solution is addressed by applying a coordinate transformation from the inertial frame of reference to a noninertial frame attached to the cylinder. Hence, in order to construct the POD modes, in both the cases of the prescribed
Figure 6.2: Reconstruction of the velocity and the pressure time histories at a point in the wake (located one and a half diameter downstream of the cylinder center and one diameter above the cylinder mid-plane) for the case of the motionless cylinder: ⭐, PDE; ○, POD; −, ODE.
cylinder motion and VIV motion of the cylinder, we utilize the cylinder motion computed from the finite-element solution. Thus, for the POD based ROM approach developed in this chapter, the prescribed motion of the cylinder and VIV of the cylinder are treated similarly. In this section, reconstructions of the pressure and the velocity fields for the case of the cylinder undergoing VIV at $Re = 100$ are discussed. Similar reconstructions of the velocity and pressure fields were obtained for the case of the prescribed cylinder motion; however the results are not discussed here.

For the case of a cylinder moving freely due to vortex induced vibration, the POD modes are constructed using the finite-element solution data after transforming them to the noninertial frame of reference. The ODE system of the ROM is obtained using the cylinder time history that is also available from the finite-element solution. The reconstruction of the velocity and pressure time series at a point in the wake (located one and a half diameter downstream of the cylinder center and one diameter above the cylinder mid-plane) are as shown
in Figs. 6.4a - 6.4c. Integration for a longer duration are as depicted in Figs. 6.5a - 6.5c, thereby verifying the long-time stability and accuracy of the fields computed using the POD based ROM.

Figure 6.4: Reconstruction of the velocity and the pressure time histories at a point in the wake (located one and a half diameter downstream of the cylinder center and one diameter above the cylinder mid-plane) for the case of the moving cylinder: ⭐, PDE; ⭕, POD; _, ODE.
Figure 6.5: Reconstruction of the velocity and the pressure time histories at a point in the wake (located one and a half diameter downstream of the cylinder center and one diameter above the cylinder mid-plane) of the cylinder undergoing VIV for a longer time duration: ★, PDE; −, ODE.
6.2 Least-squares fit of an ODE system governing pressure dynamics around the cylinder

The objective here is to construct a reduced-order model for the dynamics of the pressure distribution around a cylinder undergoing VIV. The ROM is based on the time coefficients of Karhunen-Loève (K-L) modes of the pressure distribution around the cylinder. Multiple time series of the pressure distribution are obtained from the finite-element solution with different harmonic motions prescribed to the cylinder. Use of multiple time series ensures that the sampled time series represent the phase-space of the dynamics adequately. Such a sampling results in a full rank least-squares system for the approach outlined here.

6.2.1 Formulation

The formulation to generate an orthogonal basis for the pressure around the cylinder in a laminar flow based on Karhunen-Loève (K-L) decomposition is presented in this section. The K-L modes obtained here are intended to represent the ensemble of data generated from the PDE (finite-element) solution using several prescribed motions of the cylinder in a flow with $Re = 100$. Time series obtained by finite-element solution with different prescribed cylinder motions close to the lock-in regime are used to compute the spatial correlation matrix. The K-L modes are constructed based on eigenvalue decomposition of the corresponding correlation matrix, and constitute the spatial modal basis which incorporate the spatial flow information from multiple finite-element (FE) numerical integrations. Hence, this basis is identified as global K-L modes. The temporal coefficients of the K-L modes for each case of the PDE solution are estimated by minimizing the residue between the PDE (i.e., the exact FE) data and the reconstructed pressure time series. The spatial global modes and estimated time coefficients form the basis for developing an ODE-ROM system modeling in an approximate reduced-order way the pressure distribution around the cylinder.
The $g^{th}$ PDE solution is obtained by prescribing a sinusoidal motion to the cylinder as

$$f_g(t) = A_g \sin \left( F_g \omega_s t \right)$$  \hspace{1cm} (6.17)

where $\omega_s$ is the natural shedding frequency of the stationary cylinder in a flow with $Re = 100$, and $F_g = \omega_f / \omega_s$ is the ratio of the forcing frequency to the natural shedding frequency. The forcing amplitude $A_g$ is set to be equal to $0.25D$ for this study, with $D$ being the diameter of the cylinder. For $A_g = 0.25D$ and $Re = 100$, the lock-in response is observed for $F_g$ between 0.8 and 1.2, and aperiodic lift force is observed beyond this range of $F_g$. The pressure around the cylinder is then extracted at $N_\theta$ points on the circumference of the cylinder for one cycle of oscillation of the cylinder.

The pressure around the cylinder from the $g^{th}$ PDE solution is identified as $P^*_g(t_{j,g},\theta_i)$. Pressure from the PDE solution is decomposed in terms of time-averaged and time-varying components as

$$P^*_g(t_{j,g},\theta_i) = \bar{P}_g(\theta_i) + P_g(t_{j,g},\theta_i)$$  \hspace{1cm} (6.18)

where the time-averaged component of pressure is computed as

$$\bar{P}_g(\theta_i) = \frac{1}{T_g} \int_0^{T_g} P_g(t,\theta_i)dt = \frac{\sum_{j=1}^{N_t,g} P^*_g(t_{j,g},\theta_i)}{N_{t,g}}$$  \hspace{1cm} (6.19)

The time-averaged component $\bar{P}_g(\theta_i)$ is set aside and used later in the reconstruction of the pressure field. The K-L decomposition is performed only on the time-varying component $P_g(t_{j,g},\theta_i)$. 

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The spatial correlation matrix of the time-varying pressure is computed as

\[
C_{qr,G} = \sum_{g=1}^{G} \sum_{l=1}^{N_t,g} P_g(t_{t,g}, \theta_q) P_g(t_{t,g}, \theta_r)
\]

where \( q, r = 1, 2, 3, \ldots, N_\theta \)

\( N_\theta \): Number of spatial points.
\( G \): Total number of time series used for constructing global spatial modes.
\( N_{t,g} \): Number of time points sampled from the \( g^{th} \) time series.

The eigenvalue decomposition of the spatial correlation matrix \( C_{qr,G} \) gives the spatial K-L modes \( S_n,G(\theta_i) \), with \( i, n = 1, 2, 3, \ldots, N_\theta \). The subscript \( G \) indicates the number of PDE solutions used in constructing spatial global modes.

We use the first \( N_{G,ret} \) energetic spatial global K-L modes to compute an approximation to the pressure around the cylinder corresponding to each of the PDE solution (denoted by subscript \( g \)) at each time \( t_{j,g} \) as

\[
\tilde{P}_g(t_{j,g}, \theta_i) = \bar{P}_g(\theta_i) + \sum_{n=1}^{N_{G,ret}} S_{n,G}(\theta_i) \alpha_{n,g}(t_{j,g})
\]

where the unknown temporal coefficients \( \alpha_{n,g}(t_{j,g}) \) at time \( t_{j,g} \) in (6.21) is determined by least-squares fit to minimize the error between the approximated pressure \( \tilde{P}(t_{j,g}, \theta_i) \) and the pressure obtained from PDE solution \( P_g(t_{j,g}, \theta_i) \).

**Time coefficients using global K-L modes**

For each prescribed forcing frequency, we define the residue between the PDE computed pressure and the K-L mode reconstructed pressure as

\[
R_{j,g} = \sum_{i=1}^{N_\theta} \left[ \sum_{n=1}^{N_{G,ret}} S_{n,G}(\theta_i) \alpha_{n,g}(t_{j,g}) - P_g(t_{j,g}, \theta_i) \right]^2
\]
The unknown time coefficients $\alpha_{n,g}(t_{j,g})$ are estimated using a least-squares procedure to minimize the residue $R_{j,g}$.

$$\frac{\partial R_{j,g}}{\partial \alpha_{p,g}(t_{j,g})} = \sum_{i=1}^{N_g} 2S_{p,G}(\theta_i) \left[ \sum_{n=1}^{N_{G,ret}} \alpha_{n,g}(t_{j,g})S_{n,G}(\theta_i) - P_{g}(t_{j,g}, \theta_i) \right] = 0$$  
(6.23)

$$\Rightarrow \sum_{i=1}^{N_g} \sum_{n=1}^{N_{G,ret}} \alpha_{n,g}(t_{j,g})S_{n,G}(\theta_i)S_{p,G}(\theta_i) = \sum_{i=1}^{N_g} S_{p,G}(\theta_i)P_{g}(t_{j,g}, \theta_i)$$  
(6.24)

$$1 \leq n, p \leq N_{G,ret}$$

$$1 \leq g \leq G$$

The equations (6.24) are decoupled because of the orthogonality of the spatial modes $S_{n,G}(\theta_i)$. Hence, the solution of the system of equations (6.24) for $g = 1, 2, 3, ..., G$ will give the time coefficients for each forcing frequency based on the decomposition in terms of the spatial global modes. Thus, the pressure around the cylinder is decomposed into a set of spatial global modes $S_{n,G}(\theta_i)$ and a set of time coefficients $\alpha_{n,g}(t_{j,g})$ corresponding to each of the PDE solution used to obtaining the global K-L modes.

**Least-square fit of the evolution equations for the time coefficients**

The dynamics of the pressure distribution around the cylinder is governed by the time-variation of the coefficients $\alpha_{n,m}(t_{j,m})$ determined in §6.2.1 for the $m^{th}$ forcing function. Motivated by the quadratic nonlinearity of the Navier-Stokes equations, the dynamic evolution is assumed to be quadratically nonlinear in nature. Hence, the evolution equations for the temporal coefficients of the global
K-L mode are assumed as below.

\[
\frac{d\alpha_i,M(t)}{dt} = a_i + \sum_{j=1}^{N_{dim}} b_{ij} \alpha_j,M(t) + \sum_{j=1}^{N_{dim}} \sum_{k=1}^{j} c_{ijk} \alpha_j,M(t) \alpha_k,M(t) + \frac{dV_{cyl}}{dt} \left( d_i + \sum_{j=1}^{N_{dim}} e_{ij} \alpha_j,M(t) \right)
\]  
(6.25)

\[
1 \leq i \leq N_{dim}
\]

where \( N_{dim} \leq N_{G,ret} \) is the dimension of the set of quadratic ODEs. The subscript \( M \) indicates the number of time series of the temporal coefficients of K-L modes used to fit the quadratic ODEs, hence \( M \leq G \).

The unknown ODE coefficients \( \{a_i\}, \{b_{ij}\}, \{c_{ijk}\}, \{d_i\}, \{e_{ij}\} \) in equation (6.25) are yet to be determined. The non-autonomous part of the equations (6.25) is included to reduce the data requirement. The method of least-squares (L-S) is adopted to determine the unknown ODE coefficients. The time derivatives of the temporal coefficients on the left-hand side of (6.25) are evaluated from a second-order accurate central difference scheme using the time coefficients estimated in §(6.2.1) at each time.

The least-squares system of equations is obtained by minimizing the residue between the estimated derivatives of the temporal coefficients and those given by assuming the quadratic form (6.25) as

\[
E_M = \sum_{m=1}^{M} \sum_{l=1}^{N_{t,m}} \sum_{i=1}^{N_{dim}} \left[ a_i + \sum_{j=1}^{N_{dim}} b_{ij} \alpha_j,m(t_{l,m}) + \sum_{j=1}^{N_{dim}} \sum_{k=1}^{j} c_{ijk} \alpha_j,m(t_{l,m}) \alpha_k,m(t_{l,m}) + \frac{dV_{cyl}(t_{l,m})}{dt} \left( d_i + \sum_{j=1}^{N_{dim}} e_{ij} \alpha_j,m(t_{l,m}) \right) - \frac{d\alpha_i,m(t_{l,m})}{dt} \right]^2
\]  
(6.26)

Equation (6.26) gives the sum of squares of residuals to be minimized (i.e., the penalty function) for estimating the unknown ODE coefficients \( \{a_i\}, \{b_{ij}\}, \{c_{ijk}\}, \{d_i\}, \{e_{ij}\} \).
6.2.2 Results

In this section two of systems of ODE obtained using the formulation presented in §6.2.1 are discussed.

ODE system 1

This three dimensional ODE system was obtained by constructing global K-L modes using PDE data from \( F_1 = 0.9, F_2 = 1.0, \) and \( F_3 = 1.1 \), hence, for this system \( G = 3 \). The generic quadratic system of ODEs (6.25) was fitted using all three time series of temporal coefficients, and hence \( M = 3 \). The ODE system is as given by equation (6.27), where the second subscript in \( \alpha_{i,3} \) indicates that \( M = 3 \). This system is expressed as follows:

\[
\begin{bmatrix}
\frac{d\alpha_{1,3}}{dt} \\
\frac{d\alpha_{2,3}}{dt} \\
\frac{d\alpha_{3,3}}{dt}
\end{bmatrix} = \begin{bmatrix}
-0.0848 \\
-0.0070 \\
0.0559
\end{bmatrix} + \begin{bmatrix}
-0.0072 & -3.3465 & 1.5720 \\
0.2450 & 0.0419 & 0.2294 \\
-0.0059 & 0.0846 & -0.7897
\end{bmatrix} \begin{bmatrix}
\alpha_{1,3} \\
\alpha_{2,3} \\
\alpha_{3,3}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1.0717 & -0.2047 & -2.2927 & -0.2515 & 1.3151 & 3.3832 \\
0.2997 & -0.1479 & -0.2545 & -0.2542 & -1.8978 & -3.2307 \\
-0.6627 & 4.8111 & -0.6323 & -0.1505 & -1.4808 & 0.5663
\end{bmatrix} \begin{bmatrix}
\alpha_{1,3}^2 \\
\alpha_{2,3}\alpha_{1,3} \\
\alpha_{2,3}^2 \\
\alpha_{3,3}\alpha_{1,3} \\
\alpha_{3,3}\alpha_{2,3} \\
\alpha_{3,3}^2
\end{bmatrix}
\]

\[
+ \frac{dV_{\text{cyl}}}{dt} \begin{bmatrix}
-0.1102 \\
0.0085 \\
-0.0065
\end{bmatrix} + \begin{bmatrix}
0.1839 & 0.2978 & -1.5889 \\
-0.1301 & -0.1812 & -0.7575 \\
-0.2894 & -0.9950 & -0.0927
\end{bmatrix} \begin{bmatrix}
\alpha_{1,3} \\
\alpha_{2,3} \\
\alpha_{3,3}
\end{bmatrix}
\]

(6.27)

The forward integration of the L-S fit ODE system (6.27) is shown in Fig. 6.6, and the initial conditions used for this system are those obtained from the K-L decomposition of the PDE solution with \( F = 1.0 \) as given by equation (6.28).

\[
\alpha_{1,3}(t = 125.00) = \begin{bmatrix}
-0.329943510951542 \\
-0.033583698458822 \\
+0.040883124775859
\end{bmatrix}
\]

(6.28)
The time period predicted by the ODE coefficients matches that of the PDE solution, though this system settles to a periodic orbit whose amplitude is different than that of the PDE solution. The reconstruction of $C_p$ at one instance of time is compared to that of the PDE solution as shown in Fig. 6.7. The comparison of the time series of $C_p$ generated by the ODE system and the PDE solutions on the phase plane is shown in Fig. 6.8.

**ODE system 2**

This ODE system is generated using the global K-L modes form the PDE solutions with forcing frequency ratio $F_1 = 0.79, F_2 = 0.9, F_3 = 1.0, F_4 = 1.1$, and $F_5 = 1.3$; hence $G = 5$. The least-squares fit for the coefficients of the quadratic ODE of dimension 3 was obtained using the temporal coefficients of only the time series corresponding to $F = 1.0$, hence $M = 1$ for this L-S fit ODE system.

The ODE system is given in equation (6.29). The forward integration of the ODE system, equation (6.29), is shown in Fig. 6.9; the initial conditions used were identical to (6.28). This ODE system gives a periodic solution which is very close to the PDE solution. The reconstruction of $C_p$ at one instant of time is compared to that of the PDE solution in Fig. 6.10; the corresponding RMS error is less than 5%. The phase plot shown in Fig. 6.11 confirms that
Figure 6.7: Reconstruction of $C_p$ using the temporal coefficients obtained from the forward integration of the ODE system 1 and its comparison with the PDE solution at the initial state. Since the forward integration of ODE system 1 shown in Fig. 6.6 has an initial transient during which ODE coefficients amplitudes are modulated, the reconstruction of $C_p$ is done at time ($t = 199.61$) when the ODE coefficients are periodic and they are in phase with the initial state.

Figure 6.8: Phase plot of $C_p$ from PDE and ODE system 1
Figure 6.9: Forward integration of ODE system 2.

Figure 6.10: Reconstruction of $C_p$ using the temporal coefficients obtained from the forward integration of the ODE system 2 at a time instant that is one period away from the beginning of the ODE system integration ($t = 131.11$) and its comparison with the PDE solution at the initial state.
Figure 6.11: Phase plot of $C_p$ from PDE and ODE system 2 at $\theta = 243.3329$.

the accuracy of the periodic solution predicted by this ODE system, compared to that obtained using the PDE.

$$
\begin{bmatrix}
\frac{d\alpha_{1,1}}{dt} \\
\frac{d\alpha_{2,1}}{dt} \\
\frac{d\alpha_{3,1}}{dt}
\end{bmatrix} =
\begin{bmatrix}
-0.2824 \\
0.0134 \\
-4.3589
\end{bmatrix}
+ \begin{bmatrix}
14.4612 & -3.9841 & \alpha_{1,1} \\
-0.0771 & -0.5622 & \alpha_{2,1} \\
-0.9302 & -13.3992 & -52.9735 & \alpha_{3,1}
\end{bmatrix}
+ \begin{bmatrix}
-0.1444 & 0.0558 & -1.0282 & 0.8975 & 0.2146 & 0.0327 \\
56.5144 & 13.6427 & 75.5080 & 2.9689 & -51.0953 & 7.4398
\end{bmatrix}
+ \frac{dV_{s(t)}}{dt} \begin{bmatrix}
0.0026 \\
0.1828 \\
0.9344
\end{bmatrix}
+ \begin{bmatrix}
0.0155 & -0.1839 & -0.1023 & \alpha_{1,1} \\
0.0001 & -0.0007 & -0.0001 & \alpha_{2,1} \\
-0.0140 & 0.1643 & 0.0899 & \alpha_{3,1}
\end{bmatrix}
(6.29)
$$

6.3 Conclusions

The POD-based ODE system is found to yield very good approximation for the pressure and the velocity fields for the case of the motionless cylinder. For the moving cylinder with known cylinder motion, the ODE based on POD modes
in the cylinder-fixed (non-inertial) frame are able to approximate the flow field quite accurately. However, to have a predictive ROM for VIV, we still need to devise a method to obtain POD modes in the moving frame where the cylinder motion is not known \textit{a priori}, but rather is determined by the VIV solution itself.

The global spatial K-L modes for the pressure around the cylinder are obtained utilizing the PDE solutions from several cases of prescribed motion of the cylinder. The least-squares fitted evolution equations (ODEs) for the temporal coefficients was found to be sensitive to the specific PDE solutions used in the least-squares fitting procedure. Specifically, the coefficients of the quadratic ODEs did not converge as we increased the number of PDE solutions used in the least-squares procedure. In addition, the quadratic ODE coefficients turned out to be very large in magnitude for the case of larger number of PDE solutions used in least-squares fitting procedure, thus making such an ODE-ROM system very sensitive to the numerical round-off errors. Similar to the case of the POD based ROM the least-squares based ODE-ROM for the pressure around the cylinder was developed using the PDE solutions where the cylinder motion is not known \textit{a priori}. The feasibility of extending such a method for the case of VIV of the cylinder needs to be further examined.
In this chapter, we discuss the relation between the critical Reynolds number \( (Re_c) \) for Hopf bifurcation and the stiffness of the cylinder for a sprung rigid circular cylinder. Even though \( Re_c \approx 47 \) has been well established as the critical Reynolds number for the supercritical Hopf bifurcation of the flow past a stationary cylinder, there is an underlying assumption that the cylinder is held stationary by an infinitely rigid support. In most practical applications, however, there will be a certain finite flexibility in the cylinder support system. The significance of this finite flexibility on the flow past a cylinder in the laminar regime received considerable attention of researchers, partially led by an experimental investigation by Sreenivasan (1985). In the experimental study involving a cylindrical wire, Sreenivasan (1985) reported a chaotic kind of behavior in the laminar flow regime \( 50 \leq Re \leq 100 \). However, later investigation by Van Atta and Gharib (1987) revealed that this behavior was essentially due to the influence of the stiffness of the cylinder. Due to aeroelastic coupling, the system entered into resonance interaction at commensurate combinations of the cylinder natural frequency and the Strouhal frequency of the flow, resulting in quasi-periodic response that was construed as chaotic by Sreenivasan (1985).

The possibility of VIV in flow with Reynolds number below \( Re_c \) of the stationary cylinder has been reported by Mittal and Singh (2005) for a single value of the natural frequency of the cylinder. However, to the best of our knowledge there is no study describing the effect of the finite stiffness of the cylinder support on the \( Re_c \) at which VIV can occur. In this chapter, we explore the variation of \( Re_c \) for a cylinder undergoing VIV for flows with \( Re < 47 \). First, we confirm that, for a motionless cylinder, the Hopf bifurcation occurs at \( Re_c \approx 47 \)
in our computational model. Then, we track the variation of $Re_c$ for various stiffness values to plot the bifurcation diagram for $Re_c$, with stiffness of the cylinder as the bifurcation parameter. In addition, we introduce a rotational NES into the system and study its effect on the bifurcation diagram.

### 7.1 $Re_c$ for flow past a stationary cylinder.

We undertake this initial study, before computing the bifurcation diagram for the flexibly supported cylinder, in order to confirm that our computational domain size and discretization are adequate for lower $Re$ flows and the $Re_c$ for the motionless cylinder agrees with the results available in the literature. The $Re_c$ for the flow past a motionless cylinder has been investigated by several researchers, both by experimental (Kovasznay (1949) and Sreenivasan et al. (1987)) and computational (Henderson (1995) and Kumar and Mittal (2006)) approaches. The exact value of $Re_c$ reported by various researchers exhibits considerable variation in the range $40 \leq Re_c \leq 50$. The variability in the laboratory experiments arises due to uncertainties in measurements, maintaining a 2D flow, and influence of the finite cylinder length causing end effects. In the case of computational simulations, the size of the computational domain and the mesh resolution plays a critical role; a detailed study of the effect of blockage and mesh resolution on the estimated value of $Re_c$ has been investigated by Kumar and Mittal (2006). The most widely accepted value is $Re_c \approx 47$, after which steady symmetric flow becomes temporally unstable, eventually leading
to formation of the Kármán vortex street for $Re > Re_c$.

From a dynamical system perspective, at $Re_c$ the fixed point of the wake oscillator loses its stability resulting in limit-cycle oscillation, which is a well known supercritical Hopf bifurcation. Any small disturbance in the flow past a cylinder with $Re > Re_c$ will grow in time, leading to temporal instability of the steady symmetric flow resulting in a limit-cycle oscillation. As the flow Reynolds number approaches $Re_c$, the temporal growth rate becomes extremely small and requires a very long-time integration if we rely solely on the numerical round-off error as the source of asymmetry to trigger the instability. In order to expedite the process, we introduce a small temporal disturbance in the upstream inlet boundary condition to perturb the steady-symmetric solution. The temporal perturbation of the inlet velocity profile is described below.

**Initial transient (IT):**

In the initial transient portion of the time integration, the streamwise velocity at a point on the inlet is offset from other points on the inlet boundary. The sequence of streamwise velocity prescribed on the inlet boundary is as follows:

- Uniform inlet condition is prescribed until 50 convective time units have elapsed, to allow the flow to fully develop.

- A point on the inlet located at $1.071D$ from the centerline is prescribed to have a 10% larger velocity for 20 convective time units.

- Uniform inlet velocity condition is resumed after a total of 70 convective time units.

After the perturbation, we continue the time integration long enough to assess the growth or decay of the disturbance based on the time series of the lift coefficient. We estimate the temporal growth rate of the disturbance as

$$\lambda = \frac{1}{\tau_2 - \tau_1} \ln \left( \frac{C_L(\tau_2)}{C_L(\tau_1)} \right) \quad (7.1)$$
Figure 7.2: Frequency of lift coefficient at $Re = 46.8$.

where $\tau_1$ and $\tau_2$ correspond to the peak values of $C_L$. A positive value of $\lambda$ indicates the growth of the disturbance and vice versa.

The tight range of $Re_c$ for the motionless cylinder is found to be $46.7 \leq Re_c \leq 46.8$. The time series of the lift coefficient corresponding to $Re = 46.7$ and $Re = 46.8$ are shown in Figs. 7.1a and 7.1b, respectively. The growth/decay rate of the lift coefficient is also shown in Fig. 7.1. The oscillatory behavior of the lift coefficient can be noted from Fig. 7.1. Though the amplitude of the lift coefficient has not reached its fully developed limit-cycle value, the frequency of oscillation is fully evolved. The frequency spectrum of the time series in Fig. 7.1b is shown in Fig. 7.2, with the dominant frequency representing the Strouhal number found to be $St = 0.1179$, which agrees with that reported by Cossu and Morino (2000).

7.2 $Re_c$ for a sprung cylinder.

The effect of the structural frequency of the sprung cylinder on the stability of the steady symmetric solution has been addressed by Cossu and Morino (2000), where they note that, for a small density ratio of fluid to cylinder representative of steel cylinder in water, the $Re_c$ can be much below the Hopf bifurcation $Re_c$ of the stationary cylinder. The vortex shedding at sub-critical $Re$ has been reported by Buffoni (2003) based on an experimental investigation with forced oscillation of a circular cylinder in water. The computational study by Mittal and Singh (2005) describes the possibility of vortex-induced vibration at sub-critical $Re$. Based on these results, one would expect to see dependence of $Re_c$. 
on the system parameters in the case of the sprung cylinder. In this section, we evaluate $Re_c$ for a flexibly supported cylinder with mass ratio $m^* = \rho_s/\rho_f = 10$, choose various values of stiffness for the cylinder and search for a narrow range of $Re$ that bounds $Re_c$ at each stiffness value. The dimensionless frequency $(F_n^*)$ of the cylinder is defined for this study as

$$F_n^* = \frac{2\pi \nu}{D^2} \sqrt{\frac{K_{cyl}}{M_{cyl}}} \quad (7.2)$$

Adopting the same approach as described in §7.1, we perturb the solution with a time dependent upstream inlet boundary condition. Similar to the case of the motionless cylinder, we would need a long-time integration to assess on the decay or growth of the disturbance as the flow Reynolds number approaches $Re_c$. We use the growth rate defined in (7.1) to identify the growth/decay characteristic of the response. The bifurcation diagram for the sprung cylinder is shown in Fig. 7.3, showing $Re$ vs. the inverse of the dimensionless frequency in the range $0 \leq 1/F_n^* \leq 0.5$. The narrow range of $Re_c$ for the case of a stationary cylinder is also plotted in the same figure for comparison.
As expected and in accordance with results from other researchers (Cossu and Morino (2000), Buffoni (2003), and Mittal and Singh (2005)), the $Re_c$ for the sprung cylinder is much below the stationary cylinder $Re_c$ for the greater part of the diagram shown in Fig. 7.3. As we increase the stiffness of the support, the frequency $1/F_n^*$ decreases, and the critical $Re$ approaches that of the stationary cylinder $Re_c$. However, contrary to expectation, we do not need a very stiff support to get $Re_c$ of the sprung cylinder to reach a value close to that of the motionless cylinder. There is a threshold $1/F_n^* \approx 0.12$ beyond which the bifurcation occurs at $Re_c$, the same as that of the motionless cylinder.

7.3 Synchronization regimes in sub-critical $Re$ flow

As observed in our computational validation study described in §2.2.4 and also reported by Prasanth and Mittal (2009) for a cylinder tuned for resonance at $Re = 100$, synchronization occurs over a wide range, $82.4 \leq Re \leq 134$. We also note that at $Re = 100$ the Kármán vortex street is well established. For sub-critical flow with $Re < 47$, for the motionless cylinder, the only stable solution is steady and symmetric about the cylinder mid-plane. From our computational studies in this section with sub-critical $Re$, when the cylinder is supported on an elastic foundation, we observe lock-in response over a range of $Re$. Further, we also find that the range of lock-in $Re$ is dependent on the natural frequency of the cylinder.

We illustrate this synchronization for a particular stiffness value inferred by $1/F_n^* = 0.35$. On the lower end, synchronization begins in the narrow range of $21 \leq Re \leq 22$; the corresponding time series are shown in Fig. 7.4. For the same natural frequency of the cylinder, we observe that the upper end of synchronization is located in the narrow range $29 \leq Re \leq 30$ as shown in Fig. 7.5. Thus, the lock-in regime for this stiffness spans the range $22 \leq Re \leq 29$. For similar analysis performed for $1/F_n^* = 0.3$, we find the range of synchronization to be $24 \leq Re \leq 34$. Thus, we see that the range of $Re$ over which synchronization is possible depends on the natural frequency of the cylinder.
Figure 7.4: Time series with decaying and growing solutions at the beginning of the lock-in $Re_c$ for $1/F_n^* = 0.35$ of the cylinder without NES.

Figure 7.5: Time series with decaying and growing solutions at the end of the lock-in $Re_c$ for $1/F_n^* = 0.35$ of the cylinder without NES.
Upper branch - von Kármán mode | Lower branch - structural lock-in mode
--- | --- | --- | ---
\(1/F_n^*\) | \(f_n^*\) | \(f_{C_L}^*\) | \(f_n^*\) | \(f_{C_L}^*\)
0.42 | 0.05233 | 0.11573 | 0.11337 | 0.10090
0.44 | 0.04994 | 0.11500 | 0.10820 | 0.09902

Table 7.1: Lock-in and von Kármán branches of response at sub-critical Re

The range of synchronization Re decreases as \(1/F_n^*\) is reduced; thus, based on this limited data, we can conjecture that the synchronization ceases to exist for very small \(1/F_n^*\), and the situation approaches that of the stationary cylinder.

### 7.4 VIV modes in sub-critical Re flow

Another interesting transition in the bifurcation diagram (Fig. 7.3) occurs at higher end of \(1/F_n^*\) close to 0.375. Two branches of the bifurcation curve exist for \(0.375 \leq F_n^* \leq 0.45\). The lower branch is a continuation of the lock-in curve, but the upper branch seems to be present only for very small stiffness of the cylinder.

In this section, we discuss the reasons for multiple branches in the bifurcation diagram shown in Fig. 7.3. For \(1/F_n^* > 0.4\), we observe a branch of the bifurcation curve very close to the stationary cylinder \(Re_c\), which we call the “upper branch”, and there is a corresponding “lower branch” around \(Re \approx 20\). We analyze the frequency content of the unsteady solution on both of these branches and summarize them in Table 7.1. The dimensionless natural frequency \((f_n U_0/D)\) of the cylinder is also shown. From Table 7.1, the response frequency on the lower branch is associated with a component close to the natural frequency of the cylinder; thus it represents the synchronized or lock-in response. In the synchronized case, as indicated earlier, the flow is controlled by the structure. The upper branch response frequency is very close to the frequency of the Hopf bifurcation of the stationary cylinder; the structure does not play major role in the response on this branch.

By performing linear stability analysis for flow with \(Re < 47\), Cossu and Morino (2000) classified the modes of the sprung cylinder as “structural modes”
and “von Kármán mode”. They also found that the existence of these modes strongly depends on the mass ratio and concluded that for mass ratio \( m^* < 70 \), the “structural mode” is unstable leading to VIV of the sprung cylinder well below \( Re_c \) of the stationary cylinder. Consistent with their analysis, we find that the “structural mode” causing synchronization is responsible for the “lower branch” around \( Re \approx 20 \); when we increase \( Re \), the structural mode can no longer synchronize itself with the flow resulting in the “upper branch” which is the “von Kármán mode” with frequency close to the Hopf bifurcation frequency.

### 7.5 Effect of a rotational NES on \( Re_c \) for sprung cylinder.

In this section, we discuss the effect of introducing an essentially nonlinear rotating device into the cylinder on the critical \( Re \) of the sprung cylinder. The NES parameters are fixed for all the simulations in this study at \( \bar{r}_0 = 0.3 \), \( \epsilon_r = 0.3 \), and \( \lambda_r = 0.04716 \). These NES parameter are not optimized for any specific objective; our intention is to study the general effect that TET brings to the bifurcation curves. The cylinder stiffness at which the critical \( Re \) was evaluated for the system without NES in §7.2 is used, and a similar procedure is carried out to identify a narrow range of \( Re \) at each stiffness. The resulting bifurcation diagram with the rotational NES is shown in Fig. 7.6. In plotting the bifurcation diagram in Fig. 7.6, we have used the mean of the narrow range of \( Re \) that bounds the critical value, both for the system with and without NES.

The overall effect of the rotational NES on the system with sprung cylinder is to effectively reduces the range of \( Re \) at which the lock-in occurs. The rotational NES also shifts the \( Re_c \) beyond the \( Re_c \) of the stationary cylinder for \( 0.12 \leq F_n^* \leq 0.14 \), indicating that with an appropriate choice of parameters the rotational NES can delay the onset of the unsteady solution. Also for the system with the NES, the upper branch of the bifurcation curve can be tracked for \( F_n^* \geq 0.38 \). However, the upper branch is very close to the stationary cylinder \( Re_c \), and the argument made in §7.2 for the upper branch without the NES holds for the system with the NES also. The steady and unsteady solutions with
In this section, we discuss the flow past a sprung cylinder as we successively increase the flow $Re$ while retaining the value of stiffness ($1/F^*$) of the cylinder support. In the case of flow past a motionless cylinder, a recirculation region starts forming at around $Re \approx 6.29$ (Sen et al. (2009)), which eventually detaches from the cylinder as a free vortex for $Re > 47$, leading to formation of vortex street. We discuss the development of the recirculation bubble for a sprung cylinder as we vary the flow Reynolds number in the range $10 \leq Re \leq 50$, while retaining the dimensionless inverse of frequency at $1/F^* = 0.3$. We perturb the inlet velocity boundary condition as discussed in §7.1 and study the effect of disturbance at each $Re$.

Figure 7.6: Effect of rotational NES on bifurcation curve: $\bullet$, without NES; $\star$, with NES; $\ldots$, motionless cylinder.

$F^*_n = 0.1392$ are plotted for the system without NES in Fig. 7.7 and for the system with NES in Fig. 7.8, respectively. A set of time series for $F^*_n = 0.4$ are plotted in Figs. 7.9 and 7.10 for the system without and with NES, respectively.

### 7.6 Vertical traverse in $Re$ at a particular stiffness

In this section, we discuss the flow past a sprung cylinder as we successively increase the flow $Re$ while retaining the value of stiffness ($1/F^*$) of the cylinder support. In the case of flow past a motionless cylinder, a recirculation region starts forming at around $Re \approx 6.29$ (Sen et al. (2009)), which eventually detaches from the cylinder as a free vortex for $Re > 47$, leading to formation of vortex street. We discuss the development of the recirculation bubble for a sprung cylinder as we vary the flow Reynolds number in the range $10 \leq Re \leq 50$, while retaining the dimensionless inverse of frequency at $1/F^* = 0.3$. We perturb the inlet velocity boundary condition as discussed in §7.1 and study the effect of disturbance at each $Re$. 
The cylinder displacement, lift coefficient, and streamline plot at each Re are shown in Fig. 7.11. From the bifurcation diagram for the system without NES (Fig. 7.3), we note the lock-in regime in the range $24 \leq Re \leq 34$ at $1/F^* = 0.3$. Thus, for flow with $Re = 10$ and $Re = 20$, which are below the lock-in range, we see the disturbance decay as can be observed from the displacement (Figs. 7.11a and 7.11d) and the lift coefficient (Figs. 7.11b and 7.11e), respectively. More importantly, the streamlines plot for $Re = 10$ and $Re = 20$ shown in Figs. 7.11c and 7.11f clearly depict the symmetric flow with respect to the cylinder mid-plane. As we move into the lock-in regime, for flow at $Re = 25$ and $Re = 30$, the displacement (Figs. 7.11g and 7.11j) and the lift coefficient (Figs. 7.11h and 7.11k) start to grow with time. The corresponding streamline plots within the lock-in regime shown in Figs. 7.11i and 7.11l for $Re = 25$ and $Re = 30$, respectively, clearly indicate the asymmetry in the flow across the cylinder mid-plane. For $Re = 35$ and $Re = 40$, which are out of the lock-in regime, we again observe the decay of the disturbance in the displacement (Figs. 7.11m and 7.11p) and the lift coefficient (Figs. 7.11n and 7.11q), respectively; the symmetric solutions can be observed in streamline plot shown in Figs. 7.11o and 7.11r. However, at $Re = 50$, which is above the Hopf bifurcation $Re_c (Re_c \approx 46.75)$ of the stationary
Figure 7.8: Time series with decaying and growing solutions at the beginning of the lock-in $Re_c$ for $1/F_\alpha^* = 0.1392$ of the cylinder with NES.
(a) Cylinder displacement $Re = 20$

(b) Cylinder displacement $Re = 21$

(c) Lift coefficient $Re = 20$

(d) Lift coefficient $Re = 21$

Figure 7.9: Time series with decaying and growing solutions at the beginning of the lock-in $Re_c$ for $1/F^*_n = 0.4$ of the cylinder without NES.

cylinder, we note the rapid growth in the disturbance as can be observed for the displacement (Fig. 7.11s) and the lift coefficient (Fig. 7.11t) time series, respectively; in addition the streamline plot (Fig. 7.11u) clearly indicates the formation of the vortex. The variation in the length and width of the recirculation bubble is plotted in Figs. 7.13a and 7.13b, respectively.

The effects of having a rotational NES inside the cylinder on the solutions during a vertical traverse are summarized in Fig. 7.12. The lock-in regime for the system with NES from the bifurcation diagram (Fig. 7.6) is $33 \leq Re \leq 41$. Hence, the observations for solutions in the lock-in regime and out of the lock-in regime are similar to those for the system without NES. However, an important effect of the rotational NES can be observed for flow with $Re = 50$ as shown in Fig. 7.12s, 7.12t, and 7.12u for the displacement, lift coefficient and streamline plots, respectively. The streamline plot for the system with NES in Fig. 7.12u remains quite symmetric in comparison to the system without the NES shown in Fig. 7.11u. This indicates that the rotational NES effectively delays the formation of the Kármán vortex street. The variation in the length and width of the recirculation bubble is plotted in Fig. 7.14a and 7.14b, respectively, for
Figure 7.10: Time series with decaying and growing solutions at the beginning of the lock-in $Re_c$ for $1/F_n = 0.4$ of the cylinder with NES.
Figure 7.11: Vertical traverse in Re for $1/F^* = 0.3$. 
Figure 7.12: Vertical traverse in $Re$ for $1/F^* = 0.3$ with rotational NES.
7.7 Conclusions

For a sprung cylinder VIV is possible below the Hopf bifurcation $Re_c$ of the stationary cylinder. The critical $Re$ above which VIV is possible depends on the natural frequency of the cylinder. In the sub-critical $Re$ (below the Hopf bifurcation $Re_c$) flow, lock-in occurs over a range of $Re$, with the lock-in range dependent on the natural frequency of the cylinder. Two modes of VIV solutions are found: the first mode occurs when the cylinder enters into lock-in VIV.
response by shifting the fluid’s shedding frequency from the Strouhal frequency to a value closer to the natural frequency of the cylinder. The second mode occurs when the Strouhal frequency at that $Re$ is far from the cylinder natural frequency; hence, lock-in fails to occur, but VIV is still possible and is due to a fluid mode with frequency close to the stationary cylinder’s Hopf bifurcation critical frequency. The effect of the rotational NES on the bifurcation $Re$ is to reduce the range of synchronization and delay the onset of synchronization.
In this chapter, we discuss the co-existence of two long-time solutions at a particular $Re$ for the system consisting of a sprung rigid circular cylinder in flow attached to a rotational NES inside the cylinder. We undertake this study at a $Re$ that is well below the critical Reynolds number ($Re_c$) of the stationary cylinder Hopf bifurcation, which is well documented in the literature to be around 47 (Kovasznay (1949), Sreenivasan et al. (1987), Henderson (1995), and Kumar and Mittal (2006)). For a cylinder supported on an elastic foundation, vortex shedding and VIV solutions are possible in the sub-critical flow with $Re < 47$ (Cossu and Morino (2000), Buffoni (2003), and Mittal and Singh (2005)). In addition to the cylinder being supported on the elastic foundation, we introduce a rotational NES into the system, and demonstrate that there can be multiple long-time solutions for the system. The long-time attractor for the system is shown to be dependent on the initial transient used to excite the system.

While there are a number of situations in which the Navier-Stokes equations are known to admit multiple long-time solutions that are locally stable, we are aware of no examples where both long-time solutions are oscillatory and laminar. For example, in the case of Rayleigh-Bénard convection between parallel plates, there is a region in the Rayleigh number-Prandtl number plane in which steady two-dimensional rolls and steady hexagonal (i.e., three-dimensional) convection cells (Clever and Busse (1994)) are both locally stable solutions of the governing equations, as well as regions in which steady two-dimensional rolls and steady or time-periodic rectangular convection cells (Clever and Busse (1996)) are both locally stable. In circular Poiseuille flow, there is a range of Reynolds number (roughly from 2000 to $10^5$) in which, in addition to the well-known turbu-
lent solutions, the steady, axisymmetric Poiseuille solution is also locally stable Schneider and Eckhardt (2009) and in rotating Couette flow (Andereck et al. (1986)). In the Y-shaped channel flow considered by Goodwin and Schowalter (1996), there are two steady two-dimensional solutions, each of which is apparently locally stable.

For a system without NES and for a particular set of cylinder parameters ($m^* = 10$ and $1/F^*_n = 0.24309$), we compute the critical $Re$ where the steady symmetric solution bifurcates to an unsteady solution with VIV to be in a narrow range of $26 \leq Re_c \leq 28$. Following which, we introduce a NES with parameters $\bar{r}_0 = 0.3$, $\epsilon_r = 0.3$, and $\lambda_r = 0.04716$, and find the critical $Re$ to be in the range $31 \leq Re_c \leq 32.5$. Thus, for a system with NES and for all $Re < 31$, one would expect to find a steady symmetric solution that would be persistent under small disturbances. However, based on the initial transient that is prescribed for the system we find both a steady symmetric solution and VIV with large amplitude cylinder displacement at $Re = 28$. Contrary to general expectation, depending on the initial transient the system is attracted to two different solutions. The two different initial transients are obtained by prescribing the upstream inlet velocity boundary condition as a function of time.

8.1 Time-dependent upstream inlet boundary

The upstream inlet boundary is located at a distance of $24D$ from the cylinder center. We prescribe the velocity boundary condition on the inlet as a function of time during a short period in the initial stage of simulation to obtain two different initial transients, following which a uniform streamwise velocity boundary condition is prescribed at the inlet. In this section, we briefly describe the two time dependent non-uniform streamwise velocity boundary conditions that were used to excite the system.

**Initial transient 1 (IT1):**

In this initial transient, the streamwise velocity at a point on the inlet boundary...
Figure 8.1: Small amplitude solution obtained by IT1, time history of cylinder, NES, and lift coefficient.
Figure 8.2: Streamwise velocity profile: (a) Before perturbation ($tU_0/D = 49.7$) — , Inlet; −, 1D downstream of the cylinder; (b) Perturbed inlet velocity profile ($tU_0/D = 52.535$).

Figure 8.3: Steady symmetric solution at $Re = 28$.

Figure 8.4: Time varying hyperbolic initial transient: −, $tU_0/D = 0$; −−, $tU_0/D = 10$; −., $tU_0/D = 20$. 
Figure 8.5: Large amplitude solution obtained by IT2, time history of cylinder, NES, and lift coefficient.
is offset from other points on the inlet boundary. The sequence of applying the streamwise velocity prescribed on the inlet boundary is described as below:

- Uniform inlet condition is prescribed for 50 convective time units, which will allow the solution to fully develop.
- A point on the inlet located at 1.071\(D\) from the centerline is prescribed with 10% larger velocity for 20 convective time units.
- Uniform inlet condition is resumed after a total of 70 convective time units.

**Initial transient 2 (IT2):**

In this initial transient, the inlet is prescribed with an asymmetric time dependent streamwise velocity profile

\[
 u(0, y, t) = \alpha_1 U_0 \tanh(\alpha_2 y) \left( 1 - \frac{t}{T_{trans}} \right) + U_0 \left( \frac{t}{T_{trans}} \right) \quad \forall \ t \leq T_{trans} \tag{8.1}
\]

Keeping all the system parameters constant, we solve the problem with these two initial transients at \(Re = 28\). The solution is continued for a sufficiently long duration in time so that the long-time behavior can be characterized based on the cylinder displacement amplitude.

The location of lateral boundary from the cylinder center is a critical parameter at this lower value of \(Re = 28\), as it may influence the computed results. A detailed study of the boundary requirement has been reported by Mittal and Singh (2005) and Kumar and Mittal (2006). The results presented in this chapter are obtained using a large rectangular domain of \(72D \times 96D\), discretized with 40250 nodes, and with the undisplaced center of the cylinder located at \(24D\) from the inlet and \(36D\) from the lateral boundaries, giving a blockage of around 1%.

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8.2 Co-existing solutions at $Re = 28$

For the initial transient IT1, the response of the system is shown in Fig. 8.1. Steady symmetric flow is fully developed, as shown in Fig. 8.2a, for a time instant before the inlet velocity is perturbed. The perturbed inlet velocity profile is shown in Fig. 8.2b. As one would expect, the effect of the change at the inlet is immediately propagated throughout the domain as a pressure fluctuation due to incompressibility. This effect is clearly seen in the integrated lift coefficient in Fig. 8.1c, which is composed of the pressure and viscous lift components. The unbalance force generated due to the transient perturbation excites the cylinder-NES system to undergo oscillations as shown in Fig. 8.1a. However, this being a stable solution, the motion of cylinder is dampened out, and the steady symmetric solution is restored after a few oscillations. It should be noted that, within a few oscillations, though being very small in magnitude, there results a 1:1 resonance interaction of the cylinder and NES; the dissipation in the NES results in TET leading to complete suppression of the cylinder displacement. The displacement amplitude shown in Fig. 8.1a decays to zero over long-time; the same can be observed in Figs. 8.1b and 8.1c for the NES angular displacement and the lift coefficient, respectively. The lift coefficient and cylinder displacement at $\tau = 350$ are on the order of $10^{-10}$, and the wake structure shown in Fig. 8.3 indicates the steady symmetric solution.

For the case of initial transient IT2, the inlet velocity profiles at three instants of time are shown in Fig. 8.4. The inlet streamwise velocity profile smoothly transitions to a uniform velocity profile starting from an asymmetric (about the centerline) tangent hyperbolic transient profile. The effect of such an initial transient results in a large amplitude solution at $Re = 28$ shown in Fig. 8.5. The cylinder amplitude shown in Fig. 8.5a is on the order of $0.2D$, and the maximum lift coefficient is on the order of 0.03; though the lift coefficient is very small compared to 0.328 for motionless cylinder at $Re = 100$, it drives the cylinder to large amplitude. The lift coefficient shown in Fig. 8.5c is not symmetric about zero; this is due to the fact that the initial transient makes the flow asymmetric initially. This effect of the initial transient gradually decreases
as the response approaches a quasi-periodic state in long-time. Clearly, these two different solutions obtained based on IT1 and IT2 demonstrate the possibility of multiple co-existing solutions which were realized solely due to addition of an essentially nonlinear element internal to the cylinder. It is interesting to note that the response of the NES in this case of large amplitude VIV; the NES continues to be in resonance with the cylinder, performing small amplitude oscillations. However, as the cylinder amplitude grows slowly, the amplitude of the NES motion shoots up and results in a reduction in the cylinder displacement. This interaction of the NES-cylinder and flow results in a quasi-periodic response of the coupled system. The solution based on IT2 is continued for even longer duration as compared to the IT1 solution to ascertain whether this solution is a long-time attractor of the system and not an initial transient.

8.3 Stability of multiple solutions at $Re = 28$

The solutions presented in §8.2 strongly suggest that the Navier-Stokes equations, coupled to the rigid-body equations for the cylinder and the rotational NES, admit two long-time solutions that are locally stable, and which may have different regions of attraction in the initial condition space. In this section, we study the stability of the two solutions that were observed at $Re = 28$ for two different initial transients. Small $2D$ disturbances are introduced to perturb the fully developed solution, and the solution is continued further in time long enough to assess the temporal stability of the solutions in response to the disturbances.
8.3.1 Stability of long-time steady symmetric solution

The long-time steady symmetric solution obtained by initial transient IT1 is perturbed by imposing an asymmetric inlet velocity profile for one time step after which the solution is continued with uniform inlet velocity profile. One such perturbation velocity profile is shown in Fig. 8.7. The inlet is prescribed with an asymmetric velocity profile, with the maximum velocity at the top lateral boundary 50\% larger than the free stream velocity. The response of the system including the unperturbed portion of the solution is shown in Fig. 8.8. As observed previously, the perturbation introduces fluctuations in the pressure, immediately sensed by the cylinder which undergoes a few small amplitude oscillations. However, further integration in time results in decay of the cylinder response as shown in Fig. 8.8. The perturbations smaller than that shown in Fig. 8.7 result in similar behavior where the solution decays. Thus, the steady symmetric solution with the rotational NES observed at $Re = 28$ is stable under a small disturbance.

8.3.2 Stability of long-time unsteady solution

The long-time unsteady solution at $Re = 28$ realized using initial transient IT2 is perturbed by prescribing the cylinder with a small jump in the displacement for one step, which offsets the cylinder from its observed quasi-periodic orbit. The velocity field from the previous converged step is used in conjunction with the offset cylinder position. One such perturbed solution is shown in Fig. 8.9, with the cylinder displacement at $\tau = 789.85$ perturbed by $0.05D$. The perturbed solution decays to reach the steady symmetric solution. Thus, the large amplitude VIV solution is unstable under small perturbation. Further analysis is required to study the stability of the solution.

8.4 Conclusions

Based on two different initial transients, two co-existing solutions are found at $Re = 28$, which is below the critical $Re$ of the stationary cylinder. One of the solutions is large amplitude VIV with a $Re$ out of the lock-in regime
at that stiffness of the cylinder. Small perturbation of the solutions shows that only the steady symmetric solution persists. It is of course possible that there are additional two-dimensional unsteady solutions (not necessarily time-periodic) that we have not discovered. A rigorous demonstration that both of the computed solutions are stable with respect to infinitesimal disturbances in the class of two-dimensional flows would require a Floquet analysis of these time-periodic solutions, and is beyond the scope of the present work. But the fact that we find two long-time solutions by approximate forward integration of the governing equations strongly suggests that they are locally stable, at least within the class of two-dimensional solutions.
Figure 8.8: Stability of steady symmetric solution at $Re = 28$ for small disturbance.
Figure 8.9: Stability of unsteady large amplitude solution at $Re = 28$ for small disturbance.
CHAPTER 9

CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

9.1 Conclusions

A high fidelity finite-element computational code is utilized to study the effect of an essentially nonlinear internal device in a rigid circular cylinder undergoing VIV. The computational model was subject to a rigorous validation procedure by comparing with results available in the literature. The primary objective of passive suppression of VIV in the laminar regime using a NES has been investigated with two NES configurations. An ungrounded translational NES with purely cubic stiffness restoring force; and a rotational NES with essential (i.e., nonlinearizable) nonlinear inertia coupling with the cylinder. Though complete elimination of VIV has not been achieved by either of the configurations of NES, the translational NES was found to be very effective in reducing the LCO amplitude of the cylinder by about 75%. It was able to suppress VIV by altering the rigid body dynamics of the cylinder and modifying its phase relation with respect to the driving lift force. The underlying nonlinear dynamical mechanism governing the cylinder-flow-NES interaction is passive targeted energy transfers from the flow and the cylinder to the NES caused by series of transient resonance captures. The wake structure of the system with the translational NES was found to be very similar to that of the system without the NES. More than 250 simulations were performed by varying the NES parameter to roughly span the parameter space in order to identify the beneficial regimes of the NES parameters. In summary, two kinds of passive suppression mechanism were observed for various NES parameters; the first kind corresponding to strong modulation of the response envelopes and the second characterized by suppressed amplitude of the LCO with no amplitude modulation (partial LCO
suppression). The effectiveness of the NES was compared against a linear tuned mass damper (TMD) of identical mass, optimized for $Re = 100$. The TMD was found to be very effective when its frequency is tuned to the lift frequency, as expected, since it is a narrowband device being capable of suppressing the cylinder vibration only in the vicinity of its (tuned) resonance frequency. On the contrary, the NES performance is found to be superior when the flow Reynolds number is varied over the lock-in regime. The reason for this enhanced performance is that the essentially nonlinear NES lacks a preferential resonance frequency, and as a result, its action is broadband being capable of resonance with the cylinder over broad frequency and energy ranges (in contrast to the TMD whose performance is not varied with changing energy). In fact, the NES was found to narrow down the lock-in regime which is not the case with the TMD.

About 100 simulations were performed with a rotational NES to span the considered NES parameter space. The passive suppression of VIV with the rotational NES was found to be relatively less effective when compared to the translational NES. However, the rotational NES was found to influence the wake structure significantly. Depending on the rotational NES parameters, the attached vortices were elongated up to 10 cylinder diameters downstream, while the corresponding instantaneous lift and displacement were also found to be significantly reduced. Robustness of the elongated wake response was studied by introducing small perturbations to the individual NES parameters. Existence of this interesting mechanism at different flow regimes was also established by using an NES with different parameters. An approximate analytical study was carried out which explains the observed response with elongated wake and revealed that in that regime the action of viscous effects of the surrounding fluid dominate over the inertial effects; in essence, the action of the fluid on the cylinder resembles that of a linear viscous damper. This raises the interesting prospect that in this regime that rotational NES is attempting to stabilize the steady fluid mode which is unstable above the Hopf bifurcation occurring at $Re \approx 47$. This conjecture, however, needs to be verified by further study. The significant
influence of the rotational NES on the flow field is analyzed with the help of a reduced-order model based on proper orthogonal decomposition (POD). A very good correlation has been found between angular momentum of the rotating NES mass and the observed elongation of the wake. Based on the analysis, we conjecture that the effect of the rotational NES is to (temporally) stabilize the wake.

Huge computational resource is involved in obtaining the high fidelity results, and the resulting data is also immense to analyze, understand and physically interpret. This motivated us to develop a model reduction approach with the aim of first providing a descriptive low-dimensional model and then exploring the predictive capability of such a model. In the first approach phenomenological model reduction was attempted by constructing a single-DOF self-excited oscillator to model the fluid-structure interaction. Accurate flow data available from high fidelity simulations was utilized to develop such an oscillator model. Then the NES was coupled to the self-excited oscillator model to obtain a two-DOF model for the coupled problem. This model provides good insight on the low-dimensional behavior of the problem. Existence of a slow invariant manifold (SIM) for the dynamics of this model, and its relation to different VIV suppression mechanisms for the case of translational NES were clearly explained by this model reduction approach. In the second approach, we adopted POD to get an orthogonal basis to project the governing partial differential equations (PDEs) of motion to get a system of reduced (projected) ordinary differential equations (ODEs) which can be used to approximate the solution. We were able to get a good approximation for the solution in cases when the cylinder was motionless and when it moved with prescribed motion. The case where the cylinder is undergoing VIV was not modeled by this method so there is the need to extend this approach to take into account general transient cylinder motion. In the third approach, we developed a methodology to obtain a least-squares fit ODE system to model the pressure dynamics. In this approach, we were able to get a fair approximation for the pressure variation when the cylinder motion was prescribed, but the monotonic convergence of the ODE coefficients with
increased number of sampled time series to fit the ODE was not observed and needs further investigation.

We studied the effect of flexible support of the cylinder on the bifurcation of the steady symmetric solution. We found that VIV is possible for \( Re < 47 \), which is well below the first Hopf bifurcation of the stationary cylinder. The lock-in range in the subcritical \( Re \) was found to depend significantly on the natural frequency of the oscillator. Two modes of the dynamics were found to exist for a sprung cylinder in sub-critical \( Re \), namely, the lock-in structural mode and the von Kármán mode. The bifurcation curves were plotted in the inverse of frequency-\( Re \) plane and found that the stiffness required to approach the stationary cylinder case was finite. We also introduced a rotational NES into the system at sub-critical \( Re \), and found that the rotational NES delays the onset of instability of the sprung cylinder. We also found that multiple long-time co-existing solutions are possible at a particular \( Re \) in sub-critical flow regime based on two different initial transients to excite the system. Moreover, we found the existence of chaotic dynamics of the cylinder with internal rotational NES at low \( Re \). We analyzed the stability of the two long-time solutions at \( Re = 28 \) by introducing small perturbations and found that the large amplitude unsteady solution decayed for small perturbations.

Our work can be considered as a first attempt towards studying the flow-structure interaction of a bluff body with a strongly nonlinear internal attachment. Some of the aforementioned nonlinear dynamical phenomena are reported for the first time in the literature and reveal the rich potential of using intentional strong nonlinearity in the design of such elements. Further extension of our findings, however, is needed in order to make use of such strongly nonlinear approaches in predictive designs of flow-structure interaction problems. Some suggestions for further research are provided in the next section.
9.2 Suggestions for future work

For both NES configurations, a parametric search can be performed to determine additional regions of NES parameters leading to effective and robust VIV suppression. This study will provide optimized values for the NES parameters and will reveal the different modes of VIV suppression. Such optimization was not performed in this work, since our main effort was to study the nonlinear dynamics of the integrated cylinder-NES system and reveal the interesting nonlinear dynamical phenomena that can be exhibited by this system. For the case of translational NES, we have performed our study only in the neighborhood of \( Re \approx 100 \). The lower \( Re \) and sub-critical \( Re \) VIV suppression due to the action of the translational NES needs to be explored. As shown in this work, large amplitude VIV occurs even at lower \( Re \), even though the lift force is very small; hence, the nonlinear dynamics of the system in that flow regime needs to be systematically explored. We conjecture that the third mechanism of passive suppression leading to complete elimination of LCO may be possible in lower \( Re \) (this mechanism was not detected in the present study that considered VIV suppression only in the neighborhood of \( Re = 100 \)), since closer to the Hopf bifurcation \( Re \) where LCO is originated it might be possible to achieve such complete VIV suppression using the translational NES. The reason that no complete suppression could be achieved at \( Re = 100 \) could be that the Kármán vortex street is well established and strong (based on the magnitude of the lift coefficient), making the action of the translational NES incapable of completely eliminating the LCO (in fact, as shown in (Gendelman et al. (2010)) there are dynamical regimes where a lightweight NES can be ineffective). It is clear from our presented results that the NES affects the dynamics in a broadband fashion by modifying synchronization or lock-in behavior of the cylinder response unlike the TMD whose action is narrowband. The reason for the narrow range of lock-in that persists even with the addition of the NES needs to be further explored. The ROM based on POD is a promising direction to pursue towards this goal, since we can get accurate representation of the flow field, which is important to capture the dynamics of VIV. To achieve this, however, certain technical challenges should be overcome, perhaps by adopting an iterative approach for
estimating the motion of the cylinder while simultaneously performing the POD decomposition.

In addition, our results on the influence of the rotational NES on the cylinder dynamics and on the wake of the flow using POD modes can be extended further to study the nonlinear modal energy exchanges leading to the elongated vortex street as reported herein. This result is especially surprising, given that we have shown that an internal element (the rotational NES) is capable of drastically affecting and modifying the external flow field, without having any contact with the flow. This new result should be further explored and systematically understood by techniques such as reduced-order modeling and parametric studies. Indeed, our first results can pave the way for passive control of the wake of the flow past bluff bodies through the action of internal rotating and/or oscillating elements. Moreover, this finding can contribute towards developing devices for effective energy harvesting from the flow. In fact, the rotational NES is a promising concept towards this goal, especially if the cylinder-NES system can be designed for robust and continuous rotation of the NES irrespective of flow fluctuations. In addition, two significant directions to pursue with the rotational NES are semi-active methods for wake stabilization, and exploring the feasibility of retaining or stretching the elongated wake for longer duration either by prescribing the motion of the NES or its initial conditions. Indeed, it would be worthy exploring, if the prolonging of the wake is a manifestation of the action of the rotational NES to stabilize the steady solution which is unstable after the Hopf bifurcation at Re ≈ 47. If this is indeed the case, in principle one could design internal rotational NESs (or other internal nonlinear elements) that robustly and passively stabilize the steady solution and reduces the drag coefficient of the cylinder or of a general bluff body. Such drag reduction of the cylinder in the regime of elongated vortex street has already been reported in the present work, so it would be interesting to systematically pursue the possibility of passive drag reduction through the use of internal rotating/oscillating elements. Finally, further investigations of the regimes of multiple co-existing solutions and of lower Re chaotic responses need to be performed. Our findings
revealed the existence of such regimes in the case of the cylinder with an internal rotational NES, and the domain of existence of such interesting solutions need to be systematically explored.
APPENDIX A

VARIATIONAL MULTISCALE STABILIZED FINITE-ELEMENT METHOD.

Following the computational approach developed by Calderer and Masud (2010), the Navier-Stokes equations (2.1a) and (2.1b) are solved numerically by casting them into an Arbitrary Eulerian Lagrangian (ALE) framework, which we write as

\[
\frac{\partial v}{\partial t} + (v - v^m) \cdot \nabla v - 2\nu \nabla \cdot \varepsilon(v) + \nabla p = 0 \quad \text{in } \Omega_t \times [0, T] \tag{A.1}
\]

\[
\nabla \cdot v = 0 \quad \text{in } \Omega_t \times [0, T] \tag{A.2}
\]

where \(v\) is the velocity vector, \(v^m\) is the fluid mesh velocity, \((\partial v/\partial t)_Y\) is the time derivative of the velocity in the ALE frame, \(p\) is the pressure, \(\nu\) is the kinematic viscosity, \(I\) in the identity tensor, \(\varepsilon(v)\) is the strain rate tensor, and we have taken the body force per unit mass to be zero.

The cylinder is allowed to move perpendicularly to two fixed sidewalls, thus giving rise to a moving-boundary problem, as the fluid domain depends on the position of the cylinder. The computations are performed on a time-evolving spatial domain \(\Omega_t\) and over a temporal domain defined by the open interval from 0 to \(T\). The fluid domain is an annular region bounded on its interior by the cylinder, and on its exterior by two fixed sidewalls parallel to the mean flow, and inlet and outlet surfaces perpendicular to the mean flow. We divide the boundary into two parts: \(\Gamma_g|t = \Gamma_{cyl}|t \cup \Gamma_{in}\), consisting of the cylindrical surface and the upstream inlet, on which Dirichlet conditions

\[
v = g \quad \text{on } \Gamma_g|t \times [0, T] \tag{A.3}
\]
are imposed, with $\mathbf{g}$ being the transverse velocity of the cylinder on $\Gamma_{cyl}|t$, while on $\Gamma_{in}$ we have $\mathbf{g} = U_0 \mathbf{e}_x$; and $\Gamma_{h}|t$, consisting of the sidewalls $\Gamma_{w}|t$ on which

$$\sigma \cdot \mathbf{n} = (2\nu \mathbf{e}(\mathbf{v}) - p\mathbf{I}) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{h}|t \times ]0, T[ \quad (A.4)$$

and a no-penetration condition are imposed, and the downstream outlet $\Gamma_{out}|t$, on which only (A.4) is imposed. We have taken the stress vector to be zero on each portion of $\Gamma_{h}|t$. The initial conditions are

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0 \quad \text{on } \Omega_0 \times \{0\} \quad (A.5)$$

The finite-element method used for this problem is based on the multiscale stabilized finite-element formulation developed for the Navier-Stokes equations by Masud and Calderer (2009), which makes use of an additive decomposition of the velocity $\mathbf{v}$ and weighting function $\mathbf{w}$ into coarse and fine scales, respectively given by

$$\mathbf{v}(\mathbf{x}, t) = \underbrace{\bar{\mathbf{v}}(\mathbf{x}, t)}_{\text{coarse scale}} + \underbrace{\mathbf{v}'(\mathbf{x}, t)}_{\text{fine scale}}$$

$$\mathbf{w}(\mathbf{x}) = \underbrace{\bar{\mathbf{w}}(\mathbf{x})}_{\text{coarse scale}} + \underbrace{\mathbf{w}'(\mathbf{x})}_{\text{fine scale}}$$

(A.6)

(A.7)

Substituting the decomposition of field and weighting functions into the weak form of the problem yields coarse- and fine-scale sub-problems. The fine-scale sub-problem is then solved in terms of the residual of the coarse-scale sub-problem. The solution of the fine-scale problem is variationally projected onto the coarse-scale space and is manifested in the form of a stabilization term. The stabilized form of the problem in weak form is given by

$$\left( \bar{\mathbf{w}}, \frac{\partial \mathbf{v}}{\partial t} \right) + (\mathbf{w}, (\mathbf{v} - \mathbf{v}^m) \cdot \nabla \mathbf{v}) + (\nabla^s \mathbf{w}, 2\nu \nabla^s \mathbf{v}) - (\nabla \cdot \mathbf{w}, p) + (\mathbf{q}, \nabla \cdot \mathbf{v})$$

$$+ (\mathbf{w} \nabla \cdot (\mathbf{v} - \mathbf{v}^m) + (\mathbf{v} - \mathbf{v}^m) \cdot \nabla \mathbf{w} - \mathbf{w} \cdot \nabla^T \mathbf{v} + 2\nu \Delta^s \mathbf{w} + \mathbf{v} \nabla \cdot \mathbf{q}, \tau \mathbf{r}) = 0$$

(A.8)
where \( \langle \cdot , \cdot \rangle = \int_{\Omega_t} \langle \cdot \rangle \, d\Omega \) is the \( \mathcal{L}^2(\Omega_t) \)-inner product, \( q \) is the weighting function for the pressure field, the stabilization tensor \( \tau \) is given by

\[
\tau = b^e \int b^e \, d\Omega \left[ \int (b^e)^2 \nabla^T v \, d\Omega + I \int b^e \left( v - v^m \right) \cdot \nabla b^e \, d\Omega \right]^{-1} + \nu I \int |\nabla b^e|^2 \, d\Omega + \nu I \int \nabla b^e \otimes \nabla b^e \, d\Omega
\]

(\text{A.9})

the fine scales \( v' \) and \( w' \) are represented by bubble functions \( b^e \) (Masud and Kwack (2008)), the residual \( r \) is given by eq. (17) of Calderer and Masud (2010), \( \nabla^s \mathbf{u} \) is the symmetric part of the gradient of the vector-valued function \( \mathbf{u} \), and \( \Delta^s \) is the vector Laplacian.

The mesh motion scheme for such a moving boundary problem in the Arbitrary Lagrangian-Eulerian framework is presented in Masud et al. (2007) and Kanchi and Masud (2007). Masud (2006) established a criterion for the effects of mesh motion on the accuracy and stability of ALE based numerical schemes.

The present multiscale finite-element method has been employed by Calderer and Masud (2010) to compute laminar flow past a spring-mounted rigid cylinder and around a deformable beam.

At each time step, an adaptive mesh rezoning algorithm based on a modified Laplace equation (Masud et al. (2007)) solves for the incremental mesh nodal displacements to accommodate the spatial deformation imposed by the moving cylinder boundary. In this technique (also referred to as \( r \)-refinement) the inter-element connectivity in the mesh stays unchanged and only the nodal coordinates of the mesh are updated. At each time step, an adaptive mesh rezoning algorithm based on a modified Laplace equation (Masud et al. (2007)) solves for the incremental mesh nodal displacements to accommodate the spatial deformation imposed by the moving cylinder boundary. In this technique (also referred to as \( r \)-refinement)
The computational domain is the doubly-connected area shown schematically in Fig. 2.1, with the circular boundary representing the cylinder. Since flow past a fixed cylinder is two-dimensional for Re=100, we neglect spanwise variation.

The location of the lateral boundaries relative to the cylinder plays a critical role in the accuracy of the computed results, in particular for the case of the moving cylinder. The blockage $B$ is defined as the ratio of the diameter of the cylinder to the width of the computational domain and is a measure of whether an adequate distance from the lateral boundary to the cylinder has been provided, in order to avoid wall effects. The effects of blockage, as well as the effects of the locations of upstream and downstream boundaries of the computational domain, have been studied by Prasanth et al. (2006) for Re=100 in the context of VIV with transverse and streamwise oscillation of a cylinder with $m^* = 10$. They found that $B < 0.05$ is necessary to avoid undue influence of the lateral boundaries. Taking this into account, the equilibrium position of the center of the rigid circular cylinder in our computation is located $12D$ downstream from the inlet and $36D$ upstream from the outlet, and the lateral boundaries are $24D$ apart, giving $B < 0.042$.

Spatial derivatives are large close to the cylinder boundary, so that adequate spatial resolution there is necessary to compute the drag and lift coefficients accurately. High spatial resolution around the cylinder is maintained by partitioning the computational domain into simpler sub-domains, each bounded by four curves. The moving cylinder forms the inner boundary of an annular sub-domain in which very high mesh density is maintained. When the cylinder
Figure B.1: Sub-domains to generate mapped mesh on each of the sub-domains.

Figure B.2: Sub-domains close to the cylinder.

Figure B.3: Mesh for the finite-element solution, number of nodes = 16922, number of elements = 8261.
is initially centered at the midplane in the channel, the outer boundary of this
sub-domain is a square of length $5D$. The partitioned computational domain,
with the number of elements noted on each edge, is shown in Fig. B.1. Fur-
ther, the annular sub-domain is divided by two diagonals into four four-sided
sub-domains, with the one aft of the cylinder being further partitioned into two
halves by the midline. The circular arcs of the sub-domains upstream and lateral
to the cylinder are each discretized using 27 elements. Each circular arc on the
aft boundary of the cylinder is discretized using 14 elements, corresponding to
109 divisions of the circumference of the cylinder. The element size in the radial
direction increases gradually, so that elements close to the cylinder are smaller
compared with those farther away, with a bias ratio of $1/8$ (ratio of the length
of the smallest line element on an edge to the length of the largest line element
on that edge). The remaining rectangular domains are meshed with rectangular
elements using the same bias ratio. The partitioned annular sub-domain, with
the number of mesh node seeds on each edge, is shown in Fig. B.2. The total
number of nodes for the two-dimensional mesh shown in Fig. B.3 is 8461. The
mesh used in our computations is one element thick (with $L = 0.05D$) in the
spanwise direction, thus resulting in a single layer of 8261 brick elements, having
16922 nodes. Results show that there is no flow or variation in the spanwise
direction.
APPENDIX C

VERIFICATION OF ENERGY BALANCE.

The rate of change of energy in the system is given by

\[
\frac{\partial}{\partial t} \left( \int_0^L \int_0^H \frac{\rho f}{2} \left( u^2(x, y, t) + v^2(x, y, t) \right) \, dx \, dy \right)
\]

\[+ \int_0^H \left( \frac{\partial f}{2} u(L, y, t) \left( u^2(L, y, t) + v^2(L, y, t) \right) \right) - \frac{\partial f}{2} u(0, y, t) u^2(0, y, t) \, dy
\]

\[+ \frac{\partial}{\partial t} \left( \frac{1}{2} (\hat{M}_{cyl} - \hat{M}_{nes}) \hat{y}_2^2 + \frac{1}{2} \hat{K}_{cyl} \hat{y}_1^2 + \frac{1}{2} \hat{M}_{nes} \hat{y}_2^2 + \frac{1}{2} \hat{K}_{nes} \hat{y}_2^4 \right)
\]

\[+ \frac{1}{2} \hat{C}_{nes} \hat{y}_2^2 - \int_0^H [p(0, y, t) u(0, y, t) - p(L, y, t) u(L, y, t)] \, dy
\]

\[+ \int_0^L \int_0^H \mu \left[ \frac{2}{2} \left( \frac{\partial u(x, y, t)}{\partial x} \right)^2 + 2 \left( \frac{\partial v(x, y, t)}{\partial y} \right)^2 + \left( \frac{\partial u(x, y, t)}{\partial y} \right)^2
\]

\[+ \left( \frac{\partial v(x, y, t)}{\partial x} \right)^2 + 2 \frac{\partial u(x, y, t)}{\partial y} \frac{\partial v(x, y, t)}{\partial x} \right] \, dx \, dy = \Delta_{res}(t)
\]

where $H$ is the width and $L$ is the length of the computational domain.
APPENDIX D

PREDICTOR-MULTICORRECTOR
EQUATIONS OF CYLINDER-NES

The equations of motion of the cylinder coupled to a rotating NES are given by

\[(M + m)\ddot{y}_1 + Ky_1 = F_L + mr_0 \frac{d}{dt} \left( \dot{\theta} \sin \theta \right) = F_L + N_1(\theta, \dot{\theta})\]  \hspace{1cm} (D.1)

\[mr_0^2 \ddot{\theta} + C\dot{\theta} = mr_0 \dot{y}_1 \sin \theta = N_2(\dot{y}_1, \theta)\]  \hspace{1cm} (D.2)

The generalized-α predictor-multicorrector algorithm is used to integrate the coupled cylinder-NES system under the action of the oscillating lift force. The following notations are introduced for brevity of the equations:

- $d_n$, $v_n$, and $a_n$ represent the approximations of $y_1(t_n)$, $\dot{y}_1(t_n)$, and $\ddot{y}_1(t_n)$
- $D_n$, $V_n$, and $A_n$ represent the approximations of $\theta(t_n)$, $\dot{\theta}(t_n)$, and $\ddot{\theta}(t_n)$

The equilibrium of equations D.1 and D.2 are enforced at an intermediate time between two time steps $t_n$ and $t_{n+1}$ as

\[(M + m)a^{i+1}_{n+1} + Kd^{i+1}_{n+1} = F_L(t_{n+1-\alpha_f})^{i+1} + N_1(t_{n+1-\alpha_f})^{i+1}\]  \hspace{1cm} (D.3)

\[mr_0^2 A^{i+1}_{n+1} + CV^{i+1}_{n+1} = N_2(t_{n+1-\alpha_f})^{i+1}\]  \hspace{1cm} (D.4)
Where

\[
\begin{align*}
    a_{n+1 \alpha_m}^{i+1} &= a_{m} a_{n+1}^{i+1} + (1 - a_m) a_n \\
    v_{n+1 \alpha_f}^{i+1} &= a_f v_{n+1}^{i+1} + (1 - a_f) v_n \\
    d_{n+1 \alpha_f}^{i+1} &= a_f d_{n+1}^{i+1} + (1 - a_f) d_n \\
    A_{n+1 \alpha_m}^{i+1} &= a_m A_{n+1}^{i+1} + (1 - a_m) A_n \\
    V_{n+1 \alpha_f}^{i+1} &= a_f V_{n+1}^{i+1} + (1 - a_f) V_n \\
    D_{n+1 \alpha_f}^{i+1} &= a_f D_{n+1}^{i+1} + (1 - a_f) D_n
\end{align*}
\]

Linearizing the nonlinear terms about \( |i|_{n+1} \)

\[
\begin{align*}
    N_1 \left( t_{n+1 - \alpha_f} \right)^{i+1} &= N_1 \left( t_{n+1 - \alpha_f} \right)^i + \left( \frac{\partial N_1}{\partial \theta} \right)^i \delta \theta + \left( \frac{\partial N_1}{\partial \dot{\theta}} \right)^i \delta \dot{\theta} + \left( \frac{\partial N_1}{\partial \ddot{\theta}} \right)^i \delta \ddot{\theta} \\
    N_2 \left( t_{n+1 - \alpha_f} \right)^{i+1} &= N_2 \left( t_{n+1 - \alpha_f} \right)^i + \left( \frac{\partial N_2}{\partial \theta} \right)^i \delta \theta + \left( \frac{\partial N_2}{\partial \ddot{y}_1} \right)^i \delta \ddot{y}_1
\end{align*}
\]

The variations of nonlinear functions \( N_1 \) and \( N_2 \) are obtained as

\[
\begin{align*}
    N_1 &= m r_0 \ddot{\theta} \sin \theta + m r_0 \dddot{\theta} \cos \theta \\
    \frac{\partial N_1}{\partial \theta} &= m r_0 \ddot{\theta} \cos \theta - m r_0 \dddot{\theta} \sin \theta \\
    \frac{\partial N_1}{\partial \dot{\theta}} &= 2 m r_0 \ddot{\theta} \cos \theta \\
    \frac{\partial N_1}{\partial \ddot{\theta}} &= m r_0 \sin \theta \\
    N_2 &= m r_0 \dddot{y}_1 \sin \theta \\
    \frac{\partial N_2}{\partial \theta} &= m r_0 \ddot{y}_1 \cos \theta \\
    \frac{\partial N_2}{\partial \ddot{y}_1} &= m r_0 \sin \theta
\end{align*}
\]

Substituting into equilibrium equation

\[
(M + m) \left( \alpha_m a_{n+1}^{i+1} + (1 - \alpha_m) a_n \right) + K \left( \alpha_f d_{n+1}^{i+1} + (1 - \alpha_f) d_n \right) = F_L + N_1^i + \left( \frac{\partial N_1}{\partial \theta} \right)^i \delta \theta + \left( \frac{\partial N_1}{\partial \dot{\theta}} \right)^i \delta \dot{\theta} + \left( \frac{\partial N_1}{\partial \ddot{\theta}} \right)^i \delta \ddot{\theta}
\]

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\[
\begin{align*}
    m r_0^2 (\alpha_m A_{n+1}^{i+1} + (1 - \alpha_m) A_n) + C (\alpha_f V_{n+1}^{i+1} + (1 - \alpha_f) V_n) \\
    = N_2^i + \left( \frac{\partial N_2}{\partial \theta} \right)^i \delta \theta + \left( \frac{\partial N_2}{\partial \dot{\theta}} \right)^i \delta \dot{\theta}
\end{align*}
\]

Rearranging:

\[
\begin{align*}
    (M + m) \alpha_m a_n^{i+1} + K \alpha_f a_n^{i+1} \\
    = F_L + N_1^i + \left( \frac{\partial N_1}{\partial \theta} \right)^i \delta \theta + \left( \frac{\partial N_1}{\partial \dot{\theta}} \right)^i \delta \dot{\theta}
\end{align*}
\]

\[
\begin{align*}
    - (M + m) (1 - \alpha_m) a_n - K (1 - \alpha_f) d_n
    \end{align*}
\]

\[
\begin{align*}
    m r_0^2 \alpha_m A_{n+1}^{i+1} + C \alpha_f V_{n+1}^{i+1} = N_2^i + \left( \frac{\partial N_2}{\partial \theta} \right)^i \delta \theta + \left( \frac{\partial N_2}{\partial \dot{\theta}} \right)^i \delta \dot{\theta} \\
    - m r_0^2 (1 - \alpha_m) A_n - C (1 - \alpha_f) V_n
\end{align*}
\]

Substituting

\[
\begin{align*}
    a_{n+1}^{i+1} &= a_n^i + \delta a \\
    v_{n+1}^{i+1} &= v_n^i + \delta v = v_{n+1}^i + \Delta t \gamma \delta a \\
    d_{n+1}^{i+1} &= d_n^i + \delta d = d_{n+1}^i + \Delta t^2 \beta \delta a \\
    A_{n+1}^{i+1} &= A_n^i + \delta A \\
    V_{n+1}^{i+1} &= V_n^i + \delta V = V_{n+1}^i + \Delta t \gamma \delta A \\
    D_{n+1}^{i+1} &= D_n^i + \delta D = D_{n+1}^i + \Delta t^2 \beta \delta A
\end{align*}
\]

\[
\begin{align*}
    (M + m) \alpha_m (a_{n+1}^i + \delta a) + K \alpha_f (a_{n+1}^i + \Delta t^2 \beta \delta a) \\
    = F_L + N_1^i + \left( \frac{\partial N_1}{\partial \theta} \right)^i \Delta t^2 \beta \delta A + \left( \frac{\partial N_1}{\partial \dot{\theta}} \right)^i \Delta t \gamma \delta A + \left( \frac{\partial N_1}{\partial \ddot{\theta}} \right)^i \delta \ddot{A}
\end{align*}
\]

\[
\begin{align*}
    - (M + m) (1 - \alpha_m) a_n - K (1 - \alpha_f) d_n
\end{align*}
\]

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\[ m r_0^2 \alpha_m (A_{n+1} + \delta A) + C \alpha_f (V_{n+1}^i + \Delta t \gamma \delta A) \]
\[ = N^*_2 + \left( \frac{\partial N_2}{\partial \theta} \right)^i \Delta t^2 \beta \delta A + \left( \frac{\partial N_2}{\partial \eta_1} \right)^i \delta a \]
\[ - m r_0^2 (1 - \alpha_m) A_n - C (1 - \alpha_f) V_n \]

\[ (M + m) = F_L + N^*_1 + \left( \frac{\partial N_1}{\partial \theta} \right)^i \Delta t^2 \beta \delta A + \left( \frac{\partial N_1}{\partial \eta_1} \right)^i \Delta t \gamma \delta A \]
\[ + \left( \frac{\partial N_1}{\partial \theta} \right)^i \delta A - (M + m) (1 - \alpha_m) a_n - K (1 - \alpha_f) d_n \]
\[ - (M + m) \alpha_m a_{n+1}^i - K \alpha_f d_{n+1}^i \]

\[ m r_0^2 \alpha_m \delta A + C \alpha_f \Delta t \gamma \delta A = N^*_2 + \left( \frac{\partial N_2}{\partial \theta} \right)^i \Delta t^2 \beta \delta A + \left( \frac{\partial N_2}{\partial \eta_1} \right)^i \delta a \]
\[ - m r_0^2 (1 - \alpha_m) A_n - C (1 - \alpha_f) V_n - m r_0^2 \alpha_m A_{n+1}^i - C \alpha_f V_{n+1}^i \]

\[ [(M + m) \alpha_m + K \alpha_f \Delta t^2 \beta] \delta a + \left[ \left( \frac{\partial N_1}{\partial \theta} \right)^i - \left( \frac{\partial N_1}{\partial \theta} \right)^i \Delta t \beta - \left( \frac{\partial N_1}{\partial \eta_1} \right)^i \Delta t \gamma - \left( \frac{\partial N_1}{\partial \theta} \right)^i \right] \delta A \]
\[ = F_L + N^*_1 - (M + m) (1 - \alpha_m) a_n - K (1 - \alpha_f) d_n \]
\[ - (M + m) \alpha_m a_{n+1}^i - K \alpha_f d_{n+1}^i \]

\[ \left[ - \left( \frac{\partial N_2}{\partial \eta_1} \right) \right] \delta a + \left[ m r_0^2 \alpha_m + C \alpha_f \Delta t \gamma - \left( \frac{\partial N_2}{\partial \theta} \right)^i \Delta t^2 \beta \right] \delta A \]
\[ = N^*_2 - m r_0^2 (1 - \alpha_m) A_n - C (1 - \alpha_f) V_n \]
\[ - m r_0^2 \alpha_m A_{n+1}^i - C \alpha_f V_{n+1}^i \]

Writing in matrix form

\[ M^* \delta \bar{a} = F \]
Where,

\[ M^* = \begin{bmatrix} M^*_{11} & M^*_{12} \\ M^*_{21} & M^*_{22} \end{bmatrix} \]

\[ = (M + m) \alpha_m + K \alpha_f \Delta t^2 \beta \left( -\left( \frac{\partial N_1}{\partial \theta} \right)^i \Delta t^2 \beta - \left( \frac{\partial N_1}{\partial \theta} \right)^i \Delta t \gamma - \left( \frac{\partial N_1}{\partial \theta} \right)^i \right) \]

\[ - \left( \frac{\partial N_2}{\partial y} \right)^i m r_0^2 \alpha_m + C \alpha_f \Delta t \gamma - \left( \frac{\partial N_2}{\partial \theta} \right)^i \Delta t^2 \beta \]

\[ F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \]

\[ = F_L + N_1^i - (M + m) \left( (1 - \alpha_m) a_n - \alpha_m a_n^i + 1 \right) - K (1 - \alpha_f) d_n - K \alpha_f d_{n+1}^i \]

\[ \bar{a} = \begin{bmatrix} \delta a \\ \delta A \end{bmatrix} \]

Compute the residue to check for convergence

\[ R_{n+1}^i = \sqrt{F_1^2 + F_2^2} \quad (D.5) \]

Solving the system of equations for \( \bar{a} \)

\[ \text{det} (M^*) = M^*_{11} M^*_{22} - M^*_{12} M^*_{21} \]

\[ \delta a = \frac{M^*_{22} F_1 - M^*_{12} F_2}{\text{det}(M^*)} \]

\[ \delta A = \frac{F_1 - M^*_{11} \delta a}{M^*_{12}} \]

D.1 Predictor-multicorrector algorithm

Step 1: Predictor step

At give time step \( t_{n+1} \) the predictors of displacement, velocity, and acceleration are written using the previous converged state variables at \( t_n \). Predictors for
Step 1: Multicorrection steps

Step 1: Multicorrection steps

\[
d_{n+1}^i = d_n + \Delta tv_n + \frac{\Delta t^2}{2} (1 - 2\beta) a_n
\]
\[
d_{n+1}^i = v_n + \Delta tv_n + (1 - \gamma) \Delta ta_n
\]
\[
a_{n+1}^i = 0
\]

Step 2: Multicorrector steps

Evaluate the residue \( R \) at \( i \)th iteration of time step \( t_{n+1}^i \) using equation D.5:

\[
M^\ast \delta \ddot{a} = F_{n+1}^i
\]  
(D.6)

If the residue is above tolerance limit, update the solution with corrections:

\[
a_{n+1}^{i+1} = a_{n+1}^i + \delta a
\]
\[
A_{n+1}^{i+1} = A_{n+1}^i + \delta A
\]
\[
v_{n+1}^{i+1} = v_{n+1}^i + \Delta t\gamma \delta a
\]
\[
V_{n+1}^{i+1} = V_{n+1}^i + \Delta t\gamma \delta A
\]
\[
d_{n+1}^{i+1} = d_{n+1}^i + \Delta t^2 \beta \delta a
\]
\[
D_{n+1}^{i+1} = D_{n+1}^i + \Delta t^2 \beta \delta A
\]

Repeat the step 2 until residue reduces below convergence criterion.

Step 3

Once convergence is achieved at \( t_{n+1} \), move the next time step with

\[
a_n = a_{n+1}^{i+1}, v_n = v_{n+1}^{i+1}, d_n = d_{n+1}^{i+1}
\]
\[
A_n = A_{n+1}^{i+1}, V_n = V_{n+1}^{i+1}, D_n = D_{n+1}^{i+1}
\]
In this section, the effect of performing “fluid-structure equilibrating iterations” (FSEI) at each time step on some of the key representative results are discussed. The FSEI computational results presented in this chapter were obtained by solving the fluid and cylinder-NES equations of motion in a staggered manner until convergence was achieved at each time step.

E.1 Effect of FSEI on VIV of sprung cylinder at $Re = 100$

For the case of the sprung cylinder undergoing VIV at $Re = 100$, the cylinder displacement, without and with equilibrating iterations, is shown in Figs. E.1a and E.1b, respectively. The amplitude of the cylinder displacement increases by approximately 0.8% due to FSEI. The effect of FSEI on the lift coefficient that drives the cylinder motion is examined in Figs. E.2a and E.2b, which show a 3.9% increase in the lift coefficient amplitude due to FSEI. Other than the marginal increase in the amplitude of the cylinder response, the solution obtained with FSEI remains very close to that realized without FSEI.

Figure E.1: VIV at $Re = 100$: Effect of FSEI on the cylinder displacement.
E.2 Effect of FSEI on VIV of cylinder with a translational NES

The two effective suppression mechanisms discussed in §3.2 for the system with a translational NES were recomputed performing equilibrating iterations. The effect of FSEI on both suppression mechanisms is discussed in this section.

E.2.1 Suppression Mechanism I

Suppression Mechanism I was discussed in §3.2.1 for the system with translational NES parameters $\epsilon = 0.11$, $\lambda = 0.0283$, $\kappa = 1.59$, and $Re = 100$; this solution was recomputed with FSEI, and its effects are discussed in this section. The cylinder displacement shown in Figs. E.3a and E.3b indicate the marginal influence of FSEI on the cylinder displacement. The RMS value of the reduction in the cylinder displacement computed with equilibrating iterations is about 77%, which is very close to the case without the equilibrium iteration (78%). The NES relative motion time series are compared in Fig. E.4 and the corresponding lift coefficient time series are compared in Fig. E.5. These demonstrate that the effect of FSEI on the effectiveness of the NES for passive VIV suppression is marginal; however, the solution with FSEI shows significantly less modulation in the amplitude envelope.

E.2.2 Suppression Mechanism II

The system with translational NES parameters $\epsilon = 0.066$, $\lambda = 0.0217$, $\kappa = 0.704$, and $Re = 100$, exhibited suppression mechanism II as discussed in §3.2.2;
Figure E.3: Suppression Mechanism I: Effect of FSEI on the cylinder displacement.

Figure E.4: Suppression Mechanism I: Effect of FSEI on the NES relative displacement.

Figure E.5: Suppression Mechanism I: Effect of FSEI on the lift coefficient.
this solution was recomputed with equilibrating iterations. The cylinder displacement time series are compared in Figs. E.6a and E.6b. The cylinder displacement exhibits a different transient phase due to FSEI; however, the long-time solution approaches the same suppressed LCO as the solution without FSEI. The RMS value of the reduction in the cylinder displacement computed with equilibrating iterations is approximately 75%, which is in agreement with the solution without equilibrium iterations (74%). The NES relative motion time series are compared in Fig. E.7, and the lift coefficient time series shown in Fig. E.8 are comparable, showing only minor differences.

E.3 Effect of FSEI on VIV of cylinder with a rotational NES

The system with rotational NES parameters $\bar{r}_0 = 0.5$, $\epsilon_r = 0.3$, $\lambda_r = 0.0034$, and $Re = 100$ exhibited slowly decaying amplitude response associated with elongated attached vortices and intermittent chaotic motion, as discussed in
§4.3. This solution was recomputed with equilibrating iterations, and its effect on the results is discussed in this section. The cylinder displacement time series for solutions without and with equilibrating iteration are shown in Figs. E.9a and E.9b, respectively. Time series of cylinder displacement appear different. However, the characteristic nature of the time series is preserved for solutions with FSEI. The time series shown in Fig. E.9 clearly indicates both the chaotic motion and slow-decaying portions. Time duration of the chaotic motion is the major difference between the two solutions; the chaotic motion is significantly shorter in duration for the solution with equilibrating iterations. The slowly decaying envelopes also show minor differences in temporal evolution. Similar observations can be made for the NES angle and the lift coefficient by comparing the time series shown in Figs. E.10 and E.11, respectively.

The solution with FSEI continued for longer time duration is shown in Fig. E.12, where it can be seen that the characteristic features are preserved.
in the long-time solution. Comparing the time series shown in Fig. E.12 with that of the solution without FSEI shown in Fig. 4.6, it can be seen that the repetitive slowly decaying motion separated by chaotic transitions persists in the long-time solution with FSEI. It should be noted that the chaotic solution by nature is sensitive to both initial conditions and integration error; thus, the difference in the chaotic portion of the time series is expected in the solutions shown in Fig. 4.6 and E.12. Despite the apparent differences in the time series of the solution with equilibrating iterations, the solutions with and without equilibrating iterations are qualitatively similar.

Further, the effect of temporal discretization in addition to the equilibrating iterations is examined by computing the solution with a higher temporal resolution. The solution shown in Fig. E.13 is computed with a time step of size $0.015U_0/D$ along with equilibrating iterations at each time step. The original solution shown in Fig. 4.6 obtained without equilibrating iterations and the so-
Figure E.12: Slowly decaying and elongated attached vortices solution with FSEI: Long-time solution.
Figure E.13: Slowly decaying and elongated attached vortices solution with FSEI: Higher temporal resolution.
lution with equilibrating iterations shown in Fig E.12 were both obtained with a
time step size of $0.025U_0/D$. The qualitative similarity between solutions with
higher temporal resolution can be noted by comparing time series shown in Fig.
E.13 with either the solution shown in Fig. 4.6 or that shown in Fig. E.12.
REFERENCES


Christopher, O., 2011. Study of reduced order models for vortex-induced vibration and comparison with CFD results. Master’s thesis, University of Illinois Urbana-Champaign, USA.


