RIGIDITY IN FREE GROUPS

BY

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DISSERTATION

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Abstract

Culler-Vogtmann Outer Space (denoted cv_N) is the space of all free minimal discrete isometric actions of the free group of rank N (F_N) on $\mathbb{R}$-trees, T. We say a subset $\Sigma \subseteq F_N$ is *spectrally rigid* (resp. *strongly spectrally rigid*) if whenever $T_1, T_2 \in \mathrm{cv}_N$ (resp. the closure, $\overline{\mathrm{cv}}_N$, of cv_N) are $\mathbb{R}$-trees for which $\|\sigma\|_{T_1} = \|\sigma\|_{T_2}$ for every $\sigma \in \Sigma$, then $T_1 = T_2$ in cv_N (resp. $\overline{\mathrm{cv}}_N$). Similarly, given $T \in \mathrm{cv}_N$ (resp. $\overline{\mathrm{cv}}_N$), we say that $\Sigma \subset F_N$ is *relatively rigid* (resp. *strongly relatively rigid*) for T if whenever $T' \in \mathrm{cv}_N$ (resp. $\overline{\mathrm{cv}}_N$) is such that $\|\sigma\|_T = \|\sigma\|_{T'}$ for every $\sigma \in \Sigma$, then $T = T'$ in cv_N (resp. $\overline{\mathrm{cv}}_N$). We say that $S \subset \mathrm{Curr}(F_N)$ is a *rigid set of currents* if whenever $T_1, T_2 \in \mathrm{cv}_N$ are such that $\langle \sigma, T_1 \rangle = \langle \sigma, T_2 \rangle$ for every $\sigma \in \Sigma$, then $T_1 = T_2$ in cv_N.

The general theory of (non-abelian) actions of groups on $\mathbb{R}$-trees establishes that $T \in \mathrm{cv}_N$ is uniquely determined by its translation length function $\| \cdot \|_T \colon F_N \to \mathbb{R}$, and consequently that F_N itself is spectrally rigid. Results of Smillie and Vogtmann [43], and of Cohen, Lustig, and Steiner [12] establish that no finite Σ is spectrally rigid. Capitalizing on their constructions, we prove that for any $\Phi \in \mathrm{Aut}(F_N)$ and $g \in F_N$, the set $\Sigma = \{\Phi^n(g)\}_{n \in \mathbb{Z}}$ is not spectrally rigid. We also prove that if $\{H_i\}_{i=1}^k$ is a finite collection of subgroups, each of infinite index, and $g_i \in F_N$, then $\cup_{i=1}^k g_i H_i$ is not spectrally rigid in F_N. Taking $H_i = 1$, we recover the results about finite sets. We also prove that any coset of a nontrivial normal subgroup $H \lhd F_N$ is spectrally rigid.

In [10], Carette, Francaviglia, Kapovich, and Martino prove that every $T \in \mathrm{cv}_N$ admits a finite relatively rigid set. We prove that this set actually affords strong relative rigidity. We also prove nonexistence of finite strongly relatively rigid sets for certain classes of trees on the boundary, $\partial\,\mathrm{cv}_N$, of cv_N (see Chapter 6). We also show that in volume-normalized outer space certain simplicial trees 'almost' admit finite strongly relatively rigid sets (see Chapter 7). Lastly, we prove that arational free trees admit finite strongly relatively rigid sets of currents.

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List of Abbreviations

$\mathrm{Aut}(G)$	The automorphism group of a group, G.
$\mathrm{Out}(G)$	The group of outer automorphisms of a group, G.
$\mathrm{Inn}(G)$	The group of inner automorphisms of a group, G.
$\mathrm{diam}(X)$	The diameter of a metric space, X.
$\mathrm{core}(G)$	The core of a graph, G.
CV_N	Projectivized Culler-Vogtmann Outer Space of rank N.
cv_N	Unprojectivized Culler-Vogtmann Outer Space of rank N.
$\mathrm{st}_G(v)$	The star of a vertex, v, in a graph, G.
$\mathrm{val}_G(v)$	The valence of a vertex, v, in a graph, G.
$\mathrm{PF}(M)$	The Perron-Frobenius eigenvalue of a suitable matrix, M.
$\mathrm{rk}(G)$	The rank of a group, G.
$\mathrm{ncl}_G(H)$	The normal closure of a subgroup, H, in a group, G.
$\mathrm{vol}(G)$	The volume of a graph, G.
$\mathrm{covol}(T)$	The covolume of a tree, T equipped with a group action.
$\mathrm{BCC}(f)$	The bounded cancellation constant of a map, f.
$\mathrm{BBT}(f)$	The bounded back tracking constant of a map, f.
$\mathrm{Cay}(G, S)$	The Cayley Graph of a group, G, with generating set, S.
$\mathrm{Isom}(X)$	The isometry group of a metric space, X.
$\mathrm{Cyl}(v)$	The cylinder set of a word, v.
$\mathrm{Curr}(F_N)$	The space of geodesic currents on F_N.
$\mathrm{Lip}(f)$	The Lipschitz constant of a map, f.
$\mathrm{BST}(\mathcal{G}, Y)$	The Bass-Serre Tree of a graph of groups, $(\mathcal{G}, Y)$.
$\mathrm{Fix}_T(G)$	The fixed point set, in T, under the action of a group, G.
$\mathrm{supp}(\mu)$	The support of a current, μ.
$\mathrm{Stab}_G(x)$	The stabilizer subgroup, in G, of x.

Chapter 1

Introduction

The notion of a "marked length spectrum" initially arose in the context of Riemannian and metric geometry, specifically in the setting of a group G acting by isometries on a metric space X with some negative curvature properties. We view such an action as a homomorphism $\rho\colon G \to \mathrm{Isom}(X)$. To each nontrivial $g \in G$, we associate the minimal displacement of the isometry $\rho(g)\colon X \to X$, thus obtaining a function $\|\cdot\|_\rho\colon G \to \mathbb{R}_{\geq 0}$ defined by $\|g\|_\rho := \inf\{d(x, gx), x \in X\}$. This situation often appears in the context of Riemannian geometry. If M is a closed connected Riemannian manifold of (not necessarily constant) negative curvature, then the Riemannian metric ρ determines a homomorphism $\rho\colon \pi_1(M) \to \mathrm{Isom}(\widetilde{M})$, where $\widetilde{M}$ is the universal cover of M. In this situation, the minimal displacement function $\|\cdot\|_\rho\colon \pi_1(M) \to \mathbb{R}_{\geq 0}$ is called the *marked length spectrum* of ρ. The extent to which the marked length spectrum of ρ determines the isometry type of (M, ρ) is at the center of much research (see, for example, the introduction in [26]).

The classic *marked length spectrum rigidity conjecture* asserts that knowing the marked length spectrum of ρ—where ρ is a smooth Riemannian metric of negative curvature on a closed manifold, M—does in fact determine the isometry type of (M, ρ). In settings where the marked length spectrum rigidity conjecture holds, one can ask whether or not the isometry type of (M, ρ) is determined by the restriction $\|\cdot\|_\rho\colon H \to \mathbb{R}_{\geq 0}$ where H is a *proper* subset of $\pi_1(M)$. Naturally, we say such an H is a "spectrally rigid" subset of $\pi_1(M)$.

In this thesis, we consider the setting where $G = F_N$, a finitely generated free group, and X is an $\mathbb{R}$-tree. More specifically, we work in *Culler-Vogtmann outer space*, denoted cv_N, which is the space of all free minimal discrete isometric actions of F_N on $\mathbb{R}$-trees, up to F_N-equivariant isometry. The space was introduced by Culler and Vogtmann [20] as a free group analog of Teichmuller Space. While the latter admits an action by the Teichmuller Modular Group, Outer Space admits an action by $\mathrm{Out}(F_N)$, the outer automorphism group of F_N. The space cv_N has proven to be a key tool in studying $\mathrm{Out}(F_N)$.

Every tree $T \in \mathrm{cv}_N$ can be described by means of its quotient metric graph $G = T/F_N$ (with marking isomorphism $F_N \cong \pi_1(G)$), where we record the lengths of the edges in G so that T is identified with its universal cover $\widetilde{G}$. For each $T \in \mathrm{cv}_N$, we have a *translation length function* $\|\cdot\|_T\colon F_N \to \mathbb{R}_{\geq 0}$. As with the marked length spectrum associated to Riemannian manifolds, the translation length function is a class function, and $\|g\|_T$ is equal to the translation length of the isometry $g\colon T \to T$.

Alternatively one may think of a point in the space as a triple (Γ, τ, l) where Γ is a finite graph equipped with both a marking isomorphism, $\tau\colon F_N \to \pi_1(\Gamma)$, and a function $l\colon E(\Gamma) \to \mathbb{R}_{>0}$ which assigns a length to each edge. Given such a triple, one sets $T = \widetilde{\Gamma}$ so that T is equipped with a covering space action of $F_N \cong \pi_1(\Gamma)$ which is free, minimal, discrete, and isometric. One obtains (Γ, τ, l) from T by considering the quotient graph $\Gamma = F_N \backslash T$. To each T we associate a length function $\|\cdot\|_T\colon F_N \to \mathbb{R}_{\geq 0}$ defined by $\|g\|_T = \inf_{x\in T}\{d(x, gx)\}$. Equivalently, $\|g\|_T$ is the length of the cyclically reduced loop (in Γ) which represents the conjugacy class of $\tau(g)$. The closure, $\overline{\mathrm{cv}}_N$, of cv_N is known to consist of the so called very small actions of F_N on $\mathbb{R}$-trees [2, 13]. The fact that $\|\cdot\|_T$ determines T [11] gives an injection $\overline{\ell}\colon \overline{\mathrm{cv}}_N \to (\mathbb{R}_{\geq 0})^{F_N}$; we call $\overline{\ell}(T)$ the *marked length spectrum* of T. Let $\ell = \overline{\ell}|_{\mathrm{cv}_N}$ and note that ℓ is also injective. Morally speaking, a subset $\Sigma \subset F_N$ is *spectrally rigid* (resp. *strongly spectrally rigid*) if one can replace the target of ℓ (resp. $\overline{\ell}$) by $(\mathbb{R}_{\geq 0})^{\Sigma}$ and still retain injectivity. The term was recently coined (in the free group setting) by Kapovich [30], though similarly spirited work dates back to the early 1990's (see [12, 43]).

In this thesis, we examine this notion of rigidity in several contexts. We first show, in Chapter 3, that a certain family of subsets $\Sigma \subset F_N$ are never spectrally rigid.

Theorem A. *Let $N \geq 2$. Let $\Phi \in \mathrm{Aut}(F_N)$. Let $g \in F_N$ be arbitrary. Then the set $\Sigma = \{\Phi^n(g)\}_{n\in\mathbb{Z}}$ is not spectrally rigid in F_N.*

For more detailed background information, as well as the motivation for this problem, see the beginning of Chapter 3.

In Chapter 4, we consider classical spectral rigidity from a slightly different framework. In particular, we investigate the extent to which the group theoretic structure of a given subset $\Sigma \subset F_N$ affects whether or not Σ is spectrally rigid. To that end we prove the following two theorems.

Theorem B. *Let $N \geq 2$. Let $\{H_i\}_{i=1}^k$ be a finite collection of finitely generated subgroups $H_i < F_N$. Let $\{g_i\}_{i=1}^k$ be a finite collection of elements $g_i \in F_N$. Let $\mathcal{H} = \cup_{i=1}^k g_i H_i$. Then the following are equivalent.*

1. *For every i, $[F_N\colon H_i] = \infty$.*

2. *$\mathcal{H} \subset F_N$ is not spectrally rigid in F_N.*

Theorem C. *Let $H \lhd F_N$ be a nontrivial normal subgroup. Then for any $g \in F_N$, the coset gH is strongly spectrally rigid in F_N.*

In Chapter 5 we capitalize on a result of Carette, Francaviglia, Kapovich, and Martino who prove, in [10], that given any $T \in \mathrm{cv}_N$ there exists a finite set of 'almost simple curves' that afford relative rigidity for T. We show that their result can be extended to the closure, $\overline{\mathrm{cv}}_N$, of outer space. Specifically, we prove the following.

Theorem D. *Let $T \in \mathrm{cv}_N$ be arbitrary. There exists a finite set $\mathcal{C}(T)$ (depending on T) of primitive elements of F_N with the following property: whenever $T' \in \overline{\mathrm{cv}}_N$ is such that $\|g\|_{T'} = \|g\|_T$ for every $g \in \mathcal{C}(T)$, then $T = T'$ in $\overline{\mathrm{cv}}_N$. In other words, there exists a finite strongly relatively rigid set for T.*

In Chapter 6 we continue our study of relative rigidity; however, we begin now with a tree in the boundary, $\partial\,\mathrm{cv}_N$, of outer space (note Chapter 5 proves finite strong relative rigidity for all trees in the interior of outer space). We first prove that if a simplicial tree, $T \in \partial\,\mathrm{cv}_N$ admits a graph of groups decomposition where there is a trivial edge group incident to a vertex group of rank at least two, then there does not exist a finite strongly relatively rigid set for T. Formally, we have the following.

Theorem E. *Let $(\mathcal{G}, Y)$ be a graph of groups such that $T_{\mathcal{G}} = \mathrm{BST}(\mathcal{G}, Y) \in \partial\,\mathrm{cv}_N$. Suppose there is a vertex group G_v with $\mathrm{rk}\, G_v \geq 2$ and an edge e with $o(e) = v$, $G_e = \{1\}$ and $G_{t(e)} \neq \{1\}$. Let $\Sigma \subset \pi_1(\mathcal{G}, Y) \cong F_N$ be a finite set. Then there exists a graph of groups $(\mathcal{H}, Y)$ (with $T_{\mathcal{H}} = \mathrm{BST}(\mathcal{H}, Y) \in \partial\,\mathrm{cv}_N$) so that for all $\sigma \in \Sigma$, we have*

$$\|\sigma\|_{T_{\mathcal{G}}} = \|\sigma\|_{T_{\mathcal{H}}}$$

yet $T_{\mathcal{G}} \neq T_{\mathcal{H}}$ in $\overline{\mathrm{cv}}_N$.

We also explain how a result of Guirardel [24] further enlarges our class of simplicial trees which do not admit finite strongly relatively rigid sets. Additionally, we show how previous results of Levitt and Paulin [36] can be used to show that any 'non-geometric' tree can not admit a finite strongly relatively rigid set. More specifically, say an F_N-tree is geometric [36] if it is dual to a measured foliation on a finite 2-complex with fundamental group F_N. A tree is non-geometric if it is not geometric. It is known, for example, that stable trees of certain iwip automorphisms are not geometric. We also explain how a result of Guirardel [24] further enlarges our class of simplicial trees which do not admit finite strongly relatively rigid sets. Lastly we use Skora's Duality Theorem [42] to prove that any 'relatively elliptic' $T \in \overline{\mathrm{cv}}_N$ admits a finite strongly relatively rigid set. To summarize, we have the following.

Theorem F. *Let $T \in \overline{\mathrm{cv}}_N$.*

1. *If $T \in \mathrm{cv}_N$, then there exists a finite strongly relatively rigid set $\Sigma \subset F_N$ for T.*

2. *If T is non-geometric then there does not exist a finite locally strongly relatively rigid set $\Sigma \subset F_N$ for T. In particular, there does not exist a finite strongly relatively rigid set for such T.*

3. *If T is simplicial with a sub-graph of groups containing $L *_{\{1\}} A$ or $L*_{\{1\}}$ where $\operatorname{rk} L \geq 2$ and $\operatorname{rk} A \geq 1$, then there does not exist a finite locally strongly relatively rigid set $\Sigma \subset F_N$ for T.*

4. *If $T \in \overline{\mathrm{cv}}_N$ is simplicial with vertex group $G_v \neq \{1\}$ and two edge groups $G_e, G_{e'} \cong \{1\}$ with edge origins $o(e) = o(e') = v$, then there does not exist a finite locally strongly relatively rigid set $\Sigma \subset F_N$ for T.*

5. *If T is relatively S-elliptic for some connected compact surface, S, with at least one boundary component, then there exists a finite strongly relatively rigid set for T.*

In Chapter 7 we consider a similar setting as in Chapter 6; however, we work in volume-normalized outer space. Here we show that if T has a graph of groups decomposition with all vertex groups infinite cyclic, all edge groups trivial, and underlying graph a tree, then there is a finite set Σ so that if some other simplicial T' agrees with T on Σ, then $T = T'$. That is,

Theorem G. *Let T be a very small F_N metric graph of groups with all vertex groups infinite cyclic, all edge groups trivial, and T/F_N a tree. Suppose $(\mathcal{G}, Y)$ is a very small F_N metric graph of groups. Suppose T agrees with $\mathrm{BST}(\mathcal{G}, Y)$ on $\mathcal{T}_{\mathcal{A}}$ and $\operatorname{covol} T = \operatorname{vol} Y$. Then $\mathrm{BST}(\mathcal{G}, Y) = T$.*

Interestingly enough, if one works in the usual unprojectivized outer space, the results from Chapter 6 show that for T as above, no finite set can distinguish T from all other simplicial trees.

Finally, in Chapter 8 we consider a notion of rigidity that involves *geodesic currrents*. These are measures on $\partial F_N \times \partial F_N \backslash \Delta$ (where Δ is the diagonal) which are both F_N invariant and invariant under the map that interchanges the coordinates. There is a canonical way to associate, for each class $g \in F_N$, a geodesic current η_g. Additionally, there is an *intersection form* [32] $\langle \cdot, \cdot \rangle \colon \overline{\mathrm{cv}}_N \times \mathrm{Curr}(F_N) \to \mathbb{R}$ for which $\langle T, \eta_g \rangle = \|g\|_T$. Naturally, we say that a set, S, of geodesic currents is *rigid* if whenever T, T' are such that $\langle T, \mu \rangle = \langle T', \mu \rangle$ for all $\mu \in S$, it follows that $T = T'$ in cv_N. Our result is as follows.

Theorem H. *Let $T \in \partial \mathrm{cv}_N$ be arational and free. Then there exists a finite strongly relatively rigid set of currents for T.*

Chapter 2

Preliminaries

2.1 Graphs, Fundamental Groups, and Automorphisms

As for the general objects of geometric group theory, we will follow the expositions in [7] and [23]. By a *graph*, we mean a five-tuple $G = (V(G), E(G), o, t, -)$, where $V(G)$ is a nonempty set of vertices, $E(G)$ is a set of edges, and $o, t\colon E(G) \to V(G)$, $-\colon E(G) \to E(G)$ are functions whose image of an edge is its origin, terminus, and inverse, respectively. Furthermore, we require $\overline{\overline{e}} = e, \overline{e} \neq e$, and $o(e) = t(\overline{e})$ for every $e \in E(G)$. The N*-rose*, denoted R_N, is the graph with $V(R_N) = \{*\}$ and $\#(E(R_N)) = 2N$. A *subgraph* H of a graph G is a graph for which $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and the maps $o, t, -$ are restrictions to $E(H)$. The *star* of a vertex, denoted $\mathrm{st}_G\, v$ is defined as $\mathrm{st}_G\, v := \{e \in E(G) : o(e) = v\}$; the *valency* of v, denoted $\mathrm{val}_G\, v$, is the cardinality of its star. An *orientation* of G is a decomposition $E(G) = E(G)^+ \sqcup E(G)^-$ where for each pair $\{e, \overline{e}\}$ of mutually inverse edges, exactly one edge belongs to $E(G)^+$.

By a *path*, we mean either a vertex $v \in V(G)$ (in which case p is a *degenerate* path), or a sequence $p = e_0 e_1 \cdots e_{n-1}$ of edges with $n \geq 1$ for which $t(e_i) = o(e_{i+1})$ for $0 \leq i \leq n-2$. The *inverse* of p, denoted $\overline{p}$, is defined to be either the vertex v or the path $\overline{p} = \overline{e_{n-1}} \cdots \overline{e_1}\ \overline{e_0}$, depending on whether or not p is degenerate. By a *subpath*, p' of the path $p = e_0 e_1 \cdots e_{n-1}$, we mean an path $p' = e_i e_{i+1} \cdots e_{i+m}$, where $i \in \{0, 1, \ldots, n-1\}$ and $m \leq n-1-i$; we write $p' \Subset p$. The *length* of the path $p = e_0 e_1 \cdots e_{n-1}$, denoted $|p|$, is n; a degenerate path has length zero. The set of all paths in G is denoted by $P(G)$. For $p \in P(G)$ we define $o(p) := o(e_0)$, $t(p) := t(e_{n-1})$. A *closed* path is one for which $o(p) = t(p)$; in this case, we say $o(p) = t(p)$ is the *basepoint* of the path p. By a *cyclic path*, we mean the equivalence class (under cyclic permutation) of a given closed path. When dealing with cyclic paths, we modify the definition of subpath accordingly. For $v \in V(G)$, let $P(G, v) := \{p \in P(G) : o(p) = t(p) = v\}$. An path is *reduced* if it is *degenerate* or $\overline{e_i} \neq e_{i+1}$ for $0 \leq i \leq n-2$. A closed path is *cyclically reduced* if it is reduced, and $\overline{e_0} \neq e_{n-1}$. The process by which one removes subpaths of the form $\overline{e}e$ is called *(cyclic) reduction.* A *forest* is a graph without nontrivial cyclically reduced paths; a connected forest is a *tree.* In case $T \subseteq G$ is a tree with $v, v' \in V(T)$, we denote by $[v, v']_T$ the unique path $p \subseteq T$ with $o(p) = v$ and $t(p) = v'$.

We say two paths p_1, p_2 are *homotopic (relative endpoints)* if there is a path p which can be obtained from both p_1 and p_2 by reduction. Consequently, two such paths satisfy $o(p_1) = o(p_2)$, $t(p_1) = t(p_2)$. Similarly, we say two cyclic paths p_1, p_2 are *freely homotopic* if there is a cyclic path p which can be obtained from both p_1 and p_2 by cyclic reduction. For a path p, we denote by $[p]_h$ the *homotopy class* of the path p; that is, the set of all paths homotopic to p (relative endpoints). By $[p]$ we mean the reduced form of p. In the case that p is a cyclic path, we denote its *free homotopy class* by $[[p]]_h$, and by $[[p]]$ the equivalence class containing the cyclically reduced form of p. We extend our notation and terminology to elements in F_N as follows. Given a (not necessarily reduced) word $w \in F_N$ expressed over some free basis $\mathcal{A}$ of F_N, consider now the path w in the N-rose, R_N, where each edge $e \in E(R_N)^+$ is labelled by a basis element $a \in \mathcal{A}$. Additionally, given a free basis $\mathcal{A}$ of F_N, and a (not necessarily reduced) word w, we write $[w]_{\mathcal{A}}$ for the reduced form of w where w is expressed as a word over $\mathcal{A}$, and we write $[[w]]_{\mathcal{A}}$ for the equivalence class containing the cyclically reduced form of w over $\mathcal{A}$.

The *fundamental group* of G with respect to the vertex v, $\pi_1(G, v) := \{[p]_h : p \in P(G, v)\}$. By a *graph map*, we mean a function $f\colon V(G_1) \cup E(G_1) \to V(G_2) \cup P(G_2)$ which sends $V(G_1)$ to $V(G_2)$, $E(G_1)$ to $P(G_2)$, and for which

$$f(o(e)) = o(f(e)) \quad \text{and} \quad f(\overline{e}) = \overline{f(e)}$$

for every $e \in E(G_1)$. We write $f\colon G_1 \to G_2$; if $v_1 \in V(G_1)$ with $f(v_1) = v_2 \in V(G_2)$, we sometimes write $f\colon (G_1, v_1) \to (G_2, v_2)$ for emphasis. We extend graph maps to paths $p \in P(G_1)$ as follows:

$$f(p) = f(e_0 e_1 \cdots e_{n-1}) := f(e_0) f(e_1) \cdots f(e_{n-1})$$

If for each $e \in E(G_1)$, we have that $f(e)$ is a nondegenerate reduced path, then we say that the map f is *tight*. For a graph map $f\colon (G_1, v_1) \to (G_2, v_2)$ with G_1, G_2 connected, there is an induced homomorphism $f_\#\colon \pi_1(G_1, v_1) \to \pi_1(G_2, v_2)$ given by the rule $[p]_h \mapsto [f(p)]_h$ for $p \in P(G_1, v_1)$. In the case that $f_\#$ is an isomorphism, we say f is a *homotopy equivalence*, and that the isomorphism $f_\#$ is *realized* by the homotopy equivalence f.

Given a graph map $f\colon (G, v) \to (G, f(v))$, there is a family of induced homomorphisms $f^u_\#\colon \pi_1(G, v) \to \pi_1(G, v)$ given by $[p]_h \mapsto [uf(p)\overline{u}]_h$ where u is any path with $o(u) = v$, $t(u) = f(v)$. If $f^u_\# \in \mathrm{Aut}(\pi_1(G_1, v))$ for some u, then $f^u_\# \in \mathrm{Aut}(\pi_1(G_1, v))$ for all u. Furthermore, $f^u_\#$ and $f^{u'}_\#$ differ only by conjugation and thus represent the same outer automorphism of $\pi_1(G_1, v)$; we denote this outer automorphism by $f_\circledast$.

By a *marked graph* we mean a pair (G, τ), where G is a graph and $\tau\colon R_N \to G$ is a homotopy equivalence. Choose a homotopy equivalence $\sigma\colon G \to R_N$ so that $(\tau \circ \sigma)_\# \in \mathrm{Inn}(\pi_1(G, v))$ (for some choice of v)

and $(\sigma \circ \tau)_\# \in \mathrm{Inn}(\pi_1(R_N)) = \mathrm{Inn}(F_N)$. To a given homotopy equivalence $f\colon G \to G$, we associate $(\sigma \circ f \circ \tau)_\circledast \in \mathrm{Out}(F_N)$, the outer automorphism *determined* by f. Given $\varphi \in \mathrm{Out}(F_N)$, we say that a homotopy equivalence $f\colon G \to G$ is a *topological representative* of φ with respect to τ if f determines φ and f is tight. For example, take $\Phi \in \mathrm{Aut}(F_N)$, and let $F_N = F(x_0, x_1, \ldots, x_{N-1}), E(R_N)^+ = \{e_0, e_1, \ldots, e_{N-1}\}$. Write $\Phi(x_i) = w_i(x_0, x_1, \ldots, x_{N-1})$. Let $\tau, \sigma\colon R_N \to R_N$ be the identity graph maps, and $f\colon R_N \to R_N$ be $f(e_i) = p_i(e_0, e_1, \ldots, e_{N-1})$ where p_i is obtained from w_i by replacing x_j by e_j. Evidently f is a topological representative of φ with respect to the identity marking.

2.2 Train Tracks

We follow [7], [5], and [4] for our exposition regarding relative train tracks. A *turn* in a connected graph G is an unordered pair (say e, e') of (not necessarily distinct) edges in $E(G)$, for which $o(e) = o(e')$. A turn is *degenerate* if $e = e'$ and *non-degenerate* otherwise. The set of all turns in G is denoted by $T(G)$. Given a tight graph map $f\colon G \to G$, we define a function $Df\colon E(G) \to E(G)$ by $Df(e) := e^*$ where e^* is the initial edge in the path $f(e)$. Define a function $Tf\colon T(G) \to T(G)$ by $Tf(e, e') := (Df(e), Df(e'))$. A *legal* turn is one for which $(Tf)^n(e, e')$ are non-degenerate for all $n \geq 0$; a turn is *illegal* if it is not legal. We say that an path $p = e_0 e_1 \cdots e_{n-1}$ is *legal* if the turns $\overline{e_i}, e_{i+1}$ are legal for $0 \leq i \leq n-2$. These notions culminate in the following definition.

Definition 2.2.1 (Train track map)**.** A graph map $f\colon G \to G$ is called a *train track map* if for every $e \in E(G)$, $f(e)$ is nondegenerate and legal.

Remark 2.2.2. For every $e \in E(G)$, a train track map satisfies $[f^n(e)] = f^n(e)$, for all $n \geq 1$.

Let $f\colon G \to G$ be a graph map. Choose an orientation for G, and write $E(G)^+ = \{e_0, e_1, \ldots, e_{k-1}\}$. We define a $k \times k$ matrix called the *transition matrix* of f, denoted $M(f)$, as follows: $(M(f))_{ij}$ equals the number of occurrences of e_i or $\overline{e_i}$ in $f(e_j)$. We say that $\varphi \in \mathrm{Out}(F_N)$ is *reducible* if there is a free factorization $F_N = F^0 * \cdots * F^{l-1} * H$, with $l \geq 1$ and $1 \leq \mathrm{rk}(F^0) < N$, so that φ permutes the conjugacy classes of the F^i; we allow $H = 1$. Otherwise, we say φ is *irreducible.* There is an alternate definition given in terms of irreducibility of transition matrices (see [7] or [5]). We say that $\varphi \in \mathrm{Out}(F_N)$ is *fully irreducible* (or *irreducible with irreducible powers*, *iwip* for short) if φ^n is irreducible for every $n \geq 1$. Thus $\varphi \in \mathrm{Out}(F_N)$ is not an iwip if and only if there is $n \geq 1$ so that φ^n leaves invariant the conjugacy class of a proper free factor of F_N. A result of Bestvina and Handel [5] states that every irreducible $\varphi \in \mathrm{Out}(F_N)$ is topologically represented by a train track map. Furthermore, every automorphism can be topologically represented by a improved relative train track map, a notion we recall below.

Let $f\colon G \to G$ be a topological representative for $\varphi \in \text{Out}(F_N)$. A *filtration* for f is a series of f-invariant subgraphs: $\varnothing = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_m = G$. The ith-*stratum*, denoted H_i, is defined by $H_i := G_i \backslash G_{i-1}$, whereby (for a graph G and a subgraph H) we mean the following:

$$E(G\backslash H) := E(G)\backslash E(H),$$

$$V(G\backslash H) := \{v \in V(G) : \exists e \in E(G\backslash H) \text{ for which } o(e) = v\}.$$

By abuse of notation, if $S \subset E(G)$, we define graphs $G\backslash S$ and $G \cup S$ in the obvious manner. A turn for which one edge is in H_i and the other in G_{i-1} is called *mixed*. A path $p \subseteq G_i$ is called i-*legal* if its only illegal turns lie in G_{i-1}. By only considering edges from $E(H_i)$ we obtain the *transition submatrix* for H_i, denoted $M_i(f)$. If $M_i(f)$ is the zero matrix, we say that H_i is a *zero* stratum. Recall ([7, Appendix]) that a non-negative irreducible integer matrix has an associated Perron-Frobenius eigenvalue $\lambda \geq 1$. If $\text{PF}(M_i(f)) > 1$ (resp. $= 1$) we say that H_i is *exponentially-growing* (resp. *non-exponentially-growing*). We follow [4] for our definition of a relative train track.

Definition 2.2.3 (Relative train track)**.** A topological representative $f\colon G \to G$ of $\varphi \in \text{Out}(F_N)$ is a *relative train track* with respect to the filtration $\varnothing = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_m = G$ if the following criteria hold.

1. G has no valence one vertices.

2. Each non-zero $M_i(f)$ is irreducible.

3. Each exponentially growing H_i satisfies:

   (a) If $e \in E(H_i)$, then $Df(e) \in E(H_i)$.

   (b) If p is a nontrivial path in G_{i-1} for which $o(p) \in V(H_i)$ and $t(p) \in V(H_i)$, then $[f(p)]$ is nontrivial.

   (c) If p is a legal path in H_i, then $f(p)$ is an i-legal path in G_i.

It follows immediately from Definition 2.2.3 that mixed turns are legal. Furthermore, we note the following property of relative train tracks.

Lemma 2.2.4 (Bestvina, Feighn, and Handel [4, Lemma 2.5.2])**.** *Suppose that $f\colon G \to G$ is a relative train track map, that H_r is an exponentially growing stratum and that $\sigma = a_1 b_1 a_2 \cdots b_l$ is a decomposition of an r-legal path into subpaths where each $a_i \subseteq H_r$ and each $b_j \subseteq G_{r-1}$. (Allow the possibility that a_1 or b_l is trivial, but assume that the other subpaths are nontrivial.) Then*

$$[f(\sigma)] = f(a_1)[f(b_1)]f(a_2)\cdots[f(b_l)]$$

and $[f(\sigma)]$ *is* r*-legal.* □

Furthermore, if we allow ourselves to work with an iterate of φ, we can arrange for additional properties afforded by an *improved relative train track* (see [4]). However, for our purposes we require relatively few of these properties. Accordingly, we only state the properties of interest.

Theorem 2.2.5 (Bestvina, Feighn, and Handel [4, Theorem 5.1.5])**.** *For every outer automorphism* φ *there is a relative train track map* $f\colon G \to G$ *representing a positive iterate of* φ *with the following properties.*

1. *For every exponentially growing* H_i*, we have* $M_i(f)$ *is aperiodic; that is, there exists* $k \geq 1$ *for which all entries in* $M_i^k(f)$ *are positive.*
2. *If* H_i *is a zero stratum, then* H_i *is not a top stratum (i.e.* $G_i \neq G$*).*
3. *If* H_i *is a non-exponentially-growing stratum, then:*
   - *(a)* $E(H_i) = e_0$*, a single edge.*
   - *(b)* $f(e_0) = e_0 u$ *where* u *is a closed path in* G_{i-1} *whose basepoint* $t(e_0) = o(u) = t(u)$ *is fixed by* f*.* □

Definition 2.2.6 (Good relative train track)**.** A relative train track satisfying the conclusion of Theorem 2.2.5 will be called a *good relative train track*.

Remark 2.2.7. If $f\colon G \to G$ is a good relative train track representing φ^n for $n \geq 1$, then $f^k\colon G \to G$ is a good relative train track representing φ^{nk}.

By an *EG-automorphism* we mean an automorphism φ for which there exists $n \geq 1$ so that φ^n can be equipped with a good relative train track with exponentially-growing top stratum. We similarly define a *NEG-automorphism*. Note that iwip automorphisms are EG-automorphisms, but the converse does not hold.

2.3 Culler-Vogtmann Outer Space

There are multiple equivalent descriptions of Culler-Vogtmann outer space. Our subsequent definition of spectral rigidity treats outer space as a space of actions on trees. The expositions in [43] and [12], however, lend themselves nicely to the description of outer space in terms of marked metric graphs. We will briefly describe both notions, explaining how one can move easily between the two. We will follow [45] and [26].

Culler-Vogtmann outer space, denoted by cv_N, is the space of free minimal discrete isometric actions of F_N on $\mathbb{R}$-trees, up to equivalence, where $T_1 \sim T_2$ if there exists an F_N-equivariant isometry $f\colon T_1 \to T_2$.

By *projectivized Culler-Vogtmann outer space*, we mean the subset $\mathrm{CV}_N \subseteq \mathrm{cv}_N$ consisting of those trees T for which the quotient metric graph T/F_N has volume 1. For each $g \in F_N$ and $T \in \mathrm{cv}_N$, we define *translation length*, denoted $\|g\|_T$, by $\|g\|_T := \inf\{d(x, gx) \colon x \in T\}$. This gives a *translation length function* $\|\cdot\|_T \colon F_N \to \mathbb{R}_{\geq 0}$. We remark that $T \in \mathrm{cv}_N$ is uniquely determined (up to F_N-equivariant isometry) by $\|\cdot\|_T \colon F_N \to \mathbb{R}_{\geq 0}$ (see [11, §3, Theorem 4.1]).

Alternatively, a point in cv_N is the equivalence class of a triple (G, τ, l), where

1. G is a marked graph with respect to the marking $\tau \colon R_N \to G$.

2. For every $v \in V(G)$, we have $\mathrm{val}_G\, v \geq 3$.

3. Each $e \in E(G)$ is assigned a positive real number, its *length*, denoted by $l(e)$. We require $l(e) = l(\bar{e})$.

We refer to such a triple as a *marked metric graph structure*. Condition 3 allows us to treat G as a metric space via the path metric. The equivalence relation is as follows: $(G, \tau, l) \sim (G', \tau', l')$ if there is an isometry $h \colon G \to G'$ for which the following holds: there exists an path u with $o(u) = (h \circ \tau)(*)$ and $t(u) = \tau'(*)$ for which $(h \circ \tau)_\# = c_u \circ \tau'_\#$. Here $c_u \colon \pi_1(G', \tau'(*)) \to \pi_1(G', (h \circ \tau)(*))$ is $[p]_h \mapsto [up\bar{u}]_h$.

Given such a (G, τ, l), the marking gives an isomorphism $F_N \cong \pi_1(G)$ and thus induces an action of F_N on $\widetilde{G}$ by deck transformation. This action is free, minimal, discrete, and by isometries, the metric on $\widetilde{G}$ being that lifted from G. Furthermore, $\|g\|_T$ is the length of the shortest loop in the free homotopy class determined by g in the quotient metric graph T/F_N.

2.4 Bounded Cancellation

Suppose $f \colon G \to G$ and $f' \colon G' \to G'$ are relative train tracks for φ and φ^{-1}, respectively. In Section 3.4 we construct a primitive $a \in F_N$, so that the realizations $[[\tau(a)]]$ and $[[\tau'(a)]]$ in G and G', respectively, satisfy certain properties. Verification of these properties relies on our ability to control cancellation in a manner we make precise in Proposition 3.4.6. A result of Cooper [14] ensures this is possible. We state a version best fit for our purposes.

Lemma 2.4.1 (Bestvina, Feighn, and Handel [4, Lemma 2.3.1])**.** *For any homotopy equivalence $f \colon G \to G'$ of marked graphs there is a constant $\mathrm{BCC}(f) \geq 0$ for which the following holds. If $p = \alpha\beta$ is a path in G, then $[f(p)]$ is obtained from $[f(\alpha)]$ and $[f(\beta)]$ by concatenating and by cancelling $c \leq \mathrm{BCC}(f)$ edges from the terminal end of $[f(\alpha)]$ with c edges from the initial end of $[f(\beta)]$.* □

2.5 Spectral Rigidity

We follow the exposition in [26].

Definition 2.5.1 (Spectrally rigid)**.** We say $\Sigma \subseteq F_N$ is *spectrally rigid* if whenever $T_1, T_2 \in \mathrm{cv}_N$ are such that $\|g\|_{T_1} = \|g\|_{T_2}$ for every $g \in \Sigma$, then $T_1 = T_2$ in cv_N.

In [43], Smillie and Vogtmann proved for $N \geq 3$ that no finite subset of F_N is spectrally rigid. It was remarked by Kapovich [26] that their arguments apply to a more general $\Sigma \subseteq F_N$, not necessarily finite. In particular, if $\Sigma \subseteq F_N$ satisfies a"weak aperiodicity" property, then Σ is not spectrally rigid. We will refer to this as property $\mathcal{W}$ (see Definition 2.5.2 and Proposition 2.5.3). The argument in [43] involved constructing a particular free basis $\mathcal{A}$ of F_N, and a particular marked metric graph G with $\pi_1(G) \cong F_N$, so that the realization (in G) of each $\sigma \in \Sigma$ satisfy certain properties. These properties allow us to perturb the length data of G in a way that does not change the translation lengths of elements in Σ, yet does produce different trees in outer space. We remark that these marked metric graphs have volume 1, and so represent points in CV_N.

Definition 2.5.2 (Property $\mathcal{W}$)**.** Let $\Sigma \subseteq F_N$. We say Σ has *property* $\mathcal{W}$ if there exist a free basis $\mathcal{A}$ of F_N, $a \in \mathcal{A}$, and $M \geq 1$ so that for any $\sigma \in \Sigma$, if $a^k \Subset [[\sigma]]_{\mathcal{A}}$ then $|k| \leq M$.

Recall that $p' \Subset p$ indicates that p' is a subpath of the path p, and that $[[\sigma]]_{\mathcal{A}}$ is the equivalence class containing the cyclically reduced (over the basis $\mathcal{A}$) form of σ. In view of the argument in [43], the following is immediate.

Proposition 2.5.3. *Suppose $N \geq 3$, and $\Sigma \subseteq F_N$ has property $\mathcal{W}$. Then Σ is not spectrally rigid.* □

Cohen, Lustig, and Steiner [12] showed that no finite subset of F_2 is spectrally rigid. Broadly speaking, their argument was similar to that in [43]; however, it was necessary to perturb the volumes of the quotient graphs. Definition 2.5.1 allows for such a modification since $(G, \tau, l) \neq (G, \tau, \lambda l)$ in cv_N for $\lambda \neq 1$. Our investigation of the argument in [12] reveals that property $\mathcal{W}$ does indeed suffice for non spectral rigidity in the case $N = 2$. To make this connection clear, we introduce an intermediate property (property $\mathcal{W}^*$ (see Definition 3.6.1)). We defer the details to Section 3.6.

2.6 Stallings Subgroup Graphs

Given a point $T = (\Gamma, \tau, l) \in \mathrm{cv}_N$, and a finitely generated subgroup $H \leq F_N$, we consider the (possibly infinite) cover of Γ corresponding to the subgroup $\tau(H)$, denoted $(X_H^T)_r$. Its core, denoted X_H^T, is finite (since H is finitely generated) and represents the conjugacy class of $\tau(H)$ in $\pi_1(\Gamma)$. Note that $X_H^T = (X_H^T)_r$ (i.e. X_H^T is $E(\Gamma)$-regular) if and only if $[F_N : H] < \infty$ if and only if $X_H^T \to \Gamma$ is a covering. If H is of infinite index, then one obtains $(X_H^T)_r$ from X_H^T by attaching "hanging trees" at vertices which are not $E(\Gamma)$-regular. Note that varying the basepoint in $(X_H^T)_r$ corresponds to conjugation of $\tau(H)$ by elements of $\pi_1(\Gamma)$. In the event the basepoint, b, in $(X_H^T)_r$ lies outside X_H^T, we let $(X_H^T)_b$ denote the graph $X_H^T \cup [X_H^T, b]$, where $[X_H^T, b]$ is the bridge between X_H^T and b in $(X_H^T)_r$. For more information see, e.g., [7, 34, 44].

2.7 Bestvina Feighn Handel Laminations

Fix a free basis $\mathcal{A}$ of F_N. Let ∂F_N be the *Gromov boundary* of the word hyperbolic group, F_N; that is, $\partial F_N := \{a_1 a_2 \cdots \mid a_i \in \mathcal{A}^{\pm}, a_i^{-1} \neq a_{i+1}\}$. The *double boundary* of F_N, denoted by $\partial^2 F_N$, is defined by $\partial^2 F_N := (\partial F_N \times \partial F_N)\backslash\Delta$, where $\Delta \subset \partial F_N \times \partial F_N$ consists of those pairs (ζ_1, ζ_2) for which $\zeta_1 = \zeta_2$. More generally, given a marked graph (Γ, τ), one defines $\partial^2\widetilde{\Gamma}$ by considering instead one sided infinite reduced edgepaths in Γ.

Definition 2.7.1 (Bestvina-Feighn-Handel lamination)**.** Let $\varphi \in \mathrm{Out}(F_N)$ be a fully irreducible automorphism equipped with a train track map $f\colon \Gamma \to \Gamma$. The *Bestvina-Feighn-Handel lamination* associated to $\varphi \in \mathrm{Out}(F_N)$, denoted by $\mathcal{L}_{BFH}(\varphi, f, \Gamma)$, is the set of pairs $(\zeta_1, \zeta_2) \in \partial^2\widetilde{\Gamma}$ which have the following property: for every finite subpath $[z_1, z_2] \Subset (\zeta_1, \zeta_2)$ there exists an $e \in E(\Gamma)$ and an $n \geq 1$ so that $f^n(e) \Supset \pi([z_1, z_2])$. Here $\pi\colon \partial^2\widetilde{\Gamma} \to \Gamma$ is the "labelling" map which, on each coordinate, coincides with projection from the universal covering, $\pi\colon \widetilde{\Gamma} \to \Gamma$. Such a pair (ζ_1, ζ_2) is called a *leaf* of the lamination.

If $f\colon \Gamma \to \Gamma$ is a train track representing an iwip, then given any pair of edges $e, e' \in E(\Gamma)$, there exists an $n \geq 1$ so that either $f^n(e) \Supset e'$ or $f^n(e^{-1}) \Supset e'$. In what follows, we will always assume that we have passed to a power of φ so that this is indeed the case. Definition 2.7.1 can be formulated (equivalently) in terms of a fixed edge $e \in E(\Gamma)$. Furthermore, the leaves of the Bestvina-Feighn-Handel lamination satisfy a certain "recurrence" property, which we now formulate precisely.

Definition 2.7.2 (Quasiperiodicity)**.** A leaf (ζ_1, ζ_2) of $\mathcal{L}_{BFH}(\varphi, f, \Gamma)$ is said to be *quasiperiodic* if for every $L > 0$, there exists $L' > L$ so that the following holds: if $[z_1, z_2]$, and $[w_1, w_2]$ are subpaths of (ζ_1, ζ_2) for which $|[z_1, z_2]| > L'$ and $|[w_1, w_2]| < L$, then $\pi([w_1, w_2]) \Subset \pi([z_1, z_2])$ (here $\pi\colon \widetilde{\Gamma} \to \Gamma$ is the labeling map).

Proposition 2.7.3 (Bestvina, Feighn, and Handel [3, Proposition 1.8])**.** *Every leaf* (ζ_1, ζ_2) *of* $\mathcal{L}_{BFH}(\varphi, f, \Gamma)$ *is quasiperiodic.* □

Definition 2.7.4 (Carrying a leaf)**.** Let $H \leq F_N$ be a finitely generated subgroup. Let $\varphi \in \mathrm{Out}(F_N)$ be a fully irreducible automorphism equipped with a train track map $f\colon \Gamma \to \Gamma$ (the marking on Γ being $\tau\colon R_N \to \Gamma$). Let $T = (\Gamma, \tau, l)$. Let X_H^T be the Stallings subgroup graph corresponding to the (conjugacy class of the) subgroup $\tau(H) \leq \pi_1(\Gamma, \tau(*))$. We say that (the conjugacy class of) H *carries* the leaf (ζ_1, ζ_2) of $\mathcal{L}_{BFH}(\Phi, f, \Gamma)$ if for every finite subpath $[z_1, z_2]$ of (ζ_1, ζ_2), the map

$$\pi|_{[z_1,z_2]}\colon [z_1, z_2] \to \Gamma$$

factors through X_H^T as a map $[z_1, z_2] \to X_H^T \to \Gamma$.

Of particular interest to us is the following proposition.

Proposition 2.7.5 (Bestvina, Feighn, and Handel [3, Lemma 2.4])**.** *Let $\varphi \in \mathrm{Out}(F_N)$ be a fully irreducible automorphism equipped with a train track $f\colon \Gamma \to \Gamma$. If a finitely generated subgroup $H \leq F_N$ carries a leaf of $\mathcal{L}_{BFH}(\varphi, f, \Gamma)$, then $[F_N : H] < \infty$.* □

2.8 Stability of Quasi-Geodesics

We will also need some basic results about stability of quasi-geodesics.

Proposition 2.8.1 (Bridson and Haefliger [8, III.H Theorem 1.7])**.** *For all $\delta > 0, \lambda \geq 1, \epsilon \geq 0$ there exists a constant $R = R(\delta, \lambda, \epsilon)$ with the following property: If X is a δ-hyperbolic geodesic space, c is a (λ, ϵ)-quasi-geodesic in X and $[p, q]$ is a geodesic segment joining the endpoints of c, then the Hausdorff distance between $[p, q]$ and the image of c is less than R.*

Proposition 2.8.2. *Let X, Y be δ-hyperbolic geodesic metric spaces equipped with a (λ, ϵ)-quasi-isometry $f\colon X \to Y$. Suppose A, B are collections of geodesic paths in X with the following property, there exists a constant C such that for any $\alpha \in A$ and $\beta \in B$, $\mathrm{diam}_X(\alpha \cap \beta) < C$. Let $\alpha' = [f(\alpha)], \beta' = [f(\beta)]$ be geodesics in Y obtained by reducing the images $f(\alpha), f(\beta)$, respectively. The there exists a constant C' such that for all $\alpha \in A, \beta \in B$, we have $\mathrm{diam}_Y(\alpha' \cap \beta') < C'$.*

Proof. Let $R = R(0, \lambda, \epsilon)$ be the constant afforded by Proposition 2.8.1. Let $\{\alpha_i\}, \{\beta_i\} \subset X$ and and $\{\alpha_i'\}, \{\beta_i'\} \subset Y$ be arbitrary sequences of geodesics (with α_i', β_i' contained in the R-neighborhood of $f(\alpha_i), f(\beta_i)$, respectively). Suppose without loss that $l_X(\alpha_i), l_X(\beta_i) \to \infty$ (the proposition is obvious otherwise). Since R is finite and since f is a quasi-isometry, we must have that $l_X(\alpha_i), l_X(\beta_i) \to \infty$. Similarly, the distance (in Y) between $f(\alpha_i \cap \beta_i)$ and $\alpha_i' \cap \beta_i'$ is uniformly bounded (independent of i). Suppose for contradiction that no such C' exists; that is, $\mathrm{diam}_Y(\alpha_i' \cap \beta_i') \to \infty$. Then (by passing to a subsequence and reindexing) we may find a sequence of points $a_i \in \alpha_i, b_i \in \beta_i$ with $d_X(a_i, \alpha_i \cap \beta_i), d_X(b_i, \alpha_i \cap \beta_i) \to \infty$ for which $d_Y(f(a_i), f(b_i))$ is uniformly bounded. This contradicts the fact that f is a quasi-isometry. $\square$

Corollary 2.8.3. *Let (Γ, τ) be a marked graph. Suppose z is a loop in G which represents a primitive element of $\pi_1(\Gamma)$. Let A be an arbitrary collection of loops in G. Suppose there exists a number M such that for all $a \in A$ whenever $z^k \Subset a$ it follows that $|k| \leq M$. Then for any free basis $\mathcal{B}$ with $z \in \mathcal{B}$ and a number M' such that for all $a \in A$ whenever $z^k \Subset [a]_{\mathcal{B}}$ it follows that $|k| \leq M'$.*

Proof. Let $Z = \{z^k \mid k \in \mathbb{Z}\}$. Let $f\colon \widetilde{\Gamma} \to \mathrm{Cay}(F_N, \mathcal{B})$ be a quasi-isometry with $\mathcal{B}$ any free basis containing z. We may think of Z, A as defining a collection of geodesics in $\widetilde{\Gamma}$. The assumption on z affords a constant, C, such that for any $\zeta \in Z$ and any $\alpha \in A$, we have $\mathrm{diam}_{\widetilde{\Gamma}}(\zeta \cap \alpha) < C$. We now apply Proposition 2.8.2 and conclude that (in $\mathrm{Cay}(F_N, \mathcal{B})$) there is a bound, C' on $\mathrm{diam}_{\mathrm{Cay}(F_N, \mathcal{B})}(\zeta', \alpha')$. Thus there is a bound M' so that if $z^k \Subset [a]_{\mathcal{B}}$, then $|k| \leq M'$. $\square$

2.9 Geodesic Currents

A *geodesic current* on F_N is a positive Radon measure (a Borel measure that is finite on compact sets) on $\partial^2 F_N$ which is invariant under the diagonal action of F_N and under the map which interchanges the coordinates of $\partial^2 F_N$. The space of all currents is denoted $\mathrm{Curr}(F_N)$. If $\nu, \mu \in \mathrm{Curr}(F_N)$ are (nontrivial) currents such that $\nu = \lambda\mu$ for some $\lambda \in \mathbb{R}^*$ then we write $[\nu] = [\mu]$, where $[\cdot]$ denotes the projective class of a given current. To each root free conjugacy class $g \in F_N$ we associate a *counting curent*, η_g as follows. Let $\mathcal{A}$ be a free basis of F_N. Write g as a cyclically reduced word over $\mathcal{A}$. We can thus think of g as the label of a directed graph, Γ_g, which is topologically homeomorphic to $\mathbb{S}^1$ and has $\|g\|_{\mathcal{A}}$ edges (and hence $\|g\|_{\mathcal{A}}$ vertices), each edge labelled by the appropriate $a_i \in \mathcal{A}$. Now let v be any freely reduced word. By the number of occurrences of v or v^{-1} in g, denoted $n_g(v^{\pm})$, we mean the number of vertices of Γ_g at which one may read the word v or v^{-1} along Γ_g (going 'with' the oriented edges) without leaving Γ_g. Note that $n_u(v^{\pm}) = n_{u^{-1}}(v^{\pm})$ by definition. Now let $\widetilde{v}$ be a lift of v to the Cayley tree of F_N with respect to $\mathcal{A}$. Let $\mathrm{Cyl}(v) \subset \partial^2 F_N$ be the set of all $(\zeta_1, \zeta_2) \subset \partial^2(F_N)$ such that the bi-infinite geodesic representing (ζ_1, ζ_2) passes through $\widetilde{v}$. Then $\eta_g(Cyl(v)) = n_g(v^{\pm})$. If $g = h^k$ for h a root free conjugacy class, we define $n_g(v^{\pm}) = kn_h(v^{\pm})$. Currents of the form $\lambda\eta_g$ for $\lambda > 0$ and $g \in F_N$ form a dense subset of $\mathrm{Curr}(F_N)$ [28, Corollary 3.5]. For more information, see [29, 32, 33, 37]. We will need the notion of an intersection form $\langle \cdot, \cdot \rangle\colon \overline{\mathrm{cv}}_N \times \mathrm{Curr}(F_N) \to \mathbb{R}$.

Proposition 2.9.1 (Kapovich and Lustig [32, Theorem A]). *There exists a map* $\langle \cdot, \cdot \rangle \colon \overline{\mathrm{cv}}_N \times \mathrm{Curr}(F_N) \to \mathbb{R}$ *which is continuous,* $\mathrm{Out}(F_N)$*-invariant, and linear with respect to the second argument. Furthermore for every* $g \in F_N$*, and* $T \in \overline{\mathrm{cv}}_N$ *we have* $\langle T, \eta_g \rangle = \|g\|_T$.

2.10 Dual Laminations

Our exposition follows [17, 18, 19]. Let ∂F_N be the *Gromov boundary* [8] of the word hyperbolic group F_N. Let $\partial^2 F_N = \partial F_N \times \partial F_N \backslash \Delta$ where Δ is the diagonal; that is, $\Delta = \{(\zeta, \zeta) \mid \zeta \in \partial F_N\}$. An *algebraic lamination* is a subset of $\partial^2 F_N$ which is: nonempty, closed (via the topology inherited from the topology of the cantor set ∂F_N), and invariant under the map which interchanges the factors of $\partial^2 F_N$.

Given $T \in \mathrm{cv}_N$, we consider – for each $\epsilon > 0$ – the set

$$\Omega_\epsilon(T) = \{w \in F_N \mid \|w\|_T < \epsilon\} \subset F_N$$

Now for each $w \in \Omega_\epsilon(T)$, we let $L^2(w)$ denote the *minimal rational lamination* of w; that is,

$$L^2(w) = \{(vw^{-\infty}, vw^{\infty}) \mid v \in F_N\} \cup \{(vw^{\infty}, vw^{-\infty}) \mid v \in F_N\}$$

We now define

$$\Omega^2_\epsilon(T) = \bigcup_{w \in \Omega_\epsilon(T)} L^2(w) \subset \partial^2 F_N$$

and set

$$L^2_\epsilon(T) = \overline{\Omega^2_\epsilon(T)} \subset \partial^2(F_N)$$

If $T \in \mathrm{cv}_N$ we define $L^2_\Omega(T) = \varnothing$, otherwise (for $T \in \partial\,\mathrm{cv}_N$) we define

$$L^2_\Omega(T) = \bigcap_{\epsilon > 0} L^2_\epsilon(T)$$

Evidently in the case $T \in \partial\,\mathrm{cv}_N$, $L^2_\Omega(T)$ is an algebraic lamination which we call the *dual lamination* to T.

Chapter 3

Non-Rigidity of Cyclic Automorphic Orbits

3.1 Background

Following Kapovich [26], we define a subset $\Sigma \subseteq F_N$ to be *spectrally rigid* if whenever $T_1, T_2 \in \mathrm{cv}_N$ are such that $\|\sigma\|_{T_1} = \|\sigma\|_{T_2}$ holds for every $\sigma \in \Sigma$, it follows that $T_1 = T_2$ in cv_N. Results of Smillie and Vogtmann [43] and Cohen, Lustig, and Steiner [12] show that no finite subset of F_N is spectrally rigid; however, since the action of F_N on T is free it is non-abelian ($\|ghg^{-1}h^{-1}\|_T \neq 0$, for some $g, h \in F_N$), and so the translation length function uniquely determines the action (see [11, §3, Theorem 4.1]). Thus F_N itself is spectrally rigid. Motivated by these results, Kapovich [26] initiated the search for (non) spectrally rigid infinite but "sparse" subsets of F_N, proving in [26] for $N \geq 2$ that almost every trajectory of a simple non-backtracking random walk is spectrally rigid. Furthermore, Carette, Francaviglia, Kapovich, and Martino [10], show that for $N \geq 2$, the set of primitive elements in F_N is spectrally rigid. They also show in [10] that for any subgroup $H \leq \mathrm{Aut}(F_N)$ ($N \geq 3$) which projects to an infinite normal subgroup in $\mathrm{Out}(F_N)$, the orbit Hg of any nontrivial $g \in F_N$ is spectrally rigid. Kapovich [26] remarks without proof that in the case of an atoroidal fully irreducible automorphism $\Phi \in \mathrm{Aut}(F_N)$ ($N \geq 3$), one can show—using a previous result of Smillie and Vogtmann [43] together with train track techniques—that for any nontrivial $g \in F_N$, the orbit $\langle\Phi\rangle g$ is not spectrally rigid. These results naturally lead to the following question, posed in [10]: Is it true that for any $H \leq \mathrm{Aut}(F_N)$, either for all nontrivial $g \in F_N$ the orbit Hg is spectrally rigid, or for all nontrivial $g \in F_N$ the orbit Hg is not spectrally rigid. Carette, Francaviglia, Kapovich, and Martino conjectured [10, Conjecture 7.5] that in the case $H \leq \mathrm{Aut}(F_N)$ is cyclic, the orbit Hg is never spectrally rigid. Our main result, Theorem A, verifies this conjecture.

Theorem A. *Let $N \geq 2$. Let $\Phi \in \mathrm{Aut}(F_N)$. Let $g \in F_N$ be arbitrary. Then the set $\Sigma = \{\Phi^n(g)\}_{n\in\mathbb{Z}}$ is not spectrally rigid.*

The proof of Theorem A requires two main ingredients: improved relative train track machinery established by Bestvina, Feighn, and Handel (see [4], [5]); and previous results regarding outer space by Smillie and Vogtmann [43], and by Cohen, Lustig, and Steiner [12]. The latter results show that no finite subset of

F_N is spectrally rigid ([43] handles $N \geq 3$, and [12] $N = 2$). A critical observation of Kapovich [26] reveals that these arguments hold for more general subsets of F_N, not necessarily finite. In particular, we use the argument in [43] to show if $\Sigma \subseteq F_N$ ($N \geq 3$) satisfies Property $\mathcal{W}$ (see Definition 2.5.2), then Σ is not spectrally rigid. Similarly, we formulate Property $\mathcal{W}^*$ (see Definition 3.6.1) and use the arguments in [12] to show that any subset $\Sigma \subseteq F_2$ which satisfies property $\mathcal{W}^*$ is not spectrally rigid. We further show that Property $\mathcal{W}^*$ follows from Property $\mathcal{W}$ (see Proposition 3.6.3). As mentioned above, obtaining the result for iwip automorphisms is straightforward. It is the extension of these ideas to non-iwip automorphisms which occupies the majority of this paper.

The main outline of our argument is as follows. We define (for a given $\Phi \in \mathrm{Aut}(F_N)$) a property (Property $\mathcal{P}$, see Definition 3.2.1) so that if $\Phi \in \mathrm{Aut}(F_N)$ has Property $\mathcal{P}$, then for any $g \in F_N$ the set $\Sigma = \{\Phi^n(g)\}_{n\in\mathbb{Z}}$ will have Property $\mathcal{W}$, and therefore be non-spectrally rigid. Verification of Property $\mathcal{P}$ for a given $\Phi \in \mathrm{Aut}(F_N)$ will require detailed analysis of a (relative) train track $f\colon G \to G$ representing $[\Phi] \in \mathrm{Out}(F_N)$. In Section 3.4, we handle those $[\Phi]$ that can be equipped with relative train tracks with exponentially growing top stratum; Section 3.5 handles those with non-exponentially growing top stratum. In Section 3.3 we cover the iwip case in order to elucidate the arguments which are central to the main result. In particular, we show that given an iwip $\Phi \in \mathrm{Aut}(F_N)$ with train track $f\colon G \to G$, $g \in F_N$, and a primitive $a \in F_N$, there is a uniform bound M so that if $(\tau(a))^k$ occurs as a subword in the reduced form of $f^n(\tau(g))$ (for any $n \geq 1$), then $|k| \leq M$. Here $\tau\colon R_N \to G$ is the marking. In Section 3.4 and Section 3.5, we show that appropriately modified versions of the above statement hold. Handling the case $N = 2$ involves showing that Property $\mathcal{W}$ suffices for Property $\mathcal{W}^*$, and is deferred to Section 3.6.

3.2 Property $\mathcal{P}$

We begin with a definition which is central. For convenience, we write $\mathbb{N} = \{0, 1, 2, \dots\}$.

Definition 3.2.1 (Property $\mathcal{P}$)**.** Given $\Phi \in \mathrm{Aut}(F_N)$, we say Φ has *property* $\mathcal{P}^+(\mathcal{A}, a)$ if there exist a free basis $\mathcal{A}$ of F_N and $a \in \mathcal{A}$, so that for any $g \in F_N$, there exists $M = M_g \geq 1$ so that if $a^k \Subset h \in \{[[\Phi^n(g)]]_{\mathcal{A}}\}_{n\in\mathbb{N}}$ then $|k| \leq M$. We say Φ has property $\mathcal{P}^-(\mathcal{A}, a)$ if the above holds with $\mathbb{N}$ replaced by $-\mathbb{N}$, and we say Φ has property $\mathcal{P}(\mathcal{A}, a)$ if the above holds with $\mathbb{N}$ replaced by $\mathbb{Z}$. We say Φ has $\mathcal{P}$, (respectively $\mathcal{P}^+, \mathcal{P}^-$) if Φ has $\mathcal{P}(\mathcal{A}, a)$, (respectively $\mathcal{P}^+(\mathcal{A}, a), \mathcal{P}^-(\mathcal{A}, a)$) for some pair $(\mathcal{A}, a)$. We say $\varphi \in \mathrm{Out}(F_N)$ has $\mathcal{P}(\mathcal{A}, a)$ (respectively $\mathcal{P}^+(\mathcal{A}, a), \mathcal{P}^-(\mathcal{A}, a)$) if for some (equivalently, for all) $\Psi \in \varphi$, we have that Ψ has $\mathcal{P}(\mathcal{A}, a)$ (respectively $\mathcal{P}^+(\mathcal{A}, a), \mathcal{P}^-(\mathcal{A}, a)$). Finally, we say φ has $\mathcal{P}$, (respectively $\mathcal{P}^+, \mathcal{P}^-$) if φ has $\mathcal{P}(\mathcal{A}, a)$, (respectively $\mathcal{P}^+(\mathcal{A}, a), \mathcal{P}^-(\mathcal{A}, a)$) for some pair $(\mathcal{A}, a)$.

Remark 3.2.2. If Φ has both $\mathcal{P}^+(\mathcal{A}, a)$, and $\mathcal{P}^-(\mathcal{A}, a)$, then Φ has $\mathcal{P}(\mathcal{A}, a)$.

Definition 3.2.1 and Definition 2.5.2 immediately yield the following proposition.

Proposition 3.2.3. *If $\varphi \in \mathrm{Out}(F_N)$ has $\mathcal{P}$, then for any $\Phi \in \varphi$ and for any $g \in F_N$ the set $\Sigma = \{\Phi^n(g)\}_{n\in\mathbb{Z}}$ has $\mathcal{W}$.* □

As indicated in Theorem 2.2.5, good relative train tracks often represent a positive iterate of a given outer automorphism. The following lemma and its corollaries ensure that this does not pose a problem.

Lemma 3.2.4. *Suppose $\Psi \in \mathrm{Aut}(F_N)$ has $\mathcal{P}$ and $\Phi^l = \Psi$ for some $l \geq 1$. Then Φ has $\mathcal{P}$.*

Proof. For any $m \in \mathbb{Z}$, write $m = ql + r$ with $0 \leq r < l$. Then for $g \in F_N$, we have

$$\Phi^m(g) = \Phi^{ql+r}(g) = \Phi^{ql}(\Phi^r(g)) = \Psi^q(\Phi^r(g)).$$

Let $\Gamma = \{\Phi^r(g)\}_{r=0}^{l-1}$. Say Ψ has $\mathcal{P} = \mathcal{P}(\mathcal{A}, a)$. For each $\Phi^r(g) \in \Gamma$, we obtain M_r for which

$$\text{if} \quad a^k \Subset h \in \{[[\Psi^n(\Phi^r(g))]]_{\mathcal{A}}\}_{n\in\mathbb{Z}}, \quad \text{then} \quad |k| \leq M_r.$$

Set $M' := \max_{0\leq r<l-1}\{M_r\}$. Thus,

$$\text{if} \quad a^k \Subset h \in \{[[\Phi^n(g)]]_{\mathcal{A}}\}_{n\in\mathbb{Z}}, \quad \text{then} \quad |k| \leq M'.$$ □

Lemma 3.2.4 immediately implies the following.

Corollary 3.2.5. *Suppose $\Psi \in \mathrm{Aut}(F_N)$ has $\mathcal{P}^+$ (respectively $\mathcal{P}^-$) and $\Phi^l = \Psi$ for some $l \geq 1$. Then Φ has $\mathcal{P}^+$ (respectively $\mathcal{P}^-$).* □

Lemma 3.2.4 and Definition 3.2.1 imply the following two corollaries.

Corollary 3.2.6. *Suppose $\psi \in \mathrm{Out}(F_N)$ has $\mathcal{P}$ and $\varphi^l = \psi$ for some $l \geq 1$. Then φ has $\mathcal{P}$.* □

Corollary 3.2.7. *Suppose $\psi \in \mathrm{Out}(F_N)$ has $\mathcal{P}^+$ (respectively $\mathcal{P}^-$) and $\varphi^l = \psi$ for some $l \geq 1$. Then φ has $\mathcal{P}^+$ (respectively $\mathcal{P}^-$).* □

As $1 \in \mathrm{Aut}(F_N)$ has $\mathcal{P}$, we also obtain the following.

Corollary 3.2.8. *If $\varphi \in \mathrm{Out}(F_N)$ is such that $\varphi^l = 1 \in \mathrm{Out}(F_N)$, then φ has $\mathcal{P}$.* □

We'll prove Theorem A by showing that for every $\Phi \in \mathrm{Aut}(F_N)$ there is some $l \geq 1$ so that Φ^l has $\mathcal{P}$ and then apply Lemma 3.2.4, Proposition 3.2.3 and Proposition 2.5.3. Incidently, Corollary 3.2.8 establishes

Theorem A for those $\Phi \in \mathrm{Aut}(F_N)$ for which $[\Phi]^l = 1 \in \mathrm{Out}(F_N)$ for some $l \geq 1$. In view of Theorem 2.2.5, Remark 2.2.7, and Lemma 3.2.4 we establish the following convention.

Convention 3.2.9. For an EG-automorphism with good relative train track f, we will assume that the f image of any edge from H_t contains all edges in H_t. This is achieved by passing to a power as per Condition 1 in Theorem 2.2.5. Furthermore, we will speak of a good relative train track "representing φ" when we actually mean a good relative train track representing an *iterate* of φ.

The following proposition enables us to use topological representatives and their associated marked graphs in order to verify property $\mathcal{P}$ for a given automorphism.

Proposition 3.2.10. *Let $\tau\colon R_N \to G$ and $\tau'\colon R_N \to G'$ be marked graphs. Let $a \in F_N$, and $\Sigma \subseteq F_N$. Then the following are equivalent.*

1. $\sup\{|k| : [[\tau(a)]]^k \Subset [[\tau(\sigma)]], \sigma \in \Sigma\} < \infty$.
2. $\sup\{|k| : [[\tau'(a)]]^k \Subset [[\tau'(\sigma)]], \sigma \in \Sigma\} < \infty$.

Sketch of proof. Let $f\colon G \to G'$ be a homotopy equivalence which induces the isomorphism $\tau'_\# \circ \tau_\#^{-1}$. Let k, a, and σ be so that $[[\tau(a)]]^k \Subset [[\tau(\sigma)]]$; i.e., there exist $\gamma_a \in [[\tau(a)]]$ and $\gamma_\sigma \in [[\tau(\sigma)]]$, so that $\gamma_a^k \Subset \gamma_\sigma$. Write $\gamma_\sigma = \alpha\, \gamma_a^k\, \beta$. As γ_a is homotopically nontrivial, write $[f(\gamma_a)] = u\gamma'_a\overline{u}$, where γ'_a is both cyclically reduced and nontrivial. Then

$$\begin{aligned}
[[f(\gamma_\sigma)]] &= [[f(\alpha\, \gamma_a^k\, \beta)]] \\
&= [[\ [f(\alpha)]\, [f(\gamma_a^k)]\, [f(\beta)]\ ]] \\
&= [[\ [f(\alpha)]\, u(\gamma'_a)^k\overline{u}\, [f(\beta)]\ ]].
\end{aligned}$$

Since cancellation in the above expression is controlled by $\mathrm{BCC}(f)$ (see Lemma 2.4.1), we conclude that the power, k', of γ'_a which remains after cancellation is so that $k' \geq k - 2\,\mathrm{BCC}(f)$. Thus if k is unbounded, so is k'. □

Corollary 3.2.11. *Let $f\colon G \to G$ be a topological representative for $\varphi \in \mathrm{Out}(F_N)$. Suppose there exist a free basis $\mathcal{A}$ of F_N and $a \in \mathcal{A}$ such that for any $g \in F_N$, there exists $M \geq 1$ so that if $[[\tau(a)]]^k \Subset h \in \{[f^n(\tau(g))]\}_{n\in\mathbb{N}(resp.\, -\mathbb{N})}$ then $|k| \leq M$. Then φ has $\mathcal{P}^+(\mathcal{A}, a)$ (respectively $\mathcal{P}^-(\mathcal{A}, a)$).*

Proof. Let $\tau\colon R_N \to G$ be the marking for G, and $\tau'\colon R_N \to R_N$ be a homotopy equivalence realizing the basis $\mathcal{A}$ and invoke Proposition 3.2.10. □

The following lemma will be useful in our analysis of forward images of top stratum edges for EG-automorphisms.

Lemma 3.2.12. *Let $N \geq 2$. Let $\varphi \in \mathrm{Out}(F_N)$ be an EG-automorphism. Then there is $L \geq 1$ so that for sufficiently large n and for all $e_i, e_j \in E(H_t)$, we have the following: if $w \Subset [f^n(e_i)]$ is such that w contains more than L edges (counted with repetition) from $E(H_t)$, then $e_j \Subset w$.*

Proof. For any $w \Subset [f^n(e)]$, let $|w|_t$ denote the number of top stratum edges (counted with repetition) in w. Let $K := \max_{e \in E(H_T)}\{|[f(e)]|_t\}$. Let $L := 2K$. Then for $w \Subset [f^n(e_i)]$ with $|w|_t \geq L$, we have that $[f^{-1}(w)] \Subset [f^{n-1}(e_i)]$, and $[f^{-1}(w)]$ contains an edge from $E(H_t)$. Thus w contains all edges from $E(H_t)$. □

3.3 Fully Irreducible Automorphisms

Although the iwip case follows from the result for EG-automorphisms, obtaining the result for EG-automorphisms is more involved, mainly on account that the arguments used in Theorem 3.3.4 only apply to iwip automorphisms. Fortunately, Lemmas 3.3.1 and 3.3.2 generalize to the case of an EG-automorphism insofar as we omit the proof of the analogue of Lemma 3.3.1. Aside from this reference to Lemma 3.3.1, Section 3.4 is self-contained, as is Section 3.5. Nevertheless, the section at hand serves to shed light on the arguments which are central to our analysis.

Lemma 3.3.1. *Let $N \geq 2$. Let $\varphi \in \mathrm{Out}(F_N)$ be fully irreducible. Let $g \in F_N$ be arbitrary. Let $a \in F_N$ be any primitive element. Let $\gamma := [[\tau(g)]]$, and $\alpha := [[\tau(a)]]$. Let $f\colon G \to G$ be a train track map for φ. Let $e_0 \in E(G)$ be arbitrary. Suppose for all $l \geq 1$ there exists $m = m_l \geq 1$ for which $\alpha^l \Subset [f^{m_l}(\gamma)]$. Then for all $k \geq 1$, there exists $n = n_k$ for which $\alpha^k \Subset f^{n_k}(e_0)$.*

Proof. We may assume that as $j \to \infty$, we have $|[f^j(\gamma)]| \to \infty$. Otherwise there exists a uniform bound on $|[f^j(\gamma)]|$ and so the set $\{[f^j(\gamma)]\}_{j \in \mathbb{N}}$ is finite and the lemma follows vacuously. Write $\gamma = e_0 e_1 \cdots e_{h-1}$. We have

$$[f^j(\gamma)] = [f^j(e_0) f^j(e_1) \cdots f^j(e_{h-1})]$$

by the train track property. Note that reduction can only occur between the subpaths $f^j(e_i)$ and $f^j(e_{i+1 \bmod h})$. For each j, write

$$[f^j(\gamma)] = p^j_{i_1} p^j_{i_2} \cdots p^j_{i_l},$$

where $0 \leq i_1 < i_2 < \cdots < i_r \leq h-1$, and $p^j_{i_q} \Subset f^j(e_q)$. We may assume j is large enough so that the set $I = \{i_1, i_2, \ldots, i_r\}$ is fixed, as this is eventually true. For $q \in I$, let M_q be so that $|p^j_{i_q}| \leq M_q$ for all

j; if no such bound exists, set $M_i := 0$. Note since $|[f^j(\gamma)]| \to \infty$, there is a q for which $|p^j_{i_q}| \to \infty$. Set $M := \sum_q M_q$.

Note that if $w \Subset [f^j(\gamma)]$ is such that $|w| > M$, then w meets some $p^j_{i_q}$ for which $|p^j_{i_q}| \to \infty$. Furthermore, if we let $C := \#\{q \in I : |p^j_{i_q}| \to \infty\}$, then any w with $|w| = M' > M$ meets at least $(M' - M)/C$ of a $p^j_{i_q}$ for which $|p^j_{i_q}| \to \infty$. As M, C are fixed, given k we choose l so that

$$\frac{|\alpha^l| - M}{C} \geq |\alpha|^k.$$

Then $\alpha^k \Subset p^j_{i_q} \Subset f^j(e_{i_q})$ for some q. Furthermore for a fixed $e_0 \in E(G)$ and any $q \in I$, if $a^k \Subset f^{n_k}(e_{i_q})$, then $a^k \Subset f^{n_k+1}(e_0)$. $\square$

Lemma 3.3.2. *Let $N \geq 2$. Let $\varphi \in \mathrm{Out}(F_N)$ be fully irreducible. Let $a \in F_N$ be any primitive element. Let $\alpha := [[\tau(a)]]$. Let $f\colon G \to G$ be a train track map for φ. Let $e_0 \in E(G)$ be arbitrary. Then there exists $k \geq 1$, such that for all $n \geq 1$, we have $\alpha^k \not\Subset f^n(e_0)$.*

Proof. Suppose for contradiction that for all $k \geq 1$, there exists $n = n_k \geq 1$ such that $a^k \Subset f^{n_k}(e_0)$. Choose $k_1 \geq |\alpha| + 1$. Let L be as per Lemma 3.2.12. Then if $w \Subset f^j(e_0)$ is so that $|w| \geq L$, then w contains all edges in $E(G)$. Let $\lambda_- := \min_{e \in E(G)}\{|f(e)|\}$, and $\lambda_+ := \max_{e \in E(G)}\{|f(e)|\}$. Choose k_2 so that

$$\lambda_+^{-(n_{k_1}+1)} |\alpha^{k_2}| \geq L.$$

Then $\alpha^{k_2} \Subset f^{n_{k_2}}(e_0)$ is so that

$$e_0 \Subset f^{-(n_{k_1}+1)}(\alpha^{k_2}) \Subset f^{n_{k_2}-(n_{k_1}+1)}(e_0).$$

As $\alpha^{k_1} \Subset f^{n_{k_1}}(e_0)$, we have

$$f(\alpha^{k_1}) \Subset \alpha^{k_2} \Subset f^{n_{k_2}}(e_0).$$

Since we are working in the forward image of an edge, the train track property gives

$$f(\alpha^{k_1}) = f(\alpha)^{k_1}.$$

By our choice of k_1, the Pigeonhole Principle gives $m_1, m_2 \geq 1$ for which

$$f(\alpha)^{m_1} = (\sim \alpha)^{m_2}$$

where $\sim \alpha$ denotes a cyclic permutation of α. As α represents a primitive $a \in F_N$, we conclude that $m_2 = \pm m_1$. We now note that $f(\alpha)^{m_1} = (\sim \alpha)^{\pm m_1}$ is impossible since f expands path lengths by at least $\lambda_- \geq \#E(G) \geq 2$. □

We immediately obtain the following.

Corollary 3.3.3. *Let $N \geq 2$. Let $\varphi \in \mathrm{Out}(F_N)$ be fully irreducible. Then for any free basis $\mathcal{A}$ of F_N, and for any $a \in \mathcal{A}$, φ has $\mathcal{P}^+(\mathcal{A}, a)$.*

Proof. Let $f\colon G \to G$ be a train track map for φ. By Lemma 3.3.2, for any free basis $\mathcal{A}$ of F_N, and any $a \in \mathcal{A}$, there is $M \geq 1$ so that if $[[\tau(a)]]^k \Subset f^n(e_0)$ for any $n \geq 1$, then $|k| \leq M$. Let $g \in F_N$ be arbitrary. Then by Lemma 3.3.1, there is $M' \geq 1$ so that if $[[\tau(a)]]^k \Subset [f^n([[\tau(g)]])]$ for any $n \geq 1$, then $|k| \leq M'$. Thus by Corollary 3.2.11 and Corollary 3.2.7, φ has $\mathcal{P}^+(\mathcal{A}, a)$. □

We can now prove that iwip automorphisms have property $\mathcal{P}$.

Theorem 3.3.4. *Let $N \geq 2$. Let $\varphi \in \mathrm{Out}(F_N)$ be fully irreducible. Then φ has $\mathcal{P}$.*

Proof. Note that φ is an iwip if and only if φ^{-1} is an iwip. Thus by Corollary 3.3.3 for any free basis $\mathcal{A}$, and for any $a \in \mathcal{A}$, both φ and φ^{-1} have $\mathcal{P}^+(\mathcal{A}, a)$; that is, φ has both $\mathcal{P}^+(\mathcal{A}, a)$ and $\mathcal{P}^-(\mathcal{A}, a)$, and so has $\mathcal{P}(\mathcal{A}, a)$. □

3.4 EG-Automorphisms

The arguments in Lemma 3.3.1 also apply, *mutadis mutandis*, to the case where φ is an EG-automorphism. We therefore omit the proof of the following lemma.

Lemma 3.4.1. *Let $N \geq 2$. Let $\varphi \in \mathrm{Out}(F_N)$ be an EG-automorphism and $f\colon G \to G$ a good relative train track. Let $g \in F_N$ be arbitrary. Let $a \in F_N$ be any primitive element for which $\alpha := [[\tau(a)]]$ crosses the exponentially growing top stratum. Let $\gamma := [[\tau(g)]]$. Let $e_0 \in E(H_t)$ be arbitrary. Suppose $\forall l \geq 1, \exists m = m_l \geq 1$ for which $\alpha^l \Subset [f^{m_l}(\gamma)]$. Then $\forall k \geq 1, \exists n = n_k$ for which $\alpha^k \Subset [f^{n_k}(e_0)]$.* □

Lemma 3.4.2 is the analogue of Lemma 3.3.2.

Lemma 3.4.2. *Let $N \geq 2$. Let $\varphi \in \mathrm{Out}(F_N)$ be an EG-automorphism and $f\colon G \to G$ a good relative train track. Let $a \in F_N$ be any primitive element for which $\alpha := [[\tau(a)]]$ crosses the exponentially growing top stratum. Let $e_0 \in E(H_t)$ be arbitrary. Then there exists $k \geq 1$, such that for all $n \geq 1$, we have $\alpha^k \not\Subset [f^n(e_0)]$.*

Proof. As in Lemma 3.3.2, we suppose for contradiction that for all $k \geq 1$, there exists $n = n_k \geq 1$ for which $\alpha^k \Subset [f^{n_k}(e_0)]$. Choose $k_1 \geq |\alpha|+1$. Let L be as per Lemma 3.2.12. Thus if $w \Subset [f^j(e_0)]$ is so that $|w|_t \geq L$, then w contains all edges in $E(H_T)$. Choose k_2 so that $|[f^{-(n_{k_1}+1)}(\alpha^{k_2})]|_t \geq L$. Then $\alpha^{k_2} \Subset [f^{n_{k_2}}(e_0)]$ is so that

$$e_0 \Subset [f^{-(n_{k_1}+1)}(\alpha^{k_2})] \Subset [f^{n_{k_2}-(n_{k_1}+1)}(e_0)]$$

As $\alpha^{k_1} \Subset [f^{n_{k_1}}(e_0)]$, we have

$$[f(\alpha^{k_1})] \Subset \alpha^{k_2} \Subset [f^{n_{k_2}}(e_0)]$$

Write $[f(\alpha)] = uvu^{-1}$, so that $v = [f(\alpha)]_{c\#}$ as cyclic paths. Then

$$[f(\alpha^{k_1})] = (uvu^{-1})^{k_1} = uv^{k_1}u^{-1}$$

By our choice of k_1, the Pigeonhole Principle gives $m_1, m_2 \geq 1$ for which

$$(\sim [f(\alpha)])^{m_1} = v^{m_1} = (\sim \alpha)^{m_2}$$

where $\sim$ denotes a cyclic permutation. As α represents a primitive $a \in F_N$, we conclude that $m_2 = \pm m_1$. We now note that $(\sim [f(\alpha)])^{m_1} = (\sim \alpha)^{\pm m_1}$ is impossible since f expands path lengths in H_t by at least $\lambda_- := \min_{e \in E(H_t)}\{|f(e)|_t\} \geq 2$. □

The proof of Theorem 3.3.4 does directly not carry over to non-iwip automorphisms. A priori there is no guarantee that the realization (under the marking) of a given primitive $a \in F_N$ in good relative train tracks for φ and φ^{-1} must cross top strata in both. However, we can arrange this for a particular choice of a, as is outlined below.

Proposition 3.4.3. *Let $\mathcal{A} = \{a_0, a_1, \ldots, a_{n-1}\}$ be a free basis of F_N. Let $w \in F_N$ be so that there is exactly one occurrence of a_{n-1} (or $\overline{a_{n-1}}$) in $[w]_{\mathcal{A}}$. Then w is a primitive element.*

Proof. Note that $\mathcal{A}' = \{a_0, a_1, \ldots, a_{n-2}\}$ is a free basis of F_{N-1}, and write $[w]_{\mathcal{A}} = \alpha a_{n-1} \beta$ with $\alpha = [\alpha]_{\mathcal{A}'}$, $\beta = [\beta]_{\mathcal{A}'}$. Given an arbitrary element $[v]_{\mathcal{A}} \in F_N$, if we replace all occurrences of a_{n-1} by $\alpha^{-1} w \beta^{-1}$, then we obtain an expression for v over the set $\mathcal{A}' \cup \{w\}$. Thus $\mathcal{A}' \cup \{w\}$ generates F_N. As $\#(\mathcal{A}' \cup \{w\}) = N$, it is a free basis. □

Corollary 3.4.4. *Let G be a finite graph. Let $e_0 \in E(G)$. If α is a cyclicly reduced path G that crosses e_0 (in either direction) exactly once, then α is a primitive element in $\pi_1(G) \cong F_N$.*

Proof. Note first that e_0 can not be a separating edge. Choose a maximal tree $T \subset G \backslash \{e_0\}$. Then T is also a maximal tree for G with $(G\backslash\{e_0\})/T \cong R_{N-1}$ and $G/T \cong R_N$ as graphs. Furthermore $(G\backslash\{e_0\})/T \subset G/T$, the only difference being the additional loop associated to the edge e_0. The cyclic path $[[\tau(\alpha)]]$ in G/T crosses the loop e_0 exactly once (in either direction); the remainder of the edge path lies in $(G\backslash\{e_0\})/T$. □

The proof Lemma 3.4.7 will require us to choose loops in G with very specific properties. The following two lemmas ensure this is possible.

Proposition 3.4.5. *Let $N \geq 2$. Let G be a finite connected graph with $\pi_1(G) \cong F_N$ for which G has no valence one vertices. Let $H \subsetneq G$ be a proper subgraph with $\#E(H) \geq 1$. Let $M \geq 1$, $e_0 \in E(H)$ be given. Then there is a cyclically reduced path α such that α crosses e_0 at least M times (in either direction), and α represents a primitive element in $\pi_1(G)$.*

Proof. Suppose first that $e_0 \in E(H)$ is a G-non-separating edge. Choose a maximal tree $T \subset G\backslash\{e_0\}$. Then T is a maximal tree for G with $\#E(G) - \#E(T) = N \geq 2$, and $e_0 \notin E(T)$. Choose $e \neq e_0 \in E(G)\backslash E(T)$. Choose $v \in V(G)$. By Corollary 3.4.4 the cyclically reduced form of the following path suffices

$$\{[v, o(e_0)]_T \; e_0 \; [t(e_0), v]_T\}^M \; [v, o(e)]_T \; e \; [t(e), v]_T$$

since all M occurrences of e_0 remain after cyclic reduction, along with the occurrence of e. Now suppose that $e_0 \in E(H)$ is a G-separating edge. Let Γ, Γ' be the two components of $G\backslash\{e_0\}$. There are two cases to consider. Note first that neither of Γ, Γ' are trees, else G would have valence one vertices. If either $\pi_1(\Gamma) \cong F_M$ or $\pi_1(\Gamma') \cong F_M$ with $M \geq 2$, then we proceed as follows. Without loss suppose $\pi_1(\Gamma) \cong F_M$ with $M \geq 2$. Choose maximal trees T, T' in Γ, Γ' respectively. Note that $T^* := T \cup T' \cup \{e_0\}$ is a maximal tree for G. Choose $e_1 \in E(\Gamma)\backslash E(T)$ and $e_2 \neq e_1 \in E(\Gamma)\backslash E(T)$. Choose $e' \in E(\Gamma')\backslash E(T')$. Choose $v \in V(\Gamma)$. Then by Corollary 3.4.4, the cyclically reduced form of the following path suffices

$$\{[v, o(e_1)]_T \; e_1 \; [t(e_1), o(e_0)]_T \; e_0 \; [t(e_0), o(e')]_{T'} \; e' \; [t(e'), t(e_0)]_{T'} \\ \overline{e_0} \; [o(e_0), v]_T\}^{\lceil \frac{M}{2} \rceil} \; [v, o(e_2)]_T \; e_2 \; [t(e_2), v]_T$$

since the M occurrences of e_0, and $\overline{e_0}$, along with the single occurrence of e_2 remain after cyclic reduction. Now suppose that $\pi_1(\Gamma) \cong \pi_1(\Gamma') \cong \mathbb{Z}$. Choose maximal trees T, T' in Γ, Γ' and let $T^* := T \cup T' \cup \{e_0\}$ be a maximal tree for G. Choose $e_1 \in E(\Gamma)\backslash E(T)$ and $e' \in E(\Gamma')\backslash E(T')$. Choose $v \in V(\Gamma)$. As the map

$G \to G/T^*$ is a homotopy equivalence, write $\pi_1(G) \cong F_2 = F(l_1, l')$, where

$$l_1 := [v, o(e_1)]_T \ e_1 \ [t(e_1), v]_T,$$
$$l' := [v, o(e_0)]_T \ e_0 \ [t(e_0), o(e')]_{T'} \ e' \ [t(e'), t(e_0)]_{T'} \ \overline{e_0} \ [o(e_0), v]_T.$$

Let $\eta_1, \eta' \colon F(l_1, l') \to F(l_1, l')$ be the Nielsen transformations

$$\eta_1 := \begin{cases} l_1 \mapsto l_1 l' \\ l' \mapsto l' \end{cases} \quad \text{and} \quad \eta' := \begin{cases} l_1 \mapsto l_1 \\ l' \mapsto l' l_1 \end{cases}.$$

In the automorphic image of l_1 under $(\eta' \circ \eta_1)^{M'}$, the number of occurrences of $l_1 l'$ and $l' l_1$ is unbounded in M'. Each such occurrence crosses e_0, as the e_0 do not cancel during reduction. Thus choosing M' large enough, the closed path $(\eta' \circ \eta_1)^{M'}(l_1)$ suffices. □

Proposition 3.4.6. *Let $N \geq 2$. Let G be a finite connected graph with $\pi_1(G) \cong F_N$ for which G has no valence one vertices. Let $H \subsetneq G$ be a proper subgraph with $\#E(H) \geq 1$. Let α be a cyclically reduced path in $G \backslash H$. Then there is a closed path α' for which the following hold:*

1. *$\alpha' = \eta \ \alpha$ as a cyclic path for some closed path η.*

2. *$[[\alpha']]$ crosses H; in particular $[\eta]$ crosses H and all occurrences of H-edges in $[\eta]$ remain after cyclic reduction to $[[\alpha']]$.*

3. *$[[\alpha']]$ represents a primitive element in $\pi_1(G)$.*

4. *There is a bound depending only on G for the amount of cyclic reduction in cyclically reducing $[\eta]\alpha$ to $[[\alpha']]$.*

Proof. Let Γ_1 be the component of $G \backslash H$ for which $\alpha \subset \Gamma_1$. Let $\Delta = G \backslash \{\Gamma_1 \cup H\}$, and write $\Delta = \cup_{i=2}^{k} \Gamma_i$, where the Γ_i are the components of Δ. Let $\Delta^* \subset \Delta$ consist of those components which satisfy the following: there is an edge $e \in E(H)$ for which $\Gamma_i \cup \{e\}$ is connected and meets Γ_1. If there is $e_0 \in E(H)$ for which $o(e_0) \in E(\Gamma_1)$ and $t(e_0) \in E(\Gamma_1)$, then we proceed as follows. Let T_1 be a maximal tree in Γ_1. Choose a vertex $v \Subset \alpha$ and write $o(\alpha) = t(\alpha) = v$. Then the cyclically reduced form of the following path suffices

$$\alpha' = \underbrace{[v, o(e_0)]_{T_1} \ e_0 \ [t(e_0), v]}_{\eta} \ \alpha.$$

Note the amount of cyclic reduction between $[\eta]$ and α is at most $2 \operatorname{diam} \Gamma_1$, and $e_0 \Subset [[\alpha']]$ occurs exactly once (see Corollary 3.4.4). Suppose now that no such $e_0 \in E(H)$ exists; that is if $e \in E(H)$ is such that $o(e) \in V(\Gamma_1)$, then $t(e) \in V(\Delta^*)$. Let $e_0 \in E(H)$ be such that $o(e_0) \in V(\Gamma_1)$ and write $t(e_0) \in V(\Gamma_j)$ for $j \geq 2$. Suppose further that Γ_j is not a tree. Let T_j be a maximal tree in Γ_j, and let $e_j \in E(\Gamma_j) \backslash E(T_j)$. Then the cyclically reduced form of the following path suffices

$$\alpha' = \underbrace{[v, o(e_0)]_{T_1}\ e_0\ [t(e_0), o(e_j)]_{T_j}\ e_j\ [t(e_j), t(e_0)]_{T_j}\ \overline{e_0}\ [o(e_0), v]_{T_1}}_{\eta}\ \alpha.$$

Note the amount of cyclic reduction between $[\eta]$ and α is at most $2 \operatorname{diam} \Gamma_1$. Also, $[[\alpha']]$ crosses e_j exactly once (see Corollary 3.4.4). Finally suppose that for every $e \in E(H)$ for which $o(e) \in V(\Gamma_1)$, we have that $t(e)$ lies in $V(\Delta^*)$, and Δ^* is a forest. Since G is connected, either there is an edge $e^* \in E(H)$ for which $o(e^*) \in V(\Gamma_i)$, $t(e^*) \in V(\Gamma_j)$ with $i \neq j$ and both Γ_i, Γ_j are trees in Δ^* or there is a pair of edges e_0, e_1 with $o(e_0), o(e_1) \in V(\Gamma_1)$ and $t(e_0), t(e_1) \in V(\Gamma_i)$ with Γ_i a tree in Δ^*. In the former case, the cyclically reduced form of the following path suffices

$$\alpha' = \underbrace{[v, o(e_i)]_{T_1}\ e_i\ [t(e_i), o(e^*)]_{\Gamma_i}\ e^*\ [t(e^*), t(e_j)]_{\Gamma_j}\ \overline{e_j}\ [o(e_j), v]_{T_1}}_{\eta}\ \beta.$$

In the latter case, the cyclically reduced form of the following path suffices

$$\alpha' = \underbrace{[v, o(e_0)]_{T_1}\ e_0\ [t(e_0), t(e_1)]_{T_i}\ \overline{e_2}\ [o(e_2), v]_{T_1}}_{\eta}\ \beta. \qquad \square$$

Lemma 3.4.7. *Let $N \geq 2$. Let $\varphi \in \operatorname{Out}(F_N)$ be such that neither φ nor φ^{-1} is an NEG-automorphism. Then there is a free basis $\mathcal{A}$ and an $a \in \mathcal{A}$ for which $\alpha := [[\tau(a)]]$ crosses H_t and $\alpha' := [[\tau'(a)]]$ crosses $H'_{t'}$.*

Proof. Let $f \colon G \to G$, $f' \colon G' \to G'$ be as per hypothesis. By Proposition 3.4.5, there exists a cyclically reduced path α in G such that α crosses $e_0 \in E(H_t)$ at least M times (M to be chosen later), and represents a primitive element in F_N. Write $\alpha = [[\tau(a)]]$ for $a \in F_N$. If $\alpha' := [[\tau'(a)]]$ crosses $H'_{t'}$, then we are done. Else assume that $\alpha' \subseteq G'_{t'-1}$. Let $v \Subset \alpha'$ be a vertex, and write $o(\alpha') = t(\alpha') = v$. Define $\alpha'' := \eta\ \alpha'$ where η is as per Proposition 3.4.6. Let $\upsilon \colon G' \to G$ be the difference of markings. Note $[[\upsilon(\alpha')]] = \alpha$. Let $\operatorname{BCC}(\upsilon)$ be the bounded cancellation constant for υ (see Lemma 2.4.1). Choose $M > 2(\operatorname{diam} G' + \operatorname{BCC}(\upsilon))$ so as to ensure enough of $[\upsilon(\alpha')]$ remains when we cyclically reduce to $[[\upsilon(\alpha'')]]$. More precisely, write

$$[[\upsilon(\alpha'')]] = [[\upsilon(\eta\ \alpha')]] = [[\ [\upsilon(\eta)]\ [\upsilon(\alpha')]\ ]]$$

By Lemma 3.4.6, cyclic reduction between $[\eta]$ and α is at most $2\operatorname{diam} G'$. By Lemma 2.4.1, cyclic reduction between $[\upsilon(\eta)]$ and $[\upsilon(\alpha')]$ is at most $2\operatorname{BCC}(\upsilon)$. Since the number of occurrences of e_0 in $\alpha = [[\upsilon(\alpha')]]$ is greater than M, at least one occurrence of e_0 must remain in $[[\upsilon(\alpha'')]]$. □

Remark 3.4.8. The conclusion of Lemma 3.4.7 holds for weaker hypotheses, however we only make use of it in its current form.

Theorem 3.4.9. *Let $N \geq 2$. Let $\varphi \in \operatorname{Out}(F_N)$ be such that neither φ nor φ^{-1} is an NEG-automorphism. Then φ has $\mathcal{P}$.*

Proof. Let $\mathcal{A}$, and $a \in \mathcal{A}$ be as in the conclusion of Lemma 3.4.7. Let $f\colon G \to G, f'\colon G' \to G'$ be good relative train tracks for φ, φ^{-1}, respectively. Let $g \in F_N$ be arbitrary. By Lemma 3.4.2 and Lemma 3.4.1, there is $M \geq 1$ so that if $[[\tau(a)]]^k \Subset [f^n([[\tau(g)]])]$ for any $n \geq 1$, then $|k| \leq M$. Thus by Corollary 3.2.11 and Corollary 3.2.7, φ has $\mathcal{P}^+(\mathcal{A}, a)$. Similarly, Lemma 3.4.2, Lemma 3.4.1, Corollary 3.2.11, and Corollary 3.2.7 imply that φ^{-1} has $\mathcal{P}^+(\mathcal{A}, a)$, that is, φ has $\mathcal{P}^-(\mathcal{A}, a)$. Thus φ has $\mathcal{P}(\mathcal{A}, a)$. □

3.5 NEG-Automorphisms

Lemma 3.5.1. *Let $N \geq 2$. Let $\varphi \in \operatorname{Out}(F_N)$ be an NEG-automorphism and $f\colon G \to G$ a good relative train track for φ. Let $g \in F_N$ be arbitrary. Let $\gamma := [[\tau(g)]]$. Let $a \in F_N$ be any primitive element for which $\alpha := [[\tau(a)]]$ crosses the top strata, H_t. Then $\exists k \geq 1$, such that $\forall n \geq 1$, we have $\alpha^k \not\Subset [f^n(\gamma)]$.*

Proof. By Theorem 2.2.5, we may assume that H_t consists of a single edge, e_0, and that $f(e_0) = e_0 u$ where $u \subseteq G_{t-1}$ is a closed path whose basepoint is fixed by f. Thus

$$\begin{aligned}
[f^j(e_0)] = [f^{j-1}(e_0 u)] &= [f^{j-1}(e_0)][f^{j-1}(u)] \\
&= [f^{j-2}(e_0 u)][f^{j-1}(u)] \\
&= [f^{j-2}(e_0)][f^{j-2}(u)][f^{j-1}(u)] \\
&\vdots \\
&= e_0 u [f(u)][f^2(u)] \cdots [f^{j-1}(u)].
\end{aligned}$$

Therefore, the number of occurrences of e_0 in $[f^j(\gamma)]$ is bounded by the number of occurrences of e_0 in γ. The lemma follows. □

Lemma 3.5.2. *Let $N \geq 2$. Let $\varphi \in \mathrm{Out}(F_N)$ be so that at least one of φ, φ^{-1} is an NEG-automorphism. Then there is a good relative train track $f\colon G \to G$ and a topological representative $f'\colon G \to G$ representing φ and φ^{-1}, not necessarily respectively, so that the following hold.*

1. $f'|_{G\backslash\{e_0\}}\colon G\backslash\{e_0\} \to G\backslash\{e_0\}$ *is a homotopy inverse for* $f|_{G_{t-1}}\colon G_{t-1} \to G_{t-1}$.

2. $f'(e_0) = e_0u'$ *for some* $u' \Subset G\backslash\{e_0\}$.

Proof. Without loss of generality, assume that φ is an NEG-automorphism equipped with $f\colon G \to G$. By Theorem 2.2.5, we may assume $E(H_t) = e_0$, a single edge and that $f(e_0) = e_0u$ where $u \subseteq G_{t-1}$ is a closed path whose basepoint is fixed by f. Note also that $o(e_0)$ is also fixed by f. Let us first consider the case where e_0 is a G-separating edge. Pass to the square of f so that each component of $G_{t-1} = G_1 \sqcup G_2$ is fixed. Without loss, say $o(e_0) \in V(G_1)$ and $t(e_0) \in V(G_2)$. Let $\eta\colon \pi_1(G_1, o(e_0)) \to \pi_1(G_1, o(e_0))$ and $\zeta\colon \pi_1(G_2, t(e_0)) \to \pi_1(G_2, t(e_0))$ be the isomorphisms induced by f. Let $u' \in \pi_1(G_2, t(e_0))$ be so that $\zeta(u') = \overline{u}$. Let $f'_{\eta^{-1}}, f'_{\zeta^{-1}}$ be topological representatives for η^{-1} and ζ^{-1} which fix the basepoints $o(e_0), t(e_0)$, respectively. Put

$$f'\colon e \mapsto \begin{cases} f'_{\eta^{-1}}(e) & \text{if} \quad e \in \Gamma_1 \\ f'_{\zeta^{-1}}(e) & \text{if} \quad e \in \Gamma_2 \\ eu' & \text{if} \quad e = e_0 \end{cases}.$$

Then

$$(f' \circ f) : e_0 \mapsto e_0u' \mapsto e_0u\overline{u},$$

which reduces to e_0. Evidently f, f' are the desired maps. The other case follows similarly. □

Remark 3.5.3. Using the topological representative from Lemma 3.5.2, one can now obtain the conclusion of Lemma 3.5.1 for φ^{-1}.

Theorem 3.5.4. *Let $N \geq 2$. Let $\varphi \in \mathrm{Out}(F_N)$ be so that at least one of φ, φ^{-1} is an NEG-automorphism. Then φ has $\mathcal{P}$.*

Proof. Without loss, say φ is an NEG-automorphism, and let $a \in F_N$ be as per Lemma 3.5.1. Then by Corollary 3.2.11 and Corollary 3.2.7 φ has $\mathcal{P}^+(\mathcal{A}, a)$ where $\mathcal{A}$ is a free basis of F_N with $a \in \mathcal{A}$. Corollary 3.2.11, Corollary 3.2.7 and the topological representative constructed in Lemma 3.5.2 ensure that φ has $\mathcal{P}^-(\mathcal{A}, a)$ (see Remark 3.5.3). Thus φ has $\mathcal{P} = \mathcal{P}(\mathcal{A}, a)$. □

3.6 The case of F_2

We now handle the case where $N = 2$. As mentioned earlier, Cohen, Lustig, and Steiner [12] showed that no finite subset of F_2 is spectrally rigid. In fact, their argument works for any $\Sigma \subseteq F_2$ (possibly infinite) which satisfies a property, denoted $\mathcal{W}^*$, which we define below.

Definition 3.6.1 (Property $\mathcal{W}^*$)**.** Let $\Sigma \subseteq F_2$. We say Σ has property $\mathcal{W}^*$ if there is a free basis $F_2 = F(a,b)$ and $k \geq 1$ so that for all $\sigma \in \Sigma$, the following two conditions hold.

1. We have $[[\sigma]]_{\{a,ba^k\}} \notin \langle ba^k \rangle$.

2. If $(ba^k)^t \Subset [[\sigma]]_{\{a,ba^k\}}$, then $t = \pm 1$.

In view of the argument in [12], the following is immediate.

Proposition 3.6.2. *Suppose $\Sigma \subseteq F_2$ has property $\mathcal{W}^*$. Then Σ is not spectrally rigid.* □

The following proposition allows us to extend our earlier arguments to the case of F_2.

Proposition 3.6.3. *Suppose $\Sigma \subseteq F_2$ has property $\mathcal{W}$. Then Σ has property $\mathcal{W}^*$.*

Proof. Let $\mathcal{A}, a \in \mathcal{A}$ and M be as per property $\mathcal{W}$. Write $\mathcal{A} = \{a, b\}$. Choose $k > M$. We first verify property 1 of Definition 3.6.1. Suppose $[[\sigma]]_{\{a,ba^k\}} = (ba^k)^t$ for $t \neq 0$. Then $[[\sigma]]_{\mathcal{A}}$ contains a^k (or a^{-k}) as a subword. As $k > M$, this violates property $\mathcal{W}$. Thus $[[\sigma]]_{\{a,ba^k\}} \notin \langle ba^k \rangle$. Note that if $[\sigma]_{c\mathcal{A}} = a^l$, then $[\sigma]_{c\{a,ba^k\}} = a^l$, and so condition 2 is vacuously satisfied. So write $[[\sigma]]_{\mathcal{A}} = a^{k_0^a} b^{k_0^b} \cdots a^{k_{n-1}^a} b^{k_{n-1}^b}$ where $k_{n-1}^b \neq 0$. Then

$$[[\sigma]]_{\{a,ba^k\}} = a^{k_0^a}((ba^k)(a^{-k}))^{k_0^b} \cdots a^{k_{n-1}^a}((ba^k)(a^{-k}))^{k_{n-1}^b}$$

Property $\mathcal{W}$ gives that $k_i^a \leq M$ for all $1 \leq i \leq n-1$. Since $k > M$, we have that $|k_i^a - k| \neq 0$ for all i. Thus we conclude that a nontrivial power of a remains between successive occurrences of ba^k. Thus we have property 2 of Definition 3.6.1. □

3.7 Proof of the Main Theorem

We can now prove Theorem A, whose statement we recall for convenience.

Theorem A. *Let $N \geq 2$. Let $\Phi \in \mathrm{Aut}(F_N)$. Let $g \in F_N$ be arbitrary. Then the set $\Sigma = \{\Phi^n(g)\}_{n \in \mathbb{Z}}$ is not spectrally rigid.*

Proof of Theorem A. Suppose that $N \geq 2$ and let $\Phi \in \mathrm{Aut}(F_N)$ be arbitrary. Let $[\Phi] = \varphi \in \mathrm{Out}(F_N)$. If neither φ nor φ^{-1} is an NEG-automorphism, then φ has $\mathcal{P}$ by Theorem 3.4.9. If at least one of φ or φ^{-1} is an NEG-automorphism, then φ has $\mathcal{P}$ by Theorem 3.5.4. Thus φ has $\mathcal{P}$. Therefore, by Proposition 3.2.3, for any $\Psi \in \varphi$ and for any $g \in F_N$ the set $\Sigma = \{\Psi^n(g)\}_{n\in\mathbb{Z}}$ has $\mathcal{W}$. Since $\Phi \in \varphi$, the set $\Sigma = \{\Phi^n(g)\}_{n\in\mathbb{Z}}$ has $\mathcal{W}$. If $N \geq 3$, then by Proposition 2.5.3, Σ is not spectrally rigid. If $N = 2$, then by Proposition 3.6.3, Σ has $\mathcal{W}^*$. Thus by Proposition 3.6.2, Σ is not spectrally rigid. $\square$

Chapter 4

Spectral Rigidity and Subgroups of Free Groups

4.1 Background

Definition 4.1.1 ((Strongly) Spectrally rigid). We say $\Sigma \subseteq F_N$ is (*strongly*) *spectrally rigid* if whenever $T_1, T_2 \in \mathrm{cv}_N$ (resp. $\overline{\mathrm{cv}}_N$) are such that $\|g\|_{T_1} = \|g\|_{T_2}$ for every $g \in \Sigma$, then $T_1 = T_2$ in cv_N (resp $\overline{\mathrm{cv}}_N$).

Recall that for $N \geq 3$ Smillie and Vogtmann [43] prove that no finite set is spectrally rigid in F_N. More specifically, they show that given a finite set, Σ, of conjugacy classes there exists a one parameter family of trees, $T_t \in \mathrm{cv}_N$, whose translation length functions all agree on Σ. For $N = 2$, Cohen, Lustig and Steiner [12] provide a similar argument as to why no finite set is spectrally rigid in F_N. Recall

Theorem 4.1.2 (CLS, R, SV[12, 39, 43]). *For $N \geq 2$, if a subset $\Sigma \subset F_N$ has Property $\mathcal{W}$, then Σ is not spectrally rigid.*

We prove the following.

Theorem B. *Let $N \geq 2$. Let $\{H_i\}_{i=1}^k$ be a finite collection of finitely generated subgroups $H_i < F_N$. Let $\{g_i\}_{i=1}^k$ be a finite collection of elements $g_i \in F_N$. Let $\mathcal{H} = \cup_{i=1}^k g_i H_i$. Then the following are equivalent.*

1. *For every i, $[F_N : H_i] = \infty$.*

2. *$\mathcal{H} \subset F_N$ is not spectrally rigid in F_N.*

In the above theorem, we recover the results of Smillie, Vogtmann, Cohen, Lustig, and Steiner by setting $H_i = \{1\}$. The proof by Smillie and Vogtmann uses the fact that in a finite set of conjugacy classes, there is a universal bound on the exponent with which a given primitive element may occur. Their argument, however, works for *any* set Σ which satisfies the following property: there exists a triple $(\mathcal{A}, a, M)$ with $\mathcal{A}$ a free basis, $a \in \mathcal{A}$, and $M \geq 1$, so that for any $g \in \Sigma$, if a^k occurs as a subword of the cyclically reduced form (over $\mathcal{A}$) of g, then $|k| \leq M$ (see Definition 2.5.2 and Theorem 4.1.2). To prove Theorem **??**, we use *laminations associated to a fully irreducible automorphism.* Introduced by Bestvina, Feighn, and Handel in [3] these laminations are defined – for a given fully irreducible $\varphi \in \mathrm{Out}(F_N)$ – in terms of a train track representative $f\colon \Gamma \to \Gamma$. Leaves of the lamination are "generated" by edgepaths of the form $f^n(e)$ for any $e \in E(\Gamma)$ and $n \in \mathbb{N}$ (see [3, 27]). It is a theorem of Bestvina, Feighn, and Handel [3] that only finitely generated subgroups of finite index may "carry a leaf" of such a lamination. It follows that, for finitely generated infinite index subgroups, edgepaths of the form $f^n(e)$ (for suitably large n) can not be "read" in H. Thus if a primitive loop contains such an $f^n(e)$, then it cannot be a subword of any $h \in H$ (compare Definition 2.5.2).

Theorem B fails if we allow infinite unions. Indeed, consider $\mathcal{H} = \cup_{g \in F_N} g\{1\}$. Then $\mathcal{H} = F_N$ and so is spectrally rigid. Since translation length functions satisfy $\|g^n\| = n\|g\|$, any finite index subgroup is spectrally rigid. More generally, if $\pi\colon F_N \to G$ is a presentation homomorphism ($G \cong F_N/\ker\pi$) for a torsion group, G, then $\ker\pi$ is spectrally rigid. Note that this applies to (normal) subgroups of infinite index (in the case G is infinite), and these subgroups are necessarily infinitely generated. Thus the finite generation assumption in Theorem B is also essential.

Carette, Francaviglia, Kapovich, and Martino prove [10] that if $H < \mathrm{Aut}(F_N)$ is such that the image of H in $\mathrm{Out}(F_N)$ contains an infinite normal subgroup, then for any $g \in F_N$ (so long as $N \neq 2$ or g not conjugate to a power of $[a,b]$ in $F_2 = F(a,b)$) the orbit Hg is spectrally rigid. This shows, for example, that the commutator subgroup, $[F_N, F_N]$, of F_N is spectrally rigid, as it contains the orbit $\mathrm{Aut}(F_N)[g,h]$ for a suitable commutator $[g,h]$. More generally, any nontrivial verbal or marginal subgroup is spectrally rigid since such groups are known to be characteristic [41, §2.3].

Motivated by the above result regarding characteristic subgroups, Ilya Kapovich asked the following question: Is it true that for any nontrivial normal subgroup $H \lhd F_N$, H is a spectrally rigid set? We prove the following.

Theorem C. *Let $H \lhd F_N$ be a nontrivial normal subgroup. Then for any $g \in F_N$, the coset gH is strongly spectrally rigid.*

The proof uses *geodesic currents* on free groups (see [29, 32, 33, 37]). These are positive Radon measures on $\partial^2 F_N = \partial F_N \times \partial F_N \backslash \Delta$ (where Δ is the diagonal) which are invariant under the diagonal action of F_N and under the map which exchanges the coordinates. The space of all such currents is denoted $\mathrm{Curr}(F_N)$. The spaces $\mathrm{Curr}(F_N)$ and $\overline{\mathrm{cv}}_N$ are related by an *intersection form* [32], $\langle \cdot, \cdot \rangle \colon \overline{\mathrm{cv}}_N \times \mathrm{Curr}(F_N) \to \mathbb{R}_{\geq 0}$. The main feature of the intersection form that we use is as follows: given a conjugacy class $g \in F_N$, there is a canonically associated current $\eta_g \in \mathrm{Curr}(F_N)$ such that for any $T \in \overline{\mathrm{cv}}_N$, we have $\langle T, \eta_g \rangle = \|g\|_T$. Now given any nontrivial $g, r \in F_N$, we show that the normal closure, $\mathrm{ncl}(r)$, of r contains a sequence of words w_n such that large powers of g exhaust w_n. Linearity of the intersection form then reveals that the associated counting currents converge to the counting current of g, and that if two trees T, T' agree on $\mathrm{ncl}(r)$ (and hence on the w_n), then they must agree on g. Since g is arbitrary, we conclude $T = T'$. Our result then follows from the fact that any nontrivial normal subgroup contains $\mathrm{ncl}(r)$ for some nontrivial r.

4.2 Proof of Theorem B

In what follows, whenever we speak of an iwip $\varphi \in \mathrm{Out}(F_N)$ equipped with train track map $f \colon \Gamma \to \Gamma$ we assume that we have passed to a suitable power of φ so that for any topological edge e in Γ, $f(e)$ crosses each topological edge in Γ. We will, however, continue to write φ, f.

Proposition 4.2.1. *Let $H \leq F_N$ be a finitely generated subgroup of infinite index. Let $\varphi \in \mathrm{Out}(F_N)$ be fully irreducible with train track map $f \colon \Gamma \to \Gamma$ on the marked graph (Γ, τ). Let $T = (\Gamma, \tau, l)$ in cv_N. Let X_H^T be the Stallings subgroup graph corresponding to the conjugacy class of $\tau(H)$ in $\pi_1(\Gamma)$. Then there exists a power m, such that for all $n \geq m$, and for any $e \in E(\Gamma)$, the edgepath $f^n(e)$ does not factor through X_H^T. Furthermore, given a basepoint b in $(X_H^T)_r$, we may choose m so that for all $n \geq m$, $f^n(e)$ does not factor through $(X_H^T)_b$.*

Proof. By Proposition 2.7.5, H can not carry a leaf of $\mathcal{L}_{BFH}(\varphi, f, \Gamma)$. Thus there is a finite subpath $[z_1, z_2]$ (of a leaf $(\zeta_1, \zeta_2) \subset \partial^2 \widetilde{\Gamma}$) which does not factor through X_H^T. Let $L = |[z_1, z_2]|$. Since leaves are quasiperiodic, there exists L' so that for any subpath $[w_1, w_2]$ of length at least L', we have $[w_1, w_2] \Supset [z_1, z_2]$. Choose a basepoint $v \in \Gamma$ and let b be the basepoint in $(X_H^T)_r$ so that the image of $\pi_1((X_H^T)_b)$ in $\pi_1(\Gamma, v)$ is equal to $\tau(H)$. Let l be the length of the bridge $[X_H^T, b]$. Set $L'' = L' + 2l + 2\,\mathrm{diam}(X_H^T) + 1$. Now let $m \geq 1$ be so that the length of $f^m(e)$ is at least L'' for any $e \in E(\Gamma)$. Then for any $n \geq m$, $f^n(e) \Supset [z_1, z_2]$ for any $e \in E(\Gamma)$. Thus for any $n \geq m$, and for any $e \in E(\Gamma)$, we have that $f^n(e)$ does not factor through X_H^T, nor can it factor through $(X_H^T)_b$ since at least $2l + 1$ edges in $f^n(e)$ must lie outside of X_H^T. □

Corollary 4.2.2. *Let $\mathcal{H} = \cup H_i$ be a finite union of subgroups $H_i < F_N$ of infinite index. Then there exists a power m such that for all $n \geq m$ and for any $e \in E(\Gamma)$ and for all i, the edgepath $f^n(e)$ does not factor through $(X_{H_i}^T)_b$. Furthermore, there exists a primitive element $z \in \pi_1(\Gamma)$ so that z contains $f^m(e)$ and hence z does not factor through $(X_{H_i}^T)_b$ for any i.*

Proof. Fix a fully irreducible $\varphi \in \mathrm{Out}(F_N)$ equipped with train track map $f \colon \Gamma \to \Gamma$. Fix an edge $e \in E(\Gamma)$. Now apply Proposition 4.2.1 to each H_i and obtain a collection $\{m_i\}_{i=1}^k$ of exponents for which the following holds for each i: for all $n \geq m_i$ the edgepath $f^n(e)$ does not factor through $(X_{H_i}^T)_b$. Set $m = \max_i\{m_i\}$. Then for all $n \geq m$ and for all i we have that $f^n(e)$ does not factor through $(X_{H_i}^T)_b$.

Now let a be any primitive loop in $\pi_1(\Gamma)$. Since φ is fully irreducible, z cannot be a periodic conjugacy class, and hence $l_\Gamma([[f^k(a)]]) \to \infty$. Thus we can find an edge e' in suitable iterate, $f^k(a)$, of a so that in all further iterates of a by f there is no cancellation in the images of e'. Thus for all $n \geq k+m+1$, we have that $f^m(e)$ occurs in $f^n(a)$ and hence $f^n(a)$ cannot factor through $(X^T_{H_i})_b$ for any i. Set $z = f^n(a)$ for some $n \geq k+m+1$. □

Proposition 4.2.3. *Let $\mathcal{H} = \cup H_i$ be a finite union of subgroups $H_i < F_N$ of infinite index. Then there exists a primitive element z and a free basis $\mathcal{B}$ with $z \in \mathcal{B}$ and a number M so that for all i and for all $h \in H_i$, if $z^k \Subset [h]_\mathcal{B}$, then $|k| \leq M$.*

Proof. Let z be the primitive element afforded by Corollary 4.2.2. Let $Z = \{z^k \mid k \in \mathbb{Z}\}$. Let $B = \{\tau(h) \mid h \in H\}$. The previous proposition gives the following: there exists a constant C so that for any $\zeta \in Z$ and for any $\beta \in B$ and for any choice of lifts $\widetilde{\zeta}, \widetilde{\beta}$ in $\widetilde{\Gamma}$, the intersection $\mathrm{diam}(\widetilde{\zeta} \cap \widetilde{\beta}) < C$. Let $f\colon \widetilde{\Gamma} \to \mathrm{Cay}(F_N, \mathcal{B})$ be a quasi-isometry to the Cayley graph of F_N with respect to a basis $\mathcal{B}$ with $z \in \mathcal{B}$. Now apply Proposition 2.8.1 using Z, B and f. We conclude there exists a bound M so that for any $h \in \mathcal{H}$, if $z^k \Subset [h]_\mathcal{B}$, then $|k| \leq M$. □

Theorem 4.2.4. *Let $\mathcal{H} = \cup g_i H_i$ be a finite collection of cosets of finitely generated subgroups of infinite index, $H_i \leq F_N$. Then the set $\mathcal{H}$ has property $\mathcal{W}$.*

Proof. Let $z, \mathcal{B}$ be the primitive element and free basis afforded by Proposition 4.2.3. Thus there exists an M' such that for all i and all $h \in H_i$ if $z^k \Subset [h_i]_\mathcal{B}$, then $|k| \leq M'$. Since $\{g_i\}_{i=1}^n$ is finite, there is M_g so that for all i, if $z^k \Subset [g_i]_\mathcal{B}$, then $|k| \leq M_g$. Thus there is an M so that for any i and any $h_i \in H_i$, if $z^k \Subset [[g_i h_i]]_\mathcal{B}$, then $|k| \leq M$. Thus H has property $\mathcal{W}$ with respect to $(z, \mathcal{B}, M)$. □

Since $\|g^n\| = n\|g\|$ it is clear that any (coset of a) subgroup of finite index is spectrally rigid. We thus have the following.

Theorem B. *Let $N \geq 2$. Let $\{H_i\}_{i=1}^k$ be a finite collection of finitely generated subgroups $H_i < F_N$. Let $\{g_i\}_{i=1}^k$ be a finite collection of elements $g_i \in F_N$. Let $\mathcal{H} = \cup_{i=1}^k g_i H_i$. Then the following are equivalent.*

1. *For every i, $[F_N\colon H_i] = \infty$.*

2. *$\mathcal{H} \subset F_N$ satisfies property $\mathcal{W}$.*

3. *$\mathcal{H} \subset F_N$ is not spectrally rigid in F_N.*

Proof. Theorem 4.2.4 gives (1) implies (2). Theorem 4.1.2 gives (2) implies (3). Finally, if at least one subgroup is of finite index (i.e. not (1)), then $\mathcal{H}$ is spectrally rigid (i.e. not (3)). □

4.3 Proof of Theorem C

Recall that the space, $\mathrm{Curr}(F_N)$, of geodesic currents consists of positive radon measures on $\partial^2 F_N$ which are F_N and flip invariant. Also recall that $\mathrm{Curr}(F_N)$ and $\overline{\mathrm{cv}}_N$ are related by an intersection form $\langle \cdot, \cdot \rangle$ for which $\langle \eta_g, T \rangle = \|g\|_T$ (see §2.9 and Proposition 2.9.1).

Proposition 4.3.1 (Kapovich and Lustig [31, Lemma 2.10]). *Let $\mathcal{A}$ be a free basis of F_N. Then for any cyclically reduced words $w_n, w \in F_N$, we have*

$$\lim_{n\to\infty} [\eta_{w_n}] = [\eta_w] \qquad \textit{if and only if} \qquad \lim_{n\to\infty} \frac{\eta_{w_n}}{\|w_n\|_{\mathcal{A}}} = \frac{\eta_w}{\|w\|_{\mathcal{A}}}$$

Proposition 4.3.2. *For any nontrivial $r \in F_N$, the set $\mathrm{ncl}(r)$ is strongly spectrally rigid.*

Proof. Let $u \in F_N$ be an arbitrary cyclically reduced word. We will show that $\|u\|_T = \|u\|_{T'}$ from which it will follow that $T = T'$ (since $\|\cdot\|$ is constant on conjugacy classes). To that end, we first construct a sequence of cyclically reduced words $\{w_i\} \subset \mathrm{ncl}(r)$ with the property that

$$[\eta_{w_i}] \to [\eta_u]$$

To that end, let

$$w_i = \alpha u^i \beta r \beta^{-1} u^{-i} \alpha^{-1} \gamma u^i \delta r \delta^{-1} u^{-i} \gamma^{-1}$$

Evidently w_i is a product of conjugates of r, so is in $\mathrm{ncl}(r)$. It is clear that we may choose $\alpha, \beta, \delta, \gamma$ so that w_i is freely (and so cyclically, by inspection) reduced. Now let $v \in F_N$ be arbitrary. By Proposition 4.3.1 it is enough to show that

$$\lim_{i\to\infty} \frac{n_{w_i}(v^{\pm})}{\|w_i\|_{\mathcal{A}}} = \frac{n_u(v^{\pm})}{\|u\|_{\mathcal{A}}}$$

Note that there is a uniform bound on the number of occurrences of v or v^{-1} in w_i which do not occur entirely within the u^i or u^{-i} subwords of w_i. Furthermore the difference between $\|w_i\|_{\mathcal{A}}$ and $4\|u^i\|_{\mathcal{A}}$ is uniformly bounded. We now have that

$$\begin{aligned}
\lim_{i\to\infty} \frac{n_{w_i}(v^{\pm})}{\|w_i\|_{\mathcal{A}}} &= \lim_{i\to\infty} \frac{2n_{u^i}(v^{\pm}) + 2n_{u^{-i}}(v^{\pm}) + D_i}{4i\|u\|_{\mathcal{A}} + C_i} \\
&= \lim_{i\to\infty} \frac{2in_u(v^{\pm}) + 2in_{u^{-1}}(v^{\pm}) + D_i}{4i\|u\|_{\mathcal{A}} + C_i} \\
&= \lim_{i\to\infty} \frac{2in_u(v^{\pm}) + 2in_u(v^{\pm}) + D_i}{4i\|u\|_{\mathcal{A}} + C_i} \\
&= \lim_{i\to\infty} \frac{4in_u(v^{\pm}) + D_i}{4i\|u\|_{\mathcal{A}} + C_i} = \frac{n_u(v^{\pm})}{\|u\|_{\mathcal{A}}}
\end{aligned}$$

as desired.

Now suppose that $T, T' \in \overline{\text{cv}}_N$ agree on $\text{ncl}(r)$. Since $[\eta_{w_i}] \to [\eta_u]$ there exists a sequence λ_i such that

$$\lim_{i\to\infty} \lambda_i \eta_{w_i} = \eta_u$$

Then we have that

$$\begin{aligned}
\|u\|_T = \langle T, \eta_u \rangle = \langle T, \lim_{i\to\infty} \lambda_i \eta_{w_i} \rangle &= \lim_{i\to\infty} \lambda_i \langle T, \eta_{w_i} \rangle \\
= \lim_{i\to\infty} \lambda_i \|w_i\|_T = \lim_{i\to\infty} \lambda_i \|w_i\|_{T'} &= \lim_{i\to\infty} \lambda_i \langle T', \eta_{w_i} \rangle \\
= \langle T', \lim_{i\to\infty} \lambda_i \eta_{w_i} \rangle = \langle T', \eta_u \rangle = \|u\|_{T'}
\end{aligned}$$

Recall that u was an arbitrary conjugacy class. Since length functions are class functions, we conclude that $\|\cdot\|_T = \|\cdot\|_{T'}$ and so $T = T'$ in $\overline{\text{cv}}_N$. □

Corollary 4.3.3. *Let $r \in F_N$ be nontrivial and consider $g\,\text{ncl}(r)$. Then $g\,\text{ncl}(r)$ is strongly spectrally rigid.*

Proof. In the proof of the above proposition, we may choose $\alpha, \beta, \gamma, \delta$ so that gw_i is cyclically reduced. Then – with perhaps different constants C_i, D_i – we obtain that $[\eta_{gw_i}] \to [\eta_u]$. We conclude as above that $g\,\text{ncl}(r)$ is strongly spectrally rigid. □

Theorem C. *Let $H \lhd F_N$ be a nontrivial normal subgroup. Then for any $g \in F_N$, the coset gH is strongly spectrally rigid.*

Proof. Since H is nontrivial and normal, $H \supset \text{ncl}(r)$ for some nontrivial r. By the above proposition, for any $g \in F_N$, $g\,\text{ncl}(r)$ is strongly spectrally rigid. Thus so is gH. □

Chapter 5

Almost Simple Curves Afford Strong Relative Rigidity

5.1 Background

In [10], Carette, Francaviglia, Kapovich, and Martino prove the following theorem.

Theorem 5.1.1 (CFKM [10, Theorem B]). *Let $T \in \mathrm{cv}_N$ be arbitrary. There exists a finite set $\mathcal{C}(T)$ (depending on T) of primitive elements of F_N with the following property: whenever $T' \in \mathrm{cv}_N$ is such that $\|g\|_{T'} = \|g\|_T$ for every $g \in \mathcal{C}(T)$, then $T = T'$ in cv_N.*

Such a set is said to afford *relative rigidity* for T. If the same property holds for all T' in the *closure*, $\overline{\mathrm{cv}}_N$, of cv_N, then we say that the set affords *strong* relative rigidity for T.

Question 5.1.2 (CFKM [10, Problem 6.6]). *Let $T \in \mathrm{cv}_N$ be arbitrary. Does there exist a finite strongly relatively rigid set for T?*

We show that the set $\mathcal{C}(T)$ provided by Theorem 5.1.1 does, in fact, afford strong relative rigidity. More precisely,

Theorem D. *Let $T \in \mathrm{cv}_N$ be arbitrary. There exists a finite set $\mathcal{C}(T)$ (depending on T) of primitive elements of F_N with the following property: whenever $T' \in \overline{\mathrm{cv}}_N$ is such that $\|g\|_{T'} = \|g\|_T$ for every $g \in \mathcal{C}(T)$, then $T = T'$ in $\overline{\mathrm{cv}}_N$. In other words, there exists a finite strongly relatively rigid set for T.*

5.2 Proof of Theorem D

Given $T \in \mathrm{cv}_N$, Carette, Francaviglia, Kapovich, and Martino [10] proved the existence of a finite relatively rigid set for T. We denote this set by $\mathcal{C}(T)$ and recall that it consists of those $g \in F_N$ for which $g|T$ is an embedded loop, an immersed figure 8, or a "barbell" curve. For $T, T' \in \overline{\mathrm{cv}}_N$ define $\tau(T, T') = \sum_{g \in \mathcal{C}(T)} |\|g\|_T - \|g\|_{T'}|$. Note for $T, T' \in \mathrm{cv}_N$ we have

$$\tau(T, T') = 0 \iff \|g\|_T = \|g\|_{T'} \text{ for every } g \in \mathcal{C}(T) \iff T = T' \text{ in } \mathrm{cv}_N$$

$$\tau(T',T) = 0 \iff \|g\|_T = \|g\|_{T'} \text{ for every } g \in \mathcal{C}(T') \iff T = T' \text{ in } \mathrm{cv}_N$$

but $\tau(T,T') \neq \tau(T',T)$ in general since the sets $\mathcal{C}(Y)$ depend on $Y \in \mathrm{cv}_N$.

For $T \in \mathrm{cv}_N$ and $T' \in \overline{\mathrm{cv}}_N$ denote

$$\Lambda_R(T,T') := \sup_{1 \neq g \in F_N} \frac{\|g\|_{T'}}{\|g\|_T}$$

We will need the following two results.

Proposition 5.2.1 (Francaviglia and Martino [21, Proposition 3.15])**.** *Let $T, T' \in \mathrm{cv}_N$ be arbitrary. Then*

$$\Lambda_R(T,T') := \max_{g \in \mathcal{C}(T)} \frac{\|g\|_{T'}}{\|g\|_T}$$

Proposition 5.2.2 (CFKM [9, Proposition 3.5])**.** *Let $T, T' \in \mathrm{cv}_N$. Let $L = \Lambda_R(T,T')$. Then there exists an F_N-equivariant L-Lipschitz map $f\colon T \to T'$ such that:*

1. *On each edge e of T, the map f is a linear map with constant stretch $L_e \geq 0$.*
2. *There exists $g \in \mathcal{C}(T)$ such that for every edge $e \in A_g \subset T$, we have that $L_e = L$ and $f|_{A_g}$ is an L-homothety.*

We now prove the main technical result needed to establish Theorem D.

Proposition 5.2.3. *Suppose $T, T_n \in \mathrm{cv}_N$ are such that $\tau(T,T_n) \to 0$. Then $T_n \to T$.*

Proof. Let $\Gamma = T/F_N$ and $\Gamma_n = T_n/F_N$. By $g|\Gamma$ we mean the cyclically reduced edgepath in Γ corresponding to $g \in F_N$ under the marking for T. Recall

$$\begin{aligned}\Lambda_R(T,T_n) &= \sup_{1\neq g} \frac{\|g\|_{T_n}}{\|g\|_T} \\ &= \max_{\mathcal{C}(T)} \frac{\|g\|_{T_n}}{\|g\|_T}\end{aligned}$$

by Proposition 5.2.1. Now $\{\|g\|_T \mid g \in \mathcal{C}(T)\}$ is finite and positive. Thus

$$\tau(T,T_n) \to 0 \quad \Rightarrow \quad \Lambda_R(T,T_n) \to 1$$

Let $f_n\colon T \to T_n$ be an F_N-equivariant $\Lambda_R(T,T_n)$-Lipschitz map as given by Proposition 5.2.2. Note first that

$$\begin{aligned}\|g\|_{T_n} &\leq \sum_{e_i \in g|\Gamma} L_{f_n,e_i} l_\Gamma(e_i)\\ &\leq \mathrm{Lip}(f_n) l_\Gamma(g|\Gamma)\\ &= \Lambda_R(T,T_n)\|g\|_T\end{aligned}$$

Note that the summation above is over a uniformly bounded number of terms; that is, there is a uniform bound on the simplicial lengths of the various $g|\Gamma$ for $g \in \mathcal{C}(T)$. Thus since $\tau(T,T_n) \to 0$ (and so $\|g\|_{T_n} \to \|g\|_T$ and $\Lambda_R(T,T_n) \to 1$) we have that

$$\sum_{e_i \in g|\Gamma} L_{f_n,e_i} l_\Gamma(e_i) \to \|g\|_T$$

and thus

$$L_{f_n,e_i} \to 1$$

for each $e_i \in g|\Gamma$. Since $\{g|\Gamma \mid g \in \mathcal{C}(T)\} \supset E(\Gamma)$, we have $L_{f_n,e} \to 1$ for each $e \in E(\Gamma)$.

Now let $\Delta(T, f_n) \subset \Gamma$ be the tension graph; that is, the subgraph of Γ consisting of edges stretched exactly by $\Lambda_R(T,T_n)$ under the map f_n. Let $T' = T'(n)$ be the tree obtained from T by changing lengths of the $e \in E(\Gamma)$ so that each edge is stretched exactly by $\Lambda_R(T,T_n)$. Note that since $L_{f_n,e} \to 1$ for each $e \in E(\Gamma)$ we have that T' differs from T only by length data. Furthermore as $n \to \infty$ we have that

$$T' \to T$$

Let $f'_n \colon T' \to T_n$ be the $\Lambda_R(T,T_n)$-homothety. Suppose we are to fold the turn $\{e_i, e_j\} \subset E(\Gamma')$. Let $g|\Gamma'$ be so that $g|\Gamma'$ takes the turn $\{e_i, e_j\}$ and $g \in \mathcal{C}(T') = \mathcal{C}(T)$. Let $p_n(e_i, e_j)$ be the length (in Γ') of the fold (so that $p(e_i, e_j) \in (0, \min\{l_{\Gamma'}(e_i), l_{\Gamma'}(e_j)\})$. We claim that

$$p_n(e_i, e_j) \to 0$$

Indeed, we have

$$\begin{aligned}\|g\|_{T_n} &\leq \sum_{e_i \in g|\Gamma'} L_{f'_n, e_i} l_{\Gamma'}(e_i) - 2\Lambda_R(T, T_n) p_n(e_i, e_j) \\ &= \sum_{e_i \in g|\Gamma'} \Lambda_R(T, T_n) l_{\Gamma'}(e_i) - 2\Lambda_R(T, T_n) p_n(e_i, e_j) \\ &\leq \Lambda_R(T, T_n) \|g\|_{T'}\end{aligned}$$

Now since $\tau(T, T_n) \to 0$ we have that $\|g\|_{T_n} \to \|g\|_T$, $\Lambda_R(T, T_n) \to 1$, $T' \to T$, and finally, $\sum_{e_i \in g|T} \Lambda_R(T, T_n) l_T(e_i) \to \|g\|_T$. Thus $p_n(e_i, e_j) \to 0$. Thus the isometric folding line from T' to T_n has arbitrarily small length. And since $T' \to T$, we now have that $T_n \to T$. □

Theorem D. *Let $T \in \mathrm{cv}_N$. Then $\mathcal{C}(T)$ is a finite strongly relatively rigid set for T.*

Proof. Suppose $T' \in \overline{\mathrm{cv}}_N$ is such that $\|g\|_T = \|g\|_{T'}$ for every $g \in \mathcal{C}(T)$. Then $\tau(T, T') = 0$. Let $T_n \to T'$ with $T_n \in \mathrm{cv}_N$. So for each $g \in F_N$ we have that

$$\|g\|_{T_n} \to \|g\|_{T'}$$

Thus

$$\begin{aligned}\tau(T, T_n) &= \sum_{g \in \mathcal{C}(T)} |\|g\|_T - \|g\|_{T_n}| \\ &\to \sum_{g \in \mathcal{C}(T)} |\|g\|_T - \|g\|_{T'}| \\ &= \tau(T, T') \\ &= 0\end{aligned}$$

and so $\tau(T, T_n) \to 0$ and hence by Proposition 5.2.3, we have that $T_n \to T$. Thus $T = T'$. □

Chapter 6

Some Trees Which Do Not Admit Finite Strongly Relatively Rigid Sets

6.1 Background

In this head we prove non-existence of finite strongly relatively rigid sets for various classes of trees in the boundary of Outer Space via Bass-Serre Theory and related results of Guirardel [24], and Levitt and Paulin [36]. We also prove existence of finite strongly relatively rigid sets for certain 'relatively elliptic' trees as an application of Skora's Duality Theorem [42].

Let $(\mathcal{G}, Y)$ be a graph of groups with $T \subset Y$ some maximal tree. Let $F(\mathcal{G}, Y)$ denote the factor group of the free product of all vertex groups with the free group with basis $\{t_e \mid e \in E(Y)\}$ by the normal closure of the set of elements $t_e^{-1}\alpha_e(g)t_e\omega_e(g)^{-1}$ and $t_e t_{\overline{e}}$. Then $\pi_1(\mathcal{G}, Y, T) = F(\mathcal{G}, Y)/\operatorname{ncl}\{t_e \mid e \in E(T)\}$. Let $c = y_1 y_2 \ldots y_n$ be a (non necessarily reduced) path in Y with $y_i \in E(Y)$. By a *word of type* c we mean a pair, (c, μ), where c is a path in Y and $\mu = (r_0, r_1, \ldots, r_n)$ is a sequence of elements $r_i \in G_{v_i}$ (here G_{v_i} are vertex groups). Let $|c, \mu| = r_0 t_1 r_1 t_2 \ldots t_n r_n$ be the element of $F(\mathcal{G}, Y)$ associated to (c, μ). Let $\pi_1(\mathcal{G}, Y, p)$ be the set of all $|c, \mu|$ such that c is a closed path at $p \in V(Y)$. Evidently $\pi_1(\mathcal{G}, Y, p) < F(\mathcal{G}, Y)$. Furthermore we have $\pi_1(\mathcal{G}, Y, T) \cong \pi_1(\mathcal{G}, Y, p)$. By a *reduced word* we mean a word of type (c, μ) where $c = y_1 y_2 \ldots y_n$ is a closed path in Y, $\mu = (r_0, r_1, \ldots, r_n)$ and such that

1. $r_0 \neq 1$ if $n = 0$
2. $r_i \notin \omega(G_{y_i})$ whenever $y_{i+1} = \overline{y_i}$.

We have

Lemma 6.1.1 (Baumslag [1, Proposition 2, §VII.6]). *If (c, μ) is a reduced word, then $|c, \mu| \neq 1$.*

A *reduced word* is *cyclic* if all cyclic permutations of c (i.e. change of basepoint of the closed path c) yield reduced words.

Proposition 6.1.2. *Let $(\mathcal{G}, Y)$ be a graph of groups and (c, μ) a cyclic reduced word of length n. Then*

$$\| |c, \mu| \|_{\mathrm{BST}(\mathcal{G}, Y)} = n$$

More generally, if Y is a metric graph, we have

$$\||c, \mu|\|_{\mathrm{BST}(\mathcal{G},Y)} = 2l_Y(c)$$

Proposition 6.1.3. *Let $N \geq 2$. Suppose $\{\langle x_i \rangle\}_{i=1}^m$ is a finite collection of cyclic subgroups of F_N where each $x_i \in F_N$ is not a proper power. Suppose that $\Sigma \subset F_N$ is a finite set with $1 \in \Sigma$. Then there exists $x \in F_N$ such that x is not a proper power and*

$$\langle x \rangle \cap \Sigma = \langle x \rangle \cap \{\langle x_i \rangle\}_{i=1}^m = \{1\}$$

Proof. Suppose there exists i so that $\langle x \rangle \cap \langle x_i \rangle \neq \{1\}$ and write $x^p = x_i^q$. Note that $\langle x, x_i \rangle < F_N$ is free of rank at most 2, and is of rank 2 if and only if $\{x, x_i\}$ is a basis, which is not the case since $x^p x_i^{-q} = 1$. Thus $\langle x, x_i \rangle = \langle c \rangle$ is cyclic and hence both x, x_i are powers of c. But since neither x nor x_i are proper powers, we have $x_i = x^{\pm}$. Thus $\langle x \rangle \cap \langle x_i \rangle \neq \{1\}$ implies $x_i = x^{\pm}$. Thus if we choose x so that $x \neq x_i^{\pm}$ (for any i) and x is not a proper power, then $\langle x \rangle \cap \langle x_i \rangle = 1$ for all i. Fix some free basis $\mathcal{A}$ of F_N and note that for $1 \neq x \in F_N$, we have $|x|_{\mathcal{A}} < |x^2|_{\mathcal{A}} < \dots$. Thus the proposition reduces to finding a non proper power x so that $x \notin (\Sigma \cup \{x_i^{\pm}\}_{i=1}^m)$. So by choosing x to be some primitive element of length larger than the length of any element from $(\Sigma^{\pm} \cup \{x_i^{\pm}\}_{i=1}^m)$, we obtain the proposition. □

6.2 Large Vertex Groups

Here we consider $T \in \partial\,\mathrm{cv}_N$ simplicial, with a sub-graph of groups of the form $L *_{\{1\}} A$ or $L*_{\{1\}}$ where $\mathrm{rk}\, L \geq 2$ and $\mathrm{rk}\, A \geq 1$; that is, there is a vertex group of rank at least two, and an edge, with trivial edge group, which is incident to L and another nontrivial vertex group (possibly L if it is a loop edge).

Theorem E. *Let $(\mathcal{G}, Y)$ be a graph of groups such that $\mathrm{BST}(\mathcal{G}, Y) \in \partial\,\mathrm{cv}_N$. Suppose there is a vertex group G_v with $\mathrm{rk}\, G_v \geq 2$ and an edge e with $o(e) = v$, $G_e = \{1\}$ and $G_{t(e)} \neq \{1\}$. Let $\Sigma \subset \pi_1(\mathcal{G}, Y) \cong F_N$ be a finite set. Then there exists a graph of groups $(\mathcal{H}, Y)$ (with $\mathrm{BST}(\mathcal{H}, Y) \in \partial\,\mathrm{cv}_N$) so that for all $\sigma \in \Sigma$, we have*

$$\|\sigma\|_{\mathrm{BST}(\mathcal{G},Y)} = \|\sigma\|_{\mathrm{BST}(\mathcal{H},Y)}$$

yet $\mathrm{BST}(\mathcal{G}, Y) \neq \mathrm{BST}(\mathcal{H}, Y)$ in $\overline{\mathrm{cv}}_N$.

Proof. Suppose first that the edge e is not a loop edge. Write $o(e) = v$ with $G_e \cong \{1\}$, $\mathrm{rk}\, G_v \geq 2$, and $t(e) = v' \neq v$ with $\mathrm{rk}\, G_{v'} \geq 1$. Since $\mathrm{BST}(\mathcal{G}, Y)$ is very small, any edge e' with $o(e') = v$ is such that $G_{e'}$ is either trivial or infinite cyclic. Let $\{x_i\}$ be a finite collection of generators, one for each nontrivial $G_{e'}$ for which $o(e') = v$. For each $\sigma \in \Sigma$, we write σ as a cyclic reduced word of type (c, μ) over $(\mathcal{G}, Y)$. Let

$$\Sigma_v = \{r_i \mid r_i \in \mu, r_i \in G_v, r \in (c, \mu), \sigma = |c, \mu|\}$$

By Proposition 6.1.3 there exists $x \in G_v$ such that x is not a proper power and

$$\langle x \rangle \cap \Sigma_v = \langle x \rangle \cap \{\langle x_i \rangle\} = \{1\}$$

Consider the graph of groups $(\mathcal{H}, Y)$ defined as follows. Note the underlying graph, Y, is the same as the underlying graph for $(\mathcal{G}, Y)$. For all $w \in V(Y)$ with $w \neq v'$, set $H_w = G_w$. For all $f \in E(Y)$ with $f \neq e$, set $H_f = G_f$. Set $H_{v'} = G_{v'} * \langle s \rangle$. Set $H_e = \mathbb{Z}$ where $x \in H_v (= G_v)$ is amalgamated to $s \in H_{v'}$. Amalgamation along edges not incident to v' is exactly as in $(\mathcal{G}, Y)$. If, in $(\mathcal{G}, Y)$, φ is an amalgamation monomorphism with domain (resp. range) contained in $G_{v'}$, then, in $(\mathcal{H}, Y)$, the domain (resp. range) of φ is contained in $G_{v'}$, where $G_{v'} < G_{v'} * \langle s \rangle = H_{v'}$ in the obvious way. By Proposition 6.1.3, $\mathrm{BST}(\mathcal{H}, Y) \in \partial\, \mathrm{cv}_N$ since x is not a proper power and the image of H_e in H_v is not conjugate to any other amalgamated subgroup. Furthermore, one obtains a presentation for $\pi_1(\mathcal{G}, Y)$ by performing a Tietze Transformation on $\pi_1(\mathcal{H}, Y)$.

Now write $\sigma = |c, \mu|$ with (c, μ) as per $(\mathcal{G}, Y)$. Since the underlying graphs are the same for each graph of groups, and since for each v, $G_v < H_v$ the word (c, μ) corresponds to an element $\sigma' \in \pi_1(\mathcal{H}, Y)$. We claim that $\sigma' = |c', \mu'|$ for some cyclically reduced word of type (c', μ') over $(\mathcal{H}, Y)$ where $c = c'$. From this, the result follows by Proposition 6.1.2. The result is clear if $l(c) = 0$ since each vertex group in $(\mathcal{G}, Y)$ is a subgroup of some vertex group in $(\mathcal{H}, Y)$. So assume $l(c) \geq 1$. If c is a cyclically reduced path, then there is nothing to check. So suppose now that there is a subpath $f_i \overline{f_i}$ in c. By definition of cyclic reduced word, we know that $r_i \notin \omega(G_{f_i})$. Thus the path will remain cyclically reduced in $(\mathcal{H}, Y)$ if we can show that $r_i \notin \omega(H_{f_i})$. Recall that $G_f = H_f$ for all $f \neq e$ and that edge monomorphisms in $(\mathcal{H}, Y)$ are the same as in $(\mathcal{G}, Y)$ so long as the edge is not incident to v'. Now if $f_i \neq e$ is such that $t(f_i) = v'$, recall by construction that $\omega(H_{f_i}) = \omega(G_{f_i}) < G_{v'} < G_{v'} * \langle s \rangle = H_{v'}$, and so if $r_i \notin \omega(G_{f_i})$ then $r_i \notin \omega(H_{f_i})$. Assume now that $f_i = e$. We need to show that $r_i \notin \omega(H_e) = \langle s \rangle$. This is clear since $r_i \in G_{v'} < G_{v'} * \langle s \rangle = H_{v'}$. So assume lastly that $f_i = \overline{e}$. We need to show that $r_i \notin \omega(H_{\overline{e}}) = \langle x \rangle$. But this is guaranteed by our choice of x as per Proposition 6.1.3.

Suppose now that e is a loop edge. The proof follows the above argument very closely. Write $o(e) = t(e) = v$ with $G_e = \{1\}$ and $\mathrm{rk}\, G_v \geq 2$. Let $\{x_i\}$ be a finite collection of generators one for each nontrivial

$G_{e'}$ where $o(e') = v$. Let

$$\Sigma_v = \{r_i \mid r_i \in \mu, r_i \in G_v, r \in (c, \mu), \sigma = |c, \mu|\}$$

By Proposition 6.1.3 there exists $x \in G_v$ such that x is not a proper power and

$$\langle x \rangle \cap \Sigma_v = \langle x \rangle \cap \{\langle x_i \rangle\} = \{1\}$$

Consider the graph of groups $(\mathcal{H}, Y)$ defined as follows. For all $w \in V(Y)$ with $w \neq v$, set $H_w = G_w$. For all $f \in E(Y)$ with $f \neq e$, set $H_f = G_f$. Set $H_v = G_v * \langle s \rangle$. Set $H_e = \mathbb{Z}$ with HNN relation $t_e^{-1} x t_e = s$. As in the case of the free product, we argue that the underlying path for $\sigma = |c, \mu|$ is the same in $(\mathcal{H}, Y)$ as in $(\mathcal{G}, Y)$. As above, we reduce to the case when there is a subpath $e\overline{e}$ or $\overline{e}e$ in c. In the former case, it is clear that $r_i \notin \langle s \rangle$. In the latter case, our choice of x ensures that $r_i \notin \langle x \rangle$. □

6.3 Related Results

A result of Levitt and Paulin [36] reveals that given any $T \in \overline{\mathrm{CV}}_N$, which is "non-geometric" – that is, it does not arise as the tree dual to a measured foliation on a finite 2-complex with fundamental group free – then there exists a sequence of very small "geomteric" trees, T_n, which "strongly approximate" T. In particular, given a finite set $\Sigma \subset F_N$, we eventually have that $\|\sigma\|_{T_n} = \|\sigma\|_T$ for all $\sigma \in \Sigma$ and hence for such a T no finite strongly relatively rigid set can exist. We now make this observation more precise.

Definition 6.3.1 (geometric action [36, Definition 0.1])**.** An action of a group G on an $\mathbb{R}$-tree, T is *geometric* if there exists a triple $(\Sigma, \rho, \mathcal{F})$ (here Σ a connected finite simplicial complex with an epimorphism $\rho\colon \pi_1\Sigma \to G$, and $\mathcal{F}$ is a measured foliation on Σ), such that

- T is G-equivariantly isometric to $T(\overline{\mathcal{F}})$, where $\overline{\mathcal{F}}$ is the pullback of $\mathcal{F}$ to the covering $\overline{\Sigma}$ of Σ associated to ρ and $T(\overline{\mathcal{F}})$ is the "leaf space made Hausdorff".
- every edge of $\overline{\Sigma}$ that is transverse to $\overline{\mathcal{F}}$ is mapped isometrically into $T(\overline{\mathcal{F}})$ by the canonical map $\pi\colon \overline{\Sigma} \to T(\overline{\mathcal{F}})$.

Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of finitely generated groups with epimorphisms $G_n \to G_{n+1}$ and G be its direct limit.

Definition 6.3.2 (strong convergence [36, §1.1])**.** We say a sequence of $\mathbb{R}$-trees T_n with actions of G_n *converges strongly* to an $\mathbb{R}$-tree T with action of G if there exists surjective morphisms f_{np} from T_n to T_p (for $n < p$), and f_n from $T_n \to T$, such that $f_p \circ f_{np} = f_n$, and furthermore for every $n \in \mathbb{N}$ and $x, y \in T_n$

there exists $p \geq n$ such that the distance between $f_{np}(x)$ and $f_{np}(y)$ in T_p equals the distance between $f_n(x)$ and $f_n(y)$ in T.

Note that if $G_n = G$ for all n, then – under the assumption of strong convergence – given any finite set, $\Sigma \subset G$, eventually $\| \cdot \|_{T_n}$ and $\| \cdot \|_T$ agree on Σ.

Theorem 6.3.3 (Levitt and Paulin [36, Theorem 0.2])**.** *Let T be an $\mathbb{R}$-tree with a minimal action of G.*

- *If the action is not geometric, then it is a strong limit of geometric actions on $\mathbb{R}$-trees T_n.*
- *If the action is geometric, then it may only be a strong limit in a nontrivial (i.e. stationary) way.*

As an immediate corollary, we see that any non-geometric $T \in \partial \operatorname{cv}_N$ can not admit a finite strongly relatively rigid set.

We now consider now a family of simplicial trees for which the hypotheses of Theorem E does not apply. More specifically, suppose that a simplicial $T \in \partial \operatorname{cv}_N$ admits a graph of groups decomposition with vertex group $G_v \neq \{1\}$ and two edge groups $G_e, G_{e'} \cong \{1\}$ with edge origins $o(e) = o(e') = v$. We aim to show that such a tree can not admit a finite strongly relatively rigid set via a folding construction of Guirardel [24].

Definition 6.3.4 (H condition [24, §2.1])**.** Let T be a simplicial action of F_N without inversion. Let α, β be two embedded edge-paths in T starting from the same point x. We assume that α and β run through the same number of edges and we denote by $\alpha_1, \ldots, \alpha_p$ and by $\beta_1, \ldots, \beta_p$ the edges of α and β. We say that α and β satisfy the hypothesis (H) if

- $(H1)$ for all $i = 1, \ldots, p$, α_i and β_i have the same length.
- $(H2)$ α_1 and β_1 are distinct edges.
- $(H3)$ there exists an equivariant orientation of the edges of T such that α_i and β_i are positively orieinted for $i = 1, \ldots, p$.

Lemma 6.3.5 (Guirardel [24, Folding To Approximate Lemma])**.** *Let T be a simplicial action of F_N without inversion. Let α, β be two paths in T with origin x satisfying the (H) condition such that* $\operatorname{Stab} x$ *is infinite. Let w_k be a sequence of distinct elements in* $\operatorname{Stab} x$ *and let $T^{(k)} = T/\alpha \sim w_k.\beta$. Assume that each intermediate fold is a fold between edges with trivial stabilizer. Then $T^{(k)}$ converges to T as $k \to \infty$. In particular, any finite subtree of T isometrically embeds in $T^{(k)}$ for large enough k.*

Consider now our vertex group G_v and write $\operatorname{Stab}_T x = G_v$ for the appropriate $x \in T$. Let α be the edgepath (consisting of a single edge!) from x to $t(e)$. Let β be the edgepath (again, a single edge) from x

to $t(e')$. Recall that $G_e, G_{e'} \cong \{1\}$. Thus we may apply the previous lemma. Note that we have $T \neq T^k$ for all k, yet given any finite subtree $K \subset T$, there exists a k so that K embeds in T^k. Thus T can not admit a finite strongly relatively rigid set.

We now present a positive result regarding the existence of finite strongly relatively rigid sets. The main tool of use is Skora's Duality Theorem [42]. Our exposition follows Kapovich [35]. Given a connected compact surface, S, equipped with a measured geodesic lamination, λ, we consider the tree, $T = T_\lambda$, which is *dual to the lamination* λ (see [35, §11]). For such a surface, S, with an identification $\pi_1(S) = G$, we say that the action of G on an $\mathbb{R}$-tree, T, is *relatively S-elliptic* if each peripheral element of G has a fixed point in T.

Theorem 6.3.6 (Skora [42], Kapovich [35, Lemma 11.31])**.** *Suppose that T is a small minimal relatively S-elliptic G-tree. Then there exists a measured geodesic lamination, λ, so that the dual tree, T_λ, is equivariantly isometric to T.*

Corollary 6.3.7 (Kapovich [35, Corollary 11.32])**.** *There is a finite collection of elements, $\gamma_1, \ldots, \gamma_k$ of G so that if T, T' are any small minimal relatively S-elliptic G-trees for which $\|\gamma_i\|_T = \|\gamma_i\|_{T'}$ for every i, then T, T' are G-equivariantly isometric.*

Now let S be a connected compact surface with at least one boundary component. We fix an identification $\pi_1(S) = F_N$. Accordingly, let $\{u_i\} \subset F_N$ be the peripheral elements of F_N. Suppose that $T \in \overline{\text{cv}}_N$ is such that $\|u_i\|_T = 0$ for each peripheral element, u_i. Then the action of F_N on T is very small (hence small) and relatively S-elliptic. By Theorem 6.3.6, T is F_N-equivariantly isometric to T_λ for some measured geodesic lamination, λ, on S. It is worth noting that such a T is (by definition) geometric. Furthermore, by Corollary 6.3.7, there is a finite set, Γ, of conjugacy classes which distinguish T from any other relatively elliptic T' in $\overline{\text{cv}}_N$.

Theorem 6.3.8. *Let S be a connected compact surface with at least one boundary component. Fix an identification $F_N = \pi_1(S)$ with $\{u_i\} \subset F_N$ the peripheral classes. Let $T \in \overline{\text{cv}}_N$ be so that $\|u_i\|_T = 0$ for each i. Then there exists a finite strongly relatively rigid set for T.*

Proof. Let Γ be the set afforded to T by Corollary 6.3.7. Let $\Sigma = \Gamma \cup \{u_i\}$. Suppose now that T' agrees with T on Σ. Since $\|u_i\|_{T'} = 0$ for each i, T' is relatively S-elliptic. Thus by Corollary 6.3.7, Γ distinguishes T from T'. □

Theorems D and E, along with the above related results, allow us to paint a current picture of strong relative rigidity in Outer Space.

Theorem F. *Let* $T \in \overline{\text{cv}}_N$*.*

1. *If* $T \in \text{cv}_N$*, then there exists a finite strongly relatively rigid set* $\Sigma \subset F_N$ *for* T*.*

2. *If* T *is non-geometric then there does not exist a finite locally strongly relatively rigid set* $\Sigma \subset F_N$ *for* T*.*

3. *If* T *is simplicial with a sub-graph of groups containing* $L *_{\{1\}} A$ *or* $L*_{\{1\}}$ *where* $\operatorname{rk} L \geq 2$ *and* $\operatorname{rk} A \geq 1$*, then there does not exist a finite locally strongly relatively rigid set* $\Sigma \subset F_N$ *for* T*.*

4. *If* $T \in \overline{\text{cv}}_N$ *is simplicial with vertex group* $G_v \neq \{1\}$ *and two edge groups* $G_e, G_{e'} \cong \{1\}$ *with edge origins* $o(e) = o(e') = v$*, then there does not exist a finite locally strongly relatively rigid set* $\Sigma \subset F_N$ *for* T*.*

5. *If* T *is relatively* S*-elliptic for some connected compact surface,* S*, with at least one boundary component, then there exists a finite strongly relatively rigid set for* T*.*

Chapter 7

Distinguishing Certain Simplicial Trees in Volume-Normalized Outer Space

In this head we prove the following theorem.

Theorem G. *Let T be a very small F_N metric graph of groups with all vertex groups infinite cyclic, all edge groups trivial, and T/F_N a tree. Suppose $(\mathcal{G}, Y)$ is a very small F_N metric graph of groups. Suppose T agrees with* $\mathrm{BST}(\mathcal{G}, Y)$ *on $\mathcal{T}_{\mathcal{A}}$ and* $\operatorname{covol} T = \operatorname{vol} Y$*. Then* $\mathrm{BST}(\mathcal{G}, Y) = T$.

Proposition 7.0.9. *Let $(\mathcal{G}, Y)$ be an very small F_N-graph of groups with* $\operatorname{rk} \pi_1(Y) \geq 1$*. Then there does not exist an elliptic generating set for $\pi_1(\mathcal{G}, Y)$.*

Proof. We have the epimorphism $\pi\colon \pi_1(\mathcal{G}, Y) \to \pi_1(Y)$. Assuming $\pi_1(Y) \neq \{1\}$, if some generating set of $\pi_1(\mathcal{G}, Y)$ does not contain a reduced word (c, μ) with c not homotopic to a point, then $\pi(\pi_1(\mathcal{G}, Y)) = \{1\}$ and hence π is not a surjection. Thus every generating set contains such a word. Since the c for this word is not homotopic to a point, it is not freely homotopic to a point, and hence the translation length of this element is positive since it cannot be 'HNN-pinched' to a word of length zero. □

Proposition 7.0.10. *If T agrees with* $\mathrm{BST}(\mathcal{G}, Y)$ *in $\mathcal{A}$, then $(\mathcal{G}, Y)$ has an elliptic generating set.*

Corollary 7.0.11. *If T agrees with* $\mathrm{BST}(\mathcal{G}, Y)$ *on $\mathcal{A}$, then Y is a tree.*

Definition 7.0.12 (proper n-filling)**.** Let $(\mathcal{G}, Y)$ be a minimal very small F_N-tree of groups. A *proper n-filling* of Y is a collection of n edgepaths c_i with $\#E(c_i) \geq 1$ for all i and $E(c_i \cap c_j) = \varnothing$ for $i \neq j$ and for which

1. If $v \in V(c_i)$ with $G_v \cong \{1\}$, then $\operatorname{val}_{c_i} v \geq 2$.

2. If $e \in E(c_i)$ with $G_e \cong \mathbb{Z}$ and $G_{o(e)} \cong \mathbb{Z}$ (resp. $t(e)$), then $\operatorname{val}_{c_i} o(e) \geq 2$ (resp. $t(e)$).

Proposition 7.0.13 (Paulin [38, Proposition 1.8])**.** *If g, h are two elliptic isometries of an $\mathbb{R}$-tree, T, then*

$$\|gh\|_T = 2 d_T(\operatorname{Fix}(g), \operatorname{Fix}(h))$$

Recall that if $a_i = {}^{g_i}b_i$, then

$$\mathrm{Fix}(a_i) = \mathrm{Fix}({}^{g_i}b_i) = g_i \,\mathrm{Fix}(b_i)$$

Note also that if the conjugacy class of $a =_{\pi_1(\mathcal{G},Y)} |c,\mu|$, where (c,μ) is cyclically reduced word, then

$$\|a\|_{\mathrm{BST}(\mathcal{G},Y)} = l_Y(c)$$

Thus for elliptic isometries, a_i, a_j, we have

$$\|a_i a_j\|_{\mathrm{BST}(\mathcal{G},Y)} = 2d_{\mathrm{BST}(\mathcal{G},Y)}(g_i \,\mathrm{Fix}(b_i), g_j \,\mathrm{Fix}(b_j)) = l_Y(c_{a_i a_j})$$

The segment, c', from $g_i \,\mathrm{Fix}(b_i)$ to $g_j \,\mathrm{Fix}(b_j)$ in $\mathrm{BST}(\mathcal{G},Y)$ projects to the (not necessarily reduced) closed path $c_{a_i a_j}$. Hence we conclude that $c_{a_i a_j}$ contains the bridge (in Y) from $\mathrm{Fix}(b_i)$ to $\mathrm{Fix}(b_j)$.

Proposition 7.0.14 (Higman [25, Lemma 7])**.** *Let H be a subgroup of a proper free factor of F_N. Then $F_N/\,\mathrm{ncl}_{F_N}(H)$ has a nontrivial free group as a free factor.*

Corollary 7.0.15. *The subgroup $S = \langle b_i\rangle_{i=1}^N < F_N$ is not contained in a proper free factor.*

Proof. Indeed, $\mathrm{ncl}_{F_N}(\{b_i\}_{i=1}^N) \supset \mathcal{A}$ so the quotient described above is trivial. □

Proposition 7.0.16. *The set of all c_i constitute a proper $N-1$ filling of Y.*

Proof. First we show that the desired collection fills Y. Suppose not. Let $e = G_v *_{G_e} G_{v'}$ be the edge which is not crossed. Recall that $a_i = {}^{g_i}b_i$ where b_i lies in a vertex group. Note that if we write

$$a_i a_j =_{\pi_1(\mathcal{G},Y)} |c,\mu|$$

where (c,μ) is cyclically reduced, then c contains the bridge (in Y) from $\mathrm{Fix}(b_i)$ to $\mathrm{Fix}(b_j)$. Write $Y = Y^1 - e - Y^2$ and note that if e is not crossed, then all b_i lie (perhaps via amalgamation along e) in (without loss) Y^1. Note that at most one b_i can be amalgamated across the edge e. Now if $G_e \cong \mathbb{Z}$, then each of the two components of $Y\backslash e$ has rank at least two, and if $G_e \cong \{1\}$, then each component has rank at least one. In any case, we conclude that $\langle b_i\rangle_{i=1}^N$ lie in a proper free factor of F_N, which contradicts the above corollary.

The assumption that $\mathrm{covol}\, T = \mathrm{covol}\, T'$ forces the filling to be without overlap. Indeed, since the paths fill, we have we have

$$\mathrm{covol}(T) = \sum_{\mathcal{E}_\mathcal{A}} \frac{1}{2}\|t\|_T = \sum_{\mathcal{E}_\mathcal{A}} \frac{1}{2}\|t\|_T \geq \mathrm{covol}\, T'$$

and if $E(c_i \cap c_j) \neq \varnothing$ for some $i \neq j$, then the inequality is strict. Thus if the inequality is an equality, there can be no such overlap.

The remaining properties of a proper n filling follow from the fact that each c_i arises from a cyclically reduced word of type (c_i, μ_i) □

Corollary 7.0.17. *If T agrees with T' on $\mathcal{E}_{\mathcal{T}_\mathcal{A}}$, then T' admits a proper $N-1$ filling.*

Definition 7.0.18 (good subtree)**.** A subtree $Y' \subset Y$ is deemed a *good subtree* if the induced graph of groups $(\mathcal{G}, Y')$ has the following properties:

- If $v \in V(Y')$ is such that $G_v \cong \{1\}$, then $\operatorname{val}_{Y'} v \geq 2$.
- If $e \in E(Y')$ is such that $G_e \cong \mathbb{Z}$ and $G_{o(e)} \cong \mathbb{Z}$ (resp $t(e)$), then $\operatorname{val}_{Y'} o(e) \geq 2$ (resp $t(e)$).

Remark 7.0.19.

- Y is a good subtree of Y.
- A proper n-filling of a good subtree makes sense.
- If Y admits a proper n-filling, then it admits a proper n'-filling for $1 \leq n' \leq n$.

Proposition 7.0.20. *There is a chain of subtrees, $Y^1 \subsetneq Y^2 \subsetneq \cdots \subsetneq Y^l = Y$ with the following properties:*

1. *The Y^i are good subtrees.*
2. *$Y^{i+1} = Y^i \cup E$ where E is a segment.*
3. *We have*
$$\operatorname{rk} \pi_1(\mathcal{G}, Y^i \cup E) > \operatorname{rk} \pi_1(\mathcal{G}, Y^i)$$
and for those E with at least two edges we have that
$$\operatorname{rk} \pi_1(\mathcal{G}, Y^i \cup \{E \backslash e\}) = \operatorname{rk} \pi_1(\mathcal{G}, Y^i)$$
where e is the terminal edge of E which does not meet Y^i.
4. *We have*
$$\operatorname{rk} \pi_1(\mathcal{G}, Y^1) \geq 3$$
and Y^1 does not admit a proper 2-filling.
5. *$l \leq N-2$.*

6. If $(\mathcal{G}, Y^i)$ does not admit a proper $n+1$-filling, then $(\mathcal{G}, Y^{i+1})$ does not admit a proper $n+2$-filling.

Furthermore, assuming that $(\mathcal{G}, Y)$ contains an L-vertex, 1-vertex, or $\mathbb{Z}$-edge, we have that $(\mathcal{G}, Y)$ does not admit a proper $N-1$ filling.

Proof. Assume that Y^i has been constructed and that Y^1 is a good subtree (we subsequently show how to construct a suitable Y^1, and that Y^1 is a good subtree). In the various E considered below, assume that the left most vertex lies in Y^i. If E is one of

- $L *_{\mathbb{Z}} L'$
- $\mathbb{Z} *_{\mathbb{Z}} L$
- $L * L'$
- $\mathbb{Z} * L$ (or $L * \mathbb{Z}$, or $\mathbb{Z} * \mathbb{Z}$)
- $\{1\} * \mathbb{Z}$

then $Y^i \cup E$ satisfies (1, 2, 3). Now suppose we see $C *_{\mathbb{Z}} \mathbb{Z}$ where $C \in \{\mathbb{Z}, L\}$. If E is one of the following

- $C *_{\mathbb{Z}} \mathbb{Z} *_{\mathbb{Z}} L$
- $C *_{\mathbb{Z}} \mathbb{Z} * L$
- $C *_{\mathbb{Z}} \mathbb{Z} * \mathbb{Z}$

then $Y^i \cup E$ satisfies (1, 2, 3). So suppose now that E is one of the following

- $C *_{\mathbb{Z}} \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$
- $C *_{\mathbb{Z}} \mathbb{Z} * \{1\}$

In the latter case, any further edges which do not contribute to the rank of the fundamental group have the form $\{1\} * \{1\}$. Eventually, we must see an edge of the form $\{1\} * C$ where $C \in \{\mathbb{Z}, L\}$. If the former occurs, note that if we see $\mathbb{Z} *_{\mathbb{Z}} L$, or $\mathbb{Z} * C$ for $C \in \{\mathbb{Z}, L\}$, then we are done. Thus we see either $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ or $\mathbb{Z} * \{1\}$. In the latter case, we proceed as before. Note that the former case can not lead us to a terminal vertex, and hence eventually we'll see either $\mathbb{Z} *_{\mathbb{Z}} L$, or $\mathbb{Z} * C$ for $C \in \{\mathbb{Z}, L\}$. Thus $Y^i \cup E$ satisfies (1, 2, 3).

Now given Y^i with $i < l$, there is some edge $e \in E(Y)$ for which $o(e) \in V(Y^i)$ and $t(e) \notin V(Y^i)$, and this edge is the starting edge for some E listed above.

As for Y^1, suppose first that $(\mathcal{G}, Y)$ has an L-vertex. Then at least one of the E from the above list is a subgraph of Y. If $(\mathcal{G}, Y)$ contains no L-vertices, but has $\mathbb{Z}$-edges, let e be such a $\mathbb{Z}$-edge, and take Y^1 to be any segment containing e with $\operatorname{rk}\pi_1(E) \geq 3$. If $(\mathcal{G}, Y)$ contains no L-vertices and no $\mathbb{Z}$-edges, but contains 1-vertices, let Y^0 be the star of such a 1-vertex. Evidently all Y^1 satisfy (1, 4). Thus in light of the previous paragraphs, we have established claims (1, 2, 3, 4).

(5) follows immediately from (3) and (4).

Now for any proper n-filling of $(\mathcal{G}, Y^i \cup E)$, which ever path crosses an edge in E must cross all edges in E. So let c be the path which crosses E and let $c' = c \cap Y^i$.

To establish (6) suppose that $Y^i \cup E$ admits a proper $n+2$-filling.

If $c' = v$, then it is clear that there exists a proper $n+1$ filling of Y^i. So suppose that c' contains at least one edge.

Note first that if $G_v = L$, then c' along with the remaining $n+1$ edges of the proper $n+2$-filling of $Y^i \cup E$ constitute a proper $n+2$-filling of Y^i. Hence Y^i admits a proper $n+1$ filling.

More generally, if $\#E(c' \cap \operatorname{st}_{Y^i} v) \geq 2$ then c' along with the remaining $n+1$ edges of the proper $n+2$-filling of $Y^i \cup E$ constitute a proper $n+2$-filling of Y^i. Hence Y^i admits a proper $n+1$ filling.

So we safely suppose that G_v is cyclic and that $\#E(c' \cap \operatorname{st}_{Y^i} v) = 1$.

Suppose first that $E(\operatorname{st}_{Y^i} v)) = 1$, and call this edge e. Then it cannot be that $G_v \cong \{1\}$ (else our good subtree Y^i has a 1-vertex, v with valence one). So $G_v \cong \mathbb{Z}$. Then it cannot be that $G_e \cong \mathbb{Z}$ (else our good subtree Y^i has a $\mathbb{Z}$-edge incident to a valence one $\mathbb{Z}$-vertex, v). Thus if $G_v \cong \mathbb{Z}$, then $G_e \cong \{1\}$. And then c' along with the remaining $n+1$ edges of the proper $n+2$-filling of $Y^i \cup E$ constitute a proper $n+2$-filling of Y^i. Hence Y^i admits a proper $n+1$ filling.

So suppose that $E(\operatorname{st}_{Y^i} v)) \geq 2$. Since $\#E(c' \cap \operatorname{st}_{Y^i} v) = 1$ there is an edge incident to v which is not crossed by c'. This edge is crossed by some path d which is part of the proper $n+2$-filling of $Y^i \cup E$. Then the path $c' \cup d$ is such that $\operatorname{val}_{c' \cup d} v \geq 2$. Hence this path along with the remaining n edges of the proper $n+2$-filling of $Y^i \cup E$ constitute a proper $n+1$-filling of Y^i. Hence Y^i admits a proper $n+1$ filling.

As for the furthermore part. Start with (4); that is, Y^1 does not admit a proper 2-filling. By (6), Y^2 does not admit a proper 3-filling. Eventually we see that Y^l does not admit a proper $l+1$-filling. But $Y^l = Y$, and $l \leq N-2$. Hence Y does not admit a proper $N-1$ filling. $\square$

Thus we may assume that $(\mathcal{G}, Y)$ is such that Y is a tree, all vertex groups are infinite cyclic, and all edge groups are trivial. Let $\mathcal{B}$ be the free basis associated with $(\mathcal{G}, Y)$. Note first that since a_i is primitive, it is not a proper power. Furthermore, a conjugate of a primitive element is again primitive. Thus if ${}^{g_i}a_i \in \langle b_i \rangle$, then ${}^{g_i}a_i = b_i^{\pm}$. Since $\langle b \rangle = \langle b_i^{-1} \rangle$ we may assume without loss that ${}^{g_i}a_i = b_i$. Suppose now that $a_i \neq a_j$ are both conjugate into $G_{v_{b_i}} = \langle b_i \rangle$. Then by above, we have

$$^{g_i}a_i = (^{g_j}a_j)^{\pm}$$

hence

$$^{g_j^{-1}g_i}a_i = a_j^{\pm}$$

This contradicts the fact that $\mathcal{A}$ was a free basis of F_N. Indeed, form the Stallings Subgroup Graph for $\mathcal{A}$ and replace the petal for a_j with the word $^{g_j^{-1}g_i}a_i$. Since this graph still represents $\mathcal{A}$, it must fold to the $\mathcal{A}$-rose. But then we will see two loops labelled by a_i, and folding reduces the rank of the graph, contradiction. Thus we have established that for each $b_i \in \mathcal{B}$, we have

$$b_i = {}^{g_i}a_i$$

for some $g_i \in F_N$ and a unique $a_i \in \mathcal{A}$. Note that since $\mathcal{B}$ is a free basis, so is $\{^{g_i}a_i\}_{i=1}^N$

Since $\#E(Y) = N-1 = \#\mathcal{E}_{\mathcal{A}}$ and each $t_i \in \mathcal{E}_{\mathcal{A}}$ has positive translation length, we conclude that $\#E(c_i) = 1$ for each c_i. Let $t = a_ia_j$. We know that $[[a_ia_j]]_{\mathcal{B}}$ contains the letters $^{g_i}a_i$ and $^{g_j}a_j$. Furthermore, since $[[a_ia_j]]_{\mathcal{B}} = 2$, then $^{g_i}a_i, {}^{g_j}a_j$ are the only two letters in $[[a_ia_j]]_{\mathcal{B}}$. This implies that $^{g_i}a_i{}^{g_j}a_j = [[a_ia_j]]_{\mathcal{B}}$ which implies that $g_i = g_j$. Since T is connected, it follows that all g_i are equal. Thus $\mathcal{A}$ is conjugate to $\mathcal{B}$.

Now conjugate $\mathcal{B}$ (which does not change T') so that $\mathcal{B} = \mathcal{A}$. Fix a vertex v in T/F_N with vertex group $\langle a \rangle$. This is also a vertex group in T'/F_N, at the vertex, say v'. The edges incident to v_i (with $G_{v_i} \cong \langle a_i \rangle$) in T/F_N correspond to elements $t \in \mathcal{E}_{\mathcal{T}_{\mathcal{A}}}$ for which $a_i \Subset t$ and hence to edges in T'/F_N incident to v'. Thus the (metric) star of v' in T'/F_N is the same as the (metric) star of v in T/F_N. Continue until we have reconstructed T'/F_N exactly as T/F_N. Thus we have established Theorem G.

The assumption on volumes is essential. Indeed, consider

$$T = \mathrm{BST}(\langle a \rangle * \langle b \rangle * \langle c \rangle)$$

with both edges of length 1. Now consider

$$T' = \mathrm{BST}(\langle a, {}^{b}c \rangle * \langle b \rangle)$$

with the edge having length 1. Both $T, T' \in \partial\,\mathrm{cv}_3$. Now a, b, c are elliptic in T and T'. Now

$$
\begin{aligned}
ab &=_{\pi_1(T')} a\ e\ b\ \bar{e}\ 1 \\
bc &=_{\pi_1(T')} {}^{b}c\ e\ b\ \bar{e}\ 1 \\
ac &=_{\pi_1(T')} a\ e\ b^{-1}\ \bar{e}\ {}^{b}c\ e\ b\ \bar{e}\ 1 \\
ab^{-1} &=_{\pi_1(T')} a\ e\ b^{-1}\ \bar{e}\ 1 \\
bc^{-1} &=_{\pi_1(T')} ({}^{b}c)^{-1}\ e\ b\ \bar{e}\ 1 \\
ac^{-1} &=_{\pi_1(T')} a\ e\ b^{-1}\ \bar{e}\ ({}^{b}c)^{-1}\ e\ b\ \bar{e}\ 1
\end{aligned}
$$

and so

$$
\begin{gathered}
\|ab\|_{T'} = 2, \quad \|bc\|_{T'} = 2, \quad \|ac\|_{T'} = 4 \\
\|ab^{-1}\|_{T'} = 2, \quad \|bc^{-1}\|_{T'} = 2, \quad \|ac^{-1}\|_{T'} = 4
\end{gathered}
$$

and also

$$
\begin{gathered}
\|ab\|_{T} = 2, \quad \|bc\|_{T} = 2, \quad \|ac\|_{T} = 4 \\
\|ab^{-1}\|_{T} = 2, \quad \|bc^{-1}\|_{T} = 2, \quad \|ac^{-1}\|_{T} = 4
\end{gathered}
$$

Thus T agrees with T' on two letter words over $\mathcal{A}$. However, $T \neq T'$ since

$$
\|a\ {}^{b}c\|_T = 4, \qquad \|a\ {}^{b}c\|_{T'} = 0,
$$

Note that $\operatorname{covol} T = 2$, yet $\operatorname{covol} T' = 1$.

Note that if $(\mathcal{G}, Y)$ is an F_N metric graph of actions with elliptic generating set, then Y is a tree. Indeed, we use the same argument involving the surjection $\pi_1(\mathcal{G}, Y) \to \pi_1(Y)$. However, there is a obstruction to generalizing the argument used in the case of a metric graph of groups. One can argue, as in the case with a graph of groups, that the paths, c_i from $\mathcal{E}_{\mathcal{A}}$ must fill Y. However, we may no longer assume that each c_i is a non-degenerate path in Y. For example, if $\mathrm{Fix}(a_i) \neq \mathrm{Fix}(a_j)$ both lie in T_v for the vertex group $G_v \curvearrowright T_v$ with nontrivial vertex action, then the projection (to Y) of the arc between them is trivial.

Chapter 8

Arational Free Trees Admit Finite Strongly Relatively Rigid Sets of Currents

8.1 Background

In this head we prove the following theorem.

Theorem H. *Let $T \in \partial \operatorname{cv}_N$ be arational and free. Then there exists a finite strongly relatively rigid set of currents for T.*

In the process of proving our theorem, we will cite several known results regarding geodesic currents and dual laminations to trees in $\overline{\operatorname{cv}}_N$. Recall Sections 2.9 and 2.10. The reader is invited to skip directly to Section 8.2, referring to the results below only when cited.

Theorem 8.1.1 (Kapovich and Lustig [33, Theorem 1.1])**.** *Let T be a minimal very small F_N-tree and $\mu \in \operatorname{Curr}(F_N)$. Then $\langle T, \mu \rangle = 0$ if and only if $\operatorname{supp}(\mu) \subseteq L^2(T)$.*

Theorem 8.1.2 (Coulbois et al. [15, Theorem A])**.** *Let T be a free, minimal F_N-tree with dense orbits. Then T is indecomposable if and only if $L(T)$ is minimal up to diagonal leaves; that is, there is a unique minimal sublamination whose complement consists of finitely many F_N-orbits of leaves each of which is diagonal over the sublamination. In this case the unique minimal sublamination is the derived sublamination, $L'(T)$, the subset of non-isolated leaves which is also equal to the regular sublamination, $L_r(T)$.*

Corollary 8.1.3 (Coulbois et al. [15, Corollary 1.4])**.** *Let T be a free, indecomposable F_N-tree and $\mu \in \operatorname{Curr}(F_N)$. Then the following are equivalent.*

1. $\operatorname{supp}(\mu) \subseteq L(T)$.
2. $\langle T, \mu \rangle = 0$.
3. $\operatorname{supp}(\mu) = L'(T)$.

Theorem 8.1.4 (Reynolds [40, Theorem 1.1])**.** *Let $T \in \partial \operatorname{cv}_N$. The following are equivalent.*

1. *T is arational.*

2. *T is indecomposable, and if T is not free, then T is dual to a measured geodesic lamination on a once-punctured surface with minimal and filling support.*

Proposition 8.1.5 (Bestvina and Reynolds [6, Proposition 3.1]). *Let $T \in \mathrm{cv}_N$ be indecomposable. If $U \in \overline{\mathrm{cv}}_N$ satisfies $L(T) \subseteq L(U)$, then $L(T) = L(U)$.*

Theorem 8.1.6 (Coulbois et al. [16, Theorem I, II]). *Let $T_0, T_1 \in \partial\,\mathrm{cv}_N$ have dense orbits. Then TFAE:*

1. *$L(T_0) = L(T_1)$*
2. *$\hat{T_0}^{obs}$ and $\hat{T_1}^{obs}$ are F_N-equivariantly homeomorphic via a homeomorphism that restricts to an F_N-equivariant bijection between T_0 and T_1.*

Furthermore (1) or (2) imply that the projectivized image of the segment $[T_0, T_1] \subset \mathbb{R}^{F_N}$ of convex combinations of T_0 and T_1 is contained in $\overline{\mathrm{CV}}_N$.

Proposition 8.1.7 (Gaboriau et al. [22, §3]). *Let $\mathcal{A}$ be a free basis of F_N ($N \geq 2$) and $T \in \overline{\mathrm{cv}}_N$. Then $\mathrm{BBT}_{T,p}(\mathcal{A}) < \infty$.*

Proposition 8.1.8 (Coulbois et al. [18, Lemma 3.1]). *Let $\mathcal{A}$ be a free basis of F_N ($N \geq 2$) and $T \in \overline{\mathrm{cv}}_N$. Let $p \in T$.*

1. *For any cyclically reduced word $w \in F(\mathcal{A})$ we have $d(p, wp) \leq 2\,\mathrm{BBT}_{T,p}(\mathcal{A}) + \|w\|_T$.*
2. *Any subword, u, of a cyclically reduced word w satisfies $d(p, up) \leq 2\,\mathrm{BBT}_{T,p}(\mathcal{A}) + \|w\|_T$.*

Proposition 8.1.9. *Let $T \in \overline{\mathrm{cv}}_N$. Then $L(T) \neq \varnothing$ if and only if $T \in \partial\,\mathrm{cv}_N$.*

Proposition 8.1.10. *If $T \in \partial\,\mathrm{cv}_N$, then there exists $\mu \in \mathrm{Curr}(F_N)$ such that $\langle T, \mu \rangle = 0$.*

Proof. Let $l \in L(T) \neq \varnothing$. For each n, let v_n be a subword of l with $|v_n|_{\mathcal{A}} = n$. Then v_n is a subword of a cyclically reduced word w_n with $\|w_n\|_T \leq E$, for any E. [18, Remark 4.2]. Write $w_n = v_n v_n'$ with v_n, v_n' freely reduced and w_n cyclically reduced. If v_n is not cyclically reduced, there exists $a_n \in \mathcal{A}^{\pm}$ such that $v_n a_n$ is cyclically reduced. Let $z_n = v_n a_n$. Let $A = \max\{\|a\|_T : a \in \mathcal{A}\}$. Now

$$\begin{aligned}
\|z_n\|_T \leq d(p, z_n p) &\leq d(p, v_n p) + d(v_n p, z_n p) \\
&= d(p, v_n p) + d(p, a_n p) \\
&\leq (2\,\mathrm{BBT}_{T,p}(\mathcal{A}) + \|w_n\|_T) + (2\,\mathrm{BBT}_{T,p}(\mathcal{A}) + A) \\
&\leq 4\,\mathrm{BBT}_{T,p}(\mathcal{A}) + E + A
\end{aligned}$$

Let $\mu = \lim_{n\to\infty} \frac{1}{\|z_n\|_\mathcal{A}} \eta_{z_n}$. Note $\mu \neq 0 \in \mathrm{Curr}(F_N)$ since we can pair it (via the intersection form) with $\mathrm{Cay}(F_N, \mathcal{A})$ for any free basis $\mathcal{A}$ and get a nonzero intersection number. Then we have

$$\begin{aligned}\langle T, \mu\rangle &= \langle T, \lim_{n\to\infty} \frac{1}{\|z_n\|_\mathcal{A}} \eta_{z_n}\rangle \\ &= \lim_{n\to\infty} \frac{1}{\|z_n\|_\mathcal{A}} \|z_n\|_T \\ &= 0\end{aligned}$$

since the $\|z_n\|_T$ are uniformly bounded. □

8.2 Proof of Theorem H

Let T be a very small arational free F_N-tree. Then T is also indecomposable (and so has dense orbits) by Theorem 8.1.4. Let μ be such that $\langle T, \mu\rangle = 0$. Recall such a μ exists by Proposition 8.1.10. Let T' be any tree such that $\langle T', \mu\rangle = 0$. Then $\mathrm{supp}(\mu) \subseteq L(T')$ by Theorem 8.1.1. Also, $\mathrm{supp}(\mu) = L'(T)$ by Corollary 8.1.3. Thus $L'(T) \subseteq L(T')$. Laminations dual to trees are always diagonally closed [18]. Thus $L(T')$ is diagonally closed, and contains $L'(T)$. Thus $L(T')$ also contains any leaf diagonal over $L'(T)$.

Since T is indecomposable, $L(T)$ is minimal up to diagonal leaves; that is, there exists a unique minimal sublamination $L'(T) \subseteq L(T)$ and $L(T)\backslash L'(T)$ consists of finitely many F_N orbits of leaves, each of which is diagonal over $L'(T)$ (see Theorem 8.1.2). Since $L(T')$ contains any leaf diagonal over $L'(T)$ and all leaves in $L(T)\backslash L'(T)$ are diagonal over $L'(T)$, we conclude that $L(T) \subseteq L(T')$. Finally, since T is arational and $L(T) \subseteq L(T')$ we have that $L(T) = L(T')$ (Proposition 8.1.5). Since T is arational, so is T' (by definition: their dual laminations are both arational laminations). Thus T' has dense orbits so we may apply Theorem 8.1.6 to T, T'.

Proposition 8.2.1. *Let $T, T' \in \partial\,\mathrm{cv}_N$ have dense orbits. If $L(T) = L(T')$ then $M_0(T) = M_0(T')$.*

Proof. Suppose $L(T) = L(T')$. Then $[T, T'] \subset \overline{\mathrm{CV}}_N$ by Theorem 8.1.6. Now T, T' are equipped with metrics $(d_T, d_{T'})$ which, by restricting to segments, we can view as non-atomic length measures. Thus restricting any metric in $[T, T']$ to segments yields a non-atomic length measure. Furthermore, the metric for $T'' \in [T, T']$ is obtained changing the metric on the underlying set T. Thus the length measure on T'' is obtained by changing the length measure on the underlying set T. Thus $T \in M_0(T')$. Similarly, $T' \in M_0(T)$ and thus $M_0(T) = M_0(T')$. □

$M_0(T)$ admits a linear, continuous, and injective map to cv_N, which has finite dimension [24]. Furthermore, $M_0(T)$ is convex [24]. Let $\Delta(T)$ be the image of this map. The (at most) $3n-4$ ergodic length measures in $M_0(T)$ correspond to trees $T_1, \ldots, T_{3n-4}$ such that if $T' \in \Delta(T)$, then $T' = \sum_{i=1}^{n} c_i T_i$, and if $T_i = T_j + T_k$, then T_j, T_k are homothetic to T_i.

If $T' \in \Delta(T)$ is such that $T' = \sum_{i=1}^{n} c_i T_i$, then there is a unique vector (c_i) such that

$$\begin{pmatrix} \|g_1\|_{T_1} & \|g_1\|_{T_2} & \cdots & \|g_1\|_{T_{3n-4}} \\ \|g_2\|_{T_1} & \|g_2\|_{T_2} & \cdots & \|g_2\|_{T_{3n-4}} \\ \vdots & \vdots & \vdots & \vdots \\ \|g_k\|_{T_1} & \|g_k\|_{T_2} & \cdots & \|g_k\|_{T_{3n-4}} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{3n-4} \end{pmatrix} = \begin{pmatrix} \|g_1\|_{T'} \\ \|g_2\|_{T'} \\ \vdots \\ \|g_k\|_{T'} \\ \vdots \end{pmatrix}$$

Given T', the vector (c_i) is obtained by forming the augmented matrix and row reducing until a $3n-4 \times 3n-4$ minor is row reduced. Note that the row reducing operations involve only finitely many rows. By interchanging rows, we may assume that the first k rows are needed in order to row reduce a $3n-4 \times 3n-4$ minor. Then knowing $\|g_j\|_{T_i}$ for $1 \leq i \leq 3n-4$ and $1 \leq j \leq k$ allows us, given $T' \in \Delta(T)$, to determine the unique vector (c_i) (and hence the unique T') for which $T' = \sum c_i T_i$.

proof of Theorem H. Let $\mu \in \mathrm{Curr}(F_N)$ be such that $\langle T, \mu \rangle = 0$. Let g_j be the finite set of conjugacy classes whose "rows" are involved in the row reduction operation described above. Let $\Sigma = \{\eta_{g_j}\}_{j=1}^{3n-4} \cup \{\mu\}$ and suppose $T' \in \overline{\mathrm{cv}}_N$ is such that $\langle T', \sigma \rangle = \langle T, \sigma \rangle$ for each $\sigma \in \Sigma$. Then by the above analysis, we have that $L(T) = L(T')$ and further $M_0(T) = M_0(T')$. Since

$$\|g_j\|_{T'} = \langle T', \eta_{g_j} \rangle = \langle T, \eta_{g_j} \rangle = \|g_j\|_T$$

the solution vector, (c_i) to the above matrix equation is the same for T as for T'. Indeed, the augmented matrices that one row reduces are identical up to the first k rows, and only the first k rows are used in row reduction. Thus $\sum c_i T_i = T = T'$ and so $T = T'$. $\square$

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