WAVELET ADAPTIVE AND PREDICTIVE CONTROL WITH APPLICATIONS TO CHEMICAL LOOPING SYSTEM

BY

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DISSERTATION

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ABSTRACT

Chemical looping process is a novel technology to separate oxygen from nitrogen using solid oxygen carrier to facilitate carbon dioxide capture in the design of next generation clean coal power plants. Application of the available control techniques to this process to guarantee its tight and robust operation and control has not resulted in satisfactory performance due to its highly uncertain (200% actuator uncertainty) nonlinear multiphase multiscale behavior, bringing in a need for developing novel model-based nonlinear multiscale control solutions – the subject of this thesis. An evolving structure wavelet network adaptive robust state and output feedback control, based on multiresolution analysis, is proposed for a class of nonlinear uncertain dynamical system. The robust control technique has been introduced to attenuate the effects of external disturbances. To meet transient performance requirements, wavelet network technique has been incorporated into $\mathcal{L}_1$ adaptive control architecture, extending the latter to a class of nonlinear infinite dimensional systems subject to bounded input operator and unknown Lipschitz nonlinearities. Projection based adaptation law recently introduced for linear infinite dimensional systems with constant and time-varying matched uncertainties has been extended to this class. The proposed extension inherits $\mathcal{L}_1$ adaptive control guaranteed transient performance for both input and output signals attained through the use of a low-pass filter in the feedback loop. Next, the multistep adaptive generalized predictive control (GPC) scheme based on online identification of multi-resolution wavelet model structure is designed for the single-input-single-output nonlinear autoregressive exogenous (NARX) models without state or input constraints. The control inputs and wavelet model parameters are calculated by optimizing the cost function using
gradient descent method. The convergence and stability of the proposed GPC scheme are proved using Lyapunov stability theorem. Then, the wavelet MRA modeling formalism is combined with design of robust nonlinear non-ad-hoc constrained MPC with guaranteed stability under mild assumptions. The benefits of all the proposed control techniques, such as fast approximation rate and stable tracking of reference trajectories are demonstrated by simulation. Finally, the real-time implementation results of the wavelet MRA based predictive control law with rate constraint on the single loop configuration of Alstom’s cold gas/solid flow chemical loop test facility in Windsor, Connecticut, are presented.
To mother – Xueling Zhou, father – Yan Zhang and wife – Ziying Li
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CHAPTER 1: INTRODUCTION

Developing technology for harnessing power from coal in an efficient and environmentally friendly way is a challenge that has to be met for sustainable use of coal. To meet this challenge a novel technology of chemical looping process in which multiple interacting loops of flowing reactive gas/solid mixtures produce energy via non-oxygen-based combustion has been developed at Alstom Power Inc. [1], [2]. In order to obtain and maintain optimal conditions for operation with reduced waste stream volume and minimum required energy, advanced optimizing control systems for chemical looping process are required. As such, process control development is needed to operate the system in a safe, integrated, and optimized fashion and is viewed as critical for enhancing the performance of the chemical looping system. However, the process, based on the multi-phase gas-solid flow, has a highly challenging uncertain nonlinear multi-scale dynamics with jumps. This dynamics is not captured well by traditional discrete-time models used in system identification, and as a result it is not amenable to tight control through techniques based on traditional models. As a consequence, there exists a need for developing model-based advanced control solutions to tightly and robustly operate and control the chemical looping process.

It is well known that the neural network (NN) is an effective system identification tool to characterize the complex nonlinear relationship between inputs and outputs [3]. Several works have successfully applied NN to control design of some real nonlinear processes [4]-[6]. It has been discovered in [7] that the performance of NN based control scheme depends heavily on the approximation precision of a chosen NN model. It is thus necessary to have an accurate multilayer function approximation NN structure to meet the desired control performance. However, constructing a suitable NN structure as well as
guaranteeing fast error convergence rate for backpropagation training algorithm with acceptable computation cost for real-time control is still an active research area and a challenging task for practitioners. In recent years, wavelet multiresolution analysis (MRA) has become a promising model building block for nonlinear system approximation [8]-[13]. It has been proved in [14] that wavelet based multiresolution decomposition is a universal approximator for a wide range of function spaces in terms of linear combination of scaling and wavelet functions. Compared to neural network, wavelet approximation has several advantages, including single training layer structure, near-optimal approximation rate, and localization both in time and frequency, making it a more attractive candidate for identification of complex nonlinear system with multi-scale fast time varying dynamics.

In order to continuously tune the network parameters to improve the control system performance, adaptive NN and wavelet network controls have been proposed for controller synthesis of nonlinear systems in [15]-[21] and references therein in which the stability of the closed-loop systems can be guaranteed via the Lyapunov’s direct method. In the design of NN or wavelet network, the precision of function approximation depends heavily on the selection of the network structure. Choosing an appropriate network structure itself is very challenging task and a trial and error off-line determination is often employed because the training set of data [19] or the prior estimate of smoothness of unknown function may not be available [17]. Therefore the capability of tuning the network structure in an online fashion is highly desirable. Several evolving structure NN or wavelet networks have been reported in [13], [22], [23]. Especially, a constructive MRA wavelet networks based adaptive control was proposed in [13] where the structure of the nonlinear adaptive controller can be adjusted online in a constructive manner by
gradually increasing the wavelet network resolution. The stability is proved by means of Lyapunov method when the closed-loop system undergoes parametric and structural changes concurrently. However, the control strategies in [13] don’t consider disturbances and the availability of the system states is required which restricts the applicability of the proposed strategies when only the system outputs are available. This problem is solved in the present work in which we propose evolving structure wavelet network adaptive robust state and output feedback controllers removing a key obstacle in applicability of the self-organizing wavelet network adaptive control. The output feedback controller employs high-gain observer to estimate the tracking error and the singularity perturbed form has been introduced to facilitate the stability analysis.

The $L_1$ adaptive control has been demonstrated in the finite-dimensional case to enable enforcement of the transient performance for both control signal and plant state through incorporating a low-pass system in the feedback loop [29]-[34]. Recently, adaptive observers and controllers for infinite dimensional systems have been proposed in [24]-[28], [73] and references therein. Particularly, adaptive projection-based observers have been formulated for infinite dimensional systems, and using these observers $L_1$ adaptive control architecture has been extended to a class of linear infinite dimensional systems subject to constant and time-varying matched uncertainties and disturbances [73]. However this technique doesn’t encompass unknown nonlinearities and are not easily extended to deal with such systems. To address the later problem, a $L_1$ wavelet network adaptive controller is developed for a class of infinite dimensional systems with bounded input operator and full state measurement under unknown matched Lipschitz nonlinearities. Under certain assumptions on the transfer function and on the solution to the Lyapunov inequality, the $L_1$ architecture is analyzed and uniform bounds for the
state and control signal are derived. The methodology in this work is an extension of the finite-dimensional $L_1$ adaptive controller, presented in [35], for general infinite-dimensional framework using the approach of [73]. The wavelet network $L_1$ adaptive control explores the projection based adaptive observer to update the wavelet network parameters and couples the identified version of this $L_1$ wavelet network controller with plant dynamics via a filtered feedback signal. The uniform performance bounds for the plant state and control signal are then derived under certain assumptions and systematic selection of the low-pass filter.

Input and state constraints are ubiquitous in practical process control engineering applications. The adaptive control alone is not well suited to coping with hard constraints on controls and states. Model predictive control (MPC) has been one of the most popular topics both in academic research and process control engineering in the past few years due to its capability of easily involving inherent process constraints into control strategy development. The successful industrial applications of MPC technology has been proven in a wide range of process control problems [36], [37]. Since optimization of finite horizon doesn’t guarantee stability necessarily especially when constraints are being considered, a great effort of research has been devoted to finding sufficient conditions for stability of MPC with and without constraints. [38] provides a general framework of designing stabilizing constrained nonlinear MPC. It indicates that the terminal cost $F(\cdot)$, terminal constraint set $X_f$ and a local stabilizing controller $k_f(\cdot)$ are crucial to ensure the constrained MPC is stabilizing. It also discusses the approaches to study the robustness of the MPC when uncertainty exists. However it doesn’t consider much detail of adaptive model predictive control and its associated robust stability. Variants of the framework mentioned above have been proposed to provide stabilizing constrained or
unconstrained MPC for both continuous and discrete time systems [39]-[45]. [46] provides another approach to guaranteeing stability of linear MPC with the use of contraction mapping and state estimation.

When system dynamics is not known and only nominal model is available, the model mismatch error will raise the question of robustness against the stability of MPC in the presence of modeling uncertainty. Several methods have been proposed to address the robustness issue of MPC with and without constraint. [48] presents a dual-mode variable horizon robust receding horizon controller using a local stabilizing controller around the origin. [49] discusses a robust MPC for constrained nonlinear system with additive uncertainties and proves the input-to-state stability is guaranteed under bounded admissible uncertainties. [50] analyzes the sufficient conditions under which the autonomous constrained system is ensured to be stable and ultimately bounded. All of these methods employ robust MPC to deal with the uncertainties explicitly. However, since robust MPC can’t learn the uncertainties in the plant, their performance is limited by the quality of the nominal model provided and the knowledge of the uncertainties a priori. In contrast, adaptive control is one of the few suitable methods which possess the ability to improve control system performance as it keeps learning the system uncertain dynamics by means of the measurement information from the real plant.

In this work, the framework of classical generalized predictive control (GPC), proposed by Clark et al. [52], [53], is used to develop a nonlinear wavelet network MRA unconstrained predictive control law first. GPC is based on the finite horizon open-loop unconstrained or constrained optimization of the quadratic objective function and moving horizon implementation. Due to its good performance, several researchers have successfully applied GPC in many control areas [54]-[56]. Since GPC uses linear
dynamic model to make predictions of process outputs over the prediction horizon, its performance will significantly degrade when the real process has severe nonlinearities and runs in a wide range of operating conditions - as is the case for a chemical looping process. Therefore, it is imperative to properly incorporate high fidelity identified nonlinear dynamic model into GPC. Accordingly, wavelet MRA framework is employed to model the nonlinear single loop cold flow of chemical looping process in the GPC scheme. Specifically, a single-input-single-output (SISO) NARX model based on wavelet MRA is trained on-line for real-time GPC application of chemical looping process. The gradient descent (GD) algorithm is developed for training both the weighting parameters of wavelet MRA model and the control sequence in the GPC scheme. Then Lyapunov stability theorems are used to guarantee the convergence of the wavelet MRA identified model and stability of the proposed GPC scheme without constraint.

The stability analysis of wavelet MRA GPC doesn’t take constraints into account. In practice, all process inputs are subject to certain constraints due to actuation limit. The design of adaptive nonlinear MPC has to address the issue of guarantee robust stability of MPC with constraints while the adaptation law is evolving. [47] proposes a method for designing a stable adaptive MPC for constrained continuous nonlinear system with parameter uncertainty by combining a parameter adjustment mechanism with robust MPC algorithms and proves the asymptotically stability of the overall close-loop system around the origin. Here, we extended the work of wavelet MRA based GPC control [51] to take constraints into consideration. We proposed a wavelet MRA based adaptive MPC strategy for regulation of constrained unknown nonlinear system subject to input and state constraints. The identified wavelet MRA model is trained online to estimate the nonlinear dynamic characteristics. The adaptation gain of identification is synthesized
using the Lyapunov function theorem, so that decay of prediction error can be guaranteed. Afterwards, the wavelet MRA identified nominal model is incorporated into the robustly stabilizing nonlinear MPC framework to address the issue of state and input constraints. The design is based on the computation of the bound on the mismatch between real system states and prediction states of the nominal model. Feasibility and stability are provided in the context of Lyapunov theorem with the inclusion of terminal cost and terminal constraint.

Finally, the proposed wavelet MRA model based GPC control scheme without taking constraint into consideration is implemented on the single loop cold flow CL process developed at Alstom Power Inc [61]. The NARX model, nonlinear in the wavelet basis, but linear in parameters, is employed as a model identifier that well characterizes the nonlinear dynamics of single loop gas/solid flow behavior. To reduce the excessive aggressiveness of the resulting control signals, ad hoc input constraints are designed and applied to limit the rate of control signal. The real-time implementation results of the resulting adaptive predictive control law with rate constraint on the single loop configuration of Alstom’s cold solid flow chemical loop test facility in Windsor, Connecticut, are presented. Next, we developed SISO spatio-temporal-wavelet models and controllers by combining 2-Partial Differential Equations (PDE) model of the riser and temporal wavelet NARX model. The response time of the 2-PDE model, which is typically less than 1 second, is much shorter than that for the NARX model we developed, for which the sampling time is 1 second. Therefore, we considered using the impulse response of the PDE model to approximate the faster dynamics of the system. The PDE model was simulated to get an impulse response, and the result was used in a convolution to get a model of the transients. To simplify the calculations, the impulse
response was first decomposed using Gaussian spatial and temporal wavelets. Simulation and experimental results verified the validity of the spatio-temporal wavelet models and controllers.

Key contributions presented in the thesis can be summarized as follows. First, state and output feedback evolving structure wavelet network adaptive robust control has been proposed for a class of uncertain nonlinear system. Second, the $\mathcal{L}_1$ wavelet network adaptive controllers incorporating the projection-based observers have been extended from finite dimensional systems to infinite dimensional systems. Third, adaptive wavelet MRA based predictive control framework without and with constraints has been developed for control of unknown nonlinear systems. The stability analysis is established for all the control strategies above. Fourth, the wavelet MRA model based generalized predictive control was successfully applied to single loop cold-flow CL process and achieved satisfactory tracking performance.
CHAPTER 2 : MATHEMATICAL PRELIMINARIES

2.1 Wavelet Multiresolution Analysis

Wavelet multiresolution analysis is a powerful function approximation tool to represent function details at different scales of resolution in both time and frequency domains in terms of shifted and dilated scaling and wavelet functions. The multiresolution analysis (MRA) [11] consists of a sequence of successive approximation closed subspaces \( V_j \in L_2(\mathbb{R}), j \in \mathbb{Z} \) satisfying

\[
\cdots V_{-1} \subset V_0 \subset V_1 \cdots
\]

\[
\bigcup_{j \in \mathbb{Z}} V_j \text{ is dense in } L_2(\mathbb{R}); \quad \bigcap_{j \in \mathbb{Z}} V_j = \{0\}
\]

\[
f(x) \in V_j \iff f(2x) \in V_{j+1}
\]

\[
f(x) \in V_j \iff f(x - 2^{-j}k) \in V_j, k \in \mathbb{Z}
\]

\[
V_j = \text{span} \{\phi_{j,k}, k \in \mathbb{Z}\}
\]

where \( \mathbb{Z} \) is the set of all integers, \( \phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) \) is an orthonormal basis for \( V_j \) and \( L_2(\mathbb{R}) \) is the space of square integrable functions of scalar real variable.

If \( W_j \) is defined to be the orthogonal complement of \( V_j \) in \( V_{j+1} \), then

\[
V_{j+1} = V_j \oplus W_j, V_j \perp W_j
\]

\[
W_j = \text{span} \{\psi_{j,k}, k \in \mathbb{Z}\}
\]

where \( \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \) is an orthonormal basis for \( W_j \). \( W_j \) provides a stable orthogonal split of \( V_{j+1} \) into low and high frequency parts \( V_j \) and \( W_j \) respectively. It then follows from Error! Reference source not found. and (2.6) that any \( V_j \) can be written for any \( l < j \) as

\[
V_j = V_l \oplus W_l \oplus W_{l+1} \oplus W_{l+2} \oplus \cdots \oplus W_j
\]

where all the subspaces are orthogonal. By virtue of (2.2) and (2.8) this implies that
\[ L_2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j \]

The functions \( \phi_{j,k} \) and \( \psi_{j,k} \) will be referred to as scaling and wavelet functions respectively which are derived using shift-invariance and dyadic dilation. The choice of the wavelet basis functions is determined by the space where the function resides. For instance, Haar wavelets and scaling functions can be chosen to represent a function with discontinuities. This yields a more concise representation of the latter than that obtainable through the use of a smoother basis because combination of infinite number of smooth functions such as cosine is required to represent discontinuity. Following (2.8) and (2.9), any \( f(x) \in L_2(\mathbb{R}) \) can be represented as

\[
f(x) = \sum_n c_{j,n} \phi_{j,n} + \sum_{j \geq j,n} w_{j,n} \psi_{j,n}
\]

where \( c_{j,n} = \langle f, \phi_{j,n} \rangle \) and \( w_{j,n} = \langle f, \psi_{j,n} \rangle \) are the coefficients of the expansion, namely weights of the wavelet network. \( \langle \cdot, \cdot \rangle \) is the inner product in \( L_2(\mathbb{R}) \), \( j \) is an integer parameter indicating scale (or dilation) while \( k \) is an integer of translation parameter representing the position of the basis function. The approximation could start from some lower resolution level \( J \) and can be truncated at certain higher resolution level \( K \), then (2.10) can be written as

\[
f(x) = \sum_n c_{j,n} \phi_{j,n} + \sum_{j=j,n}^{j=K} w_{j,n} \psi_{j,n} + e(K)
\]

where \( e(K) \) is the approximation error at \( K \)th resolution including truncation error. As \( j \to \infty \), \( V_j \to L_2(\mathbb{R}) \) and \( e(j) \to 0 \). It is also known from multiresolution property that \( \|e(j+1)\| < \|e(j)\| \).

Multidimensional wavelet basis function can be constructed using the tensor product method. A \( n \)-dimensional multiresolution approximation can be formed using 1 scalar scaling and \( 2^{n-1} \) scalar wavelet basis functions in different dimensions in the tensor
product [59]. For instance, two dimensional scaling and wavelet functions can be implemented in terms of the translates and dilates of the scalar scaling and wavelet basis functions $\phi$ and $\psi$ respectively as follows:

\[
\begin{align*}
\phi_{j,k}(x) &= \phi_{j,k_1}(x_1)\phi_{j,k_2}(x_2) \\
\psi_{j,k}^1(x) &= \phi_{j,k_1}(x_1)\psi_{j,k_2}(x_2) \\
\psi_{j,k}^2(x) &= \psi_{j,k_1}(x_1)\phi_{j,k_2}(x_2) \\
\psi_{j,k}^3(x) &= \psi_{j,k_1}(x_1)\psi_{j,k_2}(x_2)
\end{align*}
\]

(2.12)

where $x = [x_1, x_2]^T \in \mathbb{R}^2$ and $k = [k_1, k_2]^T \in \mathbb{Z}^2$. Another popular approach is to choose the wavelets to be radial functions [10]. For example, the $n$-dimensional Gaussian type wavelet function can be constructed as

\[
\psi^n(x) = x_1x_2 \cdots x_ne^{-\frac{1}{2}\|x\|^2}
\]

(2.13)

where $x = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$ and $\|x\|^2 = \sum_{i=1}^{n} x_i^2$. Similarly, the $n$-dimensional Marr wavelet function can be expressed as

\[
\psi^n(x) = (n - \|x\|^2)e^{-\frac{1}{2}\|x\|^2}
\]

(2.14)

Other wavelets functions include Mayer wavelets, Daubechies wavelets and B-spline wavelets, etc. If the wavelet basis functions could be selected to match the main a priori features of the unknown system dynamics, one could obtain a low order high fidelity approximation model characterized by significantly reduced computational demand.

2.2 Nonlinear Dynamical System Representation

Most control systems in real world contain some nonlinearities and linear model can’t capture the complex dynamic behavior associated with nonlinear systems. Therefore it’s important to investigate some accurate nonlinear dynamical models for both modeling and control purposes. NARMAX (Non-linear Autoregressive Moving Average with Exogenous Inputs) model [57] is well established input/output representation in nonlinear
system identification. Under some mild assumptions a discrete-time stochastic nonlinear SISO system can be expressed as NARMAX model of the following form:

\[
y(t) = f(y(t-1), \ldots, y(t-n_y), u(t-1), \ldots, u(t-n_u), \\
e(t-1), \ldots, e(t-n_e)) + e(t)
\]  

(2.15)

where \( y(t) \), \( u(t) \), \( e(t) \) are the system output, input, and noise, and \( t \) is discrete time, respectively. \( n_y \) and \( n_u \) are the maximum lags in the output and input respectively. \( e(t) \) with maximum lag \( n_e \) is not measurable and is assumed to be zero mean independent bounded noise variable and \( f(\cdot) \) is some nonlinear function. NARX (Nonlinear Autoregressive with Exogenous Inputs) model can be viewed as a special case of the NARMAX model as:

\[
y(t) = f(y(t-1), \ldots, y(t-n_y), u(t-1), \ldots, u(t-n_u)) + e(t)
\]  

(2.16)

The unknown nonlinear mapping \( f(\cdot) \) can be well approximated by several approaches including polynomials, neural network and other complex models. Unless some prior knowledge of system dynamics is available, most methods use nonparametric regression to estimate the nonlinear function \( f \) from the data.

### 2.3 Definitions and Theorems of \( \mathcal{L}_1 \) Preliminaries

**Definition 2.1** [35]: For a signal \( \xi(t) = [\xi_1(t), \ldots, \xi_n(t)]^T \in \mathbb{R}^n \), \( t \geq 0 \), the truncated \( \mathcal{L}_\infty \) norm and \( \mathcal{L}_\infty \) norm are defined as

\[
\|\xi_t\|_{\mathcal{L}_\infty} = \max_{i=1,\ldots,n} \left( \sup_{0 \leq \tau \leq t} |\xi_i(\tau)| \right)
\]  

(2.17)

\[
\|\xi\|_{\mathcal{L}_\infty} = \max_{i=1,\ldots,n} \left( \sup_{\tau \geq 0} |\xi_i(\tau)| \right)
\]  

(2.18)

**Definition 2.2** [35]: The \( \mathcal{L}_1 \) gain of a stable proper single-input-single-output (SISO) system \( H(s) \) is defined as

\[
\|H(s)\|_{\mathcal{L}_1} = \int_0^\infty |h(t)| \, dt
\]  

(2.19)
where \( h(t) \) is the impulse response of \( H(s) \).

**Lemma 2.1 [35]:** For a stable proper MIMO system \( H(s) \) with input \( r(t) \in \mathbb{R}^m \) and output \( x(t) \in \mathbb{R}^n \), we have

\[
\|x_t\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1} \|r_t\|_{\mathcal{L}_\infty} \tag{2.20}
\]

**Corollary 2.1 [35]:** For a stable proper MIMO system \( H(s) \), if the input \( r(t) \in \mathbb{R}^m \) is bounded, then the output \( x(t) \in \mathbb{R}^n \) is also bounded as \( \|x_t\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1} \|r_t\|_{\mathcal{L}_\infty} \).

**Lemma 2.2 [35]:** For a cascade system \( H(s) = H_2(s)H_1(s) \) where \( H_1(s) \) is a stable proper system with \( m \) inputs and \( l \) outputs and \( H_2(s) \) is a stable proper system with \( l \) inputs and \( n \) outputs, we have

\[
\|H(s)\|_{\mathcal{L}_1} \leq \|H_2(s)\|_{\mathcal{L}_1} \|H_1(s)\|_{\mathcal{L}_1} \tag{2.21}
\]

Consider a linear time invariant system

\[
\dot{x}(t) = A_m x(t) + b u(t)
\]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}, b \in \mathbb{R}^n \) and \( A_m \in \mathbb{R}^{n \times n} \) is Hurwitz, and assume that the transfer functions \((sI - A_m)^{-1}b\) is strictly proper an stable. Let \((sI - A_m)^{-1}b = n(s)/d(s)\) where \(d(s) = \det(sI - A_m)\) is nth order stable polynomial and \(n(s)\) is a \(n \times 1\) vector with its \(i\)th element being a polynomial function \(n_i(s) = \sum_{j=1}^n n_{ij}s^{j-1}\).

**Lemma 2.3 [35]:** If \((A_m \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n)\) is controllable, the matrix \(N\) with entries \(n_{ij}\) is full rank.

**Lemma 2.4 [35]:** If \((A_m, b)\) is controllable and \((sI - A_m)^{-1}b\) is strictly proper and stable, there exists \(c_0 \in \mathbb{R}^n\) such that the transfer function \(c_0(sI - A_m)^{-1}b\) is minimum phase with relative degree one, i.e. all its zero are located in the left half plane, and its denominator is one order larger than its numerator.
CHAPTER 3: EVOLVING STRUCTURE WAVELET NETWORK ADAPTIVE ROBUST CONTROL OF UNCERTAIN NONLINEAR SYSTEM

In this chapter, evolving structure wavelet network adaptive robust state and output feedback control are proposed for a class of nonlinear uncertain dynamical system. The structure of the wavelet network is capable of evolving itself online according to the tracking performance. It effectively avoids the time consuming effort of trial and error way of off-line determining the network structure. The multiresolution property ensures the improvement of approximation accuracy with inclusion of higher resolution. On the other hand, the orthonormal property of scaling and wavelet basis functions guarantees that the introduction of new resolution will not affect the training of existing network weights. The robust control technique has been incorporated to attenuate the effects of external disturbance. The $H_{\infty}$ design is adopted to address the situation when the exact upper bound of the uncertainty cannot be obtained in general. When the system states are not available for control design, an adaptive wavelet robust output feedback control scheme is presented for the output tracking. The method uses control saturation, high-gain observer to achieve uniform ultimate boundedness. The effectiveness of the proposed methods is demonstrated through simulations.

3.1 Problem Formulation

Consider the single-input-single-output (SISO) nonlinear uncertain dynamic systems modeled by

$$\begin{cases}
\dot{x}_1(t) = x_{i+1}(t) \\
\dot{x}_n(t) = f(x) + bu(t) + d(t) \\
y(t) = x_1(t)
\end{cases}
$$

where $x = [x_1 \cdots x_n]^T \in D_x \subset R^n$ is the state vector assumed to be measurable and $D_x$ is a compact set in $R^n$. $u \in R$ is the control input and $b \in R$ is known constant. $y \in R$
is the system output. \( f(x) : R^n \rightarrow R \) is unknown function and \( d(t) \in R \) is the external disturbance.

Rewrite the system dynamics (3.1) in the controllable canonical form as:
\[
\begin{aligned}
\dot{x} &= Ax + B(f(x) + bu + d) \\
y &= Cx
\end{aligned}
\]
\[A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{n \times n}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{1 \times n}, \quad C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}_{1 \times n}
\]
where \((A, B)\) is the canonical controllable pair that represents a chain of \(n\) integrators.

The control objective is to design an evolving structure wavelet adaptive controller to enforce the system output \(y\) to track the reference trajectory \(y_d\). Assume the reference trajectory \(y_d \in R\) has continuous derivatives up to \(n\)th order. Define reference vector \(x_d\) as
\[
x_d = [y_d, y_d^{(1)}, \cdots, y_d^{(n-1)}]^T \quad \text{and} \quad y_d^{(n)} \in \mathcal{D}_{x_d} \subset R^n
\]
where \(y_d^{(k)}, k = 0, \ldots, n\), is the \(k\)th derivative of \(y_d\) and \(\mathcal{D}_{x_d}\) is a compact set in \(R^n\).

Define tracking error vector \(e\) as
\[
e = x - x_d = [y - y_d, y^{(1)} - y_d^{(1)}, \cdots, y^{(n-1)} - y_d^{(n-1)}]^T
\]
Then the derivative of tracking error is defined as
\[
\dot{e} = Ae + B(f(x) + bu - y_d^{(n)} + d)
\]
Therefore control objective is now rigorously stated in terms of error vector \(e\) as follows:

Design an adaptive wavelet control algorithm \(u(t)\) so that for any specified tracking error bound \(\rho > 0\), there exists a finite time \(T > 0\) such that \(\forall t > T, \|e\| < \rho\), where \(\| \cdot \|\) is used for the Euclidean norm unless otherwise specified.
If the system dynamics $f(x)$ and external disturbance $d$ are known a priori, the nominal control law can be obtained as

$$u = \frac{1}{b}(-f(x) + y_d^{(n)} - ke - d)$$

(3.6)

where $k \in R^n$ is chosen design parameter such that $A_m = A - Bk$ is Hurwitz.

Substitution of (3.6) into (3.5) yields

$$\dot{e} = A_m e$$

(3.7)

which means that the tracking error $e$ converges to zero asymptotically. Thus the error $\|e\|$ will meet the control objective in finite time interval. However, since dynamic function $f(x)$ and disturbance $d$ are usually not known in practice, the nominal controller (3.6) can’t be obtained precisely. To address this problem, an evolving structure wavelet network identifier is utilized to approximate the unknown system dynamics and a robust controller is designed to attenuate the effect of external disturbance.

### 3.2 State Feedback Wavelet Adaptive Robust Controller

The proposed state feedback controller consists of a wavelet adaptive controller $u_w$ and a robust controller $u_r$, as shown in Figure 3.1. The control law is developed as follows:

$$u = u_w + u_r$$

(3.8)

The wavelet network based adaptive controller $u_w$ has the following form

$$u_w = \frac{1}{b}(-\hat{f}(x) + y_d^{(n)} - ke)$$

(3.9)

where $\hat{f}(x)$ is the wavelet network based approximator of the unknown dynamics $f(x)$ which will adaptively update its own structure based on required approximation.
resolution. \( k \in R^n \) is selected design parameter such that \( A_m = A - Bk \) is Hurwitz. Let \( P_m \) be the positive definite solution to the following Lyapunov matrix equation

\[
A_m^T P_m + P_m A_m = -2Q_m
\]  

(3.10)

where \( Q_m > 0 \) is positive definite matrix. We choose \( Q_m \) such that \( \lambda_{\text{min}}(Q_m) = 1 + \lambda_s \) where \( \lambda_s > 0 \) is a positive constant. Define \( \mu_R = \lambda_{\text{max}}(P_m) \). \( \lambda_{\text{min}}(\cdot) \) and \( \lambda_{\text{max}}(\cdot) \) are the minimum and maximum eigenvalue of the corresponding matrix respectively.

Figure 3.1 State feedback wavelet adaptive robust controller for nonlinear system

From wavelet multiresolution approximation (MRA) properties [11], any \( f(x) \in L^2(R^n) \) can be expressed as

\[
f(x) = \sum_{k \in Z^n} c_{j,k}^* \phi_{j,k}(x) + \sum_{j=J}^{\infty} \sum_{k \in Z^n} w_{j,k}^* \psi_{j,k}(x)
\]

\[
= \sum_{k \in Z^n} c_{j,k}^* \phi_{j,k}(x) + \sum_{j=J}^{\infty} \sum_{k \in Z^n} w_{j,k}^* \psi_{j,k}(x) + \varepsilon(J_k)
\]

(3.11)
where $\phi_{j,k}(x) : R^n \to R$ is the $j$th level $n$-dimensional scaling function and $\psi_{j,k} : R^n \to R$ is the $j$th level $n$-dimensional wavelet function. $c_{j,k}^* = \langle f(x), \phi_{j,k}(x) \rangle \in R$ is the $j$th level scaling coefficients and $w_{j,k}^* = \langle f(x), \psi_{j,k}(x) \rangle \in R$ is the $j$th level wavelet coefficients where $\langle \cdot, \cdot \rangle$ is the inner product of $L^2(R^n)$. $j = J, J+1, \ldots, J_K, \ldots$ represents the $j$th level scale resolution and $k \in Z^n$ is the $n$-dimensional translation parameter indicating position of the scaling and wavelet functions. $\varepsilon(j_k) \in R$ is the $J_K$th level approximation error including projection error and truncation error due to the adoption of finite number of bases functions. Because of the multiresolution property of wavelet network [11], it has the following property:

$$|\varepsilon(j)| \geq |\varepsilon(j + 1)| \geq \cdots \geq |\varepsilon(j_K)| \geq \cdots \text{ and } \lim_{j \to \infty} |\varepsilon(j)| = 0 \quad (3.12)$$

The multiresolution property guarantees that with the increase of levels of resolutions, the approximation precision will improve accordingly. Making the similar assumption as in [13] that the wavelet network approximator is of finite size defined as

**Assumption 3.1:** Let $x$ belongs to a compact set $D_x$, there exists a finite but unknown integer $K \geq J$ such that at the $K$th resolution, the approximation error $\varepsilon_K$ of the wavelet network identifier satisfies the specified precision bound $\rho_w = \rho/\mu_R$ i.e.

$$f(x) = (c_j^*)^T \phi_j(x) + \sum_{j=J}^{J_K} (w_j^*)^T \psi_j(x) + \varepsilon_K \quad (3.13)$$

where $|\varepsilon_K| < \rho_w$ and

$$c_j^* = [c_{j,k_1}^*, \ldots, c_{j,k_{N_j}}^*]^T$$
$$w_j^* = [w_{j,k_1}^*, \ldots, w_{j,k_{N_j}}^*]^T$$
$$\phi_j(x) = [\phi_{j,k_1}(x), \ldots, \phi_{j,k_{N_j}}(x)]^T$$
$$\psi_j(x) = [\psi_{j,k_1}(x), \ldots, \psi_{j,k_{N_j}}(x)]^T$$
where \( k_i \in \mathbb{Z}^n, i = 1, \cdots, N_j, j = J, \cdots, K \). \( N_j \in \mathbb{N} \) denotes the number of scaling and wavelet functions at \( j \)th level resolution. \( c^*_j \in \mathcal{D}_c \subseteq \mathbb{R}^{N_j} \) and \( w^*_j \in \mathcal{D}_w \subseteq \mathbb{R}^{N_j} \) denote the optimal scaling and wavelet weight vectors at each resolution level only for analytical purposes. \( \mathcal{D}_c \) and \( \mathcal{D}_w \) are some known compact sets. In addition, there exists finite constant \( c_w > 0 \), such that

\[
\sum_{j=K}^{\infty} (w^*_j)^T \gamma_{w,j}^{-1}(w^*_j) \leq c_w
\]

(3.14)

where \( \gamma_{w,j} > 0, j = J, J+1, \cdots \) are constant positive definite adaptation gain matrices to be chosen later in the parameter update law.

Because the optimal weights are not known and difficult to determine a priori, the fixed optimal parameters \( c^*_j \) and \( w^*_j \) are replaced by their time varying estimates \( \hat{c}_j \) and \( \hat{w}_j \) which will be adapted on line. The approximated nonlinear dynamics \( \hat{f}(x) \) in (3.9) is then represented as

\[
\hat{f}(x) = (\hat{c}_j)^T \phi_j(x) + \sum_{j=J}^{J_K(t)} (\hat{w}_j)^T \psi_j(x)
\]

(3.15)

where \( J_K(t) \in \mathbb{Z} \) is the current resolution level at time \( t \) which is determined automatically online in terms of the tracking error \( e \). There are two parameters characterizing the wavelet network identifier \( \hat{f}(x) \), the required scaling level \( J_K(t) \) and translation vector \( k_i \), where \( i = 1, \cdots, N_j \) and \( j = J, \cdots, J_K(t) \). We present below an evolving structure wavelet network that is capable of determining the \( J_K(t) \) by itself in real time to guarantee the desired approximation accuracy while keeping the network complexity. Our wavelet network approximator employs the so called compact supported orthonormal wavelets as scaling and wavelet basis functions. Examples of such functions includes Harr scaling and wavelet functions, Daubechies scaling and wavelet functions
The compact support property substantially reduces the computational effort for training and evaluation. Only a small number of wavelets at each resolution level which cover the data range need to be considered during approximation. For high dimensional problems, the saving of computational time will be more dramatic which provides an effective solution to deal with the curse of dimensionally problem.

Given the compact set $\mathcal{D}_x$ in which the nonlinear dynamics $f(x)$ resides, we first need to choose the type of scaling and wavelet basis functions for the wavelet network approximator. Then following the idea introduced in [13], we utilize the predetermined tracking error threshold $\rho = \mu_R \rho_w$ and a designed parameter $T_R > 0$ called observation time to adaptively determine the desired resolution level $J_K(t)$. The detailed description of the evolving structure wavelet approximation procedure is stated below.

We start with the coarse level $J$ for wavelet network at time zero. During each following time interval $[(i-1)T_R, iT_R)$, $i = 1,2, \ldots$, we need to check if the Euclidean norm of the tracking error $\|e(t)\|$ is ever greater than the threshold $\rho$. If the track error $\|e(t)\|$ is kept lower than $\rho$ throughout the whole time interval $t \in [(i-1)T_R, iT_R)$ which means that the closed loop system has satisfied the tracking performance requirement, we will not add new resolution level in the next time interval $[iT_R, (i + 1)T_R)$. Otherwise, if the tracking error $\|e(t)\|$ exceeds the precision bound $\rho$, we will add a new resolution level during next time interval. That is, if the current resolution level is $J_K(t) = J_i$ for $t \in [(i-1)T_R, iT_R)$, then $J_K(t) = J_i + 1$ for $t \in [iT_R, (i + 1)T_R)$. The multiresolution property will ensure the improvement of control performance due to the reduction of approximation error with increasing resolution level, i.e. $|\epsilon(J_i)| \leq |\epsilon(J_i+1)|$. On the other hand, since all scaling and wavelet basis functions are orthonormal, the tuning of scaling and wavelet weight vectors $\hat{e}_j$ and $\hat{w}_j$ are
independent of each other. Therefore adding new resolution level and new scaling and wavelet basis functions will not affect the tuning of existing parameter set. The adaptation steps of resolution level and basis function will repeat until required tracking performance is satisfied. The evolving structure wavelet network algorithm is now summarized as follows: In each sampling period,

1) Measure the state vector $x$ and obtain the tracking error $e$ according to (3.4).

2) Update the scaling and wavelet weight vectors adaptively and concurrently by certain adaptation laws.

3) During each time interval $[(i - 1)T_R, iT_R)$, check if $\|e\| > \rho$. If yes, add a new resolution level to the current wavelet network in the next period $[iT_R, (i + 1)T_R)$, i.e. $J_K(t) = J_i + 1$.

4) Repeat the process above until specified control objective has been satisfied.

The wavelet network weight parameter adaptation law is chosen similarly as in [13]. Define

$$
\hat{c}_j = \hat{c}_j - c_j^* \\
\hat{w}_j = \hat{w}_j - w_j^*
$$

(3.16)

Since $c_j^*$ and $w_j^*$ are constants, we have $\dot{\hat{c}}_j = \hat{c}_j$ and $\dot{\hat{w}}_j = \hat{w}_j$. We want to design such adaptation rules that the scaling and wavelet weight estimates $\hat{c}_j$ and $\hat{w}_j$ belong to some compact sets $\hat{D}_c$ and $\hat{D}_w$ respectively. Define $\mathcal{D}_c$ and $\mathcal{D}_w$ as

$$
\mathcal{D}_c = \{c_j^* : \underline{\xi} \leq c_j^{\star, k_l} \leq \bar{\xi} , \quad i = 1, \cdots, N_j \} \\
\mathcal{D}_w = \{w_j^* : \underline{\omega}_j \leq w_j^{\star, k_l} \leq \bar{\omega}_j , \quad i = 1, \cdots, N_j, j = J, \cdots, J_K(t) \} 
$$

(3.17)

where $\underline{\xi}$, $\bar{\xi}$, $\underline{\omega}_j$ and $\bar{\omega}_j$, $j = J, \cdots, J_K(t)$ are the lower bound and upper bound design parameters for wavelet network approximator. We adopt the smooth projection operator in [18] as follows:
\[
\begin{align*}
\dot{e}_j &= \begin{cases} 
\text{proj} \left( \hat{e}_j, Y_c B^T P_m e \phi_j(x) \right), & t \in \Omega_1 \\
0, & t \in \Omega_2 
\end{cases}, \\
\hat{w}_j &= \begin{cases} 
\text{proj} \left( \hat{w}_j, Y_{w,j} B^T P_m e \psi_j(x) \right), & t \in \Omega_1, j = J, \ldots, J_K(t) \\
0, & t \in \Omega_2 
\end{cases}
\end{align*}
\] (3.18)

where \( Y_c > 0 \) and \( Y_{w,j} > 0 \) are positive definite adaptation gain matrices with appropriate dimensions. We define the similar notations as in [13]:

\[
\begin{align*}
\Omega_1 &\triangleq \{ t | t \in [0, \infty), \| e \| \geq \rho = \mu_R \rho_w \} \\
\Omega_2 &\triangleq \{ t | t \in [0, \infty), \| e \| < \rho = \mu_R \rho_w \} \\
\Omega_a^b &\triangleq [a, b] \cap \Omega_1, \Omega_a &\triangleq [a, b] \cap \Omega_2 
\end{align*}
\] (3.20)

Let \( T_j, \forall j = J, J + 1, \ldots \) denote the time instants when a new resolution is added to the wavelet network approximator and \( T_j = 0 \). Set \( \hat{w}_j(T_j) = 0, \forall j = J, J + 1, \ldots \).

Let

\[
\begin{align*}
D_{c\delta} &= \{ \hat{e}_j : c - \delta_{c,j} \leq \hat{e}_j, k_i \leq \bar{c} + \delta_{c,j} \} \\
D_{w\delta} &= \{ \hat{w}_j : w_j, \delta_{w,j} \leq \hat{w}_j, k_i \leq \bar{w}_j + \delta_{w,j} \} \\
i &= 1, \ldots, N_j, j = J, \ldots, J_K(t) 
\end{align*}
\] (3.21)

where \( \delta_{c,j} > 0 \) and \( \delta_{w,j} > 0 \) are such that \( D_c \subset D_{c\delta} \subset \bar{D}_c \) and \( D_w \subset D_{w\delta} \subset \bar{D}_w \).

For \( t \in \Omega_1 \), we define \( \text{proj} \left( \hat{e}_j, Y_c B^T P_m e \phi_j(x) \right)_i \) and \( \text{proj} \left( \hat{w}_j, Y_{w,j} B^T P_m e \psi_j(x) \right)_i \) as
The adaptation law in (3.22) and (3.23) will ensure that the weight vectors \( \hat{c}_j \) and \( \hat{\omega}_j \) will belong to the compacts \( \mathcal{D}_{c\delta} \) and \( \mathcal{D}_{w\delta} \) respectively if their initial values resides in \( \mathcal{D}_{c\delta} \) and \( \mathcal{D}_{w\delta} \) for \( t \in \Omega_1 \). The weight estimates will also satisfy the following property for \( t \in \Omega_1 \):

\[
\hat{c}_j^T \gamma_c^{-1} \left( \hat{c}_j - \gamma_c e^T p_m B \phi_j(x) \right) \leq 0 \tag{3.24}
\]

\[
\hat{\omega}_j^T \gamma_{wj}^{-1} \left( \hat{\omega}_j - \gamma_{wj} e^T p_m B \phi_j(x) \right) \leq 0 \tag{3.25}
\]

We first consider the case when no disturbance is present that is \( d(t) = 0 \) in (3.2). Then the wavelet adaptive controller which guarantees asymptotic stability of the uncertain nonlinear system in (3.2) can be obtained from Theorem 3.1 below.
**Theorem 3.1:** Consider the uncertain nonlinear dynamic system in (3.2) with \( d(t) = 0 \) satisfying Assumption 3.1 and the evolving structure wavelet network adaptive controller chosen as (3.8), (3.9), (3.15) with \( u_r = 0 \) and adaptation laws (3.18) - (3.19), the tracking error is uniformly bounded and converges to the prespecified error bound \( \rho = \mu_R \rho_w \) in a finite time interval.

Proof: Define the following piecewise differentiable Lyapunov function candidate:

\[
V(t) = \begin{cases} 
\frac{1}{2} e^T P_m e + \frac{1}{2} \tilde{c}_j^T \gamma_c^{-1} \tilde{c}_j + \frac{1}{2} \sum_{j=1}^{J_K(t)} \tilde{w}_j^T \gamma_{w,j}^{-1} \tilde{w}_j, & \forall t \in \Omega_1 \\
\frac{1}{2} \rho^T P_m \rho + \frac{1}{2} \tilde{c}_j^T \gamma_c^{-1} \tilde{c}_j + \frac{1}{2} \sum_{j=1}^{J_K(t)} \tilde{w}_j^T \gamma_{w,j}^{-1} \tilde{w}_j, & \forall t \in \Omega_2 
\end{cases}
\]

(3.26)

We first need to show the boundness of the \( V(t) \) for any finite resolution level \( J_K(t) \). Then we will show the finite time convergence of the tracking error \( e \) as the resolution increases.

To prove the boundness of \( V(t) \), we need to consider three cases for the finite resolution level \( J_K(t) \) respectively as follows.

1. \( J_K(t) = J \). Only the coarsest resolution level is added into the wavelet network approximator. Evaluating the time derivative of \( V(t) \) along the tracking error dynamics (3.5), we obtain

\[
\text{If } t \in \Omega_{T_{J+1}}^{T_{J+1}}
\]
\[ \dot{V}(t) = e^T P_m \dot{e} + \hat{e}_j^T Y_c^{-1} \hat{\dot{e}}_j + \hat{\tilde{w}}_j^T Y_{w,j}^{-1} \hat{\tilde{w}}_j \]
\[ = e^T P_m \left( A e + B \left( f(x) + bu - y_d^{(n)} \right) \right) + \hat{e}_j^T Y_c^{-1} \hat{\dot{e}}_j + \hat{\tilde{w}}_j^T Y_{w,j}^{-1} \hat{\tilde{w}}_j \]
\[ = e^T P_m A_m e + e^T P_m B \left( f(x) - \hat{f}(x) \right) + \hat{e}_j^T Y_c^{-1} \hat{\dot{e}}_j + \hat{\tilde{w}}_j^T Y_{w,j}^{-1} \hat{\tilde{w}}_j \]
\[ = -e^T Q_m e - e^T P_m B \hat{e}_j^T \phi_j(x) - e^T P_m B \hat{\tilde{w}}_j^T \psi_j(x) + e^T P_m B \epsilon(J) \]
\[ + \hat{\epsilon}_j^T Y_c^{-1} \hat{\epsilon}_j + \hat{\tilde{w}}_j^T Y_{w,j}^{-1} \hat{\tilde{w}}_j \]
\[ = -e^T Q_m e + \hat{\epsilon}_j^T Y_c^{-1} \left( \hat{\epsilon}_j - \gamma_c e^T P_m B \phi_j(x) \right) \]
\[ + \hat{\tilde{w}}_j^T Y_{w,j}^{-1} \left( \hat{\tilde{w}}_j - \gamma_{w,j} e^T P_m B \psi_j(x) \right) + e^T P_m B \epsilon(J) \] (3.27)

Using the adaptation laws (3.18) - (3.19) and taking into account (3.24) - (3.25), it could be derived that
\[ \dot{V}(t) \leq -e^T Q_m e + e^T P_m B \epsilon(J) \] (3.28)

It then follows that
\[ \dot{V}(t) \leq -e^T Q_m e + e^T P_m B \epsilon(J) \]
\[ \leq -\lambda_{\text{min}}(Q_m) \| e \|^2 + \lambda_{\text{max}}(P_m) \| e \| \| \epsilon(J) \| \]
\[ = -(1 + \lambda_c) \| e \|^2 + \lambda_{\text{max}}(P_m) \| e \| \| \epsilon(J) \| \]
\[ = -\lambda_s \| e \|^2 - \| e \| ( \| e \| - \lambda_{\text{max}}(P_m) \| \epsilon(J) \| ) \]
\[ = -\lambda_s \| e \|^2 - \| e \| ( \| e \| - \mu_R \| \epsilon(J) \| ) \] (3.29)

where we have used \( \| B \| = 1 \) because of its definition.

If \( \| e \| > \mu_R \| \epsilon(J) \| \), it follows that
\[ \dot{V}(t) \leq 0, \forall t \in \Omega_{T_j}^{T_{j+1}} \] (3.30)

Because of the positive definiteness of \( V(t) \) and (3.30), we have that \( \| e \| \) is bounded for all \( t \in \Omega_{T_j}^{T_{j+1}} \).

If \( t \in \Omega_{T_j}^{T_{j+1}} \), the definition of Lyapunov function (3.26) and adaptation laws (3.18) - (3.19) implies that \( \dot{V}(t) = 0 \). Therefore for any time instants \( t \in [T_j, T_{j+1}] \), from (3.29)
and finiteness of $\mu_R = \lambda_{\max}(P_m)$ and $\|\varepsilon(j)\|$, there exists a finite constant $M_j > 0$ such that

$$V(t) = V(0) + \int_0^t \dot{V}(\tau)d\tau \leq V(0) + \int_{\tau \in \Omega_{T_j}} -\lambda_s \|e\|^2 - \|e\|(\|e\| - \mu_R \|\varepsilon(j)\|)d\tau \leq V(0) + \int_{\tau \in \Omega_{T_j}} \mu_R \|e\|\|\varepsilon(j)\|d\tau \leq M_j < \infty \quad (3.31)$$

Then from the Lyapunov function definition (3.26), the tracking error $e$, the wavelet weight estimate $\hat{e}_j$ and $\hat{w}_j$ are all bounded for $t \in [T_j, T^{-}_{j+1}]$. If the desired tracking performance is not satisfied at $t = T^{-}_{j+1}$, a new resolution level $j = J + 1$ will be added to the wavelet approximator at $t = T^+_{j+1}$ which implies the discontinuity of $V(t)$ at $t = T_{j+1}$. Because of the smooth projection operator (3.22) and (3.23), $\hat{e}_j$ and $\hat{w}_j$ are continuous at $t = T_{j+1}$. Then at $t = T^+_{j+1}$, we could obtain

$$V(T^+_{j+1}) = V(T^{-}_{j+1}) + \frac{1}{2} \widetilde{w}_{j+1}^T(T_{j+1})Y_{w,j+1}^{-1}\widetilde{w}_{j+1}(T_{j+1}) = V(T^{-}_{j+1}) + \frac{1}{2}(w^*_{j+1})^TY_{w,j+1}^{-1}w^*_{j+1} \leq M_j + \frac{1}{2}(w^*_{j+1})^TY_{w,j+1}^{-1}w^*_{j+1} < \infty \quad (3.32)$$

where we have used the property that $\widetilde{w}_{j+1}(T_{j+1}) = 0$. Since both $w^*_{j+1}$ and $Y_{w,j+1}^{-1}$ are constants, it follows that $V(T^+_{j+1})$ is bounded too.

(2) $J < J_K(t) = J_l < K$. Now $J_l$ levels of resolution have been added to the wavelet network and the approximation error $\varepsilon(J_l) \geq \varepsilon_K$.

If $t \in \Omega_{T_{J_l}}$, (3.27) becomes
\[ \dot{V}(t) \leq -e^T Q_m e + e^T P_m B \varepsilon(j_i) \]
\[ \leq -\lambda_s \|e\|^2 - \|e\|(\|e\| - \mu_R \|\varepsilon(j_i)\|) \]  \hspace{1cm} (3.33)

Following the similar arguments in (1), we could prove that \(\|e\|\) is bounded for all \(t \in \Omega_{t_{j_l}}^{\text{II}+1}\). Since \(\dot{V}(t) = 0\) for \(t \in \overline{\Omega}_{t_{j_l}}^{\text{II}+1}\), it could be proved that \(V(t)\) is bounded for all \(t \in [T_{j_l}, T_{j_{l+1}}^-]\). It follows that \(V(t)\) is bounded for \(t \in [0, T_{j_{l+1}}^-]\) and any \(j_l < K\).

Let \(j_l\) be \(K - 1\). Then we have \(V(T_{K}^-) \leq M_K\). Similarly as (3.32),
\[ V(T_{K}^+) = V(T_{K}^-) + \frac{1}{2} \tilde{w}_K^T(T_K) \gamma_{w,K}^{-1} \tilde{w}_K(T_K) \]
\[ \leq M_K + \frac{1}{2} (w_K^*)^T \gamma_{w,K}^{-1} w_K^* < \infty \]  \hspace{1cm} (3.34)

(3) \(J_K(t) = l \geq K\). From Assumption 3.1, we have that \(\varepsilon(l) \leq |\varepsilon_K| < \rho_w = \rho / \mu_R\).

If \(t \in \Omega_{t_l}^{T_{l+1}^+}\), then \(\|e\| \geq \rho\) and (3.27) becomes
\[ \dot{V}(t) \leq -e^T Q_m e + e^T P_m B \varepsilon(l) \]
\[ \leq -\lambda_s \|e\|^2 - \|e\|(\|e\| - \mu_R \|\varepsilon(l)\|) \]
\[ \leq -\lambda_s \|e\|^2 \]
\[ < 0 \]  \hspace{1cm} (3.35)

Since \(\dot{V}(t) = 0\) when \(t \in \overline{\Omega}_{t_l}^{T_{l+1}^+}\), we could derive that for \(t \in [T_l^+, T_{l+1}^-]\)
\[ V(t) = V(T_i^+) + \int_{T_i^+}^{t} \dot{V}(\tau) d\tau \]
\[ = V(T_i^+) + \int_{\tau \in \Omega_{T_i^+}^{\tau}} \dot{V}(\tau) d\tau \]
\[ \leq V(T_i^-) + \frac{1}{2} (w_i^*)^T \gamma_w^{-1} w_i^* - \int_{\tau \in \Omega_{T_i^+}^{\tau}} \lambda_s \| e \|^2 d\tau \]
\[ \vdots \]
\[ \leq V(T_K^+) + \sum_{j=K+1}^{l} \frac{1}{2} (w_j^*)^T \gamma_w^{-1} w_j^* - \int_{\tau \in \Omega_{T_K^+}^{\tau}} \lambda_s \| e \|^2 d\tau \]
\[ \leq M_K + \sum_{j=K}^{l} \frac{1}{2} (w_j^*)^T \gamma_w^{-1} w_j^* \]
\[ \leq M_K + \sum_{j=K}^{l} \frac{1}{2} (w_j^*)^T \gamma_w^{-1} w_j^* \quad (3.36) \]

which implies that \( V(t) \) is finite for \( t \in [T_i^+, T_{i+1}^-] \) and \( l \geq K \).

Now we will prove that there exists finite \( T(\rho) > 0 \) such that the tracking error \( \| e \| < \rho \) for \( t \geq T(\rho) \) by contradiction. Suppose no such \( T(\rho) \) exists, then \( \Omega_1 \to \infty \).

It follows that the resolution level \( f_K(t) \to \infty \) as well because of the finite response time \( T_R \). Then from (3.36) we could have

\[ \lim_{t \to \infty} V(t) \leq M_K + \lim_{l \to \infty} \sum_{j=K}^{l} \frac{1}{2} (w_j^*)^T \gamma_w^{-1} w_j^* - \lim_{t \to \infty} \int_{\tau \in \Omega_{T_K^+}^{\tau}} \lambda_s \| e \|^2 d\tau \quad (3.37) \]

From Assumption 3.1, we have that \( \sum_{j=K}^{\infty} \frac{1}{2} (w_j^*)^T \gamma_w^{-1} (w_j^*) \leq \frac{1}{2} c_w \). And \( \| e \| \geq \rho \) for \( \tau \in \Omega_{T_K^+}^{\tau} \). (3.37) now becomes

\[ \lim_{t \to \infty} V(t) \leq M_K + \frac{1}{2} c_w - \lim_{t \to \infty} \int_{\tau \in \Omega_{T_K^+}^{\tau}} \lambda_s \rho^2 d\tau \quad (3.38) \]

Since \( \Omega_1 \to \infty \), \( \lim_{t \to \infty} \Omega_{T_K^+}^{\tau} \to \infty \). Then \( \lim_{t \to \infty} V(t) \leq M_K + \frac{1}{2} c_w - \lambda_s \rho^2 \to 0 \) which contradicts \( V(t) \geq 0 \). Therefore there exists a finite time \( T(\rho) \) such that tracking error \( e \) will satisfy \( \| e \| < \rho \) for \( t \geq T(\rho) \) and the proof completes. \( \blacksquare \)
Now, let’s consider the robust adaptive control strategy for system (3.2) in the presence of additional disturbance signal \( d(t) \). In the design of robust controller, the upper bound of unknown external disturbance usually needs to be determined in advance to construct the robustifying control input \( u_r \) [22]. In practice, however, the a priori knowledge of the exact upper bound \( d^u \) may not be available in general. In order to relax this restriction, we propose a new wavelet network adaptive control law using \( H_\infty \) tracking design technique based on more relaxed assumption. In the new control law, the adaptive component \( u_w \) is identical as in (3.9) while we assume the wavelet network has \( J_K(t) \) levels of resolution at time \( t \). Denote the wavelet network approximation error as \( \epsilon(K) \) when there are \( J_K(t) \) levels of resolution in the wavelet network similar to (3.11). Define the lumped uncertainty \( \omega \) as

\[
\omega = \epsilon(K) + d
\]

(3.39)

*Assumption 3.2*: The lumped uncertainty \( \omega \) is assumed such that \( \omega \in L_2[0,T], \forall T \in [0,\infty) \).

Now we can prove the guarantee of \( H_\infty \) tracking performance for the overall system without prior knowledge on the upper bound of the lumped uncertainties of the uncertain nonlinear system.

*Theorem 3.2*: Consider the uncertain nonlinear dynamic system in (3.2) satisfying Assumption 3.2 and the evolving structure wavelet network adaptive controller chosen as (3.8), (3.9), (3.15), and adaptation laws (3.18) - (3.19). Choose the robustifying control component \( u_r \) as

\[
u_r = -\frac{\sigma}{2b\rho_v^2} \]

(3.40)
where $\rho_r > 0$ is a design constant representing the prescribed attenuation level and $\sigma = e^T P_m B$. Then the tracking performance for the overall system satisfies the following properties:

$$
\int_0^T e^T Q_m e d\tau \leq \frac{1}{2} e(0)^T P_m e(0) + \frac{1}{2} \tilde{c}_f^T(0) \gamma_c^{-1} \tilde{c}_f(0) + \frac{1}{2} \tilde{w}_f^T(0) \gamma_{w_f}^{-1} \tilde{w}_f(0) + \frac{1}{2} \rho_r^2 \int_0^T \omega^2(\tau) d\tau
$$  \hspace{1cm} (3.41)

**Proof:** Using the same Lyapunov function (3.26), if $t \in \Omega_0^T$, $\forall T' \in [0, \infty)$, the derivative of $V(t)$ is

$$
\dot{V}(t) \leq -e^T Q_m e + e^T P_m B e(\dot{K}) + e^T P_m B d + e^T P_m B b u_r
$$

$$
= -e^T Q_m e + \sigma \omega + \sigma b u_r
$$

$$
= -e^T Q_m e - \frac{1}{2} \left( \frac{\sigma}{\rho_r} - \rho_r \omega \right)^2 + \frac{1}{2} \rho_r^2 \omega^2
$$

$$
\leq -e^T Q_m e + \frac{1}{2} \rho_r^2 \omega^2
$$  \hspace{1cm} (3.42)

By Assumption 3.2, integrating (3.42) from $t = 0$ to $t = T$ and noting $V(T') \geq 0$, we obtain

$$
\int_{t \in \Omega_0^T} e^T Q_m e d\tau \leq V(0) + \frac{1}{2} \rho_r^2 \int_{t \in \Omega_0^T} \omega^2(\tau) d\tau
$$

$$
\leq V(0) + \frac{1}{2} \rho_r^2 \int_0^T \omega^2(\tau) d\tau
$$  \hspace{1cm} (3.43)

Since $\|e\| < \rho$ when $t \in \Omega_0^T$, it follows that
\[
\int_0^T e^T Q_m e d\tau \\
= \int_{\tau \in \Omega_0^T} e^T Q_m e d\tau + \int_{\tau \in \Omega_0^T} e^T Q_m e d\tau \\
\leq V(0) + \frac{1}{2} \rho_\tau^2 \int_0^T \omega^2(\tau) d\tau + \int_{\tau \in \Omega_0^T} e^T Q_m e d\tau \\
\leq V(0) + \frac{1}{2} \rho_\tau^2 \int_0^T \omega^2(\tau) d\tau + \rho^2 \lambda_{\text{max}}(Q_m)T 
\] (3.44)

Because \( \rho \) could be any positive constant and \( T \) is finite, (3.44) becomes
\[
\int_0^T e^T Q_m e d\tau \leq V(0) + \frac{1}{2} \rho_\tau^2 \int_0^T \omega^2(\tau) d\tau \\
\leq \frac{1}{2} e(0)^T P_m e(0) + \frac{1}{2} \tilde{c}_f(0) \tilde{y}_c^{-1} \tilde{c}_f(0) \\
+ \frac{1}{2} \tilde{w}_f(0) \gamma_{w_f}^{-1} \tilde{w}_f(0) + \frac{1}{2} \rho_\tau^2 \int_0^T \omega^2(\tau) d\tau 
\] (3.45)

The proof completes.

\[\square\]

\textbf{Remark 3.1:} The constraint to estimating the upper bound \( d^u \) of the disturbance is therefore removed by presuming the lumped uncertainty belongs to \( L_2[0,T] \). If \( e(0) = 0, \tilde{c}_f(0) = 0, \tilde{w}_f(0) \) and \( Q_m = I \), then the control performance of the overall system satisfies
\[
\int_0^T e^T e d\tau \leq \frac{1}{2} \rho_\tau^2 \int_0^T \omega^2(\tau) d\tau \iff \frac{\|e\|_2^2}{\|w\|_2^2} \leq \frac{1}{2} \rho_\tau^2 
\] (3.46)
where \( \|e\|_2 = \left( \int_0^T e^T e d\tau \right)^{1/2} \) and \( \|w\|_2 = \left( \int_0^T \omega^2(\tau) d\tau \right)^{1/2} \) which means that the \( L_2 \) gain from the uncertainty to state error must be equal to or less than \( \frac{1}{\sqrt{2}} \rho_\tau \).

\subsection*{3.3 Output Feedback Wavelet Adaptive Robust Controller}

In order to implement the state feedback controller in previous section, we need to assume all the state variables are measurable, which restricts its applicability when only system outputs are available. To overcome this limitation, we extend the results in
previous section to an output feedback framework. The proposed output feedback controller is shown in Figure 3.2.

In the output feedback wavelet adaptive robust control, we replace the error states $e$ with their estimates $\hat{e}$ using a high-gain observer that would allow asymptotic recovery of the performance achieved under state feedback. The high-gain observer we adopted here is same as used in [18], [19], [22]. Define $e_y = y_d - y$ and

$$\dot{\hat{e}} = A\hat{e} + L(e_y - C\hat{e}) \tag{3.47}$$

where $L = \left[ \frac{\alpha_1}{\epsilon} \cdots \frac{\alpha_n}{\epsilon^n} \right]^T$ is the observer gain vector. $\epsilon \in (0,1)$ is a small design parameter to be specified. The positive constants $\alpha_i$, $i = 1, \cdots, n$ are chosen such that the roots of the polynomial

$$s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n = 0 \tag{3.48}$$

has negative real parts. Substituting the error states $e$ with their estimates $\hat{e}$, the adaptive robust control law (3.8) becomes
\[ \hat{u} = \hat{u}_w + \hat{u}_r \]  

(3.49)

where

\[ \hat{u}_w = \frac{1}{b} \left( -f(\hat{x}) + y_d^{(n)} - k\hat{e} \right) \]  

(3.50)

with \( \hat{x} = x_d + \hat{e} \) are the estimates of the states \( x \) and \( \hat{u}_r \) is the robustifying component to be designed. In order to eliminate the peaking phenomena associated with the high-gain observer, we introduce the similar saturation technique as in [22], [63]. Since states \( x \), references \( x_d \), wavelet network weights \( \hat{e}_j \) and \( \hat{w}_j \) all belong to compact sets \( D_x \), \( D_{x_d} \), \( D_c \), \( D_w \) respectively and control input \( \hat{u} = \hat{u}(x, x_d, y_d^{(n)}, \hat{e}_j, \hat{w}_j) \) is continuous, we could then define \( S \) as

\[ S \geq \max \left| \hat{u}(\hat{e}, x_d, y_d^{(n)}, \hat{e}_j, \hat{w}_j) \right| \]  

(3.51)

where the maximization is take over \( \hat{e} \in D_{\hat{e}} \), \( x_d, y_d^{(n)} \in D_{x_d} \), \( \hat{e}_j \in D_c \) and \( \hat{w}_j \in D_w \). \( D_{\hat{e}} \) is a compact set that \( \hat{e} \) belongs to. Now we define the saturated output feedback wavelet adaptive robust controller as follows

\[ u^s = Sat \left( \frac{\hat{u}}{S} \right) = Sat \left( \frac{\hat{u}_w + \hat{u}_r}{S} \right) \]  

(3.52)

where \( Sat(\cdot) \) is the saturation function. Next, the adaptation laws for the wavelet network weight vectors (3.18) and (3.19) have become

\[ \dot{\hat{e}}_j = \begin{cases} proj \left( \hat{e}_j, y_c b^T P_m \hat{\phi}_j(\hat{x}) \right), & t \in \Omega_1 \\
0, & t \in \Omega_2 \end{cases} \]  

(3.53)

\[ \dot{\hat{w}}_j = \begin{cases} proj \left( \hat{w}_j, y_{w,j} b^T P_m \hat{\psi}_j(\hat{x}) \right), & t \in \Omega_1, j = j, \ldots, J_K(t) \\
0, & t \in \Omega_2 \end{cases} \]  

(3.54)

where \( proj(\cdot, \cdot) \) is defined same as in (3.22) and (3.23) with \( e \) replaced by \( \hat{e} \) and \( \Omega_1 \) are \( \Omega_2 \) are defined as
\[ \Omega_1 \triangleq \{ t | t \in [0, \infty), \| \hat{e} \| \geq \rho = \mu_R \rho_w \} \]
\[ \Omega_2 \triangleq \{ t | t \in [0, \infty), \| \hat{e} \| < \rho = \mu_R \rho_w \} \]
\[ \Omega_a^b \triangleq [a, b] \cap \Omega_1, \bar{\Omega}_a \triangleq [a, b] \cap \Omega_2 \] (3.55)

The adaptation laws defined above has to satisfy
\[ c_j^T y_c^{-1} \left( \hat{e}_j - y_c \hat{e}_T P_m B \phi_j(\hat{x}) \right) \leq 0 \] (3.56)
\[ \hat{w}_j^T y_{w,j}^{-1} \left( \hat{w}_j - y_{w,j} \hat{e}_T P_m B \psi_j(\hat{x}) \right) \leq 0 \] (3.57)

The wavelet network approximator adopted in the output feedback control is essentially identical to the state feedback case except that the tracking error state \( e \) is replaced by its estimate \( \hat{e} \). During each waiting time period \( T_R \), we need to check if the Euclidean norm of the tracking error estimate \( \| \hat{e}(t) \| \) is ever greater than the threshold \( \rho \). If the tracking error estimate \( \| \hat{e}(t) \| \) is kept lower than \( \rho \) throughout the whole time interval \( t \in [(i-1)T_R, iT_R) \), we will not add new resolution level in the next time interval \( [(iT_R, (i+1)T_R) \). Otherwise, if the tracking error estimate \( \| \hat{e}(t) \| \) exceeds the precision bound \( \rho \), we will add a new resolution level during next time interval. That is, if the current resolution level is \( J_K(t) = J_i \) for \( t \in [(i-1)T_R, iT_R) \), then \( J_K(t) = J_i + 1 \) for \( t \in [(iT_R, (i+1)T_R) \). The output feedback evolving structure wavelet network algorithm is now summarized as follows: In each sampling period,

1) Measure the tracking error estimate \( \hat{e} \) and state estimate \( \hat{x} \) according to (3.47).
2) Update the scaling and wavelet weight vectors adaptively and concurrently by adaptation laws of (3.53) and (3.54).
3) During each time interval \( [(i-1)T_R, iT_R) \), check if \( \| \hat{e} \| > \rho \). If yes, add a new resolution level to the current wavelet network in the next period \( [(iT_R, (i+1)T_R) \), i.e. \( J_K(t) = J_i + 1 \).
4) Repeat the process above until specified control objective has been satisfied.
To facilitate the stability analysis, define
\[
\hat{x}_i = \frac{e_i - \hat{e}_i}{\varepsilon_{n-i}}, \quad 1 \leq i \leq n
\] (3.58)
and \( \xi = [\xi_1, \ldots, \xi_n]^T \). Then we can represent the estimation error dynamics in the standard singularly perturbed form
\[
e \hat{\xi} = A_c \xi + \varepsilon B(f(x) + bu^s - y_d^{(n)} + d)
\] (3.59)
where \( A_c = \varepsilon D^{-1}(A - LC)D \) is a constant Hurwitz matrix and \( D \) is a diagonal matrix with \( \varepsilon^{n-i} \) as the \( i \)th diagonal element and \( \|D\| = 1 \) such that \( e - \hat{e} = D\xi \) following from (3.58). Define
\[
\eta_c = \max_{\hat{\xi}_j, \xi_j \in D_c} \frac{1}{2} \hat{\xi}_j^T \gamma_c^{-1} \hat{\xi}_j
\] (3.60)
\[
\eta_w = \max_{\hat{\omega}_j, \omega_j \in D_w} \frac{1}{2} \sum_{j=1}^{J_{x(t)}} \hat{\omega}_j^T \gamma_{w,j}^{-1} \hat{\omega}_j
\] (3.61)
Let
\[
\eta_{e_0} = \frac{1}{2} e(t_0)^T P_m e(t_0)
\] (3.62)
\[
D_e = \left\{ e : \frac{1}{2} e^T P_m e \leq \eta_e, \eta_e = \max \left( \eta_{e_0}, \max \frac{1}{2} e^T P_m e \right) \right\}
\] (3.63)
where
\[
\gamma_e = \frac{\lambda_{\text{max}}(P_m)}{\sqrt{\lambda_{\text{min}}(P_m)}} \frac{\lambda_{\text{max}}(P_m) \|e(f)\|^2 + \eta_c + \eta_w}{\lambda_{\text{min}}(P_m)}
\] (3.64)
Then
\[
D_x = \left\{ x : x = e + x_d, e \in D_e, x_d \in D_{x_d} \right\}
\] (3.65)
\[
D_{\hat{e}} = \left\{ \hat{e} : \frac{1}{2} \hat{e}^T P_m \hat{e} \leq \eta_{\hat{e}} \right\}
\] (3.66)
\[
D_{\hat{x}} = \left\{ \hat{x} : \hat{x} = \hat{e} + x_d, \hat{e} \in D_{\hat{e}}, x_d \in D_{x_d} \right\}
\] (3.67)
where \( \eta_{\hat{e}} > \eta_e \).
Applying the methods from [62], the following proposition can be proved.

**Proposition 3.1[22]:** There exists $\epsilon_1^* \in (0, 1)$ such that if $\epsilon \in (0, \epsilon_1^*)$, then $\|\xi(t)\| \leq \beta \epsilon$ with some $\beta > 0$ for $t \in [t_0 + T_1(\epsilon), t_0 + T_3)$, where $t_0$ is any initial time, $T_1(\epsilon)$ is a finite time and $t_0 + T_3$ is the moment when the tracking error $e(t)$ leaves the compact set $\mathcal{D}_e$ for the first time. Moreover, we have $\lim_{\epsilon \to 0^+} T_1(\epsilon) = 0$ and $\eta_{e_1} = \frac{1}{2} e(t_0 + T_1(\epsilon))^T P_m e(t_0 + T_1(\epsilon)).$

We first consider the case when no disturbance is present that is $d(t) = 0$ in (3.2). Then the output feedback wavelet adaptive controller which guarantees asymptotic stability of the uncertain nonlinear system in (3.2) can be obtained from Theorem 3.2 below. Here we need to choose $Q_m$ such that $\lambda_{\text{min}}(Q_m) = 1 + \lambda_s + \lambda_r$ where $\lambda_r, \lambda_s > 0$ are positive constants.

**Theorem 3.3:** Consider the uncertain nonlinear dynamic system in (3.2) with $d(t) = 0$ satisfying Assumption 3.1 and the evolving structure wavelet network adaptive controller chosen as (3.47), (3.49), (3.52) with $\hat{u}_r = 0$ and adaptation laws (3.53) - (3.54), the tracking error is uniformly bounded and converges to the prespecified error bound $\rho = \mu_R \rho_w$ in a finite time interval.

**Proof:** Using the same Lyapunov function as in (3.26). We first need to show the boundness of the $V(t)$ for any finite resolution level $J_K(t)$. Then we will show the finite time convergence of the tracking error $e$ as the resolution increases.

To prove the boundness of $V(t)$, we need to consider three cases for the finite resolution level $J_K(t)$ respectively as follows.

(1) $J_K(t) = J$. Only the coarsest resolution level is added into the wavelet network approximator. Let $\eta_e \geq \eta_{e_1} + \eta_e + \eta_w$. By the proposition 3.1, if $\epsilon \in (0, \epsilon_1^*)$, then
\(e(t) \in D_e\) and \(\|\xi(t)\| \leq \beta \varepsilon\) for \(t \in [t_0 + T_1(\varepsilon), t_0 + T_3]\) where \(t_0 = T_j\). Thus, we have \(\|e(t) - \dot{e}(t)\| \leq \|D\| \|\xi(t)\| \leq \beta \varepsilon\) for \(t \in [t_0 + T_1(\varepsilon), t_0 + T_3]\). There exists \(\varepsilon_2\) such that if \(\|e(t) - \dot{e}(t)\| \leq \beta \varepsilon_2\), then \(\dot{e}(t) \in D_{\dot{e}}\). Let \(\varepsilon_2^t = \min(\varepsilon_1^t, \varepsilon_2)\). If \(\varepsilon \in (0, \varepsilon_2^t)\), the saturation of the wavelet adaptive control law is not effective, that is \(u^s = \hat{u}_w\) for \(t \in [t_0 + T_1(\varepsilon), t_0 + T_3]\).

For \(t \in \Omega_{T_j+1}^{T_j+1}\), since \(e \in D_e\), \(\dot{e} \in D_{\dot{e}}\), \(x_d, y_d^{(n)} \in D_{x_d}\), \(\dot{\xi}_j \in D_c\) and \(\hat{w}_j \in D_w\) for \(t \in [t_0 + T_1(\varepsilon), t_0 + T_3]\), if \(T_j+1 \leq t_0 + T_3\), \(e\) is bounded and \(V(t)\) is bounded. The proof is trivial.

Consider \(t_0 + T_3 \leq T_j+1\), evaluating the time derivative of \(V(t)\) along the tracking error dynamics (3.5) for \(t \in [t_0 + T_1(\varepsilon), t_0 + T_3]\), we obtain

\[
\dot{V}(t) = e^T P_m \dot{e} + \dot{c}_j^T Y_c^{-1} \dot{\dot{c}}_j + \hat{\hat{w}}_j^T Y_{w,j}^{-1} \hat{\hat{w}}_j \\
= e^T P_m \left( A e + f(x) + bu^s - y_d^{(n)} + d \right) + \dot{c}_j^T Y_c^{-1} \dot{\dot{c}}_j \\
+ \hat{\hat{w}}_j^T Y_{w,j}^{-1} \hat{\hat{w}}_j \\
= -e^T Q_m e + e^T P_m B k(e - \dot{e}) + e^T P_m B \left( f(x) - \ddot{f}(\hat{x}) \right) \\
+ \dot{c}_j^T Y_c^{-1} \dot{\dot{c}}_j + \hat{\hat{w}}_j^T Y_{w,j}^{-1} \hat{\hat{w}}_j \\
= -e^T Q_m e + e^T P_m B k(e - \dot{e}) \\
+ \dot{e}^T P_m B \left( f(x) - (c_j^T) \phi_j(\hat{x}) - \left( w_j^* \right)^T \psi_j(\hat{x}) \right) \\
+ \ddot{c}_j^T Y_c^{-1} \left( \dot{\dot{c}}_j - \gamma_c \dot{e}^T P_m B \phi_j(\hat{x}) \right) \\
+ \hat{\hat{w}}_j^T Y_{w,j}^{-1} \left( \hat{\hat{w}}_j - \gamma_w \dot{e}^T P_m B \psi_j(\hat{x}) \right) \\
+ (e - \dot{e})^T P_m B \left( f(x) - \ddot{f}(\hat{x}) \right) \\
(3.68)
\]

For \(t \in [t_0 + T_1(\varepsilon), t_0 + T_3]\), if \(e \in (0, \varepsilon_2^t)\), then \(\|\xi(t)\| \leq \beta \varepsilon\), \(\|e - \dot{e}\| \leq \beta \varepsilon\), \(e \in D_e\), \(\dot{e} \in D_{\dot{e}}\), \(x_d, y_d^{(n)} \in D_{x_d}\), \(\dot{\xi}_j \in D_c\) and \(\hat{w}_j \in D_w\). Hence \(e^T P_m B\), \(\dot{e}^T P_m B\), \(x = e + x_d\), \(\hat{x} = \dot{e} + x_d\), \(\hat{u}_w\) are all bounded for \(t \in [t_0 + T_1(\varepsilon), t_0 + T_3]\). Thus there exist some \(r_1, r_2 > 0\) such that
\[ \|e^T P_m B (e - \hat{e})\| \leq r_1 \epsilon \quad (3.69) \]

and
\[ \| (e - \hat{e})^T P_m B \left( f(x) - \bar{f}(\hat{x}) \right) \| \leq r_2 \epsilon \quad (3.70) \]

From the Lipschitz continuity of scaling and wavelet basis functions, we could have that
\[ \left\| (c_j^*)^T \phi_j(x) - (c_j^*)^T \phi_j(\hat{x}) \right\| \leq L_c \| x - \hat{x} \| \]
\[ \left\| (w_j^*)^T \psi_j(x) - (w_j^*)^T \psi_j(\hat{x}) \right\| \leq L_w \| x - \hat{x} \| \quad (3.71) \]

for some Lipschitz constants \( L_c, \ L_w > 0 \). It then follows that
\[
\begin{align*}
\| \hat{e}^T P_m B \left( f(x) - (c_j^*)^T \phi_j(x) \right) - (w_j^*)^T \psi_j(x) \|
&= \| \hat{e}^T P_m B \left( f(x) - (c_j^*)^T \phi_j(x) - (w_j^*)^T \psi_j(x) \right) + (c_j^*)^T \phi_j(x) \\
&\quad + (w_j^*)^T \psi_j(x) - (c_j^*)^T \phi_j(\hat{x}) - (w_j^*)^T \psi_j(\hat{x}) \|
\leq \| \hat{e}^T P_m B \left( f(x) - (c_j^*)^T \phi_j(x) - (w_j^*)^T \psi_j(x) \right) \|
&\quad + \| \hat{e}^T P_m B \left( (c_j^*)^T \phi_j(x) - (c_j^*)^T \phi_j(\hat{x}) \right) \|
&\quad + \| \hat{e}^T P_m B \left( (w_j^*)^T \psi_j(x) - (w_j^*)^T \psi_j(\hat{x}) \right) \|
\leq \| \hat{e}^T P_m B \epsilon(j) \| + (L_c + L_w) \| e - \hat{e} \| \| \hat{e}^T P_m B \|
\leq \| \hat{e}^T P_m B \epsilon(j) \| + r_3 \epsilon \quad (3.72) \]

for some constant \( r_3 > 0 \) where we have used the fact that \( \| x - \hat{x} \| = \| e - \hat{e} \| \). It follows from (3.56), (3.57) and (3.69) - (3.72) that (3.68) becomes
\[ \dot{V}(t) \leq -e^T Q_m e + \| \dot{e}^T P_m B \varepsilon (J) \| + (r_1 + r_2 + r_3) \varepsilon \]
\[ = -(1 + \lambda_s + \lambda_r) \| e \|^2 + \mu_R \| \dot{e} \| \| \varepsilon (J) \| + (r_1 + r_2 + r_3) \varepsilon \]
\[ \leq -(1 + \lambda_s + \lambda_r) \| e \|^2 + \mu_R (\| e \| + \beta \varepsilon) \| \varepsilon (J) \| \]
\[ + (r_1 + r_2 + r_3) \varepsilon \]
\[ \leq -\lambda_s \| e \|^2 - \| e \| (\| e \| - \mu_R \| \varepsilon (J) \|) - \lambda_r \| e \|^2 \]
\[ + (r_1 + r_2 + r_3 + r_4) \varepsilon \]
\[ \leq -\lambda_s \| e \|^2 - \| e \| (\| e \| - \mu_R \| \varepsilon (J) \|) - \lambda_r \| e \|^2 + r \varepsilon \]
(3.73)

for some \( r_4 > 0 \) and \( r = r_1 + r_2 + r_3 + r_4 \). Since \( t \in \Omega_{T_j}^{T_{j+1}} \), we have \( \| \dot{e} \| \geq \rho \). There exists \( \varepsilon_3 \) such that if \( \| e - \dot{e} \| \leq \beta \varepsilon_3 \), then \( \| e \| \geq \rho \). Let \( \varepsilon_3^* = \min \{ \varepsilon_2^*, \varepsilon_3 \} \). If \( \varepsilon \leq \frac{\lambda_r \rho^2}{r} \), then \(-\lambda_r \| e \|^2 + r \varepsilon \leq 0 \). Let \( \varepsilon_4^* = \min \left\{ \varepsilon_3^*, \frac{\lambda_r \rho^2}{r} \right\} \) and \( \varepsilon \in (0, \varepsilon_4^*) \), (3.73) becomes

\[ \dot{V}(t) \leq -\lambda_s \| e \|^2 - \| e \| (\| e \| - \mu_R \| \varepsilon (J) \|) \] (3.74)

If \( \| e \| > \mu R \| \varepsilon (J) \| \), then

\[ \dot{V}(t) \leq 0, \ \forall t \in [t_0 + T_1(\varepsilon), t_0 + T_3) \] (3.75)

It is clear that

\[ \| e(t) \| \leq \gamma_e, \ \forall t \in \tilde{\Omega}_{t_0 + T_1(\varepsilon)}^{T_{j+1}} \] which implies \( T_3 \rightarrow T_{j+1} \). Because \( \| e \| \in \mathcal{D}_e \) for \( t \in [t_0, t_0 + T_1(\varepsilon)] \), we have \( e \) is bounded and \( \| e \| \in \mathcal{D}_e \) \( \forall t \in \Omega_{T_j}^{T_{j+1}} \). If \( t \in \tilde{\Omega}_{T_j}^{T_{j+1}} \), the definition of Lyapunov function (3.26) and adaptation laws (3.53) - (3.54) implies that \( \dot{V}(t) = 0 \). Therefore for any time instants \( t \in [T_j, T_{j+1}] \), there exists a finite constant \( M_j > 0 \) such that

\[ V(t) \leq \eta_e + \eta_c + \eta_w = M_j < \infty \] (3.76)

Then from the Lyapunov function definition (3.26), the tracking error \( e \), the wavelet weight estimate \( \hat{c}_j \) and \( \hat{w}_j \) are all bounded for \( t \in [T_j, T_{j+1}] \). If the desired tracking
performance is not satisfied at $t = T_{j+1}^-$, a new resolution level $j = \bar{j} + 1$ will be added to the wavelet approximator at $t = T_{j+1}^+$ which implies the discontinuity of $V(t)$ at $t = T_{j+1}$. Because of the smooth projection operator (3.22) and (3.23), $\hat{c}_j$ and $\hat{w}_j \bar{w}$ are continuous at $t = T_{j+1}$. Then at $t = T_{j+1}^+$, we could obtain

$$V(T_{j+1}^+) = V(T_{j+1}^-) + \frac{1}{2} \sum_{i=1}^{\infty} (\hat{w}_{j+1})^T \gamma_{w,j+1}(T_{j+1}) w_{j+1}^* < \infty$$

where we have used the property that $\hat{w}_{j+1} = 0$. Since both $w_{j+1}^*$ and $\gamma_{w,j+1}$ are constants, it follows that $V(T_{j+1}^+)$ is bounded too.

(2) $J < J_K(t) = J_l < K$. Now $J_l$ levels of resolution have been added to the wavelet network and the approximation error $\varepsilon(J_l) \geq \varepsilon_K$. Let $t_0 = T_{J_l}$. Consider the nontrivial case where $T_3 \leq T_{J_l+1}$. Similarly, for $t \in [t_0 + T_1(\varepsilon), t_0 + T_3)$, (3.73) becomes

$$\dot{V}(t) \leq -\lambda_s \|e\|^2 - \|e\|((\|e\| - \mu_R \|\varepsilon(J_l)\|) - \lambda_r \|e\|^2 + re)$$

Following the similar arguments in (1) and noting that $\|\varepsilon(J_l)\| \leq \|\varepsilon(J)\|$, we could prove that $\|e\|$ is bounded and $\|e\| \in D_e$ for all $t \in \Omega_{T_{J_l+1}}^{T_{J_l+1}}$. Since $\dot{V}(t) = 0$ for $t \in \Omega_{T_{J_l+1}}^{T_{J_l+1}}$, it could be proved that $V(t)$ is bounded for all $t \in [T_{J_l}, T_{J_l+1}^-]$. It follows that $V(t)$ is bounded for $t \in [0, T_{J_l+1}^-]$ and any $J_l < K$. Let $J_l$ be $K - 1$. Then we have $V(T_K) \leq M_K$. Similarly as (3.77),

$$V(T_K^+) = V(T_K^-) + \frac{1}{2} \sum_{i=1}^{\infty} (w^*_{K})^T \gamma_{w,K}^{-1} w^*_{K} < \infty$$

(3) $J_K(t) = l \geq K$. From Assumption 3.1, we have that $\varepsilon(l) \leq |\varepsilon_K| < \rho_w = \rho/\mu_K$. If $t \in \Omega_{T_{l+1}}^{T_{l+1}+1}$, then $\|e\| \geq \rho$. Let $t_0 = T_{l+1}^+$. Consider the nontrivial case where $T_3 \leq T_{l+1}^+$.
Similarly for $t \in [t_0 + T_1(\varepsilon), t_0 + T_3)$, since we could prove $\|e\| \geq \rho$ for $\varepsilon \in (0, \varepsilon_4)$, then (3.73) becomes
\[
\dot{V}(t) \leq -\lambda_s \|e\|^2 - \|e\| (\|e\| - \mu_R \|e(l)\|) - \lambda_r \|e\|^2 + r \varepsilon \\
\leq -\lambda_s \|e\|^2 - \|e\| (\|e\| - \mu_R \|e(l)\|) \\
\leq -\lambda_s \|e\|^2 \\
< 0
\]
(3.80)
which implies that $T_3 \to T_{l+1}^-$. Following the similar arguments, we could prove that $\|e\|$ is bounded and $\|e\| \in D_e$ for all $t \in \Omega_{T_{l+1}^-}^T$. Since $\dot{V}(t) = 0$ when $t \in \Omega_{T_{l+1}^-}^T$, we could prove that for $t \in [T_{l+1}^+, T_{l+1}^-]$, $V(t)$ is bounded and there exists $M_l > 0$ such that
\[
V(t) \leq \eta_e + \eta_c + \eta_w = M_l < \infty
\]
(3.81)
for any $l \geq K$.

Now we will prove that there exists finite $T(\rho) > 0$ such that the tracking error $\|e\| < \rho$ for $t \geq T(\rho)$ by contradiction. Suppose no such $T(\rho)$ exists, then $\Omega_1 \to \infty$.

It follows that the resolution level $J_K(t) \to \infty$ as well because of the finite observation time $T_R$. For any $t \in [T_{l+1}^-, T_{l+1}^-]$, we could prove that from (3.80)
\[
V(t) = V(T_{l+1}^+ + T_1(\varepsilon)) + \int_{T_{l+1}^+ + T_1(\varepsilon)}^t \dot{V}(\tau) d\tau \\
\leq V(T_{l+1}^+ + T_1(\varepsilon)) - V(T_{l+1}^+) + V(T_{l+1}^+) - \int_{t \in \Omega_{T_{l+1}^-}^T} \lambda_s \|e\|^2 d\tau \\
\leq V(T_{l+1}^+ + T_1(\varepsilon)) - V(T_{l+1}^+) + V(T_{l+1}^-) + \frac{1}{2} (w_i^*)^T \gamma_{w,l}^{-1} w_i^* \\
- \int_{t \in \Omega_{T_{l+1}^-}^T} \lambda_s \rho^2 d\tau
\]
(3.82)
From Proposition 3.1 that $\lim_{\varepsilon \to 0^+} T_1(\varepsilon) = 0$ and the continuity of $V(t)$ for $t \in [T_{l+1}^+, T_{l+1}^-]$, for any $\varepsilon > 0$, there exists $\varepsilon_5 > 0$, if $\varepsilon \in (0, \varepsilon_5)$ such that $\|V(T_{l+1}^+$
\[ T_1(\epsilon) - V(T_1^+) \leq \epsilon \quad \text{and} \quad \|T_1(\epsilon)\| \leq \epsilon . \] Let \( \epsilon_5^* = \min\{\epsilon_5, \epsilon_4^*\} \) and \( \epsilon \in (0, \epsilon_5^*) \), (3.82) becomes

\[
V(t) \leq \epsilon + V(T_1^-) + \frac{1}{2} (w_i^*)^T \gamma_{w_i}^{-1} w_i^* - \left( \int_{\tau \in \Omega_{T_1^+}^t} \lambda_s \rho^2 d\tau - \int_{\tau \in \Omega_{\tau_1^+ + \tau_2^-}^t} \lambda_s \rho^2 d\tau \right) \\
\leq \epsilon + V(T_1^-) + \frac{1}{2} (w_i^*)^T \gamma_{w_i}^{-1} w_i^* - \int_{\tau \in \Omega_{T_1^+}^t} \lambda_s \rho^2 d\tau + \lambda_s \rho^2 \epsilon
\]

(3.83)

Because the \( \epsilon \) is arbitrary positive, we could have that

\[
V(t) \leq V(T_1^-) + \frac{1}{2} (w_i^*)^T \gamma_{w_i}^{-1} w_i^* - \int_{\tau \in \Omega_{T_1^+}^t} \lambda_s \rho^2 d\tau
\]

(3.84)

for any \( l \geq K \) and any \( t \in [T_1^+, T_{l+1}^-] \). Then for any \( t \in [T_1^+, T_{l+1}^-] \), it follows from (3.84) that

\[
V(t) \leq V(T_1^-) + \frac{1}{2} (w_i^*)^T \gamma_{w_i}^{-1} w_i^* - \int_{\tau \in \Omega_{T_1^+}^t} \lambda_s \rho^2 d\tau
\]

\[\vdots\]

\[\leq V(T_K^+) + \sum_{j=K+1}^{l} \frac{1}{2} (w_j^*)^T \gamma_{w_j}^{-1} w_j^* - \int_{\tau \in \Omega_{T_1^+}^t} \lambda_s \rho^2 d\tau
\]

\[\leq M_K + \sum_{j=K}^{l} \frac{1}{2} (w_j^*)^T \gamma_{w_j}^{-1} w_j^* - \int_{\tau \in \Omega_{T_1^+}^t} \lambda_s \rho^2 d\tau
\]

(3.85)

If \( \Omega_1 \to \infty \), it follows from (3.85) that

\[
\lim_{t \to \infty} V(t) \leq M_K + \lim_{l \to \infty} \sum_{j=K}^{l} \frac{1}{2} (w_j^*)^T \gamma_{w_j}^{-1} w_j^* - \lim_{t \to \infty} \int_{\tau \in \Omega_{T_1^+}^t} \lambda_s \rho^2 d\tau
\]

(3.86)

From Assumption 3.1, we have that \( \sum_{j=K}^{\infty} \frac{1}{2} (w_j^*)^T \gamma_{w_j}^{-1} (w_j^*) \leq \frac{1}{2} c_w \). (3.86) now becomes
\[
\lim_{t \to \infty} V(t) \leq M_K + \frac{1}{2} c_w - \lim_{t \to \infty} \int_{\tau \in \Omega_K^{+}} \lambda_s \rho^2 d\tau
\]  
(3.87)

Since \( \Omega_1 \to \infty \), \( \lim_{t \to \infty} \Omega_K^{+} \to \infty \). Then \( \lim_{t \to \infty} V(t) \leq M_K + \frac{1}{2} c_w - \lambda_s \rho^2 \to 0 \) which contradicts \( V(t) \geq 0 \). Therefore there exists a finite time \( T(\rho) \) such that tracking error \( e \) will satisfy \( \|e\| < \rho \) for \( t \geq T(\rho) \) and the proof completes.

Now, let’s consider the robust adaptive control strategy for system (3.2) in the presence of additional disturbance signal \( d(t) \). Similarly to the state feedback robust controller, we designed the robustifying control input \( \hat{u}_r \) to achieve \( L_2 \) tracking performance by attenuating the effects of the approximation error caused by wavelet network and the state estimation error from high gain observer.

From (3.68) and adaptation laws (3.56), (3.57), the derivative of Lyapunov function satisfies

\[
\dot{V}(t) \leq -e^T Q_m e + e^T P_m B k(e - \hat{e})
+ \hat{e}^T P_m B \left( f(x) - (c_j^\top \phi_j(\hat{x}) - (w_j^\top \psi_j(\hat{x})) \right)
+ \hat{\epsilon}^T P_m B d + \hat{e}^T P_m B \hat{u}_r
+ (e - \hat{e})^T P_m B \left( f(x) - \hat{f}(\hat{x}) \right)
+ (e - \hat{e})^T P_m B (d + \hat{u}_r)
\]  
(3.88)

Here we adopt \( \omega \) to represent the lumped uncertainty of the approximation error and observation error together. Define \( \omega \) as

\[
\omega = \frac{e^T P_m B (e - \hat{e}) + (e - \hat{e})^T P_m B \left( f(x) - \hat{f}(\hat{x}) \right) + (e - \hat{e})^T P_m B (d + \hat{u}_r)}{\hat{e}^T P_m B}

\left( c_j^\top \phi_j(\hat{x}) - (w_j^\top \psi_j(\hat{x})) \right) + d
\]  
(3.89)

**Assumption 3.3:** The lumped uncertainty \( \omega \) in (3.89) is assumed such that \( \omega \in L_2[0,T], \forall T \in [0,\infty). \)

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Now we can prove the guarantee of $H_{\infty}$ tracking performance for the overall system similarly as the state feedback robust control strategy.

**Theorem 3.4:** Consider the uncertain nonlinear dynamic system in (3.2) satisfying Assumption 3.3 and the evolving structure wavelet adaptive controller chosen as (3.47), (3.49), (3.52), and adaptation laws (3.53) - (3.54). Choose the robustifying control component $\hat{u}_r$ as

$$
\hat{u}_r = -\frac{\sigma}{2b\rho_r^2}
$$

(3.90)

where $\rho_r > 0$ is a design constant representing the prescribed attenuation level and $\sigma = \hat{e}^TP_mB$. Then the tracking performance for the overall system satisfies the following properties:

$$
\int_0^T e^TQ_me^{0} = e(0)^TP_m e(0) + \frac{1}{2} \hat{e}_j^T(0)\gamma_c^{-1}\hat{e}_j(0)
+ \frac{1}{2} \hat{\omega}_j^T(0)\gamma_{\omega,j}\hat{\omega}_j(0) + \frac{1}{2}\rho_r^2\int_0^T \omega^2(\tau) d\tau
$$

(3.91)

**Proof:** It follows easily that $\hat{V}(\tau) \leq -e^TQ_m e + \sigma \omega + \sigma bu_r$. The rest of the proof is essentially same as theorem 3.2, so (3.91) follows directly.

---

### 3.4 Simulations

To illustrate the effectiveness of the proposed state and output feedback wavelet network adaptive robust control, we consider the nonlinear system from [35]:

$$
\begin{align*}
\dot{x} &= Ax + B(f(x) + bu + d) \\
y &= Cx
\end{align*}
$$

(3.92)

where $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $C = [1 \ 0]$. Let us first assume $x = [x_1 \ x_2]^T \in R^2$ is the measurable state vector, $u \in R$ is the control input, $f(x)$ is unknown
nonlinear function to be approximated and \( d \) is the unknown additional disturbance. In the following simulations, consider the following nonlinear function \( f(x) \) from [35]:

\[
f(x) = x_1 + (0.5 + 0.1x_1^2)x_2 - \frac{2}{1 + 0.05x_1^2}
\]  

(3.93)

The control objective is to ensure that system output \( y \) tracks the reference input \( r \). We chose \( k = [2 \ 3]^T \) to implement wavelet adaptive controller (3.9). \( Q_m \) is chosen as \( 50I_{2 \times 2} \) where \( I_{2 \times 2} \) is the identity matrix of dimension two. Then the positive definite solution to (3.10) is solved as

\[
P = \begin{bmatrix} 62.5 & 12.5 \\ 12.5 & 12.5 \end{bmatrix}
\]

The nonlinear function \( f(x) \) would be estimated online by self-organizing wavelet network. Two-dimensional orthonormal Daubechies wavelet (db3) is employed to construct the scale and wavelet basis functions for wavelet MRA model. The coarse resolution is \( J = -4 \) and \( N_j = 10 \). \( N_j \) doubles as when resolution is increased by 1. The tracking error threshold \( \rho \) and design parameter \( T_R \) to determine if additional resolution is needed are chosen as

\[
\rho = 0.5 \quad \text{and} \quad T_R = 100s
\]

The parameters for adaptation laws are chosen as follows:

\[
\gamma_c = 1, \gamma_w = 1, \varphi = w_j = -30, \bar{\varphi} = \bar{w}_j = 30 \quad \text{for} \quad j = J, J + 1, \ldots
\]

The reference input to be tracked is \( r = \cos(0.2t) \). The initial condition for states vector and wavelet network parameters are all set to 0. Both state and output feedback wavelet adaptive robust control are implemented to control the system in (3.92).

State feedback control is first applied to control the nonlinear system above provided all states variables are measurable. We first consider the case when \( d = 0 \), that is in the
absence of additional disturbance. The tracking performances of the wavelet adaptive controller are shown in Figure 3.3 - Figure 3.5. From the figure, we can see that after 200s and two more resolutions are added, the tracking error is below the threshold value. It is shown that wavelet adaptive controller performs well while adjusting its structure dynamically online. Next, the external disturbance \( d = 1 - \cos(0.3t) \) is added to the plant and the attenuation level \( \rho_r \) for the robustifying component \( u_r \) is set to 0.5. Under the same initial condition, the plant response and corresponding control signal are plotted in Figure 3.6 and Figure 3.7. It is shown that the wavelet adaptive robust controller can achieve favorable tracking performance.

We then implement the output feedback control to the above system. The parameters of the observer are chosen as

\[
\varepsilon = 0.1, \quad \alpha_1 = 10, \quad \alpha_2 = 25.
\]

The controller parameter \( S \) is 50. All other parameters are chosen the same as state feedback controller except that we use observed states \( \hat{e} \) instead. We first consider the situation when external disturbance is not present, i.e. \( d = 0 \) and no robust component \( \hat{u}_r \). Under the same reference inputs and initial conditions, the plant response and corresponding control signal are plotted in Figure 3.8 - Figure 3.10. We can see from the figures that the self-organizing wavelet network determines the appropriate structure dynamically as the output tracking error trajectory evolves with time and the plant output being able to track the reference input pretty well after 200s and two more resolutions added. Next, the same external disturbance \( d = 1 - \cos(0.3t) \) is provided and the robust controller (3.90) is then implemented to account for it. Figure 3.11 and Figure 3.12 show that the favorable performance is achieved for the closed-loop system under output feedback.
Figure 3.3 Sinusoidal reference input and corresponding plant output

Figure 3.4 Tracking error between plant output and sinusoidal reference input
Figure 3.5 Control signal generated by the wavelet adaptive controller to track sinusoidal references

Figure 3.6 Sinusoidal reference input and corresponding plant output
Figure 3.7 Control signal generated by the wavelet adaptive robust controller to track sinusoidal references

Figure 3.8 Sinusoidal reference input and corresponding plant output
Figure 3.9 Tracking error between plant output and sinusoidal reference input

Figure 3.10 Control signal generated by the wavelet adaptive robust controller to track sinusoidal references
3.5 Conclusions

In this chapter, a self-organizing state and output feedback wavelet adaptive robust controllers have been proposed for a class of nonlinear system with disturbances. The wavelet network is able to adjust its structure dynamically if incrementing new wavelet resolution is considered to be necessary so as to keep the network size small. When
disturbance is present and a prior knowledge of its upper bound is not available, a $H_{\infty}$ robustifying controller is added to achieve the tracking performance with desired attenuation level. The stability of the closed-loop system under state and output feedback control is guaranteed by means of Lyapunov method and singular perturbation theory. Simulation results verify the effectiveness of the proposed adaptive robust feedback controller and the advantage of the self-organizing wavelet network.
CHAPTER 4: \( L_1 \) WAVELET NETWORK ADAPTIVE CONTROLLERS FOR INFINITE DIMENSIONAL SYSTEM

In this chapter, we combine wavelet network approximation technique with \( L_1 \) adaptive control architecture for a class of infinite dimensional systems subject to bounded input operator and unknown matched Lipschitz nonlinearities. The \( L_1 \) adaptive control framework has been extended from finite dimensional setting to infinite dimensional systems to provide guaranteed transient performance via incorporating a low-pass filter in the feedback loop. The wavelet multiresolution analysis technique is used as the building block to approximate the unknown nonlinear system dynamics by virtue of the promising function approximation capability of wavelet networks. Under certain assumptions on the transfer function and on the solutions to the Lyapunov inequality, the framework for wavelet network adaptive controllers is theoretically analyzed and the uniform bounds on the state and control signal are established. Simulation example of heat equation with unknown nonlinearities is presented to illustrate the \( L_1 \) architecture and demonstrate the tracking performance.

4.1 Notation and Definition

In this section, we provide some definitions and notations from [73] that will be referred to subsequently.

Let \( \mathbb{R}/\mathbb{C} \) be the space of real/complex numbers. Denote \( C_\alpha \) as the set of complex numbers with real parts greater or equal than \( \alpha \in \mathbb{R} \), that is \( C_\alpha := \{ s \in C : real(s) \geq \alpha \} \). Consider the Hilbert spaces \( X \) and \( Y \). Let the space of bounded linear operators from \( X \) to \( Y \) be \( L(X,Y) \) and denote \( L(X,X) \) be \( L(X) \). For any \( x, y \in X \), denote \( \langle x, y \rangle_X \) and \( \| x \|_X^2 = \langle x, x \rangle_X \) as the inner product between \( x \) and \( y \) and the norm of \( x \) respectively (subscripts are omitted when \( X \) is clear). The image of \( x \in X \) under the
map \( L \in L(X,Y) \) is defined as \([L;x]\). Define \( L^\infty([0,T],X) \) as the space of bounded functions from \([0,T)\) to \(X\) with the supremum norm and denote \( L^\infty([0,T),X) \) as \( L^\infty(X) \). The product space of \( X\) \( m\)-times is denoted as \( X^m \) and for \( f = (f_1, \cdots, f_m) \in X^m\), its \( L^2\) norm is defined as \( \|f\|_X^2 = \sum_{i=1}^m \|f_i\|_X^2 \). For an operator \( A \) defined on \( X\), let \( D(A) \) and \( \rho(A) \) denote its domain and resolvent set, respectively. The notation \( P > 0 \) is representing a self-adjoint positive operator, i.e. \( \langle Px, x \rangle_X > 0 \) for all nonzero \( x \in X \) and \( P > Q \) is equivalent to \( P - Q > 0 \). Denote \( I_X \) as the identity operator on space \( X \) and \( g \in C^1 \) as continuously differentiable function.

### 4.2 Problem Formulation

Consider the following infinite dimensional system

\[
\begin{align*}
\dot{x} &= Ax + Bu - Bf(x), \quad x(0) = x_0 \\
y &= cx
\end{align*}
\]

(4.1)

where \( x \in X \) is the system state and \( u \in \mathbb{R} \) is the control input with \( X \) is a Hilbert space. The state operator \( A \) is assume to be the infinitesimal generator of an exponential stable \( C_0 \) - semigroup \( T_t \) on \( X \) and input operator \( B \in \mathcal{L}(\mathbb{R},X) \). Therefore \( B \) is equivalent to a constant \( b \in X \). Assume all states are measureable and the output operator \( c \in \mathcal{L}(X,\mathbb{R}) \) generates the output to be regulated. Suppose the operators \( A, B \) and \( c \) are all known. \( f:X \rightarrow \mathbb{R} \) is an unknown Lipschitz \( C^1 \) function, i.e., there exists \( L \) such that for any \( x_1, x_2 \in X \)

\[
\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\| \quad (4.2)
\]

We also assume that \( f(0) \) is known and

\[
f(0) \leq d_f \quad (4.3)
\]
The control objective is to design a wavelet network adaptive controller which guarantees boundedness of all control and state variables of the closed-loop system and tracking of a given reference signal $r(t)$ both in transient and steady states.

By the universal approximation theorem [10], there exists an $K$ level resolution wavelet network identifier of $f(x)$ such that in compact set $D_x \subseteq X$

$$f(x) = (c_j^*)^T \phi_j(x) + \sum_{j=1}^{K} (w_j^*)^T \psi_j(x) + \varepsilon_K(x)$$

$$= \sum_{i=1}^{N} \theta_i^* g_i + \varepsilon_K(x)$$

$$= (\theta^*)^T g(x) + \varepsilon_K(x), \quad x \in D_x$$

where

$$c_j^* = [c_{j,k_1}^*, \ldots, c_{j,k_N}^*]^T$$

$$w_j^* = [w_{j,k_1}^*, \ldots, w_{j,k_N}^*]^T$$

$$\phi_j(x) = [\phi_{j,k_1}(x), \ldots, \phi_{j,k_N}(x)]^T$$

$$\psi_j(x) = [\psi_{j,k_1}(x), \ldots, \psi_{j,k_N}(x)]^T$$

$\theta^* = \{\theta_i^* \in \{c_{j,m}, w_{j,n}\}, i = 1 \cdots N\} \in \mathbb{R}^N$ is the constant weight vector and $g = \{g_i \in \{\phi_{j,m}, \psi_{j,n}\}, i = 1 \cdots N\}$ is the vector of scaling or wavelet basis functions acting on $X$. $N$ is the number of basis function included in the approximator to meet the satisfactory modeling accuracy requirement. $\varepsilon_K(x)$ is the $K$th level approximation error. Further assume that the approximation error $\varepsilon_K(x)$ has a uniform bound $d_{\varepsilon}$ such that

$$\|\varepsilon_K(x)\| \leq d_{\varepsilon}, \quad x \in D_x$$

and the weight vector $\theta^*$ is in a compact set $D_{\theta}$ known a priori such that

$$\|\theta^*\|_{\mathbb{R}^N} \leq M$$
where $M > 0$ is a known constant.

4.3 Projection Adaptive Observer and $L_1$ Adaptive Controller

For any $z_1 \in X$, define the function

$$h(z_1) = \frac{\langle z_1, z_1 \rangle_X - M^2}{\varepsilon M^2}$$

(4.7)

where $\varepsilon > 0$. The Frechet derivative $Dh$ of $h$, is an element of $X$, denoted by $h'$, satisfying

$$[Dh(z_1), z_2] = \langle h'(z_1), z_2 \rangle_X = \frac{\langle z_1, z_2 \rangle_X}{\varepsilon M^2}, \quad \forall z_2 \in X$$

(4.8)

For $z_1, z_2 \in X$, define the projection operator $\Pi: X \times X \rightarrow X$

$$\Pi(z_1, z_2) = \begin{cases} 
  z_2, & h(z_1) \leq 0 \text{ or } \langle h'(z_1), z_2 \rangle \leq 0 \\
  z_2 - \frac{h'(z_1)}{\|h'(z_1)\|_X} \langle \frac{h'(z_1)}{\|h'(z_1)\|_X}, z_2 \rangle h(z_1), h(z_1) > 0 \text{ and } \langle h'(z_1), z_2 \rangle > 0 
\end{cases}$$

(4.9)

Lemma 4.1 [73]: The operator $\Pi(z_1, z_2)$ as defined is Lipschitz on the Hilbert space $X \times X$.

Let $Q$ be a self-adjoint, boundedly invertible operator on $X$, with $D(A) \subseteq D(Q)$ and $\langle Qz, z \rangle_X > 0$ for all $z \in D(A)$. Assume $Q$ is coercive, i.e., $Q \geq \varepsilon_Q I_X$ with $\varepsilon_Q > 0$. Suppose that there exists an operator $P \in \mathcal{L}(X)$ with $P > 0$ satisfying the following Lyapunov inequality

$$\langle (A + \mu I)z, Pz \rangle + \langle Pz, (A + \mu I)z \rangle \leq -\langle Qz, z \rangle$$

(4.10)

where $\mu > 0$ is a sufficiently small so that the $A + \mu I$ is exponentially stable. Note that if $Q \in \mathcal{L}(X)$, such that $P$ can always be found. $\mu$ can be chosen such that $Q + 2\mu P$ is boundedly invertible.

Consider the following state observer for (4.1)
\[
\dot{x} = A\hat{x} + B\left(u - (\hat{\theta}(t))^T g(x)\right), \quad \hat{x}(0) = x_0
\]
\[
\dot{\hat{y}} = c\hat{x}
\]
(4.11)
along with the adaptation law for \(\hat{\theta}(t)\)
\[
\dot{\hat{\theta}}(t) = \gamma_\theta \Pi \left(\hat{\theta}, (\hat{x} - x, Pb)g(x)\right)
\]
(4.12)
where \(\gamma_\theta > 0\) is the adaptation gain and initial condition satisfies \(\|\hat{\theta}(0)\| < \sqrt{1 + \epsilon M}\).

Define the observation error \(e\) as \(e = \hat{x} - x\) and the parameter error \(\tilde{\theta}\) as \(\tilde{\theta} = \hat{\theta} - \theta^*\).

Then the following error dynamics can be derived from (4.1) and (4.11)
\[
\dot{e} = Ae - B \left((\tilde{\theta}(t))^T g(x) + B\epsilon_k(x), \quad e(0) = 0
\]
(4.13)

In \(L_1\) theory, the control signal \(u\) in (4.1) is generated using the adaptive controller
\[
\dot{x}_u = A_u x_u + B_u \left((\hat{\theta}(t))^T g(x) + k_g r\right)
\]
\[
u = C_u x_u
\]
(4.14)
where \(A_u\), \(B_u\) and \(C_u\) are \(n \times n\), \(n \times 1\) and \(1 \times n\) dimensional matrices respectively. \(r \in L^{\infty}(\mathbb{R})\) is the reference input and \(k_g \in \mathbb{R}\) is a chosen constant gain.

Typically, in model reference adaptive control, the \(\left((\hat{\theta}(t))^T g(x) + k_g r\right)\) is not filtered, whereas the introduction of the filter in (4.14) is a key component of the \(L_1\) theory.

The complete \(L_1\) wavelet network adaptive controller consists of (4.1), (4.11), (4.12) and (4.14) whose architecture is show in Figure 4.1. Now we need to prove the boundedness and asymptotic properties of \(e\).
Figure 4.1 wavelet network $\mathcal{L}_1$ adaptive control architecture

**Lemma 4.2:** Consider the system (4.1), (4.13) and (4.14). If $P$ is coercive, i.e. $P \geq \varepsilon P I_X$ with $\varepsilon_P > 0$, then it follows that for all time $t$, $\|e(t)\|_X \leq 2M_0 \sqrt{1/\varepsilon P y_\theta}$

where $M_0 = \sqrt{M_S^2 + y_\theta \left( d_\varepsilon \|Pb\|_X \right)^2}$ and $M_S = \sqrt{1 + \varepsilon M}$.

**Proof:** The linear operator associated with the set of equations (4.1), (4.13) and (4.14) generates a $C_0$-semigroup and the nonlinear part is Lipschitz from Lemma 4.1. It follows that there exists a unique mild solution $(x(t), e(t), \tilde{\theta}(t), x_u(t))$ to this set of equations on a maximal time interval $[0, T_{max})$ from Theorem 1.4 in Chapter 6 of [64] provided the nonlinearity is continuous function of time and locally Lipschitz in the states. Since mild solutions are continuous, it is easily verified that $\tilde{\theta}(t) \in C^1$. The projection law ensures that $\tilde{\theta}(t)$ remains bounded and therefore can be treated as a bounded exogenous $C^1$ signal in (4.14). Therefore (4.1) together (4.12)-(4.14) can be
treated as a linear time varying system with bounded input $r$ wherein the time varying components $B_u \left( \bar{\theta}(t) \right)^T g(\cdot)$ and $Bf(\cdot)$ are bounded perturbations to the time invariant component. Hence the unique mild solution $x$, $x_u$ of (4.1) and (4.12)-(4.14) can be represented in terms of a mild evolution operator and shown to be bounded on the interval $[0, T_{max})$. As a consequence $e(t)$ is bounded for any $T_{max} < \infty$. Hence $T_{max} = \infty$ by contradiction of the Theorem 1.4 in Chapter 6 of [64].

Assume that $x_0 \in D(A)$ and $r(t)$ is continuous. Then, $x_u(t) \in C^1$ and it follows that $x(t) \in C^1$ from theorem 1.5 in Chapter 6 of [64] which along with $\bar{\theta}(t) \in C^1$ implies that $e(t) \in C^1$. Hence $e(t)$ and $\bar{\theta}(t)$ are classical solutions of (4.13). Consider the following Lyapunov function:

$$V(t) = \langle e, Pe \rangle_x + \frac{1}{Y_\bar{\theta}} \langle \hat{\theta}, \bar{\theta} \rangle_{\mathbb{R}^n}$$

(4.15)

Denote $\hat{h} = h(\hat{\theta})$ and $\hat{h}' = h'(\hat{\theta})$. We omit the subscripts for $\langle \cdot, \cdot \rangle$ for simplicity unless otherwise specified. Taking the derivative along the solution trajectory yields

$$\dot{V}(t) = \langle \hat{e}, Pe \rangle + \langle e, P\hat{e} \rangle + \frac{2}{Y_\bar{\theta}} \langle \hat{\theta}, \bar{\theta} \rangle$$

$$= \langle Ae - B \left( \bar{\theta}(t) \right)^T g(x) + B\varepsilon_K(x), Pe \rangle + \langle e, P \left( Ae - B \left( \bar{\theta}(t) \right)^T g(x) + B\varepsilon_K(x) \right) \rangle$$

$$= \langle -(Q + 2\mu P)e, e \rangle - 2 \langle e, PB \left( \bar{\theta}(t) \right)^T g(x) \rangle$$

$$+ 2 \langle \Pi \left( \hat{\theta}, -(e, Pb)g(x) \right), \bar{\theta} \rangle + 2 \langle e, PB\varepsilon_K(x) \rangle$$

$$= \langle -(Q + 2\mu P)e, e \rangle - 2 \langle (e, Pb)g(x), \bar{\theta}(t) \rangle$$

$$+ 2 \langle \Pi \left( \hat{\theta}, (e, Pb)g(x) \right), \bar{\theta} \rangle + 2 \langle e, Pb\varepsilon_K(x) \rangle$$

when $\hat{h} \leq 0$ or $\langle \hat{h}', (e, Pb)g(x) \rangle \leq 0$,

$$2 \langle \Pi \left( \hat{\theta}, (e, Pb)g(x) \right), \bar{\theta} \rangle - 2 \langle (e, Pb)g(x), \bar{\theta}(t) \rangle$$
\[
\dot{V}(t) = \langle -(Q + 2\mu P)e, e \rangle - \frac{1}{\|\hat{h}'\|} \langle \hat{h}', \hat{h} \rangle \langle \hat{\theta}, \langle e, Pb \rangle g(x) \rangle \dot{h} \\
+ 2\langle e, Pb \varepsilon_K(x) \rangle \\
= \langle -(Q + 2\mu P)e, e \rangle - \frac{1}{\varepsilon M^2} \|\hat{h}'\| \langle \hat{\theta}, \langle e, Pb \rangle g(x) \rangle \dot{h} \\
+ 2\langle e, Pb \varepsilon_K(x) \rangle
\] (4.16)

Since the terms in the braces is either 0 or when \(\hat{f} > 0\), \(\langle \hat{f}', \langle e, Pb \rangle g(x) \rangle > 0\) we need to consider \(\langle \hat{\theta}, \hat{\theta} \rangle = \langle \hat{\theta}, \hat{\theta} - \hat{\theta}^* \rangle\). Because when \(\hat{f} > 0\) we have \(\|\hat{\theta}\| > M\) together with (4.6) yields \(\langle \hat{\theta}, \hat{\theta} \rangle \geq 0\). Therefore (4.16) becomes

\[
\dot{V}(t) \leq \langle -(Q + 2\mu P)e, e \rangle + 2\langle e, Pb \varepsilon_K(x) \rangle
\] (4.17)

Since \(\langle Qe, e \rangle \geq \varepsilon_Q \|e\|^2_X\) and \(\langle Pe, e \rangle \geq \varepsilon_P \|e\|^2_X\), then from (4.5) we have \(\dot{V}(t) \leq 0\) if

\[
\|e\| \geq \frac{2d_\varepsilon \|Pb\|}{\varepsilon_Q + 2\varepsilon_P}
\] (4.18)

It has been proved in [73] that if \(h(\hat{\theta}(0)) < 1\), then \(h(\hat{\theta}(t)) \leq 1\) for all \(t\). Since \(\|\hat{\theta}(0)\| < \sqrt{1 + \varepsilon M}\), then we can have \(\|\hat{\theta}(t)\| \leq \sqrt{1 + \varepsilon M}\). Noting that \(M_s = \sqrt{1 + \varepsilon M}\), we could then derive

\[
\frac{1}{\gamma_\theta} \langle \hat{\theta}, \hat{\theta} \rangle = \frac{1}{\gamma_\theta} \|\hat{\theta}(t)\|^2 = \frac{1}{\gamma_\theta} \|\hat{\theta} - \theta^*\|^2 \leq \frac{1}{\gamma_\theta} 4M_s^2
\] (4.19)

It can then be shown that

\[
\|\langle e, Pe \rangle\|_{L^\infty(X)} \leq 2M_0 \sqrt{1/\gamma_\theta}
\] (4.20)

If \(x_0 \notin D(A)\) and \(r(t)\) is not continuous, by the continuous dependence of mild solutions on initial conditions and input [64], (4.20) continues to hold. If \(P\) is coercive, then for all \(t\), then we have
\[ \|e(t)\|_X \leq 2M_0 \sqrt{1/\epsilon_p} \gamma_\theta \]  \hspace{1cm} (4.21)

The proof completes.

### 4.4 $L_1$ Topology Analysis in the Infinite Dimensional Setting

To analyze the $L_1$ adaptive control architecture introduced in section 4.3, a stable reference system is considered, following the finite dimensional development. The errors between, respectively, the control signal and the state of the plant and their reference system counterparts are shown to be uniformly bounded in time by a constant inversely proportional to the adaptation gain. Hence for a large adaptive gain, guaranteed transient performance is guaranteed.

The reference system is designed using the unknown optimal wavelet weight vector and therefore cannot be employed to specify performance requirements. A desired system with the necessary performance requirements is thus specified, and error between the control signal and states of the desired system and their reference system counterparts are respectively shown to be uniformly bounded, with a tunable bound that must be minimized. Hence the control signal and the state of the plant remain close to those of the reference system, whose control signal and state are in turn close to those of the desired system, and consequently scale with the reference input and initial condition akin to linear time invariant (LTI) system. Note that the reference system and the desired system are used for designing the controller in (4.14) and for obtaining bounds on plant responses, but play no role in the generation of the control signal as seen in Figure 4.1.

Let $C(s)$ be the transfer function (TF) for the filter in (4.14) where $C(s) = C_u(sI - A_u)^{-1}B_u$. In $L_1$ theory $C(s)$ is chosen to be a low pass filter. Assume that $C(s)$ is stable and strictly proper and satisfies $C(0) = 1$. The gain $k_g$ in (4.14) is chosen to produce the desired steady state for the output $y = cx$ when the reference
input \( r \) is a step. Let the bounded-input-bounded-output (BIBO) gain of the system with TFs \( H(s) = (sI - A)^{-1}B \), \( C(s) \) and \( G(s) = H(s)(C(s) - 1) \) be \( H_B \), \( C_B \) and \( G_B \) respectively. These gains are calculated using the Euclidean norm on \( \mathbb{R} \).

The \( \mathcal{L}_1 \) adaptive control architecture is analyzed in infinite dimensional systems under the following assumptions:

**Assumption 4.1[73]:** The solution \( P \) to the Lyapunov inequality (4.10) is coercive, i.e., satisfies \( P \geq \varepsilon_P I_X \) for some \( \varepsilon_P > 0 \).

**Assumption 4.2[73]:** There exists \( c_0 \in X \) such that the \( F(s) \) with \( F(s) = \langle c_0, H(s) \rangle \) is invertible on \( C_\alpha \) with \( \alpha < 0 \) with \( C(s)F(s)^{-1} \) being a bounded analytic function on \( C_\alpha \). Moreover \( C(s)F(s)^{-1} \) corresponds to an exponentially stable LTI system with BIBO gain \( F_B \). Let \( c_X = \| c_0 \|_X \).

It has been mentioned in [73] that in finite dimensions Assumption 4.1 holds trivially, but is not obvious in infinite dimensions. For instance as shown in [66], for some infinite dimensional systems, the solution \( P \) to the Lyapunov equation cannot be coercive if \( Q \) is bounded (impossible in finite dimensions). Assumption 4.1 is necessary to derive uniform bounds on the state error, which in turn give uniform bounds on the error in the control signal, one of the main features of the finite dimensional \( \mathcal{L}_1 \) theory. Assumption 4.2, needed to establish uniform bounds on the error in the control signal, follows in finite dimensional if \( (A, B) \) is controllable [35]. Verifying Assumption 4.2, difficult in general, is easier if a complete set of eigenfunctions exists. In the rest of this section, we assume Assumptions 4.1 and 4.2 hold.
Now we need to consider the reference system that the $L_1$ wavelet adaptive controller tracks in both transient and steady state, and this tracking is valid for system’s both input and output signals. Towards that end, consider the following reference system

\[
\begin{align*}
\dot{x}_{ref} &= Ax_{ref} + Bu_{ref} - Bf(x_{ref}), \quad x_{ref}(0) = x_0 \\
y &= cx_{ref} \\
u_{ref}(s) &= C(s)\left(\eta(s) + k_g r(s)\right)
\end{align*}
\]

where $\eta(s)$ is the Laplace transformation of $(\theta^*)^T g(x_{ref})$.

**Lemma 4.3:** Assume that $\lambda = G_{B}L < 1$. Then the reference system (4.22) is BIBO stable, where the input is the signal $r$. The reference system state $x_{ref}$ satisfies

\[
\|x_{ref}\|_{L^\infty(\mathcal{X})} \leq \gamma_r
\]

Where

\[
\gamma_r = \frac{(H_BC_Bk_gr\|r\|_{L^\infty(\mathbb{R})} + \|G_B(d_f + d_\varepsilon) + H_Bd_\varepsilon\|_1)}{(1 - \lambda)} + \|x_i\|_{L^\infty(\mathcal{X})} 
\]

**Proof:** The system (4.22) has a semigroup associated with it and hence has a mild solution for $x_{ref}(t) \in D(A)$ existing for all time $t$. Let $x_i(t)$ be the response of $\dot{x}_i = Ax_i$ to initial condition $x_0$. For the mild solution $x_{ref}(t)$, we need to prove

\[
\|x_{ref}\|_{L^\infty(\mathcal{X})} \leq \gamma_r
\]

We prove this by contradiction. Assume this is not true and for the first time $T > 0$ and there exists $N_r$ such that $N_r > 1$

\[
\|x_{ref}(T)\|_X = N_r\gamma_r
\]

It follows that

\[
\|x_{ref}(\tau)\|_X \leq N_r\gamma_r, \quad \forall \tau \in [0, T]
\]

and hence that
\[ x_{\text{ref}}(\tau) \in D_x, \quad \forall \tau \in [0, T] \] (4.27)

and then we have

\[ \| \varepsilon_K(x_{\text{ref}}(\tau)) \| \leq d_\varepsilon, \quad \forall \tau \in [0, T] \] (4.28)

Consider the closed-loop system (4.22), we can derive

\[ \| x_{\text{ref}}(T) \|_X \leq H_B C_B k_g \| r \|_{L^\infty(\mathbb{R})} + G_B \| \eta(T) \|_{\mathbb{R}^N} + H_B d_\varepsilon \]

\[ + \| x_i \|_{L^\infty(X)} \] (4.29)

Notice that \( \eta(T) \) can be represented as

\[ \eta(T) = (\theta^*(T))^T g(x_{\text{ref}}(T)) \]

\[ = f(x_{\text{ref}}(T)) - \varepsilon_K(x_{\text{ref}}(T)) \]

\[ \leq \| f(0) \| + L \| x_{\text{ref}}(T) \|_X + d_\varepsilon \] (4.30)

Hence

\[ \| \eta(T) \|_{\mathbb{R}^N} \leq \| f(0) \| + L \| x_{\text{ref}}(T) \|_X + d_\varepsilon \] (4.31)

which leads to

\[ \| x_{\text{ref}}(T) \|_X \leq H_B C_B k_g \| r \|_{L^\infty(\mathbb{R})} + G_B \left( d_f + L \| x_{\text{ref}}(T) \|_X + d_\varepsilon \right) \]

\[ + H_B d_\varepsilon + \| x_i \|_{L^\infty(X)} \]

\[ \Leftrightarrow \]

\[ \| x_{\text{ref}}(T) \|_X \leq \frac{H_B C_B k_g \| r \|_{L^\infty(\mathbb{R})} + G_B (d_f + d_\varepsilon) + H_B d_\varepsilon}{1 - \lambda} \]

\[ + \| x_i \|_{L^\infty(X)} \]

(1 - \lambda) (4.32)

Then it follows that

\[ \| x_{\text{ref}}(T) \|_X < N_r \left( \frac{H_B C_B k_g \| r \|_{L^\infty(\mathbb{R})} + G_B (d_f + d_\varepsilon) + H_B d_\varepsilon}{1 - \lambda} \right) \]

\[ + N_r \frac{\| x_i \|_{L^\infty(X)}}{1 - \lambda} = N_r y_r \] (4.33)
which contradicts (4.25). Hence the reference system (4.22) is BIBO stable and (4.23) holds. The proof completes.

In (4.13), \( e \) is bounded since \( P \) is coercive by Assumption 4.1. Using this fact along with boundedness of \( \hat{\theta} \), we can prove \( \hat{x}(t) \) and consequently \( x(t) \) are bounded. And \( \varepsilon_{\text{ref}}(s) \) is the Laplace transformation of \( \varepsilon_K(x_{\text{ref}}) \).

**Theorem 4.1:** The states and control signals of (4.1) and (4.22) satisfy the following inequalities

\[
\|x_{\text{ref}} - x\|_{L^\infty(\chi)} \leq \gamma_1 \tag{4.34}
\]

\[
\|u_{\text{ref}} - u\|_{L^\infty(\chi)} \leq \gamma_2 \tag{4.35}
\]

where

\[
\gamma_1 = \frac{2M_0H_BF_Bc_0}{(1 - \lambda)} + \frac{3C_Bd_e}{1 - \lambda} \tag{4.36}
\]

\[
\gamma_2 = \frac{2F_Bc_0M_0}{\sqrt{\varepsilon_P}\sqrt{\lambda}} + C_B L \gamma_1 + C_B d_e \tag{4.37}
\]

**Proof:** From (4.11) and (4.22), it follows that

\[
\dot{x}_{\text{ref}} - \dot{x} = A(x_{\text{ref}} - x) + B(u_{\text{ref}} - u) - B(f(x_{\text{ref}}) - f(x)) \tag{4.38}
\]

Using (4.14) and (4.22)

\[
u_{\text{ref}} - u = \mathcal{C}(s)(\eta(s) - \eta_2(s)) \tag{4.39}\]

where \( \eta_2(s) \) is the Laplace transformation of \( \left( \hat{\theta} \right)^T g(x) \). Recalling wavelet network approximation in (4.4), we can derive

\[
\left( \theta^* \right)^T g(x_{\text{ref}}) - \left( \hat{\theta} \right)^T g(x) = -\left( \left( \hat{\theta} \right)^T (x) - \left( \theta^* \right)^T g(x_{\text{ref}}) \right)
\]

\[
= -\left( \left( \hat{\theta} \right)^T g(x) + (\theta^*)^T \left( g(x) - g(x_{\text{ref}}) \right) \right)
\]

\[
= -\left( \left( \hat{\theta} \right)^T g(x) + f(x) - f(x_{\text{ref}}) + \varepsilon_K(x_{\text{ref}}) - \varepsilon_K(x) \right) \tag{4.40}
\]
From (4.13),
\[ e(s) = H(s)(-\eta_3(s) + \varepsilon(s)) \]  
(4.41)
where \( \eta_3(s) \) and \( \varepsilon(s) \) are the Laplace transformations of \( (\hat{\theta})^T g(x) \) and \( \varepsilon_K(x) \) respectively. Using Assumption 4.2 it follows that
\[
C(s)\eta_3(s) = C(s)F(s)^{-1}F(s)\eta_3(s) = C(s)F(s)^{-1}(c_0, H(s))\eta_3(s) = -C(s)F(s)^{-1}(c_0, \varepsilon(s)) + C(s)\varepsilon(s)
\]  
(4.42)
Also from (4.2) we have
\[
\|f(x_{ref}) - f(x)\| \leq L\|x_{ref} - x\|
\]  
(4.43)
Hence it follows from (4.39) to (4.43) and Lemma 4.2 that
\[
\|x_{ref} - x\|_{L^\infty(\Xi)} \leq H_B F_B c_0 \|e\|_{L^\infty(\Xi)} + G_B L \|x_{ref} - x\|_{L^\infty(\Xi)} + 3C_B d_\varepsilon
\]
\[
\Rightarrow \|x_{ref} - x\|_{L^\infty(\Xi)} \leq \frac{H_B F_B c_0 \|e\|_{L^\infty(\Xi)}}{1 - \lambda} + \frac{3C_B d_\varepsilon}{1 - \lambda}
\]
\[
\leq \frac{2M_0 H_B F_B c_0}{(1 - \lambda)\sqrt{\varepsilon_P \gamma_\theta}} + \frac{3C_B d_\varepsilon}{1 - \lambda}
\]  
(4.44)
Therefore it follows that
\[
\|u_{ref} - u\|_\Xi \leq F_B c_0 \|e\|_X + C_B L \|x_{ref} - x\|_{L^\infty(\Xi)} + 3C_B d_\varepsilon
\]
\[
\leq \frac{2F_B c_0 M_0}{\sqrt{\varepsilon_P \gamma_\theta}} + C_B L \gamma_1 + 3C_B d_\varepsilon
\]  
(4.45)
The proof completes. 

**Remark 4.1:** From the relationship in (4.23) and (4.34), it’s straightforward to verify that
\[
\|x\|_{L^\infty(\Xi)} = \|x - x_{ref} + x_{ref}\|_{L^\infty(\Xi)}
\]
66
\[ \|x - x_{ref}\|_{L^\infty(X)} + \|x_{ref}\|_{L^\infty(X)} \leq \gamma_1 + \gamma_r \]

which implies that \( x(t) \in D_x \) for all \( t \geq 0 \)

**Corollary 4.1:** Consider the system (4.1) and the wavelet controller (4.11), (4.12) and (4.14), we have

\[
\begin{align*}
\lim_{\gamma \to \infty, d \to 0} (x_{ref}(t) - x(t)) &= 0, \quad \forall t \geq 0 \\
\lim_{\gamma \to \infty, d \to 0} (y_{ref}(t) - y(t)) &= 0, \quad \forall t \geq 0 \\
\lim_{\gamma \to \infty, d \to 0} (u_{ref}(t) - u(t)) &= 0, \quad \forall t \geq 0
\end{align*}
\]

Corollary 4.1 states that \( x(t), y(t) \) and \( u(t) \) follow \( x_{ref}(t), y_{ref}(t) \) and \( u_{ref}(t) \) not only asymptotically but also during the transient, provided the adaptation gain is chosen sufficiently large and the resolution level \( K \) of the wavelet network is selected high enough to ensure accurate approximation.

The desired system through which transient response can be specified has the following TF representations:

\[
x_{des}(s) = H(s)C(s)k_g r(s) + x_i(s)
\]

\[
u_{des}(s) = C(s)k_g r(s) + C(s)\eta_4(s)
\]

\[
y_{des}(s) = cx_{des}(s)
\]

(4.46)

where \( \eta_4(s) \) is the Laplace transformation of \( f(x_{des}(t)) \). The filter \( C(s) \) and \( k_g \) must be chosen (and when feasible \( H(s) \) must be modified) such that the output \( y_{des} \) has a desired response.

**Lemma 4.4:** The following bounds hold:

\[
\|x_{ref} - x_{des}\|_{L^\infty(X)} \leq \gamma_3
\]

(4.47)

\[
\|u_{ref} - u_{des}\|_{L^\infty(\mathbb{R})} \leq C_B L \gamma_3 + C_B d \epsilon
\]

(4.48)
where
\[ \gamma_3 = G_B (d_f + L \gamma_r + d_\varepsilon) + H_B d_\varepsilon \]  (4.49)

**Proof:** It follows from (4.22) and (4.46) that
\[ x_{\text{ref}}(s) - x_{\text{des}}(s) = G(s)\eta(s) - H(s)\varepsilon_{\text{ref}}(s) \]  (4.50)
and hence
\[ \|x_{\text{ref}} - x_{\text{des}}\|_{L^\infty(X)} \leq G_B \|\mathbf{g}(x_{\text{ref}})\|_{L^\infty(\mathbb{R})} + H_B d_\varepsilon \]  (4.51)

Since
\[ (\theta^*)^T g(x_{\text{ref}}) = f(x_{\text{ref}}) - \varepsilon_K(x_{\text{ref}}) \]  (4.52)
and from (4.2) we have
\[ \|f(x_{\text{ref}})\| \leq \|f(0)\| + L\|x_{\text{ref}}\| = d_f + L\|x_{\text{ref}}\| \]  (4.53)
It then follows that
\[ \|\mathbf{g}(x_{\text{ref}})\|_{L^\infty(\mathbb{R})} \leq d_f + L\|x_{\text{ref}}\|_{L^\infty(X)} + d_\varepsilon \]  (4.54)

From Lemma 4.3 we have
\[ \|x_{\text{ref}} - x_{\text{des}}\|_{L^\infty(X)} \leq G_B (d_f + L \gamma_r + d_\varepsilon) + H_B d_\varepsilon = \gamma_3 \]  (4.55)
Using (4.22) and (4.46) again, we have
\[ \|u_{\text{ref}} - u_{\text{des}}\|_{L^\infty(\mathbb{R})} \leq C_B \|\eta_5 - \eta_4\|_{L^\infty(\mathbb{R})} + C_B d_\varepsilon \]  (4.56)
where \( \eta_5(t) = f\left(x_{\text{ref}}(t)\right) \). From (4.2) we have
\[ \|f(x_{\text{ref}}) - f(x_{\text{des}})\| \leq L\|x_{\text{ref}} - x_{\text{des}}\| \]  (4.57)
It follows that
\[ \|u_{\text{ref}} - u_{\text{des}}\|_{L^\infty(\mathbb{R})} \leq C_B L \gamma_3 + C_B d_\varepsilon \]  (4.58)
The proof completes.
Remark 4.2: From Theorem 4.1 it follows that by choosing the adaptation gain $\gamma_\theta$ sufficiently large and $\lambda$ sufficiently small, the error between the states of the plant and the LTI desired system can be bounded uniformly in time by a small constant. Hence the plant state responds to reference inputs like a LTI system which is independent of any uncertain parameters. The value of $\lambda$ can be made arbitrarily small by increasing the bandwidth of $C(s)$. But increasing the bandwidth of $C(s)$ to reduce $\lambda$ could decrease robustness margin. Alternatively, the bandwidth of $H(s)$ can be reduced to obtain a small $\lambda$. Hence the choice of $C(s)$ and $H(s)$ leads to a performance and robustness tradeoff. The matrix $k$ is selected to ensure that the output $y_{des}$ reaches the required steady state value for step inputs.

4.5 Simulations

Consider the following heat equation with full state measurement

$$ \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + b(u - f(z)) $$

$$ x \in (0,1), z(0, t) = z(1, t) = 0, z(x, 0) = 0 $$

(4.59)

where $b$ is a square integrable function on $[0,1]$, i.e. $L^2(0,1)$, and $u \in \mathbb{R}$ is the control input. $f(z)$ is an unknown nonlinear function of system states and considered as:

$$ f(z) = (5 + z)\sin(z) $$

Let $H^2(0,1)$ be the usual Sobolev space of order 2 and $H^1_0(0,1)$ be the Sobolev space of order 1, such that for $g \in H^1_0(0,1), g(0) = g(1) = 0$. The operator $A = \frac{\partial^2}{\partial x^2}$, with $D(A) = H^2(0,1) \cap H^1_0(0,1)$ generates an exponentially stable $C_0$-semigroup on the space $Z = L^2(0,1)$; in fact $A$ generates a holomorphic semigroup [65]. For Assumption 4.1 to hold for (4.59), choose $Q = -2A$. Then the Lyapunov inequality (4.10) is satisfied with $P = I_Z$, where $I_Z$ is the identity operator on $Z$. Obviously, $P$ is coercive and Assumption 4.1 is satisfied. The eigenvalues of $A$ are $-n^2\pi^2$ with corresponding
eigenvectors $\phi_n(x) = \sqrt{2}\sin(n\pi x)$ for all integers $n \geq 1$. The eigenvectors can be shown to form an orthonormal basis for $Z$. Hence $b = \sum_{n=1}^{\infty} b^n \phi_n(x)$. For any complex numbers $s \in \rho(A)$, it follows that $(sl - A)^{-1}\phi_n = \frac{\phi_n}{s + n^2 \pi^2}$. Therefore

$$(sl - A)^{-1}b = \sum_{n=1}^{\infty} b^n \frac{\phi_n}{s + n^2 \pi^2}$$ \hspace{1cm} (4.60)

Choose a positive integer $p$ and let $c_1 = \phi_p$, then

$$F = \frac{b^p}{s + p^2 \pi^2}, F^{-1} = \frac{s + p^2 \pi^2}{b^p}$$ \hspace{1cm} (4.61)

Then for any diagonal $C(s)$ with strictly proper entries it follows that $C(s)F(s)^{-1}$ corresponds to an exponentially stable system with a finite BIBO gain. Hence, Assumption 4.2 is satisfied.

The operator $b$ is chosen similarly as in [73]:

$$b = \begin{cases} 
0, & x \notin [0.24, 0.26] \\
10, & x \in [0.24, 0.26]
\end{cases} \hspace{1cm} (4.62)$$

The output to be regulated is $y(t) = Cz$, where $C$ is defined as $Cg = 50 \int_{0.49}^{0.51} g(x)dx$, $\forall g \in L^2(0,1)$. The input and output operators are chosen to approximate point control at $x = 0.25$ and point observation at $x = 0.5$. The control objective is to design a $L_1$ wavelet network adaptive controller to ensure that output $y$ tracks any desired reference $r$ both in transient and steady states.

The projection operator’s parameters are chosen as:

$$\varepsilon = 0.01 \text{ and } M = 10$$

The nonlinear function $f(z)$ would be estimated online by wavelet network. Daubechies wavelet (db3) is employed to construct the scale and wavelet basis functions for wavelet MRA model. The coarse resolution is $J = -2$ and $N_j = 10$. $N_j$ doubles as when resolution is increased by 1. Two resolutions are included in the wavelet network to
achieve satisfactory approximation precision. All initial wavelet weights are set to 0. The law-pass filter \( C(s) \) is chosen as:

\[
C(s) = \frac{\omega}{s + \omega}
\]  

(4.63)

where \( \omega = 100\pi \). We can numerically check that the \( \lambda = G_BL < 1 \) with \( L = 10 \) and hence the stability condition in Lemma 4.3 is satisfied. The control signal \( u \) is obtained using the filter in (4.14) whose transfer function is \( C(s) \) in (4.63). The positive constant gain \( k_g \) is set to 32 so that the steady state of step response of the reference system is 1. The heat equation is discretized in space using a finite element scheme with step size of 0.005. The initial and boundary conditions for observer are set same as (4.59). The adaptation gain \( \gamma_\theta \) is chosen as \( \gamma_\theta = 500000 \).

For a constant reference input \( r = 1 \), the system response and control signal are plotted in Figure 4.2 and Figure 4.3. The performance of the \( L_1 \) wavelet network adaptive controller for sinusoidal reference input \( r(t) = 3\sin(0.4\pi t) \) is plotted in Figure 4.4 and Figure 4.5. The figures show that the \( L_1 \) wavelet network adaptive controller has guaranteed both transient and steady response without generating high frequency oscillations in control signal.
Figure 4.2 Reference input and corresponding plant output of the $L_1$ wavelet network controller for $r = 1$

Figure 4.3 Control signal of the $L_1$ wavelet network controller subject to step reference input
Figure 4.4 Sinusoidal reference input and corresponding plant output of the $L_1$ wavelet network controller

Figure 4.5 Control signal generated by the $L_1$ wavelet network controller to track sinusoidal references

### 4.6 Conclusions

Adaptive observers using projection based parameter update laws and a wavelet network $L_1$ adaptive controller for a class of infinite dimensional systems are considered. The assumptions for extending the $L_1$ theory to infinite dimensions are outlined. The proposed control architecture admits fast adaptation without high oscillations in the control signal. When the wavelet network goes to finer resolutions, the bound of both transient and steady state errors will keep decreasing until the overall
tracking performance is satisfied. The validity of the proposed control strategy has been demonstrated by the simulation results of heat equations.
CHAPTER 5: WAVELET MULTiresOLUTION MODEL BASED ADAPTIVE PREDICTIVE CONTROL FOR NONLINEAR SYSTEMS

The classic generalized predictive control (GPC), first proposed by Clark et al. [52], [53], is based on the finite horizon open-loop unconstrained or constrained optimization of the quadratic objective function and moving horizon implementation. Since GPC uses linear dynamic model to make predictions of process outputs over the prediction horizon, its performance will significantly degrade when the real process has severe nonlinearities and operating in a wide range of operating conditions. Therefore, it is imperative to incorporate high fidelity identified nonlinear dynamic model into GPC. Accordingly, we employ wavelet multiresolution analysis (MRA) framework to model nonlinear dynamic process in the GPC scheme. Specifically, a single-input-single-output (SISO) nonlinear autoregressive with exogenous (NARX) based on wavelet MRA model is trained on-line for real-time GPC application. The gradient descent (GD) algorithm is developed for training both the weighting parameters of wavelet MRA model and the control sequence in the GPC scheme. Then Lyapunov stability theorems have been derived to guarantee the convergence of the wavelet MRA identified model and stability of the proposed GPC. The proposed wavelet MRA based predictive controller is successfully implemented to control a novel hybrid combustion-gasification chemical looping process in real test rig. The chemical looping process and corresponding experimental results will be presented in next chapter. In the second part, the identified wavelet network nominal model is then combined within nonlinear model predictive control framework to address the adaptive constrained MPC problem. The asymptotical stability of the proposed adaptive MPC technique has been proved using Lyapunov stability theorem with terminal cost and terminal constraint.
5.1 Wavelet MRA based Unconstrained Adaptive Predictive Control for Nonlinear Systems

5.1.1 Problem Formulation

In this section, we consider the following discrete-time stochastic nonlinear SISO system can be expressed as:

\[ y(t) = f\left(y(t-1), \ldots, y(t-n_y), u(t-1), \ldots, u(t-n_u)\right) + e(t) \]  \hspace{1cm} (5.1)

where \( y(t), u(t), e(t) \) are the system output, input and noise of discrete time \( t \) respectively. \( n_y \) and \( n_u \) are the maximum lags in the output and input. \( e(t) \) is assumed to be zero mean independent bounded noise variable and \( f(\cdot) \) is some nonlinear function. Unless some a prior knowledge of system dynamics is available, most methods use nonparametric regression to estimate the nonlinear function \( f \) from the data. In our case, \( f \) is implemented as a linear expansion in terms of multiresolution wavelet network model

\[ f(x) = \sum_n (f, \phi_{j,n}) \phi_{j,n} + \sum_{j \geq J, n} (f, \psi_{j,n}) \psi_{j,n} \]  \hspace{1cm} (5.2)

where \( f(x) \) is any \( L_2(R) \) function. \( \phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k) \) and \( \psi_{j,k} = 2^{j/2} \psi(2^j x - k), \ j = J, J + 1, \ldots, k \in \mathbb{Z} \) are the chosen scaling and wavelet basis functions respectively. \((\cdot, \cdot)\) is the inner product of \( L_2(R) \) space. \( j \) is the scaling parameter and \( k \) is the translation parameter. The approximation starts from some lower resolution level \( J \) and can be truncated at certain higher resolution level \( N \) when

\[ \left\| f(x) - \sum_n (f, \phi_{j,n}) \phi_{j,n} + \sum_{j = J}^N \sum_n (f, \psi_{j,n}) \psi_{j,n} \right\| < \varepsilon \]  \hspace{1cm} (5.3)

for any predefined small error \( \varepsilon > 0 \). The results of one dimensional case can be extended to high dimensional wavelets. One commonly approach is to construct separable wavelets by the tensor product of several one-dimensional wavelets. Another
popular alternative is to choose the wavelets to be some radial functions. In our settings, \( f \) is approximated as one-layer wavelet network and linear in the parameter space in terms of the scaling and wavelet functions of regressors \( g_i \) selected from \( \mathcal{G} \) such that
\[
    f = \sum_{i=1}^{m} \theta_i g_i
\]
minimizes a pre-specified approximation adequacy criterion, where \( \theta = \{\theta_i\} \) is parameter vector trained online, \( g_i \in \{\phi_{j,k}, \psi_{j,k}\} \) is a multivariable scaling or wavelet basis function of past inputs and outputs, \( \mathcal{G} \) is the set of all candidate basis functions and \( m \) is the number of required basis functions to meet satisfactory modeling accuracy requirement.

5.1.2 Wavelet MRA based GPC Scheme

The basic methodology of GPC is to calculate the current control actions on-line at each sampling instant to solve a finite horizon open-loop optimal control problem where the first control in the optimal control sequence is applied to the plant. In this section, we explain both the online wavelet MRA system identification algorithm and the GPC based predictive control strategy. To clearly illustrate the ideal of the proposed control scheme, we derive the algorithm for SISO nonlinear dynamic system. The extension to multi-input- multi-output setting is fundamentally straightforward.

According to section 5.1.1, denote actual system output as \( y \) and control input as \( u \). Denote \( \hat{y} \) as the approximated system output. Then the identified wavelet MRA based model is defined as follows:
\[
    \hat{y}(t) = f \left( y(t-1), \ldots, y(t-n_y), u(t-1), \ldots, u(t-n_u) \right)
\]
where \( f \) is defined in (5.4). Then the error between real plant output \( y \) and model estimated output \( \hat{y} \) is defined as
\[ e(n) = y(n) - \hat{y}(n) \] (5.6)

The weighting parameters \( \theta \) in (5.4) are trained online in such a way that the defined loss function

\[ J_1(n) = \frac{1}{2} e^2(n) \] (5.7)

is minimized where \( n \) is discrete time. To make \( J_1 \) small, we design the parameter adaptation law using gradient decent (GD) algorithm to change the weighting gains \( \theta \) in the direction of the negative gradient of \( J_1 \), that is,

\[ \theta(n + 1) = \theta(n) - \gamma_\theta \frac{\partial J_1}{\partial \theta}(n) \]
\[ = \theta(n) - \gamma_\theta e(n) \frac{\partial e}{\partial \theta}(n) \] (5.8)

where \( \gamma_\theta \) is the adaptation gain, \( \frac{\partial J_1}{\partial \theta}(n) \) is the partial derivative of \( J_1 \) with respect to \( \theta \) at discrete time \( n \) and \( \frac{\partial e}{\partial \theta}(n) \) is the so called sensitivity derivative at time \( n \) indicating how the error is influenced by weighting parameters \( \theta \). From (5.4) to (5.6), the sensitivity derivative \( \frac{\partial e}{\partial \theta} \) can be derived as follows:

\[ \frac{\partial e}{\partial \theta} = -\frac{\partial \hat{y}}{\partial \theta} = -\frac{\partial f}{\partial \theta} = -g \Rightarrow \theta(n + 1) = \theta(n) + \gamma_\theta e(n) g(n) \] (5.9)

where subscript \( i \) is the \( i \)-th component of the corresponding vectors.

Suppose future set-point signals \( y_m(t + k), k = 1, 2, \ldots \) are available. In context of GPC settings, define another loss function as follows:

\[ J_2 = \frac{1}{2} \left\{ \sum_{k=N_1}^{N_2} (y_m(n + k) - \hat{y}(n + k))^2 + \sum_{k=1}^{N_u} \rho_k \Delta u(n + k - 1)^2 \right\} \] (5.10)

where \( N_1 \) and \( N_2 \) are the minimum and maximum output prediction horizon respectively. \( N_u \) is the control horizon. \( \Delta \) is the difference operator and \( \Delta u(n) = u(n) - u(n - 1) \). \( \rho_k \) is the \( k \)-th control weighting factor. Assume \( N_1 = 1 \), \( N_2 = L \) and identical control weighing factor \( \rho_k = \rho \), (5.10) can be rewritten in the vector form as
\[ J_2 = \frac{1}{2} \left\{ \| Y_m(n + 1) - \hat{Y}(n + 1) \| ^2 + \rho \| \Delta U(n) \| ^2 \right\} \]  

(5.11)

where

\[ Y_m(n + 1) = [y_m(n + 1), y_m(n + 2), \ldots, y_m(n + L)]^T \]
\[ \hat{Y}(n + 1) = [\hat{y}(n + 1), \hat{y}(n + 2), \ldots, \hat{y}(n + L)]^T \]
\[ U(n) = [u(n), u(n + 1), \ldots, u(n + N_u - 1)]^T \]
\[ \Delta U(n) = [\Delta u(n), \Delta u(n + 1), \ldots, \Delta u(n + N_u - 1)]^T \]

and \( \| \cdot \| \) is the \( L_2 \) norm of \( n \)-dimensional real vectors.

The loss function \( J_2 \) is now minimized to drive the system output \( y \) to the reference signal \( y_m \) given that the wavelet MRA identifier accurately approximates the real process dynamics online. At each sampling instant, an optimal control sequence is calculated using future predicted output values of the identified model, but the only the first one is applied to the system. To minimize \( J_2 \), the GD method is implemented again to recursively calculate the \( N_u \)-dimensional control increment sequence \( \Delta U \) as follows:

\[ \Delta U(n) = -\gamma_u \frac{\partial J_2}{\partial U}(n) \]  

(5.12)

where \( \gamma_u \) is the adaptation gain for control input vector \( U \). The partial derivative of loss function \( J_2 \) w.r.t. \( U \) can be obtained as:

\[ \frac{\partial J_2}{\partial U}(n) = -G \left( Y_m(n + 1) - \hat{Y}(n + 1) \right) + \rho H \Delta U(n) \]  

(5.13)

where

\[ G = \begin{bmatrix}
\frac{\partial \hat{y}(n + 1)}{\partial u(n)} & \frac{\partial \hat{y}(n + 2)}{\partial u(n)} & \ldots & \frac{\partial \hat{y}(n + N_u)}{\partial u(n)} & \frac{\partial \hat{y}(n + L)}{\partial u(n)}
0 & \frac{\partial \hat{y}(n + 2)}{\partial u(n + 1)} & \ldots & \frac{\partial \hat{y}(n + N_u)}{\partial u(n + 1)} & \frac{\partial \hat{y}(n + L)}{\partial u(n + 1)}
0 & 0 & \frac{\partial \hat{y}(n + N_u)}{\partial u(n + N_u - 1)} & \ldots & \frac{\partial \hat{y}(n + L)}{\partial u(n + N_u - 1)}
\end{bmatrix}_{N_u \times N_p} \]  

(5.14)
The control law is then designed as follows:

\[
\Delta U(n) = (I + \gamma_u \rho H)^{-1}\gamma_u G (Y_m - \hat{Y})
\]  

(5.16)

where \( I \) is the \( N_u \times N_u \) identity matrix. \( G \) can be computed from the chosen wavelet MRA model structure. The proposed wavelet MRA model based GPC control schematic is shown in Figure 5.1.

![Figure 5.1 Schematic of wavelet MRA based unconstrained GPC control system](image)

5.1.3 Convergence and Stability

In this section, we provide stability analysis for wavelet MRA model identification algorithm and the proposed GPC control strategy. The adaptive identification and control laws both have one parameter, the adaptation gain, to be chosen by user. It has been shown [21] that the adaptation gain is very crucial to adaptive control system stability and performance. Therefore, we have provided analytic guidelines in selecting those gains properly.
A. Convergence of Wavelet MRA identifier

Define a discrete type Lyapunov function as

\[ V_1(n) = \frac{1}{2} e^2(n) \] (5.17)

where \( e(n) \) defined in (5.6) is the error representing the modeling error. Then the change of Lyapunov function is obtained by

\[ \Delta V_1(n) = V_1(n + 1) - V_1(n) = \frac{1}{2} \left( e^2(n + 1) - e^2(n) \right) \] (5.18)

The error difference can be represented by [58]

\[ \Delta e(n) = e(n + 1) - e(n) = \left[ \frac{\partial e(n)}{\partial \theta(n)} \right]^T \Delta \theta(n) \] (5.19)

where \( \Delta \theta(n) = \{ \Delta \theta_i(n) \}_{i=1}^m \) represents a change of arbitrary component of the weighting gain vector \( \theta \). From (5.9), \( \Delta \theta_i(n) \) can be obtained by

\[ \frac{\partial e(n)}{\partial \theta(n)} = - \frac{\partial \bar{y}(n)}{\partial \theta(n)} = -g(n) \] (5.20)

where \( g(n) = \{ g_i(n) \}_{i=1}^m \). Now we can prove the following theorem:

**Theorem 5.1:** Let \( \gamma_\theta \) be the adaptation gain for the weights of wavelet MRA identified model and \( g_{\text{max}} \) be defined as \( g_{\text{max}} := \max_n \| g(n) \| \), where \( g \) is the wavelet MRA basis functions, \( n \) is discrete time and \( \| \cdot \| \) is the \( L_2 \) norm of real vectors. Then the convergence of the wavelet MRA identifier is guaranteed if \( \gamma_\theta \) is chosen as

\[ 0 < \gamma_\theta < \frac{2}{g_{\text{max}}^2} \] (5.22)

**Proof:** From (5.18) to (5.21), \( \Delta V_1(n) \) can be represented as:
\[ \Delta V_1(n) = \Delta e(n) \left[ e(n) + \frac{1}{2} \Delta e(n) \right] \]
\[ = \left[ \frac{\partial e(n)}{\partial \theta} \right]^T \gamma_\theta e(n) g(n) \]
\[ \times \left\{ e(n) + \frac{1}{2} \left[ \frac{\partial e(n)}{\partial \theta} \right]^T \gamma_\theta e(n) g(n) \right\} \]
\[ = -\gamma_\theta e^2(n) \| g(n) \|^2 + \frac{1}{2} \gamma_\theta^2 e^2(n) \| g(n) \|^4 \]
\[ = -\lambda e^2(n) \]  \hspace{1cm} (5.23)

where
\[ \lambda = \frac{1}{2} \gamma_\theta e^2(n) \| g(n) \|^2 (2 - \gamma_\theta \| g(n) \|^2) \]
\[ \geq \frac{1}{2} \gamma_\theta e^2(n) \| g(n) \|^2 (2 - \gamma_\theta g^2_{\text{max}}) \]
\[ \geq 0 \]  \hspace{1cm} (5.24)

and the last inequality follows from (5.22). It then shows that \( \Delta V_1(n) \leq 0 \) and the proof completes. \( \blacksquare \)

**B. Stability Analysis of wavelet MRA based GPC**

Define a second discrete Lyapunov function as
\[ V_2(n) = \frac{1}{2} \| E(n+1) \|^2 \]  \hspace{1cm} (5.25)

where \( E(n+1) = Y_m(n+1) - \hat{Y}(n+1) \). Then the change of Lyapunov function is obtained by
\[ \Delta V_2(n) = V_2(n+1) - V_2(n) = \frac{1}{2} (\| E(n+2) \|^2 - \| E(n+1) \|^2) \]  \hspace{1cm} (5.26)

Similar to (5.19), the error difference can be represented by
\[ \Delta E(n+1) = E(n+2) - E(n+1) = \left[ \frac{\partial E(n+1)}{\partial U(n)} \right] \Delta U(n) \]  \hspace{1cm} (5.27)
where $\Delta U(n)$ is defined in (5.16) and $\frac{\partial E(n+1)}{\partial U(n)} = -G^T$. Then (5.27) above can be expressed as:

$$
\Delta E(n + 1) = -G^T(I + \gamma_u \rho H)^{-1} \gamma_u GE(n + 1)
$$

(5.28)

**Theorem 5.2:** Let $\gamma_u$ be the adaptation gain for GPC control inputs sequence. Assume the control weighing factor $\rho > 0$. Then the stable tracking convergence of the wavelet MRA based GPC control system is guaranteed if

$$
0 < \gamma_u < \frac{2}{\lambda_{\text{max}}(GG^T)}
$$

(5.29)

where $\lambda_{\text{max}}(\cdot)$ is the maximum eigenvalue of the matrix.

**Proof:** From (5.26) - (5.28), $\Delta V_2(n)$ can be represented as

$$
\Delta V_2(n) = \frac{1}{2} \left[ (E(n + 1) + \Delta E(n + 1))^T (E(n + 1) + \Delta E(n + 1)) 
- E(n + 1)^T E(n + 1) \right]
= \Delta E^T(n + 1) \left[ E(n + 1) + \frac{1}{2} \Delta E(n + 1) \right]
= -(GE)^T \gamma_u ((I + \gamma_u \rho H)^{-1})^T
\times \left[ I - \frac{1}{2} G G^T (I + \gamma_u \rho H)^{-1} \gamma_u \right] GE
= -(GE)^T R_1 R_2 GE
$$

(5.30)

where

$$
R_1 = \gamma_u ((I + \gamma_u \rho H)^{-1})^T
$$

(5.31)

$$
R_2 = I - \frac{1}{2} \gamma_u G G^T (I + \gamma_u \rho H)^{-1}
$$

(5.32)

If $R_1$ and $R_2$ are both positive definite matrices, it follows that $\Delta V_2(n) < 0$. Together with $V_2(n) > 0$, the stable tracking of the reference signals is guaranteed. Now we need to prove the positive definiteness of both $R_1$ and $R_2$.

It can be derived from (5.15) that the eigenvalues of $H$ are $\lambda_H = \{1, \cdots, 1\}_{N_u \times 1}$. Then the eigenvalues of $R_1$ can be derived as:

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\[ \lambda_{R_1} = \{y_u(1 + y_u\rho)^{-1}, \ldots, y_u(1 + y_u\rho)^{-1}\}_{N_u \times 1} \]  

Hence, all eigenvalues of \( R_1 \) are positive if \( y_u > 0 \). It follows from (5.29) that \( R_1 > 0 \).

If \( 0 < y_u < \frac{2}{\lambda_{\text{max}}(GG^T)} \), then
\[
I - \frac{1}{2} y_u GG^T > 0
\]  

From (5.15), we have
\[ y_u \rho H > 0 \]  

Then (5.34) and (5.35) indicate that
\[ I - \frac{1}{2} y_u GG^T + y_u \rho H > 0 \]  

Similar to (5.33), we can prove \( I + y_u \rho H > 0 \), then (5.36) can be rewritten as
\[
(I + y_u \rho H)\left( I - \frac{1}{2} y_u GG^T(I + y_u \rho H)^{-1} \right) > 0
\]  

Since \( I + y_u \rho H > 0 \), we can deduce that \( \left( I - \frac{1}{2} y_u GG^T(I + y_u \rho H)^{-1} \right) > 0 \) which proves the positive definiteness of \( R_2 \). Then the proof completes.

### 5.2 Wavelet MRA Based Constrained Adaptive Predictive Control for Nonlinear Systems

In this section, a wavelet MRA model based adaptive predictive control strategy for control of unknown nonlinear systems subject to input and state constraints is presented. The wavelet multiresolution analysis framework is used as the building block to approximate the unknown nonlinear system dynamics by virtue of the promising function approximation capability of wavelet networks. The parameter estimation routine employed guarantees non-increase of the prediction error vector. The identified wavelet network nominal model is then combined within nonlinear model predictive control framework to address the adaptive constrained MPC problem. This work is an extension of the results in section 5.1 in which we presented the design of an unconstrained
nonlinear wavelet-based adaptive generalized predictive controller along with the stability proof and reported a successful application of this controller with ad hoc introduced constraints for chemical looping system with a very challenging multi-scale dynamics in Chapter 6. In this section, we combine the wavelet MRA modeling mechanism with design of robust nonlinear non-ad-hoc constrained MPC with guaranteed stability under mild assumptions.

5.2.1 Problem Formulation

Consider the discrete-time nonlinear system described by

\[ x_{k+1} = f(x_k, u_k) \tag{5.38} \]

where \( x_k \in \mathbb{R}^n \) is the measurable system state vector and \( u_k \in \mathbb{R}^m \) is the control input vector at discrete time \( k \). \( f(\cdot) \in L_2(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n) \) is the unknown differential equation representing original system dynamics and has an equilibrium point at the origin with \( f(0, 0) = 0 \). The control and state sequences are known to satisfy the following constraints

\[ u_k \in U \tag{5.39} \]
\[ x_k \in X \tag{5.40} \]

where \( U \) is a compact subset of \( \mathbb{R}^m \) and \( X \) is a closed subset of \( \mathbb{R}^n \) both containing origin in its interior. The control objective is to find an appropriate control to steer the state to the origin.

To model the unknown dynamics \( f(\cdot) \), we need to identify the nominal model \( \hat{f} \) defined as

\[ \hat{x}_{k+1} = \hat{f}(x_k, u_k) \tag{5.41} \]
in some online adaptive control design framework which is independent of the predictive control structure adopted. For the real system (5.38), we make the following assumption that \( f(x, u) \) is Lipschitz continuous function in \( x \), i.e., there exists a positive constant \( L \) such that

\[
\| f(x_1, u) - f(x_2, u) \| \leq L \| x_1 - x_2 \| \tag{5.42}
\]

for all \( x_1, x_2 \in X \) and all \( u \in U \). The proposed controller consists of two parts: One is the online identification of unknown function \( f \) using wavelet multiresolution approximation, and the other is to employ a stable MPC strategy based on the identified model \( \hat{f} \) that asymptotically drives the states of the system to the origin.

5.2.2 System Identification

A. Wavelet Approximation

In this section, we continuously use a wavelet multiresolution model to approximate the unknown function \( f(x, u) \) over \( X \) and \( U \). Wavelet multiresolution analysis [11] is a powerful function approximation tool to represent function details at different scales of resolution in both time and frequency domains in terms of shifted and dilated scaling and wavelet functions. According to one dimensional MRA illustrated in section 2.1, any \( f(x) \in L_2(\mathbb{R}) \) can be represented as (5.2). The approximation starts from some lower resolution level \( J \) and can be truncated at certain higher resolution level \( N \) when (5.3) satisfies. The choice of the wavelet properties is determined by the space where the approximated function resides. Multiple variable wavelet bases can be constructed from the tensor product or radial basis function of single dimensional wavelets as described in section 2.1. Because wavelet MRA can approximate any finite energy nonlinear function to any desired accuracy level, in this section, the wavelet MRA will be used to build the nominal model \( \hat{f}(x, u) \) for the nonlinear system. We employ the same \( \hat{f} \) as in (5.4) to
minimize a pre-specified approximation adequacy criterion where $\theta = \{\theta_i\}$ is parameter vector trained online, $g_i \in \{\phi_j, \psi_j\}$ is a multivariable scaling or wavelet basis function of states and inputs, $G$ is the set of all candidate basis functions and $m$ is the number of required basis functions to meet satisfactory modeling accuracy requirement.

**B. Parameter Adaptation**

Define the error between real system states $x$ and the predicted state $\hat{x}$ by the nominal model (5.41) at time $n$ as

$$e(n) = x(n) - \hat{x}(n) \quad (5.43)$$

We design the adaptation law to train the weighting parameters in (5.4) online in such a way that the defined loss function

$$J(n) = \frac{1}{2} \|e(n)\|^2 \quad (5.44)$$

at every time instant $n$. Assume $J$ is continuously differentiable with respect to $\theta$. Then we can design the parameter adaptation law using gradient descent (GD) algorithm to the change the weighting gains $\theta$ in the direction of negative gradient of $J$, that is

$$\theta(n+1) = \theta(n) - \gamma_\theta \frac{\partial J}{\partial \theta}(n) = \theta(n) - \gamma_\theta \frac{\partial e^T}{\partial \theta}(n)e(n) \quad (5.45)$$

where $\gamma_\theta > 0$ is the adaptation gain, $\frac{\partial J}{\partial \theta}(n)$ is the partial derivative of $J$ with respect to $\theta$ at discrete time $n$ and $\frac{\partial e^T}{\partial \theta}(n)$ is the so called sensitivity derivative at time $n$ indicating how the error is influenced by weighting parameters $\theta$. From (5.4), (5.41) and (5.43), the sensitivity derivative $\frac{\partial e^T}{\partial \theta}(n)$ can be derived as follows

$$\frac{\partial e^T}{\partial \theta} = -\frac{\partial \hat{y}^T}{\partial \theta} = -\frac{\partial f^T}{\partial \theta} = -G^T \quad (5.46)$$

where $G = [g_1, \cdots, g_m]_{n \times m}$. Then the adaptation law can be written as

$$\theta(n+1) = \theta(n) + \gamma_\theta G^T(n)e(n) \quad (5.47)$$
Next, we need to evaluate the performance of the adaptation mechanism. It has been shown in [51] that the adaptation gain is very crucial to adaptation algorithm performance. Therefore we provided a guideline in selecting this parameter properly.

Define a discrete Lyapunov function using \( J(n) \)
\[
V(n) = J(n) = \frac{1}{2} \|e(n)\|^2
\] (5.48)

Then the change of Lyapunov function is obtained by
\[
\Delta V(n) = V(n + 1) - V(n) = \frac{1}{2} (\|e(n + 1)\|^2 - \|e(n)\|^2)
\] (5.49)

The error difference can be represented [58] by
\[
\Delta e(n) = e(n + 1) - e(n) = \frac{\partial e}{\partial \theta^T} (n) \Delta \theta(n)
\] (5.50)

where \( \Delta \theta(n) = \{\Delta \theta_i(n)\}_{i=1}^m \) represents a change of arbitrary component of the weighting gain vector \( \theta \). From (5.46), \( \Delta \theta(n) \) and can be obtained by
\[
\Delta \theta(n) = \gamma_\theta G^T(n)e(n)
\] (5.51)

With (5.51), we can prove the convergence of the adaptive identification algorithm as follows:

**Theorem 5.3:** Let \( \gamma_\theta \) be the adaptation gain for the weights of wavelet MRA identified model \( \hat{f} \) and \( G(n) = [g_1(n), \cdots, g_m(n)]_{n\times m} \) is the wavelet basis functions matrix, \( n \) is discrete time. Then the convergence is guaranteed if \( \gamma_\theta \) is chosen as
\[
0 < \gamma_\theta < \frac{2}{\lambda_{\text{max}}(G(n)G^T(n))}
\] (5.52)

where \( \lambda_{\text{max}}(\cdot) \) is the maximum eigenvalue of the matrix.

**Proof:** From (5.48) to (5.52), \( \Delta V(n) \) can be represented as:
\[ \Delta V(n) = \frac{1}{2} (\|e(n+1)\|^2 - \|e(n)\|^2) \]
\[ = \frac{1}{2} \left[ (e(n) + \Delta e(n))^T (e(n) + \Delta e(n)) - e(n)^T e(n) \right] \]
\[ = \Delta e^T(n) \left[ e(n) + \frac{1}{2} \Delta e(n) \right] \]
\[ = \Delta \theta^T(n) \left[ \frac{\partial e^T}{\partial \theta}(n) \right] \left\{ e(n) + \frac{1}{2} \left[ \frac{\partial e}{\partial \theta^T}(n) \right] \Delta \theta(n) \right\} \]
\[ = \gamma_\theta e^T(n) G(n) [-G^T(n)] \left\{ e(n) + \frac{1}{2} [-G(n)] \gamma_\theta G^T(n) e(n) \right\} \]
\[ = -\gamma_\theta e^T(n) G(n) \left[ I - \frac{1}{2} \gamma_\theta G^T(n) G(n) \right] G^T(n) e(n) \] (5.53)

From (5.52), we could obtain \( I - \frac{1}{2} \gamma_\theta G^T(n) G(n) > 0 \). It follows that \( \Delta V(n) < 0 \) for all time \( n \). Since \( V(n) \geq 0 \), the convergence of the weighting parameters of the identified wavelet MRA model is guaranteed.

**Remark 5.1:** For system with fast dynamics, a big value of \( \lambda_{max}(G(n)G^T(n)) \) is expected. For such system, a relative small value of learning rate \( \gamma_\theta \) has to be chosen to guarantee the convergence of state estimation.

### 5.2.3 Adaptive Constrained Nonlinear MPC Strategy

In this section, we present a robust nonlinear constrained MPC controller to provide the robustness against modeling errors where the nominal model rather than the unknown real system dynamics is controlled while guaranteeing stability. The receding horizon control at each sample time \( k \) is finite sequence of decision variables

\[ \pi(k) = \{ u(k|k), u(k+1|k), \ldots, u(k+N-1|k) \} \] (5.54)

that solves the following finite-horizon control problem

\[ P(x(k), k) : \min_{\pi(k)} V(x(k), \pi(k)) \] (5.55)

where \( V(x(k), \pi(k)) \) is given by the optimal control problem:
\[ V(x(k), \pi(k)) = \sum_{i=0}^{N-1} \ell(\hat{x}(k + i|k), (k + i|k)) + F(\hat{x}(k + N|k)) \]  

(5.56)

such that

\[ \hat{x}(k + i + 1|k) = \hat{f}(\hat{x}(k + i|k), u(k + i|k)), \quad i = 0, \ldots, N - 1 \]  

(5.57)

\[ \hat{x}(k + i|k) \in \mathbb{X}_i, \forall i = 0, \ldots, N - 1 \]  

(5.58)

\[ u(k + i|k) \in \mathbb{U}, \forall i = 0, \ldots, N - 1 \]  

(5.59)

\[ \hat{x}(k + N|k) \in \mathbb{X}_f \subset \mathbb{X} \]  

(5.60)

where \( \hat{x}(k + i|k) \) is the predicted state of the nominal system at time \( k + i \) given the state of the real system at time \( k \) and \( \hat{x}(k|k) = x(k|k) = x(k) \). \( \mathbb{X}_f \) is the state constraint at time \( j \), \( F(\cdot) \) is the terminal cost and \( \mathbb{X}_f \) is the terminal constraint. Then the receding horizon control law is obtained by

\[ u(k) = \kappa_{mpc}(x(k)) = u^*(k|k) \]  

(5.61)

where \( \pi^*(k) = \{ u^*(k|k), \ldots, u^*(k + N - 1|k) \} \) is the optimal control sequence that solves the corresponding constrained optimization problem at time \( k \). This procedure is repeated at every sample instant.

To evaluate the model mismatch between real system and the nominal model, we prove a bound for the prediction error for any given control input sequence based on the Lipschitz continuity of the system. In order to achieve that, we first consider the following assumption:

**Assumption 5.1:** The nominal model \( \hat{f}(x, u) \) is constructed so that it is Lipschitz continuous in \( x \) on \( \mathbb{X} \times \mathbb{U} \), i.e., there exists a positive constant \( L_f \) such that

\[ \| \hat{f}(x_1, u) - \hat{f}(x_2, u) \| \leq L_f \| x_1 - x_2 \| \]  

(5.62)

for all for all \( x_1, x_2 \in \mathbb{X} \) and all \( u \in \mathbb{U} \).
In theorem 5.3, we prove that $\Delta V(n) < 0$ for all $n \geq 0$ under wavelet MRA adaptation algorithm. Therefore, $\|e(n)\|$ is decreasing for all $n \geq 0$. Let $\Omega$ denote the compact set including all possible initial tracking errors $e_0 = e(0)$ and let

$$d = \max_{e_0 \in \Omega} \|e_0\|$$

(5.63)

It then follows that

$$\|e(n)\| = \|x(n) - \hat{x}(n)\| \leq d$$

(5.64)

for all time $n \geq 0$. Now we can prove the following lemma to compute a bound for the prediction error.

**Lemma 5.1**: Consider discrete-time nonlinear system (5.38) and its nominal model (5.41), for a given control sequence $\pi(k)$ in (5.54), the prediction error between real system state $x(k+j)$ and the prediction state of the nominal model $\hat{x}(k+j|k)$ for $j = 0, \cdots, N - 1$, at any time instant $k$, under assumption 5.1 and (5.63), is bounded by

$$\|x(k+j) - \hat{x}(k+j|k)\| \leq \frac{(L_f)^j - 1}{L_f - 1} d$$

(5.65)

**Proof**: From (5.62) - (5.64), we have

$$\|x(k+1) - \hat{x}(k+1|k)\| = \|e(k+1)\| \leq d$$

(5.66)

And

$$\|x(k+2) - \hat{x}(k+2|k)\|$$

$$= \|f(x(k+1), u(k+1|k)) - \hat{f}(\hat{x}(k+1|k), u(k+1|k))\|$$

$$\leq \frac{(L_f)^j - 1}{L_f - 1} d$$

(5.67)

From (5.66) and (5.62), it follows that
\[
\begin{align*}
\|x(k + 2) - \hat{x}(k + 2|k)\| \\
\leq \|e(k + 2)\| + L_f \|x(k + 1) - \hat{x}(k + 1|k)\| \\
\leq d + L_f \|x(k + 1) - \hat{x}(k + 1|k)\| \\
\leq (L_f + 1)d
\end{align*}
\] (5.68)

Repeat this with \( j = 0, \cdots, N - 1 \), it can be proved that
\[
\|x(k + j) - \hat{x}(k + j|k)\| \leq \frac{(L_f)^j - 1}{L_f - 1} d
\] (5.69)

which completes the proof.

\textit{Remark 5.2:} The prediction error bound in (5.65) could be over-conservative due to the global Lipschitz constant \( L_f \) especially for system with fast dynamics when \( L_f > 1 \). Several methods could be employed to reduce this constant. [49] indicates that a proper norm could be chosen to reduce the Lipschitz constant effectively. Suppose \( L_s \) is the Lipschitz constant in \( s \)-norm and \( L_q \) is the Lipschitz constant in \( q \)-norm. From the equivalence of the norm [63], we could find positive constants \( m_q \) and \( M_q \) such that \( m_q \|x\|_q \leq \|x\|_s \leq M_q \|x\|_q \). Then (5.65) could be rewritten as \( \|x(k + j) - \hat{x}(k + j|k)\| \leq \frac{(L_q)^j - 1}{L_q - 1} \cdot \frac{M_q}{m_q} \cdot d \). Then it is possible to find an appropriate \( q \)-norm that leads to lower bound. Alternatively, [67] proposed a precompensator for control signal in which \( u_k = Kx_k + v_k \). Then the controller \( K \) can be designed to reduce the Lipschitz constant of the system.

To ensure the satisfaction of the state constraint for real system along trajectory depending on the precision of model identification, we consider the following definition [50]:

\textit{Definition 5.1 (Pontryagin Difference):} Given set \( A \subset \mathbb{R}^n \) and \( B \subset \mathbb{R}^n \), then the Pontryagin difference set between these two sets is define as
\begin{equation}
A \sim B = \{ x \in \mathbb{R}^n \mid x + y \in A, \forall y \in B \} \tag{5.70} 
\end{equation}

Define the state constraint set \( \mathcal{X}_i \), \( i = 0, \cdots, N - 1 \), as

\begin{equation}
\mathcal{X}_i = \mathcal{X} \sim B(d_i) \tag{5.71} 
\end{equation}

where \( B(d_i) = \left\{ x \in \mathbb{R}^n \mid \| x \| \leq d_i, d_i = \frac{(L^j)^{i-1}}{L^j - 1} \right\} \). It then follows from lemma 5.1 that if \( \hat{x}(k + i|k) \in \mathcal{X}_i \), then \( x(k + i) \in \mathcal{X}, \forall i = 0, \cdots, N - 1 \) which guarantees the satisfaction of constraint for the real system states.

The wavelet MRA model based adaptive MPC strategy can now be performed as follows:

\textbf{Algorithm 5.1}: Initialization: At time \( k = 0 \), choose an initial set of weighting parameters \( \theta(0) \) for the wavelet MRA approximation model. At each time instant \( k \),

1. Measure the current state \( x(k) \) of the system and evaluate the prediction error \( e(k) \) between real system state and predicted state of the nominal model;
2. Update the parameter estimate \( \theta \) according to (5.45) with appropriate choice of adaptation gain \( \gamma_\theta \);
3. Solve the optimization problem (5.55) - (5.60) and apply the resulting feedback control (5.61) to the plant until the next sampling instant;
4. Repeat the procedure from step 1 for the next sampling instant, increment \( k = k + 1 \).

The proposed wavelet MRA model based constrained nonlinear MPC schematic is shown in Figure 5.2.
### 5.2.4 Feasibility and Stability

In this section, we provide stability analysis for wavelet MRA model based nonlinear MPC strategy. The closed-loop stability is based upon the feasibility of the control action at each sample time. The following assumptions which are extensions of general sufficient conditions for stability of constrained MPC [38] are required to guarantee stabilization of the origin.

**Assumption 5.2**: There exists a feedback local control law $k_f : \mathbb{X}_\alpha \to \mathbb{U}$ where $\mathbb{X}_\alpha$ is a level set of terminal cost $F(x)$, i.e. $\mathbb{X}_\alpha = \{x \in \mathbb{R}^n | F(x) \leq \alpha\} \subset \mathbb{X}_{N-1}$, such that for terminal constraint set defined as $\mathbb{X}_f = \{x \in \mathbb{R}^n | F(x) \leq \frac{\alpha}{2}\} \subset \mathbb{X}_\alpha$ following assumptions are satisfied:

1. $0 \in \mathbb{X}_f \subset \mathbb{X}$, $\mathbb{X}_f$ is closed;
2. $k_f(x) \in \mathbb{U}$, $\forall x \in \mathbb{X}_f$;
3. $F(x)$ is Lipschitz continuous with respect to $x \in \mathbb{R}^n$, i.e.

\[
\|F(x_1) - F(x_2)\| \leq L_F \|x_1 - x_2\| \tag{5.72}
\]
(4) $\hat{f}(x, k_f(x)) \in \mathcal{X}_f, \; \forall x \in \mathcal{X}_\alpha$;

(5) $F(\hat{f}(x, k_f(x))) - F(x) \leq -\ell(x, k_f(x)), \; \forall x \in \mathcal{X}_f$.

Assumption 5.2: The stage cost function $\ell(x, u)$ in (5.56) satisfies

$$\ell(x, u) \geq \mu(||x(k), x(k)||)$$

and that $\ell(0,0) = 0$ where continuous function $\mu(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{K}_{\infty}$ that is $\mu(0) = 0$ and $\mu(\cdot)$ is strictly increasing on $\mathbb{R}^+$ and radially unbounded.

The local feedback control law $k_f$ is only required to exist. The linearized approximation of the nonlinear system around the origin could be used to obtain such controller as outlined in [48]. The following lemma is also useful for stability proof.

Lemma 5.2 [68]: Let $x \in \mathcal{X}_{f+1}$ and $y \in \mathbb{R}^n$ such that $||x - y|| \leq (L_f)^j d$, then $y \in \mathcal{X}_f$.

Given the preliminaries of the preceding results, we now establish the main theorem which states the asymptotical stability of the closed-loop system using constrained nonlinear MPC policy.

Theorem 5.3: Let $\mathcal{X}_0$ denote the set of initial states for which (5.55) has a feasible solution. Assuming assumption 1 and 2 are satisfied, then the origin of the closed-loop system (5.38) - (5.40), (5.45) and (5.55) - (5.61) is asymptotically stable for any $x_0 \in \mathcal{X}_0$ if (5.52) is satisfied and

$$d \leq \frac{\alpha/2}{L_F(L_f)^{N-1}}$$

where $N$ is the prediction horizon.

Proof: We first need to show the feasibility of the control action at each time instant.
Suppose $\pi^*(k) = \{u^*(k|k), \cdots, u^*(k + N - 1|k)\}$ is a feasible solution for optimization problem (5.55) at time instant $k$. Construct the feasible control sequence for time $k + 1$ as

$$\hat{\pi}(k + 1) = \{u^*(k + 1|k), \cdots, u^*(k + N - 1|k), k_f(\hat{x}(k + N|k))\} \quad (5.75)$$

From the feasibility of $\pi^*(k)$, we have $u^*(k + i|k) \in \mathcal{U}$, for $i = 1, \cdots, N - 1$. Since $\hat{x}(k + N|k) \in \mathcal{X}_f$, from Assumption 5.2, it follows that $k_f(\hat{x}(k + N|k)) \in \mathcal{U}$. Therefore $\hat{\pi}(k + 1)$ satisfies the input constraint. Next we need to prove that under $\hat{\pi}(k)$, we could derive that

$$\|\hat{x}(k + i|k + 1) - \hat{x}(k + i|k)\| \leq (L_f)^{i-1} d \quad (5.76)$$

for $i = 1, \cdots, N$.

By induction, when $i = 1$, we have $\hat{x}(k + 1|k + 1) = x(k + 1)$. Then

$$\|x(k + 1) - \hat{x}(k + 1|k)\| = \|f(x(k), u(k|k)) - \hat{f}(x(k), u(k|k))\| \leq d$$

Suppose $\|\hat{x}(k + i|k + 1) - \hat{x}(k + i|k)\| \leq (L_f)^{i-1} d$. Then

$$\|\hat{x}(k + i + 1|k + 1) - \hat{x}(k + i + 1|k)\| = \|\hat{f}(\hat{x}(k + i|k + 1), u(k + i|k)) - \hat{f}(\hat{x}(k + i|k), u(k + i|k))\|$$

$$\leq L_f \|\hat{x}(k + i|k + 1) - \hat{x}(k + i|k)\|$$

$$\leq L_f \cdot (L_f)^{i-1} d$$

$$= (L_f)^i d \quad (5.77)$$

Thus (5.76) is proved. Then the state $\hat{x}(k + N|k + 1)$ satisfies
\[ F(\hat{x}(k + N|k + 1)) \]
\[ \leq F(\hat{x}(k + N|k)) + L_F \|\hat{x}(k + N|k + 1) - \hat{x}(k + N|k)\| \]
\[ \leq \frac{\alpha}{2} + L \left( L_f \right)^{N-1} d \]
\[ \leq \alpha \]

(5.78)

which leads to \( \hat{x}(k + N|k + 1) \in \mathbb{X}_\alpha \). From assumption 5.2 (4), it follows that
\( \hat{x}(k + N + 1|k + 1) \in \mathbb{X}_f \). Since \( \hat{x}(k + 1 + i|k) \in \mathbb{X}_{i+1} \) for \( i = 0, \cdots, N - 2 \), from lemma 5.2, we could derive that \( \hat{x}(k + 1 + i|k + 1) \in \mathbb{X}_i \) for \( i = 0, \cdots, N - 2 \). Since \( \hat{x}(k + N|k + 1) \in \mathbb{X}_\alpha \subset \mathbb{X}_{N-1} \), we proved that the state constraints have been satisfied. Therefore the \( \hat{x}(k + 1) \) is a feasible solution for the optimization problem (5.55). The feasibility proof completes.

Second, the stability of the closed-loop system is established by proving decrease of the optimal cost function \( V^*(x(k), \pi(k)) \) as candidate Lyapunov function. Let \( \pi^*(k) = \{ u^*(k|k), \cdots, u^*(k + N - 1|k) \} \) be an optimal control sequence that solves the optimization problem (5.55). Denote \( \hat{x}(k + j|k) \) as the corresponding prediction state trajectory and \( V^*(x(k), \pi(k)) \) as the optimal cost value. It then follows that
\[ V^*(x(k), \pi^*(k)) \]
\[ = \sum_{i=0}^{N-1} \ell(\hat{x}(k + i|k), u^*(k + i|k)) + F(\hat{x}(k + N|k)) \]
\[ \geq \sum_{i=0}^{N-1} \ell(\hat{x}(k + i|k), u^*(k + i|k)) + F(\hat{x}(k + N|k)) \]
\[ + F \left( \hat{f} \left( \hat{x}(k + N|k), k_f (\hat{x}(k + N|k)) \right) \right) - F(\hat{x}(k + N|k)) \]
\[ + \ell \left( \hat{x}(k + N|k), k_f (\hat{x}(k + N|k)) \right) \]
\[ = \ell(\hat{x}(k|k), u^*(k|k)) + \sum_{i=1}^{N-1} \ell(\hat{x}(k + i|k), u^*(k + i|k)) \]
\[ + F \left( \hat{f} \left( \hat{x}(k + N|k), k_f (\hat{x}(k + N|k)) \right) \right) \]
\[ \geq \ell(\hat{x}(k|k), u^*(k|k)) + V^*(x(k + 1), \pi^*(k + 1)) \]  
(5.79)

In the above proof, first inequality follows from assumption 5.2 (5) and second inequality follows by the optimality of the \( V^*(x(k + 1), \pi^*(k + 1)) \). Then it renders that
\[ V^*(x(k + 1), \pi^*(k + 1)) - V^*(x(k), \pi^*(k)) \]
\[ \leq -\ell(\hat{x}(k|k), u^*(k|k)) \]
\[ \leq -\mu(\|x(k), x(k)\|) \]  
(5.80)

where \( \mu(\cdot) \) is a class \( \mathcal{K}_\infty \) function. Hence \( x(k) \to 0 \) asymptotically. The proof completes.

5.3 Simulations

To demonstrate the effectiveness of the proposed wavelet MRA based adaptive GPC control system, let’s consider the following discrete nonlinear system from [6]:
\[ x_1(k+1) = 0.1 \cdot x_1(k) + 2 \frac{u(k) + x_2(k)}{1 + (u(k) + x_2(k))^2} \]
\[ x_2(k+1) = 0.1 \cdot x_2(k) + u(k) \left( 2 + \frac{u^2(k)}{1 + x_1^2(k) + x_2^2(k)} \right) \]
\[ y(k) = x_1(k) + x_2(k) \] (5.81)

where \( x(k) = [x_1(k), x_2(k)]^T \) is the system state, \( u(k) \) is the control signal and \( y(k) \) is the plant output at discrete time \( k \). It is assumed that the state variables are not accessible and that the system identification has to be carried out using only input and output data. The system (5.81) is represented in terms of NARX model of (5.5) for which we assume

\[ n_y = 3 \quad \text{and} \quad n_u = 1 \]

The nonlinear NARX model \( \hat{y}(k) = f(y(k-1), y(k-2), y(k-3), u(k-1)) \) would be estimated online by wavelet MRA model. The radial Marr wavelet function [10] is chosen to construct the scaling and wavelet basis functions for wavelet MRA model:

\[ \phi(x) = e^{-0.5\|x\|^2}, \quad \psi(x) = (dim(x) - \|x\|^2)e^{-0.5\|x\|^2} \] (5.82)

The coarse resolution is \( J = -2 \) and \( N_j = 10 \). \( N_j \) doubles as when resolution is increased by 1. Three resolutions are included in the wavelet network to achieve satisfactory approximation precision. The reference input to be tracked is:

\[ r = \sin\left(2\pi k \frac{10}{10}ight) + \sin\left(2\pi k \frac{1}{25}\right) \] (5.83)

The control objective is to design an appropriate control input \( u(k) \) for the nonlinear system to track the desired reference trajectory.

The design parameters for GPC configuration are chosen as

\[ N_1 = 1, N_2 = 4, N_u = 2, \rho = 0.01 \]

The adaptation gains derived from Theorem 5.1 and 5.2, were chosen as
\[ y_0 = 0.01 \quad \text{and} \quad y_u = 0.1 \]

The initial conditions for system states are both set to 1 and initial wavelet network weight parameters are all set to 0. Neither state nor control constraints is imposed. The sampling time is 0.01s. The simulation is carried out in Matlab Simulink. The plant response and reference input are plotted in Figure 5.3. The error signal between plant output \( y \) and model estimated output \( \hat{y} \) is shown in Figure 5.4 and the control signal is plotted in Figure 5.5. From the figures we can see that in the initial time period when identified wavelet network MRA model doesn’t approximate the plant well, there is significant tracking error between plant output and reference signal. After the identified wavelet MRA model is trained to predict the actual plant response in good precision, the adaptive GPC is able to achieve satisfactory tracking performance.

Next, we need to discuss some guidelines of how to choose a set of design parameters satisfying the stability conditions of the wavelet MRA based constrained adaptive model predictive control and illustrate its effectiveness through simulation example. Consider the nonlinear plant from [6]:

\[
\begin{align*}
    x(k + 1) &= \sin(x(k)) + u(k) \ast \left( 5 + \cos(x(k) \ast u(k)) \right) \\
    y(k) &= x(k)
\end{align*}
\]  

(5.84)

where \( x(k) \) is the system state, \( u(k) \) is the control signal and \( y(k) \) is the plant output at discrete time \( k \). It is assumed that the state variables are measurable and control signal is subject to magnitude constraint:

\[
    \mathbb{U} = \{ u \in \mathbb{R} | |u| \leq 1 \}
\]

(5.85)

The stage cost function \( \ell(x, u) \) is chosen as:
\[ \ell(x, u) = x^T Q x + u^T R u \]  

(5.86)

where \( Q = 2I \) and \( R = I \) are positive definite matrices and \( I \) is the identify matrix of dimension 1. The prediction horizon \( N \) is chosen as \( N = 4 \). The terminal cost \( F \) could be designed as a quadratic Lyapunov function \( F(x) = x^T P x \) for the linearized system.

Denote the closed-loop system under a local stabilizing control law \( u = k_f(x) \) as:

\[ x(k + 1) = Ax(k) \]  

(5.87)

Then the matrix \( P > 0 \) is a positive definite matrix and is a solution of LMIs:

\[ P > 0 \text{ and } A^TP + PA < 0 \]  

(5.88)

Here we choose \( P = 4I \). The terminal constraint set \( \mathbb{X}_f \) is now defined as \( \mathbb{X}_f = \{ x : x^T P x \leq \alpha \} \) and \( \mathbb{X}_a = \{ x : x^T P x \leq \alpha \} \) for some suitably chosen \( \alpha = 0.25 > 0 \).

The nominal model \( \hat{f}(x, u) \) would be estimated online by wavelet MRA model. The same radial Marr wavelet function in (5.82) is chosen to construct the scaling and wavelet basis functions for wavelet MRA model. The coarse resolution is \( J = -3 \) and \( N_f = 10 \). \( N_f \) doubles as when resolution is increased by 1. Three resolutions are included in the wavelet network to achieve satisfactory approximation precision. The adaptation gain derived from Theorem 5.3 is chosen as

\[ \gamma_\theta = 0.02 \]  

(5.89)

to satisfy the stability condition for identification convergence. The same reference signal (5.83) also needs to be tracked. The initial condition for system state is set to 1 and initial wavelet network weight parameters are all set to 0. The sample time is 0.02s. The simulation is carried out in Matlab Simulink and sequential quadratic programming (SQP) is used for the controller calculations. The plant response and reference input are plotted in Figure 5.6. The error signal between plant output \( y \) and model estimated output \( \hat{y} \) is shown in Figure 5.7 and the control signal is plotted in Figure 5.8. From the
figures we can see that in the initial time period when identified wavelet network MRA model doesn’t approximate the plant well, there is some tracking error between plant output and reference signal. After the identified wavelet MRA model is well trained, the adaptive MPC is able to achieve satisfactory tracking performance without violation of control constraint.

![Figure 5.3 Plant output and reference input for wavelet MRA based GPC controller](image1)

![Figure 5.4 Error between plant output and model estimated output](image2)
Figure 5.5 Control signal generated by unconstrained wavelet MRA based GPC controller

Figure 5.6 Plant output and reference input for wavelet MRA based constrained adaptive MPC controller
5.4 Conclusions

In this chapter, we have proposed a wavelet MRA based adaptive MPC strategy for regulation of both unconstrained and constrained unknown nonlinear systems. The identified wavelet MRA model is trained online to estimate the nonlinear dynamic characteristics. The adaptation gain of identification is synthesized using the Lyapunov
function theorem, so that decay of prediction error can be guaranteed. Afterwards, the wavelet MRA identified nominal model is incorporated into the GPC framework for regulation and stable tracking of CL process. The adaptation gain of control laws of the proposed GPC method is synthesized using the Lyapunov function theorem, so that asymptotic stability of the control system can be guaranteed. Next it is combined with robustly stabilizing GPC and nonlinear MPC frameworks to address the issue of state and input constraints. The design is based on the computation of the bound on the mismatch between real system states and prediction states of the nominal model. Feasibility and stability are provided in the context of Lyapunov theorem with the inclusion of terminal cost and terminal constraint.
CHAPTER 6: WAVELET NETWORK PREDICTIVE CONTROL FOR HYBRID COMBUSTION-GASIFICATION CHEMICAL LOOPING PROCESS

Developing technology for harnessing power from coal in an efficient and environmentally friendly way is a challenge that has to be met for sustainable use of coal. To meet this challenge a novel technology of chemical looping process in which multiple interacting loops of flowing reactive gas/solid mixtures produce energy via non-oxygen-based combustion has been developing at Alstom [1], [2]. In order to obtain and maintain optimal conditions for operation with reduced waste stream volume and minimum required energy, advanced optimizing control systems for chemical looping process are required. As such, process control development is needed to operate the system in a safe, integrated, and optimized fashion and is viewed as critical for enhancing the performance of the chemical looping system. The goal of this section is to develop a real-time computational model for the single loop cold gas/solid flow of prototype chemical looping system and to design model based control strategies that can be used to operate the system.

This section presents the wavelet multiresolution based model estimation and control of the single loop cold flow chemical looping process and reveals its inherent multiscale nature. The specific control strategy we adopted here is the previously developed adaptive generalized predictive control (GPC) scheme based on multiresolution wavelet model structure that well characterizes the nonlinear dynamics of single loop gas/solid flow behavior. The wavelet based controller is implemented on the test rig and shown to control the system very well.

Next we extended temporal dynamic wavelet network paradigm to spatio-temporal-wavelet models and controllers by combining together partial differential equation (PDE)
model of the riser section and temporal wavelet model we developed. The response time of the PDE model, which is typically less than 1 second, is much shorter than that for the temporal model we developed. Therefore, we considered using the impulse response of the PDE model to approximate the faster dynamics of the system. The PDE model was simulated to get an impulse response, and the result was used in a convolution to get a model of the transients. To simplify the calculations, the impulse response was first decomposed using Gaussian spatial and temporal wavelets. Simulation and experimental results verified the validity of the spatio-temporal wavelet models and controllers.

6.1 Chemical Looping Process

The modeling and controls developed in this section will be focused on Alstom’s Hybrid Combustion-Gasification Chemical Looping (CL) process. Chemical looping is a two-step process which first separates oxygen (O$_2$) from nitrogen (N$_2$) in the air stream in an air reactor. The O$_2$ is transferred to a solid oxygen carrier. The oxygen is carried by the solid oxide and is then used to gasify or combust solid fuel in a separate fuel reactor. As shown in Figure 6.1, a metal or calcium material (oxygen carrier) is burned in air forming a hot oxide (MeOx or CaOx) in the air reactor (Oxidizer). The oxygen in the hot metal oxide is used to gasify coal in the fuel reactor (Reducer), thereby reducing the oxide for continuous reuse in the chemical looping cycle. CL coal power technology is an entirely new, ultra clean, low cost, high efficiency coal power plant technology for the future power market. This new concept offers the promise to become the technology link from today’s steam cycle power plant to tomorrow’s clean coal power plants, being highly efficient and CO$_2$ capture capability.

The CL process with its multi-phase flows and complicated chemical reactions is characterized by process nonlinearities and time delay due to mass transport and chemical
reaction rates. The specific operational characteristics are new and still being studied. Hence, there is a need for further investigation and the potential for advanced control solutions. In this section, we have focused on developing control oriented model for single loop cold gas/solid flow which doesn’t consider any chemical reaction inside and interaction with other loops. The block diagram of single loop cold flow CL process is shown in Figure 6.2. It consists of a lower level pipeline, a riser pipeline, an upper level horizontal pipeline, a cyclone, a dip leg, a seal pot control valve (SPCV), and a solid return leg. The lower level pipeline accepts air flow and solids returned from both seal pot control valves and/or manually added solids. In the riser the air-solid mixture (two-phase) flows upward, and then turns into the horizontal pipeline, and then enters into the cyclone. The cyclone separates the solids and the air. The separated solids then drop into the dip leg and then enter into the SPCV. The SPCV will split the solids between the return leg for its own loop and the return leg to other loop. The SPCV also maintains a pressure control boundary.

In our model, the manipulated variables (MV) include \( S1, S2 \) – two fluidizing air flows into the SPCV, which will change pressures in the SPCV and the flow conditions in the upstream and downstream of the SPCV. The controlled variable (CV) of interest is \( DP47 \) – the pressure drop that is measured across the riser which is a substantial indicator of solid/gas flow transport stability along the whole loop. A GPC based predictive controller using wavelet MRA model addressing explicitly the inherent process nonlinearities was designed for this system to regulate and track the reference command to evaluate its performance during base load operation as well as load cycling operation. The interaction between the different components of the system is shown in the Figure 6.3.
Figure 6.1 Alstom’s Hybrid Combustion-Gasification Process

Figure 6.2 Block diagram for single loop cold flow CL modeling
6.2 Experimental Results

In this section, the proposed wavelet MRA model based GPC scheme is implemented to the single loop cold flow CL process testbed developed at Alstom Power Inc. The system output $y$ was selected to be riser pressure drop - DP47 (inch H$_2$O). Fluidizing air flow $S1$ (standard cubic feet per hour - scfh) was used as the single control input $u$ while the other air flow $S2$ (scfh) is set to a constant value of about 20. The identified SISO NARX model characterizing the complex dynamic behavior determined by prior knowledge of the system is chosen as

$$\hat{y}(t) = f(y(t-1), \ldots, y(t-n_y), u(t-1), \ldots, u(t-n_u))$$

$$= \sum_{i=1}^{m} \theta_i g_i$$  \hspace{1cm} (6.1)

where $f$ is the unknown nonlinear mapping to be identified, $u(t)$ and $y(t)$ are sampled inputs and output sequences, $n_y$ and $n_u$ are the maximum lags in the output and input to be determined respectively. $\theta = \{\theta_i\}$ is parameter vector trained online, $g_i \in \{\phi_{j,k}, \psi_{j,k}\}$ is a multivariable scaling or wavelet basis function of past inputs and outputs, $\mathcal{G}$ is the set of all candidate basis functions and $m$ is the number of required
basis functions to meet satisfactory modeling accuracy requirement. The nonlinear mapping \( f \) is approximated by NARX wavelet multiresolution network model.

We first perform several offline experimental tests to understand the process better and to use the test results for tuning the structure and parameters in the models that we are going to adopt for online approximation. We generated \( S_1 \) input signals with pseudo random binary signal (PRBS). In this the \( S_1 \) input is changed about a nominal value using PRBS and measured the pressure drop across riser \( DP47 \) as the output. All the sequences used in the experiment are generated by the MATLAB command. To carry out experiments, a testbed of single loop cold flow chemical looping system was built by Alstom Power Inc. CL utilizes a metal oxide limestone (CaSO4), as an oxygen carrier to transfer oxygen from the combustion air to the fuel. Since direct contact between fuel and combustion air is avoided the products of combustion (CO\(_2\) and water) are kept separate from the rest of the flue gases (primarily nitrogen). CL splits combustion into separate oxidation and reduction reactions. The carrier releases oxygen in a reducing atmosphere to react with the fuel. The carrier is then recycled back to the oxidation chamber to be regenerated by contact with air. Calcination of hot solids produced in the oxidation reactor produce a concentrated stream of CO\(_2\) in lieu of the dilute CO\(_2\) stream typically found in flue gas from coal-fired power plants. The single loop cold flow testbed where we performed the experiments is the reducer of the CL process without consideration of the oxidation reaction. The experimental facility is shown in Figure 6.4.

The experimental results of PRBS test are shown in Figure 6.5 and Figure 6.6 below where the sampling period is 1 second. The data set consists of 3961 input and output samples. Based on the experimental data, we use the NARX multiresolution wavelet
MRA model to approximate the nonlinear relationship between S1 and DP47. We choose the set of regressor as:

\[ y(t - 1), y(t - 2), u(t - 1), u(t - 2), \ldots, u(t - 8) \]  

Hence \( n_y = 2, n_u = 8 \). The wavelet scaling and wavelet basis functions we chose for MRA model was radial Marr scaling and wavelet functions [10]

\[ \phi(x) = e^{-0.5\|x\|^2}, \psi(x) = (dim(x) - \|x\|^2)e^{-0.5\|x\|^2} \]  

The initial coarse layer \( J \) is chosen to be 3 and the number of basis function doubles when resolution increases by 1 starting with 10. The final resolution we adopt is \( K = 6 \).

In Figure 6.7 is shown how the model predicted output compares with the experimental results below. The one-step-ahead predicted output and test data set are shown in Figure 6.8. From the figures above we can see that the NARX wavelet MRA model we obtained has predicted the system outputs pretty well. The model was found to be sufficiently accurate and no finer resolution levels were needed to be added to the model structure.

The stability analysis in Chapter 5 doesn’t take constraints into account. In practice, all process inputs are subject to certain constraints due to actuation limit. In [60], two specific types of constraints are most often considered in the GPC design procedure - magnitude limits and rate limits on the input control signal.

The mathematical description of these two constraints is given by

\[ \Delta u_{min} \leq u(n + k) - u(n + k - 1) \leq \Delta u_{max} \]  

\[ u_{min} \leq u(n + k) \leq u_{max} \]  

where \( 0 \leq k \leq N_u - 1 \). When constraints are included, the stability properties obtained above have to be reanalyzed. The stability analysis for constrained wavelet MRA GPC architecture is currently being completed and will be presented later. In order to make the
wavelet MRA GPC applicable to the CL process, the control input $u$ is assumed to be subject to rate constraint such as

$$\Delta u = \Delta u_{target} \times \exp\left(1 - \mu \| \Delta u_{target} \| \right)$$

where $\Delta u_{target}$ is the unconstrained control signal calculated by the predictive control law and $\mu > 0$ is a design parameter to adjust the rate of the control signal. The effectiveness of such input constrained wavelet predictive controller on the CL process is demonstrated in the next section through experimental results.

The control objective of GPC design is to ensure the output of the system $y$ asymptotically tracks the reference signal $y_m$. The cost function to be minimized is defined as in (5.10). The design parameters for GPC configuration was chosen as $N_1 = 1, \ N_2 = 10, N_u = 8, \rho = 1$. The adaptation gains derived from Theorem 5.1 and 5.2, was chosen as, $\gamma_\theta = 0.01, \gamma_u = 0.1$. The system was initially stable around level of $y_0 = 13$ inch H$_2$O. Two setpoint step change experiments were performed consecutively. After 5 minutes, the setpoint was first increased to 16 inch H$_2$O and staying for about 7 minutes. Then it went back to the original level of 13 inch H$_2$O. The air flow $S2$ (scfh) is set to a constant value of about 20. The tracking response of system output and the corresponding control efforts are shown in Figure 6.9 and Figure 6.10 respectively. It can be seen from the figures above that the proposed wavelet MRA based GPC method effectively tracks the setpoint changes for single loop CL process. In the second test, we set the sinusoidal reference signal as $y_m(t) = 13 + 2 \sin(2\pi \times 0.01 \times t)$, while $S2$ (scfh) is still set to a constant value of about 20. The tracking response of system output and the corresponding control efforts shown in Figure 6.11 and Figure 6.12 demonstrate that the controller satisfy the tracking performance requirement while time
delay exists between the control signal and system output which should be addressed in future research.

Figure 6.4 Experimental facility of control testing

Figure 6.5 PRBS test - input S1 and S2 (scfh)
Figure 6.6 PRBS test - Output DP47 (inch H2O)

Comparison of simulation output and measured output

Figure 6.7 PRBS test - Simulation data vs. Experimental data
Figure 6.8 PRBS test - One-step-ahead predictions vs. Experimental data

Figure 6.9 Pressure difference response of riser (DP47) during step setpoint changes
Figure 6.10 Fluidizing air flow control (S1 and S2) during step setpoint changes

Figure 6.11 Pressure difference response of riser during sinusoidal setpoint changes
6.3 Spatio-temporal Wavelet Decomposition

Since the empirical identified wavelet network temporal model was obtained using data collected at 1 second sampling rate, some of the fast dynamics of the plant is not recorded. The fast dynamics come primarily from the riser. Hence, we considered using the impulse response of the riser PDE model to approximate the faster dynamics of the system. The PDE model was simulated first to get an impulse response, and the result was used in a convolution to get a model of the transients. We then put the empirical wavelet network temporal NARX model and the fast dynamics PDE model in parallel and design the controller by combining wavelet predictive control for temporal wavelet model and NARMA-L1 control [6] for fast transient model together. The block diagram of controller implementation with fast dynamics is shown in Figure 6.13.

The nonlinear partial differential equations (PDEs) governing the evolution of the variables (voidage and solid velocity) in the riser can be represented as [69]:

Figure 6.12 Fluidizing air flow control (S1 and S2) during setpoint changes
\[
\begin{align*}
\frac{\partial \varepsilon}{\partial t} &= (1 - \varepsilon) \frac{\partial u_s}{\partial x} - u_s \frac{\partial \varepsilon}{\partial x} \\
\frac{\partial u_s}{\partial t} &= -u_s \frac{\partial u_s}{\partial x} + C_1 \varepsilon^{-6.7} - C_2 \varepsilon^{-5.7} u_s + C_3 \varepsilon^{-4.7} u_s^2 \\
&\quad + C_4 (1 - \varepsilon)^{-0.54} - C_5
\end{align*}
\] (6.7)

where \( \varepsilon \) is the voidage and \( u_s \) is the solid velocity. The meanings of other parameters could refer to [69]. From 2-PDE model simulations, we could obtain a response \( h(x, t) \) to an impulse actuation in solid velocity with area of 0.1. Then the response to an arbitrary inlet solid velocity \( u(t) \) can be calculated as

\[
y(x, t) = \int_{-\infty}^{t} h(x, \tau) \cdot 10u(t - \tau) d\tau
\] (6.8)

where the scaling factor is necessary since the simulated input was not 1. The simulated impulse responses are shown in Figure 6.14.

Since the impulse response is uniformly zero after 0.6 seconds, (6.8) can be limited to

\[
y(x, t) = \int_{t-0.6}^{t} h(x, \tau) \cdot 10u(t - \tau) d\tau
\] (6.9)

To obtain a low-order high fidelity finite-dimensional representation of the impulse response, we use a wavelet decomposition [70], [71] to approximate \( h(x, t) \). That is, we try to decompose the impulse response \( h(x, t) \) as

\[
h(x, t) = \sum_{m=1}^{m_{\text{max}}} \sum_{n=1}^{n_{\text{max}}} \beta_m(x) c_{m,n} \alpha_n(t)
\] (6.10)

where \( \{\beta_m(x)\} \) and \( \{\alpha_n(t)\} \) are wavelet basis functions. Here we choose Gaussian wavelet functions specifically. In this case, we use 23 spatial and 22 temporal wavelets. The coefficients \( c_{m,n} \) were determined using a least-squares regression. Figure 6.15 is the resulting wavelet approximations of the impulse response.
The following notation will be used to divide the impulse response into separate parts for voidage $\varepsilon$ and velocity $u_s$:

$$h_{u_s}(x,t) = \sum_{m=1}^{m_{\text{max}}} \sum_{n=1}^{n_{\text{max}}} \beta_m(x)c_{m,n}\alpha_n(t)$$

$$h_{\varepsilon}(x,t) = \sum_{m=1}^{m_{\text{max}}} \sum_{n=1}^{n_{\text{max}}} \beta_m(x)d_{m,n}\alpha_n(t)$$

Then using the convolution form,

$$\Delta u_s(x,t) = h_{u_s}(x,\tau) * 10u(t - \tau)$$

$$= 10 \int_0^{0.6} \sum_{m=1}^{m_{\text{max}}} \sum_{n=1}^{n_{\text{max}}} \beta_m(x)c_{m,n}\alpha_n(\tau)u(t - \tau)d\tau$$

Since the online measurements are only available at 1 second intervals, assume that

$$u(t - \tau) = (1 - \tau)u(t) + \tau u(t - 1), 0 \leq \tau \leq 1$$

i.e. interpolate linearly between the measurements. Then,

$$\Delta u_s(x,t) = 10 \int_0^{0.6} \sum_{m=1}^{m_{\text{max}}} \sum_{n=1}^{n_{\text{max}}} \beta_m(x)c_{m,n}\alpha_n(\tau)[(1 - \tau)u(t)$$

$$+ \tau u(t - 1)]d\tau$$

$$= 10 \sum_{m=1}^{m_{\text{max}}} \sum_{n=1}^{n_{\text{max}}} \beta_m(x)c_{m,n} \int_0^{0.6} \alpha_n(\tau)[(1 - \tau)u(t)$$

$$+ \tau u(t - 1)]d\tau$$

$$= 10 \left[ \sum_{m=1}^{m_{\text{max}}} \sum_{n=1}^{n_{\text{max}}} \beta_m(x)c_{m,n} \int_0^{0.6} (1 - \tau)\alpha_n(\tau)d\tau \right] u(t)$$

$$+ 10 \left[ \sum_{m=1}^{m_{\text{max}}} \sum_{n=1}^{n_{\text{max}}} \beta_m(x)c_{m,n} \int_0^{0.6} \tau\alpha_n(\tau)d\tau \right] u(t - 1)$$

Denote
\[ a_{n,0} = \int_0^{0.6} (1 - \tau)\alpha_n(\tau) d\tau \]
\[ a_{n,1} = \int_0^{0.6} \tau\alpha_n(\tau) d\tau \] (6.15)

and

\[
\gamma_0(x) = 10 \sum_{m=1}^{m_{\text{max}}} \sum_{n=1}^{n_{\text{max}}} \beta_m(x)c_{m,n}\alpha_{n,0}
\]
\[
\gamma_1(x) = 10 \sum_{m=1}^{m_{\text{max}}} \sum_{n=1}^{n_{\text{max}}} \beta_m(x)c_{m,n}\alpha_{n,1}
\] (6.16)

then (6.14) simplifies to

\[
\Delta u_s(x, t) = \gamma_0(x)u(t) + \gamma_1(x)u(t - 1)
\] (6.17)

Similarly, we could have

\[
\Delta \varepsilon(x, t) = \eta_0(x)u(t) + \eta_1(x)u(t - 1)
\] (6.18)

where

\[
\eta_0(x) = 10 \sum_{m=1}^{m_{\text{max}}} \sum_{n=1}^{n_{\text{max}}} \beta_m(x)d_{m,n}\alpha_{n,0},
\]
\[
\eta_1(x) = 10 \sum_{m=1}^{m_{\text{max}}} \sum_{n=1}^{n_{\text{max}}} \beta_m(x)d_{m,n}\alpha_{n,1}
\] (6.19)

From [72], the output - DP47, the pressure drop across the riser - can be calculated as

\[
P(5) - P(0) = [\rho_g \varepsilon(0)u_g^2(0) + \rho_s(1 - \varepsilon(0))u_s^2(0)]
\]
\[
- [\rho_g \varepsilon(5)u_g^2(5) + \rho_s(1 - \varepsilon(5))u_s^2(5)]
\]
\[
- \int_0^5 g[\rho_g \varepsilon(y) + \rho_s(1 - \varepsilon(y))]dy
\] (6.20)

where the riser has a length of 5m and constant \( g \) is the gravity acceleration. \( u \) is the velocity, \( \rho \) is the density and subscripts \( s, g \) represents solid and gas respectively. \( u_g = \frac{U_g}{\varepsilon(x)} \) where \( U_g \) is the superficial gas velocity. Expanding (6.20), we obtain
\[ P(5) - P(0) = \left[ \rho_g U_g \frac{1}{\epsilon^2(0)} + \rho_s (1 - \epsilon(0)) u_s^2(0) \right] \\
- \left[ \rho_g U_g \frac{1}{\epsilon^2(5)} + \rho_s (1 - \epsilon(5)) u_s^2(5) \right] \\
- \int_0^5 g[\rho_g \epsilon(y) + \rho_s (1 - \epsilon(y))] dy \\
= \rho_g U_g \frac{1}{(\epsilon_{ss}(0) + \Delta \epsilon(0))^2} \\
+ \rho_s (1 - \epsilon_{ss}(0) - \Delta \epsilon(0))(u_s(0) + \Delta u_s(0))^2 \\
- \rho_g U_g \frac{1}{(\epsilon_{ss}(5) + \Delta \epsilon(5))^2} \\
+ \rho_s (1 - \epsilon_{ss}(5) - \Delta \epsilon(5))(u_s(5) + \Delta u_s(5))^2 \\
- g \rho_g \int_0^5 (\epsilon_{ss}(y) + \Delta \epsilon(y)) dy \\
- g \rho_s \int_0^5 (1 - \epsilon_{ss}(y) - \Delta \epsilon(y)) dy \] (6.21)

where subscript \( ss \) means steady state. Now, substituting the wavelet model gives

\[ P(5) - P(0) = \rho_g U_g \frac{1}{(\epsilon_{ss}(0) + \eta_0(0)u(t) + \eta_1(0)u(t-1))^2} \\
+ \rho_s (1 - \epsilon_{ss}(0) - \eta_0(0)u(t) - \eta_1(0)u(t) \\
- \eta_0(0)u(t-1))(u_{sss}(0) + \gamma_0(0)u(t) + \gamma_1(0)u(t-1))^2 \\
- \rho_g U_g \frac{1}{(\epsilon_{ss}(5) + \eta_0(5)u(t) + \eta_1(5)u(t-1))^2} \\
- \rho_s (1 - \epsilon_{ss}(5) - \eta_0(5)u(t) - \eta_1(5)u(t) \\
- \eta_0(5)u(t-1))(u_{sss}(5) + \gamma_0(5)u(t) + \gamma_1(5)u(t-1))^2 \\
- g \rho_g \int_0^5 (\epsilon_{ss}(y) + \eta_0(y)u(t) + \eta_1(y)u(t-1)) dy \\
- g \rho_g \int_0^5 (1 - \epsilon_{ss}(y) - \eta_0(y)u(t) - \eta_1(y)u(t) \\
- \eta_0(y)u(t-1)) dy \] (6.22)

Our goal is to use the model (6.22) to account for the high-frequency behavior of the CL system. Then, it will be useful to calculate the steady-state pressure drop:
\[ \Delta P_0 = \rho g U_g \frac{1}{(\varepsilon_{ss}(0) + \eta_0(0)u(t) + \eta_1(0)u(t))^2} \]
\[- \rho_s \left( 1 - \varepsilon_{ss}(0) - \eta_0(0)u(t) \right) \left( u_{s,ss}(0) + \gamma_0(0)u(t) + \gamma_1(0)u(t) \right)^2 \]
\[- \rho g U_g \frac{1}{(\varepsilon_{ss}(5) + \eta_0(5)u(t) + \eta_1(5)u(t))^2} \]
\[- \rho_s \left( 1 - \varepsilon_{ss}(5) - \eta_0(5)u(t) \right) \left( u_{s,ss}(5) + \gamma_0(5)u(t) + \gamma_1(5)u(t) \right)^2 \]
\[ -g\rho g \int_0^5 (\varepsilon_{ss}(y) + \eta_0(y)u(t) + \eta_1(y)u(t))dy \]
\[ -g\rho g \int_0^5 (1 - \varepsilon_{ss}(y) - \eta_0(y)u(t) - \eta_1(y)u(t))dy \]

(6.23)

This is the pressure drop predicted by this model for constant input \( u(t) \), as opposed to the linear interpolation described above. We can then use this model to approximate the transient difference and the NARX wavelet model to approximate the steady state. The difference between the transient pressure drop \( \Delta P(t) \) and the eventual steady pressure drop \( \Delta P_0(t) \) is then equal to

\[ \Delta P - \Delta P_0 = \rho g U_g \eta_1(0) \frac{(\varepsilon_{ss}(0) + \eta_0(0)u(t) + \eta_1(0)u(t) + \eta_1(0)u(t - 1))}{(\varepsilon_{ss}(0) + \eta_0(0)u(t) + \eta_1(0)u(t))^2} \]
\[ \times \left( \frac{u(t) - u(t - 1)}{(\varepsilon_{ss}(0) + \eta_0(0)u(t) + \eta_1(0)u(t - 1))^2} \right) \]
\[- \rho_s \left[ \eta_1(0) \left( u_{s,ss}(0) + \gamma_0(0)u(t) + \gamma_1(0)u(t) \right)^2 \right] \]
\[ + \rho_s \left[ -\gamma_1(0)((1 - \varepsilon_{ss}(0) - \eta_0(0)u(t) - \eta_1(0)u(t)) \]
\[ \times \left( u_{s,ss}(0) + \gamma_0(0)u(t) + \gamma_1(0)u(t) + \gamma_1(0)u(t - 1) \right) \]
\[ - u(t - 1) \right) \]

(6.24)
\[-\rho_g U_g \eta_1(5) \left( \frac{(\epsilon_{ss}(5) + \eta_0(5)u(t) + \eta_1(5)u(t) + \eta_1(5)u(t-1))}{(\epsilon_{ss}(5) + \eta_0(5)u(t) + \eta_1(5)u(t))} \right) \]
\[\times \frac{(\epsilon_{ss}(5) + \eta_0(5)u(t) + \eta_1(5)u(t-1))^2}{(\epsilon_{ss}(5) + \eta_0(5)u(t) + \eta_1(5)u(t))} \]
\[-\rho_s \left[ \eta_1(5) \left( u_{ss,ss}(5) + \gamma_0(5)u(t) + \gamma_1(5)u(t) \right) \right]^2 \]
\[\left[ -\gamma_1(5)((1 - \epsilon_{ss}(5) - \eta_0(5)u(t) - \eta_1(5)u(t)) \right] \]
\[\times \left( u_{ss,ss}(5) + \gamma_0(5)u(t) + \gamma_1(5)u(t) + \gamma_1(5)u(t-1) \right) \]
\[-u(t-1) \]
\[+ g(\rho_g - \rho_s) \int_0^5 \eta_1(y) dy(u(t) - u(t-1)) \]

Linearizing (6.24) about \( u(t) = u(t-1) \) gives

\[\Delta P - \Delta P_0 \approx f(u(t-1))(u(t) - u(t-1)) \quad (6.25)\]

where

\[f(y) = k_1 \left( \frac{k_2 + k_3 y}{(k_2 + k_4 y)^2} + k_5 (k_6 + k_7 y)^2 \right) \]
\[+ k_8(1 - k_2 - k_4 y)(k_6 + k_9 y) + k_10 \left( \frac{k_{11} + k_{12} y}{(k_{11} + k_{13} y)^2} \right) \]
\[+ k_14(k_{15} + k_{16} y)^2 + k_{17}(1 - k_{11} - k_{13} y)(k_{15} + k_{18} y) + k_{19} \quad (6.26)\]

And
\[k_1 = \rho_g U_g \eta_1(0), k_2 = \varepsilon_{ss}(0), k_3 = \eta_0(0) + 2\eta_1(0),\]
\[k_4 = \eta_0(0) + \eta_1(0), k_5 = \rho_s \eta_1(0), k_6 = u_{s,s}(0),\]
\[k_7 = \gamma_0(0) + \gamma_1(0), k_8 = -\rho_s \gamma_1(0), k_9 = \gamma_0(0) + 2\gamma_1(0),\]
\[k_{10} = -\rho_g U_g \eta_1(5), k_{11} = \varepsilon_{ss}(5), k_{12} = \eta_0(5) + 2\eta_1(5),\]
\[k_{13} = \eta_0(5) + \eta_1(5), k_{14} = -\rho_s \eta_1(5), k_{15} = u_{s,s}(5),\]
\[k_{16} = \gamma_0(5) + \gamma_1(5), k_{17} = \rho_s \gamma_1(5), k_{18} = \gamma_0(5) + 2\gamma_1(5),\]
\[k_{19} = g(\rho_g - \rho_s) \int_0^5 \eta_1(y) dy \]

(6.27)

The input to the computational model was in terms of the velocity boundary condition, so \(u(t) = \Delta u_s(0, t)\). This can be connected to the inputs \(S_1\) and \(S_2\) via the quadratic model [72] fitted to test data where

\[
u(t) \approx \frac{1}{\varepsilon_0} \left( 2a_1 S_1(t - 1) + a_3 S_2(t) + a_4 \right) (S_1(t) - S_1(t - 1))
+ \frac{1}{\varepsilon_0} \left( a_1 S_1^2(t - 1) + a_2 S_2^2(t) + a_3 S_1(t - 1) S_2(t)
+ a_4 S_1(t - 1) + a_5 S_2(t) + a_6 \right) - u_{s,s}(0)
= \frac{1}{\varepsilon_0} \left( a_1 S_1(t - 1) + a_3 S_2(t) + a_4 \right) S_1(t)
+ \frac{1}{\varepsilon_0} \left( a_2 S_2^2(t) + a_5 S_2(t) + a_6 \right) - u_{s,s}(0)
\]

(6.28)

Then

\[\Delta P - \Delta P_0 \approx \frac{1}{\varepsilon_0} f(u(t - 1)) (2a_1 S_1(t - 1) + a_3 S_2(t) + a_4) S_1(t)
+ f(u(t - 1)) \left[ \frac{1}{\varepsilon_0} \left( a_2 S_2^2(t) + a_5 S_2(t) + a_6 \right) - u_{s,s}(0) \right] - u(t - 1) \]
\[= g_{\Delta P}(S_1(t - 1), S_2(t - 1), S_2(t)) S_1(t)
+ f_{\Delta P}(S_1(t - 1), S_2(t - 1), S_2(t)) \]

(6.29)

The NARX wavelet MRA model takes the form as in (6.1)

\[y_w(t) = f \left( y(t - 1), \ldots, y(t - n_y), S_{1,w}(t - 1), \ldots, S_{1,w}(t - n_u) \right) \]

(6.30)

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where $S_{1,w}(t)$ is the control command calculated by wavelet adaptive GPC control. Then the fast transient behavior model (6.29) can be combined with (6.30) to obtain a spatiotemporal multiscale dynamic network representation of the entire CL process:

$$y(t) \approx y_w(t) - f_{\Delta P} - g_{\Delta P} S_1(t)$$  \hspace{1cm} (6.31)

The sign change is necessary because the pressure drop across the riser is negative in the model above, i.e. $P(5) - P(0) < 0$. Then the deadbeat predictive controller taking account for fast dynamics is

$$S_{1,fast}(t) = \frac{y_r(t) - y_w(t) + f_{\Delta P}}{g_{\Delta P}}$$  \hspace{1cm} (6.32)

where $y_r(t)$ is the reference signal. Hence the final spatio-temporal wavelet controller $S_1(t)$ implemented to the real CL process is taken as

$$S_1(t) = S_{1,w}(t) + S_{1,fast}(t)$$  \hspace{1cm} (6.33)

We tested the spatio-temporal wavelet controller (6.33) on the single loop cold flow CL test rig. The single input is $S_1$ and output is $DP47$, while $S_2$ is set to constant 20. The reference signal was set to 16 initially and then reduced to 13. The tracking response of system output and the corresponding control efforts are shown in Figure 6.16 and Figure 6.17 respectively. We can see that the controller stabilized the system quite well at high level operating condition.
Figure 6.13 Block diagram of controller implementation with fast dynamics

Figure 6.14 Simulated impulse response of 2PDE riser model
Figure 6.15 Wavelet-approximated impulse response $h(x, t)$

Figure 6.16 Pressure difference response of riser (DP47) during step setpoint changes
Figure 6.17 Fluidizing air flow control (S1 and S2) during setpoint changes
6.4 Conclusions

In this chapter, wavelet MRA based adaptive GPC control strategy for regulation and stable tracking of a single loop configuration of a solid flow test system is designed to investigate the chemical looping solid transport process. The data-driven wavelet MRA model was trained online to estimate the nonlinear dynamic multi-scale characteristics, effectively addressing the gap existing in high fidelity modeling of the given solid transport process using traditional single-scale models, such a NARX model. Rate limit has been introduced on the control input to take into consideration the actuator saturation. Finally, experimental results are provided to demonstrate the effectiveness of the wavelet MRA identified model and the predictive control strategy when implemented as a real-time controller to control the nonlinear solid transport processes. As seen from experimental results, the techniques proposed permitted to adequately control chemical looping process characterized by multi-phase gas-solid flows with a highly challenging uncertain nonlinear multi-scale dynamics with jumps. Next, spatio-temporal-wavelet models and controllers by combining together PDE model of the riser section and temporal wavelet model were developed and the effectiveness of which has been demonstrated by the experiment results as well.
CHAPTER 7: CONCLUSIONS AND FUTURE RESEARCH

In this thesis, we have been investigating the capability of wavelet MRA model as the building block to approximate the generic nonlinearities existing in the plant dynamics. First, a self-organizing state and output feedback wavelet adaptive robust controllers have been proposed for a class of nonlinear system with disturbances. Second, a consistent general framework for guaranteeing parameter estimate boundedness in the Hilbert space setting for adaptive observers with projection based parameter update laws is established for a class of linear infinite dimensional systems with bounded input operator and unknown nonlinearities. Third, wavelet MRA based adaptive predictive control is designed for regulation of both unconstrained and constrained unknown nonlinear system. The stability analysis has been established for all the control strategies by means of Lyapunov theorem and simulation results are provided to demonstrate their effectiveness. Finally, the wavelet based predictive controller is implemented on the test rig of single loop cold flow of chemical looping system and shown to track reference signals successfully.

The performance of the adaptive and predictive control is closely related to the approximation precision of the identified model. In order to obtain high fidelity identification model, more resolutions along with more basis functions are necessary to be included in the wavelet network. However, when system dimension exceeds $n = 2$, the wavelet network may encounter the difficulty of the so-called curse of dimensionality which exists for all kinds of control strategies requiring function approximation when system dimension is high. New theory and algorithms to find the minimum necessary wavelet network structure for a given class of functions is the next direction to pursue.
For $L_1$ adaptive control of infinite dimensional system, output feedback control with state estimation needs to be developed when only plant output is available. The stability analysis of spatio-temporal wavelet predictive control is also another challenging problem in our future research work.
REFERENCES


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