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RIGIDITY OF LENGTH FUNCTIONS OVER STRATA OF FLAT METRICS

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2014

Urbana, Illinois

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Abstract

In this thesis we consider strata of flat metrics coming from quadratic differentials (semi-translation structures) on surfaces of finite type. We provide a necessary and sufficient condition for a set of simple closed curves to be spectrally rigid over a stratum with enough complexity, extending a result of Duchin-Leininger-Rafi. Specifically, for any stratum with more unmarked zeroes than the genus, the Σ -length-spectrum of a set of simple closed curves Σ determines the flat metric in the stratum if and only if Σ is dense in the projective measured foliation space. We also prove that flat metrics in any stratum are locally determined by the Σ -length-spectrum of a finite set of closed curves Σ .

Acknowledgments

I would like to thank my advisor Chris Leininger for his guidance. He introduced the rich world of geometric topology to me and encouraged me to discuss with other mathematicians. As a result, I got to meet and befriend a lot of amazing people. I enjoyed the friendly atmosphere during our discussions and I learned a lot through his attention to details. I am particularly thankful to his time spent reading my various drafts and all his helpful comments. I would also like to thank my preliminary exam and dissertation defense committee members, Ilya Kapovich, Nathan Dunfield, Jayadev Athreya, and Spencer Dowdall.

An integral part of my graduate studies lies in seminars and conferences. Special thanks go to the group theory seminar and lunch organized by Ilya Kapovich for all the great talks as well as all the informal yet informative chats during the lunches. Other seminars and conferences that I would like to thank here are the differential geometry seminar, the ergodic theory seminar, the graduate student geometry and topology seminar, the retreats by the GEAR network, and the graduate student topology and geometry conferences.

There is a long list of individuals (in no particular order) that helped me along the journey and I apologize to anyone that I missed. Thanks to Ralph Howard, Maria Girardi, Jennifer McNeilly, John McKay, Ely Kerman, Howard Masur, Moon Duchin, Kasra Rafi, Marci, Ben, Greg, Mike, Anton, Brian, Anja, Caglar, Grace, Grant, Sam, Kenji, Matt, Ser-Geon, Joy, Jennifer, Arvin, Otter, Monica, Bruce, and the staff of the Department of Mathematics at UIUC. Finally, none of my works would be possible without my parents Hung-Lin Fu, Chin-Mei Kau, and my girlfriend Yueh-Ju Lin.

Table of Contents

- List of Figures v**
- List of Symbols vi**
- Chapter 1 Introduction 1**
 - 1.1 Notation and definition 2
 - 1.1.1 Flat metric 3
 - 1.1.2 Length functions and rigidity 4
 - 1.2 Outline of thesis 5
- Chapter 2 The space of quadratic differentials 7**
 - 2.1 Abelian differentials 8
 - 2.2 Local coordinates of strata 9
 - 2.3 Flat metrics 12
 - 2.4 Length functions over flat metrics 14
- Chapter 3 Measured foliations 17**
 - 3.1 Properties 18
 - 3.2 Train track coordinates 20
 - 3.3 Actions of the mapping class group 24
- Chapter 4 Rigidity of length functions 26**
 - 4.1 Rigidity results 26
 - 4.2 Non-rigidity over strata of flat metrics 28
- Chapter 5 Local rigidity of length functions 33**
 - 5.1 Local rigidity results 33
 - 5.2 Local rigidity over strata of flat metrics 34
 - 5.3 Strata of Euclidean cone metrics 38
- References 40**

List of Figures

1.1	Picture of $S_{0,4}$, $S_{1,1}$, and $S_{2,1}$.	2
1.2	Quadratic differentials on $S_{0,4}$, $S_{1,1}$, and $S_{2,1}$ with marked vertices.	3
1.3	Examples of curves and geodesic representatives locally.	4
1.4	Curves and geodesic representatives on polygons-representatives.	5
2.1	Deforming an Abelian differential.	9
2.2	Construction of the translation structure on the double cover DS .	10
2.3	Deforming q through the lifted Abelian differential on DS .	11
2.4	A deformation of ζ on DS that does not descend.	11
2.5	The metric in a neighborhood of a cone point obtained by gluing sectors.	12
2.6	Distinct quadratic differentials that induce the same flat metric.	13
2.7	Examples of cylinder curves on a flat metric.	14
3.1	Singularities of index -2 , $-1/2$, and $1/2$.	17
3.2	Vertical measured foliations induced by quadratic differentials.	19
3.3	Examples of train tracks.	20
3.4	The train track τ on pairs of pants.	21
3.5	The train track τ' on pairs of pants.	22
3.6	Examples of transverse curves.	23
3.7	Explicit weights and equations near a pants curve.	24
3.8	Visualizing the north-south dynamics on $\mathcal{PMF}(S)$.	25
5.1	Visualizing the constructed geodesic segment in the universal cover.	35
5.2	Constructing closed geodesics for a saddle connection between unmarked zeroes.	36
5.3	Constructing closed geodesics for a saddle connection with a marked endpoint.	38

List of Symbols

$S_{g,n}$	A closed surface with genus g with n marked points.
$\hat{S}_{g,n}$	Surface $S_{g,n}$ with the marked points punctured.
$\mathcal{T}(S)$	The Teichmüller space of S .
$\text{Mod}(S)$	The mapping class group of S .
$\mathcal{M}(S)$	The moduli space of S .
$\text{QD}(S)$	The space of quadratic differentials over S .
$\text{QD}(S, \alpha)$	The stratum of $\text{QD}(S)$ described by α .
$\text{Flat}(S)$	The space of flat metrics induced by quadratic differentials on S .
$\text{Flat}(S, \alpha)$	The stratum of $\text{Flat}(S)$ described by α .
$\mathcal{C}(\hat{S})$	Homotopy classes of closed curves on \hat{S} .
$\mathcal{S}(\hat{S})$	Homotopy classes of simple closed curves on \hat{S} .
$\ell(c, \rho)$	The length of c , a homotopy class of closed curves, with respect to a metric ρ .
$\mathcal{G}(S)$	A family of length metrics on S .
$\mathcal{H}(S)$	The space of Abelian differentials over S .
$\mathcal{H}(S, \alpha)$	The stratum of $\mathcal{H}(S)$ described by α .
\mathbb{H}_ρ	The Teichmüller disk corresponding to the flat metric ρ .
$\text{Cyl}(\rho)$	The set of cylinder curves with respect to the flat metric ρ .
$\mathcal{MF}(S)$	The space of measured foliations on S .
$i(\cdot, \cdot)$	Geometric intersection number.
i_*	The embedding from $\mathcal{S}(S)$ or $\mathcal{MF}(S)$ to $\mathbb{R}^{\mathcal{S}(S)}$ by intersection number.
$\mathcal{PMF}(S)$	The space of projective measured foliations on S .
$\nu_q(\theta)$	The vertical measured foliation of rotating q by an angle of θ .
$\mathcal{MF}_\tau(S)$	The set of measured foliations on S carried by the train track τ .
W_τ	The set of weight functions on the train track τ .

Chapter 1

Introduction

The classical work of Fricke and Klein proves that a hyperbolic metric on a surface of finite type is determined (up to isotopy) by the lengths of a finite set of simple closed curves. In this thesis, the motivating question was the extent to which the Fricke-Klein result holds for a certain family of flat metrics. Let S be a surface of finite type with genus g and n marked points. Given a set of homotopy classes of curves Σ and metrics X , say that Σ is length spectrally rigid over X if the lengths of Σ , $\lambda_\Sigma : X \rightarrow \mathbb{R}$, is injective. In [9], Duchin, Leininger, and Rafi studied the length spectral rigidity problem for the space $\text{Flat}(S)$ of flat metrics coming from quadratic differentials on surfaces. They prove the following which gives a complete description of when a set of simple closed curves is length spectrally rigid over $\text{Flat}(S)$. See Section 1.1 for definitions.

Theorem 1 (Duchin-Leininger-Rafi). *Let $(3g + n - 3) \geq 2$. A set of simple closed curves $\Sigma \subset \mathcal{S}(S)$ is length spectrally rigid over $\text{Flat}(S)$ if and only if $\overline{\Sigma} = \mathcal{PMF}(S)$.*

In this thesis we extend this result to strata of flat metrics. We prove that a set Σ of simple closed curves is length spectrally rigid over a stratum $\text{Flat}(S, \alpha)$ with sufficiently high dimension if and only if $\overline{\Sigma} = \mathcal{PMF}(S)$. To state this more precisely, recall that a stratum is determined by $\alpha = (\alpha_1, \dots, \alpha_k; \varepsilon)$, where $k = k_0 + n$, k_0 is the number of zeroes at non-marked points, $\alpha_1, \dots, \alpha_{k_0}$ are the order of the zeroes, $\alpha_{k_0+1}, \dots, \alpha_k$ are the orders at the marked points, and $\varepsilon = \pm 1$ indicates whether the holonomy is trivial or $\{\pm I\}$.

Theorem 2. *Let $(3g + n - 3) \geq 2$, $\alpha = (\alpha_1, \dots, \alpha_k; \varepsilon)$, and $(2k_0 - 2g + \varepsilon + 1) > 0$. Then a set of simple closed curves $\Sigma \subset \mathcal{S}(S)$ is length spectrally rigid over $\text{Flat}(S, \alpha)$ if and only if $\overline{\Sigma} = \mathcal{PMF}(S)$.*

The key technical result needed for this is a new, more flexible construction of deformation families of constant Σ -length-spectrum. In [9], the authors used a constructive case-by-case proof to prove the existence of such deformation families. For closed surfaces of genus g , they constructed deformation families of dimension $(2g - 3)$ in the stratum of largest dimension. Ours is an existence proof which allows for a unified treatment for all surfaces, and all applicable strata. One direction of Theorem 2 follows from Theorem 1. The other direction comes as the corollary of the following theorem.

Theorem 3. *Let $(3g + n - 3) \geq 2$, $\alpha = (\alpha_1, \dots, \alpha_k; \varepsilon)$, and $(2k_0 - 2g + \varepsilon + 1) > 0$. Suppose Σ is a set of simple closed curves with $\bar{\Sigma} \neq \mathcal{PMF}(S)$. Then there exists a deformation family $\Omega_\Sigma \subset \text{Flat}(S, \alpha)$ such that the length function λ_Σ is constant on Ω_Σ and $\dim(\Omega_\Sigma) \geq (2k_0 - 2g + \varepsilon + 1)$. Consequently, there exists a deformation family $\Omega_\Sigma \subset \text{Flat}(S)$ of dimension at least $(6g + 2n - 8)$.*

The results above show that a finite set of closed curves can never be length spectrally rigid when restricted to a single stratum with sufficiently high dimension. This is counterintuitive since these spaces of metrics are all finite dimensional. To complement Theorem 2, we prove that a finite set of closed curves can be *locally* length spectrally rigid over a stratum.

Theorem 4. *Let $\alpha = (\alpha_1, \dots, \alpha_k; \varepsilon)$. For any $\rho \in \text{Flat}(S, \alpha)$, there exists a set of closed curves $\Sigma \subset \mathcal{C}(S)$ such that Σ is locally length spectrally rigid at ρ over $\text{Flat}(S, \alpha)$ and $|\Sigma| \leq 15(2g + k - 2)$.*

There are still various directions in which we hope to extend these results. On one hand, we would like to prove a version of Theorem 2 for all strata, without imposing any restrictions on the dimensions with obvious exceptions. On the other hand, we would like to prove a version of Theorem 4 for $\text{Flat}(S)$ instead of for strata. In particular, we want to answer the following question. When $\mathcal{G} = \text{Flat}(S)$ or $\text{Flat}(S, \alpha)$, can we find a set of simple closed curves $\Sigma \subset \mathcal{S}(S)$ for $\rho \in \mathcal{G}$ such that Σ is locally length spectrally rigid at ρ over \mathcal{G} and $|\Sigma| = \dim(\mathcal{G})$?

1.1 Notation and definition

Let $S = S_{g,n}$ be a closed surface of genus g with n marked points, and \hat{S} the surface obtained by removing the marked points. In this thesis we always assume that $(3g + n - 3) \geq 1$. For example, Figure 1.1 shows the sphere $S_{0,4}$, the torus $S_{1,1}$, and $S_{2,1}$ with the marked points labeled by \times .

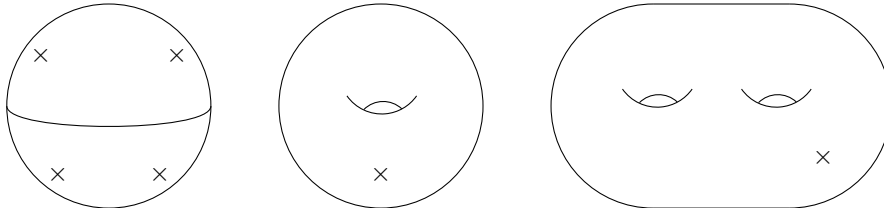


Figure 1.1: Picture of $S_{0,4}$, $S_{1,1}$, and $S_{2,1}$.

The Teichmüller space $\mathcal{T}(S)$ is the space of hyperbolic structures on S up to isotopy. The mapping class group of S , denoted $\text{Mod}(S)$, is the group of isotopy classes of orientation-preserving diffeomorphisms of S

which fix the marked points. The moduli space of Riemann surfaces $\mathcal{M}(S)$ is the quotient $\mathcal{T}(S)/\text{Mod}(S)$. We refer the reader to [5, 10, 18, 32] for a detailed discussion of their properties.

In the study of the Teichmüller space, the space of holomorphic quadratic differentials on S has played an important role. By a *quadratic differential* on S we mean a complex structure on S together with an integrable nonzero meromorphic quadratic differential. The quadratic differential is allowed to have poles of order at most one at marked points and is assumed to be holomorphic on \hat{S} . The space of all nonzero quadratic differentials, defined up to isotopy rel marked points, is denoted $\text{QD}(S)$. For more on quadratic differentials, the reader is referred to [14] and [29].

1.1.1 Flat metric

In Section 2.3 we explain how a quadratic differential on S induces a Euclidean cone metric. Denote the set of isotopy classes of unit area Euclidean cone metrics induced by quadratic differentials as $\text{Flat}(S)$. There is thus a map $\text{QD}(S) \rightarrow \text{Flat}(S)$. Any quadratic differential $q \in \text{QD}(S)$ can be obtained as *Euclidean polygons with side-gluing*, which is the following. We have a finite set of Euclidean polygons $\{\Delta_1, \dots, \Delta_m\}$ with boundary oriented counterclockwise and for every oriented side s_j of Δ_j there is an oriented side s_k of Δ_k so that s_j and s_k are parallel and of the same length. They are glued together in the opposite orientation by a semi-translation. The Euclidean polygons are marked by isometries between $\{\Delta_1, \dots, \Delta_m\}$ and the semi-translation structure determined by q that respects the side-gluing and transition functions.

The flat metric $\rho \in \text{Flat}(S)$ induced by q is obtained by scaling the Euclidean polygons to have total area 1. Figure 1.2 shows examples of quadratic differentials on $S_{0,4}$, $S_{1,1}$, and $S_{2,1}$ using Euclidean polygons with side-gluing representatives (also marking all vertices). When the marking on the Euclidean polygons is clear, we call them polygons-representatives of q or ρ .

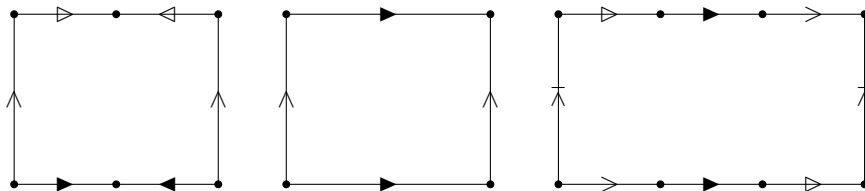


Figure 1.2: Quadratic differentials on $S_{0,4}$, $S_{1,1}$, and $S_{2,1}$ with marked vertices.

The space of quadratic differentials $\text{QD}(S)$ is naturally stratified according to the number of zeroes, the order of the zeroes, and the holonomy of a quadratic differential. Strata of $\text{QD}(S)$ are natural subspaces of interest because they are invariant under the action of $\text{Mod}(S)$ as well as an important action of $SL_2(\mathbb{R})$.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k; \varepsilon)$ where $k = k_0 + n$,

$$\alpha_1 \geq \dots \geq \alpha_{k_0} \geq 1, \alpha_{k_0+1} \geq \dots \geq \alpha_k \geq -1, \sum_{i=1}^k \alpha_i = 4g - 4,$$

and $\varepsilon = \pm 1$. We write $\text{QD}(S, \alpha)$ to denote the stratum of $\text{QD}(S)$ consisting of quadratic differentials q for which $\alpha_1, \dots, \alpha_{k_0}$ are the orders of the zeroes of q away from the marked points, $\alpha_{k_0+1}, \dots, \alpha_k$ are the orders of zeroes (or poles if $\alpha_i < 0$) at marked points, and ε records whether the holonomy is trivial or $\{\pm I\}$. This descends to a stratification of $\text{Flat}(S)$ and denote the stratum of $\text{Flat}(S)$ defined by α as $\text{Flat}(S, \alpha)$.

1.1.2 Length functions and rigidity

We are interested in *essential* closed curves on S . An essential closed curve is a closed curve on \hat{S} that is not null-homotopic or homotopic to a puncture. Let $\mathcal{C}(S)$ be the set of homotopy classes of essential closed curves on \hat{S} and $\mathcal{S}(S)$ be the set of homotopy classes of essential simple closed curves on \hat{S} . In [12] and [33], Thurston described a topology on $\mathcal{S}(S)$ and proved that the closure in this topology is naturally his space of projective measured foliations $\mathcal{PMF}(S)$. See Chapter 3 for more on the space of projective measured foliations.

Let $\mathcal{G}(S)$ be a family of isotopy classes of length metrics on S , which is either $\text{Flat}(S)$ or $\text{Flat}(S, \alpha)$ throughout most of the thesis. For $c \in \mathcal{C}(S)$, the length of c with respect to $\rho \in \mathcal{G}(S)$ is the infimum of lengths of representatives of c in a representative of ρ , denoted by $\ell(c, \rho)$. If $\rho \in \text{Flat}(S)$, then there exists a closed geodesic with respect to ρ that realizes the length $\ell(c, \rho)$ though it may lie in S rather than \hat{S} . This is obtained as the limit of representatives of c in \hat{S} with lengths approaching $\ell(c, \rho)$. Figure 1.3 shows the local picture and Figure 1.4 shows the global picture using Figure 1.2. We refer the reader to [4] for discussions on length metrics.

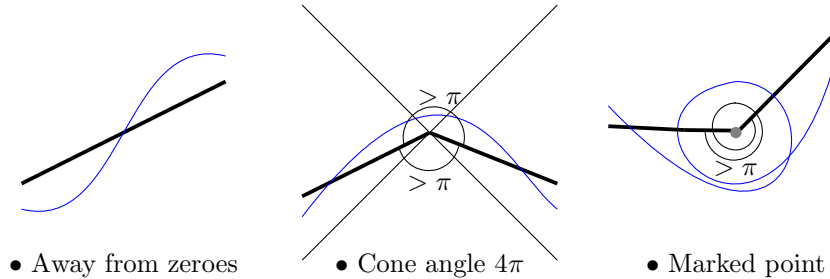


Figure 1.3: Examples of curves and geodesic representatives locally.

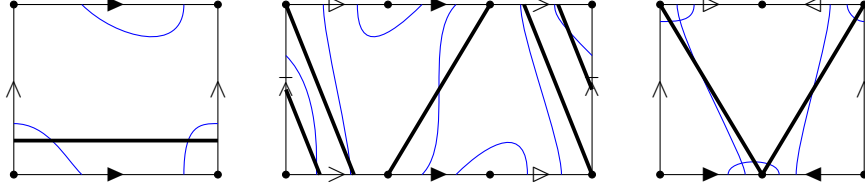


Figure 1.4: Curves and geodesic representatives on polygons-representatives.

The *length function* of $\Sigma \subset \mathcal{C}(S)$ with respect to $\mathcal{G}(S)$ is the function

$$\lambda_\Sigma : \mathcal{G}(S) \rightarrow \mathbb{R}^\Sigma, \text{ with } \lambda_\Sigma(\rho) = (\ell(c, \rho))_{c \in \Sigma}.$$

The Σ -*length-spectrum* of ρ is $\lambda_\Sigma(\rho)$, an element of \mathbb{R}^Σ . The set $\Sigma \subset \mathcal{C}(S)$ is called *length spectrally rigid* over $\mathcal{G}(S)$ if the length function λ_Σ is injective. We call the existence of a length spectrally rigid Σ over $\mathcal{G}(S)$ a *rigidity* result over $\mathcal{G}(S)$. On the other hand, a *non-rigidity* result is one that shows the existence of $\rho \neq \rho'$ in $\mathcal{G}(S)$ such that $\lambda_\Sigma(\rho) = \lambda_\Sigma(\rho')$. We call a fiber of λ_Σ a *deformation family* if it has positive dimension.

Motivated by the non-rigidity result over $\text{Flat}(S)$ in [9] and the fact that $\text{Flat}(S)$ is finite-dimensional, we study locally injective length functions. The set Σ is *locally length spectrally rigid* at ρ over $\mathcal{G}(S)$ if there exists a neighborhood $U_\rho \subset \mathcal{G}(S)$ of ρ such that the length function $\lambda_\Sigma|_{U_\rho}$ is injective.

1.2 Outline of thesis

In Chapter 2, we describe the space of quadratic differentials in more detail. Section 2.1 recalls the space of Abelian differentials and explains the relationship between the two spaces. Section 2.2 contains a description of local coordinates of strata of the space of quadratic differentials. Properties of the space of flat metrics induced by quadratic differentials are covered in Section 2.3. Section 2.4 describes properties of closed geodesics and length functions with respect to the flat metrics.

Measured foliations are important objects in Teichmüller theory, and these are discussed in Chapter 3. Some examples and properties are given in Section 3.1. Section 3.2 covers how train tracks will provide local coordinates. The mapping class group acts on the Teichmüller space, the space of quadratic differentials, homotopy classes of closed curves, and the space of measured foliations, and this is discussed in Section 3.3.

Rigidity and non-rigidity of length functions will be studied in Chapter 4. Section 4.1 overviews known results of length spectral rigidity. In particular, we sketch the proofs of rigidity and non-rigidity over the

space of flat metrics by Duchin-Leininger-Rafi in [9]. Section 4.2 contains the proof of Theorem 2. The key ingredient of the proof is the construction of a function f mapping an open set of $\text{QD}(S, \alpha)$ to the weight space W_τ of a maximal train track. The length of any curve $\gamma \in \Sigma$ with respect to a flat metric induced by $q \in \text{QD}(S, \alpha)$ is equal to the intersection number $i(f(q), \gamma)$ by construction. The fibers of f projects to deformation families of flat metrics of constant Σ -length-spectrum.

In contrast to Chapter 4, Chapter 5 considers local rigidity of length functions. Some known results are covered in Section 5.1. Theorem 4 is proved in Section 5.2, where we observe that flat metrics in a stratum are locally determined by the lengths of saddle connections. We then construct, for any saddle connection, a set of at most 5 closed curves whose lengths determine the length of the saddle connection on a sufficiently small neighborhood of the given flat metric. Section 5.3 speculates how the local rigidity result extends to strata of Euclidean cone metrics.

Chapter 2

The space of quadratic differentials

Fix a complex structure on S given by coordinate charts $\{(U_j, z_j)\}$. A collection of holomorphic functions $\{\psi_j : z_j(U_j) \rightarrow \mathbb{C}\}$ is called a *holomorphic quadratic differential* on S if it satisfies

$$\psi_k(z_k) = \psi_j \circ z_{jk}^{-1}(z_k) \cdot (z'_{jk}(z_k))^2 \text{ on } U_j \cap U_k,$$

where $z_{jk} = z_j \circ z_k^{-1}$. Accordingly, we write $\psi = \psi_j dz_j^2$ on U_j . The zeroes and the order of zeroes are well-defined (independent of coordinate chart) for a holomorphic quadratic differential. An *integrable meromorphic quadratic differential* on a Riemann surface is similarly defined except we allow poles of order at most one at the marked points of S . The space of quadratic differentials $\text{QD}(S)$ consists of all nonzero integrable meromorphic quadratic differentials on all Riemann surfaces. A point of $\text{QD}(S)$ will be denoted q , with the underlying complex structure implicit in the notation. Let P be the set of marked points of S and $C_0(q)$ be the set of zeroes of q at non-marked points. The set $C(q) = C_0(q) \cup P$ is called the set of *cone points* of q . We use C_0 and C , suppressing the dependence on the quadratic differential when it is safe to do so.

We obtain an invariant integral by taking the square root and integrating one of the two single valued branches sufficiently close to $p_0 \in S \setminus C(q)$:

$$\zeta(p) = \int_{p_0}^p \sqrt{\psi(z)} dz,$$

which is a natural coordinate ζ in which $q = d\zeta^2$. The collection of all natural coordinates determines a *semi-translation structure* on $S \setminus C(q)$. This is an open cover $\{U_\beta\}$ of $S \setminus C(q)$ along with charts $\phi_\beta : U_\beta \rightarrow \mathbb{R}^2 \equiv \mathbb{C}$ so that for every β, δ with $U_\beta \cap U_\delta \neq \emptyset$, we have

$$\phi_\beta \circ \phi_\delta^{-1}(v) = \pm v + c, \text{ where } v, c \in \mathbb{R}^2.$$

The semi-translation structure can also be described via Euclidean polygons with side-gluing representative.

Recall that Euclidean polygons with side-gluing refer to a finite set of Euclidean polygons $\{\Delta_1, \dots, \Delta_m\}$ with boundary oriented counterclockwise and for every oriented side s_j of Δ_j there is an oriented side s_k of Δ_k so that s_j and s_k are parallel and of the same length. They are glued together in the opposite orientation by a semi-translation. The Euclidean polygons are marked by isometries between $\{\Delta_1, \dots, \Delta_m\}$ and the semi-translation structure determined by q that respects the side-gluing and transition functions. Some examples were shown in Figure 1.2 where each polygon is a polygons-representative that corresponds to a semi-translation structure.

Quadratic differentials play an integral role in the proof of the famous Teichmüller's Theorem. They describe the solution to the extremal quasiconformal mapping problem, which defines the Teichmüller metric on the moduli space $\mathcal{M}(S)$. The space of quadratic differentials is identified with the cotangent space over the Teichmüller space. Quadratic differentials also show up in the study of Higgs bundles as an example of Higgs fields and the study of billiards in polygons. These appearances continue to motivate us to extend our understanding of the space of quadratic differentials.

2.1 Abelian differentials

We introduce Abelian differentials which are closely related to quadratic differentials. By an *Abelian differential* ω on S we mean a complex structure on S together with a nonzero holomorphic 1-form, that is, $\omega = \psi(z)dz$ for some holomorphic function ψ on local coordinate z . From an Abelian differential ω on S we can obtain local coordinates near any non-zero point p_0 by

$$\zeta(p) = \int_{p_0}^p \omega = \int_{p_0}^p \psi(z) dz.$$

This provides a collection of charts on the complement of the zeroes for which the transition maps are translations. Therefore an Abelian differential ω on S with zeroes $C_0(\omega)$ determines a *translation structure* on S , which is an open cover $\{U_\beta\}$ of $S \setminus C_0(\omega)$ along with charts $\phi_\beta : U_\beta \rightarrow \mathbb{R}^2$ so that for every β, δ with $U_\beta \cap U_\delta \neq \emptyset$, we have

$$\phi_\beta \circ \phi_\delta^{-1}(v) = v + c, \text{ where } v, c \in \mathbb{R}^2.$$

The translation structure on S allows a more geometric viewpoint of the space of Abelian differentials, see [24] and [36]. Let $\mathcal{H}(S)$ denote the space of Abelian differentials on S up to isotopy rel marked points. The space $\mathcal{H}(S)$ is stratified by the number and the order of the zeroes. We use $\mathcal{H}(S, \alpha)$ to denote the stratum of $\mathcal{H}(S)$ defined by α . Note that α here includes the number and order of zeroes of the Abelian differential

but not the holonomy, which is always trivial.

We have a good understanding of local structures of $\mathcal{H}(S, \alpha)$ coming from coordinate systems we now describe. Given an Abelian differential $\omega \in \mathcal{H}(S, \alpha)$, integrating ω over relative chains determines a relative cohomology class in $H^1(S, C(\omega); \mathbb{C})$. This defines a homeomorphism from a neighborhood U_ω of ω in $\mathcal{H}(S, \alpha)$ to an open set V_ω in $H^1(S, C(\omega); \mathbb{C})$. Next, given a basis of the relative homology $H_1(S, C(\omega); \mathbb{C})$, this determines an isomorphism $H^1(S, C(\omega); \mathbb{C}) \rightarrow \mathbb{C}^m$ for some m . Composing with the homeomorphism $U_\omega \rightarrow V_\omega$, we obtain *holonomy coordinates* $U_\omega \rightarrow \mathbb{C}^m$. If the polygons-representative consists of a single polygon, then a basis of the relative homology is given by all the sides of the polygon and an example of local deformation of an Abelian differential is shown in Figure 2.1.

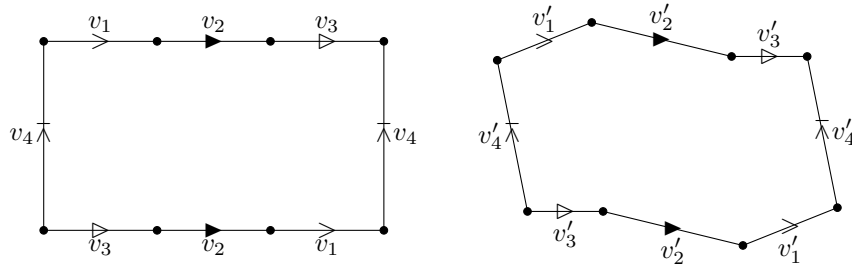


Figure 2.1: Deforming an Abelian differential.

An Abelian differential ω in $\mathcal{H}(S)$ determines a quadratic differential q with trivial holonomy in $\text{QD}(S)$ by $q = \omega \otimes \omega$. In the study of quadratic differentials, it is useful to consider the canonical double cover $DS \rightarrow S$ determined by the holonomy, where a quadratic differential $q \in \text{QD}(S)$ lifts to a quadratic differential $Q \in \text{QD}(DS)$ with trivial holonomy. See Figure 2.2 for a geometric example of lifting a quadratic differential to the double cover. Therefore it makes sense to say that each quadratic differential has a corresponding Abelian differential either on S or DS . This Abelian differential is unique up to sign.

2.2 Local coordinates of strata

We are interested in local coordinates and the dimension of a stratum of quadratic differentials. These properties depend on the holonomy and we describe them in cases.

Let $\text{QD}(S, \alpha)$ be a stratum with trivial holonomy, that is, $\varepsilon = 1$. For any $q \in \text{QD}(S, \alpha)$, there exists $\omega \in \mathcal{H}(S, \alpha')$ such that $q = \omega \otimes \omega$. This ω is unique up to sign, and we can find neighborhoods $U_q \subset \text{QD}(S, \alpha)$ of q and $U_\omega \subset \mathcal{H}(S, \alpha')$ of ω so that $U_\omega \rightarrow U_q$ defined by squaring Abelian differentials defines a homeomorphism. If U_ω is sufficiently small so that there are holonomy coordinates $U_\omega \rightarrow \mathbb{C}^m$, then inverting the homeomorphism $U_\omega \rightarrow U_q$ and composing with these coordinates defines *holonomy coordinates* $U_q \rightarrow \mathbb{C}^m$

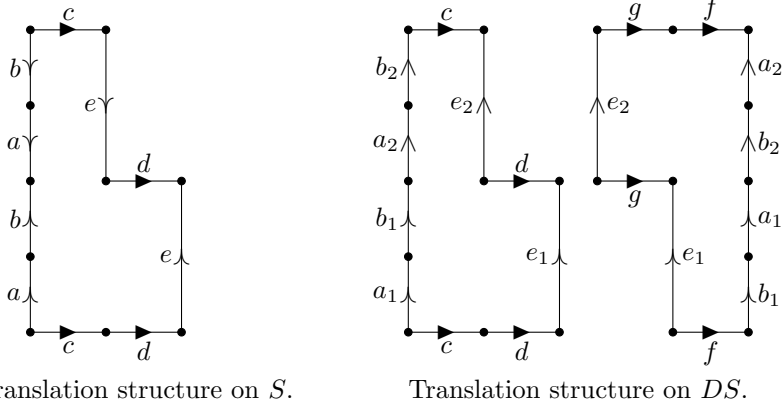


Figure 2.2: Construction of the translation structure on the double cover DS .

about q in $\text{QD}(S, \alpha)$. Figure 2.1 can be seen as a local deformation of a quadratic differential in a stratum with trivial holonomy.

When $\text{QD}(S, \alpha)$ is a stratum with nontrivial holonomy, then for $q \in \text{QD}(S, \alpha)$, we let $\zeta \in \text{QD}(DS)$ be the quadratic differential on the two-fold cover determined by the holonomy, as described in the previous setting (see Figure 2.2). Write $\zeta \in \text{QD}(DS, \alpha_{DS})$, where α_{DS} describes the stratum ζ lies in. There exists holonomy coordinates $U_\zeta \rightarrow \mathbb{C}^m$ about ζ as above since $\text{QD}(DS, \alpha_{DS})$ is a stratum with trivial holonomy.

We have two natural maps associated to this setup. The covering map $DS \rightarrow S$ and the involution $DS \rightarrow DS$ that generates the group of deck transformations. The involution induces an involution on $H^1(DS, C(\zeta); \mathbb{C})$ with eigenvalues ± 1 . If $\omega \in \mathcal{H}(DS, \alpha'_{DS})$ is such that $\omega \otimes \omega = \zeta$, then we have homeomorphisms $U_\zeta \rightarrow U_\omega$ and $U_\omega \rightarrow V_\omega \subset H^1(DS, C(\zeta); \mathbb{C})$ as previously described. This maps the subset of U_ζ consisting of quadratic differentials in U_ζ lifted from S precisely onto the intersection with the (-1) -eigenspace of the involution on $H^1(DS, C(\zeta); \mathbb{C})$. Let $W_q \rightarrow U_q \subset U_\zeta$ be a homeomorphism from W_q , a neighborhood of $q \in \text{QD}(S, \alpha)$, to U_q , a neighborhood of ζ in the (-1) -eigenspace.

Given a basis for the relative homology determining holonomy coordinates $U_\zeta \rightarrow \mathbb{C}^m$, we can choose a subset of the basis elements so that the map obtained by pairing with only these basis vectors $H^1(DS, C(\zeta); \mathbb{C}) \rightarrow \mathbb{C}^{m'}$ restricts to an isomorphism on the (-1) -eigenspace. The composition $U_\zeta \rightarrow V_\omega \rightarrow \mathbb{C}^{m'}$ is therefore a homeomorphism on the subset $U_q \subset U_\zeta$ consisting of quadratic differentials lifted from S . Composing $W_q \rightarrow U_q$ with this map, we obtain *holonomy coordinates* $W_q \rightarrow U_q \rightarrow \mathbb{C}^{m'}$. Figure 2.3 shows an example of a deformation of ζ that descends to a deformation of q , whereas Figure 2.4 shows a deformation of ζ that does not descend.

The following proposition on the dimension of a stratum of quadratic differentials can also be found in [34]. We write \dim for real dimension and $\dim_{\mathbb{C}}$ for complex dimension. It is well-known that in the case

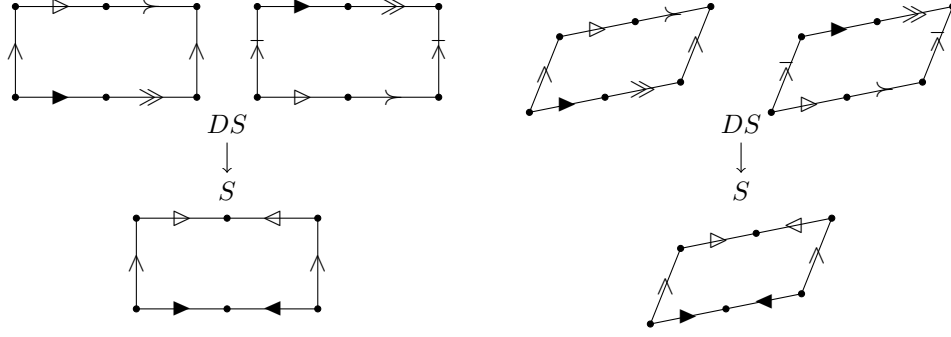


Figure 2.3: Deforming q through the lifted Abelian differential on DS .

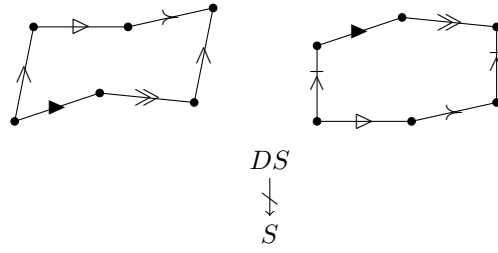


Figure 2.4: A deformation of ζ on DS that does not descend.

when $(3g + n - 3) = 1$, there is only one stratum for each of $S_{0,4}, S_{1,1}$ and $\dim(\text{QD}(S, \alpha)) = 4$.

Proposition 1. *Let $(3g + n - 3) \geq 2$ and $\alpha = (\alpha_1, \dots, \alpha_k; \varepsilon)$. Then $\dim(\text{QD}(S, \alpha)) = 4g + 2k + \varepsilon - 3$.*

Proof. If $\text{QD}(S, \alpha)$ is a stratum with trivial holonomy, that is, $\varepsilon = 1$, then

$$\dim_{\mathbb{C}} H^1(S, C(q); \mathbb{C}) = \dim_{\mathbb{C}} H_1(S, C(q); \mathbb{C}) = 2g + k - 1$$

implies that

$$\dim(\text{QD}(S, \alpha)) = 2(\dim_{\mathbb{C}}(\text{QD}(S, \alpha))) = 4g + 2k - 2.$$

For the case when $\text{QD}(S, \alpha)$ is a stratum with nontrivial holonomy, that is, $\varepsilon = -1$, fix $q \in \text{QD}(S, \alpha)$ and $\zeta \in \text{QD}(DS, \alpha_{DS})$ induced by q . Let k_1 be the number of i for which α_i is odd and k_2 be the number of i for which α_i is even. Each odd α_i corresponds to a $2(\alpha_i + 1)$ in α_{DS} and each even α_i corresponds to a pair of α_i in α_{DS} . Hence the size of the set of cone points for DS is

$$k_{DS} = k_1 + 2k_2,$$

and the genus of DS is

$$g_{DS} = 2g + \frac{k_1}{2} - 1,$$

obtained by calculating the Euler characteristic using the Riemann-Hurwitz formula. Holonomy coordinates about ζ have complex dimension $(2g_{DS} + k_{DS} - 1)$ as above. Recall that the subspace we are interested in is the (-1) -eigenspace of the involution on $H^1(DS, C(\zeta); \mathbb{C})$. The dimension of the (-1) -eigenspace is computed by subtracting the dimension of the $(+1)$ -eigenspace. The pullback of a basis of $H^1(S, C(q); \mathbb{C})$ to $H^1(DS, C(\zeta); \mathbb{C})$ is invariant under the involution and provides a basis of the $(+1)$ -eigenspace. The complex dimension of the $(+1)$ -eigenspace is equal to the complex dimension of $H^1(S, C(q); \mathbb{C})$. Therefore

$$\dim(\text{QD}(S, \alpha)) = 2[(2g_{DS} + k_{DS} - 1) - (2g + k - 1)] = 4g + 2k - 4.$$

Combine the two cases to obtain $\dim(\text{QD}(S, \alpha)) = 4g + 2k + \varepsilon - 3$. □

2.3 Flat metrics

In general, a Euclidean cone metric on S is a metric which is Euclidean on the complement of a finite set of points. In a neighborhood of the finite set, the metric is isometric to a *cone*, which is the space obtained by gluing together a union of sectors of disks in the Euclidean plane of some common radius. For example, the sectors in Figure 2.5 glue together to form a cone angle of 3π . The finite set of points are called the cone points of the Euclidean cone metric and we assume that their cone angles are at least 2π except at marked points. The flat metrics induced by quadratic differentials are Euclidean cone metrics with the property that the cone angles are all integer multiples of π .

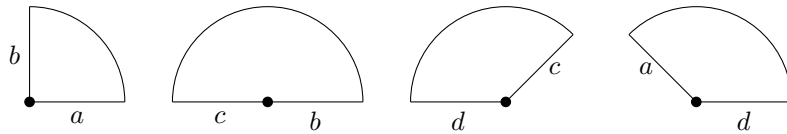


Figure 2.5: The metric in a neighborhood of a cone point obtained by gluing sectors.

The semi-translation structure from a quadratic differential q on $S \setminus C(q)$ induces a Euclidean metric on $S \setminus C(q)$. The metric extends to a Euclidean cone metric on all of S so that at a zero of order d , one has a cone point with cone angle $(2 + d)\pi$. If the point is a pole (and hence a marked point), we view this as a zero of order -1 , and then the cone angle is π . For example, in Figure 1.2, the first picture has four points with cone angle π ; the second picture has a single marked point with cone angle 2π ; the third picture has a

single cone point with cone angle 6π .

Denote the set of isotopy classes of unit area Euclidean cone metrics induced by quadratic differentials as $\text{Flat}(S)$. There is a map $\text{QD}(S) \rightarrow \text{Flat}(S)$ which is the quotient by rotation and scales to have area 1. The flat metric $\rho \in \text{Flat}(S)$ induced by q is obtained by scaling the Euclidean polygons in a polygons-representative of q to have total area 1. In Figure 2.6 we show some distinct quadratic differentials that induce the same flat metric (when the markings coincide).

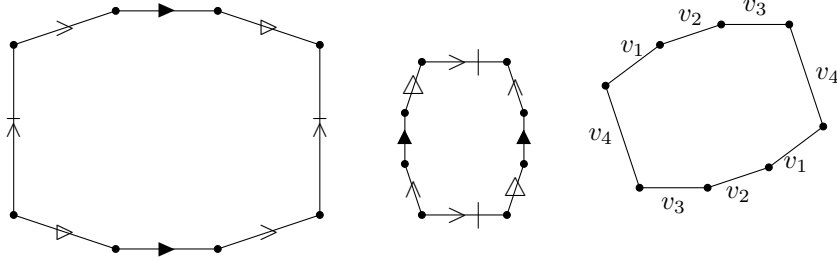


Figure 2.6: Distinct quadratic differentials that induce the same flat metric.

The stratification of $\text{QD}(S)$ descends to a stratification of $\text{Flat}(S)$. If $\alpha = (\alpha_1, \dots, \alpha_k; \varepsilon)$, then $\text{Flat}(S, \alpha)$ consists of those flat metrics from quadratic differentials with k cone points, so that the i -th cone point has cone angle $(\alpha_i + 2)\pi$, and so that the holonomy is trivial or $\{\pm I\}$ depending on whether $\varepsilon = 1$ or -1 , respectively. We denote the stratum of $\text{Flat}(S)$ induced by quadratic differentials in $\text{QD}(S, \alpha)$ as $\text{Flat}(S, \alpha)$.

We use holonomy coordinates to define local coordinates on $\text{Flat}(S, \alpha)$ as follows. Let $\rho \in \text{Flat}(S, \alpha)$ be induced by $q \in \text{QD}(S, \alpha)$. If we consider the open neighborhood $U_q \subset \text{QD}(S, \alpha)$ with holonomy coordinates $\varphi : U_q \rightarrow \mathbb{C}^m$, then there exists a real codimension two smooth submanifold (in the sense of φ coordinates) in U_q that is mapped homeomorphically onto an open set in $\text{Flat}(S, \alpha)$. The submanifold is defined by $\pi_1 \circ \varphi(q)$ being a positive real number (which effectively mods out by rotation) and the area of the quadratic differential being 1, where π_1 is the projection to the first coordinate. Therefore we can obtain from this *holonomy coordinates* about $\rho \in \text{Flat}(S, \alpha)$.

There is a specific family of flat metrics that is generated by $SL_2(\mathbb{R})$ orbits. For a fixed quadratic differential q and an induced flat metric ρ , $SL_2(\mathbb{R})$ acts naturally on q by applying matrices to the semi-translation structure. The action does not extend to ρ but the flat metrics induced by the orbit of q form a canonical set, which we denote \mathbb{H}_ρ . The image in $\text{Flat}(S)$ of an $SL_2(\mathbb{R})$ orbit is called a *Teichmüller disks*, and in fact, any stratum $\text{Flat}(S, \alpha)$ is foliated by Teichmüller disks. In general, Teichmüller disks are of particular interest in the study of Teichmüller geodesic and horocycle flows.

Let q be a quadratic differential. By definition q carries a complex structure on S , which is equivalent to

a hyperbolic structure on S . There is also the semi-translation structure induced by q . In terms of metrics, the hyperbolic metric and the flat metric coming from the structures given by q are closely related. Their relationship is studied in [28], for example, and it provides insight to a lot of open problems in the field.

2.4 Length functions over flat metrics

Recall that for a closed curve $c \in \mathcal{C}(S)$, the length of c with respect to $\rho \in \text{Flat}(S)$ is the infimum of lengths of representatives of c in a representative of ρ , denoted by $\ell(c, \rho)$.

We are interested in what geodesics look like with respect to metrics in $\text{Flat}(S)$. Let $c \in \mathcal{C}(S)$ be a closed curve on \hat{S} . For a fixed $\rho \in \text{Flat}(S)$ induced by $q \in \text{QD}(S)$, c is a geodesic with respect to ρ if and only if c is a concatenation of straight line segments connecting cone points in $S \setminus C(q)$, called *saddle connections*, and c satisfies the *angle condition* at each point in C_0 . The angle condition requires that each time c passes through a point in C_0 , the angles on both sides of c must be greater or equal to π . If c is a closed curve on S obtained as a limit of representatives of a homotopy class in \hat{S} , c is a geodesic with respect to ρ if and only if $c \setminus P$ satisfies the conditions above and at points in P the angle condition is satisfied on the “side of c opposite the point in P ”; see [29, Section 8] for a more precise description on geodesics near a marked point. Refer to Figure 1.3 and Figure 1.2 for examples of geodesics with respect to a flat metric.

In the study of quadratic differentials and flat metrics, a family of closed curves called *cylinder curves* plays an important role. We say a closed curve $c \in \mathcal{C}(S)$ is a cylinder curve with respect to a flat metric $\rho \in \text{Flat}(S)$ if c is a simple closed curve and c does not have a unique geodesic representative with respect to ρ . A cylinder curve has a closed geodesic representative that lies in $S \setminus C(q)$, where q induces ρ . The union of geodesic representatives of a cylinder curve form an annular region with the flat metric of a Euclidean cylinder. For examples of cylinder curves see Figure 2.7. We denote the set of cylinder curves with respect to a flat metric ρ by $\text{Cyl}(\rho)$. We make the observation that for any flat metric ρ' in the Teichmüller disk \mathbb{H}_ρ , the set of cylinder curves is the same, that is, $\text{Cyl}(\rho) = \text{Cyl}(\rho')$.

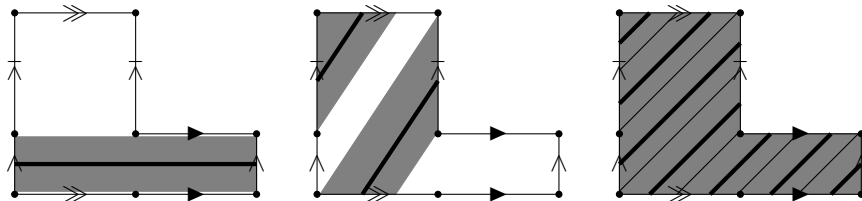


Figure 2.7: Examples of cylinder curves on a flat metric.

The description of geodesics with respect to flat metrics gives us an explicit formula of the length of

a closed curve, which is the sum of Euclidean lengths of the saddle connections making up the geodesic. Therefore the length of a closed curve changes under a perturbation of the flat metric in exactly two ways: either the lengths of the saddle connections change or the combinatorics of the saddle connections change. We say a closed geodesic is *stable* inside an open neighborhood of $\text{Flat}(S, \alpha)$ if the homotopy classes, rel endpoints, of the saddle connections making up the geodesic is constant. As a consequence, the combinatorics of the saddle connections making up the closed geodesic will be the same inside the open neighborhood. For example, every cylinder curve with respect to a flat metric ρ is stable inside a small enough neighborhood of $\rho \in \text{Flat}(S, \alpha)$.

The holonomy coordinate provides a more explicit description of open neighborhoods in $\text{Flat}(S, \alpha)$. We use holonomy coordinates to prove the following proposition that shows how closed geodesics generically remain “stable” under perturbations of the metric.

Proposition 2. *For any closed curve c , there exists an open dense subset of $\text{Flat}(S, \alpha)$ in which the geodesic representative of c is stable.*

Proof. Fix an arbitrary open set $U \subset \text{Flat}(S, \alpha)$. Let U_0 be an open neighborhood of an arbitrary $\rho \in U$ with holonomy coordinates $U_0 \rightarrow V \subset \mathbb{C}^m$. In holonomy coordinates, the length function of c becomes a function $V \rightarrow \mathbb{R}$. Using the description of geodesics, the length of the closed curve c is a finite sum of Euclidean lengths of saddle connections that belong to a geodesic representative of c .

Let $U_1 \subset U_0 \cap U$ be an open set such that the length function of c is bounded by some $r > 0$ on U_1 . Inside U_1 there is a uniform bound N on the number of relative homotopy classes of paths between cone points with length at most r and we denote the set by $\mathcal{A} = \{a_1, \dots, a_N\}$. Let $U(a_0) = U_1$ and define the following sets iteratively by

$$U(a_i) = \{\rho' \in U(a_{i-1}) \mid a_i \text{ is represented by a saddle connection in } \rho'\}$$

or $U(a_i) = U(a_{i-1})$ if the above set is empty. The terminal set $U(a_N)$ is open and nonempty. Call the subset of \mathcal{A} that is represented by a saddle connection X . By construction, all saddle connections X remain saddle connections in $U(a_N)$

For every two saddle connections s_i, s_j in X , consider the set $NP_{i,j}$ which is the subset of $U(a_N)$ on which s_i and s_j are not parallel, and $P_{i,j}$ the subset on which they are parallel. Using holonomy coordinates, we see that $P_{i,j}$ is a closed set defined by a single equation, hence $NP_{i,j}$ is open and either dense or empty. Define $E_{i,j}$ to be $NP_{i,j}$ if it is nonempty, and $P_{i,j}$ otherwise (in which case $P_{i,j}$ is all of $U(a_N)$). Then because X is finite, it follows that the intersection of all $E_{i,j}$ is an open dense subset of $U(a_N)$, and we

denote this V .

Now pick any metric ρ' in V . The geodesic representative of c in ρ' is a concatenation of saddle connections of length at most r , which therefore come from X . Any two consecutive saddle connections are either nonparallel, and remain so in V , or are parallel and remain so in V . It follows that the geodesic representative of c is a concatenation of the same set of saddle connections for every metric ρ'' in V . Therefore c is a stable closed curve in an open dense subset of $\text{Flat}(S, \alpha)$. \square

From Proposition 2, we obtain the following corollary immediately. The smoothness (with respect to holonomy coordinates) condition proved in Corollary 1 allows us to look at generic fibers of the length function, which plays a big role in the proof of non-rigidity over a stratum of flat metrics.

Corollary 1. *Let U be an arbitrary open set in $\text{Flat}(S, \alpha)$. For any closed curve $c \in \mathcal{C}(S)$ there exists an open nonempty subset $V \subset U$ such that $\ell(c, \cdot)$ is a smooth function over V with respect to holonomy coordinates.*

Chapter 3

Measured foliations

In this chapter we briefly describe measured foliations and train tracks on a surface S of finite type. For detailed and complete descriptions, see [12, 26, 31].

A *measured foliation* on S is a singular foliation on S together with a transverse measure μ of full support without atoms that is invariant under holonomy. The singularities all have negative index except possibly marked points which are allowed to have index $1/2$ (see Figure 3.1). There exist charts to \mathbb{R}^2 away from singularities so that vertical lines describe the foliation, and the transverse measure is given by $|dx|$. Two measured foliations are equivalent if they differ by an isotopy or a sequence of Whitehead move, see [12]. We use $\mathcal{MF}(S)$ to denote the set of equivalence classes of measured foliations and write $\mu \in \mathcal{MF}(S)$ with the singular foliation implicit in the notation. We should point out that an alternative way of understanding the space of measured foliations is via the space of measured laminations, see [3, 6, 21].

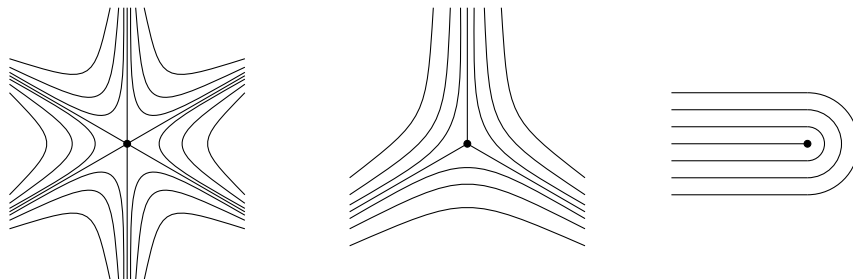


Figure 3.1: Singularities of index -2 , $-1/2$, and $1/2$.

The *geometric intersection number* $i(c_1, c_2)$ is defined for any pair homotopy classes of simple closed curves $c_1, c_2 \in \mathcal{S}(S)$ as the minimal number of intersection points among all simple representatives. For every homotopy class of closed curves and measured foliation, the geometric intersection number is the infimum over all representatives of the curve, of total variations of the measure along the representative. This is constant on equivalence classes of measured foliations, and so descends to a function on $\mathcal{MF}(S)$. Thurston defined the topology on $\mathcal{MF}(S)$ to be the weakest one for which these functions are continuous.

The geometric intersection number for curves and for measured foliations determines two maps

$$i_* : \mathcal{S}(S) \times \mathbb{R}_+ \rightarrow \mathbb{R}^{\mathcal{S}(S)} \text{ and } i_* : \mathcal{MF}(S) \rightarrow \mathbb{R}^{\mathcal{S}(S)}$$

given by

$$i_*(c, t) = \{t \cdot i(a, c)\}_{a \in \mathcal{S}(S)} \text{ and } i_*(\mu) = \{i(a, \mu)\}_{a \in \mathcal{S}(S)},$$

respectively. In [12] it is shown that i_* is an embedding and $i_*(\mathcal{S}(S) \times \mathbb{R}_+) \subset i_*(\mathcal{MF}(S))$. We use $\mathcal{PMF}(S)$ to denote the set of equivalence classes of projective measured foliations, where we projectivize $\mathcal{MF}(S)$ in the sense of $\mathbb{R}^{\mathcal{S}(S)}$.

The space of measured foliations $\mathcal{MF}(S)$ is the closure of the space of weighted simple closed curves $\mathcal{S}(S) \times \mathbb{R}_+$ under the topology introduced by Thurston. Therefore the space of projective measured foliations $\mathcal{PMF}(S)$ is the natural closure of the space of simple closed curves $\mathcal{S}(S)$. Also $\mathcal{PMF}(S)$ can be seen as the boundary of Teichmüller space $\mathcal{T}(S)$, showing that the closure of $\mathcal{T}(S)$ is homeomorphic to a closed ball of dimension $(6g + 2n - 6)$, see [12, 20]. These facts further motivate the study of the space of measured foliations.

3.1 Properties

The main focus of this section will be relating measured foliations with quadratic differentials and flat metrics. A quadratic differential on S induces a *vertical measured foliation* on S . The vertical measured foliation is obtained by foliating \mathbb{R}^2 with vertical lines and we use the coordinates from the semi-translation structure to obtain a measured foliation on the surface S . Figure 3.2 shows the induced vertical measured foliation both using polygons-representative and the surface (the foliation on the genus 2 surface is complicated).

We provide a geometric description of the geometric intersection number between a closed curve and a vertical measured foliation. Let μ be the vertical measured foliation of the quadratic differential q . For a closed curve $c \in \mathcal{C}(S)$, the intersection number $i(\mu, c)$ is the minimum horizontal distance traveled while following representatives of c . The minimum is achieved by geodesic representatives of c and hence $i(\mu, c)$ can be computed by finding saddle connections that form a geodesic representative of c . For example, the curves in Figure 1.4 intersected with the corresponding vertical measured foliations have intersection number being the width of the polygon.

Let $\varphi : \text{QD}(S) \rightarrow \mathcal{MF}(S)$ denote the map that sends a quadratic differential q to the vertical measured foliation. The map φ is in fact surjective and the fibers are sections of the projection to Teichmüller space,

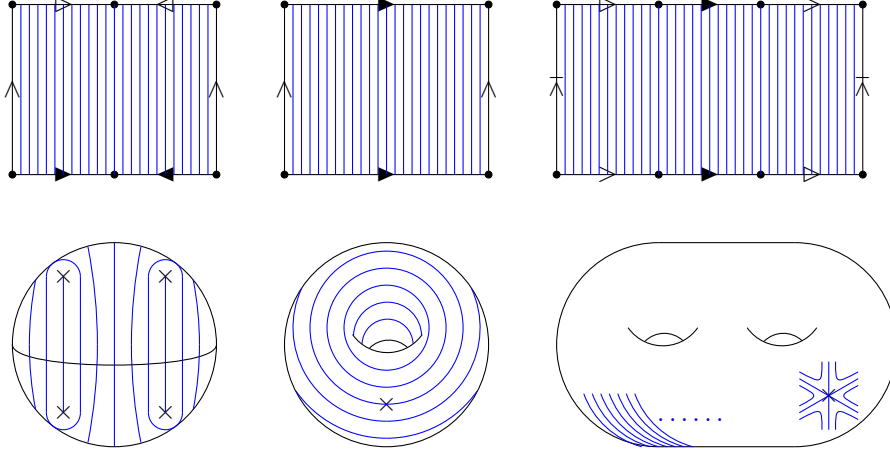


Figure 3.2: Vertical measured foliations induced by quadratic differentials.

see [17]. We can associate a circle of measured foliations to $\rho \in \text{Flat}(S)$ induced by $q \in \text{QD}(S)$ by considering the circle of all unit area quadratic differentials that induce ρ . Define the function

$$\nu_q : [0, \pi) \rightarrow \mathcal{MF}(S)$$

where

$$\nu_q(\theta) = \varphi(e^{2i\theta}q).$$

Because of how the semi-translation structure is induced by q , $e^{2i\theta}q$ corresponds to rotating the polygons in a polygons-representative of q by an angle of θ . Notice that the vertical measured foliation of the polygons rotated by an angle of π is the same as the unrotated vertical measured foliation. A key ingredient of our proof of the main theorem is [9, Lemma 9]. The lemma provides a formula to compute the length of a closed curve with respect to $\rho \in \text{Flat}(S)$ using the circle of measured foliations associated to ρ . We prove the lemma here for completeness.

Lemma 1. *For all $\rho \in \text{Flat}(S)$ induced by $q \in \text{QD}(S)$ and $c \in \mathcal{C}(S)$, we have*

$$\ell(c, \rho) = \frac{1}{2} \int_0^\pi i(\nu_q(\theta), c) d\theta.$$

Proof. Any saddle connection with respect to q can be represented by a complex number z , up to a multiple of ± 1 , so that in coordinates of the semi-translation structure, the saddle connection can be developed to a segment from 0 to z . Let s_1, \dots, s_m be saddle connections represented by $z_1, \dots, z_m \in \mathbb{C}$ that form a geodesic representative of c . The geometric description of intersection numbers with vertical measured foliations tells

us that

$$i(\nu_q(\theta), c) = i(\varphi(e^{2i\theta}q), c) = \sum_{j=1}^m |\operatorname{Re}(e^{i\theta} z_j)|.$$

A simple computation shows that

$$\int_0^\pi (|\operatorname{Re}(e^{i\theta} z_j)|) d\theta = \int_0^\pi |z_j| |\cos(\theta)| d\theta = 2|z_j|.$$

Hence

$$\ell(c, \rho) = \sum_{j=1}^m |z_j| = \frac{1}{2} \int_0^\pi \left(\sum_{j=1}^m |\operatorname{Re}(e^{i\theta} z_j)| \right) d\theta = \frac{1}{2} \int_0^\pi i(\nu_q(\theta), c) d\theta,$$

since the length of the closed curve c is the sum of the lengths of the saddle connections making up a geodesic representative. \square

3.2 Train track coordinates

One way to provide a local description of $\mathcal{MF}(S)$ is via train tracks. We follow [26] regarding this aspect of train tracks. In this thesis, a train track τ on S is an embedded trivalent graph with the following extra structure. The edges (called branches) of τ are smoothly embedded and at each vertex (called a switch) of τ , all adjacent branches have the same tangent line L . Furthermore, we require that there is at least one branch on each side of the switch and no component of $S \setminus \tau$ is a nullgon, an unmarked monogon, or an unmarked bigon. Figure 3.3 show some train tracks.

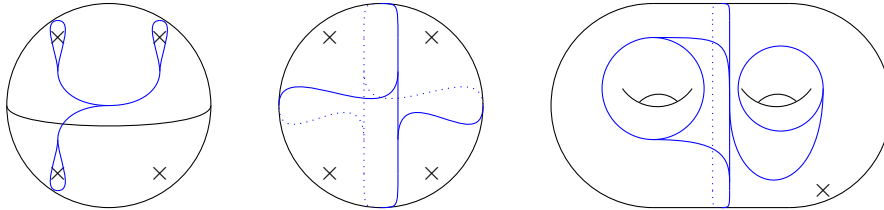


Figure 3.3: Examples of train tracks.

We say that a simple closed curve c is *carried* by a train track τ if there exists a differentiable map $f : S \rightarrow S$ homotopic to the identity rel marked points and $f|_c$ is an immersion to τ . The definition extends to measured foliations, as follows. A train track τ *carries* a measured foliation $\mu \in \mathcal{MF}(S)$ if there is a differentiable map $f : S \setminus C_\mu \rightarrow S$, where C_μ is the set of singularities of μ , homotopic to the identity, while immersing every leaf of μ to τ . Let $\mathcal{MF}_\tau(S)$ denote the set of measured foliations on S carried by τ .

A *weight function* on a train track τ is an assignment of a nonnegative real number to each branch in

such a way that the numbers satisfy the *switch condition* at each switch: the sum of the weights on incoming branches equals the sum on outgoing branches. Let W_τ denote the set of weight functions on τ . We call a function $w \in W_\tau$ a weight function. A simple closed curve c or a measured foliation μ carried by a train track τ determines a weight function w_c or w_μ , respectively, on τ .

Two simple closed curves c_1 and c_2 meet *efficiently* if they meet transversely and there are no unmarked bigon components in $S \setminus (c_1 \cup c_2)$. In this case, $|c_1 \cap c_2|$ is minimal among all simple closed curves c'_1, c'_2 homotopic to c_1, c_2 , respectively. This number is the geometric intersection number $i(c_1, c_2)$.

One can similarly define efficient intersection for a curve and a train track, or two train tracks; see [26]. When a curve c and train track τ intersect efficiently, the *geometric intersection number* between c and a weight function $w \in W_\tau$, denoted $i(c, w)$, is the sum of weights of branches over all intersection points of c and τ . If c and τ meet efficiently, and $w_{c'}$ is the weight coming from a simple closed curve c' , then $i(c, w_{c'}) = i(c, c')$. If τ and τ' meet efficiently and $w \in W_\tau, w' \in W_{\tau'}$, then $i(w, w')$ is defined as a weighted sum over all intersection points of branches. Moreover, if these weight functions correspond to curves or measured foliations, then this is the geometric intersection number of the associated objects.

We will construct a pair of train tracks τ and τ' meeting efficiently on a surface S and use them throughout the rest of the thesis. To describe these, fix a set of *pants curves*, i.e., a maximal set of pairwise disjoint, pairwise non-homotopic simple closed curves on S . Let τ be a train track that contains the set of pants curves, along with the branches shown in Figure 3.4 on each pair of pants.

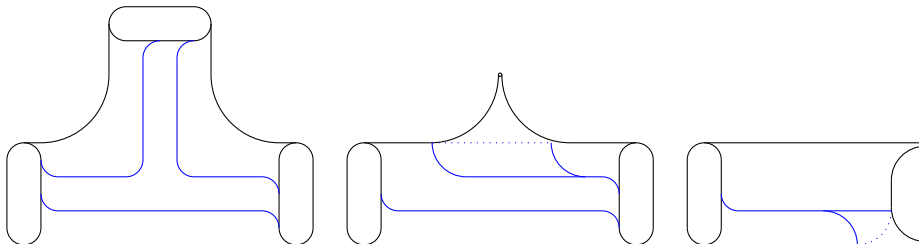


Figure 3.4: The train track τ on pairs of pants.

The train track τ has the property that every branch that is going toward a pants curve is turning to its right. We similarly construct τ' using the same pants curves, but using left turns going toward any pants curve; see Figure 3.5.

The complement of τ (and likewise τ') consists of trigons and/or marked monogons. The pair τ and τ' are constructed to be *maximal standard* train tracks as in [26]. By construction, τ and τ' satisfy the following properties.

1. By applying a suitable isotopy, we can assume that τ and τ' meet efficiently.

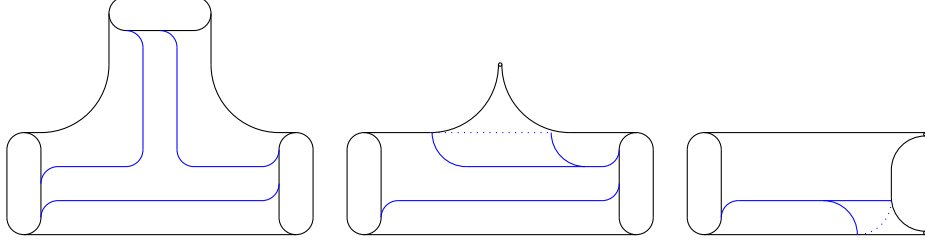


Figure 3.5: The train track τ' on pairs of pants.

2. By [26, Section 1.7], $\mathcal{MF}_\tau(S)$ and $\mathcal{MF}_{\tau'}(S)$ are both nonempty open sets homeomorphic to W_τ and $W_{\tau'}$, respectively. The homeomorphism is given by sending μ to the weight vector w_μ defined by the carrying.

The homeomorphism between $\mathcal{MF}_\tau(S)$ and W_τ is used in the key step of the proof of Theorem 3. In the next proposition, we prove that there exists a set σ of simple closed curves such that the intersection function

$$i(\sigma, \cdot) : W_\tau \rightarrow \mathbb{R}^\sigma, \text{ where } i(\sigma, w) = (i(c, w))_{c \in \sigma},$$

will provide a global coordinate for W_τ . The dimension of $\mathcal{MF}_\tau(S)$ is equal to $(6g + 2n - 6)$, which will be the same as the size of σ .

Proposition 3. *Let τ be the train track constructed above. Then there exists a set of $(6g + 2n - 6)$ simple closed curves σ such that $i(\sigma, \cdot) : W_\tau \rightarrow \mathbb{R}_{\geq 0}^\sigma$ is an injective linear map. Consequently, there exists a linear inverse $A : \text{Image}(i(\sigma, \cdot)) \rightarrow W_\tau$.*

Proof. We will find σ so that for any $w \in W_\tau$ we can solve for the weight assigned to each branch by the vector $i(\sigma, w)$. The first $(3g + n - 3)$ curves of our set σ will be the pants curves, which we denote σ_P . To determine the weights on the branches interior to the pairs of pants, we divide the analysis into three cases according to Figure 3.4.

Suppose $c_1, c_2, c_3 \in \sigma_P$ are three pants curves bounding a single pair of pants without any marked points. We can see that $i(c_j, w)$ is the sum of the weights on two of the branches interior to the pair of pants. Therefore by using the three pants curves, we obtain the weights on three branches to be of the form

$$\frac{1}{2}(i(c_{j_1}, w) + i(c_{j_2}, w) - i(c_{j_3}, w)) \text{ where } \{j_1, j_2, j_3\} = \{1, 2, 3\}.$$

For the case of a pair of pants with one marked point, we only get two pants curves c_1 and c_2 . Let us assume that in Figure 3.4, c_1 is on the left and c_2 is on the right. We see that $i(c_1, w)$ is equal to the

weight on the branch that connects the two pants curves. The weight on the branch that wrapped around the marked point is half the weight of the branch connecting it to c_2 , which is

$$i(c_2, w) - i(c_1, w).$$

The last case is a pair of pants with two marked points which has only one pants curve c_1 . The weight on the branch that wrapped around the marked point is half the weight of the branch connecting it to c_1 , which is just $i(c_1, w)$. Therefore with $(3g + n - 3)$ curves we can find the weights of w on the branches interior to each pair of pants.

Now we need to find another set of $(3g + n - 3)$ simple closed curves to determine the weights on the branches of τ that are contained in the pants curves. We take a transverse simple closed curve for each pants curve, disjoint from any other pants curves, meeting τ efficiently; see Figure 3.6 for the situation when the pairs of pants contain no marked points.

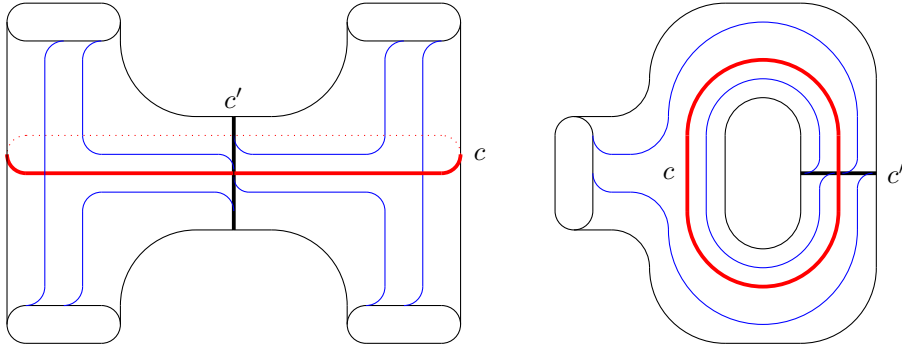


Figure 3.6: Examples of transverse curves.

Let c be such a transverse simple closed curve for the pants curve c' . We can use $i(c, w)$ to find the weights on branches on the pants curve, which are all expressed by $i(c, w)$ combined with constants determined by $i(\sigma_P, w)$. In Figure 3.7, w_5, \dots, w_{10} had been determined by σ_P , and the equations given by switch conditions allow us to solve for w_1, \dots, w_4 from these and $i(c, w)$. Explicitly we have

$$\begin{aligned} w_1 &= \frac{1}{2}(i(c, w) + w_5 - w_6 - w_9 - w_{10}), \\ w_2 &= \frac{1}{2}(i(c, w) - w_5 - w_6 - w_9 - w_{10}), \\ w_3 &= \frac{1}{2}(i(c, w) - w_5 + w_6 - w_9 - w_{10}), \\ w_4 &= \frac{1}{2}(i(c, w) - w_7 - w_8 - w_9 - w_{10}). \end{aligned}$$

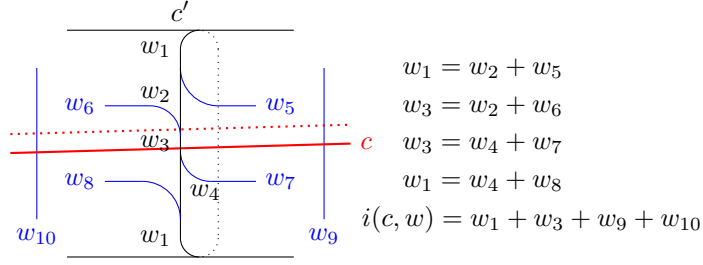


Figure 3.7: Explicit weights and equations near a pants curve.

This is true in general. For any transverse simple closed curve c of the pants curve c' , let w' be the weight on a branch on c' , then

$$w' = \frac{1}{i(c, c')} (i(c, w) + L(i(\sigma_P, w))),$$

where $i(c, c') = 1$ or 2 is the geometric intersection number and L is a linear function from \mathbb{R}^{3g+n-3} to \mathbb{R} . This completes the proof of injectivity since $w \in W_\tau$ is uniquely determined by the intersection of w with the set of $(6g + 2n - 6)$ simple closed curves.

The set of $(3g + n - 3)$ transverse curves σ_T union with the $(3g + n - 3)$ pants curves σ_P will be our σ . We make a further remark that if W_τ is seen as $\mathbb{R}^{e(\tau)}$ where $e(\tau)$ is the number of branches of τ , then there exists an $e(\tau)$ by $(6g + 2n - 6)$ matrix A that maps $i(\sigma, w)$ to $w \in W_\tau$. \square

3.3 Actions of the mapping class group

In this section we consider the action of the mapping class group $\text{Mod}(S)$. Recall that $\text{Mod}(S)$ is the group of isotopy classes of orientation-preserving diffeomorphisms of S . The mapping class group acts naturally on the Teichmüller space, which induces actions on the space of quadratic differentials, the space of Abelian differentials, and the space of flat metrics. Because the action is on the underlying complex structure, the semi-translation structure and the stratification are preserved under mapping class group actions.

The mapping class group action on the space of measured foliations can be understood in two ways. Similar to the action on homotopy classes of closed curves by a representative diffeomorphism, a mapping class element acts on a measured foliation mapping the singular foliation directly and define the measure by pulling back transversals. Alternatively, Thurston's compactification of the Teichmüller space describes how mapping class elements act on the space of projective measured foliations, seen as the boundary of Teichmüller space. From the second perspective, we have the following observation.

Pseudo-Anosov elements of the mapping class group act with *north-south dynamics* on $\mathcal{PMF}(S)$. That

is, there is a single attracting and a single repelling fixed point, and on the complement of the latter, iteration of the pseudo-Anosov element converges uniformly on compact sets to the former. Figure 3.8 illustrates the dynamics.

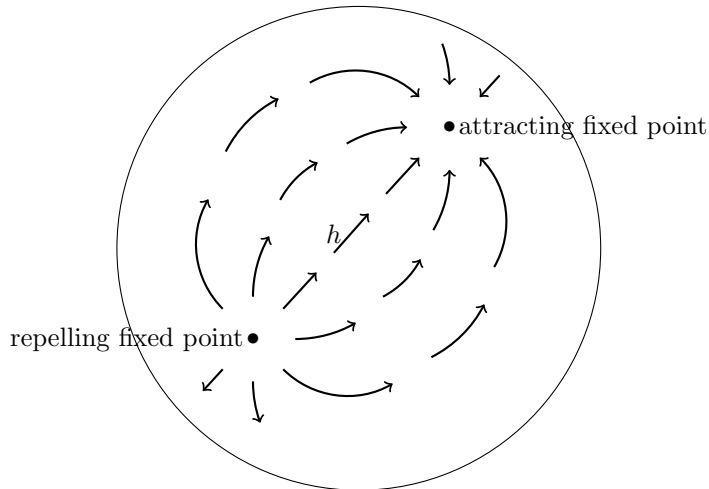


Figure 3.8: Visualizing the north-south dynamics on $\mathcal{PMF}(S)$.

In [22] it is shown that the set of attracting/repelling pairs for pseudo-Anosov elements are dense in $\mathcal{PMF}(S) \times \mathcal{PMF}(S)$. Therefore we have the following proposition.

Proposition 4. *Let τ be a train track that carries an open set in $\mathcal{PMF}(S)$. If a subset $K \subset \mathcal{PMF}(S)$ is disjoint from some open set in $\mathcal{PMF}(S)$, then there exists a mapping class h such that $K \subset h(\mathcal{PMF}_\tau(S))$.*

In particular, the train tracks τ and τ' constructed in Section 3.2 satisfy the condition of the proposition. The mapping class group action is well-defined on isotopy classes of train tracks. For $h \in \text{Mod}(S)$, if we abuse notation and let h be a representative diffeomorphism, then the set $\mathcal{PMF}_{h(\tau)}(S)$ is equal to $h \cdot \mathcal{PMF}_\tau(S)$.

The mapping class group action plays a critical role in the proof of Theorem 3. The dynamics of the actions of $\text{Mod}(S)$ provide flexibility to fit the ingredients together in the proof. Readers are referred to [10] for a more complete treatment of the mapping class group.

Chapter 4

Rigidity of length functions

Recall that a set of closed curves $\Sigma \subset \mathcal{C}(S)$ is length spectrally rigid over $\mathcal{G}(S)$ if the length function λ_Σ is injective. The motivation for studying rigidity comes from a simple question: can we determine the metric (up to isotopy, from within a family of metrics) from the length data? This question is classical and it has been answered for several classes of metrics, see below.

The notion of spectral rigidity originates from the study of eigenvalues of the Laplacian. The famous question asked by Kac in 1966 [19] of whether one can “hear the shape of a drum” served as the beginning of a rich history on spectral problems. The corresponding problem with lengths functions looks at the length spectrum, which is the nondecreasing sequence of numbers in the image of $\lambda_{\mathcal{C}(S)}$. Sunada’s work in the 1980’s ([30]) showed non-rigidity results in the case of *unmarked length spectrum* with respect to hyperbolic metrics on surfaces. In this Chapter, we consider the rigidity of the length functions, that is, keeping track of both the length spectrum and the corresponding closed curves.

A specific property that we want to study beyond rigidity is whether we can classify rigid sets of closed curves Σ . Whenever a set Σ is shown to be length spectrally rigid over $\mathcal{G}(S)$, we ask whether Σ contains a finite subset that remains length spectrally rigid. On the other hand, whenever a set Σ is shown to be non-rigid over $\mathcal{G}(S)$, we study *iso-length-spectral* subsets of $\mathcal{G}(S)$, which are fibers of the length function λ_Σ . It turns out that this classification is a hard problem that remains open in most cases. One of the main complications is the action of the mapping class group $\text{Mod}(S)$, reinforcing the importance of the study of $\text{Mod}(S)$.

4.1 Rigidity results

We begin with choices of $\mathcal{G}(S)$ such that the set of all closed curves $\mathcal{C}(S)$ is length spectrally rigid over $\mathcal{G}(S)$. By classical work of Klein and Fricke, The set $\mathcal{C}(S)$ is length spectrally rigid over the Teichmüller space $\mathcal{T}(S)$ seen as the space of complete finite-area hyperbolic (constant curvature -1) metrics on S . In fact, there exists a set of $(6g + 2n - 5)$ simple closed curves Σ that is length spectrally rigid over $\mathcal{T}(S)$. A weaker

version of this fact is the well-known $(9g - 9)$ theorem which can be found in [10].

Otal in 1990 ([25]) showed that $\mathcal{C}(S)$ is length spectrally rigid over the space of all negatively curved metrics on S . The proof used the space of geodesic currents introduced by Bonahon in [1, 2] and it was generalized by Hersch and Paulin in [16] to show that $\mathcal{C}(S)$ is length spectrally rigid over the space of negatively curved cone metrics on S . More length spectral rigidity results for $\mathcal{C}(S)$ for various qualities of nonpositively curved Riemannian metrics can be found in works of Croke [8], Fathi [11], and Croke-Fathi-Feldman [7] (see the references for more precise statements). In his thesis [13], Frazier also suggested further evidence that $\mathcal{C}(S)$ should be length spectrally rigid over the space of nonpositively curved metrics on S .

In the rest of this section we consider $\mathcal{G}(S) = \text{Flat}(S)$ and set of simple closed curves $\Sigma \subset \mathcal{S}(S)$. The case when $(3g + n - 3) = 1$ ($S_{0,4}$ and $S_{1,1}$) has a rigidity result similar to $\mathcal{T}(S)$. The following proposition is from [9, Proposition 17].

Proposition 5. *Let $(3g + n - 3) = 1$. If $\Sigma \subset \mathcal{S}(S)$ is a set of three distinct simple closed curves, then Σ is length spectrally rigid over $\text{Flat}(S)$.*

Proof. The space of flat metrics $\text{Flat}(S)$ in these cases can be identified with the hyperbolic plane. In the identification, the level sets of the length of a simple closed curve is a horocycle. The intersection of two distinct horocycles is at most two points and hence three distinct simple closed curves must be length spectrally rigid. \square

The general case of $(3g + n - 3) \geq 2$ has a different type of rigidity behavior. The set of all simple closed curves $\mathcal{S}(S)$ is length spectrally rigid over $\text{Flat}(S)$. However, instead of a finite set of simple closed curves being length spectrally rigid over $\text{Flat}(S)$, Duchin, Leininger, and Rafi prove that the only length spectrally rigid sets are the ones that is dense in $\mathcal{S}(S)$. We sketch the main ideas of the proof of Theorem 1 in [9].

Theorem 1. *Let $(3g + n - 3) \geq 2$. A set of simple closed curves $\Sigma \subset \mathcal{S}(S)$ is length spectrally rigid over $\text{Flat}(S)$ if and only if $\overline{\Sigma} = \mathcal{PMF}(S)$.*

Sketch of proof. The proof has two parts. The first part proves that $\mathcal{S}(S)$ is length spectrally rigid over $\text{Flat}(S)$, which automatically imply one direction. The other direction is proved by constructing deformation families of constant length spectrum for every set Σ that is not dense in $\mathcal{PMF}(S)$. We will provide more details for the first part and only provide a general idea for the second part since Section 4.2 contains an alternative proof.

Let $\rho, \rho' \in \text{Flat}(S)$ be such that $\lambda_{\mathcal{S}(S)}(\rho) = \lambda_{\mathcal{S}(S)}(\rho')$. The first step is to prove that $\text{Cyl}(\rho) = \text{Cyl}(\rho')$. This follows immediately with two lemmas. See [9] for details of their proof. We write T_c for the Dehn twist in $c \in \mathcal{S}(S)$.

Lemma 2 ([9] Lemma 19). *For $c_1 \in \text{Cyl}(\rho)$ and any curve $c_2 \in \mathcal{S}(S)$ with $i(c_1, c_2) \neq 0$,*

$$\ell(T_{c_1}(c_2), \rho) - \ell(c_2, \rho) < \ell(c_1) \cdot i(c_1, c_2).$$

The idea behind this lemma is simple. When Dehn twisting along a cylinder curve, the resulting geodesic representative will never make a “sharp turn” in the middle of the cylinder, but will always follow a shorter hypotenuse.

Lemma 3 ([9] Lemma 20). *If $c_1 \notin \text{Cyl}(\rho)$, then there exists $c_2 \in \mathcal{S}(S)$ with $i(c_1, c_2) \neq 0$ so that*

$$\ell(T_{c_1}(c_2), \rho) - \ell(c_2, \rho) = \ell(c_1) \cdot i(c_1, c_2).$$

If we start with any simple closed curve that intersects c_1 and apply a sufficiently large number of Dehn twists in c_1 , then we obtain a simple closed curve whose geodesic representative share saddle connections with the geodesic representative of c_1 in a specific fashion. The geodesic meets the geodesic representative of c_1 at a singularity and starts to follow along. Eventually the geodesic leaves c_1 on “the other side”. This description guarantees the equality in the lemma.

We have shown that $\text{Cyl}(\rho) = \text{Cyl}(\rho')$. The next lemma is more involved, using results in [15, 23], and we state it without proof.

Lemma 4 ([9] Lemma 22). *If $\text{Cyl}(\rho) = \text{Cyl}(\rho')$, then $\mathbb{H}_\rho = \mathbb{H}_{\rho'}$.*

The level sets of the length of a cylinder curve is a horocycle on \mathbb{H}_ρ . Hence three distinct cylinder curves provide evidence that $\rho = \rho'$ similar to the proof of Proposition 5.

The second part of the proof is a construction of “deformable magnetic train tracks”. The idea is to cut the surface into building blocks and place train tracks correspondingly. Each train track has the property that the branches are assigned lengths and simple closed curves carried by the train track would have a geodesic representative that stays on the train track. Furthermore, there exists an explicit description of how to deform the flat metric while preserving lengths of simple closed curves carried by the train track. Finally, we use the action of the mapping class group so that the set Σ is carried by the train track. See [9, Section 4] for details. □

4.2 Non-rigidity over strata of flat metrics

We will now prove our main theorem, which shows that a set Σ of simple closed curves is length spectrally rigid over a stratum $\text{Flat}(S, \alpha)$ with sufficiently high dimension if and only if $\overline{\Sigma} = \mathcal{PMF}(S)$. Our result

holds for strata $\text{Flat}(S, \alpha)$ of dimension at least half of the dimension of $\text{Flat}(S)$, which cover a large subset of strata of $\text{Flat}(S)$.

Theorem 3. *Let $(3g + n - 3) \geq 2$, $\alpha = (\alpha_1, \dots, \alpha_k; \varepsilon)$, and $(2k_0 - 2g + \varepsilon + 1) > 0$. Suppose Σ is a set of simple closed curves with $\bar{\Sigma} \neq \mathcal{PMF}(S)$. Then there exists a deformation family $\Omega_\Sigma \subset \text{Flat}(S, \alpha)$ such that the length function λ_Σ is constant on Ω_Σ and $\dim(\Omega_\Sigma) \geq (2k_0 - 2g + \varepsilon + 1)$. Consequently, there exists a deformation family $\Omega_\Sigma \subset \text{Flat}(S)$ of dimension at least $(6g + 2n - 8)$.*

Proof. We consider τ' constructed in Section 3.2. Since $\bar{\Sigma} \neq \mathcal{PMF}(S)$, the set Σ is disjoint from some open set in $\mathcal{PMF}(S)$. By Proposition 4, there exists a mapping class h_1 such that $h_1(\tau')$ is a train track that carries Σ . By replacing τ' and τ with $h_1(\tau')$ and $h_1(\tau)$, we may assume that Σ is carried by τ' .

Denote the natural projection from $\text{QD}(S, \alpha)$ to $\text{Flat}(S, \alpha)$ by

$$p : \text{QD}(S, \alpha) \rightarrow \text{Flat}(S, \alpha).$$

We consider an arbitrary $\rho \in \text{Flat}(S, \alpha)$ with $\hat{\rho} \in p^{-1}(\rho) \subset \text{QD}(S, \alpha)$. From Section 2.2 we know that there exist holonomy coordinates $U_{\hat{\rho}} \rightarrow \mathbb{C}^m$ about $\hat{\rho}$.

Recall the natural map $\varphi : \text{QD}(S) \rightarrow \mathcal{MF}(S)$ and the circle of measured foliations $\text{Image}(\nu_{\hat{\rho}})$ associated to ρ as described in Section 3.1. Let K be the union of $\nu_q(\theta)$ over all $\theta \in [0, \pi)$ and $q \in U_{\hat{\rho}}$.

Observe that by taking $U_{\hat{\rho}}$ sufficiently small, K is a small neighborhood of the circle $\nu_{\hat{\rho}}([0, \pi))$. In particular K can be assumed disjoint from some open set when projected to $\mathcal{PMF}(S)$. By Proposition 4, there exists a mapping class h_2 such that the train track $h_2(\tau)$ carries K . Thus, by replacing ρ with $h_2^{-1}(\rho)$, we may assume that K is carried by τ .

Use $\mathcal{MF}^{[0, \pi)}(S)$ to denote the set of functions that map $[0, \pi)$ into $\mathcal{MF}(S)$. We define

$$f_1 : U_{\hat{\rho}} \rightarrow \mathcal{MF}^{[0, \pi)}(S)$$

by

$$f_1(q) = \nu_q.$$

Let F_1 be the image of f_1 .

We will consider the space W_τ , which is the space of weight functions on τ as described in Section 3.2.

Let

$$\psi : \mathcal{MF}_\tau(S) \rightarrow W_\tau$$

be the homeomorphism between $\mathcal{MF}_\tau(S)$ and W_τ .

Let $\sigma = \{\sigma_1, \dots, \sigma_{6g+2n-6}\}$ be the set of curves from Proposition 3 and write $(\mathbb{R}^\sigma)^{[0, \pi]}$ for the set of functions that map $[0, \pi)$ into \mathbb{R}^σ . We define

$$f_2 : F_1 \rightarrow (\mathbb{R}^\sigma)^{[0, \pi]}$$

by

$$f_2(\nu_q)(\theta) = \{i(\sigma_j, \nu_q(\theta))\}_{\sigma_j \in \sigma}.$$

Let F_2 be the image of f_2 .

Next we let $(W_\tau)^{[0, \pi]}$ be the space of functions that map $[0, \pi)$ into W_τ and define

$$f_3 : F_2 \rightarrow (W_\tau)^{[0, \pi]}$$

by

$$f_3(h)(\theta) = A(h(\theta))$$

for all $h \in F_2$ and $\theta \in [0, \pi)$, where A is the linear map in Proposition 3. Equivalently, this is determined by

$$(f_3 \circ f_2)(\nu_q)(\theta) = \psi(\nu_q(\theta)).$$

Let F_3 be the image of f_3 .

For any $q \in U_{\hat{\rho}}$, the function $(f_3 \circ f_2 \circ f_1)(q)(\theta)$ is uniformly continuous in $\theta \in [0, \pi)$. We define

$$f_4 : F_3 \rightarrow W_\tau$$

by applying the integral $\frac{1}{2} \int_0^\pi \cdot d\theta$ to functions in F_3 . We will let

$$f = f_4 \circ f_3 \circ f_2 \circ f_1 : U_{\hat{\rho}} \rightarrow W_\tau.$$

Combining everything together we have the diagram below.

$$\begin{array}{ccccccc} U_{\hat{\rho}} & \xrightarrow{f_1} & F_1 & \xrightarrow{f_2} & F_2 & \xrightarrow{f_3} & F_3 & \xrightarrow{f_4} & W_\tau \\ & & \cap & & \cap & & \cap & & \\ & & \mathcal{MF}^{[0, \pi]}(S) & & (\mathbb{R}^\sigma)^{[0, \pi]} & & (W_\tau)^{[0, \pi]} & & \end{array}$$

If we define

$$f_5 : F_2 \rightarrow \mathbb{R}^\sigma$$

by applying the integral $\frac{1}{2} \int_0^\pi \cdot d\theta$ to functions in F_2 , then by Lemma 1 we have

$$(\lambda_\sigma \circ p)(q) = \left\{ \frac{1}{2} \int_0^\pi i(\sigma_j, \nu_q(\theta)) d\theta \right\}_{\sigma_j \in \sigma} = (f_5 \circ f_2 \circ f_1)(q),$$

where λ_σ is the length function and p is the projection from quadratic differentials to flat metrics. We obtain the following commutative diagram.

$$\begin{array}{ccccccc} U_{\hat{\rho}} & \xrightarrow{f_1} & F_1 & \xrightarrow{f_2} & F_2 & \xrightarrow{f_3} & F_3 \\ & \searrow & & & \downarrow f_5 & & \downarrow f_4 \\ & & & & \mathbb{R}^\sigma & \xrightarrow{A} & W_\tau \end{array}$$

Therefore, we have $f = A \circ \lambda_\sigma \circ p$, where A is the linear map in Proposition 3.

Recall that τ and τ' meet efficiently. For every $c \in \mathcal{S}(S)$ carried by τ' and $\mu \in \mathcal{MF}(S)$ carried by τ , $i(c, \mu) = i(c, w_\mu)$ where w_μ is the weight on τ determined by μ . Appealing to Lemma 1, we have

$$\begin{aligned} \ell(\gamma, p(q)) &= \frac{1}{2} \int_0^\pi i(\gamma, \nu_q(\theta)) d\theta \\ &= \frac{1}{2} \int_0^\pi i(\gamma, w_{\nu_q(\theta)}) d\theta \\ &= \frac{1}{2} \int_0^\pi i(\gamma, (f_3 \circ f_2 \circ f_1)(q)) d\theta \end{aligned}$$

for any $\gamma \in \Sigma$ and $q \in U_{\hat{\rho}}$. We use the fact that the intersection number of γ with any w_μ is linear in W_τ .

Hence

$$\begin{aligned} \ell(\gamma, p(q)) &= \frac{1}{2} \int_0^\pi i(\gamma, (f_3 \circ f_2 \circ f_1)(q)) d\theta \\ &= i(\gamma, (f_4 \circ f_3 \circ f_2 \circ f_1)(q)) \\ &= i(\gamma, f(q)). \end{aligned}$$

Therefore the length of $\gamma \in \Sigma$ with respect to any flat metric induced by $q \in U_{\hat{\rho}}$ is equal to the intersection number $i(\gamma, f(q))$.

From Proposition 1 we know that $U_{\hat{\rho}}$ is $(4g + 2k + \varepsilon - 3)$ -dimensional, where $k = k_0 + n$ is the sum of

the number of unmarked zeroes and the number of marked points. From Section 3.2 we know that W_τ is a $(6g + 2n - 6)$ -dimensional space. Applying Corollary 1 to each $\sigma_j \in \sigma$, it follows that there exists an open set $V \subset p(U_{\hat{\rho}})$ so that $\lambda_\sigma|_V$ is a smooth function with respect to holonomy coordinates. Since A is linear, we conclude that $f|_{p^{-1}(V)}$ is a smooth function. This implies that the generic fiber $\Omega \subset U_{\hat{\rho}}$ has dimension at least $(2k_0 - 2g + \varepsilon + 3)$. On Ω , f is constant.

Let $\gamma \in \Sigma$, $\zeta \in p(\Omega)$, and $\hat{\zeta} \in \Omega$ be a preimage of ζ . The length of γ with respect to ζ is equal to $i(f(\hat{\zeta}), \gamma)$. The length of γ with respect to metrics in $p(\Omega)$ is independent of the metric since $f|_\Omega$ is a constant function. Therefore $p(\Omega) \subset \text{Flat}(S, \alpha)$ is a deformation family of constant Σ -length-spectrum. The dimension of $p(\Omega)$ is at least $\dim(\Omega) - 2$, which is $(2k_0 - 2g + \varepsilon + 1)$.

For the case of $\text{Flat}(S)$, we maximize the dimension of the deformation family by maximizing k_0 . Consider the stratum where $\alpha_1 = \dots = \alpha_{k_0} = 1$ and $\alpha_{k_0+1} = \dots = \alpha_k = -1$. The value of k_0 is equal to $(4g + n - 4)$. Consequently, there exists a deformation family of dimension at least $(6g + 2n - 8)$. \square

The natural question to ask is what happens in strata of lower dimensions. A relevant open question whose answer would be a great tool is to provide a description of $\nu_q(\theta)$ from descriptions of $\nu_q(0)$ and $\nu_q(\pi/2)$. By the work of Gardiner and Masur in [15] we know that $\nu_q(\theta)$ is uniquely determined, but no description is available. A better understanding of $\nu_q(\theta)$ allows for a more explicit expression of the circles of measured foliations and the averaging map in the proof, therefore it will give us better leverage to the injectivity of the length functions.

Chapter 5

Local rigidity of length functions

Recall that a set of closed curves Σ is locally length spectrally rigid at ρ over $\mathcal{G}(S)$ if there exists a neighborhood $U_\rho \subset \mathcal{G}(S)$ of ρ such that the length function $\lambda_\Sigma|_{U_\rho}$ is injective. The local rigidity problem is equivalent to asking whether a local coordinate system of $\mathcal{G}(S)$ can be explicitly given by lengths of closed curves. The interest in the local problem is motivated by the fact that $\text{Flat}(S)$ is finite-dimensional. However, it turns out that local coordinates of $\text{Flat}(S)$ is transparent only in the largest stratum. The situation of a cone point splitting into two is the mysterious case where behaviors of length functions become hard to analyze. In this Chapter we show that the local rigidity problem over a stratum of flat metrics $\text{Flat}(S, \alpha)$ can be answered in a straightforward fashion.

Similar to the case of rigidity, we also ask whether a finite set of closed curves can be locally length spectrally rigid. If such a finite set exists, it is natural to ask for minimal sets that are locally length spectrally rigid. In the local picture, the size of a minimal set has a natural lower bound, which is the dimension of $\mathcal{G}(S)$. Hence whether an “optimal” locally rigid set exists for every point in $\mathcal{G}(S)$ is of interest. Another interesting direction considers the “maximal” neighborhood such that injectivity still holds. These properties illustrate further directions in the study of local rigidity.

5.1 Local rigidity results

Any set of closed curves that is length spectrally rigid over $\mathcal{G}(S)$ is also locally length spectrally rigid at every point in $\mathcal{G}(S)$. Therefore it is natural to take results discussed in Section 4.1 and ask if a finite number of closed curves suffice for local rigidity.

In the case when $\mathcal{G}(S) = \mathcal{T}(S)$, an optimal locally rigid set of simple closed curves exists for every hyperbolic metric in $\mathcal{T}(S)$. For example, let Σ be the union of a set of pants curves and a set of transverse curves, one for each pants curve, as constructed in Section 3.2. The Fenchel-Nielsen coordinate show that the preimage of the Σ -length-spectrum is a discrete set in $\mathcal{T}(S)$. The set Σ is locally length spectrally rigid at $\rho \in \mathcal{T}(S)$ if and only if the preimage of the Σ -length-spectrum is a discrete set of size $2^{(3g+n-3)}$ in

$\mathcal{T}(S)$. Equivalently, ρ is not a critical point of the length function of any transverse curve along the twisting deformation of the pants curve it intersects. This condition can be satisfied by replacing the transverse curves by Dehn twisting the transverse curve along the pants curve sufficiently large number of times. This proof extends to a proof of the existence of a set of $(6g + 2n - 5)$ simple closed curves that is length spectrally rigid over $\mathcal{T}(S)$ by finding an extra simple closed curve whose length distinguish the discrete set.

For $\mathcal{G}(S) = \text{Flat}(S_{0,4})$ or $\text{Flat}(S_{1,1})$, any pair of distinct simple closed curves is locally length spectrally rigid, which can be seen in the proof of Proposition 5. As commented above, the case $\mathcal{G}(S) = \text{Flat}(S)$ for $(3g + n - 3) \geq 2$ also remains unknown except to choice of Σ that is dense in $\mathcal{PMF}(S)$. In the next two sections, we discuss how local coordinates of a stratum of flat metrics (or a stratum of Euclidean cone metrics) helps in constructing locally length spectrally rigid sets.

5.2 Local rigidity over strata of flat metrics

In this section we prove that for any given flat metric on S with $(3g + n - 3) \geq 2$, we can find a finite set of closed curves that satisfies local length spectral rigidity over the stratum. In fact, we will explicitly construct the locally length spectrally rigid set of closed curves Σ by describing geodesic representatives of Σ .

Theorem 4. *Let $\alpha = (\alpha_1, \dots, \alpha_k; \varepsilon)$. For any $\rho \in \text{Flat}(S, \alpha)$, there exists a set of closed curves $\Sigma \subset \mathcal{C}(S)$ such that Σ is locally length spectrally rigid at $\rho \in \text{Flat}(S, \alpha)$ and $|\Sigma| \leq 15(2g + k - 2)$.*

Proof. We take a maximal collection of saddle connections with pairwise disjoint interiors with respect to ρ . By [35], this will be a triangulation of S by Euclidean triangles with each side of each triangle being a saddle connection. Locally, the set of lengths of saddle connections determines the metric, so it suffices to find curves whose lengths determine the lengths of these saddle connections.

We will now prove a lemma that allows us to construct geodesic segments, i.e., concatenations of saddle connections that satisfy the angle condition. Let C be the set of cone points of ρ . A *direction* at $a \in C$ is a geodesic segment with initial point a . Let $\angle_a(u_1, u_2)$ denote the smaller of the pair of angles between directions u_1 and u_2 at $a \in C$.

Lemma 5. *Suppose we are given $\rho \in \text{Flat}(S, \alpha)$, an initial point $a \in C$, a terminal point $b \in C$, a direction u at a , and a direction v at b . For any $\epsilon > 0$ there exists a geodesic segment γ from a to b such that*

$$\angle_a(u, \gamma) < \epsilon \text{ and } \angle_b(v, \bar{\gamma}) < \epsilon,$$

where $\bar{\gamma}$ is γ with orientation reversed.

Proof. There exists a direction u' arbitrarily close to u such that the ray from a in u' direction is minimal, that is, the ray is dense on S . Similarly there is a minimal ray beginning at b in the a direction v' arbitrarily close to v . We can pick u' and v' such that the two rays intersect infinitely many times.

Consider the curve that starts at a and follows along the ray for a long time before hitting an intersection and following the other ray backwards for a long time before reaching b . Then the geodesic segment obtained by straightening satisfies the statement as long as we use an intersection that is far enough from both a and b . Figure 5.1 shows how the curve would look in the universal cover.

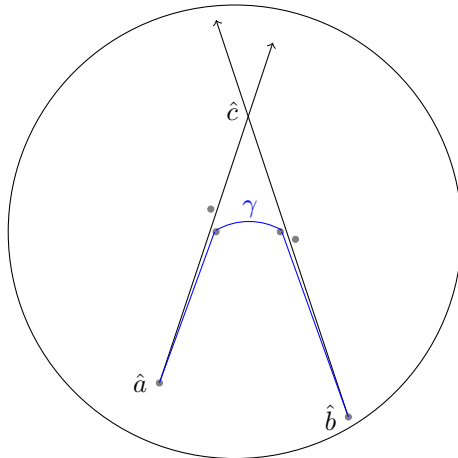


Figure 5.1: Visualizing the constructed geodesic segment in the universal cover.

On each side of the ray based at a , there will eventually be cone points at distance less than $\tan(\epsilon)$ from the ray, projecting to the ray at distance at least one from a . This means that these cone points are in a direction making angle less than ϵ with the ray. The same is true for the ray based at b . We pick an intersection point from the infinite set of intersections that is past four such cone points, i.e., so that the segments \overline{ac} and \overline{bc} have these nearby cone points on both sides.

Let \hat{a} , \hat{b} , and \hat{c} in the universal cover of S be lifts of a , b , and c on S . We consider the geodesic triangle with vertices \hat{a} , \hat{b} , and \hat{c} with the three sides being the two rays and the geodesic γ connecting \hat{a} and \hat{b} . By a Gauss-Bonnet Theorem argument (see [27, Theorem 3.3]), the geodesic triangle does not contain the lift of any point in C_0 in the interior, where C_0 is the set of unmarked zeroes. The same is true for the marked points P since γ is homotopic to the concatenation of the other two sides of the triangle.

Therefore the four cone points that are of distance less than ϵ from the rays will act as barriers as we straighten to obtain γ . Hence the resulting geodesic segment γ must satisfy the requirement:

$$\angle_a(u, \gamma) < \epsilon \text{ and } \angle_b(v, \bar{\gamma}) < \epsilon,$$

where $\bar{\gamma}$ is γ with orientation reversed. □

Now we will describe how the length of a saddle connection can be determined in various cases. Let γ_0 be an oriented saddle connection from the triangulation and let its initial and terminal points be a and b .

Case 1: $a = b$ and a is not a marked point.

This means that the saddle connection itself is a closed geodesic. We know that when γ_0 is a closed curve, the length of γ_0 with respect to ρ is well-defined. Pick an open neighborhood U_ρ of ρ small enough so that γ_0 as a closed curve is always a single saddle connection. This is possible since the condition is an open condition. Hence we add γ_0 to our set Σ . The length of γ_0 as a saddle connection is determined by the length of γ_0 as a closed curve.

Case 2: $a \neq b$ and both a and b are not marked points.

Fix a small $\epsilon > 0$. Let u_1, \dots, u_4 be directions at a and v_1, \dots, v_4 be directions at b specified by the angles made with γ_0 (measured counterclockwise) according to the following table:

direction at a	angle	vector at b	angle
u_1	$\pi + 2\epsilon$	v_1	$-\pi - 2\epsilon$
u_2	$-\pi - 2\epsilon$	v_2	$\pi + 2\epsilon$
u_3	$\pi - 4\epsilon$	v_3	$-\pi + 4\epsilon$
u_4	$-\pi + 4\epsilon$	v_4	$\pi - 4\epsilon$

We apply Lemma 5 to each pair u_j and v_j for $j = 1, 2, 3, 4$. We obtain geodesic segments γ_j for $j = 1, 2, 3, 4$. Figure 5.2 shows how everything fits together.

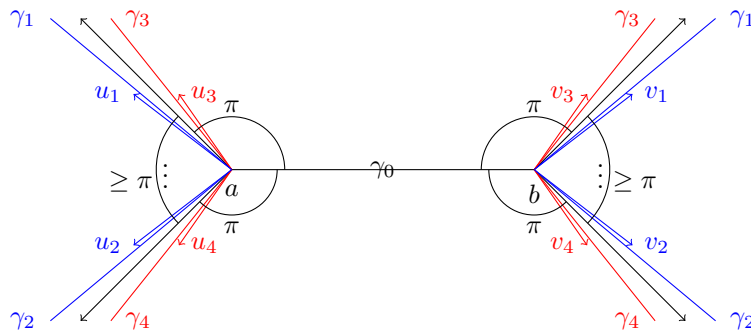


Figure 5.2: Constructing closed geodesics for a saddle connection between unmarked zeroes.

Add the following closed geodesics to Σ .

$$c_1 = \gamma_0 \cup \gamma_1, \quad c_2 = \gamma_0 \cup \gamma_2, \quad c_3 = \gamma_1 \cup \gamma_4, \quad c_4 = \gamma_2 \cup \gamma_3, \quad \text{and} \quad c_5 = \gamma_3 \cup \gamma_4.$$

They are all closed geodesics with respect to ρ because the angle condition is satisfied by construction. Furthermore, by construction the angles between saddle connections meeting at a or at b are all strictly greater than π . Hence there exists an open neighborhood U_ρ of ρ small enough so that the angle condition at a and b are strictly satisfied with respect to any ρ' in the neighborhood.

The length of γ_0 as a saddle connection is then described by

$$\frac{1}{2}[\ell(c_1, \rho') + \ell(c_2, \rho') - \ell(c_3, \rho') - \ell(c_4, \rho') + \ell(c_5, \rho')]$$

for any $\rho' \in U_\rho$ because the geodesic representatives of c_1 and c_2 both always contain γ_0 .

Case 3: Both a and b are marked points.

The argument is similar to Case 1. Consider a simple closed curve that encloses only γ_0 but no other cone points besides a and b . The geodesic representative would be $\gamma_0 \cup \overline{\gamma_0}$, where $\overline{\gamma_0}$ is γ_0 with orientation reversed. The length of γ_0 as a saddle connection is determined by the length of this closed geodesic by a factor of one half.

Case 4: Only one of a and b is a marked point.

Without loss of generality, let b be the marked point. The setting is similar to Case 2 with directions specified by the angles made with γ_0 (measured counterclockwise).

direction at a	angle
u_1	$\pi + 2\epsilon$
u_2	$-\pi - 2\epsilon$
u_3	$\pi - 4\epsilon$
u_4	$-\pi + 4\epsilon$

We apply the lemma to the pair u_1 and u_2 to obtain geodesic segment γ_1 . Use the pair u_3 and u_4 to obtain γ_2 as in Figure 5.3.

Add the following closed geodesics to Σ .

$$c_1 = \gamma_0 \cup \gamma_1 \cup \overline{\gamma_0}, \quad c_2 = \gamma_1 \cup \gamma_2, \quad \text{and} \quad c_3 = \gamma_2.$$

Once again we can find U_ρ where the angle condition at a and at b are both strict. By a similar argument, the length of γ_0 is described by

$$\frac{1}{2}[\ell(c_1, \rho') - \ell(c_2, \rho') + \ell(c_3, \rho')]$$

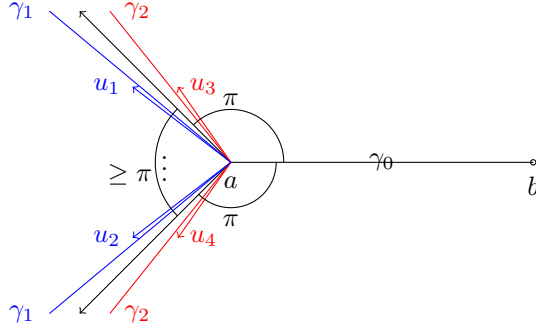


Figure 5.3: Constructing closed geodesics for a saddle connection with a marked endpoint.

for any $\rho' \in U_\rho$.

We constructed at most five closed geodesics for each saddle connection in the triangulation. One can compute the number of saddle connections in the triangulation, which is $(6g + 3k - 6)$. Hence the number of closed curves $|\Sigma|$ is at most $5(6g + 3k - 6)$. Finally we pick U_ρ that respects the triangulation and the expression of the length of each saddle connection by the lengths of closed geodesics. This means that the set of saddle connections for the triangulation form a triangulation for any $\rho' \in U_\rho$. The second part means that, for each saddle connection γ_0 , the conclusion to the case analysis is true with respect to any $\rho' \in U_\rho$. Hence we are done. \square

5.3 Strata of Euclidean cone metrics

In this section we look at aspects of the proof of Theorem 4 that might extend to a more general setting. The arguments below are not rigorously written out and the result at this point is speculative.

Recall that a Euclidean cone metric on S is a Euclidean metric on $S \setminus C$, where C is a finite set of points. The Euclidean metric extends to a Euclidean cone metric on S so that the points in C become cone points. We further require that the cone points with cone angle at most 2π must be marked points of S . A stratum of Euclidean cone metrics is obtained by specifying the number of cone points, the cone angles, and the holonomy. The geodesics with respect to a Euclidean cone metric are the concatenation of saddle connections that satisfy the angle condition.

A maximal collection of saddle connections with pairwise disjoint interiors will be a triangulation of S by Euclidean triangles. Therefore it suffices to locally determine the lengths of these saddle connections in order to extend the local rigidity result. A version of Lemma 5 for Euclidean cone metrics follows from a similar analysis on the universal cover along with a Gauss-Bonnet Theorem for Euclidean cone metrics.

We describe how the length of a saddle connection can be determined in the case when the saddle

connection connects distinct unmarked cone points. The number of closed curve we use depends on the minimum cone angle at unmarked cone points, which by definition is greater than 2π . Let the minimum cone angle at unmarked cone points be $2\pi + s\pi$, where $s > 0$. By a similar construction, we construct an odd number of closed geodesics such that the lengths determine the length of the saddle connection by a linear equation. Observe that three would suffice if $s > 1$ and five would suffice for $s = 1$ as shown in Figure 5.2. If $(2m + 1)$ is the number of closed geodesics used, then they “wrap” around the cone point m times, going a total angle of $2m\pi + ms\pi$. The construction requires that $(2m + ms)/(2m + 1) > 1$, hence m is equal to the least integer greater than $1/s$. The argument shows that there exists a set of closed curves $\Sigma \subset \mathcal{C}(S)$ such that Σ is locally length spectrally rigid at ρ over the stratum of Euclidean cone metrics and $|\Sigma| \leq (2[1 + 1/s] + 1)(6g + 3k - 6)$. This concludes our sketch of how the technique extends to strata of Euclidean cone metrics.

The extension of Theorem 4 to strata of Euclidean cone metrics is of particular interest in the context of showing that $\mathcal{C}(S)$ is length spectrally rigid over the space of Euclidean cone metrics, which is a missing piece in the direction of showing $\mathcal{C}(S)$ being length spectrally rigid over the space of nonpositively curved metrics on S .

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