SHARP ESTIMATES FOR TRILINEAR OSCILLATORY INTEGRAL FORMS

BY

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DISSERTATION

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Abstract

This dissertation consists of 4 chapters. In Chapter 1, we will briefly introduce some background of the trilinear oscillatory integrals and the motivations for their study. We also outline some key ideas in their proofs, as well as the major novelty of this dissertation.

Chapter 2 serves as analytic preparation for the proof of our main theorem. We first extend a result of Hörmander to a trilinear setting, which can be viewed as a junior version of the main result. Secondly, we establish a trilinear analogue of Phong-Stein's van der Corput Lemma, which is our major analytic ingredient in the proof of our main theorem.

Chapter 3 is the heart of this dissertation. An algorithm of resolution of singularities in $\mathbb{R}^2$ is presented in detail. The content of this chapter is self-contained and the readers who are merely interested in this algorithm may jump in this chapter directly.

Chapter 4 is designed to prove the main theorem. By using the algorithm in Chapter 3, a small neighborhood of a singular point is decomposed into finitely many curved triangular regions. In each of these regions, we can employ the analytic tools developed in Chapter 2 to obtain optimal control for the trilinear oscillatory integrals.

To my parents and to my wife.

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Chapter 1

Introduction

Oscillatory integrals have emerged as powerful analytic tools in various problems, ranging from PDEs to combinatorics and geometry. One famous example to illustrate this is that the restriction conjecture in oscillatory integrals can imply the Kakeya conjecture in geometric measure theory; see e.g. [Wol03]. Therefore, the study of various types of oscillatory integrals have been an indispensable part of harmonic analysis since its early age.

The purpose of the dissertation is to introduce a self-contained algorithm, an algorithm of resolution of singularities in $\mathbb{R}^2$, to establish sharp estimates for certain trilinear oscillatory integrals. As a natural extension of the seminal results by Phong and Stein [PS97], these estimates also answer a question raised by Christ, Li, Tao and Thiele [CLTT05] concerning sharp estimates for certain multilinear oscillatory integrals in a special setting.

1.1 The main problem to study

Let $S(x, y)$ be a real analytic function defined in a neighborhood of $(0, 0) \in \mathbb{R}^2$. Assume $a(x, y)$ is a smooth cut-off function supported in U, a sufficiently small neighborhood of $(0, 0)$, whose diameter depends on the given function S. Define a trilinear form as follows

$$\Lambda_S(f_1, f_2, f_3) = \iint e^{i\lambda S(x,y)} f_1(x) f_2(y) f_3(x + y) a(x, y)\, dxdy, \tag{1.1}$$

where λ is a nonzero parameter and $f_j \in L^2(\mathbb{R})$ for $1 \leq j \leq 3$.

The principal goal in this dissertation is to study the asymptotic behavior of the quotient

$$\frac{|\Lambda_S(f_1, f_2, f_3)|}{\prod_{j=1}^3 \|f_j\|_2}, \tag{1.2}$$

as the parameter λ tends to infinity. Here $\|\cdot\|_2 = \|\cdot\|_{L^2(\mathbb{R})}$.

Alternatively, we want to find out the best constant $m \in [0, \infty]$ such that there exists a constant $C > 0$, independent of λ, satisfying

$$|\Lambda_S(f_1, f_2, f_3)| \leq C|\lambda|^{-\frac{1}{2m}} \prod_{j=1}^{3} \|f_j\|_2, \tag{1.3}$$

as $|\lambda| \to \infty$. Since such constant C always exists when $m = \infty$, the above problem amounts to find out the smallest constant m. The exact value of C is unimportant throughout this dissertation.

1.2 A basic overview of oscillatory integrals

Oscillatory integrals of the first kind, in the terminology of Stein, can typically be written as

$$I(\lambda) = \iint e^{i\lambda S(x,y)} a(x, y) dx dy. \tag{1.4}$$

The main problem is then the asymptotic behavior of $I(\lambda)$ as the parameter λ goes to infinity.

Oscillatory integrals of the second kind, which also known as oscillatory integral operators, can be given in the form

$$T_\lambda(f)(x) = \int e^{i\lambda S(x,y)} f(y) a(x, y) dy. \tag{1.5}$$

The principal goal is then finding the estimates for the norm of the operator T_λ as λ tends to infinity.

Under certain derivative conditions on the phase $S(x, y)$, it is not difficult to establish sharp estimates for (1.4) and (1.5). The following theorem is one of the most well-known results in the theory of oscillatory integrals:

Theorem 1.1. *Assume $S(x, y)$ is a two variable smooth function defined in a neighborhood of the origin and $a(x, y)$ is a smooth cut-off function. If*

$$\nabla S(x, y) \neq (0, 0) \quad \text{for all} \quad (x, y) \in \text{supp } (a), \tag{1.6}$$

then for every $N \in \mathbb{N}$, there is a constant C_N depending on N such that

$$|I(\lambda)| \leq C_N |\lambda|^{-N}, \tag{1.7}$$

as $\lambda \to \infty$.

For high dimensional analogues, we refer the readers to Stein's book [Ste93].

The operator analogue was established by Hörmander [Hör73] in the 1970s:

Theorem 1.2 (Hörmander [Hör73])**.** *Assume $S(x, y)$ is a two variable smooth function defined in a neighborhood of the origin and $a(x, y)$ is a smooth cut-off function. If*

$$\left| \frac{\partial^2 S}{\partial x \partial y} \right| \geq 1, \quad \text{for all} \quad (x, y) \in \text{supp } (a), \tag{1.8}$$

then there is a constant $C > 0$ independent of λ, such that

$$\|T_\lambda(f)\|_2 \leq C|\lambda|^{-1/2} \|f\|_2. \tag{1.9}$$

The situation becomes very subtle but a lot more interesting when ∇S or $\partial_x \partial_y S$ vanishes at an isolated point. In this case, if $S(x, y)$ is merely a smooth function, we may need the following van der Corput type assumption: there exist integers $\alpha \geq 1$ and $\beta \geq 1$ such that

$$|\partial_x^\alpha \partial_y^\beta S(x,y)| \geq 1, \ \forall (x,y) \in \text{supp } (a). \tag{1.10}$$

Sharp estimates for $I(\lambda)$ and for the norm of T_λ can usually be characterized by the sum of α and β. Research on this topic may be found in the works of Phong-Stein [PS94a], Carbery-Christ-Wright [CCW99], Greenblatt [Gre05] and others.

If $S(x,y)$ is a non-zero real analytic function, no extra condition is required. In this setting, what is the correct notion to characterize the estimates for $I(\lambda)$ and for the norm of T_λ? Arnold had a very interesting Hypothesis: discrete invariants of a real analytic function can be characterized in terms of its Newton polyhedron.

Definition 1.1. *Let S be a two variable real analytic function defined in a neighborhood of the origin. Assume the Taylor expansion of S at $(0,0)$ is given by*

$$S(x,y) = \sum_{(p,q)\in\mathbb{N}^2} c_{p,q} x^p y^q, \quad c_{p,q} \in \mathbb{R}. \tag{1.11}$$

Then the Newton polyhedron of S is given by:

$$\mathcal{N}(S) = \text{Conv}\Bigg(\bigcup_{p,q} \Big\{(u,v) \in \mathbb{R}^2 : u \geq p, v \geq q \text{ and } c_{p,q} \neq 0\Big\}\Bigg).$$

Here, $\text{Conv}(X)$ *represents the convex hull of a set X in $\mathbb{R}^2$.*

The Newton distance $d = d(S)$ is given by

$$d = \inf\{t : (t,t) \in \mathcal{N}(S)\}.$$

The height of S is defined by

$$h(S) = \sup\{d_{(x,y)}\},$$

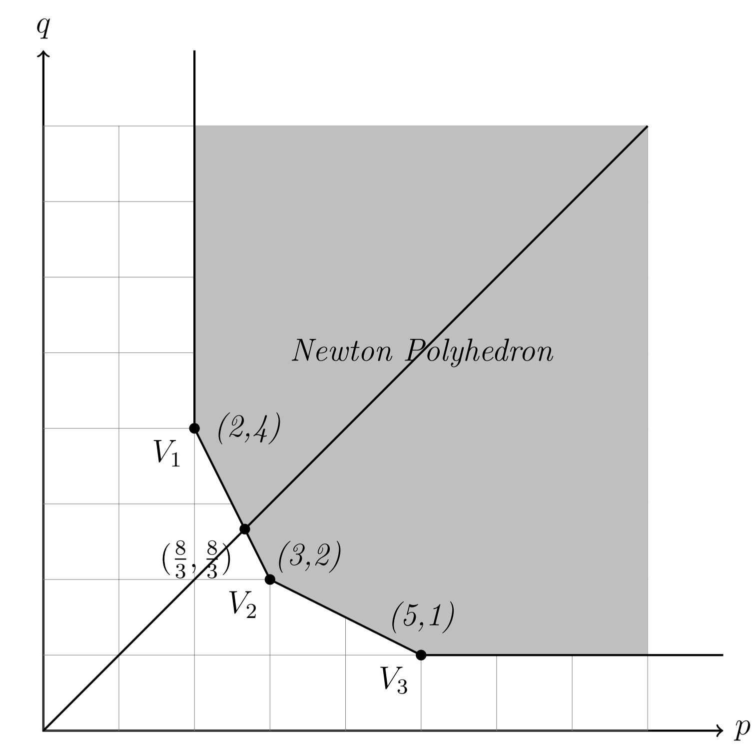
where the supremum is taking over all local analytic coordinate systems (x, y) *centered at the origin.*

Finally, a coordinate system is called adapted to S*, if the Newton distance is the same as the height of* S*, i.e.* $d = h(S)$*.*

Example 1. *Let* $S(x, y) = x^5y - x^3y^2 + x^2y^4 + Err(x, y)$*. Here* $Err(x, y) = \sum_{(p,q)\in A} c_{p,q}x^py^q$ *is the error term, i.e. for every* $(p, q) \in A$*,* $(p, q) \geq (5, 1), (3, 2)$ *and* $(2, 4)$*, where* $(p, q) \geq (p', q')$ *means* $p \geq p'$ *and* $q \geq q'$*.*

The Newton polyhedron of S *contains three vertices* $V_1 = (2, 4)$*,* $V_2 = (3, 2)$ *and* $V_3 = (5, 1)$*. The boundary of* $\mathcal{N}(S)$ *consists of two compact edges* V_1V_2 *and* V_2V_3*, and two non-compact edges; see the graph below.*

The Newton distance of S *is* $d = 8/3$*. The point* (d, d) *is indeed the intersection of the bi-sectrix* $p = q$ *and the boundary of the Newton polyhedron of* S*.*



Arnold's hypothesis was first verified by Varchenko in the case of oscillatory integrals of the first kind.

Theorem 1.3 (Varchenko [Var76]). *Let $S(x,y)$ be a real analytic function defined in a neighborhood of the origin and assume $S(0,0)=0$, $\nabla S(0,0)=(0,0)$. Then*

(i) There exist coordinate systems that are adapted to S.

(ii) Assume also the support of $a(x,y)$ is contained in a sufficiently small neighborhood of $(0,0)\in\mathbb{R}^2$, then

$$|I(\lambda)|\leq C|\lambda|^{-1/h(S)}\log(2+|\lambda|)^{\mu},$$

where $\mu=1$ if $(h(S),h(S))$ is a vertex of the Newton polyhedron of S in an adapted coordinate system, otherwise $\mu=0$.

The operator analogue of the above theorem was established by Phong and Stein in the 1990s:

Theorem 1.4 (Phong-Stein [PS97]). *Let $S(x,y)$ be a real analytic function defined in a neighborhood of the origin and assume $S(0,0)=0$. Assume also the support of $a(x,y)$ is contained in a sufficiently small neighborhood of $(0,0)\in\mathbb{R}^2$, then*

$$\|T(f)\|_2\leq C|\lambda|^{-\frac{1}{2d(\tilde{S})}}\|f\|_2,$$

where $\tilde{S}=S(x,y)-S(x,0)-S(0,y)$ and $d(\tilde{S})$ is the Newton distance of $\tilde{S}$.

1.3 Motivation

By duality, Hörmander and Phong-Stein's results can be rephrased as

$$|\Lambda_2(f,g)|\leq C|\lambda|^{-\frac{1}{2}}\|f\|_2\|g\|_2 \tag{1.12}$$

and

$$|\Lambda_2(f,g)| \le C|\lambda|^{-\frac{1}{2d(\tilde{S})}}\|f\|_2\|g\|_2 \tag{1.13}$$

respectively, where the bilinear form is defined as

$$\Lambda_2(f,g) = \iint e^{i\lambda S(x,y)} f(y)\overline{g(x)}a(x,y)dxdy. \tag{1.14}$$

One can see that the trilinear form we study here:

$$\Lambda_S(f_1,f_2,f_3) = \iint e^{i\lambda S(x,y)} f_1(x)f_2(y)f_3(x+y)a(x,y)dxdy \tag{1.1}$$

is a trilinear analogue of the bilinear forms studied by Hörmander's and Phong-Stein. It is then natural to raise a question concerning the sharp estimates in a similar manner

$$|\Lambda_S(f_1,f_2,f_3)| \le C|\lambda|^{-\frac{1}{2m}}\|f_1\|_2\|f_2\|_2\|f_3\|_2. \tag{1.3}$$

The motivation for the study of (1.1) does not only lie in Hörmander's and Phong-stein's works, but also the work by Christ, Li, Tao and Thiele [CLTT05], where certain multi-linear oscillatory integrals were studied in a very general setting.

To formulate the questions posed in [CLTT05], we need some preliminary notations. Let $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_J)$, where each $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n_j} \subset \mathbb{R}^n$ is a surjective linear projection. Let $S : \mathbb{R}^n \to \mathbb{R}$ be a polynomial and $a(X)$ be a smooth cut-off function supported in a small neighborhood of $0 \in \mathbb{R}^n$. For each j, let $f_j : \mathbb{R}^{n_j} \to \mathbb{C}$ be a measurable function. Consider the following multilinear oscillatory integrals:

$$\Lambda_{S,\boldsymbol{\pi}}(f_1,f_2,\ldots,f_J) = \int_{\mathbb{R}^n} e^{i\lambda S(X)}a(X)\prod_{j=1}^{J} f_j \circ \pi_j(X)dX. \tag{1.15}$$

Q1: For what kind of input $(S, \boldsymbol{\pi})$, the following is true

$$|\Lambda_{S,\boldsymbol{\pi}}(f_1, f_2, \ldots, f_J)| \le C|\lambda|^{-\delta} \prod_{j=1}^{J} \|f_j\|_{p_j} \tag{1.16}$$

for some $\delta > 0$, some $\boldsymbol{p} = (p_1, \ldots, p_J) \in [1, \infty]^J$ and all $f_j \in L^{p_j}(\mathbb{R}^{n_j})$?

Q2: If **Q1** could be answered affirmatively, what is the optimal exponent δ?

Giving a complete answer to **Q1** for a general input $(S, \boldsymbol{\pi})$ seems very difficult. Nevertheless, under certain dimension assumptions on $\boldsymbol{\pi}$, an affirmative answer to **Q1** was given in [CLTT05]; see also [Chr11a, Chr11b, CeS11, Gre08].

For **Q2**, some results were known before. For instance, when $J = 2$, $n_1 = n_2 = n/2$ (assume n is even) and S is smooth, Theorem 1.2 provides a sufficient characterization when the best possible decay can be obtained. Theorem 1.4 settled the case $n = J = 2$ and S is an arbitrary analytic function; see [Gre05, Ryc01, See98] for $S \in C^\infty(\mathbb{R}^2)$. For $n = J \ge 2$, almost sharp estimates (probably up to a power of $\log|\lambda|$) were known, by the work of Phong, Stein and Sturm [PSS01].

In Christ, Li, Tao and Thiele's attempt to answer **Q1**, an important step is a reduction to the trilinear setting. Thus, it is crucial to fully understand the trilinear case, in particular to determine the optimal exponent in (1.16) in this setting. This motivates us to study sharp estimates for the trilinear form (1.1), which corresponds to the case $n = 2$, $J = 3$, S is analytic and $\boldsymbol{\pi} = \boldsymbol{\pi}_0$, where

$$\boldsymbol{\pi}_0(x, y) = (\pi_{01}(x, y), \pi_{02}(x, y), \pi_{03}(x, y)) := (x, y, x + y). \tag{1.17}$$

Indeed, a more general setting

$$\Lambda_{S,\boldsymbol{\pi}}(f_1, f_2, f_3) = \iint e^{i\lambda S(x,y)} a(x, y) \prod_{j=1}^{3} f_j \circ \pi_j(x, y)\,dxdy, \tag{1.18}$$

can be reduced to (1.1) via an invertible linear transformation in $\mathbb{R}^2$ (see Chapter 2), where $\pi_j : \mathbb{R}^2 \to \mathbb{R}$ are pairwise linearly independent projections for $j = 1, 2, 3$.

One necessary condition for (1.18) to possess a decay bound is that S should be nondegenerate relative to the collection of projections $\boldsymbol{\pi}$:

Definition 1.2. *A function S is called degenerate relative to $\boldsymbol{\pi}$, if there exist one variable measurable functions S_1, S_2 and S_3 such that*

$$S(x, y) = \sum_{1 \le j \le 3} S_j \circ \pi_j(x, y).$$

If there are no such functions, then S is called nondegenerate relative to $\boldsymbol{\pi}$.

If S is degenerate relative to $\boldsymbol{\pi}$, i.e.

$$S(x, y) = \sum_{1 \le j \le 3} S_j \circ \pi_j(x, y),$$

then we can incorporate each $e^{i\lambda S_j \circ \pi_j(x,y)}$ into $f_j \circ \pi_j(x, y)$ by setting

$$\tilde{f}_j \circ \pi_j(x, y) = e^{i\lambda S_j \circ \pi_j(x,y)} f_j \circ \pi_j(x, y).$$

Since $\|\tilde{f}_j\|_{p_j} = \|f_j\|_{p_j}$, one cannot expect any decay as in (1.16).

Let $\pi_j^\perp : \mathbb{R}^2 \to \mathbb{R} \subset \mathbb{R}^2$ be linear projections s.t. $\pi_j \circ \pi_j^\perp = 0$ and $\|\pi_j^\perp\|_2 = 1$. Set $\boldsymbol{\pi}^\perp = (\pi_1^\perp, \pi_2^\perp, \pi_3^\perp)$ and $D_{\boldsymbol{\pi}^\perp} = \prod_{j=1}^{3} \pi_j^\perp \cdot \nabla$.

Definition 1.3. *The given analytic function S is called simply degenerate relative to $\boldsymbol{\pi}$ if*

$$D_{\boldsymbol{\pi}^\perp} S \equiv 0;$$

otherwise S is called simply nondegenerate relative to $\boldsymbol{\pi}$. In addition, S is simply degenerate at a point (x_0, y_0) if $D_{\boldsymbol{\pi}^\perp} S(x_0, y_0) = 0$.

Simply degeneracy implies degeneracy and the inverse is also true in our setting; see **Proposition 3.1** in [CLTT05].

1.4 Results

The following theorem, extending Theorem 1.2 to the trilinear setting, states that if S is simply nondegenerate everywhere in Conv(supp (a)), then one can obtain the optimal bound of (1.18).

Theorem 1.5. *Assume $a(x,y)$ is a smooth cut-off function supported in a neighborhood of $(0,0) \in \mathbb{R}^2$ and $S(x,y)$ is smooth s.t.*

$$|D_{\boldsymbol{\pi}^\perp} S(x,y)| \geq 1 \quad \textit{for all} \quad (x,y) \in \mathrm{Conv}(supp(a)), \tag{1.19}$$

then

$$|\Lambda_{S,\boldsymbol{\pi}}(f_1,f_2,f_3)| \leq C|\lambda|^{-1/6} \prod_{j=1}^{3} \|f_j\|_2. \tag{1.20}$$

We also extend Theorem 1.4 to the trilinear form (1.18). Different to what was expected, the characterization for the sharp exponent in this setting is not the same as the one in Phong–Stein's result. Instead, it is described by the relative multiplicity of S, which is an algebraic concept. Nevertheless, it can still be interpreted geometrically in terms of the Newton polyhedron of $D_{\boldsymbol{\pi}} S$; see Chapter 4. We shall investigate such difference in Chapter 4.

Define the multiplicity of an analytic function S as

$$\mathrm{mult}(S) = \min\{i : S_i(x,y) \not\equiv 0\},$$

where $S(x,y) = \sum_i S_i(x,y),\ S_i(x,y) = \sum_{p+q=i} c_{p,q} x^p y^q$ are homogeneous polynomials. We always assume the constant term of S is zero and also adopt the convention that $\mathrm{mult}(S) = -\infty$ if $S \equiv 0$. The multiplicity of S relative to $\boldsymbol{\pi}$ is defined as

$$\mathrm{mult}_{\boldsymbol{\pi}}(S) = \min\{i : D_{\boldsymbol{\pi}^\perp} S_i \not\equiv 0\} = \mathrm{mult}(D_{\boldsymbol{\pi}^\perp} S) + 3, \tag{1.21}$$

which is the multiplicity of the quotient of S by the class of degenerate analytic functions. Notice that if S is simply degenerate, then $\mathrm{mult}_{\boldsymbol{\pi}}(S) = -\infty$. The following theorem is the main analytic result of this dissertation:

Theorem 1.6. *Assume $S(x,y)$ is a real analytic function and the support of $a(x,y)$ is sufficiently small. Then*

$$|\Lambda_{S,\boldsymbol{\pi}}(f_1, f_2, f_3)| \leq C|\lambda|^{-\frac{1}{2\mathrm{mult}_{\boldsymbol{\pi}}(S)}} \prod_{j=1}^{3} \|f_j\|_2. \tag{1.22}$$

The result (1.22) is exact in the sense that if $a(0,0) \neq 0$, then

$$|\Lambda_{S,\boldsymbol{\pi}}(f_1, f_2, f_3)| \geq C'|\lambda|^{-\frac{1}{2\mathrm{mult}_{\boldsymbol{\pi}}(S)}} \prod_{j=1}^{3} \|f_j\|_2, \tag{1.23}$$

as $|\lambda| \to \infty$, for some $C' > 0$ and some $\{f_j\}_{1 \leq j \leq 3}$.

Remark 1.1. *The existence of a (non-sharp) decay rate in the bound of (1.22) is included in the results of [CLTT05] as a special case.*

1.5 Methods

Like Phong and Stein's proof of Theorem 1.4, the proof of Theorem 1.6 requires elaborate analysis. There are two main ingredients in their proof:

(1) The operator version of the van der Corput Lemma [PS94b]; see Theorem 2.2; and

(2) Weierstrass Preparation Theorem.

In order to extend Phong and Stein's framework to the trilinear setting, we first establish the trilinear analogue of (1):

(1') Theorem 2.1: trilinear version of Phong–Stein's van der Corput Lemma.

In addition, we develop

(2') a self-contained algorithm of resolution of singularities in $\mathbb{R}^2$,

as a substitution of Weierstrass Preparation Theorem, which is our second main result:

Theorem 1.7. *Let $P(x, y)$ be a real analytic function in $\mathbb{R}^2$ and $U = \{(x, y) : |x|, |y| < \epsilon\}$ be a neighborhood of $(0, 0)$, where $\epsilon > 0$ is sufficiently small. Then there is an algorithm, which partitions a dense open subset of U into a finite collection of regions $\{V_k\}_{1\le k\le K}$, such that P behaves almost like a "monomial" in each V_k in the following sense. There is an integer $M \in \mathbb{N}$, and for each k there is a diffeomorphism*

$$\rho_k : V_k \to \rho_k(V_k) \tag{1.24}$$

$$(x, y) \mapsto (x_k, y_k) \tag{1.25}$$

satisfying the following properties:

$$P(x, y) = P_k(x_k, y_k) = x_k^{p_k} y_k^{q_k} \cdot Q_k(x_k, y_k) \quad \textit{for all} \quad (x, y) \in V_k, \tag{1.26}$$

where

(1) $(x_k, y_k) = \rho_k(x, y)$ and $P_k = P \circ \rho_k^{-1}$;

(2) (p_k, q_k) is a vertex of the Newton polyhedron of P_k under the coordinate system $x_k - y_k$;

(3) The function Q_k is smooth and nonvanishing near 0 in $\rho_k(V_k)$, i.e.

$$\lim_{(x_k,y_k)\to(0,0)} Q_k(x_k,y_k) \neq 0 \quad \textit{inside} \quad \rho_k(V_k);$$

(4) $\rho_k(V_k)$ (as well as V_k) is a curved triangular region:

$$\rho_k(V_k) = \{(x_k,y_k) : C'_k|x_k|^{m'_k} < y_k < C_k|x_k|^{m_k} \textit{ and } 0 < |x_k| < \epsilon\},$$

for some $0 \le m_k \le m'_k \le \infty$ with $m_k M$, $m'_k M \in \mathbb{N} \cup \{\infty\}$, and C_k, C'_k are constants.

(5) $\rho_k^{-1}(x_k,y_k)$ are real analytic functions of $(|x_k|^{\frac{1}{M}}, y_k)$, more precisely

$$\begin{cases} x = x_k \\ y = \gamma_k(|x_k|^{\frac{1}{M}}) + |x_k|^{\frac{M_k}{M}} y_k, \end{cases} \tag{1.27}$$

where $M_k \in \mathbb{N}$ and γ_k is a polynomial, unless $P(x, \gamma_k(|x|^{\frac{1}{M}})) = 0$, then γ_k is a real analytic function.

Moreover, the constants m_k, m'_k, (p_k,q_k), M_k/M and the function γ_k[1] can be computed explicitly via the Newton polyhedra of $\{P_k\}_{1\le k\le K}$.

Remark 1.2. *See Theorem 3.7 in Chapter 3 for a complete version.*

The major novelty of this theorem lies in the fact that almost all the important information can be computed in an explicit manner; see Chapter 3.

The idea of employing resolution of singularities to investigate oscillatory integrals appeared in Varchenko's work [Var76], where the fundamental results from Hironaka [Hir64] played a crucial role. More recently, an algorithm of resolution of singularities in $\mathbb{R}^2$ was introduced by Greenblatt [Gre04], where an elegant proof of Theorem 1.4 was presented based on this algorithm.

[1]In the case γ_k is an infinite series, we can compute any partial sum of γ_k.

Our proof of Theorem 1.6 and the algorithm here are both inspired by the work of Greenblatt [Gre04]. Many of the ideas inside the algorithm here are very elementary and known for centuries, which may come back to Newton's algorithm for solving $S(x, y) = 0$ by a fractional power series $y = y(x^{\frac{1}{M}})$ (the Puiseux series); see [Cut04]. The philosophy of the algorithm here is similar to that of the one in [Gre04]. Here, we outline some of the major novelty as follows:

(1) The implicit function theorem (IFT) is **not** involved here. The change of variables is always of the form:

$$(x, y) = (x_1, x_1^m(r + y_1)), \tag{1.28}$$

with the possible exception at the finishing steps. In [Gre04], the IFT plays an important role. The change of variables is of the form:

$$(x, y) = (x_1, y_1 + q(x_1)), \tag{1.29}$$

where $q(x)$ is a Puiseux series obtained by the IFT and can be written as $q(x) = rx^m + O(x^{m+\nu})$. Thus, our change of variables is simpler and more explicit. As a result, we are able to switch variables between different stages of iterations; see (1.26) and (1.27). In addition, the x_1^m factor in the 2nd coordinate of (1.28) plays an important role. Namely, it "rescales" each curved triangular region (non-standard) back into a standard non-curved region, allowing one to do iterations in the same region.

(2) Our idea for the termination of the algorithm is very natural. Only performing the form of change of variables (1.28) is not sufficient to ensure the termination of the algorithm[2]. Greenblatt [Gre04] had some nice observations to overcome this barrier. The key point is to invoke the IFT to find the solution of $\partial_y^{n-1} P(x, y) = 0$, which

[2]For example, if only performing the above form of change of variables to $P(x, y) = (y - (\sum_{j=1}^{\infty} x^j))^n$, the algorithm does not stop.

corresponds to the change of variables (1.29). Roughly speaking, each such change of variables decreases certain "order" of $P(x, y)$ by at least 1, which ensures that the algorithm stops after finite steps. The cost is the resulting tail in $q_1(x)$, which is in an implicit form[3]. Can one retain the simplicity of the change of variables (1.28) and also ensure the termination of the algorithm? The answer is Yes. To do so, in the beginning, we assume the algorithm does not stop, which results in an infinite chain

$$[U, P] = [U_0, P_0] \to [U_1, P_1] \to [U_2, P_2] \to \cdots \to [U_n, P_n] \to \cdots . \tag{1.30}$$

Each U_n above can be viewed as an identical copy of U in Theorem 1.7, and P_n is obtained from P_{n-1} via the change of variables of the form (1.28). We search for some "invariants" inside this infinite chain; see Definition 3.2 and Lemma 3.5. It turns out that these "invariants" can be visualized by the Newton polyhedra: the shapes of the Newton polyhedra of P_n are unchanged after finite steps. Lemma 3.6, which describes such "invariants" analytically, is the key observation to make the algorithm stop naturally.

[3]For instance, applying this form of change of variables to $\tilde{P}(x, y) = (y - x)^n + x^n y^{2n}$, one needs to use the IFT to solve $y - x + cx^n y^{n+1} = 0$ for y, whose expansion contains a tail.

Chapter 2

Some analytic lemmas

In this section, we utilize the TT^* method to prove Theorem 1.5 and the following technical theorem which is needed in the proof of Theorem 1.6.

Theorem 2.1. *Assume $a(x, y)$ is a smooth function supported in a strip of x-width no more than δ_1 and y-width no more than δ_2, satisfying the following derivative conditions*

$$|\partial_y a(x, y)| \lesssim \delta_2^{-1} \qquad \text{and} \qquad |\partial_y^2 a(x, y)| \lesssim \delta_2^{-2}. \tag{2.1}$$

Let $\mu > 0$ and $S(x, y)$ be a smooth function s.t. for all $(x, y) \in \mathrm{Conv}(\mathrm{supp}\ (a))$:

$$|D_{\boldsymbol{\pi}_0^\perp} S(x, y)| \gtrsim \mu \qquad \text{and} \qquad |\partial_y^\beta D_{\boldsymbol{\pi}_0^\perp} S(x, y)| \lesssim \frac{\mu}{\delta_2^\beta} \quad \text{for} \quad \beta = 1, 2 \tag{2.2}$$

then for Λ_S defined as in (1.1), one has

$$|\Lambda_S(f_1, f_2, f_3)| \lesssim |\lambda\mu|^{-\frac{1}{6}} \prod_{j=1}^{3} \|f_j\|_2.$$

The above theorem can be viewed as a trilinear analogue of Phong–Stein's operator version of van der Corput Lemma [PS94b]:

Theorem 2.2. *Assume $a(x, y)$ is a smooth function supported in a strip of x-width no more than δ_1 and y-width no more than δ_2, satisfying the following derivative conditions*

$$|\partial_y a(x, y)| \lesssim \delta_2^{-1} \qquad \text{and} \qquad |\partial_y^2 a(x, y)| \lesssim \delta_2^{-2}. \tag{2.3}$$

Suppose $\mu > 0$ and $S(x,y)$ is a smooth function in $\mathbb{R}^2$ s.t. the following holds for all $(x,y) \in \text{supp }(a)$:

$$|\partial_x\partial_y S(x,y)| \gtrsim \mu \qquad \textit{and} \qquad |\partial_x\partial_y^{1+\beta} S(x,y)| \lesssim \frac{\mu}{\delta_2^{\beta}} \quad \textit{for} \quad \beta = 1,2 \tag{2.4}$$

then

$$\|T(f)\|_2 \lesssim (\lambda\mu)^{-1/2}\|f\|_2.$$

In both theorems above, we have adopted the notation $X \lesssim Y$ to denote $|X| \leq CY$ for some constant C, which depends on a and S, but is independent of δ_1, δ_2, μ and λ. It's also worth pointing out that theorem 2.2 is not exactly the same as the one employed by Phong-Stein in [PS94b]. We have adopted a more general version from Greenblatt in [Gre04]. For the proof of Theorem 2.2, we also refer the readers to [Gre04].

Now we turn to the technical details. First of all, we show that (1.18) can be reduced to (1.1). Set

$$\|\Lambda_{S,\boldsymbol{\pi}}\| = \sup\{|\Lambda_{S,\boldsymbol{\pi}}(f_1,f_2,f_3)|, \|f_j\|_2 \leq 1 \quad \text{for} \quad j = 1,2,3\} \tag{2.5}$$

and $\|\Lambda_S\|$ is defined similarly. We may assume $\pi_1(x,y) = x$, $\pi_2(x,y) = y$ and $\pi_3(x,y) = Ax + By$ where $A \neq 0$ and $B \neq 0$. Change variables $u = Ax$ and $v = By$, then

$$\begin{aligned}
\Lambda_{S,\boldsymbol{\pi}}(f_1,f_2,f_3) &= \iint e^{i\lambda S(x,y)} f_1(x)f_2(y)f_3(Ax+By)a(u,v)dxdy \\
&= \frac{1}{AB}\iint e^{i\lambda S(u/A,v/B)} f_1(u/A)f_2(v/B)f_3(u+v)a(u/A,v/B)dudv \\
&= \frac{1}{AB}\iint e^{i\lambda S_{A,B}(u,v)} f_{1,A}(u)f_{2,B}(v)f_3(u+v)a_{A,B}(u,v)dudv,
\end{aligned}$$

where $S_{A,B}(u,v) = S(u/A,v/B)$, $f_{1,A}(u) = f_1(u/A)$, $f_{2,B}(v) = f_2(v/B)$ and $a_{A,B}(u,v) =$

$a(u/A, v/B)$. Notice that

$$D_{\boldsymbol{\pi}_0^\perp} S_{A,B}(u,v) = \frac{1}{AB}((\partial_u/A - \partial_v/B)\partial_u\partial_v S)(u/A, v/B)$$

Thus $D_{\boldsymbol{\pi}_0^\perp} S_{A,B} = C D_{\boldsymbol{\pi}^\perp} S$ for an appropriate constant C. This implies if $|D_{\boldsymbol{\pi}^\perp} S(u,v)| \geq C_0$ for all $(u,v) \in \text{Conv}(\text{supp }(a))$, then $|D_{\boldsymbol{\pi}_0^\perp} S_{A,B}(u,v)| \geq C C_0$ for all $(u,v) \in \text{Conv}(\text{supp }(a_{A,B}))$. In addition $\|f_{1,A}\|_2 = \sqrt{A}\|f_1\|_2$ and $\|f_{2,B}\|_2 = \sqrt{B}\|f_2\|_2$. Therefore, for an appropriate constant C_1, one has

$$\|\Lambda_{S,\boldsymbol{\pi}}\| \leq C_1 \|\Lambda_{S_{A,B}}\|.$$

Now we turn to the proofs of Theorem 1.5 and Theorem 2.1, and we only need to consider Λ_S. For simplicity, we assume $\|f_1\|_2 = \|f_2\|_2 = \|f_3\|_2 = 1$. Applying change of variables $(u,v) = (x+y, y)$ and duality, one has

$$\|\Lambda_S(f_1, f_2, f_3)\| \leq \|B(f_1, f_2)\|_2 \|f_3\|_2 = \|B(f_1, f_2)\|_2, \tag{2.6}$$

where

$$B(f_1, f_2)(u) = \int e^{i\lambda S(u-v,v)} f_1(u-v) f_2(v) a(u-v, v) dv. \tag{2.7}$$

Employing TT^* yields

$$\|B(f_1, f_2)\|_2^2 = \iiint e^{i(\lambda S(u-v_1,v_1) - \lambda S(u-v_2,v_2))} f_1(u-v_1) \bar{f}_1(u-v_2) f_2(v_1) \bar{f}_2(v_2) \tag{2.8}$$

$$a(u-v_1, v_1) a(u-v_2, v_2) dv_1 dv_2 du. \tag{2.9}$$

Change variables: $x = u - v_1$, $y = v_1$ and $\tau = v_2 - v_1$ and set

$$S_\tau(x, y) = S(x, y) - S(x - \tau, y + \tau) \tag{2.10}$$

$$F_\tau(x) = f_1(x)\bar{f}_1(x - \tau) \tag{2.11}$$

$$G_\tau(y) = f_2(y)\bar{f}_2(y + \tau) \tag{2.12}$$

$$a_\tau(x, y) = a(x, y)a(x - \tau, y + \tau) \tag{2.13}$$

Then

$$\|B(f_1, f_2)\|_2^2 = \int \left(\iint e^{i\lambda S_\tau(x,y)} F_\tau(x) G_\tau(y) a_\tau(x, y) dxdy \right) d\tau \tag{2.14}$$

The proofs of Theorem 1.5 and Theorem 2.1 slightly diverge now and are presented in two separated sections.

2.1 Proof of Theorem 1.5

Split $\|B(f_1, f_2)\|_2^2$ into $B_1 + B_2$ according to the value of $|\tau|$ as below

- Case 1. $|\tau| \leq |\lambda|^{-1/3}$,
- Case 2. $|\tau| \geq |\lambda|^{-1/3}$.

For Case 1, we simply move the absolute value into the integrals, which yields

$$B_1 \lesssim \int_{|\tau| \leq |\lambda|^{-1/3}} \|F_\tau\|_1 \|G_\tau\|_1 d\tau \lesssim |\lambda|^{-1/3} \|f_1\|_2^2 \|f_2\|_2^2 = |\lambda|^{-1/3}. \tag{2.15}$$

For Case 2, we assume for a moment that in the support of a_τ, the following holds for some positive constant C:

$$|\partial_x \partial_y S_\tau(x, y)| \geq C|\tau|. \tag{2.16}$$

Applying Theorem 1.2 to the inner double integral, B_2 is dominated by

$$\begin{aligned}
&C\int_{|\tau|\geq|\lambda|^{-1/3}}|\lambda\tau|^{-1/2}\|F_\tau\|_2\|G_\tau\|_2 d\tau &(2.17)\\
\lesssim&|\lambda|^{-1/3}\int\|F_\tau\|_2\|G_\tau\|_2 d\tau &(2.18)\\
\lesssim&|\lambda|^{-1/3}\Big(\int\|F_\tau\|_2^2 d\tau\cdot\int\|G_\tau\|_2^2 d\tau\Big)^{1/2} &(2.19)\\
=&|\lambda|^{-1/3}\|f\|_2^2\|g\|_2^2=|\lambda|^{-1/3}. &(2.20)
\end{aligned}$$

Thus

$$\|B(f,g)\|_2^2=B_1+B_2\lesssim|\lambda|^{-1/3}. \tag{2.21}$$

It remains to verify (2.16) on the support of (2.13). Set

$$F(t)=S_{xy}(x-t,y+t), \tag{2.22}$$

then

$$|F'(t)|=|(\partial_x-\partial_y)\partial_x\partial_y S(x-t,y+t)|. \tag{2.23}$$

By the mean value theorem, there is a constant t_0 between 0 and τ, such that

$$|\partial_x\partial_y S_\tau(x,y)|=|F(0)-F(\tau)|=|\int_0^\tau F'(t)dt|=|\tau||F'(t_0)|. \tag{2.24}$$

Notice that $(x,y)\in$ supp (a) and $(x-\tau,y+\tau)\in$ supp (a). By convexity $(x-t_0,y+t_0)\in$ Conv(supp (a)). Therefore, (1.19), (2.24) and (2.23) yield (2.16).

2.2 Proof of Theorem 2.1

Similarly, we split $\|B(f_1, f_2)\|_2^2$ into $B_1 + B_2$ according to the value of $|\tau|$ as below

- Case 1. $|\tau| \le |\lambda\mu|^{-1/3}$,
- Case 2. $|\tau| \ge |\lambda\mu|^{-1/3}$.

For Case 1, we simply move the absolute value into the integrals

$$B_1 \lesssim \int_{|\tau|\le|\lambda\mu|^{-1/3}} \|F_\tau\|_1 \|G_\tau\|_1 d\tau \lesssim |\lambda\mu|^{-1/3} \|f_1\|_2^2 \|f_2\|_2^2 = |\lambda\mu|^{-1/3}. \tag{2.25}$$

For Case 2, assume at a moment that (2.3) is true for a_τ and (2.4) are true for S_τ with μ replaced by $|\lambda\mu|$. Then Theorem 2.2 implies

$$B_2 \lesssim \int_{|\tau|\ge|\lambda\mu|^{-1/3}} |\lambda\mu\tau|^{-1/2} \|F_\tau\|_2 \|G_\tau\|_2 d\tau \lesssim |\lambda\mu|^{-1/3}. \tag{2.26}$$

It remains to verify the conditions mentioned above. Indeed (2.3) follows from $a_\tau(x, y) = a(x, y)a(x - \tau, y + \tau)$. S_τ satisfies the first part of (2.4) with μ replaced by $|\lambda\mu|$ due to (2.2), (2.23), (2.24) and the convexity assumption in theorem 2.1. If we set

$$F_1(t) = \partial_x \partial_y^2 S(x - t, y + t)$$

and

$$F_2(t) = \partial_x \partial_y^3 S(x - t, y + t)$$

then the second part of (2.4) (with μ replaced by $|\lambda\mu|$) follows from (2.2), (2.23), (2.24) (with F replaced by F_1 and F_2) and convexity.

Chapter 3

An algorithm for resolution of singularities in $\mathbb{R}^2$

In order to employ Theorem 2.1 to prove Theorem 1.6, one needs to decompose supp (a) into regions such that $P(x, y)$ is well-behaved, where $P = \partial_x \partial_y (\partial_x - \partial_y) S$. Ideally, in each of such regions, one hopes $P(x, y)$ to behave almost like a "monomial". The algorithm is driven by this idea. In each stage of iteration, "good" regions (with the desired property) are obtained via vertices and edges of the Newton polyhedron of P when $P(x, y)$ is "nonvanishing", and "bad" regions are obtained when $P(x, y)$ "vanishes" on these edges. In each of these "good" regions, $P(x, y)$ behaves almost like a "monomial" and no further treatment is required; while each of those "bad" regions is carried to the next stage of iteration. A branch of iterations is created for each "bad" region. We outline the main ideas below.

3.1 Main Ideas of the algorithm

Let $P(x, y)$ be a real analytic function defined on a small neighborhood of $(0, 0)$ whose Taylor expansion is

$$P(x, y) = \sum_{(p,q) \in \mathbb{N}^2} c_{p,q} x^p y^q. \tag{3.1}$$

We drop all those coefficients $c_{p,q} = 0$ from this expression. In particular, we assume $c_{0,0} = 0$, otherwise

$$P(x, y) \sim c_{0,0}$$

given (x, y) in a sufficiently small neighborhood of $(0, 0)$, which is already a "monomial".

Recall that the Newton polyhedron of P is given by

$$\mathcal{N}(P) = \text{Conv}\Big(\cup_{p,q} \Big\{ (u,v) \in \mathbb{R}^2 : u \geq p, v \geq q \text{ with } c_{p,q} \neq 0 \Big\} \Big). \tag{3.2}$$

The Newton diagram is the boundary of $\mathcal{N}(P)$, which consists of two non-compact edges, a finite collection of compact edges $\mathcal{E}(P)$ (may be empty) and a finite collection of vertices $\mathcal{V}(P)$. The vertices and the edges are called the faces of the Newton polyhedron. We use $\mathcal{F}(P)$ to denote all the faces, including non-compact ones. The Euler formula gives

$$\#\mathcal{V}(P) - \#\mathcal{E}(P) = 1.$$

For each compact face $F \in \mathcal{F}(P)$, define P_F as the restriction of P in F:

$$P_F(x,y) = \sum_{(p,q)\in F} c_{p,q} x^p y^q. \tag{3.3}$$

Choose a vertex $(p_v, q_v) = V \in \mathcal{V}(P)$, then V lies in two edges: E_l and E_r, where E_l is left to V and E_r is right to V. Assume the slopes of E_l and E_r are $-1/m_l$ and $-1/m_r$ respectively, then $0 \leq m_l < m_r \leq \infty$. We use $m_l = 0$ to represent that E_l is the vertical non-compact edge. Consider the region $|y| \sim |x|^m$ in the following three cases:

Case (1). $m_l < m < m_r$,

Case (2). $m = m_l$, and

Case (3). $m = m_r$,

which corresponds to:

(1) the vertex V "dominates" $P(x,y)$,

(2) the edge E_l "dominates" $P(x,y)$ and

(3) the edge E_r "dominates" $P(x,y)$ respectively.

Case (2) and **Case (3)** are exactly the same and only **Case (2)** is discussed here.

In **Case (1)**, $p_v + mq_v < p + mq$ for any other $(p,q) \in \mathcal{V}(P)$. Thus in the region $|y| \sim |x|^m$

and $|x|$ sufficiently small,

$$P_V(x,y) = c_{p_v,q_v} x^{p_v} y^{q_v} \sim x^{p_v + m q_v}$$

is the dominant term in $P(x,y)$, since the remaining term

$$P(x,y) - P_V(x,y) = O(x^{p_v + m q_v + \nu})$$

has a higher x degree, which can be viewed as an error term given $|x|$ sufficiently small. Here, $\nu > 0$ is some constant. Thus

$$P(x,y) \sim P_V(x,y) = c_{p_v,q_v} x^{p_v} y^{q_v}. \tag{3.4}$$

The following is a graph illustrates this case. The given polynomial is $P(x,y) = x^5 y - x^3 y^2 + xy^4$ and the chosen vertex is $V_2 = (3,2)$.

Case (1): The vertex V_2 is dominant, where $1/2 < m < 2$

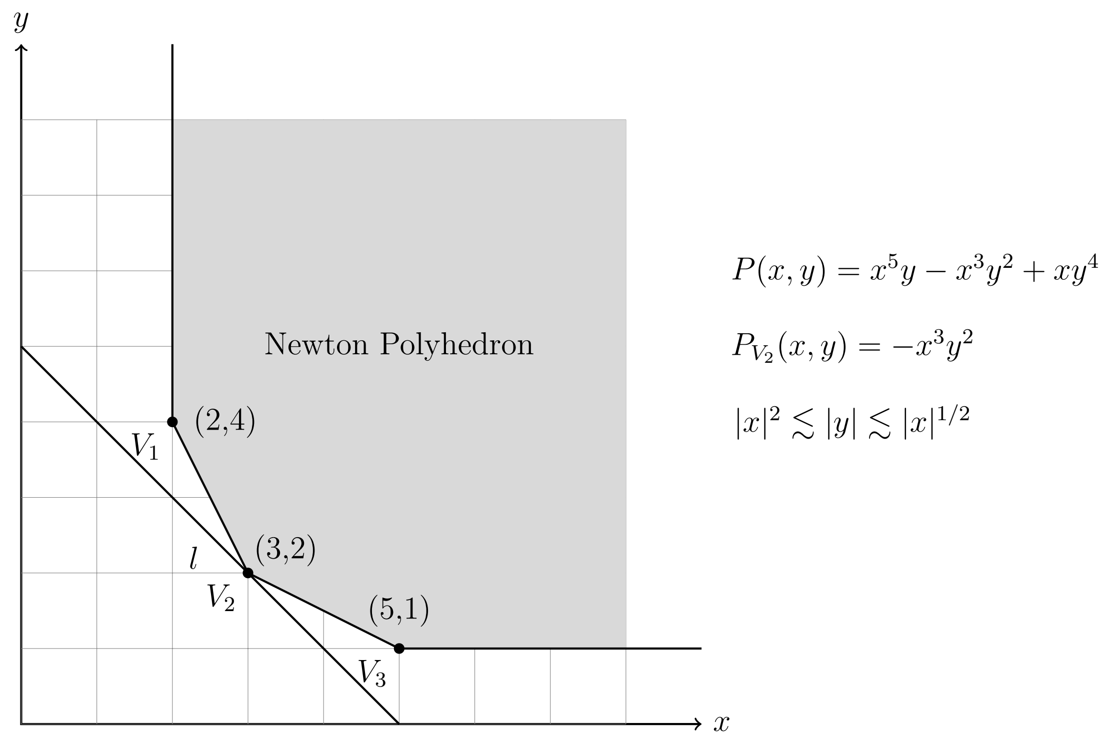


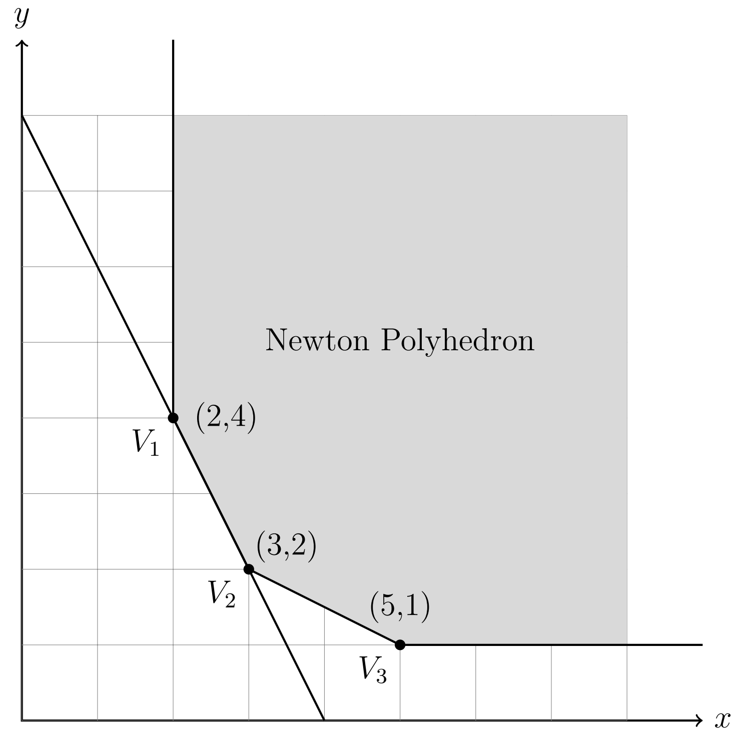
Figure 1.

Case (2) $m = m_l$ is more complicated. We see $p_v + mq_v = p + mq$ for all $(p,q) \in E_l$ and $p_v + mq_v < p + mq$ for all $(p,q) \notin E_l$. Then for all $(p,q) \in E_l$, $x^p y^q \sim x^{p_v} y^{q_v}$ in the region $|y| \sim |x|^m$ and thus

$$P_{E_l}(x,y) \sim x^{p_v} y^{q_v} \sim x^{p_v + mq_v}$$

is the dominant term of $P(x,y)$, **unless** there is **cancellation** inside $P_{E_l}(x,y)$! We call this is a "bad" situation and it demands most of the work.

Case(2): The edge V_1V_2 is dominant, where $m = 1/2$



$$P(x, y) = x^5y - x^3y^2 + x^2y^4$$

$$P_{V_1V_2}(x, y) = -x^3y^2 + x^2y^4$$

$$|y| \sim |x|^{1/2}$$

Figure 2.

We shed some light on how to handle the "bad" situation. Set $P_{E_l}(r) = P_{E_l}(1, r)$. Cancellation happens inside $P_{E_l}(x, y)$ if and only if $P_{E_l}(r) = 0$ has non-zero real roots. Each root r_j of $P_{E_l}(r) = 0$ corresponds to a region where $P_{E_l}(x, y)$ vanishes. Via change of variables $x = x'$ and $y = (r_j + y')x^{m_l}$, a new function $P'(x', y')$ is obtained. Previous discussion can be then repeated on P'. We want to emphasize two points here. Firstly, each root r_j corresponds to a new branch of iteration and thus the algorithm has a tree structure (not a linear structure). Secondly, the iterations end up essentially after finitely many steps.

3.2 The resolution algorithm Part I: A single step of Partition

Let U be a sufficiently small neighborhood of $(0,0)$ in $\mathbb{R}^2$. For simplicity, we restrict our discussion on the right half-plane $x > 0$, since the left half plane can be reduced to this case via change of variables $(x, y) \to (-x, y)$. We assume $U = \{(x, y) : 0 < x < \epsilon, -\epsilon < y < \epsilon\}$ where ϵ is sufficiently small. An inductive decomposition procedure will be performed on the pair $[U, P]$, where P is an analytic function and U defined above. Let $M \in \mathbb{N}$ be a pre-fixed large constant whose value will be chosen later.

Definition 3.1. *Given a coordinate (X, Y) and a real analytic function $Q(X, Y)$ of $(X^{\frac{1}{M}}, Y)$, if $W = \{(X, Y) : 0 < X < \epsilon,\ -\epsilon < Y < \epsilon\}$, where $\epsilon > 0$ is sufficiently small, then we call W a standard region and $[W, Q]$ is a standard pair under the coordinate (X, Y). In addition, we denote* $\mathrm{rad}(W) = \epsilon$*, the "radius" of W.*

By the definition, $[U, P]$ is a standard pair under the coordinate (x, y) and we set $[U, P] = [U_0, P_0]$ and $(x, y) = (x_0, y_0)$ to indicate the procedure is in the starting stage (0-th stage). It is worth mentioning that the algorithm will always perform on a standard region, with different analytic functions of $(x^{1/M}, y)$. Moreover, $\epsilon > 0$ denotes a sufficiently small number whose value may be varied but it is completely harmless.

Consider

$$P_E(x, y) = \sum_{(p,q)\in E} c_{p,q} x^p y^q, \quad \text{for} \quad E \in \mathcal{E}(P).$$

Let $V_{E,l} = (p_{E,l}, q_{E,l})$ and $V_{E,r} = (p_{E,r}, q_{E,r}) \in \mathcal{V}(P)$ be the left and right vertices of E. Set

$$m_E = \frac{p_{E,l} - p_{E,r}}{q_{E,r} - q_{E,l}}, \tag{3.5}$$

which is the negative reciprocal of the slope of E, i.e. $-1/m_E$ is the slope of E. The constant m_E is the most important constant assigned to each edge E. If we set

$$e_E = p_{E,l} + m_E q_{E,l}, \tag{3.6}$$

then for all $(p', q') \in E$,

$$e_E = p' + m_E q', \tag{3.7}$$

and for all $(p'', q'') \notin E$,

$$p'' + m_E q'' = e_E + \nu \tag{3.8}$$

for some $\nu > 0$. In the curve $y = rx^{m_E}$ where $r \in \mathbb{R} \setminus \{0\}$,

$$P_E(x, y) = x^{e_E} \sum_{(p,q)\in E} c_{p,q} r^q =: x^{e_E} P_E(r) \tag{3.9}$$

and the remaining term

$$P(x, y) - P_E(x, y) = O(x^{e_E+\nu}) \tag{3.10}$$

has a higher degree. Thus given $|x|$ sufficiently small, $P_E(x, y)$ dominates $P(x, y)$, unless

$$P_E(r) = \sum_{(p,q)\in E} c_{p,q} r^q \to 0. \tag{3.11}$$

Now it becomes clear that the nonzero roots of $P_E(r)$ are the trouble makers and more elaborate treatment is needed. Assume $\{r_{E,j}\}_{1\leq j\leq J_E}$, labeled in the increasing order, is the set of non-zero roots of $P_E(r) = 0$ whose orders are $\{s_{E,j}\}_{1\leq j\leq J_E}$.

Notice that

$$J_E \leq \sum_{1\leq j\leq J_E} s_j \leq q_{E,l} - q_{E,r}, \tag{3.12}$$

since

$$P_E(r) = r^{q_{E,r}} \sum_E c_{p,q} r^{q-q_{E,r}}. \tag{3.13}$$

For simplicity, we say $r_{E,j}$ is a root of E if $r_{E,j}$ is a root of $P_E(r)$. Let

$$I_j^\epsilon(E) = (r_{E,j} - \epsilon, r_{E,j} + \epsilon), \tag{3.14}$$

where

$$0 < \epsilon < 2^{-10} \cdot \min\{|r_{E,j}| : E \in \mathcal{E}(P),\, 1 \leq j \leq J_E\} \tag{3.15}$$

is a sufficiently small constant. Assign two constants c_E and C_E to each edge E such that

$$0 < c_E < 2^{-10}|r_{E,j}| < 2^{10}|r_{E,j}| < C_E, \quad \text{for all} \quad 1 \leq j \leq J_E. \tag{3.16}$$

In addition, c_E is chosen to be sufficiently small and C_E to be sufficiently large. Set

$$\begin{cases} I(E) = [c_E, C_E] \cup [-C_E, -c_E], \\ \\ I_b(E) = \cup_{1\leq j\leq J_E} I_j^\epsilon(E), \\ \\ I_g(E) = I(E) \setminus I_b(E). \end{cases} \tag{3.17}$$

Here $I_b(E)$ represents small neighborhoods of the roots $\{r_{E,j}\}$ and $I_g(E)$ represents the points away from the non-zero roots, 0 and $\pm\infty$. Then

$$|P_E(r)| \gtrsim_\epsilon 1 \qquad \text{for} \quad r \in I_g(E). \tag{3.18}$$

Thus $P_E(x, y)$ dominates $P(x, y)$ in the region $y = rx^{m_E}$ and $r \in I_g(E)$. Let

$$U_{0,g}(E) = \{(x, y) \in U_0 : y = rx^{m_E},\, r \in I_g(E)\} \tag{3.19}$$

be the "good" regions generated by the edge E. One can see that $U_{0,g}(E)$ is a disjoint union of $(J_E + 2)$ "good" regions: $U_{0,g}(E, j)$. Each "good" region $U_{0,g}(E, j)$ is a curved triangular region of the form

$$U_{0,g}(E, j) = \{(x, y) \in U_0 : b_j x^{m_E} \le y \le B_j x^{m_E}\}, \tag{3.20}$$

where $[b_j, B_j] := I_g(E, j)$ is just a connected sub-interval of $I_g(E)$ and $(J_E + 2)$ is just the number of the connected components of $I_g(E)$.

In the above notations, the subindex 0 in $U_{0,g}(E, j)$ indicates the algorithm is in the 0-th stage, g indicates the region is "good" and E indicates this "good" region is generated by the edge E. The "bad" regions are defined as:

$$U_{0,b}(E, j) = \{(x, y) \in U_0 : y = rx^{m_E},\, r \in I_j^\epsilon(E)\} \tag{3.21}$$

$$= \{(x, y) \in U_0 : (r_j - \epsilon)x^{m_E} < y < (r_j + \epsilon)x^{m_E}\}, \tag{3.22}$$

for $1 \le j \le J_E$. If $\{r_{E,j}\}$ is empty, then there is no "bad" region generated by this edge and the only two "good" regions are

$$U_{0,g}(E) = \{(x, y) \in U_0 : c_E x^{m_E} < |y| < C_E x^{m_E}\}. \tag{3.23}$$

The following lemma states that P behaves almost like a "monomial" in each "good" region $U_{0,g}(E,j)$.

Lemma 3.1. *Given any positive integers N and L, assume $\epsilon = \mathrm{rad}(U_0)$ is sufficiently small (depends on N, L and P), then for all $E \in \mathcal{N}(P)$ and $(x,y) \in U_{0,g}(E,j)$, one has*

$$|x^{p_{E,l}} y^{q_{E,l}}| \sim |P_E(x,y)| \geq 2^N |P(x,y) - P_E(x,y)|. \tag{3.24}$$

Here $(p_{E,l}, q_{E,l})$ is the left vertex of the edge E. In addition,

$$|\partial_x^\alpha \partial_y^\beta P(x,y)| < C \min\{1, |x^{p_{E,l}-\alpha} y^{q_{E,l}-\beta}|\} \tag{3.25}$$

for $0 \leq \alpha, \beta \leq L$.

Proof. In the region $y = rx^{m_E}$ where $r \in I_g(E)$,

$$|P(x,y) - P_E(x,y)| < Cx^{e_E+\nu}, \tag{3.26}$$

where ν is a positive fraction (can be computed but not necessary). By (3.18), one has $|P_E(r)| \geq C$ for $r \in I_g(E)$, where $C = C(c_E, C_E, \epsilon, P)$ is a positive constant. Thus if $|x|$ is sufficiently small, then for all $(x,y) \in U_{0,g}(E,j)$ we have

$$|P_E(x,y)| \sim |x^{p_v} y^{q_v}| \sim x^{e_E} > 2^N \cdot O(x^{e_E+\nu}) > 2^N |P(x,y) - P_E(x,y)|, \tag{3.27}$$

which proves (3.24).

Now we turn to (3.25). The bound $|\partial_x^\alpha \partial_y^\beta P(x,y)| \lesssim 1$ is trivial. In the region $y = rx^{m_E}$ where $r \in I_g(E)$, for $0 \leq \alpha, \beta \leq L$ and every $(p'', q'') \in E$, one has

$$|x^{p_{E,l}-\alpha} y^{q_{E,l}-\beta}| \sim |x|^{p''-\alpha} |y|^{q''-\beta} \sim |x|^{e_E - \alpha - m_E \beta}, \tag{3.28}$$

even for $p'' - \alpha < 0$ or $q'' - \beta < 0$. Notice $|y| \sim |x|^{m_E}$, then

$$|\partial_x^\alpha \partial_y^\beta (P(x,y) - P_E(x,y))| \lesssim |x|^{e_E + \nu - \alpha - m_E \beta}. \tag{3.29}$$

Thus given $|x|$ sufficiently small, one has

$$|\partial_x^\alpha \partial_y^\beta P(x,y)| \lesssim |x^{p_{E,l} - \alpha} y^{q_{E,l} - \beta}|. \tag{3.30}$$

This completes the the proof of (3.25). □

The above lemma handles the case when an edge E is "dominant" and no cancellation inside P_E. Another easy case is when a vertex $V = (p_v, q_v)$ plays a dominant role. In this case, let E_l and E_r be the edges left and right to V, with slopes $-1/m_{E_l}$ and $-1/m_{E_r}$ respectively. Then $0 \le m_{E_l} < m_{E_r} \le \infty$. Here $m_{E_l} = 0$ means E_l is the vertical non-compact edge and $m_r = \infty$ means E_r is the horizontal non-compact edge. Consider the following region

$$U_{0,g}(V) = \{(x,y) \in U_0 : C_{E_r} x^{m_{E_r}} < |y| < c_{E_l} x^{m_{E_l}}\}, \tag{3.31}$$

where C_{E_r} and c_{E_l} are constants that are assigned to each edges and defined in (3.16).

We can alway choose rad(U_0) sufficiently small (i.e. $|x|$ sufficiently small) s.t. the origin $(0,0)$ is the only interception of $y = C_{E_r} x^{m_{E_r}}$ and $y = c_{E,l} x^{m_{E_l}}$ inside U_0. If $m_{E_r} = \infty$, then V is the far right vertex, set

$$U_{0,g}(V) = \{(x,y) \in U_0 : |y| < c_{E_l} x^{m_{E_l}}\}, \tag{3.32}$$

where the portion of the x-axis inside U_0 is included in $U_{0,g}(V)$. Similarly, if $m_{E_l} = 0$ then V is the far left vertex. We replace $c_{E_l} x^{m_{E_l}}$ by a sufficiently small constant $\epsilon > 0$ in (3.31).

The following lemma the "vertex" analogue of Lemma 3.1. The proof is exactly the same and we omit the details.

Lemma 3.2. *For each $V \in \mathcal{V}(P)$, suppose $C_{E_r} > 0$ is sufficiently large and $c_{E_l} > 0$ is sufficiently small. Given positive numbers N and L, assume* $\mathrm{rad}(U_0) = \epsilon$ *is sufficiently small (depends on N, L, C_{E_r}, c_{E_l} and P), then for $(x, y) \in U_{0,g}(V)$ one has*

$$|x^{p_v} y^{q_v}| \sim |P_V(x, y)| \geq 2^N |P(x, y) - P_V(x, y)| \tag{3.33}$$

and

$$|\partial_x^\alpha \partial_y^\beta P(x, y)| < C \min\{1, |x^{p_v-\alpha} y^{q_v-\beta}|\} \tag{3.34}$$

for $0 \leq \alpha, \beta \leq L$.

Set

$$\mathcal{G}_0(P_0) = \mathcal{G}_0(P) = \{U_{0,g}(V) : V \in \mathcal{V}(P)\} \cup \{U_{0,g}(E, j) : E \in \mathcal{E}(P) \text{ and all } j\}, \tag{3.35}$$

which represents the collection of "good" regions in the 0-th stage. In addition, we say $U_{0,g} \in \mathcal{G}_0(P_0)$ is defined by (E, m_E) if $U_{0,g} = U_{0,g}(E, j)$ for some E and j, where $-1/m_E$ is the slope of E, or defined by an edge E for short. Similarly, $U_{0,g} \in \mathcal{G}_0(P_0)$ is defined by (V, m_l, m_r) represents $U_{0,g} = U_{0,g}(V)$ and $-1/m_l$, $-1/m_r$ are the slopes of the edges left and right to V, or defined by a vertex V for short.

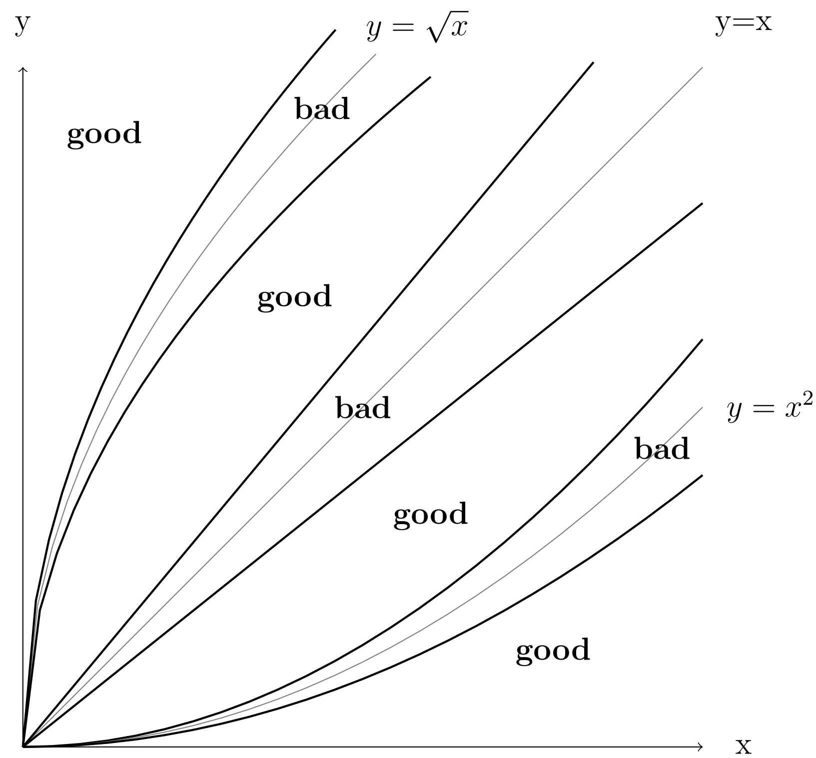
Now we focus on the "bad" regions

$$U_{0,b}(E, j) = \{(x, y) \in U_0 : (r_j - \epsilon)x^{m_E} < y < (r_j + \epsilon)x^{m_E}\} \tag{3.36}$$

and set

$$\mathcal{B}_0(P_0) = \mathcal{B}_0(P) = \{U_{0,b}(E, j) : E \in \mathcal{E}(P) \quad \text{and} \quad 1 \leq j \leq J_E\} \tag{3.37}$$

to represent the collection of "bad" regions in the 0-stage. If $U_{0,b} \in \mathcal{B}_0(P_0)$ has the form of (3.36), we say $U_{0,b}$ is defined by $(E, y = r_j x^{m_E})$ or defined by $y = r_j x^{m_E}$ for short. The following graph demonstrates a partition of U into "good" and "bad" regions, according to the analytic function $P(x, y) = xy(y^2 - x)(y - x^2)(y - \sum_{n=1}^{\infty} x^n)^2$.



"good" regions and "bad" regions of $P(x, y) = xy(y^2 - x)(y - x^2)(y - \sum_{n=1}^{\infty} x^n)^2$
in the first quadrant.

We summarize the above discussion as follows:

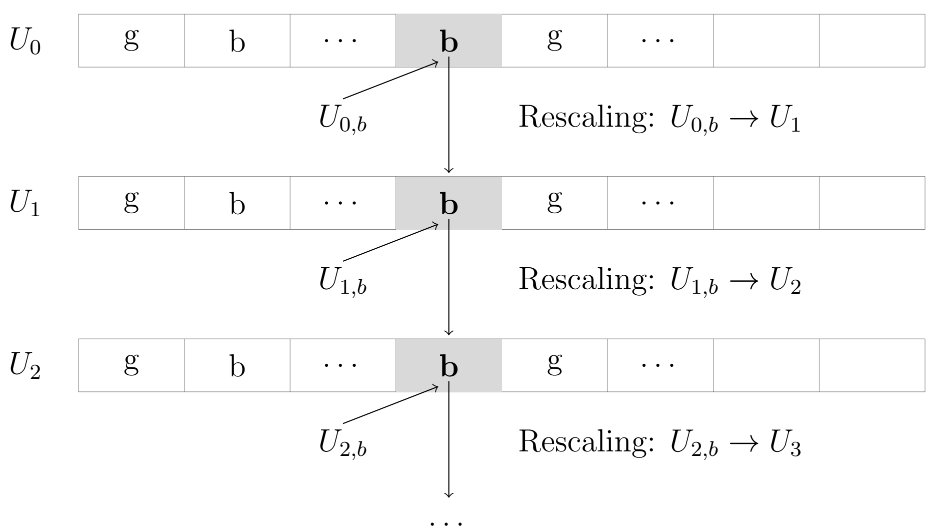
Proposition 3.3 (**A Single step of Partition**).

Let U be a standard region and P be a real analytic function. If $\mathrm{rad}(U)$ is sufficiently small, then U can be partitioned into two families of curved triangular regions: $\mathcal{G}_0(P)$ and $\mathcal{B}_0(P)$. For each $U_{0,g} \in \mathcal{G}_0(P)$, $U_{0,g}$ is defined by (3.20) or (3.31) or (3.32). The behaviors of P in $U_{0,g}$ are characterized by Lemma 3.1 or Lemma 3.2. Each $U_{0,b} \in \mathcal{B}_0(P)$ is defined by (3.36). Finally, the numbers of triangular regions in $\mathcal{G}_0(P)$ and $\mathcal{B}_0(P)$ are finite, depending on P.

3.3 The resolution algorithm Part II: Iterations

The next step is to iterate Proposition 3.3. One main problem is that $U_{0,b} \in \mathcal{B}_0(P)$ is not a standard region. Nevertheless, this difficulty can be overcome by a "‘rescaling" argument. Via an appropriate change of variables, we can always turn a **non-standard** pair $[U_{0,b}, P_0]$ to a **standard** pair $[U_1, P_1]$. Here $P_0 = P$ and P_1 is an analytic function of $(x^{1/M}, y)$. Then Proposition 3.3 is applicable to $[U_1, P_1]$ (the arguments in the previous section work equally well for analytic function of $(x^{1/M}, y)$). "Bad" regions obtained from $[U_1, P_1]$ can be rescaled to standard regions, where Proposition 3.3 can be applied again and so on.

The following graph illustrates the main ideas of how the algorithm runs. The letter "g" bellows represents a "good" region while "b" represents a "bad" region. Each time, we pick up a "bad" region, "rescale" (via change of variable) it into a standard region.



Before diving into the details, we introduce the following notations to characterize some invariants inside each stage of iteration.

Definition 3.2. *Let (p_l, q_l) and (p_r, q_r) be the far left and far right vertices of $\mathcal{N}(P)$, the*

Heights of $\mathcal{N}(P)$ or P are defined as

$$\mathrm{Hght}(\mathcal{N}(P)) = \mathrm{Hght}(P) = q_l - q_r,$$
$$\mathrm{Hght}^*(\mathcal{N}(P)) = \mathrm{Hght}^*(P) = q_l$$

For an edge $E \in \mathcal{E}(P)$, let $(p_{E,l}, q_{E,l})$ and $(p_{E,r}, q_{E,r})$ be its left and right vertices. Then the height of this edge is defined as

$$\mathrm{Hght}(E) = q_{E,l} - q_{E,r}. \tag{3.38}$$

If $\{r_{E,j}\}_{1\le j\le J_E}$ is the set of non-zero roots of $P_E(r)$ of orders $\{s_{E,j}\}_{1\le j\le J_E}$, then we define the order of E as

$$\mathrm{Ord}(E) = \sum_{j=1}^{J_E} s_{E,j} \tag{3.39}$$

and the order of P as

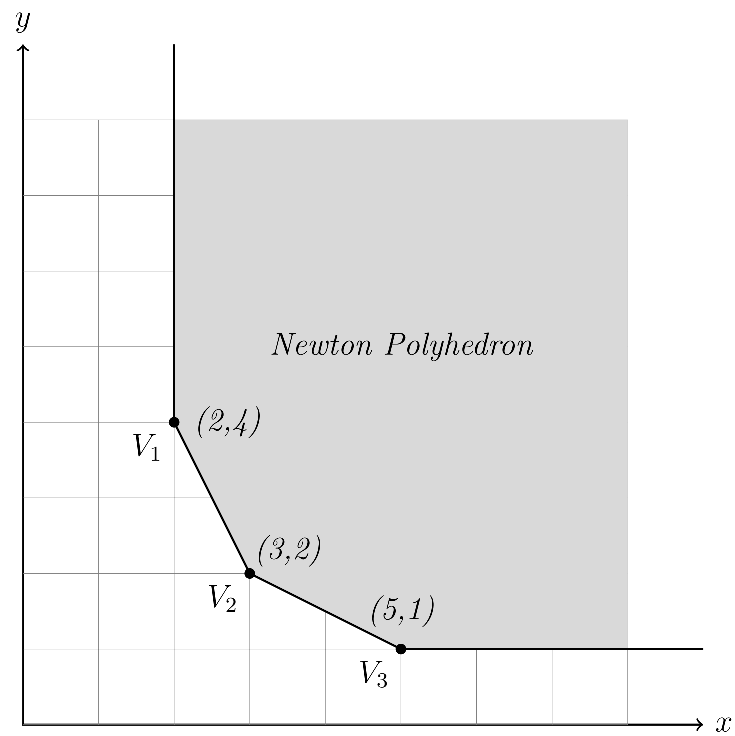
$$\mathrm{Ord}(P) = \sum_{E\in\mathcal{E}(P)} \mathrm{Ord}(E) = \sum_{E\in\mathcal{E}(P)} \sum_{j=1}^{J_E} s_{E,j}. \tag{3.40}$$

Finally, we say r is a root of $P(x,y)$ or $\mathcal{N}(P)$ if $r = r_{E,j}$ for some $E \in \mathcal{E}(P)$ and some $1 \le j \le J_E$.

Example 2. *Consider $P(x,y) = x^5y - x^3y^2 + x^2y^4$. The far left vertex of $\mathcal{N}(P)$ is $V_1 = (2,4)$*

and the far right vertex is $V_3 = (5, 1)$*. Then*

$$\begin{aligned}
&\mathrm{Hght}(\mathcal{N}(P)) = \mathrm{Hght}(P) = 4 - 1 = 3,\\
&\mathrm{Hght}^*(\mathcal{N}(P)) = \mathrm{Hght}^*(P) = 4,\\
&\mathrm{Hght}(V_1V_2) = 2 = \mathrm{Ord}(V_1V_2),\\
&\mathrm{Hght}(V_2V_3) = 1 = \mathrm{Ord}(V_2V_3),\\
&\mathrm{Ord}(P) = 2 + 1 = 3.
\end{aligned}$$



The above definition immediately implies

$$\mathrm{Ord}(E) \leq \mathrm{Hght}(E) = q_{E,l} - q_{E,r} \tag{3.41}$$

and

$$\mathrm{Ord}(P) \leq \mathrm{Hght}(P) = q_l - q_r \leq q_l = \mathrm{Hght}^*(P). \tag{3.42}$$

Since $\#\mathcal{B}_0(P) = \sum_{E\in\mathcal{E}(P)} J_E \leq \sum_{E\in\mathcal{E}(P)} \mathrm{Ord}(E) = \mathrm{Ord}(P)$, we obtained:

Lemma 3.4.

$$\#\mathcal{B}_0(P) \leq \mathrm{Ord}(P) \leq \mathrm{Hght}^*(P). \tag{3.43}$$

Choose a $U_{0,b} \in \mathcal{B}_0(P_0)$ and assume it is defined by $y = r_0 x^{m_0}$. The next step is to utilize change of variables to turn $[U_{0,b}, P]$ into a standard pair. Adopt the previous notations $[U_0, P_0] = [U, P]$ and $(x_0, y_0) = (x, y)$ and choose x to be the principal variable which will be unchanged during the iterations, i.e. $x = x_n$ for all $n \in \mathbb{N}$. Change variables

$$\begin{cases} x_0 = x_1 \\ y_0 = (r_0 + y_1)x_1^{m_0} \end{cases}$$

and set

$$\begin{cases} P_1(x_1, y_1) = P(x_1, (r_0 + y_1)x_1^{m_0}) \\ \\ U_1 = \{(x_1, y_1) : (x_0, y_0) \in U_{0,b}\}. \end{cases}$$

Notice for $(x_1, y_1) \in U_1$, one has

$$\begin{cases} 0 < x_1 < \epsilon \\ -\epsilon < y_1 < \epsilon. \end{cases}$$

Then $[U_1, P_1]$ is a standard pair under the coordinate (x_1, y_1). By applying Proposition 3.3 to $[U_1, P_1]$, a finite collection $\mathcal{G}_1(P_1)$ of "good" regions $U_{1,g}$'s and a finite collection $\mathcal{B}_1(P_1)$ of "bad" regions $U_{1,b}$'s are obtained. In a "good" region $U_{1,g}$, the function $P_1(x_1, y_1)$ behaves like a monomial of (x_1, y_1) and no further treatment is required. For the "bad" regions, choose a $U_{1,b} \in \mathcal{B}_1(P_1)$ and assume $U_{1,b}$ is defined by $y_1 = r_1 x_1^{m_1}$, i.e.

$$U_{1,b} = \{(x_1, y_1) \in U_1 : (r_1 - \epsilon)x_1^{m_1} < y_1 < (r_1 + \epsilon)x_1^{m_1}\}.$$

As we did before, we perform the following change of variables

$$\begin{cases} x_1 = x_2 \\ y_1 = (r_1 + y_2)x_2^{m_1}. \end{cases}$$

Then a new standard pair $[U_2, P_2]$ is obtained:

$$\begin{cases} U_2 = \{(x_2, y_2) :\ (x_1, y_1) \in U_{1,b}\} \\ \\ P_2(x_2, y_2) = P_1(x_2, (r_1 + y_2)x_2^{m_1}). \end{cases}$$

Then the same procedure is repeated on $[U_2, P_2]$ and so on. A collection of standard pairs $\{[U_n, P_n]\}$ is obtained from these iterations. Here the subindex n merely represents $[U_n, P_n]$ is obtained from the n-th stage of iteration (or we say a n-th generation of $[U_0, P_0]$). Notice that for a n, there can be many $[U_n, P_n]$ and the structure of $\{[U_n, P_n]\}$ is a tree (non-linear). If we want to specify the "identity" of $[U_n, P_n]$, set $[U_n, P_n] = [U_{n,\boldsymbol{\alpha}}, P_{n,\boldsymbol{\alpha}}]$ where the subindex $\boldsymbol{\alpha}$ represents the "path" from $[U_0, P_0]$ to $[U_{n,\boldsymbol{\alpha}}, P_{n,\boldsymbol{\alpha}}]$. The subindex $\boldsymbol{\alpha}$ can also be viewed as the code that compresses the genealogy information which is needed to obtain $[U_{n,\boldsymbol{\alpha}}, P_{n,\boldsymbol{\alpha}}]$ from $[U_0, P_0]$; or conversely, $P_0(x_0, y_0)$ can be "decoded" from $P_{n,\boldsymbol{\alpha}}(x_n, y_n)$ by $\boldsymbol{\alpha}$. More precisely:

$(\star)$ $\boldsymbol{\alpha}$ contains the information of the change of variables, i.e. for $0 \le k \le n-1$ the following is known if $\boldsymbol{\alpha}$ is given:

$$\begin{cases} x_k = x_{k+1} \\ y_k = (r_k + y_{k+1})x_{k+1}^{m_k}. \end{cases} \tag{3.44}$$

We also use $U_{n,b,\boldsymbol{\alpha}}$ and $U_{n,g,\boldsymbol{\alpha}}$ to represent an arbitrary "bad" and "good" regions in $U_{n,\boldsymbol{\alpha}}$. Since $U_{n,\boldsymbol{\alpha}}$ may have more than one such regions, we list them by $U_{n,b,\boldsymbol{\alpha},j}$ and $U_{n,g,\boldsymbol{\alpha},j}$ when necessary. For a fixed n, the cardinality of $(\boldsymbol{\alpha}, j)$ is uniformly bounded by $\mathrm{Ord}(P)$ (see

Lemma 3.5) and there is no need to specify its range.

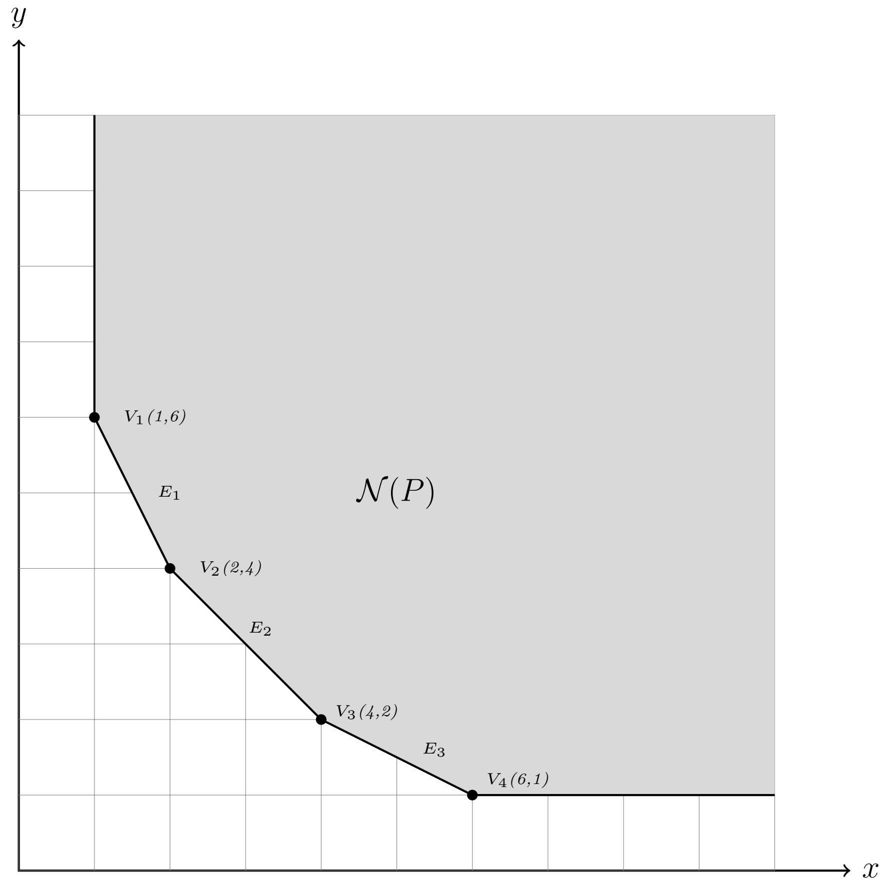
Both notations: with and without subindex $\boldsymbol{\alpha}$, are being used. Not to confuse the reader, we follow the rules below:

(1) $[U_n, P_n]$ is our priority choice, in particular for **an arbitrary** pair from the n-th stage of iteration.

(2) $[U_{n,\boldsymbol{\alpha}}, P_{n,\boldsymbol{\alpha}}]$ is a secondary choice. It is often employed when at least two different pairs from the same stage of iteration appear simultaneously.

The above conventions also apply to $U_{n,g}$'s, $U_{n,b}$'s, $U_{n,g,\boldsymbol{\alpha}}$ and $U_{n,b,\boldsymbol{\alpha}}$.

Example 3. *The following graphs demonstrate the first step of the algorithm, for the given analytic function* $P(x,y) = xy(y^2 - x)(y - x^2)(y - \sum_{n=1}^{\infty} x^n)^2$. *The Newton polyhedron* $\mathcal{N}(P)$ *has 3 compact edges:* E_1, E_2 *and* E_3, *see Figure A below.*



$\mathrm{Hght}^*(P) = 6$

$\mathrm{Hght}(P) = 5$

$\mathrm{Ord}(E_1) = 1$

$\mathrm{Ord}(E_2) = 2$

$\mathrm{Ord}(E_3) = 1$

$\mathrm{Ord}(P) = 4$

Figure A

When P is restricted in the edge E_1, $P_{E_1} = xy(y^2 - x) \cdot y \cdot y^2$. The only non-zero root is $y = x^{\frac{1}{2}}$. Change of variables: $(x, y) = (x_1, x_1^{\frac{1}{2}}(1+y_1))$ yields $P_1(x_1, y_1) = P(x_1, x^{\frac{1}{2}}(1+y_1)) = x_1^4 y_1 \cdot O(1)$. See Figure B, $\mathcal{N}(P_1)$ has only one vertex and the algorithm stops.

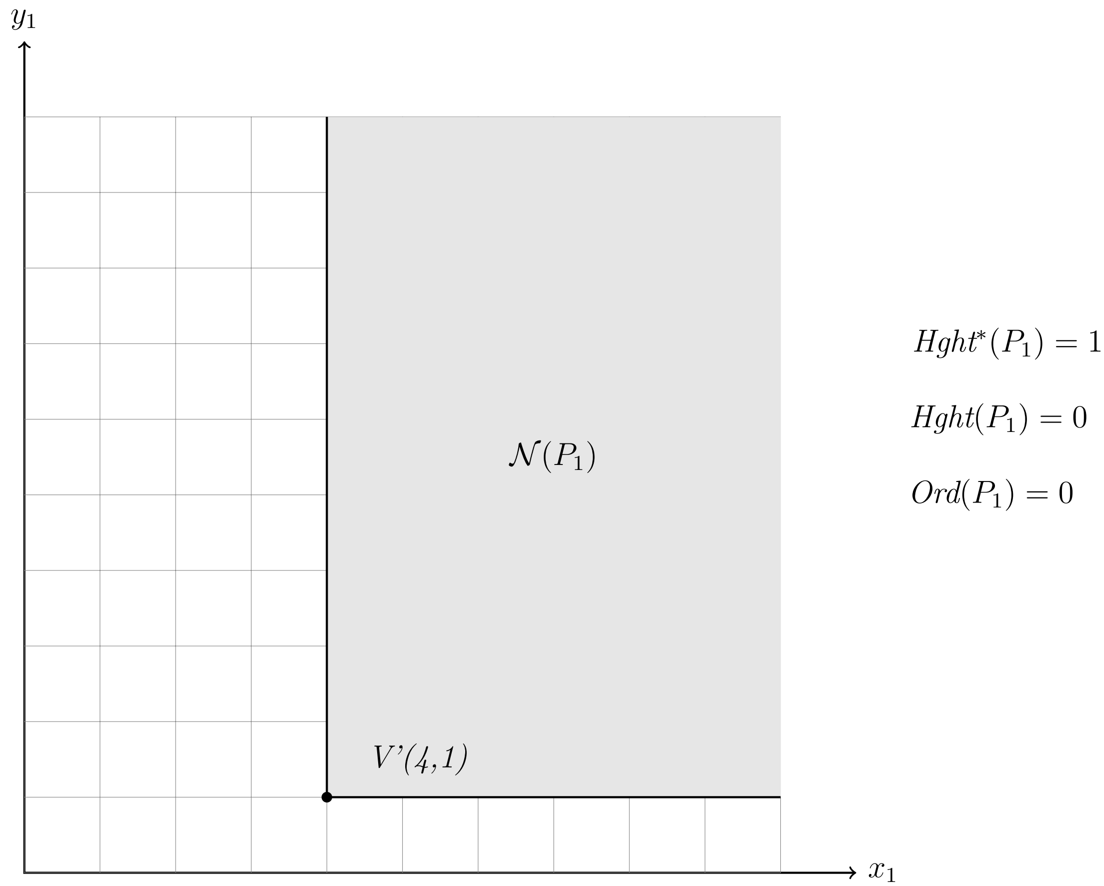
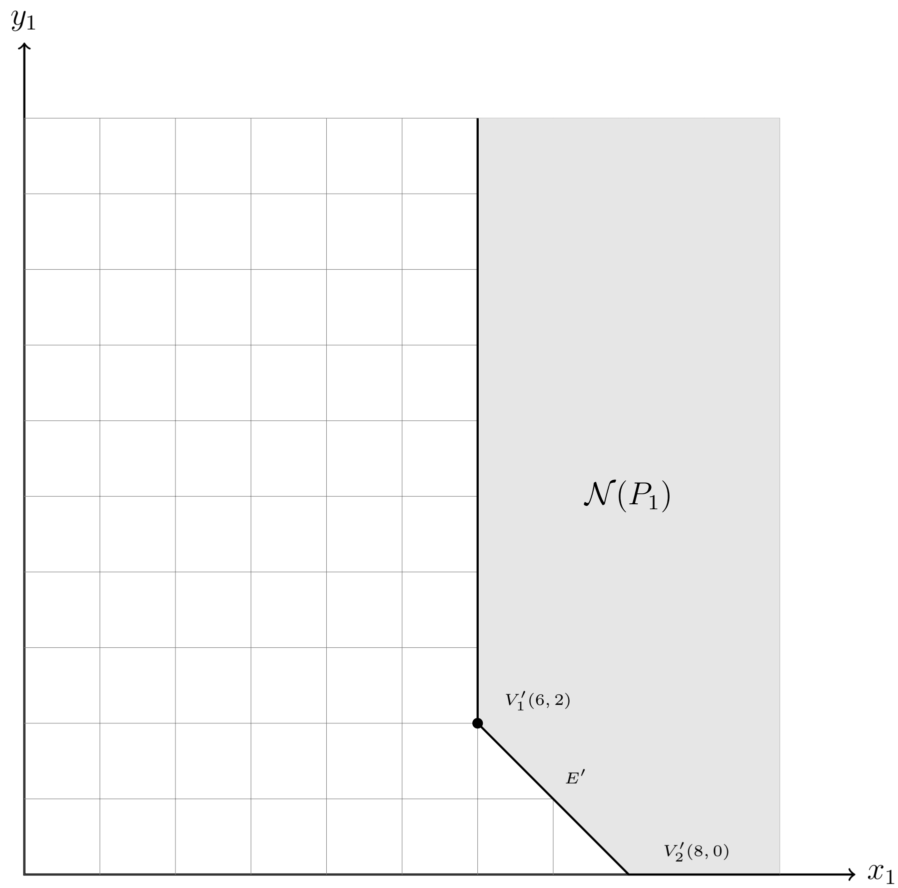


Figure B

When P is restricted in the edge E_2, $P_{E_2} = -xy \cdot x \cdot y \cdot (y-x)^2$. The only non-zero root is $y = x$. Change of variables: $(x, y) = (x_1, x_1(1+y_1))$ yields $P_1(x_1, y_1) = x_1^6(y_1 - \sum_{n=1}^{\infty} x^n)^2 \cdot O(1)$. See Figure C, $\mathcal{N}(P_1)$ has one edge. The algorithm still runs. If we keep doing the change of variables of the form $(x_{k-1}, y_{k-1}) = (x_k, x_k(1+y_k))$, we can see that $\mathcal{N}(P_k) = \mathcal{N}(P_{k-1}) + (2, 0)$ for $k \geq 2$.



$Hght^*(P_1) = 2$

$Hght(P_1) = 2$

$Ord(P_1) = 2$

$Ord(E') = 2$

Figure C

Finally, $P_{E_3} = xy(-x)(y-x^2)(-x)^2$. The only non-zero root is $y = x^2$. Change of variables: $(x, y) = (x_1, x_1^2(1 + y_1))$ yields $P_1(x_1, y_1) = x_1^8 y_1 \cdot O(1)$. See Figure D, $\mathcal{N}(P_1)$ has only one vertex and the algorithm stops.

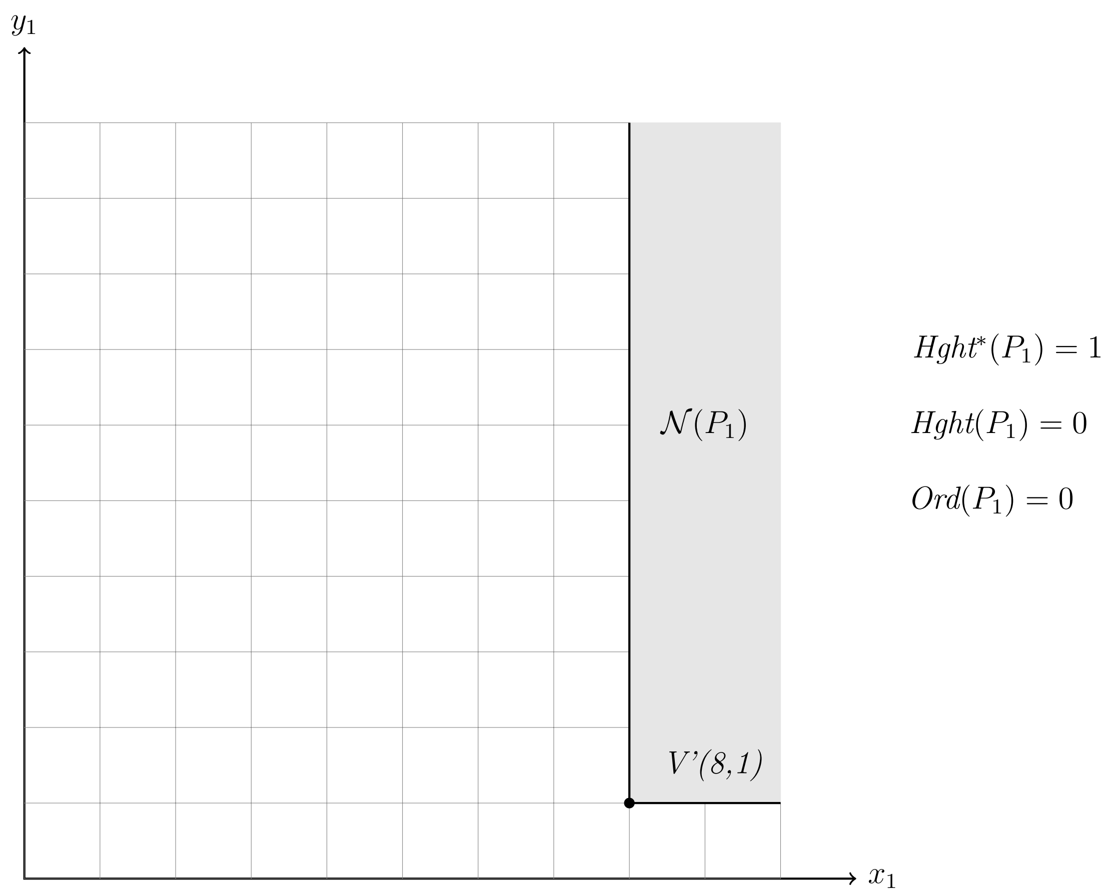


Figure D

It is worth mentioning that, the change of variables: $(x_n, y_n) \to (x_{n+1}, y_{n+1})$ acts as a diffeomorphism from $U_{n,b}$ to U_{n+1}. Thus one can diffeomorphically embed U_{n+1} into U_n and a chain of diffeomporphic embeddings is obtained:

$$\cdots \to U_{n+2} \to U_{n+1} \to U_n \to \cdots \to U_1 \to U_0.$$

In addition, if the change of variables: $(x, y) \to (x_n, y_n)$ is specified, then we are allowed to identify $(x, y) \in U_n$ (or $U_{n,b}$, $U_{n,g}$) with $(x_n, y_n) \in U_n$ (or $U_{n,b}$, $U_{n,g}$). To be precise, there is a deffeomorphism ρ_n^{-1} :

$$\rho_n^{-1} : U_n \longmapsto \rho_n^{-1}(U_n) \subset U_0 \tag{3.45}$$

$$(x_n, y_n) \longmapsto (x, y) \tag{3.46}$$

where (3.46) is defined by the composition of change of variables (3.44). More precisely, $(x, y) = \rho_n^{-1}(x_n, y_n)$ is given by

$$\begin{cases} x = x_n \\ y = r_0 x^{m_0} + r_1 x^{m_0+m_1} + \cdots + r_{n-1} x^{m_0+\cdots+m_{n-1}} + y_n x^{m_0+\cdots+m_{n-1}}. \end{cases} \tag{3.47}$$

Under this notation, $P_n = P \circ \rho_n^{-1}$.

If U_n is specified to $U_{n,\boldsymbol{\alpha}}$, then ρ_n^{-1} is also specified to $\rho_{n,\boldsymbol{\alpha}}^{-1}$. Notice that for all j, $U_{n,g,\boldsymbol{\alpha},j}$'s and $U_{n,b,\boldsymbol{\alpha},j}$'s share the same $\rho_{n,\boldsymbol{\alpha}}^{-1}$ with $U_{n,\boldsymbol{\alpha}}$. In particular, $\{\rho_{n,\boldsymbol{\alpha}}^{-1}(U_{n,g,\boldsymbol{\alpha},j})\}_{n,\boldsymbol{\alpha},j}$ are disjoint curved triangular regions in U_0. It will become clear later, $\{\rho_{n,\boldsymbol{\alpha}}^{-1}(U_{n,g,\boldsymbol{\alpha},j})\}_{n,\boldsymbol{\alpha},j}$ will form a finite disjoint partition of U_0.

The reader may now have a clear picture of how this resolution algorithm runs. Still, there are two questions need to be answered:

(i) In each stage of iteration, is the cardinality of $\{U_{n,b,\boldsymbol{\alpha},j}\}_{\boldsymbol{\alpha},j}$ bounded above uniformly?

(ii) Does this procedure end up in a finite steps?

The answer to the first question is Yes and the upper bound can be controlled by $\mathrm{Ord}(P)$; to (ii), the answer is still Yes, but a refinement of change of variables is needed!

We provide the answer to (i) first.

Lemma 3.5. *For each $n \geq 0$, the cardinality of $\{U_{n,b,\boldsymbol{\alpha},j}\}_{\boldsymbol{\alpha},j}$ is bounded by* $\mathrm{Ord}(P)$.

Proof. Indeed, there is a bijection between $\{U_{0,b}\}$ and the non-zero roots of $P_E(r)$, $E \in \mathcal{E}(P)$: each $U_{0,b} = U_{0,b}(E, j)$ is defined by $(E, y = r_{E,j} x^{m_E})$. Assume the order of $r_{E,j}$ is $s_{E,j}$. Then

$P_1(x_1,y_1) = P_{1,E,j}(x_1,y_1)$ is obtained by setting $P_1(x_1,y_1) = P_0(x_1,(y_1+r_{E,j})x^{m_E})$. Here, $P_{1,E,j}$ is used to specify that P_1 is defined by the root $r_{E,j}$. The following observation serves as an bridge between $\mathcal{N}(P_0)$ and $\mathcal{N}(P_1)$. Let $(p_{E,l}, q_{E,l})$ be the left vertex of E and $(p_{1,l}, q_{1,l})$ be the far left vertex of $\mathcal{N}(P_1)$ then

$$\begin{cases} p_{1,l} = p_{E,l} + m_E \cdot q_{E,l} \\ \\ q_{1,l} = s_{E,j}. \end{cases} \tag{3.48}$$

This implies the order $s_{E,j}$ (in the 0-stage) is equal to the $\text{Hght}^*(P_{1,E,j})$ (in the 1-stage). It is worth pointing out that the identities (3.48) are extremely useful for relating parameters of the Newton polyhedra in different stage of iterations. To prove (3.48), notice first

$$P_E(x,y) = P_E(x_1,(y_1+r_{E,j})x_1^{m_E}) = x_1^{p_{E,l}+m_E\cdot q_{E,l}} y_1^{s_{E,j}} \cdot O(1). \tag{3.49}$$

Indeed, the fact that the degree of y_1 is $s_{E,j}$ follows from the fact that $r_{E,j}$ is a root of $P_E(r)$ of order $s_{E,j}$. Moreover, every term in

$$P_1(x_1,y_1) - P_E(x_1,(y_1+r_{E,j})x_1^{m_E}) \tag{3.50}$$

has a x_1-degree strictly greater than $(p_{E,l} + m_E \cdot q_{E,l})$. Thus $(p_{E,l} + m_E \cdot q_{E,l}, s_{E,j})$ is the far left vertex of $\mathcal{N}(P_1)$.

Immediately, one obtains $\text{Ord}(P_{1,E,j}) = \text{Ord}(P_1) \le \text{Hght}^*(P_1) = s_{E,j}$. Thus the number of "bad" regions $U_{1,b}$'s coming from a single P_1 is no more than $\text{Ord}(P_1) \le s_{E,j}$. Counting all possible P_1 (coming from different roots of different edges), the number of all possible $U_{1,b}$ is thus no more than

$$\sum_{E\in\mathcal{E}(P)} \sum_{1\le j\le J_E} \text{Ord}(P_{1,E,j}) \le \sum_{E\in\mathcal{E}(P)} \sum_{1\le j\le J_E} s_{E,j} = \text{Ord}(P). \tag{3.51}$$

The cases when $n \geq 2$ follow from iterating (3.51). □

We now turn to the second question, which is the most crucial for the algorithm. Suppose the procedure does not stop. Thus we obtain an infinite chain of pairs:

$$[U_0, P_0] \to [U_1, P_1] \to [U_2, P_2] \to \cdots \to [U_n, P_n] \to [U_{n+1}, P_{n+1}] \to \dots \tag{3.52}$$

We shall find certain stable pattern inside the above chain. Specify the change of variables from $[U_n, P_n] \to [U_{n+1}, P_{n+1}]$ as

$$\begin{cases} x_n = x_{n+1} \\ y_n = (r_n + y_{n+1})x_{n+1}^{m_n} \end{cases}$$

Then r_n is a root of an edge in $\mathcal{N}(P_n)$. We assume s_n is the order of r_n. Let $(p_{n,l}, q_{n,l})$ be the **most** left vertex of $\mathcal{N}(P_n)$ and (p_n, q_n) be the left vertex of the edge in $\mathcal{N}(P_n)$ that defines $[U_{n+1}, P_{n+1}]$. By (3.48) one has

$$\begin{cases} p_{n+1} \geq p_{n+1,l} = p_n + m_n \cdot q_n \\ \\ q_{n+1} \leq q_{n+1,l} = s_n \leq q_n \end{cases} \tag{3.53}$$

and thus

$$\mathrm{Hght}^*(P_0) \geq \mathrm{Hght}(P_0) \geq s_0 = \mathrm{Hght}^*(P_1) \geq \mathrm{Hght}(P_1) \geq s_1 = \mathrm{Hght}^*(P_2) \geq \tag{3.54}$$

$$\cdots \geq s_{n-1} = \mathrm{Hght}^*(P_n) \geq \mathrm{Hght}(P_n) \geq s_n = \mathrm{Hght}^*(P_{n+1}) \cdots \tag{3.55}$$

Notice that for all n, $\mathrm{Hght}(P_n)$ and s_n must be positive integers. Otherwise, if $\mathrm{Hght}(P_n) = 0$ then $\mathcal{N}(P_n)$ has no edge and thus no root; if $s_n = 0$, then $\mathcal{N}(P_n)$ has no root. In both situations, the chain ends at the n-stage, which contradicts to our assumption.

Since (3.54) is an infinite sequence and $\mathrm{Hght}^*(P_0)$ is a finite positive integer, there is a least integer $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\mathrm{Hght}^*(P_n) = \mathrm{Hght}(P_n) = s_n = \mathrm{Hght}^*(P_{n_0}) = \mathrm{Hght}(P_{n_0}) = s_{n_0} > 0. \tag{3.56}$$

This implies that for **every** $n \geq n_0$:

(i) $\mathcal{N}(P_n)$ has only one compact edge E_n,

(ii) in this edge E_n, $P_n(x_n, y_n)$ has only one root r_n of order $s_n = s_{n_0}$,

(iii) when P_n is restricted in E_n, $P_{n,E_n}(x_n, y_n) = c_n(y_n - r_n x_n^{m_n})^{s_n}$, where c_n is some non-zero constant.

This is the exact pattern we are looking for. The following lemma shows that the chain (3.52) essentially ends up at the (n_0+1)-stage.

Lemma 3.6. *Assume we have an infinite chain (3.52) and n_0 is the constant defined in (3.56), then*

$$P_{n_0}(x_{n_0}, y_{n_0}) = x_{n_0}^{p_{n_0}}(y_{n_0} - f(x_{n_0}))^{s_{n_0}} Q_{n_0}(x_{n_0}, y_{n_0}) \tag{3.57}$$

where

$$f(x_{n_0}) = \sum_{n=n_0}^{\infty} r_n x_{n_0}^{m_{n_0}+m_{n_0+1}+\cdots+m_n} \tag{3.58}$$

is an analytic function of $x_{n_0}^{1/M}$ and $Q_{n_0}(x_{n_0}, y_{n_0})$ is an analytic function of $(x_{n_0}^{1/M}, y_{n_0})$ with $Q_{n_0}(0,0) \neq 0$, where M is a large integer depending on P.

Proof. To obtain $P_{n_0}(x_{n_0}, y_{n_0})$ from $P_0(x, y)$, we have only iterated finitely many steps. Thus $P_{n_0}(x_{n_0}, y_{n_0})$ is a real analytic function of $(x_{n_0}^{1/M}, y_{n_0})$, for some integer M. For $n \geq n_0$, the change of variables from $[U_n, P_n]$ to $[U_{n+1}, P_{n+1}]$ is $x_n = x_{n+1}$ and $y_n = (y_{n+1} + r_n)x^{m_n}$. The only compact edge E_n of P_n is of the form

$$P_{n,E_n}(x_n, y_n) = c_n x_n^{p_n}(y_n - r_n x^{m_n})^{s_n}, \tag{3.59}$$

where c_n is a nonzero constant and $s_n = s_{n_0}$. Using induction, it is not difficult to prove that $m_n M$ is an integer for all $n \geq n_0$. Thus $P_n(x_n, y_n)$ is a real analytic function of $(x_n^{1/M}, y_n)$ for all $n \in \mathbb{N}$. Since $(p_n + s_n m_n, 0)$ is the far right vertex of $\mathcal{N}(P_n)$, by setting $y_n = 0$, (3.59) yields

$$P_n(x_n, 0) = C_n x_n^{p_n+s_n m_n} + O(x_n^{p_n+s_n m_n+\nu}), \tag{3.60}$$

where $\nu > 0$. Consider the partial sum of $f(x_{n_0})$,

$$f_k(x_{n_0}) = \sum_{n=n_0}^{k} r_n x_{n_0}^{m_{n_0}+m_{n_0+1}+\cdots+m_n}, \quad k \geq n_0. \tag{3.61}$$

Then $y_{n_0} = y_n x_{n_0}^{m_{n_0}+m_{n_0+1}+\cdots+m_{n-1}} + f_{n-1}(x_{n_0})$, $n \geq n_0 + 1$. Notice

$$P_n(x_n, y_n) = P_{n_0}(x_n, y_n x_{n_0}^{m_{n_0}+m_{n_0+1}+\cdots+m_{n-1}} + f_{n-1}(x_{n_0})). \tag{3.62}$$

By (3.60), we have

$$P_{n_0}(x_{n_0}, f_{n-1}(x_{n_0})) = P_n(x_n, 0) = C_n x_n^{p_n+s_n m_n} + O(x_n^{p_n+s_n m_n+\nu}). \tag{3.63}$$

Notice $m_n \geq \frac{1}{M}$ and $s_n = s_{n_0}$ is a positive integer, thus

$$p_n + s_n m_n \to \infty \qquad \text{as} \qquad n \to \infty,$$

and

$$P_{n_0}(x_{n_0}, f(x_{n_0})) = 0. \tag{3.64}$$

This yields $P_{n_0}(x_{n_0}, y_{n_0})$ has a factor $(y_{n_0} - f(x_{n_0}))$. We still need to show its order s is exactly s_{n_0}. Assume,

$$P_{n_0}(x_{n_0}, y_{n_0}) = x_{n_0}^{p_{n_0}}(y_{n_0} - f(x_{n_0}))^s Q_{n_0}(x_{n_0}, y_{n_0}), \tag{3.65}$$

where $Q_{n_0}(x_{n_0}, f(x_{n_0})) \neq 0$. It is not difficult to see all the terms in (3.65) are analytic functions of $(x_{n_0}^{1/M}, y_{n_0})$. Notice

$$Q_{n_0}(x_{n_0}, y_{n_0}) = Q_{n_0}(x_{n_0}, f(x_{n_0})) + \Big(Q_{n_0}(x_{n_0}, y_{n_0}) - Q_{n_0}(x_{n_0}, f(x_{n_0}))\Big)$$

and $\big(Q_{n_0}(x_{n_0}, y_{n_0}) - Q_{n_0}(x_{n_0}, f(x_{n_0}))\big)$ is divisible by $(y_{n_0} - f(x_{n_0}))$. Assume the leading term of $Q_{n_0}(x_{n_0}, f(x_{n_0}))$ is $Cx_{n_0}^A$ (the term with lowest degree). Then

$$Q_{n_0}(x_{n_0}, f_{n-1}(x_{n_0})) = Cx_{n_0}^A + O(x_{n_0}^{A+\nu}) \qquad \text{as} \qquad n \to \infty.$$

Combining (3.65), one has

$$P_{n_0}(x_{n_0}, f_{n-1}(x_{n_0})) = Cx_{n_0}^{p_{n_0}+s(m_{n_0}+\cdots+m_n)+A} + O(x_{n_0}^{p_{n_0}+s(m_{n_0}+\cdots+m_n)+A+\nu}), \tag{3.66}$$

as $n \to \infty$. Notice that, for all $n > n_0$,

$$p_n = p_{n-1} + s_{n-1}m_{n-1} = p_{n-1} + s_{n_0}m_{n-1}, \tag{3.67}$$

which yields

$$p_n + s_n m_n = p_{n_0} + s_{n_0}(m_{n_0} + \cdots + m_n). \tag{3.68}$$

Comparing (3.63) and (3.66) yields

$$\begin{cases} A = 0 \\ s = s_0 \\ C \neq 0, \end{cases}$$

as desired.

□

Based on Lemma 3.6, in the n_0-stage, we refine the change of variables as follows:

$$x_{n_0} = x_{n_0+1} \quad \text{and} \quad y_{n_0} - f(x_{n_0}) = y_{n_0+1} x_{n_0}^{m_{n_0}}. \tag{3.69}$$

By (3.57) one has

$$\begin{aligned} P_{n_0+1}(x_{n_0+1}, y_{n_0+1}) :&= P_{n_0}(x_{n_0+1}, y_{n_0+1} x_{n_0+1}^{m_{n_0}} + f(x_{n_0+1})) & (3.70) \\ &= x_{n_0+1}^{p_{n_0}+s_{n_0}m_{n_0}} y_{n_0+1}^{s_{n_0}} Q_{n_0}(x_{n_0+1}, y_{n_0+1} x_{n_0+1}^{m_{n_0}} + f(x_{n_0+1})) & (3.71) \\ :&= x_{n_0+1}^{p_{n_0}+s_{n_0}m_{n_0}} y_{n_0+1}^{s_{n_0}} Q_{n_0+1}(x_{n_0+1}, y_{n_0+1}) & (3.72) \end{aligned}$$

and $Q_{n_0+1}(0,0) \neq 0$. This implies $\mathcal{N}(P_{n_0+1})$ has only one vertex! Set

$$U_{n_0+1} = \{(x_{n_0+1}, y_{n_0+1}) : (x_{n_0}, y_{n_0}) \in U_{n_0,b}\},$$

then $U_{n_0+1,g} = U_{n_0+1}$, $U_{n_0+1,b} = \emptyset$ and the procedure ends up here. Thus if we take N_P to be the maximum of all possible $n_0 + 1$, then the resolution procedure ends up at the N_P-stage.

Set

$$\mathcal{G}_n = \cup_{\boldsymbol{\alpha}} \cup_j \{U_{n,g,\boldsymbol{\alpha},j}\}, \tag{3.73}$$

$$\mathcal{V}_n = \cup_{\boldsymbol{\alpha}} \mathcal{V}(P_{n,\boldsymbol{\alpha}}), \tag{3.74}$$

$$\mathcal{E}_n = \cup_{\boldsymbol{\alpha}} \mathcal{E}(P_{n,\boldsymbol{\alpha}}) \tag{3.75}$$

which represent the "good" regions, vertices and compact edges in the n-th stage respectively. The followings represent all the "good" regions, vertices and compact edges in all stages:

$$\mathcal{G} = \cup_{0 \le n \le N_P} \mathcal{G}_n = \cup_n \cup_{\boldsymbol{\alpha}} \cup_j \{U_{n,g,\boldsymbol{\alpha},j}\}, \tag{3.76}$$

$$\mathcal{V} = \cup_{0 \le n \le N_P} \mathcal{V}_n = \cup_n \cup_{\boldsymbol{\alpha}} \mathcal{V}(P_{n,\boldsymbol{\alpha}}), \tag{3.77}$$

$$\mathcal{E} = \cup_{0 \le n \le N_P} E_n = \cup_n \cup_{\boldsymbol{\alpha}} \mathcal{E}(P_{n,\boldsymbol{\alpha}}). \tag{3.78}$$

Theorem 3.7. *Let P be a real analytic function and U be a sufficiently small neighborhood of $(0,0)$. Then U can be partitioned into a finite collection of "good" regions*

$$\{\rho_n^{-1}(U_{n,g}), U_{n,g} \in \mathcal{G}\} \tag{3.79}$$

where

$$\rho_n^{-1}(x_n, y_n) = (x, y) = (x_0, y_0) \tag{3.80}$$

is defined inductively by

$$\begin{cases} x_k = x_{k+1} \\ y_k = (r_k + y_{k+1}) x_{k+1}^{m_k} \end{cases} \tag{3.81}$$

for $k = 0, \dots, n-1$ *via the chain*

$$[U_0, P_0] \to [U_1, P_1] \to \cdots \to [U_n, P_n]. \tag{3.82}$$

However, if (3.83) is a subchain of an infinite chain:

$$[U_0, P_0] \to [U_1, P_1] \to \cdots \to [U_n, P_n] \to [U_{n+1}, P_{n+1}] \to \dots \tag{3.83}$$

and $n-1 = n_0$ *for some* n_0 *defined as in (3.56), then the last step of the change of variables is redefined by*

$$\begin{cases} x_{n_0} = x_{n_0+1} \\ y_{n_0} - f(x_{n_0}) = y_{n_0+1} x_{n_0}^{m_{n_0}} \end{cases} \tag{3.84}$$

where

$$f(x_{n_0}) = \sum_{k=n_0}^{\infty} r_k x_{n_0}^{m_{n_0}+m_{n_0+1}+\cdots+m_k}. \tag{3.85}$$

For $0 \le k \le n-1$*, let* (p_k, q_k) *be the left vertex of the edge where* $U_{k,b} \subset U_k$ *is defined by and* (p_n, q_n) *be defined as below: if* $U_{n,g}$ *is defined by a vertex* V*, then* $(p_n, q_n) = V$*; otherwise* $U_{n,g}$ *is defined by an edge for some* $E \in \mathcal{E}(P_n)$*, set* (p_n, q_n) *to be the left vertex of* E*.*

Then for any given $L \in \mathbb{N}$*, for all* $0 \le \alpha, \beta \le L$ *and* $(x, y) = \rho_n^{-1}(x_n, y_n) \in \rho_n^{-1}(U_{n,g})$ *one has*

$$|P(x, y)| = |P_n(x_n, y_n)| \sim |x_n^{p_n} y_n^{q_n}| \tag{3.86}$$

$$|\partial_{x_n}^{\alpha} \partial_{y_n}^{\beta} P_n(x_n, y_n)| \lesssim \min\{1, |x_n^{p_n-\alpha} y_n^{q_n-\beta}\}| \tag{3.87}$$

and

$$|\partial_x^\alpha \partial_y^\beta P(x,y)| \lesssim \min\{1, |x^{p_n-\alpha-\beta(m_0+\cdots+m_{n-1})} y_n^{q_n-\beta}|\}. \tag{3.88}$$

Proof. The partition in the theorem is a consequence of the algorithm. In addition, (3.86) and(3.87) come from Lemma 3.1 and Lemma 3.2. Finally, (3.88) is a an outcome of the chain rule. Indeed, notice

$$y = y_0 = r_0 x^{m_0} + r_1 x^{m_0+m_1} + \cdots + r_{n-1} x^{m_0+m_1+\cdots+m_{n-1}} + y_n x^{m_0+\cdots+m_{n-1}} \tag{3.89}$$

and

$$\partial y/\partial y_n = x^{m_0+\cdots+m_{n-1}}. \tag{3.90}$$

Then (3.87) yields

$$|\partial_x^\alpha \partial_y^\beta P(x,y)| \lesssim |x^{p_n-\alpha-\beta(m_0+\cdots+m_{n-1})} y_n^{q_n-\beta}|. \tag{3.91}$$

□

3.4 A smooth partition

In the above theorem, U is partitioned into a finite collection of **disjoint** "good" regions $\rho_n^{-1}(U_{n,g})$'s. In what follows, we enlarge each "good" regions $\rho_n^{-1}(U_{n,g})$ and allow them to **overlap**. This can help us to overcome some technical problems, e.g. the convexity assumption in Theorem 2.1.

To do so, we need enlarge $U_{n,g}$ to $U_{n,g} \subset U_{n,g}^{*} \subset U_{n,g}^{**}$ and $U_{n,b}$ to $U_{n,b} \subset U_{n,b}^{*} \subset U_{n,b}^{**}$. The first step is to enlarge $I(E)$, $I_g(E)$ and $I_b(E)$ as follows

$$I^*(E) = [\frac{1}{2}c_E, 2C_E] \cup [-2C_E, -\frac{1}{2}c_E], \tag{3.92}$$

$$I^{**}(E) = [\frac{1}{4}c_E, 4C_E] \cup [-4C_E, -\frac{1}{4}c_E], \tag{3.93}$$

$$I_g^*(E) = I^*(E) \setminus \left(\cup_{1\leq j\leq J_E} I_j^{\frac{\epsilon}{2}}(E)\right), \tag{3.94}$$

$$I_g^{**}(E) = I^{**}(E) \setminus \left(\cup_{1\leq j\leq J_E} I_j^{\frac{\epsilon}{4}}(E)\right) \tag{3.95}$$

and

$$I_b^*(E) = \cup_{1\leq j\leq J_E} I_j^{2\epsilon}(E), \qquad I_b^{**}(E) = \cup_{1\leq j\leq J_E} I_j^{4\epsilon}(E). \tag{3.96}$$

Notice that we can always choose ϵ sufficiently small such that for all E, $\{I_j^{4\epsilon}(E)\}$ does not overlap. Then we can defined the enlarged "good" regions as

$$U_{0,g}^*(E) = \{(x,y) \in U_0 : y = rx^{m_E},\ r \in I_g^*(E)\} \tag{3.97}$$

and

$$U_{0,g}^{**}(E) = \{(x,y) \in U_0 : y = rx^{m_E},\ r \in I_g^{**}(E)\}. \tag{3.98}$$

Both $U_{0,g}^*(E)$ and $U_{0,g}^{**}(E)$ consist of $(J_E + 2)$ curved triangular regions $\{U_{0,g}^*(E,j)\}$ and $\{U_{0,g}^{**}(E,j)\}$ respectively. In addition, one has

$$U_{0,g}(E,j) \subset U_{0,g}^*(E,j) \subset U_{0,g}^{**}(E,j). \tag{3.99}$$

The enlarged "good" regions defined by a vertex are:

$$U^*_{0,g}(V) = \{(x,y) \in U_0 : \frac{C_{E_r}}{2} x^{m_{E_r}} < y < 2c_{E_l} x^{m_{E_l}}\}, \tag{3.100}$$

$$U^{**}_{0,g}(V) = \{(x,y) \in U_0 : \frac{C_{E_r}}{4} x^{m_{E_r}} < y < 4c_{E_l} x^{m_{E_l}}\}, \tag{3.101}$$

and finally the enlarged "bad" regions are

$$U^*_{0,b}(E,j) = \{(x,y) \in U_0 : (r_j - 2\epsilon) x^{m_E} < y < (r_j + 2\epsilon) x^{m_E}\}, \tag{3.102}$$

$$U^{**}_{0,b}(E,j) = \{(x,y) \in U_0 : (r_j - 4\epsilon) x^{m_E} < y < (r_j + 4\epsilon) x^{m_E}\}. \tag{3.103}$$

For $n \geq 1$, $U^*_{n,g}$'s, $U^*_{n,b}$'s and $U^{**}_{n,g}$'s, $U^{**}_{n,b}$'s are defined similarly. Since $\epsilon > 0$ can be chosen arbitrary small, the above definitions do not cause any conflict. We have:

Corollary 3.8. *For all $(x_n, y_n) \in U^{**}_{n,g}$ and all $U_{n,g} \in \mathcal{G}$, the estimates (3.86), (3.87) and (3.88) in Theorem 3.7 still hold.*

We address some technical problems first. Let c be a positive constant such that neither $y = cx$ nor $y = -cx$ is a solution of $P_E(x,y) = 0$ for any $E \in \mathcal{E}(P)$. Then $y = \pm cx$ divides U into four regions: R_1, R_2, R_3 and R_4, which represents the East, North, West and South regions respectively. Let $\{\Psi_j\}_{1 \leq j \leq 4}$ be smooth functions such that

$$1 = \sum_{j=1}^{4} \Psi_j(x,y), \qquad (x,y) \neq (0,0). \tag{3.104}$$

Here $\Psi_1(x,y)$ is supported in

$$R_1^\epsilon = \{(x,y) : x > 0, \; -(c+\epsilon)x < y < (c+\epsilon)x\}$$

and $\Psi_1(x,y)=1$ if

$$-(c-\epsilon)x<y<(c-\epsilon)x. \tag{3.105}$$

The constant ϵ is chosen to be sufficiently small. In addition, Ψ_1 satisfies

$$|\partial_x^\alpha \partial_y^\beta \Psi_1(x,y)| \le C_{\alpha,\beta}|x|^{-\alpha}|y|^{-\beta}, \quad \text{for any} \quad (\alpha,\beta)\in\mathbb{N}^2. \tag{3.106}$$

The other functions Ψ_2,Ψ_3 and Ψ_4 are defined similarly in the other 3 regions.

Let W be a sufficiently small neighborhood of 0 and $\Phi(x,y)$ be a smooth function adapted to W, in a sense supp $\Phi\subset W$ and $\Phi(x,y)=1$ if $(2x,2y)\in W$. Then

$$\Phi(x,y)=\Phi(x,y)\sum_{j=1}^{4}\Psi_j(x,y), \qquad \text{for} \quad (x,y)\neq(0,0). \tag{3.107}$$

We focus on $\Phi\Psi_1$, discussions of $\{\Phi\Psi_j\}_{2\le j\le 4}$ can be reduced to this case. Let

$$U=W\cap R_1^\epsilon$$

then $\Phi\Psi_1$ is supported in U.

For a given analytic function $P(x,y)$, applying the resolution algorithm to $P(x,y)$ in the region U yields a collection of "bad" regions $\{U^*_{n,b,\boldsymbol{\alpha},j}\}_{(n,\boldsymbol{\alpha},j)}$. For a fixed $U^*_{n,b,\boldsymbol{\alpha},j}$, $\rho^{-1}_{n,\boldsymbol{\alpha}}(U^*_{n,b,\boldsymbol{\alpha},j})$ is equal to

$$U\cap\{(x,y): r_0x^{m_0}+\cdots+r_{n-1}x^{m_0+\cdots+m_{n-1}}+(r_n-2\epsilon)x^{m_0+\cdots+m_n}<y \tag{3.108}$$

$$<r_0x^{m_0}+\cdots+r_{n-1}x^{m_0+\cdots+m_{n-1}}+(r_n+2\epsilon)x^{m_0+\cdots+m_n}\}. \tag{3.109}$$

We can then define a smooth function $\Phi_{n,b,\boldsymbol{\alpha},j}$ supported in $\rho^{-1}_{n,\boldsymbol{\alpha}}(U^*_{n,b,\boldsymbol{\alpha},j})$ and $\Phi_{n,b,\boldsymbol{\alpha},j}(x,y)=1$

if $(x,y)\in\rho_{n,\boldsymbol{\alpha}}^{-1}(U_{n,b,\boldsymbol{\alpha},j})$. In addition, the following is true

$$|\partial_x^\alpha\partial_y^\beta\Phi_{n,b,\boldsymbol{\alpha},j}(x,y)|\le C_{\alpha,\beta}|x|^{-\alpha-\beta(m_0+\cdots+m_n)}\quad\forall(\alpha,\beta)\in\mathbb{N}^2. \tag{3.110}$$

Then

$$\Phi(x,y)\Psi_1(x,y)\Big(1-\sum_{\boldsymbol{\alpha}}\sum_j\Phi_{0,b,\boldsymbol{\alpha},j}(x,y)\Big) \tag{3.111}$$

can be written as

$$\sum_{\boldsymbol{\alpha}}\sum_j\Phi_{0,g,\boldsymbol{\alpha},j}(x,y) \tag{3.112}$$

where each $\Phi_{0,g,\boldsymbol{\alpha},j}(x,y)$ is supported in the "good" region $U^*_{0,g,\boldsymbol{\alpha},j}$. Similarly,

$$\Phi(x,y)\Psi_1(x,y)\Big(\sum_{\boldsymbol{\alpha}}\sum_j\Phi_{0,b,\boldsymbol{\alpha},j}(x,y)\Big)\Big(1-\sum_{\boldsymbol{\alpha}}\sum_j\Phi_{1,b,\boldsymbol{\alpha},j}(x,y)\Big) \tag{3.113}$$

can be written as

$$\sum_{\boldsymbol{\alpha}}\sum_j\Phi_{1,g,\boldsymbol{\alpha},j}(x,y) \tag{3.114}$$

where $\Phi_{1,g,\boldsymbol{\alpha},j}$ is supported in $\rho_{1,\boldsymbol{\alpha}}^{-1}(U^*_{1,g,\boldsymbol{\alpha},j})$. Then we can iterate the above procedures as in the algorithm and the process ends up after finite steps. Combining (3.110), we obtain a smooth partition version of Theorem 3.7.

Theorem 3.9. *Let Φ, Ψ_1 and P as above. Then*

$$\Phi(x,y)\Psi_1(x,y)=\sum_n\sum_{\boldsymbol{\alpha}}\sum_j\Phi_{n,g,\boldsymbol{\alpha},j}(x,y), \tag{3.115}$$

where $\Phi_{n,g,\boldsymbol{\alpha},j}(x,y)$ is a smooth function supported in $\rho_{n,\boldsymbol{\alpha}}^{-1}(U^*_{n,g,\boldsymbol{\alpha},j})$, where $\{U_{n,g,\boldsymbol{\alpha},j}\}$ is the *collection of "good" regions as in Theorem 3.7." The behaviors of $P(x,y)$ in "good" regions $\rho_{n,\boldsymbol{\alpha}}^{-1}(U_{n,g,\boldsymbol{\alpha},j}^{*})$ and $\rho_{n,\boldsymbol{\alpha}}^{-1}(U_{n,g,\boldsymbol{\alpha},j}^{**})$ are the same as Corollary 3.8. Moreover, $\Phi_{n,g,\boldsymbol{\alpha},j}(x,y)$ satisfies the following derivative conditions:*

(1) *If $U_{n,g,\boldsymbol{\alpha},j}$ is defined by an edge. We can assume $\rho_{n,\boldsymbol{\alpha}}^{-1}(U_{n,g,\boldsymbol{\alpha},j}^{*})$ is contained in a curved triangular region of the form*

$$|y-(r_0x^{m_0}+\cdots+r_{n-1}x^{m_0+\cdots+m_{n-1}})|\sim x^{m_0+\cdots+m_n}. \tag{3.116}$$

Then

$$|\partial_x^{\alpha}\partial_y^{\beta}\Phi_{n,g,\boldsymbol{\alpha},j}(x,y)|\le C_{\alpha,\beta}|x|^{-\alpha-\beta(m_0+\cdots+m_n)},\quad \forall(\alpha,\beta)\in\mathbb{N}^2. \tag{3.117}$$

(2) *Otherwise, $U_{n,g,\boldsymbol{\alpha},j}$ is defined by a vertex, then $\rho_{n,\boldsymbol{\alpha}}^{-1}(U_{n,g,\boldsymbol{\alpha},j}^{*})$ is contained in the curved triangular region of the form*

$$x^{m_0+\cdots+m_{n-1}+m_{n,r}}\lesssim|y-(r_0x^{m_0}+\cdots+r_{n-1}x^{m_0+\cdots+m_{n-1}})|\lesssim x^{m_0+\cdots+m_{n-1}+m_{n,l}}, \tag{3.118}$$

where $0\le m_{n,l}<m_{n,r}\le\infty$. In the upper portion of $\rho_{n,\boldsymbol{\alpha}}^{-1}(U_{n,g,\boldsymbol{\alpha},j}^{*})\backslash\rho_{n,\boldsymbol{\alpha}}^{-1}(U_{n,g,\boldsymbol{\alpha},j})$, one has

$$|\partial_x^{\alpha}\partial_y^{\beta}\Phi_{n,g,\boldsymbol{\alpha},j}(x,y)|\le C_{\alpha,\beta}|x|^{-\alpha-\beta(m_0+\cdots+m_{n-1}+m_{n,l})},\quad \forall(\alpha,\beta)\in\mathbb{N}^2. \tag{3.119}$$

In the lower portion of $\rho_{n,\boldsymbol{\alpha}}^{-1}(U_{n,g,\boldsymbol{\alpha},j}^{*})\backslash\rho_{n,\boldsymbol{\alpha}}^{-1}(U_{n,g,\boldsymbol{\alpha},j})$, if $m_{n,r}\neq\infty$ then

$$|\partial_x^{\alpha}\partial_y^{\beta}\Phi_{n,g,\boldsymbol{\alpha},j}(x,y)|\le C_{\alpha,\beta}|x|^{-\alpha-\beta(m_0+\cdots+m_{n-1}+m_{n,r})},\quad \forall(\alpha,\beta)\in\mathbb{N}^2; \tag{3.120}$$

else $m_{n,r}=\infty$, then $U_{n,g,\boldsymbol{\alpha},j}^{*}$ is defined by the far right vertex of $\mathcal{N}(P_{n,\boldsymbol{\alpha}})$ and $\rho_{n,\boldsymbol{\alpha}}^{-1}(U_{n,g,\boldsymbol{\alpha},j}^{*})$

can be represented as

$$-x^{m_0+\cdots+m_{n-1}+m_{n,l}} \lesssim y - (r_0 x^{m_0} + \cdots + r_{n-1} x^{m_0+\cdots+m_{n-1}}) \lesssim x^{m_0+\cdots+m_{n-1}+m_{n,l}}, \qquad (3.121)$$

one has

$$|\partial_x^\alpha \partial_y^\beta \Phi_{n,g,\boldsymbol{\alpha},j}(x,y)| \leq C_{\alpha,\beta} |x|^{-\alpha-\beta(m_0+\cdots+m_{n-1}+m_{n,l})}, \quad \forall (\alpha,\beta) \in \mathbb{N}^2. \qquad (3.122)$$

Chapter 4

Proof of Theorem 1.6

4.1 The exponent in the sharp bound

Before diving into the details, we give a brief exploration for the exponent appeared in the sharp decay rate of Theorem 1.6: $-1/(2\mathrm{mult}_{\boldsymbol{\pi}_0}(S))$. Mainly, we address the following questions:

(i) Where does this exponent come from?

(ii) Why this exponent is different from the one in Phong-Stein's result [PS97]?

(iii) Show that the exponent in Theorem 1.6 is sharp.

To answer the first one, set

$$P(x,y) = \partial_x \partial_y (\partial_x - \partial_y) S(x,y), \tag{4.1}$$

then

$$\mathrm{mult}_{\boldsymbol{\pi}_0}(S) = \mathrm{mult}(P) + 3. \tag{4.2}$$

Temporarily index the vertices of $\mathcal{N}(P)$ from left to right by $V_1 = (p_1, q_1)$, $V_2 = (p_2, q_2)$, $\ldots$, $V_k = (p_k, q_k)$ and all its compact edges by $E_1 = V_1V_2$, $E_2 = V_2V_3$, $\ldots$, $E_{k-1} = V_{k-1}V_k$, with slope $-\frac{1}{m_1},\cdots,-\frac{1}{m_{k-1}}$ respectively. Then

$$0 = m_0 < m_1 < m_2 < \cdots < m_{k-1} < m_k = \infty, \tag{4.3}$$

where $-1/m_0$ and $-1/m_k$ are the "slopes" corresponding to the perpendicular and horizontal non-compact edges. We assign a constant d_{E_j} to each E_j and a constant d_{V_j} to each V_j below:
(1) For an edge E_j, $1 \le j \le k-1$, let $d_{E_j,x}$ and $d_{E_j,y}$ be the x-intercept and y-intercept of the line containing E_j. Set

$$d_{E_j} = \min\{d_{E_j,x}, d_{E_j,y}\}; \tag{4.4}$$

(2) For a vertex V_j, $1 \le j \le k$, let E be any line containing V_j but not intersecting the interior of $\mathcal{N}(P)$. Let $d_{E,x}$ and $d_{E,y}$ be the x-intercept and y-intercept, and $d_E = \min\{d_{E,x}, d_{E,y}\}$. Then set

$$d_{V_j} = \sup_E d_E. \tag{4.5}$$

We call d_{E_j} and d_{V_j} the decay factors corresponding to the edge E_j and the vertex V_j. One can see that

$$\mathrm{mult}(P) = \max\{d_V, d_E: \quad V \in \mathcal{V}(P), \quad E \in \mathcal{E}(P)\}. \tag{4.6}$$

In addition, there is exact one vertex $V_* = (p_*, q_*)$ in $\mathcal{N}(P)$ such that

$$m_{*-1} \le 1 < m_*.$$

Then $\mathrm{mult}(P) = p_* + q_*$. Here $*$ is an integer between 1 and k.

If $j < *$, $m_j \leq 1$, one has

$$\text{mult}(P) = p_* + q_* \geq p_j + m_j q_j = d_{E_j} = d_{E_j,x}. \tag{4.7}$$

The value $|\lambda|^{-\frac{1}{2(3+d_{E_j})}}$ will correspond to the bound of Λ in Theorem 1.6, when restricting (x, y) to the 'good' regions defined by E_j.

Else $j \geq *$, $m_j \geq 1$ then

$$\text{mult}(P) = p_* + q_* \geq p_j/m_j + q_j = d_{E_j} = d_{E_j,y}. \tag{4.8}$$

When restricted to the 'good' regions defined by E_j, the corresponding bound of Λ will be $|\lambda|^{-\frac{1}{2(3+d_{E_j})}}$.

Similarly, when restricted to the 'good' region defined by the vertex V_j, the corresponding bound of Λ will be $|\lambda|^{-\frac{1}{2(3+d_{V_j})}}$.

Finally, the sharp bound $C|\lambda|^{-\frac{1}{2(3+\text{mult}(P))}}$ will be obtained via the vertex $V_* = (p_*, q_*)$, in the region $|x| \sim |y|$ (it may coincide with an edge).

Now we turn to the second question. One noticeable difference between the operator in Theorem 1.6 and the one in Theorem 1.4 is the extra term $f_3(x + y)$. If x and y vary in intervals of length δ_1 and δ_2 respectively, then the range of $(x+y)$ is of length $\sim \max\{\delta_1, \delta_2\}$. If f_3 is a characteristic function supported in this interval, then $\|f_3\|_2 \sim \max\{\delta_1^{1/2}, \delta_2^{1/2}\}$. This freezes the ratio $\log |x|/\log |y|$ to be 1, if one wants to optimize the bound of the operator. Indeed, our example which shows the sharpness of the bound in Theorem 1.6 is constructed in this flavor.

However, without the term $f_3(x + y)$, the ratio between $\log |x|$ and $\log |y|$ is totally free. The upshot of such difference on the operators is realized by the difference of the following two Schur-type's lemmas:

Lemma 4.1. *Assume $a(x,y)$ is a smooth function supported in a strip of x-width no more than δ_1 and y-width no more than δ_2. Assume $\|a\|_\infty \le 1$, then*

$$\left|\iint f_1(x)f_2(y)a(x,y)dxdy\right| \le (\delta_1\delta_2)^{1/2}\|f_1\|_2\|f_2\|_2. \tag{4.9}$$

Lemma 4.2. *Assume $a(x,y)$ is a smooth function supported in a strip of x-width no more than δ_1 and y-width no more than δ_2. Assume $\|a\|_\infty \le 1$, then*

$$\left|\iint f_1(x)f_2(y)f_3(x+y)a(x,y)dxdy\right| \le C\min\{\delta_1^{1/2},\delta_2^{1/2}\}\|f_1\|_2\|f_2\|_2\|f_3\|_2. \tag{4.10}$$

Lemma 4.1 is a directly result of Schur's Lemma, which is employed in [PS97] to control the norm of the operator in Theorem 1.4 when the phase fails to provide sufficient decay. Lemma 4.2 plays the same role in our proof. We provide the proof of Lemma 4.2 as an appetizer.

Proof. By the Cauchy-Schwarz inequality and the Fubini Theorem, one has

$$\begin{aligned}
&\left|\iint f_1(x)f_2(y)f_3(x+y)a(x,y)dxdy\right| \\
=&\left|\int f_1(x)\left(\int f_2(y)f_3(x+y)a(x,y)dy\right)dx\right| \\
\le&\left|\int\left(\int f_2(y)f_3(x+y)a(x,y)dy\right)^2dx\right|^{1/2}\cdot\|f_1\|_2 \\
\le&\left|\int\left(\int |a(x,y)|^2dy\right)\left(\int |f_2(y)f_3(x+y)|^2\,dy\right)dx\right|^{1/2}\cdot\|f_1\|_2 \\
\le&\delta_2^{1/2}\|f\|_1\|f_2\|_2\|f_3\|_2.
\end{aligned}$$

The other bound can be obtained similarly. □

Finally, we provide an example to show the sharpness of the exponent. Firstly, we write S as a sum of homogeneous polynomials:

$$S(x,y) = \sum_{n=n_0}^{\infty} S_n(x,y). \tag{4.11}$$

In addition, we assume

$$\partial_x \partial_y (\partial_x - \partial_y) S_{n_0} \neq 0. \tag{4.12}$$

Indeed, if $\partial_x \partial_y (\partial_x - \partial_y) S_n = 0$, then $S_n(x,y) = S_{n,1}(x) + S_{n,2}(y) + S_{n,3}(x+y)$ due to the equivalence between simply degeneracy and degeneracy in this case. Thus we can incorporate $e^{iS_n(x,y)}$ into the functions $f_1(x)$, $f_2(y)$ and $f_3(x+y)$, while the L^2-norms of these functions are unchanged.

Let A be a sufficiently large number, f_1 and f_2 be characteristic functions of the interval $I_A = [-\lambda^{-1/n_0}/A, \lambda^{-1/n_0}/A]$. Let f_3 be the characteristic function of the interval $[-2\lambda^{-1/n_0}/A, 2\lambda^{-1/n_0}/A]$. If A is sufficiently large (depending on S), then

$$|\lambda S(x,y)| \leq 2^{-100}, \qquad \forall \quad x, y \in I_A. \tag{4.13}$$

Thus

$$\left| \iint e^{i\lambda S(x,y)} a(x,y) f_1(x) f_2(y) f_3(x+y) dxdy \right| \sim |I_A| \times |I_A| \sim \lambda^{-\frac{2}{n_0}}. \tag{4.14}$$

Notice

$$\|f_1\|_2 \sim \|f_2\|_2 \sim \|f_3\|_2 \sim \lambda^{-\frac{1}{2n_0}}. \tag{4.15}$$

Hence, if

$$\left|\iint e^{i\lambda S(x,y)}a(x,y)f_1(x)f_2(y)f_3(x+y)dxdy\right| \lesssim C(\lambda)\prod_{j=1}^{3}\|f_j\|_2 \tag{4.16}$$

then $C(\lambda) \gtrsim \lambda^{-\frac{1}{2n_0}} = \lambda^{-\frac{1}{2\mathrm{mult}_{\boldsymbol{\pi}_0}(S)}}$, as desired.

Now, we come back to use the set of indices in Chapter 3 and get rid of the above temporary indices.

4.2 Proof of Theorem 1.6

Assume $a(x,y)$ is supported in $W/100$, where W is a sufficiently small neighborhood of 0. Same as what have been done in Theorem 3.9, we divide W into 4 regions by the lines $y = cx$ and $y = -cx$. Here c is a positive constant s.t. neither $y = cx$ nor $y = -cx$ is a solution of $P_E(x,y) = 0$, for all $E \in \mathcal{E}(P)$. We restrict our discussion in the East region

$$U = W \cap \{(x,y) : x > 0, -(c+\epsilon)x < y < (c+\epsilon)x\}, \tag{4.17}$$

since the other three regions can be reduced to U by either changing x to $-x$ or permuting x and y or both. Let $\Psi_j(x,y)$ and $\Phi(x,y)$ be smooth functions as in Theorem 3.9, then

$$a(x,y) = a(x,y)\Phi(x,y)\sum_{j=1}^{4}\Psi_j(x,y), \quad \text{for} \quad (x,y) \neq (0,0) \tag{4.18}$$

and

$$\Lambda_S(f_1,f_2,f_3) = \sum_{j=1}^{4}\Lambda_S^j(f_1,f_2,f_3) \tag{4.19}$$

where

$$\Lambda_S^j(f_1, f_2, f_3) = \iint e^{i\lambda S(x,y)} a(x,y)\Phi(x,y)\Psi_j(x,y) f_1(x) f_2(y) f_3(x+y) dxdy. \tag{4.20}$$

We focus only on $j = 1$. Theorem 3.9 yields:

$$a(x,y)\Phi(x,y)\Psi_1(x,y) = \sum_{0 \le n \le N_P} \sum_{\boldsymbol{\alpha}} \sum_{j} a_{n,g,\boldsymbol{\alpha},j}(x,y) \tag{4.21}$$

where

$$a_{n,g,\boldsymbol{\alpha},j}(x,y) = \Phi_{n,g,\boldsymbol{\alpha},j}(x,y) a(x,y). \tag{4.22}$$

Set

$$\Lambda_{n,g,\boldsymbol{\alpha},j}(f_1, f_2, f_3) = \iint e^{i\lambda S(x,y)} f_1(x) f_2(y) f_3(x+y) a_{n,g,\boldsymbol{\alpha},j}(x,y) dxdy, \tag{4.23}$$

then

$$\Lambda_S^1(f_1, f_2, f_3) = \sum_{0 \le n \le N_P} \sum_{\boldsymbol{\alpha}} \sum_{j} \Lambda_{n,g,\boldsymbol{\alpha},j}(f_1, f_2, f_3). \tag{4.24}$$

Since the sum (4.24) contains only finitely many terms, it suffices to prove

$$\|\Lambda_{n,g,\boldsymbol{\alpha},j}\| \lesssim |\lambda|^{-\frac{1}{2(3+\mathrm{mult}(P))}}, \quad \text{for every} \quad (n, g, \boldsymbol{\alpha}, j). \tag{4.25}$$

In the rest of this section, we always deal with a single $\Lambda_{n,g,\boldsymbol{\alpha},j}$ and the indices j and $\boldsymbol{\alpha}$ are unimportant. Thus we drop them and use $\Lambda_{n,g}$ to represent $\Lambda_{n,g} = \Lambda_{n,g,\boldsymbol{\alpha},j}$ for some $\boldsymbol{\alpha}$ and j. The proof is splitted into three cases (i) $n = 0$ and $U_{0,g}$ is defined by an edge E, (ii) $n = 0$ and $U_{0,g}$ is defined by a vertex V and (iii) $n \ge 1$.

Proposition 4.3. *If $U_{0,g}$ is defined by an edge (E, m), where $-1/m$ is the slope of E, then*

$$\|\Lambda_{0,g}\| \lesssim |\lambda|^{-\frac{1}{2(3+d_E)}}. \tag{4.26}$$

Proposition 4.4. *If $U_{0,g}$ is defined by a vertex V, then*

$$\|\Lambda_{0,g}\| \lesssim |\lambda|^{-\frac{1}{2(3+d_V)}}. \tag{4.27}$$

Proposition 4.5. *If $U_{n,g} \subset U_n$ comes from the following chain:*

$$[U_0, P_0] \to [U_1, P_1] \to \cdots \to [U_n, P_n] \tag{4.28}$$

and $[U_1, P_1]$ is obtained from $[U_0, P_0]$ by the edge $E \in \mathcal{E}(P_0)$. Then

$$\|\Lambda_{n,g}\| \lesssim |\lambda|^{-\frac{1}{2(3+d_E)}}. \tag{4.29}$$

One can see that Proposition 4.3, Proposition 4.4 and Proposition 4.5 imply Theorem 1.6. The rest of this chapter is to prove these propositions.

4.3 Proof of Proposition 4.3.

Let $0 < \sigma < 1$ be a dyadic number and $\phi_\sigma(x)$ be a smooth function supported in $\frac{1}{2}\sigma < |x| < 2\sigma$ such that

$$\sum_{0<\sigma<1} \phi_\sigma(x) = 1 \quad \text{for} \quad 0 < |x| < 1/10. \tag{4.30}$$

Set

$$\Lambda_{0,g,\sigma_1,\sigma_2}(f_1, f_2, f_3) = \iint e^{i\lambda S(x,y)} f_1(x) f_2(y) f_3(x+y) a_{0,g}(x,y) \phi_{\sigma_1}(x) \phi_{\sigma_2}(y) dxdy, \tag{4.31}$$

where σ_1 and σ_2 are dyadic numbers and $a_{0,g} = a_{0,g,\boldsymbol{\alpha},j}$ for some $(\boldsymbol{\alpha}, j)$. Notice that

$$\text{supp } (a_{0,g}) \subset U^*_{0,g}, \tag{4.32}$$

which is a 'good' region defined by (E, m). Thus

$$|y| \sim |x|^m \quad \text{and} \quad |\sigma_2| \sim |\sigma_1|^m. \tag{4.33}$$

This yields, for a fixed σ_1 that there is only finitely many choices of σ_2. Without loss of generality, we assume σ_2 is fixed given σ_1 is fixed. To employ Theorem 2.1, we need to verify its conditions. Let K be a large constant, equally divide the interval $(\sigma_1/2, 2\sigma_1)$ into K subintervals $\{I_k\}_{1\le k\le K}$ and set

$$U^*_{0,g,k} = \{(x, y) \in U^*_{0,g} \, : \, x \in I_k\}. \tag{4.34}$$

Lemma 4.6. *Given K large enough, depending only on P and ϵ, one has*

$$\text{Conv}(U^*_{0,g,k}) \subset U^{**}_{0,g}, \tag{4.35}$$

for all $1 \le k \le K$.

The proof of this lemma is postponed in the end of this section. Now let (p_l, q_l) be the left vertex of E. Then for every $(x, y) \in \text{Conv}(U^*_{0,g,k}) \subset U^{**}_{0,g}$, Theorem 3.7 and Corollary 3.8 yield,

$$|P(x, y)| \gtrsim |x|^{p_l}|y|^{q_l} \sim |\sigma_1|^{p_l}|\sigma_2|^{q_l}, \tag{4.36}$$

and for $\beta = 0, 1, 2$

$$|\partial_y^\beta P(x,y)| \lesssim |x|^{p_l}|y|^{q_l-\beta} \sim |\sigma_1|^{p_l}|\sigma_2|^{q_l-\beta}. \tag{4.37}$$

Theorem 3.9 together with (4.22) yields

$$|\partial_y^\beta a_{0,g}(x,y)| \lesssim |\sigma_2|^{-\beta}. \tag{4.38}$$

By invoking Theorem 2.1, one has

$$\|\Lambda_{0,g,\sigma_1,\sigma_2,k}\| \lesssim |\lambda \sigma_1^{p_l}\sigma_2^{q_l}|^{-1/6}, \tag{4.39}$$

where $\Lambda_{0,g,\sigma_1,\sigma_2,k}(f_1, f_2, f_3)$ is given by

$$\iint e^{i\lambda S(x,y)} f_1(x)\mathbf{1}_{I_k}(x) f_2(y) f_3(x+y) a_{0,g}(x,y)\phi_{\sigma_1}(x)\phi_{\sigma_2}(y)dxdy. \tag{4.40}$$

Summing over $1 \le k \le K$ (K is a constant) yields:

$$\|\Lambda_{0,g,\sigma_1,\sigma_2}\| \lesssim |\lambda \sigma_1^{p_l}\sigma_2^{q_l}|^{-1/6}. \tag{4.41}$$

Employing Lemma 4.2 and combining (4.41), one obtains:

$$\|\Lambda_{0,g,\sigma_1,\sigma_2}\| \lesssim \begin{cases} |\lambda \sigma_1^{p_l}\sigma_2^{q_l}|^{-1/6}, \\ \\ \min\{\sigma_1, \sigma_2\}^{1/2}. \end{cases} \tag{4.42}$$

Notice that our assumption on U implies $m \ge 1$ and thus $\sigma_2 \lesssim \sigma_1$. This gives

$$
\|\Lambda_{0,g,\sigma_1,\sigma_2}\| \lesssim \begin{cases} |\lambda\sigma_2^{q_l+p_l/m}|^{-1/6}, \\ \\ \qquad\qquad \sigma_2^{1/2}. \end{cases} \tag{4.43}
$$

Since for fixed σ_2, σ_1 is fixed, summing over σ_2 yields

$$
\left\| \sum_{\sigma_1,\sigma_2} \Lambda_{0,g,\sigma_1,\sigma_2} \right\| \lesssim |\lambda|^{-\frac{1}{2(3+q_l+p_l/m)}} = |\lambda|^{-\frac{1}{2(3+d_E)}}, \tag{4.44}
$$

as desired.

4.4 Proof of Proposition 4.4.

Like the proof of Proposition 4.3, insert the smooth support $\phi_{\sigma_1}(x)\phi_{\sigma_2}(y)$ into $\Lambda_{0,g}$. Set $V = (p, q)$ and assume $-1/m_l$ and $-1/m_r$ be the slopes of the edges left and right to V. Due to the following assumption on U:

$$
U = W \cap \{(x, y) : x > 0,\ -(c+\epsilon)x < y < (c+\epsilon)x\}, \tag{4.45}
$$

we may replace m_l with 1 if $m_l < 1$. Thus

$$
\infty \geq m_r > m_l \geq 1. \tag{4.46}
$$

Notice that

$$
\sigma_2^{1/m_r} \gtrsim \sigma_1 \gtrsim \sigma_2^{1/m_l} \gtrsim \sigma_2. \tag{4.47}
$$

Consider all (σ_1, σ_2) with $\sigma_2 \lesssim \lambda_2 := |\lambda|^{-\frac{1}{3+q+p/m_l}} = |\lambda|^{-\frac{1}{3+d_V}}$. By Lemma 4.2, we have

$$\left\| \sum_{\sigma_2 \lesssim \lambda_2} \left(\sum_{\sigma_1} \Lambda_{0,g,\sigma_1,\sigma_2} \right) \right\| \lesssim \sum_{\sigma_2 \lesssim \lambda_2} |\sigma_2|^{1/2} \lesssim |\lambda_2|^{1/2} = |\lambda|^{-\frac{1}{2(3+d_V)}}. \tag{4.48}$$

Now we assume $\sigma_2 \gtrsim \lambda_2$. Theorem 2.1 yields[1]

$$\|\Lambda_{0,g,\sigma_1,\sigma_2}\| \lesssim |\lambda \sigma_1^p \sigma_2^q|^{-1/6}, \tag{4.49}$$

Since $\sigma_1 \gtrsim \sigma_2^{1/m_l}$, thus

$$\sum_{\sigma_1} \|\Lambda_{0,g,\sigma_1,\sigma_2}\| \lesssim |\lambda \sigma_2^{p/m_l+q}|^{-1/6}. \tag{4.50}$$

Summing all $\sigma_2 \gtrsim \lambda_2$ we obtain the same bound as (4.48).

4.5 Proof of Proposition 4.5.

For $0 \le k \le n-1$, assume the change of variables is

$$\begin{cases} x_k = x_{k+1} \\ y_k = (r_k + y_{k+1}) x^{m_k}. \end{cases} \tag{4.51}$$

If (4.28) is a subchain of an infinite chain as (3.52) and $n = n_0 + 1$ (n_0 defined in (3.56) and hence there is no $n > n_0 + 1$), then the last step of change of variables is replaced by (3.69):

$$\begin{cases} x_{n_0} = x_{n_0+1} \\ y_{n_0} - f(x_{n_0}) = y_{n_0+1} x_{n_0}^{m_{n_0}}. \end{cases}$$

[1]Here we have invoked the same trick to meet the convexity condition of Theorem 2.1 as in the proof of Proposition 4.3: splitting $\Lambda_{0,g,\sigma_1,\sigma_2}$ into the sum of $\Lambda_{0,g,\sigma_1,\sigma_2,k}$, applying Theorem 2.1 to each $\Lambda_{0,g,\sigma_1,\sigma_2,k}$ and summing them together.

The behavior of $P_n(x_n, y_n)$ in $U_{n,g}$ (and $U^{**}_{n,g}$) is either dominant by a vertex or an edge; if by a vertex, let (p_n, q_n) be that vertex; else let (p_n, q_n) be the left vertex of that edge. In addition, let s_k be the order of r_k, (p_k, q_k) be the left vertex of the edge E_k, where E_k is the edge corresponding to $y_k = r_k x_k^{m_k}$ in $\mathcal{N}(P_k)$. Theorem 3.7 and Corollary 3.8 yield

$$|P(x,y)| = |P_n(x_n, y_n)| \sim |x_n^{p_n} y_n^{q_n}| \tag{4.52}$$

for all $(x_n, y_n) \in U^{**}_{n,g}$. Since $U^{**}_{n,g} \subset U^{**}_n$ is a 'good' region, we can find m'_n and m_n s.t.

$$|x_n|^{m'_n} \lesssim |y_n| \lesssim |x_n|^{m_n}, \tag{4.53}$$

where $0 \leq m_n \leq m'_n \leq \infty$. In addition, if $m_n = m'_n$, then $U^{**}_{n,g}$ is defined by an edge; otherwise by a vertex.

Dyadically decompose (x_n, y_n) as

$$\begin{cases} |x_n| \sim \sigma_1 \\ |y_n| \sim \sigma_2 x_n^{m_n} \sim \sigma_2 \sigma_1^{m_n} \end{cases}$$

and let $\Lambda_{n,g,\sigma_1,\sigma_2}$ denote the operator $\Lambda_{n,g}$ when (x_n, y_n) is restricted in this region. Different to the case when $n = 0$, the 'almost orthogonality' plays a crucial role when $n \geq 1$, which comes from the diagonal distribution of $U^{**}_{n,g}$. In fact, from the change of variables, we have

$$y = y_0 = r_0 x^{m_0} + r_1 x^{m_0+m_1} + \cdots + r_{n-1} x^{m_0+\cdots+m_{n-1}} + y_n x^{m_0+\cdots+m_{n-1}}. \tag{4.54}$$

In addition, the assumption on U ensures $m_0 \geq 1$.

Notice that $y_n \sim \sigma_2 \sigma_1^{m_n}$ and $|x| \sim \sigma_1$, thus in this region the width of y: $\triangle y$ is bounded by

$$|\triangle y| \lesssim \sigma_2 \sigma_1^{m_0+\cdots+m_n}. \tag{4.55}$$

In addition, (4.54) yields

$$\triangle y \sim \triangle x \frac{dy}{dx} \sim \triangle x \cdot \sigma_1^{m_0-1} \tag{4.56}$$

and thus

$$|\triangle x| \lesssim \sigma_2 \sigma_1^{m_0+\cdots+m_n-(m_0-1)}. \tag{4.57}$$

Based on the above analysis, we divide the interval $(\sigma_1/2, 2\sigma_1)$ equally into H subintervals $\{I_h\}_{1\leq h\leq H}$, where

$$H = K \cdot \sigma_1 \Big(\sigma_2 \sigma_1^{m_0+\cdots+m_n-(m_0-1)} \Big)^{-1}. \tag{4.58}$$

Here K is a large constant designed merely to treat the convexity condition in Theorem 2.1. Set

$$U^*_{n,g,h} = \{(x,y) \in U^*_{n,g} : x \in I_h\} \tag{4.59}$$

and

$$Y(I_h) = \{y(x) : x \in I_h \text{ and } y_n \sim \sigma_2 \sigma_1^{m_n}\} \tag{4.60}$$

where $y(x)$ is defined in (4.54). Then $\Lambda_{n,g,\sigma_1,\sigma_2}$ can be further decomposed into $\{\Lambda_{n,g,\sigma_1,\sigma_2,h}\}_{1\leq h\leq H}$ by restricting $x \in I_h$. By (4.54), $y = y(x)$ is monotone given $|x|$ sufficiently small. Hence, given $L = L(P, \epsilon, K)$ large enough, by (4.55) and (4.57) one has

$$Y(I_h) \cap Y(I_{h'}) = \emptyset \quad if \quad |h - h'| \geq L, \tag{4.61}$$

which implies the following 'almost orthogonality' principle:

Claim 1. *If there is a constant A s.t.*

$$\|\Lambda_{n,g,\sigma_1,\sigma_2,h}\| \leq A, \tag{4.62}$$

then

$$\|\Lambda_{n,g,\sigma_1,\sigma_2}\| \leq LA \tag{4.63}$$

Proof of Claim 1.

Consider the congruence classes modulo L in H: let $0 \leq \ell < L$ and

$$H_\ell = \{1 \leq h \leq H : h \equiv \ell \, mod \, L\}. \tag{4.64}$$

Then

$$\Big\| \sum_{1 \leq h \leq H} \Lambda_{n,g,\sigma_1,\sigma_2,h} \Big\| \leq L \sup_{0 \leq \ell < L} \Big\| \sum_{h \in H_\ell} \Lambda_{n,g,\sigma_1,\sigma_2,h} \Big\|. \tag{4.65}$$

In addition, notice that

$$\Lambda_{n,g,\sigma_1,\sigma_2,h}(f_1, f_2, f_3) = \Lambda_{n,g,\sigma_1,\sigma_2,h}(f_1 \mathbf{1}_{I_h}, f_2 \mathbf{1}_{Y(I_h)}, f_3), \tag{4.66}$$

The Cauchy-Schwarz inequality and (4.62) yield

$$\begin{aligned}\Big|\sum_{h\in H_\ell}\Lambda_{n,g,\sigma_1,\sigma_2,h}(f_1,f_2,f_3)\Big| \leq &\sum_{h\in H_\ell}\Big|\Lambda_{n,g,\sigma_1,\sigma_2,h}(f_1\mathbf{1}_{I_h},f_2\mathbf{1}_{Y(I_h)},f_3)\Big| \\ \leq & A\sum_{h\in H_\ell}\|f_1\mathbf{1}_{I_h}\|_2\|f_2\mathbf{1}_{Y(I_h)}\|_2\|f_3\|_2 \\ \leq & A\|\sum_{h\in H_\ell}f_1\mathbf{1}_{I_h}\|_2\|\sum_{h\in H_\ell}f_2\mathbf{1}_{Y(I_h)}\|_2\|f_3\|_2\end{aligned}$$

which is controlled by $A\|f_1\|_2\|f_2\|_2\|f_3\|_2$ due to (4.61). Thus (4.63) is true as desired. □

To prove (4.62), we also need the following lemma which is similar to Lemma 4.6, whose proof can be found in the end of this chapter.

Lemma 4.7. *If $K=(P,\epsilon)$ large enough, then for $1\leq h\leq H$, one has*

$$\mathrm{Conv}(\rho_n^{-1}(U^*_{n,g,h}))\subset\rho_n^{-1}(U^{**}_{n,g}). \tag{4.67}$$

Invoking Lemma 4.2, Theorem 3.7, Corollary 3.8 and Theorem 2.1 yields

$$\|\Lambda_{n,g,\sigma_1,\sigma_2,h}\|\lesssim\begin{cases}|\lambda\sigma_1^{p_n+q_nm_n}\sigma_2^{q_n}|^{-1/6}\\ \\ (\sigma_2\sigma_1^{m_0+\cdots+m_n})^{1/2},\end{cases} \tag{4.68}$$

for every $1\leq h\leq H$. By Claim 1, summing over h yields

$$\|\Lambda_{n,g,\sigma_1,\sigma_2}\|\lesssim\begin{cases}|\lambda\sigma_1^{p_n+q_nm_n}\sigma_2^{q_n}|^{-1/6}\\ \\ (\sigma_2\sigma_1^{m_0+\cdots+m_n})^{1/2}.\end{cases} \tag{4.69}$$

Summing over σ_1 yields

$$\sum_{\sigma_1} \|\Lambda_{n,g,\sigma_1,\sigma_2}\| \lesssim |\lambda|^{-\frac{1}{2(3+(p_n+q_n m_n)/(m_0+\cdots+m_n))}} \cdot \sigma_2^{\frac{p_n+m_n q_n - q_n(m_0+\cdots+m_n)}{3(m_0+\cdots+m_n)+p_n+m_n q_n}}. \tag{4.70}$$

If $p_n + m_n q_n - q_n(m_0 + \cdots + m_n) > 0$, we can sum over σ_2 in (4.70) and obtain

$$\sum_{\sigma_2}\sum_{\sigma_1} \|\Lambda_{n,g,\sigma_1,\sigma_2,h}\| \lesssim |\lambda|^{-\frac{1}{2(3+(p_n+q_n m_n)/(m_0+\cdots+m_n))}} \leq |\lambda|^{-\frac{1}{2(3+d_E)}}, \tag{4.71}$$

where the latter inequality will be proved in a moment.

Otherwise

$$p_n + m_n q_n - q_n(m_0 + \cdots + m_n) \leq 0. \tag{4.72}$$

Again, by tracking back to the change of variables (see (3.53)), one can see that for $1 \leq k \leq n$

$$p_k \geq p_{k-1} + m_{k-1}s_{k-1}, \tag{4.73}$$

and

$$q_0 \geq s_0 \geq q_1 \geq s_1 \geq q_2 \geq \cdots \geq q_{n-1} \geq s_{n-1} \geq q_n. \tag{4.74}$$

Hence, the only possibility (4.72) holds is when $p_0 = 0$ and

$$q_0 = q_1 = q_2 = \cdots = q_{n-1} = q_n. \tag{4.75}$$

Then (4.69) becomes:

$$\|\Lambda_{n,g,\sigma_1,\sigma_2}\| \lesssim \begin{cases} |\lambda\sigma_1^{q_0(m_0+\cdots+m_n)}\sigma_2^{q_0}|^{-1/6} \\ \\ (\sigma_2\sigma_1^{m_0+\cdots+m_n})^{1/2}. \end{cases} \tag{4.76}$$

Summing over σ_2 yields

$$\|\Lambda_{n,g,\sigma_1}\| = \|\sum_{\sigma_2}\Lambda_{n,g,\sigma_1,\sigma_2}\| \leq \sum_{\sigma_2}\|\Lambda_{n,g,\sigma_1,\sigma_2}\| \lesssim |\lambda|^{-\frac{1}{2(3+q_0)}}. \tag{4.77}$$

By the Cauchy-Schwarz inequality, one has

$$\sum_{\sigma_1}\|\Lambda_{n,g,\sigma_1}\| \lesssim \sup_{\sigma_1}\|\Lambda_{n,g,\sigma_1}\|, \tag{4.78}$$

where we have employed the same almost orthogonality trick as in the proof of Claim 1. Indeed, by (4.54) there is a constant L s.t. if $\sigma_1 > 2^L\sigma_1'$, then $\Lambda_{n,g,\sigma_1,\sigma_2}$ and $\Lambda_{n,g,\sigma_1',\sigma_2'}$ integrate in distinct x-region and distinct y-region, which allows us to invoke the Cauchy-Schwarz inequality to the functions f_1 and f_2.

It remains to show $q_0 \leq d_E$ and $(p_n + q_n m_n)/(m_0 + \cdots + m_n) \leq d_E$.

For the former, notice $p_0 = 0$ and $m_0 \geq 1$, hence $\text{mult}(P) = d_E = q_0$.

For the latter, notice that $V_k = (p_{k-1}+m_{k-1}s_{k-1}, s_{k-1})$ as the far left vertex of the Newton Polyhedron in the k-stage, is above the line passing (p_k, q_k) with slope $-\frac{1}{m_k}$. Thus

$$p_k + m_k q_k \leq p_{k-1} + m_{k-1}s_{k-1} + m_k s_{k-1} \leq p_{k-1} + m_{k-1}q_{k-1} + m_k q_0, \tag{4.79}$$

since $q_0 \geq q_{k-1} \geq s_{k-1}$ for all $k \geq 1$. Iterating the above formula yields

$$p_n + m_n q_n \leq p_0 + m_0 q_0 + m_1 q_0 + \cdots + m_n q_0 \tag{4.80}$$

Therefore

$$\frac{p_n + q_n m_n}{m_0 + \cdots + m_n} \le \frac{(p_0 + m_0 q_0) + q_0(m_1 + \cdots + m_n)}{m_0 + (m_1 + \cdots + m_n)} \le \frac{p_0 + m_0 q_0}{m_0} = d_E, \tag{4.81}$$

since

$$d_E = \frac{p_0 + m_0 q_0}{m_0} \ge q_0. \tag{4.82}$$

4.6 Verification of Lemma 4.6 and Lemma 4.7

We only provide the proof of Lemma 4.7, since the proof of Lemma 4.6 is similar and somewhat easier.

Firstly, notice that the upper and the lower boundaries of $U^*_{n,g,h}$ can be represented by two curves:

$$\gamma_1(x) = r_0 x^{m_0} + r_1 x^{m_0+m_1} + \cdots + r_{n-1} x^{m_0+\cdots+m_{n-1}} + r_{n,1} x^{m_0+\cdots+m_{n-1}+m_n}, \tag{4.83}$$

$$\gamma_2(x) = r_0 x^{m_0} + r_1 x^{m_0+m_1} + \cdots + r_{n-1} x^{m_0+\cdots+m_{n-1}} + r_{n,2} x^{m_0+\cdots+m_{n-1}+m'_n} \tag{4.84}$$

and the upper and the lower boundaries of $U^{**}_{n,g,h}$ by

$$\gamma^*_1(x) = r_0 x^{m_0} + r_1 x^{m_0+m_1} + \cdots + r_{n-1} x^{m_0+\cdots+m_{n-1}} + r^*_{n,1} x^{m_0+\cdots+m_{n-1}+m_n}, \tag{4.85}$$

$$\gamma^*_2(x) = r_0 x^{m_0} + r_1 x^{m_0+m_1} + \cdots + r_{n-1} x^{m_0+\cdots+m_{n-1}} + r^*_{n,2} x^{m_0+\cdots+m_{n-1}+m'_n}, \tag{4.86}$$

where $r_{n,1} < r^*_{n,1}$, $r_{n,2} > r^*_{n,2}$ and $0 \le m_n \le m'_n$. Moreover, if $m_n = m'_n$, then $r_{n,1} > r_{n,2}$. Without loss of generality, we assume $r_0 > 0$ and thus all the curves above are increasing functions of x. The assumption $m_0 \ge 1$ implies all the functions above are convex. Thus we only need to take care of the upper boundary of $U^*_{n,g,h}$.

Let $\sigma_{1,h}$ be the left end point of the interval I_h. By the definition of convexity, one needs to verify that if K is sufficiently large, then

$$t\gamma_1(\sigma_{1,h}) + (1-t)(\sigma_{1,h+1}) < \gamma_1^*(t\sigma_{1,h} + (1-t)\sigma_{1,h+1}) \tag{4.87}$$

for all $0 \le t \le 1$ and $0 < \sigma_1 < 1$. Since both γ_1 and γ_1^* are increasing, it suffices to show

$$\gamma_1(\sigma_{1,h+1}) < \gamma_1^*(\sigma_{1,h}). \tag{4.88}$$

By the Mean Value Theorem, there is a $\sigma \in I_h$ such that

$$\gamma_1(\sigma_{1,h+1}) - \gamma_1(\sigma_{1,h}) = \gamma_1'(\sigma)(\sigma_{1,h+1} - \sigma_{1,h}) = \frac{3}{2K}\sigma_2\sigma_1^{m_0+\cdots+m_n-(m_0-1)}\gamma_1'(\sigma). \tag{4.89}$$

Since $\sigma_1/2 \le \sigma \le 2\sigma_1$, then for some constant $C > 0$,

$$|\gamma_1'(\sigma)| \le C\sigma_1^{m_0-1}. \tag{4.90}$$

Therefore

$$\gamma_1(\sigma_{1,h+1}) - \gamma_1(\sigma_{1,h}) \le \frac{C}{K}\sigma_2\sigma_1^{m_0+\cdots+m_n}. \tag{4.91}$$

Notice that

$$\gamma_1^*(\sigma_{1,h}) - \gamma_1(\sigma_{1,h}) = (r_{1,n}^* - r_{1,n})\sigma_{1,h}^{m_0+\cdots+m_n} \ge (r_{1,n}^* - r_{1,n})(\sigma_1/2)^{m_0+\cdots+m_n}. \tag{4.92}$$

Since $\sigma_2 < 1$, by choosing

$$K > \frac{2^{m_0+\cdots+m_n}\cdot C}{r_{1,n}^* - r_{1,n}}, \tag{4.93}$$

the inequalities (4.91) and (4.92) yield (4.87), as desired.

References

[CCW99] Anthony Carbery, Michael Christ, and James Wright. Multidimensional van der Corput and sublevel set estimates. *J. Amer. Math. Soc.*, 12(4):981–1015, 1999.

[CeS11] Michael Christ and D. Oliveira e Silva. On trilinear oscillatory integrals. *arXiv preprint arXiv:1107.2495*, 2011.

[Chr11a] Michael Christ. Bounds for multilinear sublevel sets via Szemeredi's Theorem. *arXiv preprint arXiv:1107.2350*, 2011.

[Chr11b] Michael Christ. Multilinear oscillatory integrals via reduction of dimension. *arXiv preprint arXiv:1107.2352*, 2011.

[CLTT05] Michael Christ, Xiaochun Li, Terence Tao, and Christoph Thiele. On multilinear oscillatory integrals, nonsingular and singular. *Duke Math. J.*, 130(2):321–351, 2005.

[Cut04] Steven Dale Cutkosky. *Resolution of singularities*, volume 63 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2004.

[Gre04] Michael Greenblatt. A direct resolution of singularities for functions of two variables with applications to analysis. *J. Anal. Math.*, 92:233–257, 2004.

[Gre05] Michael Greenblatt. Sharp L^2 estimates for one-dimensional oscillatory integral operators with C^∞ phase. *Amer. J. Math.*, 127(3):659–695, 2005.

[Gre08] Michael Greenblatt. Simply nondegenerate multilinear oscillatory integral operators with smooth phase. *Math. Res. Lett.*, 15(4):653–660, 2008.

[Hir64] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2) 79 (1964), 109–203; ibid. (2)*, 79:205–326, 1964.

[Hör73] Lars Hörmander. Oscillatory integrals and multipliers on FL^p. *Ark. Mat.*, 11:1–11, 1973.

[PS94a] D. H. Phong and E. M. Stein. On a stopping process for oscillatory integrals. *J. Geom. Anal.*, 4(1):105–120, 1994.

[PS94b] D. H. Phong and E. M. Stein. Operator versions of the van der Corput lemma and Fourier integral operators. *Math. Res. Lett.*, 1(1):27–33, 1994.

[PS97] D. H. Phong and E. M. Stein. The Newton polyhedron and oscillatory integral operators. *Acta Math.*, 179(1):105–152, 1997.

[PSS01] D. H. Phong, E. M. Stein, and Jacob Sturm. Multilinear level set operators, oscillatory integral operators, and Newton polyhedra. *Math. Ann.*, 319(3):573–596, 2001.

[Ryc01] Vyacheslav S. Rychkov. Sharp L^2 bounds for oscillatory integral operators with C^∞ phases. *Math. Z.*, 236(3):461–489, 2001.

[See98] Andreas Seeger. Radon transforms and finite type conditions. *J. Amer. Math. Soc.*, 11(4):869–897, 1998.

[Ste93] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.

[Var76] A. N. Varčenko. Newton polyhedra and estimates of oscillatory integrals. *Funkcional. Anal. i Priložen.*, 10(3):13–38, 1976.

[Wol03] Thomas H. Wolff. *Lectures on harmonic analysis*, volume 29 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003.