

OPTIMAL DELEVERAGING AND LIQUIDATION OF FINANCIAL PORTFOLIOS WITH
MARKET IMPACT

BY

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DISSERTATION

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ABSTRACT

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The 2008 Financial Crisis highlighted the importance of effective portfolio deleveraging and liquidation strategies, which is critical to surviving financial distress and maintaining system stability. This thesis studies two related problems: which portion of the portfolio should be executed to relieve the financial distress and how the execution should be conducted to balance the trading cost and the trading risk. An optimal deleveraging strategy determines what portion of the portfolio needs to be liquidated to reduce leverage at the minimal trading cost. While an optimal execution strategy tells how liquidation should proceed to minimize the cost and risk. In this thesis, we formulate a one-period optimal deleveraging problem as a non-convex quadratic (polynomial) program with quadratic (polynomial) and box constraints under linear (nonlinear) market price impact functions. A Lagrangian algorithm is developed to numerically solve the NP-hard problem and estimate the quality of the solution. We further propose a two-period robust deleveraging program to account for market uncertainties. Depending on whether the portfolio contains derivative securities, the robust optimization program can be converted to either a convex semidefinite program or a convex second-order cone program, both of which are computationally tractable. We model the optimal execution problem as a stochastic control program and propose a Markov chain approximation scheme to numerically obtain the optimal trading trajectory. We also analyze theoretically how asset characteristics and market conditions affect the optimal deleveraging and execution strategies, which provides guidance on how to design trading policies from qualitative aspects.

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Chapter 1

Introduction

1.1 Optimal Portfolio Deleveraging under Financial Distress

When facing financial distress, for example, a high leverage ratio or a major margin call, a financial institution might be forced to initiate portfolio deleveraging for the purpose of avoiding insolvency risks. By analyzing the 15 trading losses with amount exceeding \$1 billion since 1990, Barth and McCarthy ([7]) find out that strong capital ratio, or equivalently small leverage ratio, could allow institutions to withstand difficult events and survive from catastrophe. Moreover, it is pointed out in the world bank report ([26]) that the excessive leverage of financial institutions is one of the main contributors to the global financial crisis. Therefore, maintaining a proper leverage ratio is unquestionably important to both individual institutions and the financial system.

For financial institutions, the sizes of their portfolios are usually very large, making the portfolio deleveraging a complex task. If an institutional investor sells large blocks of assets instantaneously, due to the finiteness of market liquidity, the investor must significantly compromise on the prices of the assets. This triggers very high trading cost. Market microstructure theory suggests that the trading cost is influenced by temporary and permanent price impact ([48]). Temporary price impact measures the instantaneous price pressure resulted from trading, while permanent price impact measures the change of the equilibrium price before and after trading.

An optimal deleveraging problem aims to determine the portion of the portfolio to liquidate so that the distress can be relieved at minimum cost. More specifically, there are two questions that need to be answered: which assets in the portfolio should be selected to sell? how many units of each assets should be sold? Developing a systemic and efficient algorithm to numerically obtain

the optimal strategy is a key issue. In addition to the quantitative method, analyzing the effect of both exogenous and endogenous factors on the optimal strategy is also meaningful from qualitative aspect.

1.2 Optimal Portfolio Execution under Market Risk

Given the portion of the portfolio to sell, a following question is how to place orders over time. Owing to the limited market liquidity, the investor may not be able to execute a large order all at once or only at high trading cost. The trading cost is influenced by market price impact that penalizes for high trading speed. More often, the investor divides the large order into smaller pieces and trades them gradually. In addition to the trading cost, the existence of market volatility also imposes trading risk to the liquidation process. Therefore, obtaining a good balance between the trading cost and trading risk becomes a critical issue for risk-averse investors. In the extant literature of portfolio liquidation, different risk measures have been considered ([2], [28], [33], [34], [50], [60], [62], [70]).

A widely-used risk measure is mean-variance. [2] is a pioneering work that introduces a mean-variance framework to account for the volatility risk. A closed-form optimal strategy in a discrete arithmetic Brownian motion model is obtained in [2]. However, it is well understood that the mean-variance framework is time-inconsistent ([8], [28]), meaning that an optimal strategy determined earlier is not necessarily optimal at a later time. The time-inconsistency could also lead to numerical difficulty in general since dynamic programming cannot be applied directly to solve the optimal control problem. Thus, it is important to select an appropriate risk measure that makes the portfolio execution problem computationally tractable and also accounts for the investor's risk concern. It should also be noticed that different investors may have different concerns regarding their trading strategy or even one investor may have different preferences in different situations. So designing a robust and efficient scheme to obtain optimal liquidation strategy under different risk measures is of great value.

1.3 Organization of Dissertation

The dissertation consists of six chapters. The remaining chapters are organized as follows:

- Chapter 2-Portfolio Deleveraging under Linear Market Impact

In this chapter, we study a non-convex optimal portfolio deleveraging problem where the objective is to decide which portion the portfolio should be sold so that the leverage ratio could be reduced to the expected level with minimum sacrifice in equity. We propose a Lagrangian algorithm to find the optimal deleveraging strategy under certain conditions. When the conditions are violated, the algorithm might return a sub-optimal strategy and an upper bound on the loss in equity caused by the approximation is obtained.

- Chapter 3-Portfolio Deleveraging under Nonlinear Market Impact

There exist ample theoretical and empirical works in favor of a strictly concave temporary price impact function. Accounting for the strict concavity, in this chapter, we further consider the portfolio deleveraging problem under a power-law temporary impact function with a general exponent between zero and one. We extend the Lagrangian algorithm to solve the non-convex constrained polynomial optimization program.

- Chapter 4-Two-Period Robust Portfolio Liquidation under Margin Requirement

In this chapter, we propose a two-period robust liquidation model, aiming to seek a robust strategy that is able to meet the margin call under a range of market conditions. Depending on the portfolio type (i.e., a simple portfolio containing only basic assets or a enriched portfolio containing derivative securities as well), the robust optimization program can be converted to either a second-order cone program or a semidefinite program, both of which are computationally tractable.

- Chapter 5-Optimal Portfolio Execution with a Markov Chain Approximation Approach

Given the portion of the portfolio to execute, how the execution should proceed within a short time horizon with the minimal trading cost and trading risk is another important issue. We develop a Markov chain approximation scheme to find the optimal liquidation trajectory under different risk measures. The convergence and efficiency of the approach is guaranteed by theoretical analysis and verified by numerical experiments.

- Chapter 6-Conclusions and Future Extensions

- Appendix

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Chapter 2

Portfolio Deleveraging under Linear Market Impact

2.1 Introduction

Financial institutions usually have high leverage ratios. For example, before their collapses, both Bear Stearns and Lehman Brothers had leverage ratios of more than 30, implying that a three percent decrease in the asset value could almost wipe out their whole equities. It is important for financial institutions to keep the leverage below a certain level to avoid the insolvency risk. When their leverage ratios are higher than the tolerance level, financial institutions might choose to unwind their portfolios to reduce the ratios. This process is called portfolio deleveraging. Most often, the deleveraging activity of one institution will further drive down the price of certain assets, which impairs the balance sheets of other institutions. As a consequence, those institutions suffering from the increase of leverage ratios start to liquidate portfolios as well. This creates a vicious cycle that may lead to massive deleveraging among financial institutions. For example, dating back to 2008, it was documented in some works ([29], [21]) that the failure of Lehman Brothers generated portfolio deleveraging and liquidation in all asset classes around the world. In 2013, the Basel III Reforms “introduced a simple, transparent, non-risk based leverage ratio to act as a credible supplementary measure to the risk-based capital requirements. The leverage ratio is intended to restrict the build-up of leverage in the banking sector to avoid destabilising deleveraging processes that can damage the broader financial system and the economy” ([9]). Therefore, leverage ratio now is a critical factor that financial institutions have to watch carefully.

Liquidating large blocks of assets is highly possible to suffer from both temporary and permanent price impact owing to the limited market liquidity. Linear price impact models have been widely

used in both academic research and practical applications (see [3], [14], [15], [58], [34], [59], [41], [45]). Here we also assume the linearity of both impact functions. It has been stated in several works that permanent price impact has to be linear to avoid dynamic arbitrage (see [38], [33]). Thus, throughout the dissertation, we only focus on the linear permanent impact function. For the temporary impact function, we will extend to the nonlinear case in Chapter 3. [14] studies optimal deleveraging strategies where the main objective is to generate cash to reduce leverage by selling a fraction of the assets in a portfolio in a short period of time. [14] first considers a one-period optimal deleveraging problem, which is formulated as a quadratic program. Under certain convexity assumptions, [14] derives analytical results regarding the optimal trading strategy. For example, more liquid assets are prioritized for selling, and it is optimal to deleverage to the margin. The convexity assumption in [14] requires the temporary price impact parameter to be greater than one half of the permanent price impact parameter for each asset in the portfolio. A similar assumption is also made in [2]. Empirical studies, however, show that this does not always hold. For example, [37] observes that permanent price impact may dominate in block transactions. A similar phenomenon is also reported in [65]. See [60] for a discussion of plastic markets where permanent price impact dominates. Therefore, in [20], we relax the restrictions on the relative magnitudes of the price impact parameters and consider the one-period optimal deleveraging problem studied in [14]. This leads to a non-convex program with a quadratic objective function and quadratic and box constraints, which is generally quite challenging.

To analyze this non-convex deleveraging problem, we notice that both the objective function and the quadratic constraint are separable in terms of the individual variables. [10] presents a way to transform a quadratic program with a separable objective function and quadratic constraint to a simpler convex program. However, their method is not directly applicable in our problem due to the box constraints. [54] and [72] study semidefinite relaxation for certain quadratic programs with quadratic and/or box constraints. However, this method is also not directly applicable in our problem due to a linear term in the quadratic constraint. On the other hand, linear time breakpoint searching algorithms have been successfully used for solving the continuous quadratic knapsack problem, which is a separable convex quadratic program with linear and box constraints. See [56] and [42]. Inspired by this, we propose a Lagrangian method for our non-convex quadratic program. In particular, we study the breakpoints of the Lagrangian problem and provide conditions

under which an optimal trading strategy can be found using the Lagrangian method. When the Lagrangian algorithm returns a suboptimal approximation, we assess the quality of the solution. By studying the original quadratic program and the corresponding Lagrangian problem, we are able to derive some analytical results on the optimal trading strategy. These results help us better understand how a portfolio is liquidated in the case of a deleveraging need.

2.2 Model Formulation

The execution price is modeled as follows:

$$p_t = p_0 + \Gamma(x_t - x_0) + \Lambda y_t, \quad (2.2.1)$$

where $p_t, x_t, y_t \in \mathbb{R}^m$ are vectors of prices, holdings and trading rates of the m assets in the portfolio at time t , $\Gamma = \text{diag}(\gamma_1, \dots, \gamma_m)$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$. For each $1 \leq i \leq m$, $\gamma_i > 0$ is the permanent price impact parameter related to the cumulative trading amount in asset i , and $\lambda_i > 0$ is the temporary price impact parameter associated with the trading rate in asset i . The initial prices and initial holdings are positive: $x_{0,i} > 0, p_{0,i} > 0, 1 \leq i \leq m$. The trading amount and trading rate satisfy $x_t - x_0 = \int_0^t y_s ds$.

Without loss of generality, we assume a finite trading horizon of length $T = 1$. Let $q = p_0 - \Gamma x_0$. The amount of cash generated during the trading period is

$$\begin{aligned} K &= \int_0^1 -p_t^T y_t dt \\ &= \int_0^1 -(q + \Gamma x_t + \Lambda y_t)^T y_t dt \\ &= \int_0^1 -(q + \Gamma x_t + \Lambda y_t)^T y_t dt. \end{aligned}$$

Assume the initial liability is l_0 and the liability after liquidation is

$$l_1 = l_0 - K = \int_0^1 (q + \Gamma x_t + \Lambda y_t)^T y_t dt + l_0.$$

Let $e_0 = p_0^T x_0 - l_0$ be the initial equity and e_1 be the equity after trading. Then

$$\begin{aligned}
e_1 &= p_1^T x_1 - l_1 \\
&= (q + \Gamma x_1)^T x_1 - \int_0^1 (q + \Gamma x_t + \Lambda y_t)^T y_t dt - l_0 \\
&= (q + \Gamma x_1)^T x_1 - \int_0^1 (q + \Gamma x_t + \Lambda y_t)^T y_t dt - l_0.
\end{aligned}$$

We seek a trading strategy that maximizes the equity e_1 subject to the constraint that the leverage ratio, defined as l_1/e_1 , does not exceed a predetermined level ρ_1 at the end of the trading period. We further impose restrictions on short selling and buying. Thus, at any time point t , we have $y_t \leq 0$, which further implies that $x_1 \geq 0$ can guarantee that there is no shortselling during the liquidation period. These lead to the following optimal control problem:

$$\begin{aligned}
&\max_{y_t} && (q + \Gamma x_1)^T x_1 - \int_0^1 (q + \Gamma x_t + \Lambda y_t)^T y_t dt - l_0 \\
&\text{subject to} && \rho_1 (q + \Gamma x_1)^T x_1 - (\rho_1 + 1) \int_0^1 (q + \Gamma x_t + \Lambda y_t)^T y_t dt \geq (\rho_1 + 1) l_0 \\
&&& \dot{x}_t = y_t \\
&&& y_t \leq 0 \\
&&& x_1 \geq 0.
\end{aligned} \tag{2.2.2}$$

We then derive the following key property regarding the optimal solution.

Proposition 2.2.1. *The optimal solution to (2.2.2) is a constant.*

According to Proposition 2.2.1, the optimal deleveraging strategy has a constant trading rate. So instead of considering the trading rate y_t , we only need to focus on the cumulative trading amount during the liquidation period, which we denote as $y = x_1 - x_0 \in \mathbb{R}^m = y_t \times T$. Now we can simplify the notations by replacing y_t with y/T :

$$K(y) = -p_0^\top y - y^\top (\Lambda + \frac{1}{2}\Gamma)y,$$

$$l_1(y) = l_0 + p_0^\top y + y^\top (\Lambda + \frac{1}{2}\Gamma)y,$$

and

$$e_1(y) = p_1^\top x_1 - l_1 = x_0^\top \Gamma y - y^\top (\Lambda - \frac{1}{2}\Gamma)y + p_0^\top x_0 - l_0 \quad (2.2.3)$$

Furthermore, the no shortselling or buying requirements reduce to $x_1 = x_0 + y \geq 0$ and $y \leq 0$, which lead to the following quadratic programming problem:

$$\begin{aligned} \max_{y \in \mathbb{R}^m} \quad & e_1(y) = -y^\top (\Lambda - \frac{1}{2}\Gamma)y + x_0^\top \Gamma y + p_0^\top x_0 - l_0 \\ \text{subject to} \quad & -y^\top \left(\rho_1 (\Lambda - \frac{1}{2}\Gamma) + \Lambda + \frac{1}{2}\Gamma \right) y + (\rho_1 \Gamma x_0 - p_0)^\top y + \rho_1 p_0^\top x_0 - (\rho_1 + 1)l_0 \geq 0 \\ & -x_0 \leq y \leq 0 \end{aligned} \quad (2.2.4)$$

The first constraint corresponds to the leverage requirement $\rho_1 e_1 - l_1 \geq 0$. The second constraint is a box constraint. We further assume that the leverage requirement is not satisfied before trading: $l_0/e_0 > \rho_1$ (that is, $\rho_1 p_0^\top x_0 - (\rho_1 + 1)l_0 < 0$). By taking derivative of the objective function in the above optimization problem, it is easy to see that trading will lead to a reduction in the equity. If the leverage requirement is satisfied before trading, the deleveraging problem becomes trivial: no trading is necessary and $y = 0$. We further assume that the above quadratic programming problem is strictly feasible.

Assumption 2.2.2. *Problem (2.2.4) is strictly feasible and $\rho_1 e_0 - l_0 < 0$.*

[14] assumes that $\Lambda - \frac{1}{2}\Gamma$ is positive definite so that the above quadratic programming problem is convex. Here we consider the general case of problem (2.2.4) without assuming convexity.

2.3 Optimization Algorithm

This section consists of two parts. In the first part, we consider a Lagrangian problem associated with (2.2.4) and study the properties of its breakpoints. In the second part, we present a Lagrangian algorithm for the deleveraging problem, and give conditions under which the Lagrangian algorithm returns an optimal solution. When the solution obtained from the Lagrangian algorithm is suboptimal, we give upper bounds on the loss in equity caused by using such a suboptimal trading strategy.

More specifically, the Lagrangian problem is formed by adding the leverage constraint $f(y) =$

$\rho_1 e_1(y) - l_1(y) \geq 0$ to the objective function. It admits closed-form solution $y^*(z)$ for any given Lagrangian multiplier z (Proposition 2.3.2). Since $y^*(z)$ may not be unique, we study the set-valued function $f^*(z) = \{f(y^*(z))\}$ and give necessary and sufficient conditions for z to be a breakpoint (i.e., $f^*(z)$ has at least two distinct values). This is Proposition 2.3.4. We then show that $f^*(z)$ is a piecewise continuous non-decreasing set-valued function with at most m breakpoints, where m is the number of assets (Theorem 2.3.6), and when there exists a non-breakpoint z^* such that $f^*(z^*) = 0$, the Lagrangian problem with parameter z^* is equivalent to the original deleveraging problem (Theorem 2.3.7). These results form the theoretical foundations for the Lagrangian algorithm presented in Section 2.3.2. When the condition of Theorem 2.3.7 is satisfied, we use bisection to find z^* such that $f^*(z^*) = 0$ and hence obtain an optimal solution to the deleveraging problem. Otherwise, we obtain a suboptimal solution and present bounds for the loss in equity caused by using such a suboptimal trading strategy (Theorem 2.3.10).

Before we proceed to the algorithm, we first show that the leverage constraint at the optimal solution is active. This property is important for our algorithm. The result on the case with the convexity assumption was reported in [14]. In the following, we provide a proof without assumptions on the relative magnitudes of the price impact.

Proposition 2.3.1. *The leverage constraint of problem (2.2.4) is active at its optimal solution.*

Proof: Since the feasible set of problem (2.2.4) is closed and bounded, there exists an optimal solution y^* . Assume to the contrary that the leverage constraint is not active at the optimal solution. Then it is easy to see that the so-called linear independence constraint qualification (LICQ) holds¹ (see Chapter 12 of [55]). Denote $y^* = (y_1^*, \dots, y_m^*)^\top$, $x_0 = (x_{0,1}, \dots, x_{0,m})^\top$. According to the first order optimality condition, there exists $\mu^* = (\mu_0^*, \mu_1^*, \dots, \mu_m^*, \mu_{m+1}^*, \dots, \mu_{2m}^*) \geq 0$ satisfying the following conditions:

$$\mu_0^*(\rho_1 e_1(y^*) - l_1(y^*)) = 0, \quad (2.3.1)$$

$$\mu_i^* y_i^* = 0, \quad i = 1, \dots, m, \quad (2.3.2)$$

$$\mu_{m+i}^*(y_i^* + x_{0,i}) = 0, \quad i = 1, \dots, m, \quad (2.3.3)$$

$$-\nabla e_1(y^*) + \mu_0^* \nabla g_0(y^*) + \sum_{i=1}^m (\mu_i^* \nabla g_i(y^*) + \mu_{m+i}^* \nabla g_{m+i}(y^*)) = 0, \quad (2.3.4)$$

¹LICQ holds if the gradients of active constraints are linearly independent.

where

$$g_0(y) = l_1(y) - \rho_1 e_1(y), \quad g_i(y) = y_i, \quad g_{m+i}(y) = -y_i - x_{0,i}, \quad i = 1, \dots, m.$$

By the assumption that the leverage constraint is not active, we have $\mu_0^* = 0$. Consequently, for any $i = 1, \dots, m$, equation (2.3.4) becomes

$$2(\lambda_i - \frac{1}{2}\gamma_i)y_i^* - \gamma_i x_{0,i} + \mu_i^* - \mu_{m+i}^* = 0, \quad i = 1, \dots, m. \quad (2.3.5)$$

Since $x_0 > 0$, it is easy to see from (2.3.2) and (2.3.3) that μ_i^* and μ_{m+i}^* , $1 \leq i \leq m$, cannot be positive simultaneously. We consider the following cases:

1. $\mu_i^* = 0, \mu_{m+i}^* \neq 0$: From equation (2.3.3), we obtain $y_i^* = -x_{0,i}$. From Equation (2.3.5), we get $\mu_{m+i}^* = -2\lambda_i x_{0,i} < 0$, which contradicts to the requirement that $\mu^* \geq 0$.
2. $\mu_i^* = \mu_{m+i}^* = 0$: By Equation (2.3.5), we obtain $2(\lambda_i - \frac{1}{2}\gamma_i)y_i^* = \gamma_i x_{0,i} > 0$. Since $y_i^* \leq 0$, we must have $\lambda_i - \frac{1}{2}\gamma_i < 0$ and $y_i^* = \frac{\gamma_i x_{0,i}}{2\lambda_i - \gamma_i} < 0$. But then $y_i^* + x_{0,i} = \frac{2\lambda_i x_{0,i}}{2\lambda_i - \gamma_i} < 0$, which contradicts to the requirement that $y^* + x_0 \geq 0$.

Since for any $1 \leq i \leq m$, the above cases are not possible, we must have $\mu_i^* \neq 0, \mu_{m+i}^* = 0$ for all $1 \leq i \leq m$. But then we have $y^* = 0$ from equation (2.3.2). The leverage requirement in problem (2.2.4) then becomes $\rho_1 e_0 - l_0 \geq 0$, which contradicts Assumption 2.2.2. This finishes the proof of the proposition. \square

Intuitively, Proposition 2.3.1 states that the optimal deleveraging strategy precisely achieves the maximal allowed leverage ratio. It is suboptimal to further reduce the leverage ratio due to the trading cost caused by market impact.

2.3.1 Lagrangian Problem and Breakpoints

Denote $f(y) = \rho_1 e_1(y) - l_1(y)$. The Lagrangian problem maximizes $e_1(y) + zf(y)$ subject to the box constraint $-x_0 \leq y \leq 0$ for some $z \geq 0$. Equivalently, we have

$$\begin{aligned} \max_{y \in \mathbb{R}^m} \quad & -y^\top \left((1 + z\rho_1)(\Lambda - \frac{1}{2}\Gamma) + z(\Lambda + \frac{1}{2}\Gamma) \right) y + ((1 + z\rho_1)x_0^\top \Gamma - zp_0^\top) y \\ \text{subject to} \quad & -x_0 \leq y \leq 0. \end{aligned} \quad (2.3.6)$$

The following proposition gives the optimal solution of the Lagrangian problem with parameter z , which we denote by $y^*(z)$.

Proposition 2.3.2. *The optimal solution to the Lagrangian problem (2.3.6) is given by*

$$y_i^*(z) = \begin{cases} \max(-x_{0,i}, \min(0, \frac{q_{2,i}}{2q_{1,i}})), & q_{1,i} > 0 \\ 0, & q_{1,i} \leq 0, q_{2,i} + x_{0,i}q_{1,i} > 0 \\ -x_{0,i}, & q_{1,i} \leq 0, q_{2,i} + x_{0,i}q_{1,i} < 0 \\ 0 \text{ or } -x_{0,i}, & q_{1,i} < 0, q_{2,i} + x_{0,i}q_{1,i} = 0 \\ \text{any point in } [-x_{0,i}, 0], & q_{1,i} = q_{2,i} = 0 \end{cases}$$

for $1 \leq i \leq m$, where $q_{1,i} = (1 + z\rho_1)(\lambda_i - \frac{1}{2}\gamma_i) + z(\lambda_i + \frac{1}{2}\gamma_i)$, and $q_{2,i} = (1 + z\rho_1)\gamma_i x_{0,i} - zp_{0,i}$.

Proof: The objective function of the Lagrangian problem is simply $\sum_{i=1}^m (q_{2,i}y_i - q_{1,i}y_i^2)$. It suffices to derive the optimal solution for the following problem:

$$\begin{aligned} \max_{y_i \in \mathbb{R}} \quad & q_{2,i}y_i - q_{1,i}y_i^2 \\ \text{subject to} \quad & -x_{0,i} \leq y_i \leq 0. \end{aligned}$$

This is a univariate quadratic problem whose optimal solution is given as in the proposition. \square

Note that the solution of the Lagrangian problem (2.3.6) is analytically available due to the separability of its objective function, which results from the diagonal structures of the price impact matrices Λ and Γ . This allows us to design an efficient algorithm to solve the original deleveraging problem. We would like to point out that the algorithm we will present may not easily extend to the case when there is cross impact, that is, when the impact matrices have nonzero off-diagonal entries.

We note that the Lagrangian problem may not have a unique solution. For any $z \geq 0$, denote $f^*(z)$ the set of the values of f at all possible optimal solutions of the Lagrangian problem with parameter z . That is, f^* is a set-valued function (see [6]). We analyze the properties of $f^*(z)$ below. We start with the definition of breakpoint.

Definition 2.3.3. *z is said to be a breakpoint of f^* if $f^*(z)$ has at least two distinct values.*

In the following, we give necessary and sufficient conditions for z to be a breakpoint.

Proposition 2.3.4. $z \geq 0$ is a breakpoint of f^* if and only if the following holds for some $1 \leq i \leq m$:

$$\left((1 + \rho_1) \left(\lambda_i + \frac{1}{2} \gamma_i \right) x_{0,i} - p_{0,i} \right) z + \left(\lambda_i + \frac{1}{2} \gamma_i \right) x_{0,i} = 0, \quad (2.3.7)$$

$$p_{0,i} \geq (6 + 4\sqrt{2}) x_{0,i} \lambda_i, \quad (2.3.8)$$

and $\gamma_i \in [\theta_1, \theta_2]$, where θ_1, θ_2 are the two roots of $x^2 + (2\lambda_i - \frac{p_{0,i}}{x_{0,i}})x + \frac{2p_{0,i}\lambda_i}{x_{0,i}} = 0$.

Proof: According to Proposition 2.3.2, for z to be a breakpoint of f^* , it is necessary that $q_{1,i} \leq 0, q_{2,i} + x_{0,i}q_{1,i} = 0$ for some $1 \leq i \leq m$. Here $q_{1,i}, q_{2,i}$ are defined in Proposition 2.3.2. On the other hand, for a given z , if $q_{1,i} \leq 0, q_{2,i} + x_{0,i}q_{1,i} = 0$, then there exist at least two optimal solutions to the Lagrangian problem, $y^*(z)$ and $y^{**}(z)$, such that $y_i^*(z) = 0, y_i^{**}(z) = -x_{0,i}, y_j^*(z) = y_j^{**}(z), j \neq i$, and $e_1(y^*(z)) + zf(y^*(z)) = e_1(y^{**}(z)) + zf(y^{**}(z))$. Since

$$e_1(y^{**}(z)) - e_1(y^*(z)) = -\gamma_i x_{0,i}^2 - \left(\lambda_i - \frac{1}{2} \gamma_i \right) x_{0,i}^2 = -\left(\lambda_i + \frac{1}{2} \gamma_i \right) x_{0,i}^2 < 0,$$

and $z \geq 0$, we have $f(y^*(z)) < f(y^{**}(z))$. Thus, $f^*(z)$ has at least two distinct values and z is thus a breakpoint. Therefore, for $z \geq 0$ to be a breakpoint of f^* , it is necessary and sufficient that $q_{1,i} \leq 0, q_{2,i} + x_{0,i}q_{1,i} = 0$ for some $1 \leq i \leq m$.

It is easy to verify that $q_{2,i} + x_{0,i}q_{1,i} = 0$ if and only if (2.3.7) holds. Since $z \geq 0$, we must have $(1 + \rho_1) \left(\lambda_i + \frac{1}{2} \gamma_i \right) x_{0,i} - p_{0,i} < 0$. On the other hand, $q_{1,i} \leq 0$ if and only if

$$\left(\rho_1 \left(\lambda_i - \frac{1}{2} \gamma_i \right) + \lambda_i + \frac{1}{2} \gamma_i \right) z + \lambda_i - \frac{1}{2} \gamma_i \leq 0.$$

Substitute z in the above with the solution from (2.3.7), we obtain

$$\gamma_i^2 + \left(2\lambda_i - \frac{p_{0,i}}{x_{0,i}} \right) \gamma_i + \frac{2p_{0,i}\lambda_i}{x_{0,i}} \leq 0. \quad (2.3.9)$$

Since $\gamma_i > 0$, (2.3.9) holds if and only if $2\lambda_i - \frac{p_{0,i}}{x_{0,i}} < 0$,

$$\left(2\lambda_i - \frac{p_{0,i}}{x_{0,i}} \right)^2 - \frac{8p_{0,i}\lambda_i}{x_{0,i}} \geq 0, \quad (2.3.10)$$

and $\gamma_i \in [\theta_1, \theta_2]$, where θ_1, θ_2 are the two roots of $x^2 + (2\lambda_i - \frac{p_{0,i}}{x_{0,i}})x + \frac{2p_{0,i}\lambda_i}{x_{0,i}} = 0$. (2.3.10) becomes

$$\frac{p_{0,i}^2}{x_{0,i}^2 \lambda_i^2} - \frac{12p_{0,i}}{x_{0,i} \lambda_i} + 4 \geq 0.$$

Since $2\lambda_i - \frac{p_{0,i}}{x_{0,i}} < 0$, the above holds if and only if $p_{0,i} \geq (6 + 4\sqrt{2})x_{0,i}\lambda_i$. This finishes the proof.

□

Corollary 2.3.5. *If $p_0 < (6 + 4\sqrt{2})\Lambda \mathbf{x}_0$, then there is no breakpoints for f^* .*

This is a useful result which gives us conditions under which the optimal solution of the quadratic programming problem (2.2.4) can be obtained by using our Lagrangian algorithm, as will be seen. Since λ_i is the temporary price impact corresponding to the trading rate for asset i , $\lambda_i x_{0,i}$ can be taken as the temporary price impact caused by selling all of asset i in one unit of time. Then the above condition states that, as long as such price impact is larger than 8.6% of the initial asset price, the optimal solution of the deleveraging problem can be obtained by solving a sequence of very simple Lagrangian problems. Now we state a key property of f^* .

Theorem 2.3.6. *$f^*(z)$ is a piecewise continuous non-decreasing set-valued map² with at most m breakpoints. In particular, $f^*(0) < 0$, and $\exists z' > 0$ such that $f^*(z') > 0$.*

Proof: From Proposition 2.3.2, it can be seen that the solution of the Lagrangian problem $y^*(z)$ and f^* are continuous in z except at breakpoints. From Proposition 2.3.4, z is a breakpoint only if z solves (2.3.7) for some $1 \leq i \leq m$. Therefore, there could be at most m breakpoints.

To prove the monotonicity, consider two different parameters z_1 and z_2 . Let $\xi_1 \in f^*(z_1)$, and $y^*(z_1)$ be an optimal solution of the Lagrangian problem with parameter z_1 that satisfies $\xi_1 = f(y^*(z_1))$. Define $\xi_2 \in f^*(z_2)$ and $y^*(z_2)$ similarly. By the optimality of $y^*(z_1)$ and $y^*(z_2)$, we have

$$e_1(y^*(z_1)) + z_1 \xi_1 \geq e_1(y^*(z_2)) + z_1 \xi_2,$$

$$e_1(y^*(z_2)) + z_2 \xi_2 \geq e_1(y^*(z_1)) + z_2 \xi_1.$$

² $f^*(z)$ is a non-decreasing set-valued map if $(z_1 - z_2)(\xi_1 - \xi_2) \geq 0, \forall \xi_i \in f^*(z_i), i = 1, 2$. For details of set-valued maps, refer to [6].

Adding the above two inequalities, we obtain

$$z_1\xi_1 + z_2\xi_2 \geq z_1\xi_2 + z_2\xi_1.$$

Thus, we have $(z_1 - z_2)(\xi_1 - \xi_2) \geq 0$. Therefore, f^* is non-decreasing.

When $z = 0$, for any $1 \leq i \leq m$, using the notations in Proposition 2.3.2,

$$q_{2,i} = \gamma_i x_{0,i} > 0, \quad q_{2,i} + x_{0,i}q_{1,i} = (\lambda_i + \frac{1}{2}\gamma_i)x_{0,i} > 0.$$

From Proposition 2.3.2, the solution of the Lagrangian problem is $y^*(0) = 0$. Therefore, $f^*(0) = f(0) = \rho_1 e_0 - l_0 < 0$. Let \bar{y} be an optimal solution to

$$\max_{y \in \mathbb{R}^m} f(y) \quad \text{subject to} \quad -x_0 \leq y \leq 0,$$

and \hat{y} an optimal solution to

$$\max_{y \in \mathbb{R}^m} e_1(y) \quad \text{subject to} \quad -x_0 \leq y \leq 0.$$

Note that $f(\bar{y}) > 0$. Let $z > 0$ be larger than all the m breakpoints and satisfy

$$z > \frac{e_1(\hat{y}) - e_1(\bar{y})}{f(\bar{y})}.$$

Denote the optimal solution of the Lagrangian problem corresponding to z by $y^*(z)$. Then we have

$$e_1(\hat{y}) + zf(y^*(z)) \geq e_1(y^*(z)) + zf(y^*(z)) \geq e_1(\bar{y}) + zf(\bar{y}).$$

From the above, we immediately obtain $f^*(z) = f(y^*(z)) > 0$. This finishes the proof. \square

2.3.2 Lagrangian Algorithm

We present a Lagrangian algorithm for the deleveraging problem, give conditions under which the algorithm returns an optimal solution, and give an upper bound on the loss of equity when a suboptimal solution returned from the algorithm is adopted.

Theorem 2.3.7. *Suppose there exists a non-breakpoint $z^* > 0$ such that $f^*(z^*) = 0$. Then the Lagrangian problem with parameter z^* is equivalent to the deleveraging problem (2.2.4).*

Proof: Denote the optimal solution of the Lagrangian problem by $y^*(z^*)$, and an optimal solution of the deleveraging problem by y^* . Since $f^*(z^*) = f(y^*(z^*)) = 0$, $y^*(z^*)$ is a feasible solution of the deleveraging problem. Obviously, y^* is also a feasible solution of the Lagrangian problem. According to Proposition 2.3.1, $f(y^*) = 0$. By the optimality of $y^*(z^*)$,

$$e_1(y^*(z^*)) = e_1(y^*(z^*)) + z^* f(y^*(z^*)) \geq e_1(y^*) + z^* f(y^*) = e_1(y^*).$$

This shows that $y^*(z^*)$ is also optimal to the deleveraging problem. As a result, $e_1(y^*(z^*)) = e_1(y^*)$. Consequently, $e_1(y^*(z^*)) + z^* f(y^*(z^*)) = e_1(y^*) + z^* f(y^*)$. But this implies that y^* is also optimal to the Lagrangian problem. This finishes the proof. \square

When $z^* > 0$ is a breakpoint and zero is one of the values of $f^*(z^*)$, we have the following result. Denote the optimal solution to the Lagrangian problem that corresponds to this zero value by $y^*(z^*)$. That is, $f(y^*(z^*)) = 0$. Then using similar arguments, one can show that $y^*(z^*)$ is also optimal to the deleveraging problem. In this case, however, we don't have equivalence between the Lagrangian problem and the deleveraging problem. In the following, we give a condition under which Theorem 2.3.7 is immediately applicable.

Corollary 2.3.8. *Suppose $p_0 < (6 + 4\sqrt{2})\Lambda\mathbf{x}_0$. Then f^* has no breakpoint. There exists $z^* > 0$ such that $f^*(z^*) = 0$. The Lagrangian problem with parameter z^* is equivalent to the deleveraging problem (2.2.4).*

Proof. This follows from Corollary 2.3.5, Theorem 2.3.6, and Theorem 2.3.7. \square

Based on the above theoretical results, we design a Lagrangian algorithm for solving the problem (2.2.4). Roughly speaking, when $f^*(z^*) = 0$ can be achieved at a non-breakpoint z^* , we use binary search to find z^* and consequently an optimal solution to the deleveraging problem. When zero falls between different values of $f^*(z^*)$ for a certain breakpoint z^* , we find a feasible approximation to the optimal solution of the deleveraging problem through the Lagrangian problem with parameter z^* . In this case, we give an upper bound on the loss of equity caused by using a suboptimal solution. Before we present the algorithm, we make the following assumption which simplifies the algorithm.

Assumption 2.3.9. For any $1 \leq i \leq m$,

$$(p_{0,i} - \rho_1 \gamma_i x_{0,i})(\lambda_i - \frac{1}{2} \gamma_i) + \gamma_i x_{0,i}(\rho_1(\lambda_i - \frac{1}{2} \gamma_i) + \lambda_i + \frac{1}{2} \gamma_i) \neq 0.$$

For any $i \neq j$,

$$x_{0,i}(\lambda_i + \frac{1}{2} \gamma_i)((1 + \rho_1)(\lambda_j + \frac{1}{2} \gamma_j)x_{0,j} - p_{0,j}) \neq x_{0,j}(\lambda_j + \frac{1}{2} \gamma_j)((1 + \rho_1)(\lambda_i + \frac{1}{2} \gamma_i)x_{0,i} - p_{0,i}).$$

This is a rather weak assumption since it would be rare that the parameters of the deleveraging problem precisely satisfy the equalities instead. The first requirement excludes cases where the Lagrangian problem has a constant objective function and hence admits infinitely many solutions (see Proposition 2.3.2). The second requirement guarantees that breakpoints determined from different assets using Proposition 2.3.4 do not overlap. Consequently, at any breakpoint z^* , $f^*(z^*)$ has exactly two different values.

When there exists a breakpoint z_i , corresponding to the i th asset, such that 0 falls in the open interval defined by the two values of $f^*(z_i)$, we construct a feasible solution for (2.2.4) as follows. Denote the two optimal solutions of the Lagrangian problem with parameter z_i by $y^*(z_i)$ and $y^{**}(z_i)$. By Proposition 2.3.2 and Assumption 2.3.9, $y^*(z_i)$ and $y^{**}(z_i)$ differ only at the i th entry. Suppose $y_i^*(z_i) = 0$ and $y_i^{**}(z_i) = -x_{0,i}$. Denote the feasible approximation we are seeking by $y^L = (y_1^L, \dots, y_m^L)^\top$. Let $y_k^L = y_k^*(z_i) = y_k^{**}(z_i)$ for $k = 1, \dots, m$, $k \neq i$, and y_i^L be the zero of $f(y^L)$ as a function of y_i^L on $(-x_{0,i}, 0)$. From the proof of Proposition 4, we know that $f(y^*(z_i)) < 0 < f(y^{**}(z_i))$. Moreover, $f(y^L)$ as a function of y_i^L is quadratic. Its zero on $(-x_{0,i}, 0)$ can thus be uniquely determined. Then y^L is accepted as the approximate solution. It is obviously a feasible solution for the deleveraging problem (2.2.4).

In fact, $y^{**}(z_i)$ could also be used as a feasible approximate solution. But it can be easily verified that the objective function of the deleveraging problem (2.2.4) is non-decreasing in y_i for $y_i \in [-x_{0,i}, 0]$ (see the proof of Theorem 2.3.10). Therefore, y^L constructed in the above outperforms $y^{**}(z_i)$ and is hence adopted.

We now present the following algorithm in seeking a solution y^L of the problem (2.2.4). Here the superscript L refers to Lagrangian. It will be clear soon when y^L is optimal, and when it is suboptimal. Let $\epsilon > 0$ be a small enough tolerance level.

Lagrangian Algorithm

1. Using Proposition 2.3.4, find all the breakpoints $z_1 < z_2 < \dots < z_k$.
2. If $k = 0$ (no breakpoints), let $a = 0$, and find a large enough b such that $f^*(b) > 0$. Go to Step 7.
3. Using Proposition 2.3.2, for any z_i , compute the two values of $f^*(z_i)$, denoted by $f_1^*(z_i) = f(y^*(z_i)) < f_2^*(z_i) = f(y^{**}(z_i))$, where $y^*(z_i)$ and $y^{**}(z_i)$ are the two optimal solutions to the Lagrangian problem with parameter z_i .
4. If there exists z_i such that one of the values of $f^*(z_i)$ is zero, then let $y^L = y^*(z_i)$ if $f(y^*(z_i)) = 0$ and $y^L = y^{**}(z_i)$ otherwise, and stop.
5. Otherwise, if there exists z_i , corresponding to the i th asset, such that $f_1^*(z_i) < 0 < f_2^*(z_i)$, then let $y_k^L = y_k^{**}(z_i)$ for $k = 1, \dots, m$, $k \neq i$, and y_i^L be the zero of $f(y^L)$ as a function of y_i^L on $(-x_{0,i}, 0)$, and stop.
6. Otherwise, determine a and b in the following way: if the values of $f^*(z_k)$ are negative, let $a = z_k$, and find a large enough b such that $f^*(b) > 0$; if the values of $f^*(z_1)$ are positive, let $a = 0$ and $b = z_1$; otherwise, find z_i such that $f_2^*(z_i) < 0 < f_1^*(z_{i+1})$, and let $a = z_i, b = z_{i+1}$.
7. For the given a and b ,

While $(|f^*(\frac{a+b}{2})| > \epsilon)$ {

If $f^*(\frac{a+b}{2}) > \epsilon$, then $b \leftarrow \frac{a+b}{2}$

Else $a \leftarrow \frac{a+b}{2}$

}

Let $z^* = \frac{a+b}{2}$, $y^L = y^*(z^*)$, where $y^*(z^*)$ is the optimal solution of the Lagrangian problem with parameter z^* .

According to our theoretical results, it is clear that the above Lagrangian algorithm returns a suboptimal solution only when Step 5 is executed. Otherwise, it returns an optimal solution to the deleveraging problem (at least near optimal when ϵ is small).

Note that when zero is contained in an open interval defined by the two values of $f^*(z_i)$ for some breakpoint z_i (see Step 5 of the Lagrangian algorithm), we obtain a suboptimal solution to the problem (2.2.4). In the following, we quantify the loss caused by choosing such a suboptimal solution.

Theorem 2.3.10. *Suppose that there exists a breakpoint $z_i > 0$, corresponding to the i th asset, so that 0 is contained in the open interval defined by the two values of $f^*(z_i)$, and the Lagrangian algorithm is used to solve the problem (2.2.4). Then the loss in equity caused by using such obtained suboptimal trading strategy is bounded by*

$$(\lambda_i + \frac{1}{2}\gamma_i)x_{0,i}^2.$$

The above can further be bounded by

$$p_{0,i}x_{0,i} \min\left(\frac{1}{1+\rho_1}, \frac{1+\gamma_i/(2\lambda_i)}{6+4\sqrt{2}}\right).$$

Proof: Denote an optimal solution to the original deleveraging problem by y^* , the approximate solution obtained from the Lagrangian algorithm by y^L , and the two optimal solutions to the Lagrangian problem with parameter z_i by $y^*(z_i)$ and $y^{**}(z_i)$, with $f(y^*(z_i)) < 0 < f(y^{**}(z_i))$. Since y^* is feasible to the Lagrangian problem, and $y^{**}(z_i)$ is optimal to the Lagrangian problem, we have

$$e_1(y^{**}(z_i)) + z_i f(y^{**}(z_i)) \geq e_1(y^*) + z_i f(y^*). \quad (2.3.11)$$

According to the proof of Proposition 2.3.4, $y^{**}(z_i)$ is the one whose i th element is $-x_{0,i}$. Moreover,

$$z_i f(y^{**}(z_i)) < z_i (f(y^{**}(z_i)) - f(y^*(z_i))) = e_1(y^*(z_i)) - e_1(y^{**}(z_i)) = (\lambda_i + \frac{1}{2}\gamma_i)x_{0,i}^2.$$

Combining the above with (2.3.11), and because $f(y^*) = 0$ from Proposition 2.3.1, we obtain

$$e_1(y^*) - e_1(y^{**}(z_i)) < (\lambda_i + \frac{1}{2}\gamma_i)x_{0,i}^2.$$

Note that $y_k^L = y_k^{**}(z_i)$ for $k = 1, \dots, m$, $k \neq i$, and $y_i^L > -x_{0,i} = y_i^{**}(z_i)$. From equation (3.2.3), we have $\partial e_1 / \partial y_i = \gamma_i(x_{0,i} + y_i) - 2\lambda_i y_i \geq 0$ for $-x_{0,i} \leq y_i \leq 0$. Thus, $e_1(y^L) \geq e_1(y^{**}(z_i))$ and

$$e_1(y^*) - e_1(y^L) \leq e_1(y^*) - e_1(y^{**}(z_i)) < (\lambda_i + \frac{1}{2}\gamma_i)x_{0,i}^2.$$

We may relax the right hand side in the above. According to (2.3.7), $p_{0,i} > (1 + \rho_1)(\lambda_i + \frac{1}{2}\gamma_i)x_{0,i}$ must hold for z_i to be a breakpoint. Together with (2.3.8), we obtain

$$x_{0,i} \leq \min \left(\frac{p_{0,i}}{(6 + 4\sqrt{2})\lambda_i}, \frac{p_{0,i}}{(1 + \rho_1)(\lambda_i + \frac{1}{2}\gamma_i)} \right).$$

The conclusion of the Theorem follows immediately. \square

Theorem 2.3.10 shows that the loss in equity caused by using a suboptimal solution obtained from the Lagrangian algorithm is bounded by a fraction of the initial total value of a particular asset. In particular, the loss is small when ρ_1 is large, or when the ratio of the permanent to temporary price impact parameters is small.

2.4 Trading Properties

In this section, we derive some properties of the optimal deleveraging strategy. In particular, we examine the factors that influence the optimal strategy.

Proposition 2.4.1. *Suppose assets i and j have the same initial price and holding: $p_{0,i} = p_{0,j}$, $x_{0,i} = x_{0,j}$. If $\gamma_i \leq \gamma_j$, $\lambda_i \leq \lambda_j$, $\gamma_i < 2\lambda_j$, then the i th asset is prioritized for selling, i.e., $y_i^* \leq y_j^*$.*

Proof: Assume to the contrary that $-x_{0,i} = -x_{0,j} \leq y_j^* < y_i^* \leq 0$. Consider a direction $\Sigma = (\Sigma_1, \dots, \Sigma_m)$ with $\Sigma_i = -1$, $\Sigma_j = 1$, $\Sigma_k = 0$, $k \neq i, j$. Note that

$$\nabla l_1(y) = p_0 + (2\Lambda + \Gamma)y, \quad \nabla e_1(y) = \Gamma x_0 - (2\Lambda - \Gamma)y.$$

Since $p_{0,i} = p_{0,j}$, $\lambda_i \leq \lambda_j$, $\gamma_i \leq \gamma_j$, we have

$$\Sigma^\top \nabla l_1(y^*) = (2\lambda_j + \gamma_j)y_j^* - (2\lambda_i + \gamma_i)y_i^* \leq (2\lambda_j + \gamma_j)(y_j^* - y_i^*) < 0.$$

We next show that

$$\Sigma^\top \nabla e_1(y^*) = (\gamma_j - \gamma_i)x_{0,i} + (2\lambda_i - \gamma_i)y_i^* - (2\lambda_j - \gamma_j)y_j^* > 0.$$

Let us consider the following three cases:

Case 1. $2\lambda_i - \gamma_i > 0$. When $\lambda_i < \lambda_j$ strictly,

$$\begin{aligned} \Sigma^\top \nabla e_1(y^*) &\geq (\gamma_j - \gamma_i)x_{0,i} + (2\lambda_i - \gamma_i)y_j^* - (2\lambda_j - \gamma_j)y_j^* \\ &= (\gamma_j - \gamma_i)(x_{0,j} + y_j^*) + 2(\lambda_i - \lambda_j)y_j^* > 0. \end{aligned}$$

When $\lambda_i = \lambda_j$,

$$\begin{aligned} \Sigma^\top \nabla e_1(y^*) &= (\gamma_j - \gamma_i)x_{0,i} + (2\lambda_i - \gamma_i)y_i^* - (2\lambda_i - \gamma_j)y_j^* \\ &\geq -(\gamma_j - \gamma_i)y_j^* + (2\lambda_i - \gamma_i)y_i^* - (2\lambda_i - \gamma_j)y_j^* \\ &= (2\lambda_i - \gamma_i)(y_i^* - y_j^*) > 0. \end{aligned}$$

Case 2. $2\lambda_i - \gamma_i \leq 0$, $2\lambda_j - \gamma_j > 0$.

$$\Sigma^\top \nabla e_1(y^*) \geq -(2\lambda_j - \gamma_j)y_j^* > 0.$$

Case 3. $2\lambda_i - \gamma_i \leq 0$, $2\lambda_j - \gamma_j \leq 0$. From the assumption of the proposition, $2\lambda_j - \gamma_i > 0$. Then,

$$\begin{aligned} \Sigma^\top \nabla e_1(y^*) &\geq (\gamma_j - \gamma_i)x_{0,i} + (2\lambda_i - \gamma_i)y_i^* + (2\lambda_j - \gamma_j)x_{0,i} \\ &= (2\lambda_j - \gamma_i)x_{0,i} + (2\lambda_i - \gamma_i)y_i^* > 0. \end{aligned}$$

Therefore, there exists $\sigma > 0$ that is small enough so that $-x_0 \leq y^* + \sigma\Sigma \leq 0$, and the new trading policy $y^* + \sigma\Sigma$ leads to strictly larger equity and smaller liability after trading. This contradicts the optimality of y^* . \square

If we regard the i th asset as more liquid than the j th asset when the above conditions on the price impact parameters are satisfied, Proposition 2.4.1 then states that more liquid assets are prioritized for selling. This is intuitive since more liquid assets with smaller price impact incur smaller trading cost.

According to Theorem 2.3.7, when there exists a non-breakpoint z^* such that $f^*(z^*) = 0$, the deleveraging problem (2.2.4) is equivalent to the Lagrangian problem with parameter z^* . In this case, we may derive more analytical results regarding the optimal trading strategy. We show how the initial price p_0 and the initial liability l_0 affect the optimal trading strategy.

Proposition 2.4.2. *Suppose there exists a non-breakpoint z^* such that $f^*(z^*) = 0$. For assets i and j with the same initial holding and price impact parameters, i.e., $x_{0,i} = x_{0,j}$, $\lambda_i = \lambda_j$, $\gamma_i = \gamma_j$, the one with higher initial price is prioritized for selling. That is, $y_i^* \leq y_j^*$ when $p_{0,i} \geq p_{0,j}$.*

Proof. Denote the optimal solution of the deleveraging problem by y^* . By assumption, $y^* = y^*(z^*)$, where $y^*(z^*)$ is the unique optimal solution of the Lagrangian problem with parameter z^* and is given in Proposition 2.3.2. If $q_{1,i} \leq 0$ and $q_{2,i} + x_{0,i}q_{1,i} < 0$, then $y_i^* = -x_{0,i} = -x_{0,j} \leq y_j^*$. If $q_{1,i} \leq 0$ and $q_{2,i} + x_{0,i}q_{1,i} > 0$, then $y_i^* = 0$. Since $x_{0,i} = x_{0,j}$, $\lambda_i = \lambda_j$, $\gamma_i = \gamma_j$, $p_{0,i} \geq p_{0,j}$, we have $q_{1,j} = q_{1,i} \leq 0$, $q_{2,j} \geq q_{2,i}$. Consequently, $q_{2,j} + x_{0,j}q_{1,j} \geq q_{2,i} + x_{0,i}q_{1,i} > 0$. Therefore, $y_j^* = 0 = y_i^*$. If $q_{1,i} > 0$, then $q_{1,j} = q_{1,i} > 0$ as well. Since $q_{2,j} \geq q_{2,i}$,

$$y_j^* = \max(-x_{0,j}, \min(0, \frac{q_{2,j}}{2q_{1,j}})) \geq \max(-x_{0,i}, \min(0, \frac{q_{2,i}}{2q_{1,i}})) = y_i^*.$$

Therefore, $y_j^* \geq y_i^*$ in all cases. This finishes the proof. \square

This result states that, everything else being equal, we prefer to sell assets with higher prices so that we can generate more cash to reduce the leverage. The following result shows how the set of actively traded assets changes when l_0 changes. An asset i is said to be actively traded in a trading strategy y^* if $y_i^* \neq 0$. We consider two scenarios: one with initial liability l_0 , and the other with initial liability \bar{l}_0 . f^* and \bar{f}^* are defined as before, corresponding to these two scenarios.

Proposition 2.4.3. *Suppose there exist non-breakpoints z^* and \bar{z}^* such that $f^*(z^*) = 0$ and $\bar{f}^*(\bar{z}^*) = 0$ respectively under the following two scenarios: one with initial liability l_0 , and the other with initial liability $\bar{l}_0 \leq l_0$. Then the set of actively traded assets in the optimal trading*

strategy corresponding to \bar{l}_0 is a subset of the set of actively traded assets in the optimal trading strategy corresponding to l_0 .

Proof. Let y^* and z^* be the optimal solution and the optimal Lagrangian multiplier when the initial liability is l_0 . Similarly, \bar{y}^* and \bar{z}^* are the optimal solution and Lagrangian multiplier corresponding to \bar{l}_0 . Recall that:

$$f(y) = \rho_1 e_1(y) - l_1(y) = -y^\top \left(\rho_1 \left(\Lambda - \frac{1}{2} \Gamma \right) + \Lambda + \frac{1}{2} \Gamma \right) y + (\rho_1 \Gamma x_0 - p_0)^\top y + \rho_1 p_0^\top x_0 - (\rho_1 + 1) l_0.$$

Define $\bar{f}(y)$ similarly, by replacing l_0 in the above by \bar{l}_0 . We first show that $z^* \geq \bar{z}^*$ when $l_0 \geq \bar{l}_0$. By assumptions of the proposition, $f(y^*) = f^*(z^*) = 0 = \bar{f}^*(\bar{z}^*) = \bar{f}(\bar{y}^*)$. Therefore,

$$\bar{f}^*(\bar{z}^*) = 0 = f(y^*) \leq \bar{f}(y^*).$$

Note that the Lagrangian problem (2.3.6) does not contain l_0 . It is the same for different initial liabilities. By definition, y^* is the optimal solution for the Lagrangian problem with parameter z^* in the first scenario with initial liability l_0 . It is also the optimal solution for the Lagrangian problem with parameter z^* in the second scenario with initial liability \bar{l}_0 . That is, $\bar{f}(y^*) = \bar{f}^*(z^*)$. We thus have $\bar{f}^*(\bar{z}^*) \leq \bar{f}^*(z^*)$. From Theorem 2.3.6, we have $z^* \geq \bar{z}^*$.

Next, we show that $\bar{y}_i^* = 0$ when $y_i^* = 0$. We define $q_{1,i}, q_{2,i}$ that are associated with z^* and $\bar{q}_{1,i}, \bar{q}_{2,i}$ that are associated with \bar{z}^* as in Proposition 2.3.2. Since $z^* \geq \bar{z}^*$, we have

$$\frac{z^*}{1 + z^* \rho_1} \geq \frac{\bar{z}^*}{1 + \bar{z}^* \rho_1}.$$

Then, $q_{2,i} \geq 0$ implies $\bar{q}_{2,i} \geq 0$. Similarly,

$$\frac{z^*}{1 + z^* + \rho_1 z^*} \geq \frac{\bar{z}^*}{1 + \bar{z}^* + \rho_1 \bar{z}^*}.$$

Then, $q_{2,i} + x_{0,i} q_{1,i} > 0$ implies $\bar{q}_{2,i} + x_{0,i} \bar{q}_{1,i} > 0$. Suppose $y_i^* = 0$. Since z^* is a non-breakpoint, according to Proposition 2.3.2 and the proof of Proposition 2.3.4, $y_i^* = 0$ is possible only if one of the following occurs:

1. $q_{1,i} > 0$ and $q_{2,i} \geq 0$;

2. $q_{1,i} \leq 0, q_{2,i} + x_{0,i}q_{1,i} > 0$.

In either case, we must have $q_{2,i} \geq 0$ and $q_{2,i} + x_{0,i}q_{1,i} > 0$. According to the above analysis, $\bar{q}_{2,i} \geq 0$ and $\bar{q}_{2,i} + x_{0,i}\bar{q}_{1,i} > 0$. If $\bar{q}_{1,i} > 0$, then according to Proposition 2.3.2, we have $\bar{y}_i^* = 0$. If $\bar{q}_{1,i} \leq 0$, since $\bar{q}_{2,i} + x_{0,i}\bar{q}_{1,i} > 0$, we still have $\bar{y}_i^* = 0$ according to Proposition 2.3.2. This finishes the proof. \square

This proposition shows that an asset that is retained (not sold) will still be retained at a smaller l_0 , i.e., when the leverage requirement is relaxed. Similarly, an asset that is actively sold will still be actively sold at a larger l_0 , i.e., when the leverage requirement is more restrictive.

2.5 Numerical Examples

In this section, we present a few numerical examples illustrating the performance of the Lagrangian algorithm. All numerical experiments are conducted on a Dell N5010 laptop with 2.26 GHz CPU and 4 GB memory using C++.

Example 1. In this example, we numerically examine the loss in equity resulted from a suboptimal solution. $x_{0,i} = 1$ million, $1 \leq i \leq 10$, and $l_0 = \$48$ million. The initial asset prices and the price impact parameters are given in Table 2.1. The initial equity is \$2 million. The initial leverage ratio is $\rho_0 = 24$ and the desired ratio is $\rho_1 = 18$. From the Lagrangian algorithm, we obtain $z^* = 0.0074$, which is a breakpoint where 0 is included in the interval defined by the two values of $f^*(z^*)$. The algorithm thus returns a suboptimal solution. The corresponding trading rates are reported in Table 2.1. The equity after trading with the suboptimal strategy given in Table 2.1 is

Asset no.	Initial price	Temporary price impact λ_i	Permanent price impact γ_i	Suboptimal trading rate
1	5	0.026	0.076	0
2	5	0.016	0.042	0
3	5	0.012	0.041	-0.1900
4	5	0.018	0.037	0
5	5	0.020	0.040	0
6	5	0.028	0.018	-0.3823
7	4.6	0.017	0.037	0
8	4.8	0.017	0.048	0
9	5.2	0.017	0.013	-0.9866
10	5.4	0.017	0.013	-1

Table 2.1: Numerical parameters: $x_{0,i} = 1$ million, $1 \leq i \leq 10$, $l_0 = 48$ million, $\rho_1 = 18$.

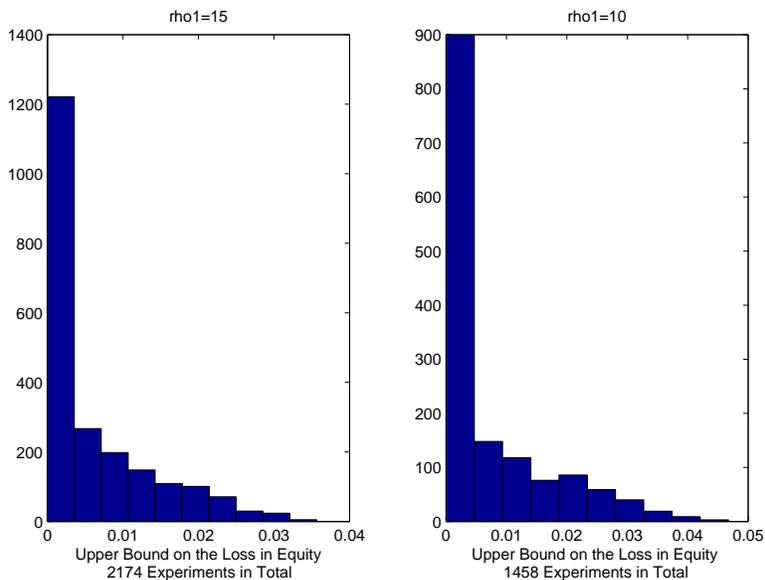


Figure 2.1: Histograms for the Upper Bound on the Loss in Equity

$e_1 = 1.9363$. To better understand the loss in equity caused by using such an approximate solution, we observe that the optimal objective value of the corresponding Lagrangian problem is 1.9392. It is well known that this provides an upper bound on the optimal value of the original deleveraging problem. That is, the loss in equity of using the suboptimal solution is bounded from the above by $1.9392 - 1.9363 = 0.0029$, which is around 0.145% of the initial equity. The first bound in Theorem 2.3.10 gives 0.0325, which is around 1.6% of the initial equity. The second bound in Theorem 2.3.10 gives 0.2632. We note that the actual loss in equity is much smaller than the theoretical bounds in this example. That is, the actual performance of the Lagrangian algorithm could be much better than what Theorem 2.3.10 characterizes.

Example 2. In this example, we numerically illustrate how often the Lagrangian algorithm returns an optimal solution. We consider a portfolio consisting of 8 assets. $x_{0,i} = 1$ million, $p_{0,i} = \$5$, $1 \leq i \leq 8$, and $l_0 = \$38$ million. The initial equity is \$2 million. The initial leverage ratio is $\rho_0 = 19$. We randomly generate 5000 problems. For each problem, the temporary and permanent price impact parameters are simulated uniformly from $[0, 0.1]$. When the desired leverage ratio is $\rho_1 = 15$, 553 of the 5000 problems do not have breakpoints (among these problems, 284 are convex) and thus are solved optimally. Among the remaining 4447 problems with at least one breakpoint, 2273 are solved optimally. Therefore, $(553 + 2273)/5000 = 56.5\%$ of the 5000 problems are solved

optimally. We keep the randomly generated price impact parameters the same and repeat the above experiment with $\rho_1 = 10$, and find that 553 problems do not have breakpoints (again, 284 of these problems are convex), and 2989 problems have at least one breakpoint but are solved optimally. In total, $(553 + 2989)/5000 = 70.8\%$ of the 5000 problems are solved optimally. For each problem with a suboptimal solution, we also calculate the Lagrangian based upper bound on the loss in equity caused by adopting a suboptimal strategy. When $\rho_1 = 15$, 2174 problems are solved suboptimally. The upper bound on the loss in equity ranges from 0.0014% to 3.56%. The mean and median loss in equity are 0.57% and 0.035% of the initial equity, respectively. When $\rho_1 = 10$, 1458 problems are solved suboptimally. The upper bound on the loss in equity ranges from 0.0039% to 4.68%. The mean and median loss in equity are 0.72% and 0.025% of the initial equity, respectively. Finally, Figure 2.1 presents the corresponding histograms for the upper bound on the loss in equity as a fraction of the initial equity.

2.6 Summary

Market impact affects the trading price when a large portfolio is liquidated in a short period of time. It is important for investors to take price impact into account when determining appropriate trading strategies. In this chapter, we study an optimal deleveraging problem without assumptions on the relative magnitudes of the price impact parameters. The objective is to maximize equity while meeting a prescribed leverage ratio requirement. The resulting non-convex quadratic program is analyzed. Analytical results regarding the optimal trading strategy are obtained. An efficient Lagrangian method is proposed and studied for the numerical solution of the non-convex problem. In particular, by studying the breakpoints of the Lagrangian problem, we are able to obtain conditions under which the Lagrangian method returns an optimal solution of the deleveraging problem. On the other hand, when the Lagrangian algorithm returns a suboptimal solution, we give upper bounds on the loss in equity caused by using the suboptimal trading strategy.

Chapter 3

Portfolio Deleveraging under Nonlinear Market Impact

3.1 Introduction

In this chapter, we investigate the optimal deleveraging strategy under general nonlinear temporary price impact ([17]). Our primary motivation is the fact that many theoretical and empirical works are in support of nonlinear temporary impact function. For example, Lillo, Farmer and Mantegna ([47]) show that the temporary price impact function is strictly concave by carrying out fits of impact curves. The concavity is also explained in [31], [16] and [12]. In particular, some empirical works are in favor of a square-root power-law temporary impact function (see [68], [40], [30], [27], [52]). In the meanwhile, Almgren et. al. ([4]) demonstrate that a power law function with an exponent of 0.6 is more plausible by analyzing data from Citigroup equity trading. Tóth et. al. [69] report an exponent around 0.5 for small tick contracts and 0.6 for large tick ones by observing proprietary trading data from Capital Fund Management. In order to be consistent with the theoretical and empirical results and to allow for higher adaptability, we propose a power-law temporary impact function with a general rational order between zero and one.

With a power-law temporary price impact function, the optimal deleveraging problem becomes a non-convex polynomial program with polynomial and box constraints. In the absence of convexity, the optimization problem is very challenging. To approach the problem, we extend the Lagrangian method proposed in Chapter 2 to the case of non-convex polynomial optimization program. An efficient algorithm is then developed by studying the connections between the optimal deleveraging problem and the corresponding Lagrangian problem. Similar as in Chapter 2, we characterize the conditions under which the convergence of the algorithm can provide a global optimal solution.

When those conditions are not satisfied, the algorithm returns an approximate solution and an upper bound on the loss in equity induced by adopting the suboptimal strategy is derived. These results indicate that the Lagrangian method we proposed is robust in the sense that it can be applied to deal with the optimal deleveraging problem efficiently under generic temporary price impact function.

As mentioned above, there are many theoretical and empirical works that are in favor of nonlinear temporary price impact function (see [68], [40], [47], [30], [4], [27], [31], [16], [12], [52], [69]). However, in the existing optimal deleveraging works [14] and [20], linearity of temporary price impact function is assumed for the ease of mathematical convenience. The difference between the deleveraging strategies obtained under the linear and nonlinear price impact functions draws our attention. To start, we first note that the permanent price impact outweighs the temporary counterpart in some situations. For instance, Schöneborn and Schied ([63]) point out that in markets that do not suffer from extreme cash outflows, the impact of stock sales is predominantly permanent. It's also mentioned in [63] that Holthausen et. al. ([37]) observe from their data sample that the permanent impact accounts for 85% of the total impact. In addition, Coval and Stafford ([22]) show that in markets except open-ended mutual fund, the permanent price impact is strongly dominant. In a permanent-impact-dominant market, we find that the linear price impact assumption leads to an extreme trading strategy: only one asset in the portfolio is partially liquidated; all the other assets are either sold or retained completely. But under nonlinear price impact function, the optimal strategy is comparatively moderate and complicated. This illustrates a clear distinction in optimal trading strategies under two different assumptions on the price impact.

3.2 Model Formulation

This section introduces the problem formulation. The asset price dynamics is given by

$$p_t = p_0 + \Gamma(x_t - x_0) - \Lambda y_t^k, \quad (3.2.1)$$

where p_t , x_t , $y_t \geq \mathbf{0} \in \mathfrak{R}^m$ are vectors of price, holdings and trading rates at time t , y_t^k ($0 < k \leq 1$) is a vector with $y_{i,t}^k$ as its i th entry, Γ and Λ are positive definite diagonal matrices with permanent and temporary price impact coefficients (i.e. $\gamma_i > 0$ and $\lambda_i > 0$) on diagonals respectively. The

minus sign associated with y_t^k indicates that only selling strategy is of interest, since the primary goal is to reduce liability. The permanent price impact coefficients are related to the cumulative trading amounts while the temporary price impact coefficients are related to the sizes of trades in unit time. For simplicity, denote $q = p_0 - \Gamma x_0$ and the price dynamics becomes

$$p_t = q + \Gamma x_t - \Lambda y_t^k. \quad (3.2.2)$$

The price process (3.2.2) contains the price dynamics in [14] and [20] as a specific case (i.e., $k = 1$). The temporary price impact here is no longer restricted to a linear function as in [14] and [20], but a power-law function with a general exponent between zero and one. As discussed in the introduction, such an assumption is more consistent with what has been observed in numerous empirical works on price impact.

We adopt the same notations as in Chapter 2. The holding dynamics is $x_t = x_0 - \int_0^t y_s ds$ and the trading period $T = 1$. The asset price is $p_0 = q + \Gamma x_0$ before trading and $p_1 = q + \Gamma x_1$ after trading. The amount of cash generated is given by

$$K = \int_0^1 p_t^\top y_t dt.$$

After paying for the debt, the remaining liability is $l_1 = l_0 - K = l_0 - \int_0^1 p_t^\top y_t dt$, where l_0 is the initial liability. Denote the value of the portfolio after liquidation as $a_1 = p_1^\top x_1$. The net equity at time 1 is hence given by

$$\begin{aligned} e_1 &= a_1 - l_1 \\ &= p_1^\top x_1 - l_0 + \int_0^1 p_t^\top y_t dt \\ &= q^\top x_1 + x_1^\top \Gamma x_1 + \int_0^1 (q^\top y_t + x_t^\top \Gamma y_t - y_t^\top \Lambda y_t^k) dt - l_0. \end{aligned} \quad (3.2.3)$$

It can be shown in a similar way as Proposition 2.2.2 that y_t^* is a constant. Thus, replacing y_t by y , we get

$$l_1 = \frac{1}{2} y^\top \Gamma y + y^\top \Lambda y^k - p_0^\top y + l_0, \quad (3.2.4)$$

$$e_1 = \frac{1}{2} y^\top \Gamma y - y^\top \Lambda y^k - x_0^\top \Gamma y + e_0, \quad (3.2.5)$$

where $e_0 = p_0^\top x_0 - l_0$ is the initial net equity.

Let ρ_1 be the required leverage ratio. The optimization problem is given as follows

$$\begin{aligned} \max_{y \in \mathbb{R}^m} \quad & e_1 \\ \text{subject to} \quad & \rho_1 e_1 \geq l_1 \\ & 0 \leq y \leq x_0. \end{aligned} \tag{3.2.6}$$

Rewrite the problem explicitly as follows

$$\begin{aligned} \max_{y \in \mathbb{R}^m} \quad & y^\top \left(\frac{1}{2} \Gamma \right) y - y^\top \Lambda y^k - x_0^\top \Gamma y + e_0 \\ \text{subject to} \quad & (\rho_1 - 1) \frac{1}{2} y^\top \Gamma y - (\rho_1 + 1) y^\top \Lambda y^k + (p_0 - \rho_1 \Gamma x_0)^\top y + \rho_1 e_0 - l_0 \geq 0 \\ & 0 \leq y \leq x_0. \end{aligned} \tag{3.2.7}$$

We assume that (3.2.7) is strictly feasible and the initial condition doesn't meet the leverage requirement (i.e. $\rho_1 e_0 < l_0$). If this is not the case, the investor will not conduct any liquidation since liquidation is costly. This can be easily verified by calculating the gradient of the objective function with respect to the decision variable. We next point out in the following proposition that the investor will deleverage to the level that the leverage requirement is immediately satisfied. The property also holds in the linear temporary price impact case as shown in Proposition 2.3.1.

Proposition 3.2.1. *The leverage constraint of the optimal deleveraging problem (3.2.7) is active at the optimal solution.*

3.3 Lagrangian Method

In this section, we modify the Lagrangian method proposed in Chapter 2 to solve the polynomial optimization problem numerically. Let α be a non-negative Lagrangian multiplier and the

corresponding Lagrangian program is given by

$$\begin{aligned}
& \max_{y \in \mathbb{R}^m} && \frac{1}{2}(1 + \alpha\rho_1 - \alpha)y^\top \Gamma y - (1 + \alpha\rho_1 + \alpha)y^\top \Lambda y^k + [\alpha p_0 - (1 + \alpha\rho_1)\Gamma x_0]^\top y \\
& && + (1 + \alpha\rho_1)p_0^\top x_0 - (1 + \alpha\rho_1 + \alpha)l_0 \\
& \text{subject to} && 0 \leq y \leq x_0.
\end{aligned} \tag{3.3.1}$$

For each Lagrangian multiplier α , let $y^*(\alpha)$ denote the set of optimal solutions to the Lagrangian problem (3.3.1). Let the corresponding sets of the values of the objective function and constraint be $e_1^*(\alpha) = e_1(y^*(\alpha))$ and $f_1^*(\alpha) = f_1(y^*(\alpha))$ respectively. Note that $k = \frac{k_1}{k_2}$ is a rational number between 0 and 1. Thus, we can make a change of variable ($\tilde{y} \leftarrow y^{\frac{1}{k_2}}$) and the objective function becomes a sum of three terms with orders $2k_2$, $k_1 + k_2$ and k_2 respectively. Then we are able to conclude that for any give Lagrangian multiplier α , there exist a finite number of optimal solutions.

Definition 3.3.1. α is said to be a breakpoint of $f_1^*(\alpha)$ if $f_1^*(\alpha)$ has distinct values.

Remark 3.3.2. There's no breakpoint of $f_1^*(\alpha)$ if $\gamma_i < k(k+1)\lambda x_{0,i}^{k-1}$, for $1 \leq i \leq m$.

If $\gamma_i < k(k+1)\lambda x_{0,i}^{k-1}$ for each asset i , the Lagrangian problem (3.3.1) is strictly concave and there exists a unique optimal solution for any given α . Hence, no breakpoint exists.

Since the price impact matrices Λ and Γ are both diagonal, the Lagrangian problem (3.3.1) can be regarded as a sum of m univariate polynomial optimization programs with box constraints.

$$\begin{aligned}
(P_i) \quad \max_{y_i \in \mathbb{R}} &&& \frac{1}{2}(1 + \alpha\rho_1 - \alpha)\gamma_i y_i^2 - (1 + \alpha\rho_1 + \alpha)\lambda_i y_i^{1+k} + [\alpha p_{0,i} - (1 + \alpha\rho_1)\gamma_i x_{0,i}]y_i \\
&&& + (1 + \alpha\rho_1)p_{0,i}x_{0,i} \\
& \text{subject to} && 0 \leq y_i \leq x_{0,i}.
\end{aligned} \tag{3.3.2}$$

For each subproblem (P_i) , $i = 1, \dots, m$, let B_i denote the set of Lagrangian multipliers α under which (3.3.2) has multiple optimal solutions. We have the following assumption.

Assumption 3.3.3. For any two assets i and j , the corresponding sets B_i and B_j have no intersection, i.e., $B_i \cap B_j = \emptyset$.

For any $\alpha \in B_i$, for some $1 \leq i \leq m$, α is a breakpoint according to Definition 3.3.1. Assumption 3.3.3 implies that there is no overlapping breakpoint for different assets.

Next we study some key properties of $f_1^*(\alpha)$, which are the foundations of the algorithm.

Theorem 3.3.4. $f_1^*(\alpha)$ is a piecewise non-increasing map. In particular, $f_1^*(0) > 0$, and $\exists \alpha' > 0$ such that $f_1^*(\alpha') < 0$.

Proof: The monotonicity of f_1^* can be shown in a similar manner as Theorem 2.3.6 in Chapter 2. We do not describe the details here. Note that in this chapter, $f_1^* = l_1 - \rho_1 e_1$; while in Chapter 2, $f^* = \rho_1 e_1 - l_1$. Let $\xi_i \in f^*(\alpha_i)$, $i = 1, 2$.

When $\alpha = 0$, we only maximize e_1 . Since e_1 is decreasing with respect to each y_i according to (3.2.5), the optimal solution is $y^* = \mathbf{0}$. Then we have $f^*(0) = f(\mathbf{0}) = \rho_1 e_0 - l_0 < 0$. The remaining part of the proof follows similarly as Theorem 2.3.6. \square

Theorem 3.3.4 implies there exists a α^* such that the zero value is contained in either $f_1^*(\alpha^*)$ or an interval defined by the two different values of $f_1^*(\alpha^*)$. If $0 \in f_1^*(\alpha^*)$, we may obtain the optimal solution using binary search.

Theorem 3.3.5. If there exists a α^* such that $0 \in f_1^*(\alpha^*)$, the optimal solution $y^*(\alpha^*)$ to the Lagrangian problem (3.3.1) satisfying $f_1^*(y^*(\alpha^*)) = 0$ is also optimal to (3.2.6).

Refer to Theorem 2.3.7 in Chapter 2 for a detailed proof.

Theorem 3.3.5 shows that when there exists a Lagrangian multiplier α^* such that $0 \in f_1^*(\alpha^*)$, we can obtain the optimal deleveraging strategy by solving the corresponding Lagrangian problem. Otherwise, if $\exists \alpha_i^*$ such that 0 falls in the open interval defined by the two values of $f_1^*(\alpha_i^*)$, we construct an approximate solution in a similar way as in [20]. Let $Y^*(\alpha_i^*) = \{Y^{1*}(\alpha_i^*), \dots, Y^{n*}(\alpha_i^*)\}$, $n \in \mathbb{N}$ be the set of the optimal solutions. Then from Assumption 3.3.3, we know that the optimal solutions are the same in each components except the i th one. That is, $Y_j^{1*}(\alpha_i^*) = Y_j^{2*}(\alpha_i^*) = \dots = Y_j^{n*}(\alpha_i^*)$, for $j = 1, \dots, m$, $j \neq i$.

Denote \bar{Y}^* as the feasible approximation we are seeking. For $j = 1, \dots, m$, $j \neq i$, let $\bar{Y}_j^* = Y_j^{1*}(\alpha_i^*) = \dots = Y_j^{n*}(\alpha_i^*)$. Then let \bar{Y}_i^* be a zero of $f_1(\bar{Y}^*)$, which is a function of \bar{Y}_i^* , on $(-x_{0,i}, 0)$. Note that $f_1(\bar{Y}^*)$ is a polynomial function of \bar{Y}_i^* that may have multiple zeros on $(-x_{0,i}, 0)$. We select the smallest one since the net equity is decreasing with respect to the trading amount. The approximate solution constructed in this way is feasible to (3.2.7).

For a given α , let $\mathbf{Y}^*(\alpha) = \{Y^{1*}(\alpha), \dots, Y^{n*}(\alpha)\}$, $n \in \mathbb{N}$ denote the set of optimal solutions to the Lagrangian problem (3.3.1) with parameter α . These solutions are arranged in an order such

that

$$e_1(Y^{1*}(\alpha)) \geq \cdots \geq e_1(Y^{n*}(\alpha)),$$

$$f_1(Y^{1*}(\alpha)) \geq \cdots \geq f_1(Y^{n*}(\alpha)).$$

We are now ready to present the algorithm.

Algorithm

1. Choose $\hat{\alpha}$ large enough such that $f_1(Y^{1*}(\hat{\alpha})) > \epsilon$.

Let $a = 0$ and $b = \hat{\alpha}$.

2. **If** $\exists Y^{i*}(b)$ with $|f_1(Y^{i*}(b))| < \epsilon$,

let $\alpha^* = b$ and $y^* = Y^{i*}(b)$. Stop.

Else If $f_1(Y^{n*}(b)) < -\epsilon$, and b is a breakpoint corresponding to the i th asset,

let $\alpha^* = b$, $y_k^* = Y_k^{1*}(b)$ for $k = 1, \dots, m$, $k \neq i$ and y_i^* be the smallest zero of $f_1(y^*)$ as a function of y_i^* on $(-x_{0,i}, 0)$, and stop.

Else

3. **While** $(|f_1(Y^{1*}(\frac{a+b}{2}))| > \epsilon)$

{

If $f_1(Y^{1*}(\frac{a+b}{2})) > \epsilon$,

let $b \leftarrow \frac{a+b}{2}$ and go to step 3.

Else let $a \leftarrow \frac{a+b}{2}$ and go to step 2.

}

For the case that there is no breakpoint or there exists a breakpoint $\hat{\alpha}$ such that $0 \in f_1^*(\hat{\alpha})$, the optimal deleveraging problem can be solved via the algorithm efficiently. For the remaining case that an approximate solution is obtained, we estimate the loss in the objective value.

Theorem 3.3.6. *If the zero value of f_1^* lies in the interval generated by two optimal solutions of the Lagrangian problem with breakpoint α for asset i (i.e. $0 \in (f_1(\tilde{y}^*), f_1(\hat{y}^*))$), we construct an approximate solution \bar{y}^* with $f_1(\bar{y}^*) = 0$ according to the above Lagrangian algorithm. The loss in*

the net equity caused by adopting such a suboptimal strategy is bounded by $\frac{\gamma_i}{2}x_{0,i}^2 + \lambda_i x_{0,i}^{1+k}$, the price impact of fully liquidating asset i .

Proof: Note that $f_1(\check{y}^*) < 0$, $f_1(\hat{y}^*) > 0$ and $f_1(\bar{y}^*) > 0$. \check{y}^* , \hat{y}^* and \bar{y}^* differ only in the i th component. Let y^* be the optimal solution to (3.2.7). Since y^* is feasible to Lagrangian problem (3.3.1), we have

$$e_1(\check{y}(\alpha^*)) - \alpha^* f_1(\check{y}(\alpha^*)) = e_1(\hat{y}(\alpha^*)) - \alpha^* f_1(\hat{y}(\alpha^*)) \geq e_1(y^*) - \alpha^* f_1(y^*) = e_1(y^*), \quad (3.3.3)$$

where the last equality is due to Proposition 3.2.1. Similarly, we have

$$e_1(\check{y}(\alpha^*)) - \alpha^* f_1(\check{y}(\alpha^*)) = e_1(\hat{y}(\alpha^*)) - \alpha^* f_1(\hat{y}(\alpha^*)) \geq e_1(\bar{y}^*) - \alpha^* f_1(\bar{y}^*) = e_1(\bar{y}^*). \quad (3.3.4)$$

Thus,

$$e_1(\check{y}(\alpha^*)) < e_1(y^*) < e_1(\hat{y}(\alpha^*)), \quad e_1(\check{y}(\alpha^*)) < e_1(\bar{y}^*) < e_1(\hat{y}(\alpha^*)). \quad (3.3.5)$$

Then we have

$$\| e_1(\bar{y}(\alpha^*)) - e_1(y^*) \| \leq \| e_1(\check{y}(\alpha^*)) - e_1(\hat{y}(\alpha^*)) \|. \quad (3.3.6)$$

According to (3.2.5), $e_1(y)$ is decreasing with respect to y_i , for $1 \leq i \leq m$. Thus,

$$\begin{aligned} \| e_1(\check{y}(\alpha^*)) - e_1(\hat{y}(\alpha^*)) \| &\leq 0 - \left(\frac{\gamma_i}{2}x_{0,i}^2 - \lambda_i x_{0,i}^{1+k} - \gamma_i x_{0,i}^2 \right) \\ &= \frac{\gamma_i}{2}x_{0,i}^2 + \lambda_i x_{0,i}^{1+k}. \end{aligned}$$

Therefore, $\bar{y}(\alpha^*)$ is an approximate solution with loss bounded by $\frac{\gamma_i}{2}x_{0,i}^2 + \lambda_i x_{0,i}^{1+k}$. For a multi-asset portfolio, it's normal that the optimal equity is much greater than the price impact of one asset. Otherwise, liquidation is too costly that the investor may not choose to do so.

3.4 Nonlinear V.S. Linear Price Impact

When $k = 1$, we have a linear temporary price impact function. The optimal deleveraging problem then becomes as follows

$$\begin{aligned}
& \max_{y \in \mathbb{R}^m} && e_1(y) = -y^\top \left(\Lambda - \frac{1}{2} \Gamma \right) y - x_0^\top \Gamma y + p_0^\top x_0 - l_0 \\
\text{subject to} &&& -y^\top \left(\rho_1 \left(\Lambda - \frac{1}{2} \Gamma \right) + \Lambda + \frac{1}{2} \Gamma \right) y + (p_0 - \rho_1 \Gamma x_0)^\top y + \rho_1 p_0^\top x_0 - (\rho_1 + 1) l_0 \geq 0 \\
&&& 0 \leq y \leq x_0.
\end{aligned} \tag{3.4.1}$$

(3.4.1) is the same as the deleveraging problem (2.2.4) in Chapter 2. As explained in the Introduction, we could have a permanent-impact-dominant market under certain circumstances ([37], [22], [62]). In a market where $\Lambda - \frac{1}{2} \Gamma \prec 0$, we have the following property.

Proposition 3.4.1. *When $\Lambda \prec \frac{\rho_1 - 1}{2(\rho_1 + 1)} \Gamma$, there exists at most one asset i such that the optimal solution to (3.4.1) satisfies $0 < y_i^* < x_{0,i}$. For the rest assets j ($j \neq i$) in the portfolio, $y_j^* = x_{0,j}$ or $y_j^* = 0$.*

Proof: Denote y^* as the optimal solution to (3.4.1). Assume to the contrary that there are two assets i and j such that $0 < y_i^* < x_{0,i}$ and $0 < y_j^* < x_{0,j}$. Fix the rest of the portfolio and treat them as constant. (3.4.1) then becomes

$$\begin{aligned}
& \max_{y \in \mathbb{R}^m} && A_1 x^2 + B_1 x + C_1 y^2 + D_1 y + E \\
\text{subject to} &&& A_2 x^2 + B_2 x + C_2 y^2 + D_2 y + F \geq 0 \\
&&& 0 \leq x \leq x_{0,i}, 0 \leq y \leq x_{0,j},
\end{aligned} \tag{3.4.2}$$

where $x = y_i$, $y = y_j$, $\{A_1 > 0, B_1, A_2 > 0, B_2\}$ and $\{C_1 > 0, D_1, C_2 > 0, D_2\}$ are the two sets of coefficients associated with assets i and j in (3.4.1) respectively and E and F are two constants related to the rest of the portfolio.

Denote z^* as the Lagrangian multiplier associated with the quadratic constraint. Let $e(x, y)$ denote the objective function and $f(x, y)$ denote the quadratic constraint. (x^*, y^*) satisfies the KKT condition of (3.4.2), which means that (x^*, y^*) is also a KKT point of the corresponding Lagrangian problem. Then (x^*, y^*) is the unique global minimum of $e(x, y) + z^* f(x, y)$, since

$A_1 + z^*A_2 > 0$ and $C_1 + z^*C_2 > 0$. $f(x, y) = 0$ is an ellipsoid that intersects with box $0 \leq x \leq x_{0,i}$, $0 \leq y \leq x_{0,j}$ at (x^*, y^*) . Since (x^*, y^*) is an interior point of the box, there exists another point $(\bar{x}, \bar{y}) \in B^\delta(x^*, y^*)$ such that $f(\bar{x}, \bar{y}) = 0$, where $B^\delta(x^*, y^*) = \{(x, y) | (x - x^*)^2 + (y - y^*)^2 < \delta\}$. Since $e(\bar{x}, \bar{y}) + z^*f(\bar{x}, \bar{y}) > e(x^*, y^*) + z^*f(x^*, y^*)$, we have $e(\bar{x}, \bar{y}) > e(x^*, y^*)$. Thus, we have found a feasible solution with higher objective value. This contradicts to the optimality of (x^*, y^*) . Therefore, there can be at most one optimal solution that is not boundary point. \square

Proposition 3.4.1 indicates that under linear price impact function, there can be at most one asset that is partially sold. The other assets in the portfolio are either completely sold or retained. Such a trading strategy is straightforward and simple. Proposition 2.4.1 indicates that we prefer to sell more liquid assets when other parameters are the same (i.e., initial price and holding). In this case, the exact optimal deleveraging strategy can be found very efficiently. More specifically, for a portfolio consisting of m assets with the same initial price and holding, there can be a total of m candidate solutions if the assets can be ordered according to their liquidity as suggested in Proposition 2.4.1.

In this case, first assume that asset i ($1 \leq i \leq m$) is partially liquidated. Then assets that are more (less) liquid than i will be completely sold (retained). According to Proposition 3.2.1, we can obtain the trading amount of asset i by solving $f_1 = 0$. Repeat this approach m times and obtain n ($n \leq m$) feasible solutions. Compare the objective value corresponding to the n feasible solutions and select the one with highest value, which gives the optimal deleveraging strategy.

Under the nonlinear temporary price impact, the optimal trading strategies are more complicated, which may not be as straightforward as those in the linear case. The numerical results in next section will verify our theoretical results.

3.5 Numerical Examples

Example 1. In this example, we consider a portfolio containing six assets. $x_{0,i} = 1$ million, $1 \leq i \leq 6$. Initial liability $l_0 = 220\$$. The temporary price impact function is assumed to have a power of $k = 0.5$. $\rho_1 = 11$ and $l_0 = 222$. For these assets, $\gamma_i < k(k + 1)\lambda_i x_{0,i}^{k-1}$, $1 \leq i \leq 6$. It implies the concavity of the optimal deleveraging problem. Solving the problem via the Lagrangian algorithm, we get $\alpha^* = 0.0545$. The corresponding optimal trading amounts are in Table 3.1. We

Initial asset price $p_{0,i}$	Temporary price impact λ_i	Permanent price impact γ_i	Optimal trading rate y_i^*
39	1	0.5	0.6778
39	1.2	0.8	0.3295
39	2	1	0.1100
40	1	0.5	0.7166
40	1.2	0.8	0.3551
40	2	1	0.1218

Table 3.1: Numerical parameters: $x_{0,i} = 1$, $1 \leq i \leq 6$, $l_0 = 222$, $\rho_1 = 11$

also consider another portfolio where the relationship between the magnitudes of permanent and temporary price impact no longer exists. The parameters are given in Table 3.2. The Lagrangian algorithm returns $\alpha^* = 0.0689$ and an approximate solution with objective value $e_1 = 12.7336$. The optimal objective value of the corresponding Lagrangian problem is 12.8288, giving an upper bound on the optimal deleveraging problem. Thus, the loss in equity due to the suboptimality of the strategy is bounded by 0.0952, which is around 0.63% of the initial equity.

Example 2. In this example, we compare the optimal deleveraging strategies obtained under

Initial asset price $p_{0,i}$	Temporary price impact λ_i	Permanent price impact γ_i	Optimal trading rate y_i^*
39	1	0.9	0.6097
39	1.2	0.8	0.4873
39	2	1.6	0.0000
40	1	0.9	0.7021
40	1.2	0.8	0.5231
40	2	1.6	0.4053

Table 3.2: Numerical parameters: $x_{0,i} = 1$, $1 \leq i \leq 6$, $l_0 = 222$, $\rho_1 = 11$

linear and nonlinear temporary price impact respectively. Consider a two-asset portfolio. The initial asset holding $x_{0,i} = 0.05$ million and the initial asset price $p_{0,i} = 2\$$, $1 \leq i \leq 2$. The initial liability $l_0 = 0.185$ million. The initial leverage ratio $\rho_0 = 12.65$ and the desired leverage ratio $\rho_1 = 11$. We solve the problem (3.2.7) under linear ($k=1$) and square root ($k=0.5$) temporary price impact functions respectively. See Table 3.3 for the optimal liquidation strategies.

It can be seen from the price impact given in Table 3.3 that it is a permanent-impact-dominant market. Under the linear temporary price impact, there is only asset that needs to be partially liquidated. While under the square root temporary price impact, both assets are partially sold. The numerical observation is consistent with our theoretical result, i.e., Proposition 3.4.1.

Initial asset price $p_{0,i}$	Temporary price impact λ_i	Permanent price impact γ_i	Optimal trading rate y_i^* (k=1)	Optimal trading rate y_i^* (k=0.5)
2	0.36	1	0	0.0091
2	0.32	1	0.0165	0.0133

Table 3.3: Numerical parameters: $x_{0,i} = 0.05$, $1 \leq i \leq 2$, $l_0 = 0.185$

3.6 Summary

In summary, we study the optimal deleveraging problem under general and realistic nonlinear temporary price impact function, in particular, a power-law function with a general exponent between zero and one. The resulting optimization problem is a non-convex polynomial program with polynomial and box constraints. An efficient algorithm is proposed to find the global optimal solution numerically under certain conditions. If the conditions are violated, a suboptimal strategy is obtained and an upper bound on the equity loss is derived. Differences between the trading strategies obtained under linear and nonlinear market impact are analyzed and compared.

Chapter 4

Two-Period Robust Portfolio Deleveraging under Margin Call

4.1 Introduction

The Securities Act of 1934 empowered the Federal Reserve Board (FRB) to “set initial, maintenance, and short sale margin requirements on all securities traded on a national exchange” for regulatory purpose ([43]). According to Regulation T of FRB, the initial margin is currently 50 percent. However, there is no imposed “lower limits on equity capital as a percentage of the portfolio value subsequent to the date the position is initially established” ([61]). Usually, it is the broker that set the margin requirements. As codified in Financial Industry Regulatory Authorization (FINRA) Rule 4210, the maintenance margin requirements imposed by brokers for long positions in equity markets are 25 percent. While for other financial instruments, the maintenance requirements can be higher or lower than this value. A decrease in portfolio value may trigger the violation of the requirement, which may further lead to a margin call. Investors facing a margin call are required to “either deposit additional funds (or securities) into the account or initiate the partial liquidation of the positions in the portfolio” ([25]). In this chapter, we consider the second case where the margin call is satisfied by reducing the sizes of the positions, that is, by portfolio liquidation.

The portfolio deleveraging problems studied in Chapter 2 and 3 are in a continuous-time setting. Initially, they are formulated as optimal control programs where the optimal solutions are proved to be constant functions. Thus, the optimal deleveraging program is reduced to a quadratic or

polynomial optimization program, depending on whether the temporary price impact function is linear or not. In this chapter, we consider a two-period distressed liquidation problem in a discrete-time setting ([19]). We study the problem in two cases: simple portfolios containing only basic assets; enriched portfolios containing both basic assets and derivative securities. In the two-period setting, the investors do not know the asset price in the second period at the beginning of trading. The information they typically have is an estimation of the first and second moments of the basic asset return from the historical data. For the derivative return, the scarcity of time series data makes it difficult to estimate (see [73]). So we treat the derivative return as a quadratic function of the underlying return by adopting the delta-gamma approximation, which is a second-order Taylor approximation of asset returns that has been used extensively in risk management and portfolio hedging (see [39], [49], [73]). The basic asset return here is assumed to belong to an ellipsoidal set determined by the means and covariances. The margin call is a hard requirement that needs to be satisfied no matter what the market condition is, which makes a robust strategy desirable. Namely, we seek a trading strategy that is able to meet the margin requirement for any return within the ellipsoidal set.

In this chapter, we assume that the price impact functions are linear for the purpose of computational tractability. In addition, we assume that the temporary price impact is larger than half of the permanent price impact as in [2] and [14]. In other words, we focus on the market where the asset price has high resilience after large transactions. Some investors, especially those with buy-and-hold strategy, are unwilling to make big change to their already constructed portfolios. Thus, our objective is to minimize the change of asset positions, that is, the total liquidation amounts of the assets. Such an objective helps avoid high transaction cost. Owing to the fact that an immediate margin requirement is to be satisfied, we restrict our attention to selling assets in the long positions and we do not allow for shortselling. Therefore, we come up with a robust optimization program with a linear objective function, quadratic and box constraints. We further prove that the liquidation program can be converted to either a second-order cone program for portfolios containing only basic assets or a semidefinite program for portfolios containing both basic assets and derivative securities. In both cases, the program can be solved efficiently via certain optimization softwares.

Another focus of this chapter is the analysis of the trading properties. We try to see how the

liquidation strategy is affected by different factors. First, we find that the investors will liquidate to the point that the immediate margin requirement is satisfied. They are unwilling to do any further liquidation, since the objective in our setting is to minimize the change of the portfolio positions. A similar phenomenon has been discussed in Chapter 2 and 3. Moreover, we analyze the effect of the empirical return and variance on the optimal liquidation strategy. We find that assets with lower return but higher variance are prioritized for selling. We prefer to retain assets with higher return to take advantage of the price movement and prefer to deplete assets with higher variance to maintain a safer portfolio. It is pointed out in Chapter 2 that assets with smaller price impact are given liquidation preference. In the two-period robust strategy, we no longer simply give preference to more liquid assets. The trading strategy becomes more complex. Sometimes we prefer to aggressively sell the more liquid assets in the beginning so that the asset price thereafter will not be affected dramatically. Sometimes it is better to conservatively retain the more liquid assets at first to prevent the drainage of liquidity in the second period. Which type of strategy (i.e., the aggressive or the conservative) performs better is determined by the interaction of those influencing factors, such as initial price, holdings, price impact, etc. Though the more liquid assets may not be prioritized for selling in both periods, it is shown that the optimal strategy should give preference to them in at least one period. Moreover, we demonstrate that in a highly resilient market where the permanent price impact is ignorable, the more liquid assets are preferred for sale in both periods.

For derivative securities in the portfolio, the trading priority is also affected by the Greeks of the derivatives. We consider vanilla options with the same underlying asset and study the effect of Delta on liquidation priority for call and put options respectively. If the underlying asset has a negative return in the first period and the options are call options, we prefer to sell the one with lower Delta in the second period when other parameters are the same. If the underlying asset has a positive return in the first period and the options are put options, we prefer to sell the one with higher Delta in the second period when other parameters are the same. When the underlying asset price increases, the call option price responses positively while the put option price responses negatively. Delta measures the rate of change of option value with respect to changes in the underlying asset price. So the call option with lower Delta suffers from smaller price decrease under negative underlying return and hence is prioritized for selling. Similarly, the put option with

higher Delta suffers from smaller price drop under positive underlying return and is prioritized for liquidation. Moreover, we find that at the optimal strategy, the trading preference is given to options with high Theta and Gamma, where Theta measures the change of the derivative value with respect to the passage of time and Gamma is the second derivative of the value function with respect to the underlying price. Giving liquidation priority to options with high Theta and Gamma is to keep the portfolio more stable over time and ensure the delta-hedge effective over a wider range of underlying price movements.

4.2 Model Formulation

Let p_i , $i = 1, 2$, denote the asset price at the beginning of the i th trading period. As explained earlier that when trading illiquid assets, there is a compromise on the executed price which is the temporary price impact. Denote $\bar{p}_i = p_i + \Lambda y_i$, $i = 1, 2$, as the executed price in the i th period, where y_i , $i = 1, 2$, is the trading amount in the i th period and Λ is the temporary price impact matrix. We assume that Λ is positive definite. Note that in this chapter, we no longer require the price impact matrix to be diagonal. We allow for cross-asset price pressure, or equivalently, cross-impact. Cross-impact is the impact suffered by one asset which is caused by trading another asset. The cross-impact exists when the two assets are related in some way. A simple example is that trading the underlying asset influences the derivative price. Since our goal is to liquidate portfolio to meet margin requirement, we only focus on long-position assets and we are restricted from shortselling or buying. Mathematically, $x_0 > 0$, $-x_0 \leq y_1 \leq 0$ and $-(x_0 + y_1) \leq y_2 \leq 0$. Let r be a random vector of asset return with a mean vector μ and covariance matrix Σ . Here μ and Σ can be empirically estimated from the historical data. Note that the act of trading in the first period can also influence the price in the second period. Such an influence is called permanent price impact, which is also assumed to be a linear function of the trading amount in the first period. Thus, $p_2 = p_1 \circ (1 + r) + \Phi y_1$, where $p_1 \circ (1 + r)$ is the entry-wise product of p_1 and $1 + r$ and Φ is the permanent price impact matrix which is assumed to be positive definite.

The amount of cash generated during liquidation is given by

$$\begin{aligned}
K &= -\bar{p}_1 y_1 - \bar{p}_2 y_2 \\
&= -(p_1 + \Lambda y_1)^\top y_1 - (p_1 \circ (1+r) + \Phi y_1 + \Lambda y_2)^\top y_2 \\
&= - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^\top \begin{bmatrix} \Lambda & \frac{1}{2}\Phi \\ \frac{1}{2}\Phi & \Lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_1 \circ (1+r) \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{4.2.1}
\end{aligned}$$

It is mentioned in the Introduction that we consider the “elastic” market where the temporary price impact dominates over its permanent counterpart. Mathematically, we assume $\Lambda \succ \frac{1}{2}\Phi$.

Then from Schur complement, it can be proved that $\begin{bmatrix} \Lambda & \frac{1}{2}\Phi \\ \frac{1}{2}\Phi & \Lambda \end{bmatrix} \succ 0$. During liquidation, we want to generate enough cash to satisfy the margin requirement. In particular, we have the following robust constraint,

$$K \geq A, \quad \forall r \in \mathcal{U}_\epsilon, \tag{4.2.2}$$

where A is the minimum cash required and $\mathcal{U}_\epsilon = \{r : \sqrt{(r - \mu)^\top \Sigma^{-1} (r - \mu)} \leq \epsilon\}$ is the uncertainty set for the asset return vector. The intuition underlying this uncertainty set is that the random return vector r is assumed to be close to the empirical return vector μ , with a deviation scaled by the inverse of the covariance matrix, since the deviation could be larger under a higher variance. ϵ represents the restriction on the amount of scaled deviations against which the investor would like to be protected. Therefore, the robust portfolio liquidation problem with margin requirement is defined as

$$\begin{aligned}
&\min_{y_1, y_2} && -1^\top (y_1 + y_2) \\
&\text{subject to} && K \geq A, \quad \forall r \in \mathcal{U}_\epsilon \\
&&& -x_0 \leq y_1 \leq 0, \quad -x_0 - y_1 \leq y_2 \leq 0. \tag{4.2.3}
\end{aligned}$$

Rewrite (4.2.3) as follows

$$\begin{aligned}
& \min_{y_1, y_2} && -1^\top (y_1 + y_2) \\
\text{subject to} &&& - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^\top \begin{bmatrix} \Lambda & \frac{1}{2}\Phi \\ \frac{1}{2}\Phi & \Lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_1 \circ (1+r) \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq A \\
&&& \sqrt{(r - \mu)^\top \Sigma^{-1} (r - \mu)} \leq \epsilon, \forall r \\
&&& -x_0 \leq y_1 \leq 0, -x_0 - y_1 \leq y_2 \leq 0.
\end{aligned} \tag{4.2.4}$$

The main approach is to first fix the original variables y_1, y_2 and then try to find the worst-case value for the constraint containing the uncertain return vector r . To start, we formulate the following optimization program

$$\begin{aligned}
& \min_r && K(r) \\
\text{subject to} &&& \sqrt{(r - \mu)^\top \Sigma^{-1} (r - \mu)} \leq \epsilon.
\end{aligned} \tag{4.2.5}$$

Fix y_1 and y_2 and rewrite (4.2.5) as follows

$$\begin{aligned}
& \min_r && -(p_1 \circ y_2)^\top r \\
\text{subject to} &&& \|\Sigma^{-1/2}(r - \mu)\| \leq \epsilon.
\end{aligned} \tag{4.2.6}$$

This is a second-order cone program and its dual program is given by

$$\begin{aligned}
& \max_{l, v} && -l^\top (-\Sigma^{-1/2}u) - v\epsilon \\
\text{subject to} &&& \Sigma^{-1/2}l = -p_1 \circ y_2 \\
&&& \|l\| \leq v.
\end{aligned} \tag{4.2.7}$$

The above equality constraint indicates that $l = -\Sigma^{1/2}(p_1 \circ y_2)$. Therefore, (4.2.7) can be rewritten

as

$$\begin{aligned} \max_v \quad & -u^\top(p_1 \circ y_2) - v\epsilon \\ \text{subject to} \quad & \|\Sigma^{1/2}(p_1 \circ y_2)\| \leq v. \end{aligned}$$

It is obvious that the strong duality holds in this case. Thus the optimal value of the primal problem is the same as that of the dual problem, which is $-u^\top(p_1 \circ y_2) - \|\Sigma^{1/2}(p_1 \circ y_2)\|\epsilon$. Now we are able to give the equivalent deterministic formulation of original robust liquidation problem (4.2.4)

$$\begin{aligned} \min_{y_1, y_2} \quad & -1^\top(y_1 + y_2) \\ \text{subject to} \quad & - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^\top \begin{bmatrix} \Lambda & \frac{1}{2}\Phi \\ \frac{1}{2}\Phi & \Lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_1 \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - u^\top(p_1 \circ y_2) - \|\Sigma^{1/2}(p_1 \circ y_2)\|\epsilon \geq A \\ & -x_0 \leq y_1 \leq 0, -x_0 - y_1 \leq y_2 \leq 0. \end{aligned} \quad (4.2.8)$$

It can be seen from (4.2.8) that when $\epsilon = 0$, (4.2.3) reduces to a deterministic program with linear objective function, quadratic and box constraints. When $\epsilon \neq 0$, we introduce an auxiliary variable a to convert (4.2.8) to the following tractable form

$$\begin{aligned} \min_{y_1, y_2} \quad & -1^\top(y_1 + y_2) \\ \text{subject to} \quad & - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^\top \begin{bmatrix} \Lambda & \frac{1}{2}\Phi \\ \frac{1}{2}\Phi & \Lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_1 \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - u^\top(p_1 \circ y_2) - a \geq A, \\ & \|\Sigma^{1/2}(p_1 \circ y_2)\| \leq a/\epsilon, \\ & -x_0 \leq y_1 \leq 0, -x_0 - y_1 \leq y_2 \leq 0. \end{aligned} \quad (4.2.9)$$

4.3 Trading Properties

Next we discuss the properties of the robust liquidation strategy. We first present the theoretical results and then verify the analytical findings using numerical examples.

4.3.1 Theoretical Results

Similar as in the portfolio deleveraging cases studied in Chapter 2 and 3, the margin constraint is active at the optimal solution.

Proposition 4.3.1. *At the robust optimal solution, the margin constraint of (4.2.8) is active.*

Proof: See the Appendix.

Our primary focus is how asset characteristics (e.g., price impact) and market conditions (e.g., return and volatility) influence the liquidation order of different assets. To facilitate discussion, we assume that there is no interaction between assets. More specifically, we assume the permanent impact matrix, temporary impact matrix and the covariance matrix are all diagonal. Before we state our main results, we start with a straightforward and useful lemma.

Lemma 4.3.2. (1). *If $a \geq c$ and $b \geq d$, then we have $ab + cd \geq ad + bc$. The strict inequality holds if $a > c$ and $b > d$.*

(2). *If $a \geq c$ and $0 \geq b \geq d$, then we have $ab^2 + cd^2 \leq ad^2 + cb^2$. The strict inequality holds if $a > c$ and $b > d$.*

We assume that the price impact matrices Λ and Φ are both diagonal in order to better investigate the effect of asset price impact on the liquidation priority, since the diagonality excludes interactions between assets.

Proposition 4.3.3. *For assets i and j with the same parameters except for price impact, we prefer to liquidate the one with smaller price impact in either the first or second period, i.e., $y_{1i}^* \leq y_{1j}^*$ or $y_{2i}^* \leq y_{2j}^*$ if $\Lambda_{ii} \leq \Lambda_{jj}$ and $\Phi_{ii} \leq \Phi_{jj}$.*

Proof: Assume to the contrary that $y_{1i} > y_{1j}$ and $y_{2i} > y_{2j}$. Let

$$M(y_1, y_2) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^\top \begin{bmatrix} \Lambda & \frac{1}{2}\Phi \\ \frac{1}{2}\Phi & \Lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_1 \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + u^\top (p_1 \circ y_2) + \|\Sigma^{1/2}(p_1 \circ y_2)\| \epsilon + A$$

be the margin constraint value. Then we have

$$\nabla M(y_1^*, y_2^*) = \begin{bmatrix} \Lambda & \frac{1}{2}\Phi \\ \frac{1}{2}\Phi & \Lambda \end{bmatrix} \begin{bmatrix} y_1^* \\ y_2^* \end{bmatrix} + \begin{bmatrix} p_1 \\ p_1 \end{bmatrix} + \begin{bmatrix} 0 \\ u \circ p_1 + \frac{\Sigma(p_1 \circ y_2^*)}{\|\Sigma^{1/2}(p_1 \circ y_2^*)\|} \epsilon \end{bmatrix}.$$

Let $\bar{p}_1 = p_{1i} = p_{1j}$, $\bar{u} = u_{1i} = u_{1j}$, $\bar{\Sigma} = \Sigma_{ii} = \Sigma_{jj}$. Consider a direction $\delta = (\delta_1, \dots, \delta_m, \delta_{m+1}, \dots, \delta_{2m})$ with $\delta_i = -1, \delta_j = 1, \sigma_k = 0, k \neq i, j$.

$$\delta^\top \nabla M(y_1^*) = -(2y_{1,i}^* \Lambda_{ii} + y_{2,i}^* \Phi_{ii}) + (2y_{1,j}^* \Lambda_{jj} + y_{2,j}^* \Phi_{jj}) < 0. \quad (4.3.1)$$

For τ small enough, $y^* + \tau\delta$ is still feasible. In addition, $y^* + \tau\delta$ has the same objective value as the optimal solution but with an inactive constraint, which contradicts with Proposition 4.3.1. This finishes the proof of the proposition. \square

Proposition 4.3.3 indicates that liquid assets should be given trading preference in at least one period. Sometimes it is optimal that we adopt an aggressive liquidation strategy, which is to sell the more liquid assets immediately so that we do not suffer from severe price drop in the second period. While in some other scenarios, it is better to first reserve the more liquid assets as a preparation for future liquidity needs. Which type of strategy to choose depends on the composite effect of other factors. However, when the market is highly elastic, that is, the asset price has a high resilience after early transactions, the liquid assets are preferred to sell in both periods.

Proposition 4.3.4. *When permanent price impact can be ignored, we prefer to liquidate the assets with smaller temporary price impact in both the first and second period when all the other parameters are the same, i.e., $y_{1i}^* \leq y_{1j}^*$ and $y_{2i}^* \leq y_{2j}^*$ if $\Lambda_{ii} \leq \Lambda_{jj}$.*

Proof: If $\Lambda_{ii} = \Lambda_{jj}$, then it is trivial that $y_{1i}^* = y_{1j}^*$ and $y_{2i}^* = y_{2j}^*$. So we only consider the case that $\Lambda_{ii} < \Lambda_{jj}$. Assume to the contrary that $y_{2i}^* > y_{2j}^*$. Then consider another point $\bar{y} = \{\bar{y}_1, \bar{y}_2\}$ such that $\bar{y}_1 = y_1^*$ and $\bar{y}_2 = y_2^*$ except that $\bar{y}_{2i} = y_{2j}^*$ and $\bar{y}_{2j} = y_{2i}^*$. Then we have

$$\Lambda_{ii} \bar{y}_{2i}^2 + \Lambda_{jj} \bar{y}_{2j}^2 = \Lambda_{ii} y_{2j}^{*2} + \Lambda_{jj} y_{2i}^{*2} < \Lambda_{ii} y_{2i}^{*2} + \Lambda_{jj} y_{2j}^{*2}$$

The last inequality is due to Lemma 4.3.2. Thus, \bar{y} is also feasible to (4.2.8) with the same objective value as the optimal solution but an inactive constraint, contradicting with Proposition 4.3.1. We can show similar result for y_1^* . This finishes the proof of the proposition. \square

Apart from the asset price impact, the asset return and volatility are also important that can influence the sale order distribution over the two periods.

Proposition 4.3.5. *We prefer to sell assets with the lower expected return in the first period but*

prefer to sell assets with the higher expected return in the second period when other parameters are the same.

Proof: Assume $u_i \geq u_j$. When $u_i = u_j$, we have the trivial case that $y_{1i}^* = y_{1j}^*$ and $y_{2i}^* = y_{2j}^*$. So we only consider the case $u_i > u_j$. Let y_2^* be the optimal solution. We next show that $y_{2i}^* \leq y_{2j}^*$. Assume to the contrary that $y_{2i}^* > y_{2j}^*$. Then consider another point $\bar{y} = y^*$ except that $\bar{y}_{1i} = y_{1j}^*$, $\bar{y}_{1j} = y_{1i}^*$, $\bar{y}_{2i} = y_{2j}^*$ and $\bar{y}_{2j} = y_{2i}^*$. Then we have

$$u_i \bar{y}_{2i} + u_j \bar{y}_{2j} = u_i y_{2j}^* + u_j y_{2i}^* < u_i y_{2i}^* + u_j y_{2j}^*,$$

where the inequality is due to Lemma 4.3.2. Thus, \bar{y} is also optimal but with an inactive margin constraint, contradicting with Proposition 4.3.1.

Next we show that assets with lower expected return are prioritized for selling in the first period. That is, $y_{1j}^* \leq y_{1i}^*$. Assume to the contrary $y_{1i}^* < y_{1j}^*$. Again consider a new point $\tilde{y} = y^*$ except that $\tilde{y}_{2i} = y_{2j}^*$ and $\tilde{y}_{2j} = y_{2i}^*$. Then

$$\tilde{y}_{1i}^* \tilde{y}_{2i}^* + \tilde{y}_{1j}^* \tilde{y}_{2j}^* = y_{1i}^* y_{2j}^* + y_{1j}^* y_{2i}^* \leq y_{1i}^* y_{2i}^* + y_{1j}^* y_{2j}^*$$

Thus, the new point is also a optimal solution but with $\tilde{y}_{2i} \geq \tilde{y}_{2j}$, contradicting with the result regarding y_2^* . \square

Proposition 4.3.5 shows that in the first period, we reduce the sale of the assets with higher expected return to make preparation for the second-period liquidation so that in the second period we can sell more assets with higher return to take advantage of the price movement.

Proposition 4.3.6. *We prefer to sell assets with higher variance in the first period but prefer to sell assets with lower variance in the second period when other parameters are the same.*

Proof: The proof is parallel to Proposition 4.3.5 that we omit the details here. \square

Proposition 4.3.6 shows that it is optimal to first sell more volatile assets and retain less volatile assets for the second period. This is consistent with our intuition. It is safe to get rid of risky assets quickly so that there is less uncertainty in the second period.

4.3.2 Numerical Examples

In this section, we use small examples to visualize how asset price impact, return and volatility affect the trading strategy.

Example 1: Asset Price Impact

Consider a two-asset portfolio with the following set of parameters: $p_{11} = p_{12} = 5$, $x_{01} = x_{02} = 1$, $A = 5$, $\Sigma = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$, $u = \begin{bmatrix} -0.01 \\ -0.01 \end{bmatrix}$ and $\epsilon = 0.1$. Let $\Lambda = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.25 \end{bmatrix}$ and $\Phi = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$. We can see that asset 1 is more liquid than asset 2. The optimal solution is $y_{11}^* = -0.5149$, $y_{12}^* = -0.4071$, $y_{21}^* = -0.0446$ and $y_{22}^* = -0.0561$. In the first period, asset 1 is preferred for selling while the opposite holds in the second period.

Consider another slightly different set of price impact parameters: $\Lambda = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.21 \end{bmatrix}$, $\Phi = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$. The optimal solution is $y_{11}^* = -0.133$, $y_{12}^* = -0.1332$, $y_{21}^* = -0.3784$ and $y_{22}^* = -0.3647$. We can see that asset 1 is still more liquid. But the sale of asset 1 is smaller in the first period but larger in the second period. These numerical results are consistent with Proposition 4.3.4. When permanent impact is negligible, i.e., $\Phi = 0$, the optimal solution becomes $y_{11}^* = -0.3314$, $y_{12}^* = -0.3156$, $y_{21}^* = -0.1884$ and $y_{22}^* = -0.1801$. The more liquid asset is preferred for selling in both periods.

Example 2: Asset Return and Volatility

Consider a two-asset portfolio where $p_{11} = p_{12} = 5$, $x_{01} = x_{02} = 1$, $A = 5$, $\Lambda = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$, $\Phi = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$ and $\epsilon = 0.1$. We first examine the effect of the mean vector. Consider the return vector $u = \begin{bmatrix} 0.01 \\ 0.015 \end{bmatrix}$ and the variance matrix $\Sigma = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$.

It can be seen that asset 2 has a higher expected return compared with asset 1. The optimal solution to (4.2.8) is $y_{11}^* = -0.1330$, $y_{12}^* = -0.0937$, $y_{21}^* = -0.3514$ and $y_{22}^* = -0.4300$. The sale of asset 1 is larger in the first period but smaller in the second period. It is consistent with Proposition 4.3.5.

Similarly, we examine the effect of volatility. Let $\Sigma = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.15 \end{bmatrix}$ and $u = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}$. The optimal solution is $y_{11}^* = -0.1447$, $y_{12}^* = -0.1555$, $y_{21}^* = -0.3658$ and $y_{22}^* = -0.3443$. We can see that asset 1 has smaller variance. The sale of asset 1 is smaller in the first period but larger in the second period, which is consistent with Proposition 4.3.6.

4.4 Derivative Trading

In this section, we consider the robust liquidation of portfolios containing derivatives. Assume there are m basic assets and $n - m$ derivatives. Zymler et. al. ([73]) model the derivative return as a quadratic function of the underlying return. In particular, the quadratic function is formulated by the delta-gamma approximation.

4.4.1 Robust Deleveraging Formulation

We begin with introducing the delta-gamma approximation proposed in [73].

Let ξ be the vector of basic asset return, p_1 and $p_2 = p_1 \circ (1 + \xi)$ be the unaffected price of basic assets at the beginning of the first and second period. The ‘‘unaffected’’ price refers to the price when there is no trading activity. Denote $v_i(p_t, t)$, $i = 1, \dots, n$, as the value of asset i (basic or nonbasic), where $v_i : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be a twice continuously differentiable function of time t and basic asset price p_t . Denote the time length of the first trading period as T . Then according to the second-order Taylor expansion,

$$v_i(p_2, T) - v_i(p_1, 0) \approx \bar{\theta}_i T + \bar{\Delta}_i^\top (p_2 - p_1) + \frac{1}{2} (p_2 - p_1)^\top \bar{\Gamma}_i (p_2 - p_1),$$

where $\bar{\theta}_i = \partial_t v_i(p_1, 0)$, $\bar{\Delta}_i = \nabla_p v_i(p_1, 0)$ and $\bar{\Gamma}_i = \nabla_p^2 v_i(p_1, 0)$ are the Greeks of the assets. In particular, for basic asset i , $\bar{\theta}_i$ is a zero vector, $\bar{\Delta}_i$ is a unit vector and $\bar{\Gamma}_i$ is a zero matrix. Define $\theta_i = \frac{\bar{\theta}_i T}{v_i(p_1, 0)}$, $\Delta_i = \frac{\text{diag}(p_1) \bar{\Delta}_i}{v_i(p_1, 0)}$ and $\Gamma_i = \frac{\text{diag}(p_1)^\top \bar{\Gamma}_i \text{diag}(p_1)}{v_i(p_1, 0)}$. Then we can represent any asset return (basic or nonbasic) as

$$r_i \approx f_i(\xi) = \theta_i + \Delta_i^\top \xi + \frac{1}{2} \xi^\top \Gamma_i \xi.$$

Denote $(p_1 \circ r)^\top y_2 = \theta(y_2) + \Delta(y_2)^\top \xi + \frac{1}{2} \xi^\top \Gamma(y_2) \xi$, where $\theta(y_2) = \sum_1^n y_{2,i} p_{1,i} \theta_i$, $\Delta(y_2) = \sum_1^n y_{2,i} p_{1,i} \Delta_i$

and $\Gamma(y_2) = \sum_1^n y_{2,i} p_{1,i} \Gamma_i$. Thus, the liquidation problem (4.2.8) becomes

$$\begin{aligned}
& \min_{y_1, y_2, \xi} && -1^\top (y_1 + y_2) \\
\text{subject to} & - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^\top \begin{bmatrix} \Lambda & \frac{1}{2}\Phi \\ \frac{1}{2}\Phi & \Lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_1 \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \theta(y_2) - \Delta(y_2)^\top \xi - \frac{1}{2} \xi^\top \Gamma(y_2) \xi \geq A \\
& && (\xi - \mu)^\top \Sigma^{-1} (\xi - \mu) \leq \epsilon, \forall \xi \\
& && -x_0 \leq y_1 \leq 0, -x_0 - y_1 \leq y_2 \leq 0.
\end{aligned} \tag{4.4.1}$$

To solve (4.4.1), we consider a subproblem for fixed y , which is given by

$$\begin{aligned}
& \min_{\xi} && -\theta(y_2) - \Delta(y_2)^\top \xi - \frac{1}{2} \xi^\top \Gamma(y_2) \xi \\
\text{subject to} & && (\xi - \mu)^\top \Sigma^{-1} (\xi - \mu) \leq \epsilon.
\end{aligned} \tag{4.4.2}$$

The constraint can be rewritten as $\xi^\top \Sigma^{-1} \xi - 2(\Sigma^{-1} \mu)^\top \xi + \mu^\top \mu \leq \epsilon$. It can be seen that (4.4.2) may not be convex. But according to [24], the optimal solution to (4.4.2) can still be found by solving the following dual problem

$$\begin{aligned}
& \max_{\zeta, \eta} && \zeta + \eta(\mu^\top \mu - \epsilon) - \theta(y_2) \\
\text{subject to} & && \begin{bmatrix} -\frac{1}{2}\Gamma(y_2) + \eta\Sigma^{-1} & -\frac{\Delta(y_2) + 2\eta\Sigma^{-1}\mu}{2} \\ -\frac{\Delta(y_2) + 2\eta\Sigma^{-1}\mu}{2} & -\zeta \end{bmatrix} \succeq 0, \eta \geq 0.
\end{aligned} \tag{4.4.3}$$

In this case, strong duality holds and there is zero duality gap between (4.4.2) and (4.4.3). Thus,

(4.4.1) is equivalent to

$$\begin{aligned} \min_{y_1, y_2, \zeta, \eta} \quad & -1^\top (y_1 + y_2) \\ \text{s.t.} \quad & - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^\top \begin{bmatrix} \Lambda & \frac{1}{2}\Phi \\ \frac{1}{2}\Phi & \Lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_1 \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \theta(y_2) \\ & + \zeta + \eta(\mu^\top \mu - \epsilon) \geq A \end{aligned} \tag{4.4.4}$$

$$\begin{aligned} & \begin{bmatrix} -\frac{1}{2}\Gamma(y_2) + \eta\Sigma^{-1} & -\frac{\Delta(y_2) + 2\eta\Sigma^{-1}\mu}{2} \\ -(\frac{\Delta(y_2) + 2\eta\Sigma^{-1}\mu}{2})^\top & -\zeta \end{bmatrix} \succeq 0, \eta \geq 0 \\ & -x_0 \leq y_1 \leq 0, -x_0 - y_1 \leq y_2 \leq 0. \end{aligned} \tag{4.4.5}$$

(4.4.5) is a convex semidefinite program that is computationally tractable.

4.4.2 Derivative Trading Properties

This subsection studies how the Greeks affect the trading preference between derivatives. We start with the following assumption.

Assumption 4.4.1. (1). At the optimal solution y^* , $y_1^* \neq 0$; (2). The price impact matrices Λ , Φ and the covariance matrix Σ are diagonal; (3). $p_1 > \Phi x_0$; (4). For each derivative security, there is only one underlying asset.

When $y_1^* = 0$, all the liquidation is done in the second period, which is a rare case. So we assume that $y_1^* \neq 0$. The diagonality assumption is for the ease of discussing the effect of derivative Greeks on the trading priority. The existence of off-diagonal elements indicates the interaction between assets. If this is the case, it becomes more difficult to understand the pure effect of the derivative Greeks. (3) imposes an upper bound on the permanent price impact suffered by liquidating the portfolio. The permanent price impact is assumed to be less than the asset price, which is a mild assumption. Otherwise, the trading cost is too high that the investor may not choose to do so. Under (4), $\bar{\Gamma}_i$ becomes a matrix with only one non-zero element on the diagonal which facilitates our discussion.

Proposition 4.4.2. The margin constraint (4.4.4) is active at the optimal solution y^* .

Proof: Assume to the contrary that the margin constraint is not active. Denote

$$K = - \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^\top \begin{bmatrix} \Lambda & \frac{1}{2}\Phi \\ \frac{1}{2}\Phi & \Lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_1 \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \theta(y_2) + \zeta + \eta(\mu^\top \mu - \epsilon),$$

then $K(y_1^*, y_2^*) > A$. Calculating the derivative of K with respect to y_1 , we get $\partial K / \partial y_1 = -2\Lambda y_1 - \Phi y_2 - p_1 < \Phi(y_1 + y_2) \leq 0$, according to Assumption 4.4.1. Without loss of generality, we assume $y_{1,i}^* < 0$. Then there exists a small enough $\delta > 0$ such that $y_{1,i}^* + \delta \leq 0$ and $K(\bar{y}_1^*, y_2^*) \geq A$, where $\bar{y}_{1,i}^* = y_{1,i}^* + \delta$ and $\bar{y}_{1,j}^* = y_{1,j}^*$, for $1 \leq j \leq n, j \neq i$. Thus, we have found another feasible trading strategy (\bar{y}_1^*, y_2^*) with smaller objective value, contradicting with the optimality of y^* . \square

Proposition 4.4.3. *For two derivatives with the same underlying asset, we prefer to sell the one with higher θ in the second period when other parameters are the same.*

Proof: Assuming $\theta_i \geq \theta_j$, we would like to prove that $y_{2,i}^* \leq y_{2,j}^*$. When $\theta_i = \theta_j$, it is obvious that we have $y_{2,i}^* = y_{2,j}^*$. Next we only consider the case that $\theta_i > \theta_j$. Assume to the contrary $y_{2,i}^* > y_{2,j}^*$. Then from Lemma 4.3.2, we have $\theta_i y_{2,i}^* + \theta_j y_{2,j}^* > \theta_i y_{2,j}^* + \theta_j y_{2,i}^*$. Consider the trading strategy (\bar{y}_1, \bar{y}_2) where $\bar{y}_1 = y_1^*$, $\bar{y}_{2,i} = y_{2,j}^*$, $\bar{y}_{2,j} = y_{2,i}^*$ and $\bar{y}_{2,k} = y_{2,k}^*$ for $1 \leq k \leq n, k \neq i, j$. Then (\bar{y}_1, \bar{y}_2) is a feasible solution with the same objective value as the optimal strategy. Namely, \bar{y} is also optimal to (4.4.5). But at \bar{y} , the margin constraint (4.4.4) is not active, contradicting with Proposition 4.4.2. This finishes the proof of the proposition. \square

Proposition 4.4.4. *For two derivatives with the same underlying asset, we prefer to sell the one with higher initial price in both periods when other parameters are the same.*

Proof: The proof is similar to Proposition 4.4.3. \square

Proposition 4.4.5. *For two derivatives with the same underlying asset and positive Γ , we prefer to sell the one with higher Γ in the second period when other parameters are the same.*

Proof: Let y^* be the optimal solution to (4.4.5). Assume that derivatives i and j have the same underlying asset k and $\gamma_i \geq \gamma_j$. Then by the definition of Γ , both Γ_i and Γ_j only have one non-zero elements, which is the k th diagonal entry with a value of γ_i and γ_j respectively. We would like to show that $y_{2,i}^* \leq y_{2,j}^*$. If $\gamma_i = \gamma_j$, it's obvious that $y_{2,i}^* = y_{2,j}^*$. So we only consider the case that $\gamma_i > \gamma_j$. Assume to the contrary that $y_{2,i}^* > y_{2,j}^*$. Then we have $\gamma_i y_{2,i}^* + \gamma_j y_{2,j}^* > \gamma_i y_{2,j}^* + \gamma_j y_{2,i}^*$ according

to Lemma 4.3.2. Consider the trading strategy (\bar{y}_1, \bar{y}_2) where $\bar{y}_1 = y_1^*$ and $\bar{y}_{2,i} = y_{2,j}^*$, $\bar{y}_{2,j} = y_{2,i}^*$ and $\bar{y}_{2,k} = y_{2,k}^*$ for $1 \leq k \leq n$, $k \neq i, j$. Note that

$$\begin{bmatrix} -\frac{1}{2}\Gamma(y_2^*) + \eta^* \Sigma^{-1} & -\frac{\Delta(y_2^*) + 2\eta^* \Sigma^{-1} \mu}{2} \\ -(\frac{\Delta(y_2^*) + 2\eta^* \Sigma^{-1} \mu}{2})^\top & -\zeta^* \end{bmatrix} \succeq 0.$$

According to Theorem 4.3 in [32], we have $A \succeq 0$ and $-\zeta^* \geq B^\top A^\dagger B$ where $A = -\frac{1}{2}\Gamma(y_2^*) + \eta^* \Sigma^{-1}$ and $B = -\frac{\Delta(y_2^*) + 2\eta^* \Sigma^{-1} \mu}{2}$. According to the definition of $\Gamma(y_2^*)$ and Assumption 4.4.1, both Γ and Σ are diagonal matrices. Thus, A and A^\dagger are also diagonal matrices. In particular, we have

$$A_{i,i}^\dagger = \begin{cases} \frac{1}{A_{i,i}} & , A_{i,i} \neq 0 \\ 0 & , A_{i,i} = 0 \end{cases}$$

$\Gamma(y_2^*)_{k,k} - \Gamma(\bar{y}_2)_{k,k} \propto (\gamma_i y_{2,i}^* + \gamma_j y_{2,j}^*) - (\gamma_i y_{2,j}^* + \gamma_j y_{2,i}^*) > 0$. Thus, $0 < A(y_2^*)_{k,k} < A(\bar{y}_2)_{k,k}$. Therefore $A(\bar{y}_2) \succ 0$. Moreover, $A^\dagger(y_2^*)_{k,k} > A^\dagger(\bar{y}_2)_{k,k}$. Thus, $B^\top A^\dagger(\bar{y}_2)B < B^\top A^\dagger(y_2^*)B \leq -\zeta^*$. Therefore, \bar{y} is also feasible. In addition, there exists $\bar{\zeta}$ such that $-\bar{\zeta} < -\zeta^*$ and $B^\top A^\dagger(\bar{y})B \leq -\bar{\zeta}$. So far we have found another optimal solution, i.e., $(\bar{y}, \bar{\zeta}, \eta^*)$, but with inactive margin constraint, contradicting to Proposition 4.4.2. This finishes the proof of the proposition. \square

Proposition 4.4.6. *Consider derivatives i and j with the same underlying asset k . If $\mu_k \leq 0$ and $\delta_i \geq \delta_j \geq 0$, then $y_{2,i}^* \geq y_{2,j}^*$ when other parameters are the same; if $\mu_k \geq 0$ and $\delta_i \geq \delta_j \geq 0$, then $y_{2,i}^* \geq y_{2,j}^*$ when other parameters are the same.*

Proof: Let y^* be the optimal solution to (4.4.5). The proof follows in a similar manner as the proof of 4.4.5 and we continue to use the notations there. Note that we only provide proof for the first case. The second case can be shown by the same token. When $\delta_i = \delta_j$, it is trivial that $y_{2,i}^* = y_{2,j}^*$. So we consider the case that $\delta_i > \delta_j$. Assume to the contrary that $y_{2,i}^* < y_{2,j}^*$. Then we have $\delta_i y_{2,i}^* + \delta_j y_{2,j}^* < \delta_i y_{2,j}^* + \delta_j y_{2,i}^*$. Similar as in Proposition 4.4.5, we consider another trading strategy (\bar{y}_1, \bar{y}_2) where $\bar{y}_1 = y_1^*$ and $\bar{y}_{2,i} = y_{2,j}^*$, $\bar{y}_{2,j} = y_{2,i}^*$ and $\bar{y}_{2,k} = y_{2,k}^*$ for $1 \leq k \leq n$, $k \neq i, j$. This time we have $A(y_2^*) = A(\bar{y}_2^*)$ and $A^\dagger(y_2^*) = A^\dagger(\bar{y}_2^*)$. $B(y_2^*)_{k,k} - B(\bar{y}_2)_{k,k} \propto (\delta_i y_{2,i}^* + \delta_j y_{2,j}^*) - (\delta_i y_{2,j}^* + \delta_j y_{2,i}^*) > 0$. The non-positiveness of μ implies the non-negativeness of $B(y_2^*)_{k,k}$ and $B(\bar{y}_2)_{k,k}$. Thus, $B(y_2^*)_{k,k} > B(\bar{y}_2)_{k,k} > 0$. Thus, $B^\top A^\dagger(\bar{y}_2)B < B^\top A^\dagger(y_2^*)B \leq -\zeta^*$. The remaining follows the same argument as in the proof of Proposition 4.4.5. \square

4.5 Summary

In this chapter, we propose a two-period robust optimization model for portfolio liquidation under margin requirement. The objective is to meet the requirement with the least change in asset positions. We study the liquidation problem in two cases: portfolios consisting of only basic assets and portfolios containing both basic assets and derivative securities. For basic assets, the first period return is assumed to belong to a scaled ellipsoid. For derivative securities, the return is assumed to be a quadratic function (i.e., a delta-gamma approximation) of the underlying asset return. Properties regarding how the optimal trading strategy is affected by asset characteristics and market conditions are derived and analyzed.

Chapter 5

Portfolio Execution with a Markov Chain

Approximation Approach

5.1 Introduction

This chapter considers the problem of executing a large multi-asset portfolio in a short time period where the objective is to find an optimal trading strategy that minimizes both the trading cost and the trading risk ([18]). In particular, we study two risk measures: quadratic variation and time-averaged value at risk. [28] recognizes that variance “measures only the end result with no concern of how liquidation proceeds during the whole trading horizon” ([28] Section 1). It proposes quadratic variation as a risk measure instead and studies a continuous-time optimal execution problem. In contrast to variance, quadratic variation takes into account the full trajectory of the liquidation process. In particular, [28] derives an analytical solution in a single-asset arithmetic Brownian motion model with zero drift. For the geometric Brownian motion case, [28] computes the optimal strategy by numerically solving the corresponding Hamilton-Jacobian-Bellman equation. Other risk measures have also been proposed. Motivated by “the practice at investment banks of imposing a daily risk capital charge on trading portfolios proportional to value at risk” ([34] Section 2.1), [34] proposes time-averaged value at risk as the risk measure and obtains an analytical optimal trading strategy in a single-asset geometric Brownian motion model with zero drift.

Both [28] and [34] have focused on the liquidation of a single asset. In practice, it is often the case that multiple assets must be liquidated simultaneously. The liquidation of multiple assets not only makes theoretical analysis and numerical solution of the optimal liquidation problem more

challenging, but also elicits important questions about how the correlation and cross impact among the assets influence the optimal trading strategy. Cross impact measures the price impact on one asset that is caused by the trading of another. Strong evidence of cross-stock price pressure has been reported in many recent studies ([36], [35], [5], [67], [57]). In this chapter, we study the problem of liquidating a multi-asset portfolio using quadratic variation as the risk measure. More specifically, we are interested in solving the portfolio liquidation problem in multi-dimensional arithmetic and geometric Brownian motions models. Analytical solutions to the corresponding stochastic control problems are generally not available and efficient numerical methods must be used. In addition to obtaining the optimal trading strategy, we are also interested in studying how risk measure, price impact, risk aversion, correlation and cross impact affect the optimal strategy.

We propose a Markov chain approximation approach for the optimal liquidation problem to achieve the above goals. The general theory of the Markov chain approximation approach can be found in [44]. The application of the Markov chain approximation approach for financial problems can be found, e.g., in [53] and [51]. The main step in such an approach is to construct a discrete-time discrete-state Markov chain to approximate the continuous stochastic process in the original stochastic control problem. To ensure the convergence of the approach, the Markov chain needs to be locally consistent to the original stochastic process. Inspired by the success of the binomial method for American options valuation in the Black-Scholes-Merton model, for optimal liquidation in the geometric Brownian motion model, we construct a locally consistent Markov chain by adapting the Cox-Ross-Rubinstein binomial method of [23] and the Boyle-Evnine-Gibbs multi-dimensional binomial method of [13]. The binomial method allows us to easily compute the optimal trading trajectory via backward induction. Numerical results show that the binomial method converges at rate $O(1/N)$, where N is the number of time steps. We prove that at each time step we need only to solve a strictly convex quadratic program whose solution can be identified explicitly. This further enables us to theoretically analyze the influence of factors such as risk measure, price impact and risk aversion on the optimal strategy in one-dimensional models. As for the multi-dimensional arithmetic Brownian motion model, the optimal liquidation strategy can be shown to be static. This makes the model numerically tractable for high dimensions.

[62] considers infinite-horizon multi-asset liquidation in a mean-variance setting and derives a closed-form relationship between the trading rate and the holding amount when there is no cross

impact. We allow nonzero cross impact and solve a finite-horizon problem under the quadratic variation risk measure and observe that positive cross impact slows down the liquidation while negative cross impact speeds up the process. Intuitively, with positive cross-impact, selling one asset adversely influences the liquidation of the other asset, prompting a smoother strategy to reduce the contagion effect on the trading cost. In addition to the cross-impact, the correlation among the assets also influence the optimal trading strategy. We find that the correlation affects the liquidation trajectory in an opposite way as compared to the cross-impact. That is, a positive correlation accelerates the liquidation process while a negative one slows down the liquidation. Intuitively, having negatively correlated assets helps diversify the portfolio and reduce the risk exposure. Consequently, the liquidation process leans toward controlling the trading cost and a smoother strategy is hence preferred. These observations suggest that, in a real application, one must take factors such as cross impact and correlation into consideration when designing a good liquidation strategy.

Throughout this chapter, we have focused on quadratic variation as the risk measure. Nevertheless, our method can be easily adapted to time-averaged value at risk in one-dimensional problems. This allows us to compare the optimal strategies derived under these two different risk measures. We show that short selling and buying are never optimal during the liquidation process in one-dimensional models with zero drift and quadratic variation as the risk measure. However, this doesn't hold when the time-averaged value at risk is used.

The optimal liquidation strategy obviously depends on the magnitude of the price impact. With everything else being equal, we show that the optimal execution is smoother with larger price impact in one-dimensional models with zero drift. Intuitively, assets with larger price impact are less liquid. When selling illiquid assets, our main consideration is the large price sacrifice that needs to be made in compensation for the lack of market liquidity. Thus, it is preferred to trade more smoothly to avoid high trading cost. But for liquid assets, we are less worried about the trading cost. Instead, our main concern would be to reduce the trading risk. This leads to a more rapid liquidation strategy.

5.2 Model Formulation

In this section, we introduce the formulation of the portfolio liquidation problem. Let m be the number of assets to be liquidated, and $S_t = (S_t^1, \dots, S_t^m)^\top$ the vector of unaffected asset prices, i.e., the prices of the assets assuming no impact from the liquidation. We assume that the unaffected asset prices are governed by the following:

$$dS_t = \mu_t dt + \sigma_t dB_t,$$

where $\mu_t = (\mu_t^1, \dots, \mu_t^m)^\top$ is the drift coefficient, $\sigma_t = \text{diag}(\sigma_t^1, \dots, \sigma_t^m)$ is the diffusion coefficient, and $B_t = (B_t^1, \dots, B_t^m)^\top$ is an m -dimensional Brownian motion. The components B_t^i and B_t^j are correlated with correlation coefficient ρ_{ij} , $1 \leq i, j \leq m$. When $\mu_t^i = \mu_i S_t^i$ and $\sigma_t^i = \sigma_i S_t^i$, $1 \leq i, j \leq m$, we have a multi-dimensional geometric Brownian motion (GBM). When $\mu_t^i = \mu_i S_0^i$ and $\sigma_t^i = \sigma_i S_0^i$, $1 \leq i, j \leq m$, we obtain a multi-dimensional arithmetic Brownian motion (ABM).

Denote the liquidation time horizon by $[0, T]$ for some $T > 0$. Let X_t be an m -dimensional vector with its i th element being the holding amount of the i th asset at time t . Since all assets must be liquidated by time T , the process $\{X_t, 0 \leq t \leq T\}$ represents the trading trajectory with initial value $X_0 > 0$ and terminal value $X_T = 0$. We assume that X_t is absolutely continuous with $dX_t = \xi_t dt$, where ξ_t is the m -dimensional vector of trading rates at time t . Equivalently,

$$X_t = X_0 + \int_0^t \xi_s ds.$$

The absolute continuity assumption has been made in many works. See [62] and the reference cited therein.

For block liquidation, price impact cannot be neglected according to the market microstructure theory ([48]). Liquidating large blocks of assets suffers from both permanent (information effect) and temporary (liquidity effect) price impact which are related to cumulative trading amounts and instantaneous trading rates, respectively. We assume that both the permanent and temporary components are linear functions. We adopt an additive model where the execution prices, the prices

at which the assets are actually liquidated, admit the following structure:

$$\hat{S}_t = S_t + \Lambda(X_t - X_0) + \Gamma\xi_t. \tag{5.2.1}$$

where Λ and Γ represent the permanent and temporary price impact matrices respectively and are assumed to be positive definite.

The objective of a portfolio liquidation problem is to balance between the revenue received from the liquidation process and the trading risk. As mentioned earlier, we focus on two types of risk measures, quadratic variation and time-averaged value at risk, as proposed in [28] and [34] respectively. According to [28], the quadratic variation is given by the following formally:

$$QV = \int_0^T (X_t^\top dS_t)^2.$$

More specifically, we have the following expressions for the quadratic variation under the geometric and arithmetic Brownian motion models:

$$\text{GBM : } QV = \int_0^T X_t^\top \text{diag}(S_t) \Sigma \text{diag}(S_t) X_t dt$$

$$\text{ABM : } QV = \int_0^T X_t^\top \text{diag}(S_0) \Sigma \text{diag}(S_0) X_t dt$$

Here, for any $x \in \mathbb{R}^m$, $\text{diag}(x) \in \mathbb{R}^{m \times m}$ refers to the diagonal matrix generated by x , with its $(i \times i)$ th entry being the i th element of x . Σ is a semi-positive definite matrix with $\Sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$, $1 \leq i, j \leq m$. While we mainly focus on the quadratic variation in the paper, other risk measures such as time-averaged value at risk can be handled very similarly. We also would like to see how the choice of the risk measure influences the optimal liquidation strategy in the one-dimensional geometric Brownian motion model. The time-averaged value at risk in this case is given by

$$\text{GBM}(m = 1) : VaR = \alpha \int_0^T X_t S_t dt,$$

where $\alpha < 1$ is the risk-aversion parameter that depends on the volatility of the asset and the confidence level used to compute the value at risk. An investor that is more risk-averse will choose a higher confidence level and hence a larger α . For details of time-averaged value at risk, we refer

to [34] and [59].

The total revenue generated by the liquidation strategy $\{\xi_t, 0 \leq t \leq T\}$ is given by

$$\mathcal{R} = - \int_0^T \hat{S}_t^\top \xi_t dt.$$

We wish to maximize the expected revenue and minimize the expected risk exposure. In the geometric Brownian motion model, when quadratic variation is used as the risk measure, we minimize

$$-E[\mathcal{R}] + \alpha E[QV] = E \left[\int_0^T (\xi_t^\top \Gamma \xi_t + S_t^\top \xi_t + \alpha X_t^\top \text{diag}(S_t) \Sigma \text{diag}(S_t) X_t) dt \right] + \frac{1}{2} X_0^\top \Lambda X_0, \quad (5.2.2)$$

where $\alpha \geq 0$ is the risk-aversion parameter. When time-averaged value at risk is used as the risk measure in a one-dimensional geometric Brownian motion model, we minimize

$$-E[\mathcal{R}] + E[VaR] = E \left[\int_0^T (\Gamma \xi_t^2 + S_t \xi_t + \alpha X_t S_t) dt \right] + \frac{1}{2} \Lambda X_0^2. \quad (5.2.3)$$

Note that in (5.2.3), all quantities are scalars. In the arithmetic Brownian motion, when quadratic variation is used, we minimize

$$-E[\mathcal{R}] + \alpha E[QV] = E \left[\int_0^T (\xi_t^\top \Gamma \xi_t + S_t^\top \xi_t + \alpha X_t^\top \text{diag}(S_0) \Sigma \text{diag}(S_0) X_t) dt \right] + \frac{1}{2} X_0^\top \Lambda X_0.$$

The time-averaged value at risk could also be used in the arithmetic Brownian motion model. We have omitted this case to reduce redundancy.

Note that the permanent price impact component (i.e. $\frac{1}{2} X_0^\top \Lambda X_0$) in the above objective functions is constant. We will drop this term from now on since it will not affect the optimal solution. In the geometric Brownian motion model with quadratic variation as the risk measure, define the following cost function:

$$J(X_t, S_t, t, \xi) = E \left[\int_t^T (\xi_u^\top \Gamma \xi_u + S_u^\top \xi_u + \alpha X_u^\top \text{diag}(S_u) \Sigma \text{diag}(S_u) X_u) du \middle| X_t, S_t \right]. \quad (5.2.4)$$

Let $V(X_t, S_t, t)$ be the value function at time t defined by $V(X_t, S_t, t) = \inf_{\{\xi_u, t \leq u \leq T\}} J(X_t, S_t, t, \xi)$.

We want to compute $V(X_0, S_0, 0)$ and the corresponding optimal control $\xi^* = \{\xi_t^*, 0 \leq t \leq T\}$.

When $m = 1$ and time-averaged value at risk is used, V is defined similarly with the following cost

function:

$$J(X_t, S_t, t, \xi) = E \left[\int_t^T (\Gamma \xi_u^2 + S_u \xi_u + \alpha X_u S_u) du \middle| X_t, S_t \right].$$

For the arithmetic Brownian motion case with quadratic variation as the risk measure, we have the following cost function:

$$J(X_t, S_t, t, \xi) = E \left[\int_t^T (\xi_u^\top \Gamma \xi_u + S_u^\top \xi_u + \alpha X_u^\top \text{diag}(S_0) \Sigma \text{diag}(S_0) X_u) du \middle| X_t, S_t \right]. \quad (5.2.5)$$

The optimal liquidation problem has thus been reduced to a stochastic control problem. Analytical solutions are rare except in special cases. In the remaining of the chapter, we study efficient numerical solution of the stochastic control problem and properties of the optimal trading strategy.

5.3 The Geometric Brownian Motion Model

We use a Markov chain approximation approach to solve the portfolio liquidation problem in the geometric Brownian motion model. The idea of the Markov chain approximation approach is to approximate the original continuous time stochastic process by a locally consistent discrete time discrete state Markov chain and reduce the original continuous time stochastic control problem to a simpler problem in discrete time. To construct a locally consistent Markov chain approximation for the geometric Brownian motion, we extend the one-dimensional binomial method of [23] and the multi-dimensional binomial method of [13], where the asset price evolves along a binomial tree. We then obtain the optimal trading strategy by solving the resulting dynamic programming equation through backward induction.

5.3.1 Markov Chain Approximation

The Markov chain approximation approach starts with a locally consistent Markov chain approximation for the geometric Brownian motion $S = \{S_t, 0 \leq t \leq T\}$. Recall that

$$dS_t^i = \mu_i S_t^i dt + \sigma_i S_t^i dB_t^i, \quad \text{corr}(B_t^i, B_t^j) = \rho_{ij}, \quad 1 \leq i, j \leq m. \quad (5.3.1)$$

Divide the liquidation horizon $[0, T]$ into N equal subintervals, each with length $\delta = T/N$. Let $\bar{S} = \{\bar{S}_n \in \mathbb{R}^m, 0 \leq n \leq N\}$ be a discrete time discrete state Markov chain. Define $\Delta \bar{S}_n =$

$\bar{S}_{n+1} - \bar{S}_n \in \mathbb{R}^m$. \bar{S} is locally consistent with the geometric Brownian motion S if it satisfies the following local consistency conditions:

$$E[\Delta \bar{S}_n^i | \bar{S}_n] = \mu_i \bar{S}_n^i \delta + o(\delta), \quad 1 \leq i \leq m, \quad (5.3.2)$$

$$E[(\Delta \bar{S}_n^i - E[\Delta \bar{S}_n^i | \bar{S}_n])(\Delta \bar{S}_n^j - E[\Delta \bar{S}_n^j | \bar{S}_n]) | \bar{S}_n] = \rho_{ij} \sigma_i \sigma_j \bar{S}_n^i \bar{S}_n^j \delta + o(\delta), \quad 1 \leq i, j \leq m. \quad (5.3.3)$$

The next step is to approximate the cost function and reduce the original stochastic control problem in continuous time to a simpler problem in discrete time. Consider a piecewise constant approximation of the trading rate that takes value $\bar{\xi}_t = y_n/\delta$ for $t \in [n\delta, (n+1)\delta)$, $0 \leq n < N$. Note that $-y_n$ is the amount of assets that are sold from $n\delta$ to $(n+1)\delta$. Denote $y = \{y_n, 0 \leq n < N\}$. For any $0 \leq n \leq N$, let \bar{X}_n be the holding amount at time $n\delta$ corresponding to the trading rate process $\bar{\xi} = \{\bar{\xi}_t, 0 \leq t \leq T\}$. Denote $\bar{X} = \{\bar{X}_n \in \mathbb{R}^m, 0 \leq n \leq N\}$. For any $0 \leq n < N$, we have $\bar{X}_{n+1} = \bar{X}_n + y_n$, with initial holding amount $\bar{X}_0 = X_0$. In particular, $\bar{X}_N = \bar{X}_{N-1} + y_{N-1} = 0$ must be satisfied. Suppose quadratic variation is used as the risk measure. We approximate the integral in the cost function in (5.2.4) using a left endpoint rule and solve the resulting discrete time stochastic control problem:

$$V_n(\bar{X}_n, \bar{S}_n) = \min_{\{y_k, n \leq k < N\}} E \left[\sum_{k=n}^{N-1} \left(\frac{1}{\delta} y_k^\top \Gamma y_k + \bar{S}_k^\top y_k + \alpha \delta \bar{X}_k^\top \text{diag}(\bar{S}_k) \Sigma \text{diag}(\bar{S}_k) \bar{X}_k \right) \middle| \bar{X}_n, \bar{S}_n \right].$$

Based on Bellman's principle of optimality, we obtain the following dynamic programming equation: for $0 \leq n < N$,

$$\begin{aligned} V_n(\bar{X}_n, \bar{S}_n) &= \min_{y_n} \left(E [V_{n+1}(\bar{X}_n + y_n, \bar{S}_{n+1}) | \bar{X}_n, \bar{S}_n] \right. \\ &\quad \left. + \frac{1}{\delta} y_n^\top \Gamma y_n + \bar{S}_n^\top y_n + \alpha \delta \bar{X}_n^\top \text{diag}(\bar{S}_n) \Sigma \text{diag}(\bar{S}_n) \bar{X}_n \right). \end{aligned} \quad (5.3.4)$$

The terminal value is $V_N(\bar{X}_N, \bar{S}_N) = 0$. Similarly, when $m = 1$ and time-averaged value at risk is used as the risk measure, the dynamic programming equation becomes

$$V_n(\bar{X}_n, \bar{S}_n) = \min_{y_n} \left(E [V_{n+1}(\bar{X}_n + y_n, \bar{S}_{n+1}) | \bar{X}_n, \bar{S}_n] + \frac{1}{\delta} \Gamma y_n^2 + \bar{S}_n y_n + \alpha \delta \bar{S}_n \bar{X}_n \right) \quad (5.3.5)$$

for $0 \leq n < N$ and with terminal value $V_N(\bar{X}_N, \bar{S}_N) = 0$. The dynamic programming equations

(5.3.4) and (5.3.5) are solved by backward induction.

5.3.2 Multi-Dimensional Binomial Method

To construct a locally consistent Markov chain for the geometric Brownian motion, we adapt the one-dimensional Cox-Ross-Rubinstein (CRR) binomial method and the multi-dimensional Boyle-Evnine-Gibbs (BEG) binomial method. Let's first consider the single asset case where the asset price follows a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t. \quad (5.3.6)$$

In the CRR model, the above geometric Brownian motion is approximated by a binomial tree with the following structure: for any $0 \leq n < N$, $\bar{S}_{n+1} = \exp(\sigma\sqrt{\delta})\bar{S}_n$ with probability $p_1 = (1 + \mu\sqrt{\delta}/\sigma)/2$ and $\bar{S}_{n+1} = \exp(-\sigma\sqrt{\delta})\bar{S}_n$ with probability $p_2 = 1 - p_1$. Unfortunately, it can be verified easily that the CRR binomial model violates the first local consistency condition (5.3.2). Instead of the CRR model, we propose the following binomial structure:

$$\bar{S}_{n+1} = \begin{cases} u\bar{S}_n, & \text{("up" jump) with probability } p_1 \\ d\bar{S}_n, & \text{("down" jump) with probability } p_2 = 1 - p_1 \end{cases}$$

$$u = 1 + \sigma\sqrt{\delta}, \quad d = 1 - \sigma\sqrt{\delta}, \quad p_1 = \frac{1}{2} \left(1 + \sqrt{\delta} \frac{\mu}{\sigma} \right). \quad (5.3.7)$$

We select $\delta < 1/\sigma^2$ and $\delta < \sigma^2/\mu^2$ so that $d = 1 - \sigma\sqrt{\delta} > 0$ and $0 < p_1 < 1$. It is easy to verify that both local consistency conditions (5.3.2) and (5.3.3) are satisfied.

The CRR model was extended to the multi-asset case in [13]. Suppose there are m assets. The prices are governed by the geometric Brownian motion in (5.3.1). In the BEG binomial model, for any $0 \leq n < N$, the price of i th asset, $1 \leq i \leq m$, goes from \bar{S}_n^i at time $n\delta$ to either $\bar{S}_{n+1}^i = u_i\bar{S}_n^i$ or $\bar{S}_{n+1}^i = d_i\bar{S}_n^i$ at time $(n+1)\delta$. Over each period, there are 2^m possible states. Denote the corresponding probabilities by $p_j, j = 1, 2, \dots, 2^m$. For example, p_1 is the probability that $\bar{S}_{n+1}^i = u_i\bar{S}_n^i$ for all $1 \leq i \leq m$, p_2 is the probability that $\bar{S}_{n+1}^i = u_i\bar{S}_n^i$ for all $1 \leq i \leq m-1$ and $\bar{S}_{n+1}^m = d_m\bar{S}_n^m$, p_{2^m} is the probability that $\bar{S}_{n+1}^i = d_i\bar{S}_n^i$ for all $1 \leq i \leq m$, etc. Again, we revise

the original BEG model so that it meets the consistency requirements. More specifically, we let

$$u_i = 1 + \sigma_i \sqrt{\delta}, \quad d_i = 1 - \sigma_i \sqrt{\delta}, \quad 1 \leq i \leq m.$$

The probabilities are the same as those in [13]:

$$p_j = \frac{1}{2^m} \left(1 + \sum_{\substack{k,l=1 \\ k < l}}^m \chi_{kl}(j) \rho_{kl} + \sqrt{\delta} \sum_{k=1}^m \chi_k(j) \frac{\mu_k}{\sigma_k} \right), \quad j = 1, \dots, 2^m, \quad (5.3.8)$$

where

$$\begin{aligned} \chi_{kl}(j) &= 1 \text{ if both } k\text{th and } l\text{th assets have jumps in the same direction in state } j, \\ &= -1 \text{ otherwise,} \\ \chi_k(j) &= 1 \text{ if } k\text{th asset has an up-jump in state } j, \\ &= -1 \text{ otherwise.} \end{aligned}$$

We select δ sufficiently small so that $d_i > 0, 1 \leq i \leq m$. We assume that the probabilities are between 0 and 1. It can be shown that this multi-dimensional binomial model satisfies both consistency conditions.

With the locally consistent Markov chain constructed in this section, we proceed and solve the dynamic programming equations described in Section 5.3.1.

5.3.3 Backward Induction

In this section, we analyze the solution to the dynamic programming equations (5.3.4) and (5.3.5). Denote the optimal solution by $y^* = \{y_n^*, 0 \leq n < N\}$, and the corresponding optimal trading rate by $\bar{\xi}^* = \{\bar{\xi}_t^*, 0 \leq t \leq T\}$, where $\bar{\xi}_t^* = y_n^*/\delta$ for $t \in [n\delta, (n+1)\delta)$ and $\bar{\xi}_T^* = 0$. We denote the optimal holding amount corresponding to the strategy y^* by $\bar{X}^* = \{\bar{X}_n^*, 0 \leq n \leq N\}$, where $\bar{X}_0^* = X_0$, $\bar{X}_N^* = 0$ and $\bar{X}_{n+1}^* = \bar{X}_n^* + y_n^*$, $0 \leq n < N$. In the above, optimality refers to the discrete time stochastic control problem derived in Section 5.3.1. Recall that $\xi^* = \{\xi_t^*, 0 \leq t \leq T\}$ denotes the optimal trading rate of the original continuous time stochastic control problem. For later convenience, we denote the optimal holding amount corresponding to ξ^* by $X^* = \{X_t^*, 0 \leq t \leq T\}$.

Quadratic variation

We first consider (5.3.4). Recall the terminal value $V_N(\bar{X}_N, \bar{S}_N) = 0$. Due to the requirement that $\bar{X}_N = \bar{X}_{N-1} + y_{N-1} = 0$, we have $y_{N-1}^* = -\bar{X}_{N-1}$. We immediately get the following expression for V_{N-1} :

$$V_{N-1}(\bar{X}_{N-1}, \bar{S}_{N-1}) = \frac{1}{\delta} \bar{X}_{N-1}^\top \Gamma \bar{X}_{N-1} - \bar{S}_{N-1}^\top \bar{X}_{N-1} + \alpha \delta \bar{X}_{N-1}^\top \text{diag}(\bar{S}_{N-1}) \Sigma \text{diag}(\bar{S}_{N-1}) \bar{X}_{N-1}. \quad (5.3.9)$$

For each of $n = 0, 1, \dots, N-2$, however, one must solve an optimization problem to obtain $V_n(\bar{X}_n, \bar{S}_n)$ and the corresponding optimal trading amount y_n^* . The following Proposition 5.3.1 shows that the above optimization problems are strictly convex quadratic programs and admit analytical solutions. It further establishes an affine relationship between the optimal trading amount y_n^* and the optimal holding amount \bar{X}_n^* for each $0 \leq n < N$.

For any $0 \leq s \in \mathbb{R}^m$, define the following $m \times m$ matrices $a_{N-1}(s), d_{N-1}(s)$, m -vectors $b_{N-1}(s), e_{N-1}(s)$ and scalar $c_{N-1}(s)$:

$$\begin{aligned} a_{N-1}(s) &= \frac{1}{\delta} \Gamma + \alpha \delta \text{diag}(s) \Sigma \text{diag}(s) \\ b_{N-1}(s) &= -s \\ c_{N-1}(s) &= 0 \\ d_{N-1}(s) &= -I_{m \times m} \\ e_{N-1}(s) &= (0, \dots, 0)^\top. \end{aligned} \quad (5.3.10)$$

For any $0 \leq n \leq N - 2$, define

$$\begin{aligned}
A_n(s) &= E [a_{n+1}(\bar{S}_{n+1}) | \bar{S}_n = s] \\
B_n(s) &= E [b_{n+1}(\bar{S}_{n+1}) | \bar{S}_n = s] \\
a_n(s) &= -A_n(s) \left(\frac{1}{\delta} \Gamma + A_n(s) \right)^{-1} A_n(s) + A_n(s) + \alpha \delta \text{diag}(s) \Sigma \text{diag}(s) \\
b_n(s) &= B_n(s) - A_n(s) \left(\frac{1}{\delta} \Gamma + A_n(s) \right)^{-1} (B_n(s) + s) \\
c_n(s) &= E [c_{n+1}(\bar{S}_{n+1}) | \bar{S}_n = s] - \frac{1}{4} (B_n(s) + s)^\top \left(\frac{1}{\delta} \Gamma + A_n(s) \right)^{-1} (B_n(s) + s) \\
d_n(s) &= - \left(\frac{1}{\delta} \Gamma + A_n(s) \right)^{-1} A_n(s) \\
e_n(s) &= - \frac{1}{2} \left(\frac{1}{\delta} \Gamma + A_n(s) \right)^{-1} (B_n(s) + s).
\end{aligned} \tag{5.3.11}$$

We then have the following proposition. The proof can be found in the appendix.

Proposition 5.3.1. *The optimization problems in (5.3.4) are strictly convex quadratic programs.*

For any $0 \leq n < N$,

$$V_n(\bar{X}_n, \bar{S}_n) = \bar{X}_n^\top a_n(\bar{S}_n) \bar{X}_n + \bar{X}_n^\top b_n(\bar{S}_n) + c_n(\bar{S}_n).$$

The optimal trading amount y_n^* is an affine function of the optimal holding amount \bar{X}_n^* :

$$y_n^* = d_n(\bar{S}_n) \bar{X}_n^* + e_n(\bar{S}_n).$$

The coefficients a_n, b_n, c_n, d_n and e_n can be computed recursively through equations (5.3.10)-(5.3.11).

In a m -dimensional N -step binomial model, for any $0 \leq n \leq N$, there are $(n + 1)^m$ possible combinations of asset prices at time $n\delta$. Denote \mathcal{S}_n the set of all possible asset prices at time $n\delta$. To solve the dynamic programming equation (5.3.4) to obtain the optimal initial trading amount y_0^* , one starts with computing $a_{N-1}(s)$ and $b_{N-1}(s)$ for each $s \in \mathcal{S}_{N-1}$. Equation (5.3.11) then allows one to compute $a_n(s)$ and $b_n(s)$ for each $1 \leq n \leq N - 2$ and $s \in \mathcal{S}_n$ recursively. Finally, $d_0(\bar{S}_0)$ and $e_0(\bar{S}_0)$ can be computed from $a_1(\cdot)$ and $b_1(\cdot)$. The initial optimal trading amount is then given by $y_0^* = d_0(\bar{S}_0) \bar{X}_0 + e_0(\bar{S}_0)$. Here $\bar{S}_0 = S_0$ is the initial asset price, and $\bar{X}_0 = X_0$ is the initial holding amount. Both of them are known at time 0. Using $\bar{X}_1^* = \bar{X}_0 + y_0^*$, one finds the optimal holding amount \bar{X}_1^* at time δ . Consequently, one can compute $y_1^* = d_1(s) \bar{X}_1^* + e_1(s)$ for

any $s \in \mathcal{S}_1$. Repeating in this way, we are able to compute the optimal trading trajectory along any path of the binomial tree. To compute the optimal value V_0 , one must also compute $c_n(s)$ for each $0 \leq n \leq N-2$ and $s \in \mathcal{S}_n$ using the recursion (5.3.10)-(5.3.11).

When the drift of the geometric Brownian motion is zero, it can be shown using (5.3.8) that $E[\bar{S}_{n+1} | \bar{S}_n = s] = s$ for any $0 \leq n < N$. For a risk neutral investor with $\alpha = 0$, it can then be shown from (5.3.10) and (5.3.11) that

$$a_n(s) = \frac{1}{(N-n)\delta}\Gamma, \quad b_n(s) = -s, \quad c_n(s) = 0, \quad d_n(s) = -\frac{1}{N-n}I_{m \times m}, \quad e_n(s) = 0.$$

Consequently,

$$y_0^* = -\frac{1}{N}X_0, \quad \bar{X}_1^* = X_0 + y_0^* = \frac{N-1}{N}X_0, \quad y_1^* = -\frac{1}{N-1}\bar{X}_1^* = -\frac{1}{N}X_0.$$

It can be shown by induction that for any $0 \leq n < N$,

$$y_n^* = -\frac{1}{N}X_0, \quad \bar{\xi}_t^* = -\frac{1}{N\delta}X_0 = -\frac{X_0}{T}.$$

Corollary 5.3.2. *In the geometric Brownian motion model with zero drift, the optimal trading rate is constant for a risk neutral investor: $\bar{\xi}_t^* = -X_0/T$.*

[11] studies the optimal liquidation problem by minimizing the trading cost only. In a random walk model, they show that the optimal liquidation strategy is constant. Corollary 5.3.2 shows that, for a risk neutral investor that is only interested in minimizing the trading cost, the optimal trading rate is also constant in the geometric Brownian motion model with zero drift. However, in general, the optimal trading strategy is neither constant, nor static, but dynamic. One must adjust the trading rate as market conditions change over time.

In the single-asset case, the dynamic programming equation (5.3.4) reduces to

$$V_n(\bar{X}_n, \bar{S}_n) = \min_{y_n} \left(E [V_{n+1}(\bar{X}_n + y_n, \bar{S}_{n+1}) | \bar{X}_n, \bar{S}_n] + \frac{1}{\delta}\Gamma y_n^2 + \bar{S}_n y_n + \alpha \delta \sigma^2 \bar{S}_n^2 \bar{X}_n^2 \right). \quad (5.3.12)$$

Note that all quantities in (5.3.12) are scalars. From (5.3.10), we have

$$a_{N-1}(s) = \frac{\Gamma}{\delta} + \alpha\delta\sigma^2 s^2, \quad b_{N-1}(s) = -s, \quad c_{N-1}(s) = 0, \quad d_{N-1}(s) = -1, \quad e_{N-1}(s) = 0. \quad (5.3.13)$$

For any $0 \leq n \leq N - 2$, from (5.3.11) and (5.3.7), we have

$$\begin{aligned} A_n(s) &= a_{n+1}(us)p_1 + a_{n+1}(ds)p_2 \\ B_n(s) &= b_{n+1}(us)p_1 + b_{n+1}(ds)p_2 \\ a_n(s) &= \frac{\Gamma A_n(s)}{\Gamma + \delta A_n(s)} + \alpha\delta\sigma^2 s^2 \\ b_n(s) &= B_n(s) - \frac{\delta A_n(s)(B_n(s) + s)}{\Gamma + \delta A_n(s)} \\ c_n(s) &= c_{n+1}(us)p_1 + c_{n+1}(ds)p_2 - \frac{\delta(B_n(s) + s)^2}{4(\Gamma + \delta A_n(s))} \\ d_n(s) &= -\frac{\delta A_n(s)}{\Gamma + \delta A_n(s)} \\ e_n(s) &= -\frac{\delta(B_n(s) + s)}{2(\Gamma + \delta A_n(s))}. \end{aligned} \quad (5.3.14)$$

We then have the following corollary for the single-asset case.

Corollary 5.3.3. *In a single-asset geometric Brownian motion model with quadratic variation as the risk measure, the dynamic programming equation in (5.3.12) admits the following solution:*

$$V_n(\bar{X}_n, \bar{S}_n) = a_n(\bar{S}_n)\bar{X}_n^2 + b_n(\bar{S}_n)\bar{X}_n + c_n(\bar{S}_n), \quad y_n^* = d_n(\bar{S}_n)\bar{X}_n^* + e_n(\bar{S}_n),$$

where a_n, b_n, c_n, d_n and e_n can be computed recursively through (5.3.13-5.3.14). When the drift coefficient in (5.3.6) is zero, we have $c_n(s) = e_n(s) = 0$ and $-\bar{X}_n^* \leq y_n^* < 0$ for any $0 \leq n < N$.

It can be seen from (5.3.13) and (5.3.14) that $-1 < d_n(s) < 0$ for any $0 \leq n < N - 1$ and $d_{N-1} = -1$. When $\mu = 0$ in the one-dimensional geometric Brownian motion model (5.3.6), using (5.3.7), (5.3.13) and (5.3.14), it is easy to verify that

$$b_n(s) = -s, \quad 0 \leq n < N.$$

Consequently,

$$c_n(s) = 0, \quad e_n(s) = 0, \quad 0 \leq n < N.$$

Since $-1 < d_n(s) < 0$ for any $0 \leq n < N - 1$ and $s \in \mathcal{S}_n$, and $\bar{X}_0 = X_0 > 0$, we have

$$-\bar{X}_0 < y_0^* = d_0(\bar{S}_0)\bar{X}_0 < 0$$

$$\bar{X}_1^* = \bar{X}_0 + y_0^* > 0$$

$$-\bar{X}_1^* < y_1^* = d_1(\bar{S}_1)\bar{X}_1^* < 0.$$

By induction, $-\bar{X}_n^* < y_n^* < 0$ for any $0 \leq n < N - 1$. Together with $y_{N-1}^* = -\bar{X}_{N-1}^*$, we obtain the conclusion in Corollary 5.3.3. In Section 5.3.5, we discuss the implication of this corollary when comparing the two risk measures that we consider in this chapter.

Time-averaged value at risk

When time-averaged value at risk is used as the risk measure in the one-dimensional geometric Brownian motion model, the dynamic programming equation (5.3.5) can also be solved easily, as shown in the following proposition. Define

$$a_{N-1} = \frac{\Gamma}{\delta}, \quad b_{N-1}(s) = (\alpha\delta - 1)s, \quad c_{N-1}(s) = 0, \quad d_{N-1} = -1, \quad e_{N-1}(s) = 0. \quad (5.3.15)$$

For any $0 \leq n < N - 1$, define

$$\begin{aligned} B_n(s) &= b_{n+1}(us)p_1 + b_{n+1}(ds)p_2 \\ a_n &= \frac{\Gamma}{(N-n)\delta} \\ b_n(s) &= B_n(s) + \alpha\delta s - \frac{\delta a_{n+1}(B_n(s) + s)}{\Gamma + \delta a_{n+1}} \\ c_n(s) &= c_{n+1}(us)p_1 + c_{n+1}(ds)p_2 - \frac{\delta(B_n(s) + s)^2}{4(\Gamma + \delta a_{n+1})} \\ d_n &= -\frac{1}{N-n} \\ e_n(s) &= -\frac{\delta(B_n(s) + s)}{2(\Gamma + \delta a_{n+1})}. \end{aligned} \quad (5.3.16)$$

Proposition 5.3.4. *The dynamic programming equation in (5.3.5) admits the following solution:*

$$V_n(\bar{X}_n, \bar{S}_n) = a_n \bar{X}_n^2 + b_n(\bar{S}_n) \bar{X}_n + c_n(\bar{S}_n), \quad y_n^* = d_n \bar{X}_n^* + e_n(\bar{S}_n),$$

where a_n, b_n, c_n, d_n and e_n can be computed recursively through (5.3.15)-(5.3.16). When the drift coefficient in (5.3.6) is zero, we have

$$V_n(\bar{X}_n, \bar{S}_n) = \frac{\Gamma}{(N-n)\delta} \bar{X}_n^2 + \left(\frac{\alpha}{2}(N+1-n)\delta - 1 \right) \bar{S}_n \bar{X}_n + c_n(\bar{S}_n)$$

$$y_n^* = -\frac{1}{N-n} \bar{X}_n^* - \frac{\alpha \bar{S}_n}{4\Gamma} (N-1-n)\delta^2.$$

In particular, the initial trading rate $\bar{\xi}_0^*$ obtained from the Markov chain approximation approach converges to the true optimal solution ξ_0^* at rate $1/N$: $\bar{\xi}_0^* - \xi_0^* = \alpha S_0 T / (4\Gamma N)$.

The proof can be found in the appendix. In the one-dimensional geometric Brownian motion model with zero drift and time-averaged value at risk as the risk measure, the closed-form solution of the continuous time stochastic control problem is given in [34]. The optimal holding amount is

$$X_t^* = \frac{T-t}{T} \left[X_0 - \frac{\alpha T}{4\Gamma} \int_0^t S_u du \right]. \quad (5.3.17)$$

The optimal trading rate is

$$\xi_t^* = -\frac{X_0}{T} + \frac{\alpha}{4\Gamma} \left(\int_0^t S_u du - (T-t)S_t \right).$$

In particular, the optimal initial trading rate is

$$\xi_0^* = -\frac{X_0}{T} - \frac{\alpha S_0 T}{4\Gamma}. \quad (5.3.18)$$

The optimal cost function is

$$V(X_0, S_0, 0) = \frac{\Gamma}{T} X_0^2 + \left(\frac{\alpha}{2} T - 1 \right) S_0 X_0 - \frac{\alpha^2}{8\Gamma\sigma^6} \left(e^{\sigma^2 T} - 1 - \sigma^2 T - \frac{1}{2} \sigma^4 T^2 \right) S_0^2. \quad (5.3.19)$$

From Proposition (5.3.4), we obtain

$$\bar{\xi}_0^* = \frac{y_0^*}{\delta} = -\frac{X_0}{N\delta} - \frac{\alpha S_0}{4\Gamma}(N-1)\delta = -\frac{X_0}{T} - \frac{\alpha S_0 T}{4\Gamma}\left(1 - \frac{1}{N}\right) = \xi_0^* + \frac{\alpha S_0 T}{4\Gamma N}.$$

That is, $\bar{\xi}_0^*$ converges to ξ_0^* at rate $1/N$, where N is the number of time steps we use. Moreover, a_0 matches its counterpart in the true optimal solution (5.3.19) exactly, and

$$b_0(S_0) = \left(\frac{\alpha}{2}(N+1)\delta - 1\right)S_0 = \left(\frac{\alpha}{2}T - 1\right)S_0 + \frac{\alpha T S_0}{2N}$$

differs from its counterpart in the true optimal solution by $\alpha T S_0 / (2N)$. Numerical experiments (not reported) also show that $c_0(S_0)$ that we obtain using the binomial model also converges to its counterpart in the true optimal solution at rate $1/N$.

Although the solutions in Corollary 5.3.3 and Proposition (5.3.4) are similar in structure, they could be very different qualitatively. In Section 5.3.5, we discuss some implications of the choice of the risk measure.

5.3.4 Price Impact, Risk Aversion and Initial Asset Price

The optimal liquidation strategy is affected by temporary price impact, risk aversion and initial prices of the assets, as can be seen from the problem formulation. We analyze the effects of these factors on the optimal liquidation strategy in a one-dimensional geometric Brownian motion model with zero drift when quadratic variation is used as the risk measure. As seen in the previous section, the case with time-averaged value at risk as the risk measure admits an analytical solution and can be studied directly using results from [34].

Proposition 5.3.5. *Assume a one-dimensional geometric Brownian motion model with zero drift and quadratic variation as the risk measure. The initial trading rate $\bar{\xi}_0^*$ is an increasing function of the temporary price impact parameter Γ .*

The proposition can be proved by showing that $a_n(s)/\Gamma$ is a decreasing function of Γ . See appendix for the detailed proof. According to Corollary 5.3.3, $\bar{\xi}_0^* = y_0^*/\delta < 0$ under zero drift. Proposition 5.3.5 shows that the larger Γ is, the larger $\bar{\xi}_0^*$ will be (closer to zero since it's negative), and hence the smaller the initial selling speed. Assets with larger price impact could be regarded as

more illiquid. We can see from (5.2.1) that the temporary price impact punishes for high trading speed. Therefore, when an asset is illiquid with large Γ , it is better to use a smoother trading strategy to avoid high trading cost.

Since our goal is to balance trading cost and risk, the risk aversion parameter α also plays an important role in determining the optimal liquidation strategy. [1] observes significant difference in the optimal strategy under different levels of risk tolerance in their model. Note that $a_{N-1}(s)$ in (5.3.13) is an increasing function of α . From the recursion (5.3.14), it can be seen that $a_n(s)$ is an increasing function of α for any $1 \leq n < N$ and $s \in \mathcal{S}_n$. From the expression for d_0 , it can be seen that $d_0(S_0)$ is a decreasing function of α . We therefore obtain the following proposition, stating that the initial trading rate is a decreasing function of α .

Proposition 5.3.6. *Assume a one-dimensional geometric Brownian motion model with zero drift and quadratic variation as the risk measure. The initial trading rate $\bar{\xi}_0^*$ is a decreasing function of α .*

Intuitively, an investor that is more risk-averse (with larger α) tends to liquidate faster to lower the trading risk. Conversely, an investor that is less risk-averse is more concerned about the trading cost and tends to slow down the liquidation process to reduce the trading loss caused by price impact. Proposition 5.3.6 confirms this intuition.

Finally, although less intuitive, the optimal initial trading rate $\bar{\xi}_0^*$ is a decreasing function of the initial asset price S_0 . It can be seen from (5.3.13)-(5.3.14) that $a_n(s)$ is an increasing function of s . Consequently, both $d_0(S_0)$ and $\bar{\xi}_0^*$ decrease as S_0 increases. Therefore, everything else being equal, an asset with higher initial price will be liquidated faster. This is summarized in the following proposition.

Proposition 5.3.7. *In a one-dimensional geometric Brownian motion model with zero drift and quadratic variation as the risk measure, the initial trading rate $\bar{\xi}_0^*$ is a decreasing function of S_0 .*

In the one-dimensional geometric Brownian motion model with zero drift and when time-averaged value at risk is used, we have similarly results. They can be obtained immediately from the closed-form expression for ξ_0^* in (5.3.18). It is decreasing in α and S_0 and increasing in Γ . The optimal trading strategy computed using the Markov chain approximation approach is consistent

with this. It can be seen from Proposition 5.3.4 that y_0^* and hence $\bar{\xi}_0^*$ is decreasing in α and S_0 and increasing in Γ .

5.3.5 Risk Measure

The Markov chain approximation approach allows us to solve optimal liquidation problems with very different risk measures in a unified framework. The numerical procedures are similar in structure. However, it is interesting to compare the optimal trading strategies associated with different risk measures. We assume a one-dimensional geometric Brownian motion model with zero drift in the following discussion.

When time-averaged value at risk is used as the risk measure, the optimal holding amount is given in (5.3.17) in closed-form:

$$X_t^* = \frac{T-t}{T} \left[X_0 - \frac{\alpha T}{4\Gamma} \int_0^t S_u du \right].$$

Note that X_t^* can be negative when $\int_0^t S_u du$ is large or α/Γ is positive and large (see Figure 1 in [34] for a numerical illustration). A negative X_t^* implies that the investor liquidates all assets in the portfolio before time T and continues on with borrowing and selling (short-selling). Since $X_T = 0$ is required, at some point, the investor must buy back some assets to close the short-selling position. When α is positive and large in contrast to Γ and/or when the asset price becomes large, due to the $\alpha X_t S_t$ term in (5.2.3), it could be optimal to short-sell to get a negatively large $\alpha X_t S_t$ term since we are minimizing the expression in (5.2.3). On the other hand, short-selling increases the speed of the liquidation process. When α is small in contrast to Γ and/or when the asset price is small, the $\Gamma \xi_t^2$ term in (5.2.3) may prevent the investor from short-selling.

When quadratic variation is used as the risk measure, Corollary 5.3.3 shows that

$$-\bar{X}_n^* \leq y_n^* \leq 0, \quad 0 \leq n < N.$$

This simply implies that short-selling and buying are never optimal in this case. When the drift in the geometric Brownian motion model (5.3.6) is zero, since $X_T = 0$, we have that,

$$E \left[\int_0^T S_t \xi_t dt \right] = E \left[\int_0^T S_t dX_t \right] = -S_0 X_0 - E \left[\int_0^T X_t dS_t \right] = -S_0 X_0.$$

It can then be seen from the expression in (5.2.2) that high trading rate of both positive and negative signs and high holding amount of both signs will be penalized. This leads to a no short-selling optimal policy. However, when the drift is not zero, short-selling could still be optimal.

5.4 The Arithmetic Brownian Motion Model

The computational cost of the binomial method proposed in Section 5.3 grows exponentially as the number of assets in the portfolio increases. In this section, we assume a multi-dimensional arithmetic Brownian motion instead. Since liquidation duration is normally short, the arithmetic Brownian motion model provides a reasonable substitution for the geometric Brownian motion model. In an arithmetic Brownian motion model, the optimal solution is path-independent, making the liquidation problem tractable for high dimensions. Using the arithmetic Brownian motion helps avoid the “curse of dimensionality”, which is often encountered in high-dimensional stochastic control problems.

[62] studies an infinite-horizon multi-asset liquidation problem. Assuming a diagonal temporary price impact matrix, the paper obtains a closed-form relation between the trading rate and the holding amount under the mean-variance setting. We relax the diagonal assumption on the temporary price impact matrix and solve a finite-horizon problem under the quadratic variation setting.

As described in Section 5.2, the unaffected asset prices in the arithmetic Brownian motion model are governed by

$$dS_t^i = \mu_i S_0^i dt + \sigma_i S_0^i dB_t^i, \quad \text{corr}(B_t^i, B_t^j) = \rho_{ij}, \quad 1 \leq i, j \leq m.$$

The cost function to be minimized is given in (5.2.5). Denote $R = \text{diag}(S_0) \Sigma \text{diag}(S_0)$. Divide the liquidation horizon into N equal subintervals, each with length $\delta = T/N$. As in Section 5.3.1, we get the following dynamic programming equation:

$$V_n(\bar{X}_n, \bar{S}_n) = \min_{y_n} \left(E[V_{n+1}(\bar{X}_n + y_n, \bar{S}_{n+1}) | \bar{X}_n, \bar{S}_n] + \frac{1}{\delta} y_n^\top \Gamma y_n + y_n^\top \bar{S}_n + \alpha \delta X_n^\top R X_n \right), \quad (5.4.1)$$

where $y = \{y_n, 0 \leq n < N\}$ and $\bar{X} = \{\bar{X}_n, 0 \leq n \leq N\}$ are defined in the same way as in Section

5.3.1, and $\bar{S}_n^i = S_0^i + \mu_i S_0^i n \delta + \sigma_i S_0^i B_{n\delta}^i$ for $0 \leq n \leq N$. The terminal value is $V_N(\bar{X}_N, \bar{S}_N) = 0$. In the following, we show that the optimization problems in (5.4.1) are strictly convex quadratic programs and hence admit closed-form solutions.

Define $m \times m$ matrices a_{N-1} and d_{N-1} , m -vectors b_{N-1} and e_{N-1} , and scalar c_{N-1} as follows:

$$a_{N-1} = \frac{1}{\delta} \Gamma + \alpha \delta R, \quad b_{N-1} = (0, \dots, 0)^\top, \quad c_{N-1} = 0, \quad d_{N-1} = -I_{m \times m}, \quad e_{N-1} = (0, \dots, 0)^\top. \quad (5.4.2)$$

Let $\eta = \text{diag}(S_0) \mu$, where $\mu = (\mu_1, \dots, \mu_m)^\top$. For any $0 \leq n < N - 1$, define

$$\begin{aligned} a_n &= -a_{n+1} \left(\frac{1}{\delta} \Gamma + a_{n+1} \right)^{-1} a_{n+1} + a_{n+1} + \alpha \delta R \\ b_n &= b_{n+1} - \delta \eta - a_{n+1} \left(\frac{1}{\delta} \Gamma + a_{n+1} \right)^{-1} (b_{n+1} - \delta \eta) \\ c_n &= c_{n+1} - \frac{1}{4} (b_{n+1} - \delta \eta)^\top \left(\frac{1}{\delta} \Gamma + a_{n+1} \right)^{-1} (b_{n+1} - \delta \eta) \\ d_n &= - \left(\frac{1}{\delta} \Gamma + a_{n+1} \right)^{-1} a_{n+1} \\ e_n &= - \frac{1}{2} \left(\frac{1}{\delta} \Gamma + a_{n+1} \right)^{-1} (b_{n+1} - \delta \eta). \end{aligned} \quad (5.4.3)$$

Proposition 5.4.1. *The optimization problems in the dynamic programming equation (5.4.1) are strictly convex quadratic programs that admit the following solution: for any $0 \leq n < N$,*

$$V_n(\bar{X}_n, \bar{S}_n) = \bar{X}_n^\top a_n \bar{X}_n + \bar{X}_n^\top (b_n - \bar{S}_n) + c_n.$$

The optimal trading amount is an affine function of the optimal holding amount:

$$y_n^* = d_n \bar{X}_n^* + e_n.$$

The coefficients a_n, b_n, c_n, d_n and e_n can be computed recursively from (5.4.2)-(5.4.3). The optimal trading strategy is static: y_n^ does not depend on the asset price at time $n\delta$.*

The proof can be found in the appendix. Proposition 5.4.1 enables us to find the optimal trading strategy $y^* = \{y_n^*, 0 \leq n < N\}$ easily. In particular, the optimal trading amount y_n^* does not depend on the asset price at time $n\delta$. The optimal strategy is static and can be computed at time 0. In the discrete arithmetic Brownian motion model of [2], it is shown that the optimal

trading strategy is static under the mean-variance setting. Proposition 5.4.1 shows that the optimal strategy is also static under the quadratic variation setting.

When the drift of the arithmetic Brownian motion is zero (i.e., $\mu = 0$), it is easy to see from (5.4.2) and (5.4.3) that $b_n = c_n = e_n = 0$ for any $0 \leq n < N$. In particular, for a risk neutral investor with $\alpha = 0$, it can be easily shown that

$$a_n = \frac{1}{(N-n)\delta}\Gamma, \quad d_n = -\frac{1}{N-n}I_{m \times m}, \quad y_n^* = -\frac{1}{N-n}\bar{X}_n^*.$$

As in Corollary 5.3.2, the optimal trading rate is constant: $\bar{\xi}_t^* = X_0/T$. We thus obtain the following corollary.

Corollary 5.4.2. *In the arithmetic Brownian motion model with zero drift and quadratic variation as the risk measure, the dynamic programming equation (5.4.1) admit the following solution: for any $0 \leq n < N$,*

$$V_n(\bar{X}_n, \bar{S}_n) = \bar{X}_n^\top a_n \bar{X}_n - \bar{X}_n^\top \bar{S}_n, \quad y_n^* = d_n \bar{X}_n^*,$$

where a_n and d_n can be computed recursively from (5.4.2)-(5.4.3). For a risk neutral investor, the optimal trading rate is constant: $\bar{\xi}_t^* = X_0/T$.

In the single asset case with zero drift, we have

$$-1 \leq d_n < 0, \quad y_n^* = d_n \bar{X}_n^*, \quad 0 \leq n < N.$$

It follows that $-\bar{X}_n^* \leq y_n^* \leq 0$ for any $0 \leq n < N$. Short-selling and buying are therefore never optimal. Since $R = \sigma^2 S_0^2$,

$$a_{N-1} = \frac{1}{\delta}\Gamma + \alpha\delta\sigma^2 S_0^2, \quad a_n = \frac{\Gamma a_{n+1}}{\Gamma + \delta a_{n+1}} + \alpha\delta\sigma^2 S_0^2, \quad d_n = -\frac{\delta a_{n+1}}{\Gamma + \delta a_{n+1}}, \quad 0 \leq n < N-1.$$

It can be seen immediately that for any $0 \leq n < N$, d_n decreases when α , σ or S_0 increases. Moreover,

$$\frac{a_{N-1}}{\Gamma} = \frac{1}{\delta} + \frac{1}{\Gamma}\alpha\delta\sigma^2 S_0^2, \quad \frac{a_n}{\Gamma} = \frac{a_{n+1}/\Gamma}{1 + \delta a_{n+1}/\Gamma} + \frac{1}{\Gamma}\alpha\delta\sigma^2 S_0^2, \quad d_n = -\frac{\delta a_{n+1}/\Gamma}{1 + \delta a_{n+1}/\Gamma}, \quad 0 \leq n < N-1.$$

When Γ increases, a_n/Γ decreases and d_n increases. In summary, we have the following corollary.

Corollary 5.4.3. *In the single-asset arithmetic Brownian motion model with zero drift and quadratic variation as the risk measure, short-selling and buying are never optimal. The initial trading rate $\bar{\xi}_0^*$ decreases when α , σ or S_0 increases, or when Γ decreases.*

Corollary 5.4.3 shows that, in the single-asset arithmetic Brownian motion model with zero drift, when Γ increases, $\bar{\xi}_0^*$ increases. Note that $\bar{\xi}_0^* < 0$. It implies that the liquidation process is smoother with a smaller initial trading speed for an illiquid asset. However, when α increases, the market risk becomes more of a concern, and the initial trading speed becomes larger. Similarly, when σ is larger, the asset is more volatile. The increasing market risk requires that we liquidate faster. The initial trading speed therefore also becomes larger. Finally, everything being equal, liquidation is faster with a larger initial speed for an asset that is more expensive.

In the single-asset arithmetic Brownian motion model with zero drift and quadratic variation as the risk measure, the closed-form solution of the optimal liquidation problem is derived in [28]. The optimal initial trading rate is given by

$$\xi_0^* = -\sigma S_0 \sqrt{\frac{\alpha}{\Gamma}} X_0 \coth\left(\sigma S_0 T \sqrt{\frac{\alpha}{\Gamma}}\right), \quad (5.4.4)$$

where $\coth(x) = (1 + e^{-2x})/(1 - e^{-2x})$ is the hyperbolic cotangent function. Note that $x \coth(x)$ is an increasing function of x for $x > 0$. We can thus verify directly that ξ_0^* increases when Γ increases, and decreases when α or σ or S_0 increases. The existence of the closed-form solution (5.4.4) also allows us to examine the convergence of our numerical approach as the number of time steps increases. Numerical experiments in Section 5.5 show that $\bar{\xi}_0^*$ converges to ξ_0^* at rate $1/N$.

5.5 Numerical Examples

In this section, numerical experiments are conducted to test the convergence of the algorithm, verify the analytical properties of the optimal liquidation strategy, and investigate how other factors such as cross impact and correlation affect the optimal liquidation strategy.

5.5.1 Convergence

Theoretical convergence of the Markov chain approximation approach can be found in [44] and references cited therein, as well as in some recent works such as [66]. In this section, we examine

the convergence of the method numerically. In the single-asset geometric Brownian motion model with zero drift and time-averaged value at risk as the risk measure, the optimal liquidation problem admits an analytical solution. We have seen in Section 5.3.3 that

$$\bar{\xi}_0^* - \xi_0^* = \frac{\alpha S_0 T}{4\Gamma N}$$

The initial trading rate $\bar{\xi}_0^*$ we obtain using the Markov chain approximation approach converges to the true value ξ_0^* at rate $1/N$. The analytical solution for the optimal liquidation problem is also known in the single-asset arithmetic Brownian motion model with zero drift and quadratic variation as the risk measure. This provides another rare case that allows us to examine the convergence.

Example 5.5.1. *Consider a single-asset arithmetic Brownian motion model with zero drift and quadratic variation as the risk measure. The initial asset price is $S_0 = 10$, the initial holding amount is $X_0 = 1$ (e.g., million), the volatility is $\sigma = 0.3$, the temporary price impact parameter is $\Gamma = 0.002$, the risk aversion parameter is $\alpha = 1$ and the length of the liquidation horizon is $T = 1/50$ (one week roughly). Using (5.4.4), the optimal initial trading rate can be computed to be $\xi_0^* = -76.9231$.*

N	Relative Error
10	0.0560
20	0.0286
40	0.0145
80	0.0073
160	0.0036
320	0.0018
640	0.0009
1280	0.0005

Table 5.1: Convergence of the optimal initial trading rate.

The first column in Table 5.1 contains the number of time steps. The second column contains the relative error, which is defined to be $|(\bar{\xi}_0^* - \xi_0^*)/\xi_0^*|$. From Table 5.1, it can be seen clearly that the error halves when N doubles, indicating a convergence rate of $1/N$.

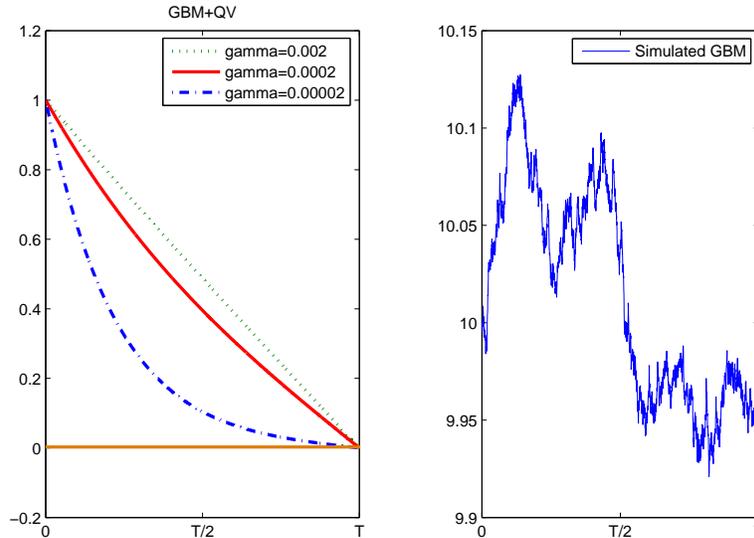


Figure 5.1: Effects of the temporary price impact.

5.5.2 Effects of Price Impact and Risk Aversion

In the following examples, we numerically illustrate the effects of price impact and risk aversion on the optimal trading strategy in the single-asset geometric Brownian motion model with zero drift and quadratic variation as the risk measure. The corresponding analytical results can be found in Section 5.3.4. In Example 5.5.2, we simulate a geometric Brownian motion and plot the corresponding optimal holding amount for different price impact parameters.

Example 5.5.2. *Consider a single-asset geometric Brownian motion model with zero drift and quadratic variation as the risk measure. The initial asset price is $S_0 = 10$, the initial holding amount is $X_0 = 1$, the volatility is $\sigma = 0.1$, the risk aversion parameter is $\alpha = 1$ and the length of the liquidation horizon is $T = 1/50$. The temporary price impact parameter varies from $\gamma = 0.002$, to 0.0002, to 0.00002. The number of time steps used in the binomial method is $N = 2560$.*

It can be seen from Figure 5.1 that the optimal trading strategy has a higher initial trading speed for a smaller temporary price impact. This result is consistent with Proposition 5.3.5. When the price impact is large, we see a nearly straight line for the optimal holding amount, implying a very smooth trading strategy with a nearly constant trading rate. That is, with everything else being equal, we prefer a smoother trading strategies for illiquid assets to reduce the trading cost.

In Example 5.5.3, we plot the optimal trading amount for different risk aversion parameters.

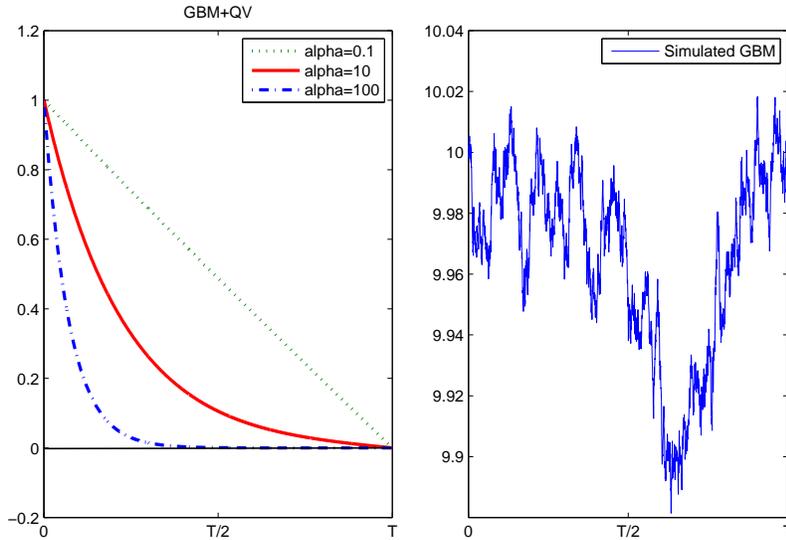


Figure 5.2: Effects of the risk aversion parameter.

Example 5.5.3. Consider a single-asset geometric Brownian motion model with zero drift and quadratic variation as the risk measure. The initial asset price is $S_0 = 10$, the initial holding amount is $X_0 = 1$, the volatility is $\sigma = 0.1$, the temporary price impact parameter is $\Gamma = 0.002$ and the length of the liquidation horizon is $T = 1/50$. The risk aversion parameter varies from $\alpha = 0.1$, to $\alpha = 10$, to $\alpha = 100$. The number of time steps used in the binomial method is $N = 2560$.

Figure 5.2 verifies Proposition 5.3.6. For an investor who is more risk-averse with larger α , the optimal liquidation strategy starts with a larger initial trading speed in order to reduce the trading risk. Conversely, for an investor that is more risk neutral, the optimal trading strategy is smoother to reduce the trading cost.

5.5.3 Effects of Cross Impact and Correlation

The trading of one asset may impact the trading of another, even when they are uncorrelated. It is thus interesting to see how the cross impact affects the optimal liquidation strategy. For correlated assets, it is also interesting to see how the correlation impact the optimal execution. We conduct two numerical experiments to visualize the effects of cross-impact and correlation on the optimal liquidation strategy.

Example 5.5.4. Consider a two-asset arithmetic Brownian motion model with zero drift and

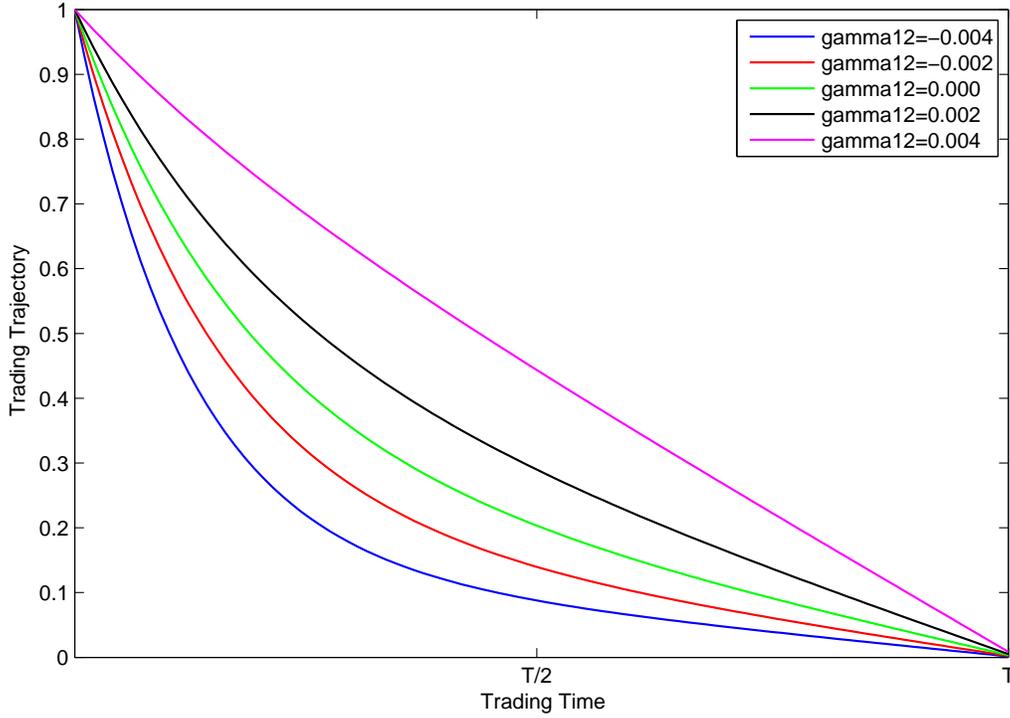


Figure 5.3: Effects of cross impact.

quadratic variation as the risk measure. The initial asset price is $S_0 = (10, 10)^\top$. The covariance matrix Σ and the temporary price impact matrix Γ are given by

$$\Sigma = \begin{pmatrix} 0.08 & 0 \\ 0 & 0.06 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0.006 & \gamma_{12} \\ \gamma_{12} & 0.008 \end{pmatrix}.$$

The risk-aversion parameter is $\alpha = 0.5$ and the length of the trading horizon is $T = 1/250$ (one day roughly). The cross-impact parameter γ_{12} varies from -0.004 to 0.004 with a step size of 0.002 . The number of time steps used is $N = 2560$.

In Figure 5.3, we plot the optimal trading trajectories of the second asset under different cross-impact parameters. The pattern for the first asset is similar. It can be seen that positive cross-impact slows down the liquidation while negative cross-impact accelerates the process. Having positive cross-impact means that selling one asset will adversely affect the price of the other asset. Everything else being equal, a smoother trading strategy is preferred to reduce the trading cost.

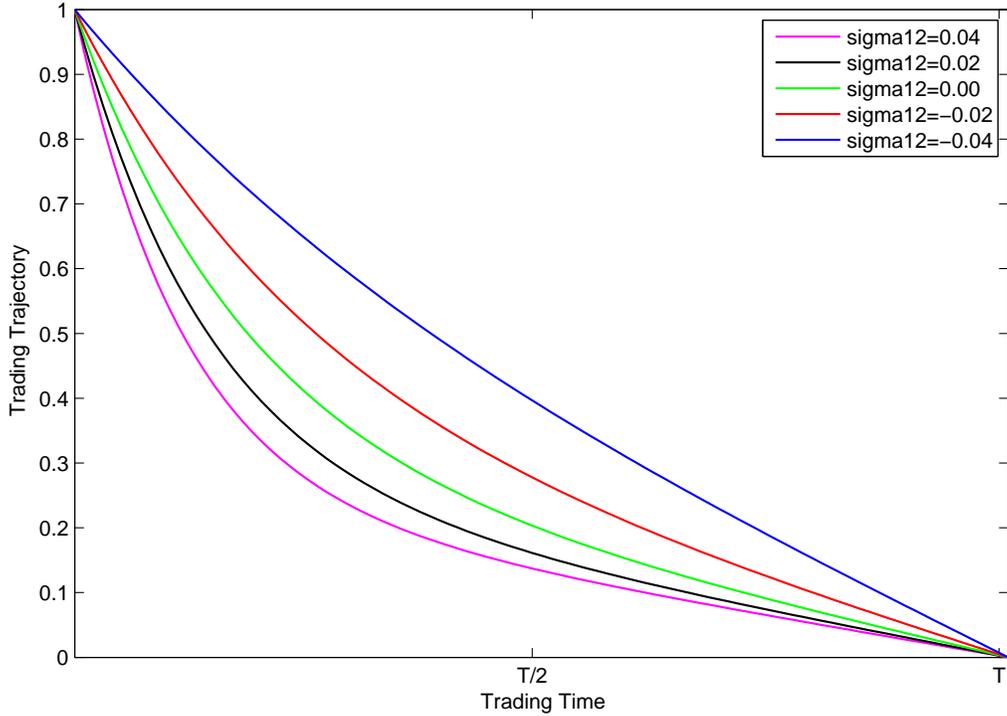


Figure 5.4: Effects of correlation.

Example 5.5.5. Consider a two-asset arithmetic Brownian motion model with zero drift and quadratic variation as the risk measure. The initial asset price is $S_0 = (10, 10)^\top$. The covariance matrix Σ and the temporary price impact matrix Γ are given by

$$\Sigma = \begin{pmatrix} 0.08 & \sigma_{12} \\ \sigma_{12} & 0.06 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0.006 & 0.000 \\ 0.000 & 0.008 \end{pmatrix}.$$

The risk-aversion parameter is $\alpha = 0.5$ and the length of the liquidation horizon is $T = 1/250$. σ_{12} varies from -0.04 to 0.04 with a step size of 0.02 . The number of time steps used is $N = 2560$.

In Figure 5.4, we plot the optimal trading trajectories of the second asset under different correlation parameters. We can see that, as opposed to the effects of cross-impact, positive correlation accelerates the liquidation while negative correlation slows down the process. Intuitively, positively correlated assets suffers from higher risk exposure. It is thus preferred to liquidate faster to reduce the trading risk. Conversely, due to the diversification effect, negatively correlated assets lead to smoother trading strategies to reduce the trading cost.

The above examples show that both cross-impact and correlation affect the liquidation of a multi-asset portfolio. It is thus important to take cross-impact and correlation into consideration in practical applications.

5.6 Summary

In this chapter, we consider the optimal liquidation of a multi-asset portfolio where the objective is to minimize trading cost and trading risk. We propose a simple and effective Markov chain approximation approach for the stochastic control problem. The optimal liquidation problem in the multi-dimensional geometric Brownian motion model can be implemented using a simple binomial method. On the other hand, the arithmetic Brownian motion model is numerically tractable for high dimensions. The Markov chain approximation approach allows us to not only numerically obtain the optimal trading strategy, but also analytically study the effects of factors such as risk measure, price impact, risk aversion and initial asset price on the optimal strategy. Numerical experiments further illustrate the effects of cross impact and correlation on the optimal liquidation of a multi-asset portfolio and the importance of taking these factors into account in practical applications.

Chapter 6

Conclusions and Future Extensions

During a time of financial difficulties, institutional investors often need to unwind their portfolios to resolve financial distress, such as reducing leverage ratio or meeting margin call. More often, for financial institutions, both the number and the amount of the assets in the portfolios are large. So a key issue is to determine which portion of the portfolio should be sold, namely, which assets should be selected for liquidation and how many shares should be liquidated. In Chapter 2-4, we propose portfolio deleveraging models that seek liquidation strategy under different concerns. For risk-averse investors, they are willing to take into account the volatility risk during the liquidation process. Thus, in Chapter 5, we introduce another portfolio liquidation model that returns liquidation trajectory under different risk measures. When liquidating a large portfolio, one must take price impact into consideration. More specifically, our theoretical and numerical results show that market price impact plays an important role in designing the optimal liquidation strategy.

In Chapter 2, we consider an optimal portfolio deleveraging problem under linear market impact functions, where the objective is to meet specified debt/equity requirements at the minimal execution cost. Mathematically, the optimal deleveraging problem is a non-convex quadratic program with quadratic and box constraints. A Lagrangian method is proposed to solve the non-convex quadratic program numerically. By studying the breakpoints of the Lagrangian problem, we obtain conditions under which the Lagrangian method returns an optimal solution of the deleveraging problem. When the Lagrangian algorithm returns a suboptimal approximation, we present upper bounds on the loss in equity caused by using such an approximation. In Chapter 3, we further extend the Lagrangian algorithm proposed in Chapter 2 to the deleveraging problem under a non-

linear temporary price impact function, more specifically, a power-law temporary impact function with an exponent between 0 and 1. Similar as in Chapter 2, we characterize conditions under which the algorithm returns an optimal liquidation strategy and derive bounds on the loss of objective value when the algorithm returns a suboptimal strategy.

In Chapter 4, we propose a two-period robust optimization model for portfolio deleveraging under margin requirement. The primary motivation is to study how the asset return and volatility influence the optimal trading strategy. The objective is to meet the margin requirement with the least change in asset positions. Margin requirement is a hard requirement that must be satisfied within a certain time allowance. Thus, a robust strategy is needed so that enough cash can be generated for a certain range of market conditions. We consider two types of portfolios respectively: portfolios consisting of only basic assets and portfolios containing both basic assets and derivative securities. The first period return of basic assets is assumed to belong to a scaled ellipsoid. For the derivative return, we assume it to be a quadratic function (i.e., a delta-gamma approximation) of the underlying asset return. Depending on whether there is derivative security in the portfolio, the robust optimization program is then converted to either a second-order cone program or a semidefinite program, both of which are computationally tractable.

In Chapter 5, we further consider a portfolio execution problem where the objective is to find an optimal trading strategy that minimizes both the trading cost and the trading risk. The asset price is assumed to follow a multi-dimensional arithmetic or geometric Brownian motion. Both quadratic variation and time-averaged value at risk are considered as risk measures. We propose a Markov chain approximation approach to obtain the optimal trading trajectory where the Markov chain is based on a multi-dimensional binomial tree. We also analyze theoretically the influence of factors such as risk measure, price impact, risk aversion and initial asset price on the optimal execution strategy. The portfolio liquidation problem in the arithmetic Brownian motion case reduces to a linear quadratic Gaussian control problem that is numerically tractable when the number of assets in the portfolio is large.

Apart from the mathematical models and computational algorithms, another important component of the dissertation is the analytical study of the trading properties, which can provide guidance on how to design trading policies. We have the following primary findings:

1. In a permanent-impact-dominant market where the earlier execution imposes large impact

on the later transaction, the linearity assumption of temporary impact function leads to an extreme trading strategy: all the assets except one are either sold or retained completely. While in the case of a nonlinear temporary price impact function, the trading strategy is comparatively moderate and complicated.

2. In the one-period liquidation case, it is optimal to give liquidation priority to the more liquid assets; in the two-period liquidation case, it might be optimal to first retain the liquid assets to avoid the drainage of the market liquidity.
3. Interactions between assets: positive cross-impact slows down the liquidation while negative cross-impact accelerates the process; positive correlation speeds up the liquidation while negative correlation slows down the process.
4. For options with the same underlying asset, we give trading priority to options with high Theta and Gamma. When the underlying asset has a negative return, we prefer to sell call options with smaller Delta; when the underlying asset has a positive return, we prefer to sell put options with larger Delta.

The financial institutions are connected. The deleveraging activity of one institution may further lead to the deleveraging processes of other institutions that hold similar assets. Therefore, it is promising to study the scenario of multi-agent simultaneous deleveraging, which helps us analyze how institutions interact with each other and how they respond to others' liquidation strategies. In addition, from the perspective of financial stability, the massive deleveraging may cause a big shock to the whole system that needs to be well investigated to prevent economic collapse. The portfolio liquidation problem studied in Chapter 5 assumes that the asset price follows diffusion process, i.e., the multi-variate geometric or arithmetic Brownian motion. Since the liquidation period is usually short, the market shock should also be taken into account. Therefore, obtaining the optimal liquidation strategy under a jump-diffusion process for the assets is also desirable.

Appendix

Proposition 2.3.1

Proof. Denote the optimal solution as y^* . Assume to the contrary that the constraint is not active at the optimal solution, then the linear independence constraint qualification (LICQ) holds at y^* .¹ According to the first order optimality condition, there exists $\mu^* = (\mu_0^*, \mu_1^*, \dots, \mu_m^*, \mu_{m+1}^*, \dots, \mu_{2m}^*) \geq 0$ satisfying the following conditions:

$$\mu_0^*(\rho_1 e_1 - l_1) = 0 \quad (6.0.1)$$

$$\mu_i^* y_i^* = 0, \quad i = 1, \dots, m \quad (6.0.2)$$

$$\mu_{m+i}^*(x_{0,i} - y_i^*) = 0, \quad i = 1, \dots, m \quad (6.0.3)$$

$$\nabla e_1(y^*) - \mu_0^* \nabla f_1(y^*) + \sum_{i=1}^m (\mu_i^* \nabla g_i(y^*) + \mu_{m+i}^* \nabla g_{m+i}(y^*)) = 0, \quad (6.0.4)$$

where

$$g_i(y) = y_i, \quad g_{m+i}(y) = x_{0,i} - y_i, \quad i = 1, \dots, m.$$

If the leverage constraint is inactive (i.e. $\rho_1 e_1 > l_1$), we have $\mu_0^* = 0$ from (6.0.1). For $i = 1, \dots, m$, equation (6.0.4) can be rewritten as

$$\gamma_i y_i^* - (k+1) \lambda_i y_i^{*k} - \gamma_i x_{0,i} + \mu_i^* - \mu_{m+i}^* = 0. \quad (6.0.5)$$

For each asset i , it can be seen from (6.0.2) and (6.0.3), μ_i^* and μ_{m+i}^* can not be non-zero simultaneously. Therefore, we only need to consider the following three possible cases.

(1) $\mu_i^* = 0, \mu_{m+i}^* \neq 0$: From (6.0.3), we obtain $y_i^* = x_{0,i}$. According to equation (6.0.5), we get $\mu_{m+i}^* = -(k+1) \lambda_i x_{0,i}^k < 0$, contradicting to $\mu^* \geq 0$.

(2) $\mu_i^* = \mu_{m+i}^* = 0$: By equation (6.0.5), we obtain $\gamma_i(x_{0,i} - y_i^*) = -(k+1) \lambda_i y_i^{*k}$. Since $\gamma_i(x_{0,i} - y_i^*) \geq 0$ and $-(k+1) \lambda_i y_i^{*k} \leq 0$, we have $\gamma_i(x_{0,i} - y_i^*) = -(k+1) \lambda_i y_i^{*k} = 0$. That is, $y_i^* = x_{0,i}$ and $y_i^* = 0$, which itself is a contradiction.

¹LICQ holds if the set of gradients of active constraints is linearly independent

(3) $\mu_i^* \neq 0, \mu_{m+i}^* = 0$ From (6.0.2), y_i^* should be 0.

Summarizing above arguments, we conclude that $y^* = \mathbf{0}$, which contradicts to the assumption that deleveraging is required. Therefore, $\mu_0^* > 0$ and $\rho_1 e_1 = l_1$. This finishes the proof of the proposition. \square

Proposition 2.2.1.

Proof. Since $(q + \Gamma x_1)^T x_1 = \int_0^1 (q^T y_t + 2x_t \Gamma y_t) dt + q^T x_0 + x_0 \Gamma x_0 = \int_0^1 (q^T y_t + 2x_t \Gamma y_t) dt + p_0^T x_0$, we rewrite the problem into a more simplified form

$$\begin{aligned} & \max && \int_0^1 (\Phi - \Psi) dt \\ & \text{subject to} && \int_0^1 [\rho_1 \Phi - (\rho_1 + 1) \Psi] dt \geq (\rho_1 + 1) l_0 - \rho_1 p_0^T x_0 \\ & && \dot{x}_t = y_t \\ & && y_t \leq 0 \\ & && x_1 \geq 0, \end{aligned}$$

where $\Phi = q^T y_t + 2x_t \Gamma y_t$ and $\Psi = (q + \Gamma x_t + \Lambda y_t)^T y_t$.

Denote $M = \Phi - \Psi$ and $N = \rho_1 \Phi - (\rho_1 + 1) \Psi$. From the first-order necessary condition (Chapter 2 in [46]), it follows that

$$\int_0^1 (M_x - \frac{d}{dt} M_y) \eta(t) dt = 0 \quad \forall \eta \text{ such that } \int_0^1 (N_x - \frac{d}{dt} N_y) \eta(t) dt \geq 0, \quad (6.0.6)$$

where $\eta(t)$ is the variation of function x_t . The above first-order necessary condition implies that

$$\int_0^1 (M_x - \frac{d}{dt} M_y) \eta(t) dt = 0 \quad \forall \eta \text{ such that } \int_0^1 (N_x - \frac{d}{dt} N_y) \eta(t) dt = 0. \quad (6.0.7)$$

Let $L = z_0 M + z_1 N$, z_0 and z_1 cannot be zero simultaneously. Then we have $L_x - \frac{d}{dt} L_y = 0$. Solving the equation, we get $\frac{d}{dt} y = 0$. Thus, $y_t^* = c$ is the solution of (6.0.7). We further verify that $N_x - \frac{d}{dt} N_y = 0$ at $y_t^* = c$. Thus, $y_t^* = c$ is also the solution of (6.0.6). Therefore, the optimal trading rate is a constant. \square

Proposition 3.2.1.

Proof. The proof is similar to that of Proposition 2.3.1. Assume to the contrary that the margin constraint is not active at the optimal solution. Then linear independence constraint qualification (LICQ) holds. According to the first order optimality condition, there exists $\mu^* = (\mu_0^*, \mu_1^*, \dots, \mu_m^*, \mu_{m+1}^*, \dots, \mu_{2m}^*) \geq 0$ satisfying the following conditions:

$$\mu_0^*(y_1^{*\top} \Lambda y_1^* + p_1^\top y_1^* + A) = 0 \quad (6.0.8)$$

$$\mu_i^* y_i^* = 0, \quad i = 1, \dots, m \quad (6.0.9)$$

$$\mu_{m+i}^*(y_i^* + x_{0,i}) = 0, \quad i = 1, \dots, m \quad (6.0.10)$$

$$-e + \mu_0^*(\Lambda y^* + p_1) + \sum_{i=1}^m (\mu_i^* \nabla g_i(y^*) + \mu_{m+i}^* \nabla g_{m+i}(y^*)) = 0, \quad (6.0.11)$$

where e is a all-one vector,

$$g_i(y) = y_i, \quad g_{m+i}(y) = -y_i - x_{0,i}, \quad i = 1, \dots, m.$$

By the assumption that the margin constraint is not active, we have $\mu_0^* = 0$. Consequently, for any $i = 1, \dots, m$, equation (6.0.11) becomes

$$-1 + \mu_i^* - \mu_{m+i}^* = 0, \quad i = 1, \dots, m. \quad (6.0.12)$$

Since $x_0 > 0$, it is easy to see from (6.0.9) and (6.0.10) that μ_i^* and μ_{m+i}^* , $1 \leq i \leq m$, cannot be positive simultaneously. Due to the non-negativity of these Lagrangian multipliers, we can only have $\mu_i^* = 1$ and $\mu_{m+i}^* = 0$, for $i = 1, \dots, m$. Thus, $y^* = 0$ from equation (6.0.9). $y^* = 0$ is not a feasible solution and hence contradiction arises. This finishes the proof of the proposition. \square

Proposition 4.3.1.

Proof. The proof is similar to Proposition 2.3.1. But due to the L-2 norm in the margin constraint, we need to show the result in two cases. Assume y_1^* and y_2^* are the optimal trading amounts in the first and second period, respectively. If $y_2^* = 0$, the problem reduces to a special case of (2.2.4). Then the margin constraint is active at the optimal solution according to Proposition 2.3.1. If $y_2^* \neq 0$, then $(p_1 \circ y_2^*)^\top \Sigma(p_1 \circ y_2^*) > 0$. Thus, the margin constraint is differentiable at y_2^* . The remaining of the proof follows in the same way as Proposition 2.3.1. \square

Lemma 6.0.1. *If A and B are real positive definite matrices, then $-A(A+B)^{-1}A+A$ is also positive definite.*

Proof. Since B is symmetric and positive definite, there exists an upper triangular matrix U^\top of the same size with strictly positive diagonal entries such that $B = UU^\top$. According to the Sherman-Morrison-Woodbury formula [64] and [71], we have

$$(A+B)^{-1} = A^{-1} - A^{-1}U(I+U^\top A^{-1}U)^{-1}U^\top A^{-1}.$$

It follows that

$$-A(A+B)^{-1}A+A = U(I+U^\top A^{-1}U)^{-1}U^\top,$$

which is positive definite since U is invertible and $(I+U^\top A^{-1}U)^{-1}$ is positive definite. \square

Proposition 5.3.1.

Proof. From (5.3.9), we have the following for V_{N-1} for any $x, 0 \leq s \in \mathbb{R}^m$:

$$V_{N-1}(x, s) = x^\top a_{N-1}(s) x + x^\top b_{N-1}(s) + c_{N-1}(s). \quad (6.0.13)$$

Moreover, since Γ is positive definite and Σ is semi-positive definite, $a_{N-1}(s)$ is positive definite.

According to (5.3.4),

$$\begin{aligned} V_{N-2}(x, s) &= \min_{y_{N-2}} \left(E [V_{N-1}(x + y_{N-2}, \bar{S}_{N-1}) | \bar{S}_{N-2} = s] \right. \\ &\quad \left. + \frac{1}{\delta} y_{N-2}^\top \Gamma y_{N-2} + s^\top y_{N-2} + \alpha \delta x^\top \text{diag}(s) \Sigma \text{diag}(s) x \right) \\ &= \min_{y_{N-2}} \left((x + y_{N-2})^\top E [a_{N-1}(\bar{S}_{N-1}) | \bar{S}_{N-2} = s] (x + y_{N-2}) \right. \\ &\quad \left. + (x + y_{N-2})^\top E [b_{N-1}(\bar{S}_{N-1}) | \bar{S}_{N-2} = s] + E [c_{N-1}(\bar{S}_{N-1}) | \bar{S}_{N-2} = s] \right. \\ &\quad \left. + \frac{1}{\delta} y_{N-2}^\top \Gamma y_{N-2} + s^\top y_{N-2} + \alpha \delta x^\top \text{diag}(s) \Sigma \text{diag}(s) x \right) \\ &= \min_{y_{N-2}} \left(y_{N-2}^\top \left(\frac{1}{\delta} \Gamma + E [a_{N-1}(\bar{S}_{N-1}) | \bar{S}_{N-2} = s] \right) y_{N-2} \right. \\ &\quad \left. + y_{N-2}^\top \left(2E [a_{N-1}(\bar{S}_{N-1}) | \bar{S}_{N-2} = s] x + E [b_{N-1}(\bar{S}_{N-1}) | \bar{S}_{N-2} = s] + s \right) \right. \\ &\quad \left. + x^\top (E [a_{N-1}(\bar{S}_{N-1}) | \bar{S}_{N-2} = s] + \alpha \delta \text{diag}(s) \Sigma \text{diag}(s)) x \right. \\ &\quad \left. + x^\top E [b_{N-1}(\bar{S}_{N-1}) | \bar{S}_{N-2} = s] + E [c_{N-1}(\bar{S}_{N-1}) | \bar{S}_{N-2} = s] \right). \end{aligned}$$

Since $a_{N-1}(s)$ is positive definite for any s , so is $E [a_{N-1}(\bar{S}_{N-1})|\bar{S}_{N-2} = s]$. The following matrix

$$\frac{1}{\delta}\Gamma + E [a_{N-1}(\bar{S}_{N-1})|\bar{S}_{N-2} = s]$$

is thus also positive definite. The objective function of the optimization problem in V_{N-2} is therefore a strictly convex quadratic function. The unique solution is given by

$$\begin{aligned} y_{N-2}^* &= -\frac{1}{2}\left(\frac{1}{\delta}\Gamma + E [a_{N-1}(\bar{S}_{N-1})|\bar{S}_{N-2} = s]\right)^{-1} \\ &\quad \left(2E [a_{N-1}(\bar{S}_{N-1})|\bar{S}_{N-2} = s] x + E [b_{N-1}(\bar{S}_{N-1})|\bar{S}_{N-2} = s] + s\right) \\ &= d_{N-2}(s) x + e_{N-2}(s). \end{aligned}$$

The corresponding optimal value is

$$\begin{aligned} V_{N-2}(x, s) &= -\frac{1}{4}(2E [a_{N-1}(\bar{S}_{N-1})|\bar{S}_{N-2} = s] x + E [b_{N-1}(\bar{S}_{N-1})|\bar{S}_{N-2} = s] + s)^\top \\ &\quad \left(\frac{1}{\delta}\Gamma + E [a_{N-1}(\bar{S}_{N-1})|\bar{S}_{N-2} = s]\right)^{-1} \\ &\quad (2E [a_{N-1}(\bar{S}_{N-1})|\bar{S}_{N-2} = s] x + E [b_{N-1}(\bar{S}_{N-1})|\bar{S}_{N-2} = s] + s) \\ &\quad + x^\top (E [a_{N-1}(\bar{S}_{N-1})|\bar{S}_{N-2} = s] + \alpha\delta \text{diag}(s) \Sigma \text{diag}(s))x \\ &\quad + x^\top E [b_{N-1}(\bar{S}_{N-1})|\bar{S}_{N-2} = s] + E [c_{N-1}(\bar{S}_{N-1})|\bar{S}_{N-2} = s] \\ &= x^\top a_{N-2}(s) x + x^\top b_{N-2}(s) + c_{N-2}(s). \end{aligned} \tag{6.0.14}$$

Since $E [a_{N-1}(\bar{S}_{N-1})|\bar{S}_{N-2} = s]$ and Γ are positive definite, Σ is semi-positive definite, by Lemma 6.0.1, $a_{N-2}(s)$ is also positive definite. Note that we have only used the fact that $a_{N-1}(s)$ is positive definite and V_{N-1} is of the form (6.0.13) in the above proof. We have shown that V_{N-2} has the same form (6.0.14) and $a_{N-2}(s)$ is positive definite. The proof of the proposition is therefore finished by induction. \square

Proposition 5.3.4.

Proof. Recall the terminal value $V_N = 0$ and $y_{N-1}^* = -\bar{X}_{N-1}$. From (5.3.5), we get

$$V_{N-1}(x, s) = a_{N-1}x^2 + b_{N-1}(s)x + c_{N-1}(s).$$

Now suppose $V_{n+1}(x, s) = a_{n+1}x^2 + b_{n+1}(s)x + c_{n+1}(s)$ for some $0 \leq n < N - 1$ with a_{n+1} given in (5.3.16). We prove the proposition by induction. From (5.3.5), we have

$$\begin{aligned} V_n(x, s) &= \min_{y_n} \left(E[V_{n+1}(x + y_n, \bar{S}_{n+1}) | \bar{S}_n = s] + \frac{1}{\delta} \Gamma y_n^2 + s y_n + \alpha \delta s x \right) \\ &= \min_{y_n} \left(\left(\frac{1}{\delta} \Gamma + a_{n+1} \right) y_n^2 + (2a_{n+1}x + E[b_{n+1}(\bar{S}_{n+1}) | \bar{S}_n = s] + s) y_n \right. \\ &\quad \left. + a_{n+1}x^2 + (E[b_{n+1}(\bar{S}_{n+1}) | \bar{S}_n = s] + \alpha \delta s)x + E[c_{n+1}(\bar{S}_{n+1}) | \bar{S}_n = s] \right). \end{aligned}$$

The optimal trading amount is given by

$$y_n^* = - \frac{2a_{n+1}x + E[b_{n+1}(\bar{S}_{n+1}) | \bar{S}_n = s] + s}{2 \left(\frac{1}{\delta} \Gamma + a_{n+1} \right)} = d_n x + e_n(s).$$

The corresponding optimal value is

$$\begin{aligned} V_n(x, s) &= - \frac{(2a_{n+1}x + E[b_{n+1}(\bar{S}_{n+1}) | \bar{S}_n = s] + s)^2}{4 \left(\frac{1}{\delta} \Gamma + a_{n+1} \right)} \\ &\quad + a_{n+1}x^2 + (E[b_{n+1}(\bar{S}_{n+1}) | \bar{S}_n = s] + \alpha \delta s)x + E[c_{n+1}(\bar{S}_{n+1}) | \bar{S}_n = s] \\ &= a_n x^2 + b_n(s)x + c_n. \end{aligned}$$

When the drift coefficient in (5.3.6) is zero, using (5.3.15), (5.3.16) and (5.3.7), it is easy to show by induction that b_n and e_n are given by the expressions in the proposition. The first part of the proof is then finished by induction. The difference between $\bar{\xi}_0^*$ and ξ_0^* can be easily computed from the analytical expression (5.3.18) for ξ_0^* . \square

Proposition 5.3.5.

Proof. From Corollary 5.3.3, $y_0^* = d_0(S_0)X_0$. It suffices to show that $d_0(S_0)$ determined in (5.3.13)-(5.3.14) is an increasing function of Γ . Note that

$$d_0(S_0) = - \frac{\delta(a_1(uS_0)p_1 + a_1(dS_0)p_2)}{\Gamma + \delta(a_1(uS_0)p_1 + a_1(dS_0)p_2)}$$

$$a_{N-1}(s) = \frac{\Gamma}{\delta} + \alpha \delta \sigma^2 s^2, a_n(s) = \frac{\Gamma(a_{n+1}(us)p_1 + a_{n+1}(ds)p_2)}{\Gamma + \delta(a_{n+1}(us)p_1 + a_{n+1}(ds)p_2)} + \alpha \delta \sigma^2 s^2, 1 \leq n < N - 1.$$

Here u, d, p_1, p_2 are given in (5.3.7). In the following, we show that $a_n(s)/\Gamma$ is a decreasing function

of Γ for any $0 \leq n < N$ and $s \in \mathcal{S}_n$. Note that

$$\frac{1}{\Gamma}a_{N-1}(s) = \frac{1}{\delta} + \frac{1}{\Gamma}\alpha\delta\sigma^2s^2$$

is a decreasing function of Γ . Suppose now that $a_{n+1}(s)/\Gamma$ is a decreasing function of Γ for some $0 \leq n < N - 1$ and any $s \in \mathcal{S}_{n+1}$. Then

$$\frac{1}{\Gamma}a_n(s) = \frac{(a_{n+1}(us)p_1 + a_{n+1}(ds)p_2)/\Gamma}{1 + \delta(a_{n+1}(us)p_1 + a_{n+1}(ds)p_2)/\Gamma} + \frac{1}{\Gamma}\alpha\delta\sigma^2s^2.$$

Since $a_{n+1}(s)/\Gamma$ is decreasing in Γ , we can see from the above that $a_n(s)/\Gamma$ is also decreasing in Γ . By induction, we have that $a_1(s)/\Gamma$ is a decreasing function of Γ . From the expression for d_0 , it is then obvious that $d_0(S_0)$ is an increasing function of Γ . This finishes the proof. \square

Proposition 5.4.1.

Proof. Since Γ is positive definite and R is semi-positive definite, we have that a_{N-1} is positive definite. By Lemma 6.0.1, a_n is positive definite for any $0 \leq n < N$. Recall the terminal value $V_N(\bar{X}_N, \bar{S}_N) = 0$ and $y_{N-1}^* = -\bar{X}_{N-1}$. From the dynamic programming equation (5.4.1), we immediately obtain

$$V_{N-1}(x, s) = x^\top a_{N-1}x + x^\top (b_{N-1} - s) + c_{N-1}.$$

Now suppose $V_{n+1}(x, s) = x^\top a_{n+1}x + x^\top (b_{n+1} - s) + c_{n+1}$ for some $0 \leq n < N - 1$. We show that $V_n(x, s) = x^\top a_nx + x^\top (b_n - s) + c_n$. Note that

$$E[\bar{S}_{n+1} | \bar{S}_n = s] = s + \delta\eta.$$

From the dynamic programming equation (5.4.1), we have

$$\begin{aligned} V_n(x, s) &= \min_{y_n} \left(E[V_{n+1}(x + y_n, \bar{S}_{n+1}) | \bar{S}_n = s] + \frac{1}{\delta}y_n^\top \Gamma y_n + y_n^\top s + \alpha\delta x^\top R x \right) \\ &= \min_{y_n} \left(y_n^\top \left(\frac{1}{\delta}\Gamma + a_{n+1} \right) y_n + y_n^\top (2a_{n+1}x + b_{n+1} - \delta\eta) \right. \\ &\quad \left. + x^\top (a_{n+1} + \alpha\delta R)x + x^\top (b_{n+1} - s - \delta\eta) + c_{n+1} \right). \end{aligned}$$

This is a strictly convex quadratic program. The optimal solution is given by

$$y_n^* = -\frac{1}{2} \left(\frac{1}{\delta} \Gamma + a_{n+1} \right)^{-1} (2a_{n+1}x + b_{n+1} - \delta\eta) = d_n x + e_n.$$

The corresponding optimal value is given by

$$\begin{aligned} V_n(x, s) &= x^\top \left(-a_{n+1} \left(\frac{1}{\delta} \Gamma + a_{n+1} \right)^{-1} a_{n+1} + a_{n+1} + \alpha \delta R \right) x \\ &\quad + x^\top (b_{n+1} - \delta\eta - a_{n+1} \left(\frac{1}{\delta} \Gamma + a_{n+1} \right)^{-1} (b_{n+1} - \delta\eta) - s) \\ &\quad + c_{n+1} - \frac{1}{4} (b_{n+1} - \delta\eta)^\top \left(\frac{1}{\delta} \Gamma + a_{n+1} \right)^{-1} (b_{n+1} - \delta\eta) \\ &= x^\top a_n x + x^\top (b_n - s) + c_n. \end{aligned}$$

The proof is finished by induction. □

Bibliography

- [1] R. Almgren and N. Chriss. Value under liquidation. *Risk*, 12(12):61–63, 1999.
- [2] R. Almgren and N. Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, 3(2):5–39, 2000.
- [3] R. Almgren and J. Lorenz. Adaptive arrival price. *Trading*, 2007(1):59–66, 2007.
- [4] R. Almgren, C. Thum, E. Hauptmann, and H. Li. Direct estimation of equity market impact. *Risk*, 18(7):57–62, 2005.
- [5] S. C. Andrade, C. Chang, and M. S. Seasholes. Trading imbalances, predictable reversals, and cross-stock price pressure. *Journal of Financial Economics*, 88(2):406–423, 2008.
- [6] J.-P. Aubin and H. Frankowska. *Set-valued analysis*. Springer, 2009.
- [7] J. Barth and D. McCarthy. Trading losses: A little perspective on a large problem. *Available at SSRN 2169660*, 2012.
- [8] S. Basak and G. Chabakauri. Dynamic mean-variance asset allocation. *Review of Financial Studies*, 23(8):2970–3016, 2010.
- [9] Basel Committee on Banking Supervision. Revised basel iii leverage ratio framework and disclosure requirements. Technical report, BCBS, 2013.
- [10] A. Ben-Tal and D. Den Hertog. *Immunizing conic quadratic optimization problems against implementation errors*. Tilburg University, 2011.
- [11] D. Bertsimas and A. W. Lo. Optimal control of execution costs. *Journal of Financial Markets*, 1(1):1–50, 1998.

- [12] J.-P. Bouchaud, J. Farmer, and F. Lillo. How markets slowly digest changes in supply and demand. *Fabrizio, How Markets Slowly Digest Changes in Supply and Demand (September 11, 2008)*, 2008.
- [13] P. Boyle, J. Evnine, and S. Gibbs. Numerical evaluation of multivariate contingent claims. *Review of Financial Studies*, 2(2):241–250, 1989.
- [14] D. B. Brown, B. I. Carlin, and M. S. Lobo. Optimal portfolio liquidation with distress risk. *Management Science*, 56(11):1997–2014, 2010.
- [15] B. I. Carlin, M. S. Lobo, and S. Viswanathan. Episodic liquidity crises: Cooperative and predatory trading. *The Journal of Finance*, 62(5):2235–2274, 2007.
- [16] G. C. Chacko, J. W. Jurek, and E. Stafford. The price of immediacy. *The Journal of Finance*, 63(3):1253–1290, 2008.
- [17] J. Chen, L. Feng, and J. Peng. Optimal deleveraging with nonlinear temporary price impact. *European Journal of Operational Research, Under review*, 2014.
- [18] J. Chen, L. Feng, and J. Peng. Portfolio execution with a markov chain approximation approach. *Working Paper*, 2014.
- [19] J. Chen, L. Feng, and J. Peng. Robust portfolio deleveraging under margin call. *Working Paper*, 2014.
- [20] J. Chen, L. Feng, J. Peng, and Y. Ye. Analytical results and efficient algorithm for optimal deleveraging with market impact. *Operations Research*, 62(1):195–206, 2014.
- [21] R. Cont, L. Wagalath, et al. Fire sales forensics: measuring endogenous risk. *Available at SSRN 2051013*, 2012.
- [22] J. Coval and E. Stafford. Asset fire sales (and purchases) in equity markets. *Journal of Financial Economics*, 86(2):479–512, 2007.
- [23] J. Cox, S. Ross, and M. Rubinstein. Option pricing: A simplified approach. *Journal of financial Economics*, 7(3):229–263, 1979.

- [24] A. dAspremont and S. Boyd. Relaxations and randomized methods for nonconvex qcqps. *EE392o Class Notes, Stanford University*, 2003.
- [25] G. Deng, T. Dulaney, and C. McCann. Optimizing portfolio liquidation under risk-based margin requirements. *Available at SSRN 2035562*, 2012.
- [26] K. D’Hulster. The leverage ratio. 2009.
- [27] R. Engle, R. Ferstenberg, and J. Russell. Measuring and modeling execution cost and risk. 2006.
- [28] P. A. Forsyth, J. S. Kennedy, S. Tse, and H. Windcliff. Optimal trade execution: a mean quadratic variation approach. *Journal of Economic Dynamics and Control*, 36(12):1971–1991, 2012.
- [29] M. Fratzscher. Capital flows, push versus pull factors and the global financial crisis. *Journal of International Economics*, 88(2):341–356, 2012.
- [30] X. Gabaix, P. Gopikrishnan, V. Plerou, and H. E. Stanley. A theory of power-law distributions in financial market fluctuations. *Nature*, 423(6937):267–270, 2003.
- [31] X. Gabaix, P. Gopikrishnan, V. Plerou, and H. E. Stanley. Institutional investors and stock market volatility. *The Quarterly Journal of Economics*, 121(2):461–504, 2006.
- [32] J. Gallier. The schur complement and symmetric positive semidefinite (and definite) matrices. *December*, 44:1–12, 2010.
- [33] J. Gatheral. No-dynamic-arbitrage and market impact. *Quantitative Finance*, 10(7):749–759, 2010.
- [34] J. Gatheral and A. Schied. Optimal trade execution under geometric brownian motion in the almgren and chris framework. *International Journal of Theoretical and Applied Finance*, 14(4):353–368, 2011.
- [35] J. Harford and A. Kaul. Correlated order flow: Pervasiveness, sources, and pricing effects. *Journal of Financial and Quantitative Analysis*, 40(01):29–55, 2005.

- [36] J. Hasbrouck and D. J. Seppi. Common factors in prices, order flows, and liquidity. *Journal of financial Economics*, 59(3):383–411, 2001.
- [37] R. W. Holthausen, R. W. Leftwich, and D. Mayers. Large-block transactions, the speed of response, and temporary and permanent stock-price effects. *Journal of Financial Economics*, 26:71–95, 1990.
- [38] G. Huberman and W. Stanzl. Price manipulation and quasi-arbitrage. *Econometrica*, 72(4):1247–1275, 2004.
- [39] S. R. Jaschke. The cornish-fisher-expansion in the context of delta-gamma-normal approximations. Technical report, Discussion Papers, Interdisciplinary Research Project 373: Quantification and Simulation of Economic Processes, 2001.
- [40] A. Kempf and O. Korn. Market depth and order size. *Journal of Financial Markets*, 2(1):29–48, 1999.
- [41] R. Kissell, M. Glantz, R. Malamut, A. M. Association, et al. *Optimal trading strategies: quantitative approaches for managing market impact and trading risk*. Amacom, 2003.
- [42] K. C. Kiwiel. Breakpoint searching algorithms for the continuous quadratic knapsack problem. *Mathematical Programming*, 112(2):473–491, 2008.
- [43] P. H. Kupiec. Margin requirements, volatility, and market integrity: What have we learned since the crash? *Journal of Financial Services Research*, 13(3):231–255, 1998.
- [44] H. Kushner and P. Dupuis. *Numerical methods for stochastic control problems in continuous time*, volume 24. Springer, 2000.
- [45] D. Leinweber. Algo vs. algo. *Institutional Investor’s Alpha Magazines*, 2:44–51, 2007.
- [46] D. Liberzon. *Calculus of variations and optimal control theory: a concise introduction*. Princeton University Press, 2012.
- [47] F. Lillo, J. D. Farmer, and R. N. Mantegna. Master curve for price-impact function. *Nature*, 421:129–130, 2003.

- [48] A. Madhavan. Market microstructure: A survey. *Journal of Financial Markets*, 3:205–258, 2000.
- [49] J. Mina and A. Ulmer. Delta-gamma four ways. Technical report, Technical report, RiskMetrics Group, 1999.
- [50] C. Moallemi and M. Saglam. Dynamic portfolio choice with linear rebalancing rules. *Available at SSRN 2011605*, 2012.
- [51] M. Monoyios. Option pricing with transaction costs using a markov chain approximation. *Journal of Economic Dynamics and Control*, 28(5):889–913, 2004.
- [52] E. Moro, J. Vicente, L. G. Moyano, A. Gerig, J. D. Farmer, G. Vaglica, F. Lillo, and R. N. Mantegna. Market impact and trading profile of hidden orders in stock markets. *Physical Review E*, 80(6):066102, 2009.
- [53] C. Munk. The markov chain approximation approach for numerical solution of stochastic control problems: experiences from mertons problem. *Applied Mathematics and Computation*, 136(1):47–77, 2003.
- [54] Y. Nesterov. Semidefinite relaxation and nonconvex quadratic optimization. *Optimization methods and software*, 9(1-3):141–160, 1998.
- [55] J. Nocedal and S. J. Wright. *Numerical Optimization*. Springer-Verlag, Berlin, Germany, 1999.
- [56] P. M. Pardalos and N. Kovoor. An algorithm for a singly constrained class of quadratic programs subject to upper and lower bounds. *Mathematical Programming*, 46(1-3):321–328, 1990.
- [57] P. Pasquariello and C. Vega. Strategic cross-trading in the us stock market*. *Review of Finance*, page rft055, 2013.
- [58] L. Rogers and S. Singh. The cost of illiquidity and its effects on hedging. *Mathematical Finance*, 20(4):597–615, 2010.
- [59] A. Schied. Robust strategies for optimal order execution in the almgren–chriss framework. *Applied Mathematical Finance*, (ahead-of-print):1–23, 2012.

- [60] A. Schied and T. Schöneborn. Risk aversion and the dynamics of optimal liquidation strategies. *Finance and Stochastics*, 13(2):181–204, 2009.
- [61] G. J. Schinasi and T. T. Smith. *Portfolio Diversification, Leverage, and Financial Contagion (EPub)*. International Monetary Fund, 1999.
- [62] T. Schöneborn. Adaptive basket liquidation. *Available at SSRN 1343985*, 2011.
- [63] T. Schöneborn and A. Schied. Liquidation in the face of adversity: stealth vs. sunshine trading. In *EFA 2008 Athens Meetings Paper*, 2009.
- [64] J. Sherman and W. J. Morrison. Adjustment of an inverse matrix corresponding to a change in one element of a given matrix. *The Annals of Mathematical Statistics*, pages 124–127, 1950.
- [65] R. Sias, L. Starks, and S. Titman. The price impact of institutional trading. In <http://ssrn.com/abstract>, volume 283779, 2001.
- [66] Q. Song. Convergence of markov chain approximation on generalized hjb equation and its applications. *Automatica*, 44(3):761–766, 2008.
- [67] H. E. Tookes. Information, trading, and product market interactions: Cross-sectional implications of informed trading. *The Journal of Finance*, 63(1):379–413, 2008.
- [68] N. Torre and M. Ferrari. *Market Impact Model Handbook*.
- [69] B. Toth, Y. Lempriere, C. Deremble, J. De Lataillade, J. Kockelkoren, and J.-P. Bouchaud. Anomalous price impact and the critical nature of liquidity in financial markets. *Physical Review X*, 1(2):021006, 2011.
- [70] J. Wang and P. Forsyth. Continuous time mean variance asset allocation: A time-consistent strategy. *European Journal of Operational Research*, 209(2):184–201, 2011.
- [71] M. A. Woodbury. Inverting modified matrices. *Memorandum report*, 42:106, 1950.
- [72] Y. Ye. Approximating quadratic programming with bound and quadratic constraints. *Mathematical programming*, 84(2):219–226, 1999.

- [73] S. Zymler, D. Kuhn, and B. Rustem. Worst-case value at risk of nonlinear portfolios. *Management Science*, 59(1):172–188, 2013.