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STABILITY OF LINEAR AUTONOMOUS SYSTEMS UNDER REGULAR
SWITCHING SEQUENCES

BY

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THESIS

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ABSTRACT

In this work, we discuss the stability of a discrete-time linear autonomous system under regular switching sequences, whose switching sequences are generated by a Muller automaton. This system arises in various engineering problems such as distributed communication and automotive engine control. The asymptotic stability of this system, referred to as regular asymptotic stability, generalizes two well-known definitions of stability of autonomous discrete-time linear switched systems, namely absolute asymptotic stability (AAS) and shuffle asymptotic stability (SAS). We also extend these stability definitions to robust versions. We prove that absolute asymptotic stability, robust absolute asymptotic stability and robust shuffle asymptotic stability are equivalent to exponential stability. In addition, by using the Kronecker product, we prove that a robust regular asymptotic stability problem is equivalent to the conjunction of several robust absolute asymptotic stability problems.

To my family and my advisor

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LIST OF ABBREVIATIONS

LS	Lyapunov stable
AAS	Absolutely asymptotically stable
ES	Exponentially stable
SAS	Shuffle asymptotically stable
RAS	Regularly asymptotically stable
rAAS	Robustly absolutely asymptotically stable
rSAS	Robustly shuffle asymptotically stable
rRAS	Robustly regularly asymptotically stable

LIST OF SYMBOLS

$\mathbb{Z}, \mathbb{Z}^+, \mathbb{R}$	Set of integers, positive integers and real numbers
$[k]$	Set of integers from 1 to k
$\mathbb{M}_k, \mathbb{I}_k$	Set of $k \times k$ real matrices and invertible real matrices
$\mathbb{M}_k^n, \mathbb{I}_k^n$	Cartesian product of n copies of \mathbb{M}_k and \mathbb{I}_k
\mathbb{K}_k	Set of $k \times k$ characteristic matrices
\mathbb{K}_k^n	Cartesian product of n copies of \mathbb{K}_k
I	Identity matrix
\mathbf{X}	A matrix tuple $\mathbf{X} = (X_1, X_2, \dots, X_n)$
X_i	The i -th matrix in tuple \mathbf{X}
$\rho(\mathbf{X})$	Joint spectral radius of \mathbf{X}
$\mathcal{P}(\mathbf{X})$	Product set of \mathbf{X}
$\text{DLI}(\mathbf{X})$	Discrete linear inclusion of \mathbf{X}
$\ X\ $	A sub-multiplicative norm of matrix X
$\ \mathbf{X}\ $	Diameter of matrix tuple \mathbf{X}
$\mathcal{C}(\mathbf{X})$	Characteristic function of \mathbf{X}
$\mathcal{B}_{\ \cdot\ }^{\mathcal{C}}(\mathbf{X}, \epsilon)$	Structured \mathcal{C} -neighborhood of \mathbf{X}
\otimes	Kronecker Product
$\Sigma^i, \Sigma^*, \Sigma^\omega$	Set of sequences of Σ of length i , finite length, and infinite length
$ \sigma $	Length of a sequence σ

$A = (S, \Sigma, T, s^{\text{init}}, F)$	Muller automaton with states S , alphabet Σ , transitions T , initial state s^{init} and accepting transitions F
$\text{Lang}(A)$	Set of sequences accepted by A
$\text{Src}(t), \text{Lbl}(t), \text{Dst}(t)$	Source, label, and destination of a transition or a sequence of transitions t
\mathcal{G}	Graph of Muller Automaton A
\mathcal{G}_f	Subgraph of \mathcal{G} generated by $f \in F$
(\mathbf{X}, A)	A discrete-time linear autonomous system under regular switching sequences
$\text{Ind}(t)$	Index of a transition or a sequence of transitions t
\mathbf{Y}	Tuple of transition matrices of system (\mathbf{X}, A)

CHAPTER 1

INTRODUCTION

Discrete-time linear switched systems naturally arise as discretizations of a hybrid automaton where the system dynamics transits between different modes according to switching sequences[1]. They appear in various engineering problems such as distributed communication [2] and automotive engine control [3]. Some studies on the subject have been conducted under the name discrete linear inclusion[4]. When a discrete-time linear switched system is autonomous and permits arbitrary switches, the trajectories generated by the sequence forms the discrete linear inclusion of a set of square matrices.

Given a autonomous discrete-time linear switched system, several notions of stability can be defined. Among them, absolute asymptotic stability is the most fundamental. Previous research shows that absolute asymptotic stability is closely related to the joint spectral radius of the set of matrices defining the system dynamics (see e.g. [5, 6]). The joint spectral radius is an extension of spectral radius. However, since the finiteness conjecture which holds for a single matrix does not hold for a set of matrices [7, 8, 9], the computation of the joint spectral radius is much harder than the computation of the common spectral radius [10, 11].

Other definitions of stability include: uniform exponential stability which re-

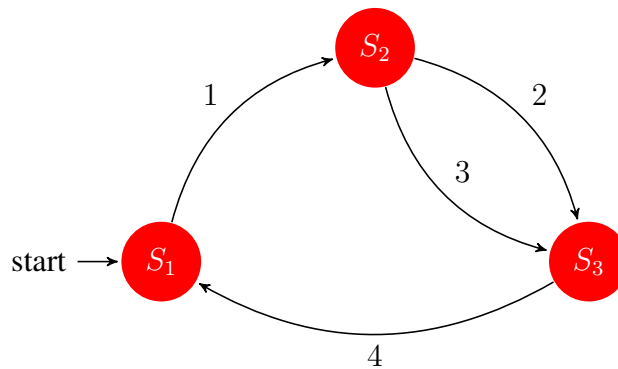


Figure 1.1: Muller automaton generating the admissible transition sequences of the switched system

quires that the system state decays exponentially [3, 12]; shuffle asymptotic stability which imposes the fairness condition that the system dynamics enter each mode infinitely often [4]; and point-wise stability which focuses on the existence, instead of the universality, of stable trajectories [13, 14].

In this paper, we generalize previous works by introducing the definition of *a linear autonomous system under regular switching sequences*. At each time, the system switches between different dynamic modes and updates its state by

$$x(t + 1) = X_{\theta(t)}x(t) \tag{1.1}$$

where the switching sequence $\theta(t)$ is generated by a Muller automaton (see Figure 1.1). Here, we use the variation of Muller automata where the fairness condition is imposed on transitions instead of states.

The asymptotic stability of this system, defined as regular asymptotic stability, generalizes the definition of absolute asymptotic stability and shuffle asymptotic stability. Furthermore, for physics-motivated reasons, we consider the robustness issues in the stability of autonomous discrete-time linear switched system [15].

The main contribution of this work is that we prove exponential stability is equivalent to absolute asymptotic stability, robust absolute asymptotic stability and robust shuffle asymptotic stability. In addition, using the Kronecker product, we show that regular asymptotic stability problems are equivalent to the conjunction of several robust absolute asymptotic stability problems.

In section 2, a brief explanation on regular languages, stability of discrete linear inclusion and the Kronecker product is given. In section 3, a general mathematical formalism for a linear autonomous system under regular switching sequences is presented, together with stability definitions. In section 4, we prove the equivalence of absolute asymptotic stability, exponential stability, robust absolute asymptotic stability and robust shuffle asymptotic stability. In section 5, we develop a method to convert regular asymptotic stability problems to shuffle asymptotic stability problems by using the Kronecker product. In addition, we demonstrate that robust regular asymptotic stability problems can be converted to the conjunction of several robust absolute asymptotic stability problems.

CHAPTER 2

PRELIMINARIES

2.1 Sequences and Automata

Denote the set of integers, positive integers and real numbers by $\mathbb{Z}, \mathbb{Z}^+, \mathbb{R}$. Let $[k]$ be the set of integers from 1 to k . Let Σ be a finite non-empty set and $\sigma = \sigma_1\sigma_2\dots$ be a sequence of elements of Σ . Denote the length of σ by $|\sigma|$. ($|\sigma| = \infty$ if σ is infinite.) In the sequence σ , denote the i -th element, and the sub-sequence starting from the i -th element to the j -th element $\sigma_i\sigma_{i+1}\dots\sigma_j$ by σ_i and $\sigma_{[i,j]}$. If σ is infinite, let $\sigma_{[i,\infty]}$ be the subsequence starting from the i -th element $\sigma_i\sigma_{i+1}\dots$. In addition, denote the set of sequences of length i , finite sequences, and infinite sequences of elements of Σ by Σ^i, Σ^* , and Σ^ω respectively. For $\sigma \in \Sigma^\omega$, let $\text{Inf}(\sigma) = \{\Sigma \in \Sigma \mid \Sigma \text{ appear infinitely often in } \sigma\}$. A sequence σ is shuffle iff $\sigma \in \Sigma^\omega$ and $\text{Inf}(\sigma) = \Sigma$. The set of shuffle sequences is denoted by Σ^{sh} .

In this work, we mainly concern sets of infinite sequences that are generated by a special kind of Muller Automata where accepting conditions are imposed on transitions. Other kinds of automata that have the same expressive power include ordinary Muller Automata, Rabin automata, and Streett automata. Readers may refer to [16] and [17] for a survey on the theory of finite state automata on infinite words and how to transfer automata in different definitions to each other.

Definition 1. *A Muller automata with accepting condition on transitions is a tuple $A = (\mathcal{S}, \Sigma, \mathcal{T}, s^{\text{init}}, \mathcal{F})$ in which*

- \mathcal{S} is a finite set of states,
- Σ is a finite alphabet,
- $\mathcal{T} \subset \mathcal{S} \times \Sigma \times \mathcal{S}$ is a set of transitions,
- $s^{\text{init}} \in \mathcal{S}$ is an initial state,

- $F \subset 2^T$ is a set of accepting sets.

For a transition $\mathfrak{t} = (\mathfrak{s}_1, \sigma_1, \mathfrak{s}_2) \in T$, we call \mathfrak{s}_1 , σ_1 , and \mathfrak{s}_2 the *source*, *label* and *destination* of \mathfrak{t} and denote them by $\text{Src}(\mathfrak{t})$, $\text{Lbl}(\mathfrak{t})$ and $\text{Dst}(\mathfrak{t})$ respectively.

Definition 2. An infinite sequence $\sigma \in \Sigma^\omega$ is accepted by A iff there exists an infinite sequence $\mathfrak{t} \in T^\omega$ such that

1. $\mathfrak{s}^{\text{init}} = \text{Src}(\mathfrak{t}_1)$,
2. $\forall i \in \mathbb{Z}^+, \text{Dst}(\mathfrak{t}_i) = \text{Src}(\mathfrak{t}_{i+1})$,
3. $\text{Inf}(\mathfrak{t}) \in F$.

For any infinite sequence $\mathfrak{t} \in T^\omega$, we call \mathfrak{t}

- a computation of A iff it satisfies the first two conditions,
- an accepting computation of A iff it satisfies all the three conditions.

The set of sequences accepted by A is denoted by $\text{Lang}(A)$.

Given a finite (or infinite) sequence of transitions \mathfrak{t}' , we call it a *fragment of computation* of A iff there exists a computation \mathfrak{t} of A and $i, j \in \mathbb{Z}^+$ (or $i \in \mathbb{Z}^+$) such that $\mathfrak{t}' = \mathfrak{t}_{[i,j]}$ (or $\mathfrak{t}' = \mathfrak{t}_{[i,\infty]}$). For a fragment of computation $\mathfrak{t} = (\mathfrak{s}_1, \sigma_1, \mathfrak{s}_2)(\mathfrak{s}_2, \sigma_2, \mathfrak{s}_3) \dots$, define the *source* and *label* by

$$\text{Src}(\mathfrak{t}) = \mathfrak{s}_1 \tag{2.1}$$

$$\text{Lbl}(\mathfrak{t}) = \text{Lbl}(\mathfrak{t}_1)\text{Lbl}(\mathfrak{t}_2) \dots \tag{2.2}$$

For a finite fragment of computation $\mathfrak{t} = (\mathfrak{s}_1, \sigma_1, \mathfrak{s}_2)(\mathfrak{s}_2, \sigma_2, \mathfrak{s}_3) \dots (\mathfrak{s}_n, \sigma_n, \mathfrak{s}_{n+1})$, define the *destination* by

$$\text{Dst}(\mathfrak{t}) = \mathfrak{s}_{n+1} \tag{2.3}$$

Definition 3. A set of infinite sequences L are called *regular* if there exists a Muller Automaton A such that $L = \text{Lang}(A)$.

From the above definitions, we can easily derive the following lemma.

Lemma 4. If \mathfrak{t} is an accepting computation of a Muller Automaton $A = (\mathcal{S}, \Sigma, T, \mathfrak{s}^{\text{init}}, F)$, then there exists $i \in \mathbb{Z}^+$ and $\mathfrak{f} \in F$ such that $\mathfrak{t}_{[i,\infty]} \in \mathfrak{f}^\omega$.

The transition relation of Muller automata is easily represented by a graph in which every node stands for a state and every edge stands for a transition.

Definition 5. $\mathcal{G}_A = (V, E)$ is a graph of an Muller Automaton $A = (\mathcal{S}, \Sigma, \mathcal{T}, s^{\text{init}}, F)$ if there exist bijections $f : \mathcal{S} \mapsto V$ and $g : \mathcal{T} \mapsto E$ such that for any $\mathfrak{t} = (s_1, \sigma_1, s_2) \in \mathcal{T}$, $g(\mathfrak{t})$ connects $f(s_1)$ and $f(s_2)$. For simplicity, we equate V to \mathcal{S} and E to \mathcal{T} , i.e. $\mathcal{G}_A = (\mathcal{S}, \mathcal{T})$.

Recall the following concepts of graphs. A path in a graph $\mathcal{G} = (\mathcal{S}, \mathcal{T})$ is a finite sequence of edges $\mathfrak{t}_1 \mathfrak{t}_2 \dots \mathfrak{t}_n \in \mathcal{T}^*$ such that for each $i \in [n - 1]$, \mathfrak{t}_i and \mathfrak{t}_{i+1} share exactly one node. A graph $\mathcal{G} = (\mathcal{S}, \mathcal{T})$ is called *connected* if any two nodes in \mathcal{G} are connected by a path. $\mathcal{G}' = (\mathcal{S}', \mathcal{T}')$ is a subgraph of $\mathcal{G} = (\mathcal{S}, \mathcal{T})$ if $\mathcal{S}' \in \mathcal{S}$ and $\mathcal{T}' \in \mathcal{T}$. The subgraph of \mathcal{G} with edges \mathcal{T}' is denoted by $\mathcal{G}_{\mathcal{T}'}$.

Let \mathcal{G} be the graph of Muller Automaton $A = (\mathcal{S}, \Sigma, \mathcal{T}, s^{\text{init}}, F)$. By Lemma 4, if there exists $f \in F$ such that \mathcal{G}_f is not reachable from s^{init} or not strongly connected, then no computation \mathfrak{t} will be able to satisfy $\text{Inf}(\mathfrak{t}) = f$. Therefore, we can remove them from F without changing $\text{Lang}(A)$. In the following, we assume that for all $f \in F$, \mathcal{G}_f is reachable from s^{init} and strongly connected. In addition, we assume that $\text{Lang}(A)$ is not empty. This, in particular, means that F is non-empty.

Example 1. As shown in Figure 2.1, consider an automaton with $\mathcal{S} = \{s\}$, $s^{\text{init}} = s$, $\Sigma = \{a, b\}$ and $\mathcal{T} = \{(s, a, s), (s, b, s)\}$. When $F = \{\{(s, a, s), (s, b, s)\}, \{(s, a, s)\}, \{(s, b, s)\}\}$, the automaton accepts all infinite sequences on $\{a, b\}$, i.e. $\text{Lang}(A) = \{a, b\}^\omega$. When $F = \{\{(s, a, s), (s, b, s)\}\}$, the automaton only accepts shuffle sequences of $\{a, b\}$, i.e. $\text{Lang}(A) = \{a, b\}^{\text{sh}}$.

In general for any set Σ , we can construct the following two Muller automata:

$$A_1 = (\{s\}, \Sigma, \mathcal{T}, s, 2^{\mathcal{T}} - \{\emptyset\}) \quad (2.4a)$$

$$A_2 = (\{s\}, \Sigma, \mathcal{T}, s, \{\mathcal{T}\}) \quad (2.4b)$$

where $\mathcal{T} = \{(s, a, s) \mid a \in \Sigma\}$, such that $\text{Lang}(A_1) = \Sigma^\omega$ and $\text{Lang}(A_2) = \Sigma^{\text{sh}}$.

Example 2. As shown in Figure 2.2, consider an automaton B with $\mathcal{S} = \{s_1, s_2, s_3\}$, $s^{\text{init}} = s_1$, $\Sigma = \{a, b, c, d\}$ and $\mathcal{T} = \{(s_1, a, s_2), (s_2, b, s_3), (s_2, c, s_3), (s_3, d, s_1)\}$ and $F = \{f\}$ where $f = \{(s_1, a, s_2), (s_2, b, s_3), (s_3, d, s_1)\}$. The automaton accepts sequences of the form $\text{Lang}(B) = ((acd)^* abd)^* (abd)^\omega$. Clearly, \mathcal{G}_f is reachable from s_1 and strongly connected. In addition, for any $\sigma \in \text{Lang}(B)$, there exists $i \in \mathbb{Z}^+$ such that $\sigma_{[i, \infty]} \in f^\omega$

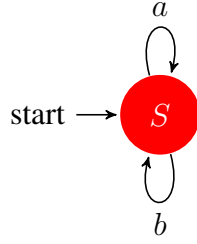


Figure 2.1: Finite State Automaton A

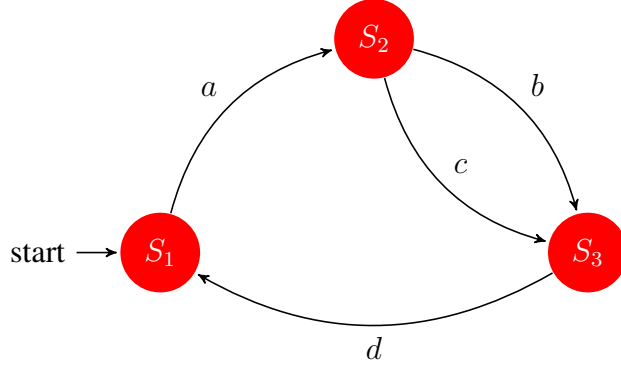


Figure 2.2: Finite State Automaton B

2.2 Matrix Analysis

2.2.1 Matrix Norms and Joint Spectral Radius

Denote the set of $k \times k$ real matrices and invertible real matrices by \mathbb{M}_k and \mathbb{I}_k . Denote the Cartesian product of n copies of \mathbb{M}_k and \mathbb{I}_k by \mathbb{M}_k^n and \mathbb{I}_k^n respectively. For a matrix tuple $\mathbf{X} = (X_1, X_2, \dots, X_n) \in \mathbb{M}_k^n$, denote i -th matrix by X_i . Here, we use the concept "tuple" to emphasis the order in \mathbf{X} . Let $\|\cdot\|$ be a norm on \mathbb{M}_k .

¹ The diameter of a set (or tuple) of matrices \mathbf{X} is defined as

$$\|\mathbf{X}\| = \max_{X \in \mathbf{X}} \|X\|. \quad (2.5)$$

For $\mathbf{X} \in \mathbb{M}_k^n$, define the product set of \mathbf{X} as

$$\mathcal{P}(\mathbf{X}) = \{X_{p_i} \cdots X_{p_1} \mid i \in \mathbb{Z}^+, X_{p_j} \in \mathbf{X}, j \in [i]\} \quad (2.6)$$

The product set of a matrix tuple has the following property.

¹In this work, norms are assumed sub-multiplicative.

Lemma 6 (Rota and Strang [18]). *Given a norm $\|\cdot\|$ on \mathbb{M}_k and $\mathbf{X} \in \mathbb{M}_k^n$, $\|\mathcal{P}(\mathbf{X})\| < +\infty$ iff there exists a norm $\|\cdot\|_*$ on \mathbb{M}_k such that $\|I\|_* = 1$ and $\|\mathbf{X}\|_* \leq 1$.*

For $x \in \mathbb{M}_k$, the spectral radius $\rho(X) = \max\{|x| \mid x \text{ is an eigenvalue of } X\}$. It satisfies that $\rho(X) = \lim_{n \rightarrow \infty} \|X^n\|^{\frac{1}{n}}$. For a matrix tuple, we can define its joint spectral radius.

Definition 7 (Rota and Strang [18]). *The joint spectral radius of $\mathbf{X} \in \mathbb{M}_k^n$ is*

$$\rho(\mathbf{X}) = \lim_{k \rightarrow \infty} (\sup \{\|X_{p_k} \cdots X_{p_1}\| \mid i \in [k], p_j \in [n]\})^{\frac{1}{k}}. \quad (2.7)$$

The computation of joint spectral radius is challenging since the "finiteness conjecture" does not hold in general [7, 19]. Readers may refer to [5] for further explanations on the subject. An important property of the joint spectral radius is as follows [4].

Lemma 8. *For $\mathbf{X} \in \mathbb{M}_k^n$, $\rho(\mathbf{X}) < 1$ iff there exists a norm $\|\cdot\|_*$ on \mathbb{M}_k such that $\|I\|_* = 1$ and $\|\mathbf{X}\|_* < 1$.*

2.2.2 Characteristic Matrices and Structured Neighborhood

To discuss robustness, we sometimes need to perturb certain entries of a matrix which are correlated with the uncertain physical parameters, while keeping others unchanged. Therefore, we introduce the concept of *characteristic matrices* below. We call $C = [c_{pq}] \in \mathbb{M}_k$ a characteristic matrix if

$$c_{pq} = \begin{cases} 1 & \text{if } p = q \\ 0 \text{ or } 1 & \text{otherwise.} \end{cases} \quad (2.8)$$

Denote the set of $k \times k$ characteristic matrices by \mathbb{K}_k and the Cartesian product of n copies of \mathbb{K}_k by \mathbb{K}_k^n . Given $C, D \in \mathbb{K}_k$, if $[C]_{pq} \leq [D]_{pq}$ for any $p, q \in [n]$, we say $C \leq D$. Given $\mathbf{C}, \mathbf{D} \in \mathbb{K}_k^n$, if for any $i \in [n]$, $C_i \leq D_i$ and, we say $\mathbf{C} \leq \mathbf{D}$.

Given $\mathbf{X} \in \mathbb{M}_k^n$, the characteristic function $\mathcal{C} : \mathbb{M}_k^n \mapsto \mathbb{K}_k^n$ is defined by $\mathcal{C}(\mathbf{X}) = (\mathcal{C}(X_1), \mathcal{C}(X_2), \dots, \mathcal{C}(X_n))$, where

$$\mathcal{C}(X)_{pq} = \begin{cases} 0 & \text{if } p \neq q \text{ and } X_{pq} = 0 \\ 1 & \text{otherwise.} \end{cases} \quad (2.9)$$

With the help of characteristic function, we define the concept of *structured neighborhood*.

Definition 9. For a norm $\|\cdot\|$ on \mathbb{M}_k , $C \in \mathbb{K}_k$ and $\mathbf{C} \in \mathbb{K}_k^n$, the structured C -neighborhood of $X \in \mathbb{M}_k$ is

$$\mathcal{B}_{\|\cdot\|}^C(X, \epsilon) = \{Y \mid \mathcal{C}(Y - X) \leq C, \|Y - X\| < \epsilon\}, \quad (2.10)$$

and the structured \mathbf{C} -neighborhood of $\mathbf{X} \in \mathbb{M}_k^n$ is

$$\mathcal{B}_{\|\cdot\|}^{\mathbf{C}}(\mathbf{X}, \epsilon) = \mathcal{B}_{\|\cdot\|}^{C_1}(X_1, \epsilon) \times \cdots \times \mathcal{B}_{\|\cdot\|}^{C_n}(X_n, \epsilon). \quad (2.11)$$

For simplicity, denote $\mathcal{B}_{\|\cdot\|}^{\mathcal{C}(\mathbf{X})}(\mathbf{X}, \epsilon)$ by $\mathcal{B}_{\|\cdot\|}(\mathbf{X}, \epsilon)$.

2.2.3 Kronecker Product

Definition 10. Let $A = [a_{ij}] \in \mathbb{M}_{k_1}$ and $B = [b_{ij}] \in \mathbb{M}_{k_2}$, The Kronecker product of A and B is

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{bmatrix}. \quad (2.12)$$

The Kronecker product is associative, non-commutative and bi-linear. In addition, the Kronecker product has the mixed-product property. For two finite sets of matrices $\{A_i\}_{i=1}^n \subset \mathbb{M}_{k_1}$ and $\{B_i\}_{i=1}^n \subset \mathbb{M}_{k_2}$, $(A_n A_{n-1} \cdots A_1) \times (B_n B_{n-1} \cdots B_1) = (A_n \times B_n)(A_{n-1} \times B_{n-1}) \cdots (A_1 \times B_1)$.

CHAPTER 3

SYSTEM FORMULATION

3.1 Discrete-time Linear Autonomous Systems Under Regular Switching Sequences

In this work, we consider a discrete-time linear autonomous switched system in which the switching sequences are ω -regular words generated by a Muller automaton. Specifically, given a tuple of matrices $\mathbf{X} \in \mathbb{M}_k^n$ and a Muller automaton A with alphabet $[n]$, a discrete-time linear autonomous system under regular switching sequences (\mathbf{X}, A) is defined as

$$x_{t+1} = X_{\sigma_t} x_t. \quad (3.1)$$

where $x_1 \in \mathbb{R}^k$ is some initial state, $t \in \mathbb{Z}^+$ and $\sigma \in \text{Lang}(A)$. For simplicity, the following conventions are made:

1. for a sequence $\sigma \in [n]^i$, let $X_\sigma = X_{\sigma_i} X_{\sigma_{i-1}} \cdots X_{\sigma_2} X_{\sigma_1}$;
2. for $\sigma \in [n]^\omega$, let $X_\sigma = \lim_{i \rightarrow \infty} X_{\sigma_i} X_{\sigma_{i-1}} \cdots X_{\sigma_2} X_{\sigma_1}$ when the limit exists;
3. for a set of sequences L , let $X_L = \{X_\sigma \mid \sigma \in L\}$.

When $\text{Lang}(A) = [n]^\omega$, the set of trajectories of the system coincides with the *discrete linear inclusion* of \mathbf{X} [4][20] defined as

$$\text{DLI}(\mathbf{X}) = \left\{ \{x_t\}_{t=1}^\infty \mid x_t \in \mathbb{R}^k, x_{t+1} = X_{\sigma_t} x_t, \sigma_t \in [n], t \in \mathbb{Z}^+ \right\}. \quad (3.2)$$

The system is called *regular asymptotic stability* if all possible trajectories of the system converges to zero.

Definition 11. *The system (\mathbf{X}, A) is regularly asymptotically stable (RAS) iff for any word $\sigma \in \text{Lang}(A)$, $X_\sigma = 0$.*

Regular asymptotic stability generalizes two definitions of stability of discrete linear autonomous system in previous literature: when $\text{Lang}(A) = [n]^\omega$, we derive absolute asymptotic stability; when $\text{Lang}(A) = [n]^{\text{sh}}$, we derive shuffle asymptotic stability [4].

Definition 12 (Gurvits [4]). *The system \mathbf{X} is absolutely (shuffle) asymptotically stable, abbreviated as AAS (SAS), iff for any sequence $\sigma \in [n]^\omega$ ($\sigma \in [n]^{\text{sh}}$), $X_\sigma = 0$.*

In addition, we introduce the definitions of Lyapunov stability and exponential stability for the case when $\text{Lang}(A) = [n]^\omega$.

Definition 13. *The system \mathbf{X} is Lyapunov stable, abbreviated as LS, iff for $\forall \delta > 0$, $\exists \epsilon > 0$, such that for any $\|x_1\| < \epsilon$, we have $\|x_t\| < \delta$ for $t \geq 1$ under any switching sequence.*

Recall the definition of product set, the following lemma holds.

Lemma 14. *The system \mathbf{X} is LS iff $\|\mathcal{P}(\mathbf{X})\| < \infty$.*

Definition 15 (Lee and Dullerud [3]). *The system \mathbf{X} is exponentially stable, abbreviated as ES, iff there exist $c > 0$ and $0 < \lambda < 1$ such that for any $t \in \mathbb{Z}^+$ and $\sigma \in [n]^\omega$, $\|X_{\sigma_{[1,t]}}\| \leq c\lambda^t$.*

Motivated by the presence of physical errors in the system operations, we extend the definitions of stability to their robust versions. The definition of regular asymptotic stability extends to robust regular asymptotic stability.

Definition 16. *Given $\mathbf{X} \in \mathbb{M}_k^n$ and a Muller automaton A with alphabet $[n]$, for $\epsilon > 0$ and $\mathbf{C} \in \mathbb{K}_k^n$, the system (\mathbf{X}, A) is called to be (\mathbf{C}, ϵ) -robustly regular asymptotically stable, abbreviated as (\mathbf{C}, ϵ) -rRAS iff for any $\mathbf{X}' \in \mathcal{B}_{\|\cdot\|}^{\mathbf{C}}(\mathbf{X}, \epsilon)$, the system (\mathbf{X}', A) is RAS. When $\mathbf{C} = \mathcal{C}(\mathbf{X})$, the system (\mathbf{X}, A) is called ϵ -robustly regular asymptotically stable, abbreviated as ϵ -rRAS. Finally, the system (\mathbf{X}, A) is called robustly regular asymptotically stable, abbreviated as rRAS if it is ϵ -robustly regular asymptotically stable for $\epsilon > 0$.*

The definition of absolute asymptotic stability and shuffle asymptotic stability extends to robust absolute asymptotic stability and robust shuffle asymptotic stability.

Definition 17. Given $\mathbf{X} \in \mathbb{M}_k^n$, for $\epsilon > 0$ and $\mathbf{C} \in \mathbb{K}_k^n$, the system \mathbf{X} is (\mathbf{C}, ϵ) -robustly absolute (shuffle) asymptotically stable, abbreviated as (\mathbf{C}, ϵ) -rAAS ((\mathbf{C}, ϵ) -rSAS) iff for any $\mathbf{X}' \in \mathcal{B}_{\|\cdot\|}^{\mathbf{C}}(\mathbf{X}, \epsilon)$, the system \mathbf{X}' is AAS (SAS). Especially, when $\mathbf{C} = \mathcal{C}(\mathbf{X})$, the system \mathbf{X} is called ϵ -robustly absolute (shuffle) asymptotically stable, abbreviated as ϵ -rAAS (ϵ -rSAS). Finally, the system \mathbf{X} is called robustly absolute (shuffle) asymptotically stable, abbreviated as rAAS (rSAS) if it is ϵ -robustly absolute (shuffle) asymptotically stable for $\epsilon > 0$.

By the definitions, we give the following statements.

Lemma 18. *The following statements holds:*

- *exponential stability implies absolute asymptotic stability*
- *absolute asymptotic stability implies shuffle asymptotic stability*
- *absolute asymptotic stability implies Lyapunov stability*

In general, shuffle asymptotic stability does not imply Lyapunov stability. For example, if $\mathbf{X} = \{0, 2I\}$, then the system \mathbf{X} is shuffle asymptotic stability but not Lyapunov stability. Finally, we introduce the following Lemmas from Gurvits [4] without proof.

Lemma 19. *The system \mathbf{X} is absolutely asymptotically stable iff $\rho(\mathbf{X}) < 1$.*

3.2 Running Example

Consider a discrete-time distributed system G where the agents are numbered $1, \dots, m$. The goal of the agents in G is to reach a common destination. But only the leader agents know what the destination is, and other agents try to reach it by communicating with the leaders. At time $t \in \mathbb{Z}^+$, the position of agent i relative to the destination is denoted by $x_i(t)$ and the ensemble state of the system is denoted by $x(t) = (x_1(t), x_2(t), \dots, x_m(t))$. The agents are connected by a communication network so that they can exchange their state information with neighboring agents. We want to check whether the multi-agent consensus algorithm described below stabilizes the system asymptotically at zero for any initial state.

At each time $t \in \mathbb{Z}^+$, a leader agent i may move closer to the destination by reducing its current state by half. In addition, two neighboring agents i, j in the

communication network can make on consensus to move closer to the average of their current states so that the followers pursue the leaders. These two actions are denoted by red_i and $\text{cons}_{i,j}$ respectively, and the set of all actions is denoted by \mathcal{A} . For individual agents, the two actions are mathematically formulated as follows.

1. red_i : Leader i reduces the state by half.

$$x_i(t+1) = \frac{1}{2}x_i(t). \quad (3.3)$$

2. $\text{cons}_{i,j}$: Two agents i and j make a consensus.

$$x_i(t+1) = \frac{2}{3}x_i(t) + \frac{1}{3}x_j(t), \quad (3.4)$$

$$x_j(t+1) = \frac{1}{3}x_i(t) + \frac{2}{3}x_j(t). \quad (3.5)$$

For the ensemble state x , the two actions are formulated by matrix multiplications. Specifically, red_i is formulated by $x(t+1) = R_i x(t)$ and $\text{cons}_{i,j}$ is formulated by $x(t+1) = S_{ij} x(t)$, where

$$(R_i)_{pq} = \begin{cases} 1, & \text{if } p = q \neq i \\ \frac{1}{2}, & \text{if } p = q = i \\ 0, & \text{otherwise} \end{cases} \quad (3.6)$$

$$(S_{ij})_{pq} = \begin{cases} 1, & \text{if } p = q \neq i, j \\ \frac{2}{3}, & \text{if } (p, q) = (i, i), (j, j) \\ \frac{1}{3}, & \text{if } (p, q) = (i, j), (j, i) \\ 0, & \text{otherwise} \end{cases} \quad (3.7)$$

$(R_i)_{pq}$ and $(S_{ij})_{pq}$ stands for the element in the p -th row and the q -th column of R_i and S_{ij} respectively.

We will also consider a robust version of this example. This issue arises when the real system deviates to some extent from our mathematical model. In this example, we assume that each agent is suffering from an error proportional to the

current states, then

$$(R_i)_{pq} = \begin{cases} 1 + \zeta_{pq}^k & \text{if } p = q \neq i \\ \frac{1}{2} + \zeta_{pq}^k & \text{if } p = q = i \\ 0 & \text{otherwise.} \end{cases} \quad (3.8)$$

$$(S_{ij})_{pq} = \begin{cases} 1 + \zeta_{pq}^k & \text{if } p = q \neq i, j \\ \frac{2}{3} + \zeta_{pq}^k & \text{if } (p, q) = (i, i), (j, j) \\ \frac{1}{3} + \zeta_{pq}^k & \text{if } (p, q) = (i, j), (j, i) \\ 0 & \text{otherwise.} \end{cases} \quad (3.9)$$

where $\{\zeta_{pq}^k \mid p, q \in [m], k \in \mathcal{A}\}$ are constants chosen from neighborhoods of 0.

Furthermore, in the multi-agent consensus algorithm, the sequences of actions may be restricted by the constraints imposed by the communication protocol the agents follow as well as possible physical restrictions like the communication network. For example, we may want to make a consensus on the state of a leader agent i with another agent j immediately after reducing the state of agent i in an attempt to improve efficiency of the algorithm, or reduce the state of the leader agent i infinitely often in an attempt to guarantee that the agent i constantly go to zero. These constraints are the sequences of actions are adequately modeled by automata (see Section 2.1 for formal definition of automata).

CHAPTER 4

STABILITY

4.1 Absolute Stability and Shuffle Stability

In this section, we focus on absolute asymptotic stability and shuffle asymptotic stability and their robust extensions. The results presented in this chapter serve as the foundation for handling regular asymptotic stability and robust regular asymptotic stability. To begin with, we show that absolute asymptotic stability, exponential stability and robust absolute asymptotic stability are equivalent.

Theorem 20. *For $\mathbf{X} \in \mathbb{M}_k^n$, the following statements are equivalent:*

1. *the system \mathbf{X} is absolutely asymptotically stable*
2. *the system \mathbf{X} is exponentially stable*
3. *there exists $\epsilon > 0$ such that the system \mathbf{X} is ϵ -robustly absolutely asymptotically stable*

For simplicity, we define

$$a\mathbf{X} = (aX_1, aX_2, \dots, aX_n), \quad (4.1)$$

$$\mathbf{X} + a = (X_1 + aI, X_2 + aI, \dots, X_n + aI), \quad (4.2)$$

where $a \in \mathbb{R}$, $\mathbf{X} \in \mathbb{M}_k^n$ and I is the identity matrix.

Proof. By Lemma 18, (2) \Rightarrow (1).

(1) \Rightarrow (3). By Lemma 8, there exists a norm $\|\cdot\|_1$ on \mathbb{M}_k such that $\|\mathbf{X}\|_1 < 1$. Let $\epsilon' = 1 - \|\mathbf{X}\|_1$. For any $\mathbf{X}' \in \mathcal{B}_{\|\cdot\|_1}(\mathbf{X}, \epsilon')$, since $\|\mathbf{X}'\|_1 < \epsilon' + \|\mathbf{X}\|_1 = 1$, \mathbf{X}' is AAS. Thus \mathbf{X} is ϵ' -rAAS under norm $\|\cdot\|_1$. Since all norms are equivalent on \mathbb{M}_k , there exists $\epsilon > 0$ such that \mathbf{X} is ϵ -rAAS under norm $\|\cdot\|$.

(3) \Rightarrow (2). Clearly, the claim holds for $\|\mathbf{X}\| = 0$. For $\|\mathbf{X}\| > 0$, let $\mathbf{X}' = c\mathbf{X}$, where $c = 1 + \frac{\epsilon}{2\|\mathbf{X}\|}$. Since $\mathbf{X}' \in \mathcal{B}_{\|\cdot\|}(\mathbf{X}, \epsilon)$, \mathbf{X}' is AAS. Again by Lemma 8, there

exists a norm $\|\cdot\|_2$ such that $\|\mathbf{X}'\|_2 < 1$. Under this norm, $\|\mathbf{X}\|_2 < \frac{1}{c}$. Thus, \mathbf{X} is ES. \square

In the previous section, we show that SAS does not imply LS. However, if the matrices in \mathbf{X} are invertible, then the above statement holds.

Theorem 21. *If a system $\mathbf{X} \in \mathbb{I}_k^n$ is shuffle asymptotically stable, then it is Lyapunov stable, and $\|\mathcal{P}(\mathbf{X})\| < \infty$,*

Proof. Let $\mathbf{Z} = \mathcal{P}(\mathbf{X})$ and $P = X_1 X_2 \cdots X_n$. If $\mathbf{Y}_1 = \mathbf{Z}P$ is bounded, since P is invertible, \mathbf{Z} will be bounded. Otherwise, there exists $U_1 \in \mathbf{Z}$ such that $\|U_1 P\| > 1$. Then let $\mathbf{Y}_2 = \mathbf{Z}P U_1 P$. If \mathbf{Y}_2 is bounded, since $P U_1 P$ is invertible, \mathbf{Z} will be bounded. Otherwise, there exists $U_2 \in \mathbf{Z}$ such that $\|U_2 P U_1 P\| > 2$. By repeating the above procedure, for some finite l , $\mathbf{Y}_{l+1} = \mathbf{Z}P U_l P U_{l-1} \cdots P U_1 P$ should be bounded. Otherwise, we will have a divergent infinite shuffle product $\cdots P U_l \cdots P U_2 P U_1 P$. Finally by invertibility of $P U_l P U_{l-1} \cdots P U_1 P$, \mathbf{Z} is bounded. By definition, the system \mathbf{X} is LS. \square

Furthermore, we can show that robust shuffle asymptotic stability implies absolute asymptotic stability. Due to the denseness of invertible matrices, the requirement that the matrices in \mathbf{X} are invertible can be removed.

Lemma 22. *If a system $\mathbf{X} \in \mathbb{M}_k^n$ is ϵ -robustly shuffle asymptotically stable for some $\epsilon > 0$, then it is absolutely asymptotically stable.*

Proof. Clearly, the lemma holds for $\|\mathbf{X}\| = 0$. For $\|\mathbf{X}\| \neq 0$, let

$$\Lambda = \{|x| \mid x \neq 0, x \in \text{Eig}(X), X \in \mathbf{X}\} \quad (4.3)$$

and

$$\lambda = \begin{cases} \infty & \text{if } \Lambda \text{ is empty} \\ \inf \Lambda & \text{otherwise} \end{cases} \quad (4.4)$$

where $\text{Eig}(X)$ is the set of eigenvalues of X . For $0 < \delta < \min\{\epsilon, \lambda\}$, let $\mathbf{X}' = \{X' \in \mathbb{M}_k \mid X' = X + \delta I, X \in \mathbf{X}\}$. Then $\mathbf{X}' \in \mathbb{I}_k^n$ and $\mathbf{X}' \in \mathcal{B}_{\|\cdot\|}(\mathbf{X}, \epsilon)$. By Theorem 21, there exists $K > 0$ such that $\max\{\|\mathcal{P}(\mathbf{X}')\|, \|I\|\} < K$. For any $\sigma \in [n]^m$ and $0 < \delta < \min\{\epsilon, \lambda\}$, recalling the convention made in Section 3.1,

we have

$$\begin{aligned}
\|X_\sigma\| &= \|(X'_{\sigma_m} - \delta I)(X'_{\sigma_{m-1}} - \delta I) \cdots (X'_{\sigma_1} - \delta I)\| \\
&= \|X'_\sigma - \delta \sum_{1 \leq i \leq m} X'_{\sigma_{[i+1, m]}} X'_{\sigma_{[1, i-1]}} \\
&\quad + \delta^2 \sum_{1 \leq i < j \leq m} X'_{\sigma_{[j+1, m]}} X'_{\sigma_{[i+1, j-1]}} X'_{\sigma_{[1, i-1]}} + \cdots + (\delta)^m I\| \\
&\leq \|X'_{\sigma_{[1, m]}}\| + \delta \sum_{1 \leq i \leq m} \|X'_{\sigma_{[i+1, m]}} X'_{\sigma_{[1, i-1]}}\| \\
&\quad + \delta^2 \sum_{1 \leq i < j \leq m} \|X'_{\sigma_{[j+1, m]}} X'_{\sigma_{[i+1, j-1]}} X'_{\sigma_{[1, i-1]}}\| + \cdots + \delta^m \\
&\leq K + Km\delta + \frac{m(m-1)}{2} K\delta^2 \cdots + K\delta^m \\
&= K(1 + \delta)^m.
\end{aligned} \tag{4.5}$$

Thus, the joint spectral radius $\rho(\mathbf{X}) = \lim_{m \rightarrow \infty} (\sup_{|\sigma|=m} \|X_\sigma\|) \leq 1 + \delta$. As $\delta \searrow 0$, we have $\rho(\mathbf{X}) \leq 1$.

We have shown that $\rho(\mathbf{X}) \leq 1$ if \mathbf{X} is ϵ -rSAS for some $\epsilon > 0$. Now consider $\mathbf{X}' = c\mathbf{X}$, where $c = 1 + \frac{\epsilon}{2\|\mathbf{X}\|}$. Since $\mathcal{B}_{\|\cdot\|}(\mathbf{X}', \frac{1}{2}\epsilon) \subset \mathcal{B}_{\|\cdot\|}(\mathbf{X}, \epsilon)$, \mathbf{X}' is $\frac{1}{2}\epsilon$ -rSAS, hence $\rho(\mathbf{X}') \leq 1$. Therefore, $\rho(\mathbf{X}) \leq \frac{1}{c} < 1$. Recalling Lemma 19, we prove that the system \mathbf{X} is AAS. \square

By Lemma 22 and Theorem 20, ϵ -shuffle asymptotic stability implies absolute asymptotic stability and absolute asymptotic stability implies ϵ' -robust absolute asymptotic stability for some $\epsilon' > 0$. The following theorem shows that actually we can choose $\epsilon' = \epsilon$.

Theorem 23. *Given $\epsilon > 0$ and $\mathbf{X} \in \mathbb{M}_k^n$, the system \mathbf{X} is ϵ -robustly shuffle asymptotically stable iff the system \mathbf{X} is ϵ -robustly absolutely asymptotically stable.*

Proof. It suffices to prove necessity. For any $\mathbf{Y} \in \mathcal{B}_{\|\cdot\|}(\mathbf{X}, \epsilon)$, there exists ϵ' such that $\mathcal{B}_{\|\cdot\|}(\mathbf{Y}, \epsilon') \subset \mathcal{B}_{\|\cdot\|}(\mathbf{X}, \epsilon)$, hence \mathbf{Y} is ϵ' -rSAS. By Lemma 22, \mathbf{Y} is AAS. Therefore, \mathbf{X} is ϵ -rAAS. \square

Combining Theorem 21 and Theorem 23, we derive the following corollary.

Corollary 24. *The following statements are equivalent*

1. *the system \mathbf{X} is exponentially stable*
2. *the system \mathbf{X} is absolutely asymptotically stable*

3. there exists $\epsilon > 0$ such that the system \mathbf{X} is ϵ -robustly shuffle asymptotically stable

Remark 25. For $\mathbf{C} \in \mathbb{K}_k^n$ satisfying that $\mathbf{C} \geq \mathcal{C}(\mathbf{X})$, Theorem 20 Lemma 22, Theorem 23 and Corollary 24 still hold after replacing $\mathcal{C}(\mathbf{X})$ with \mathbf{C} .

Example 3. Recall the running example. Suppose there are three agents $x = [x_1, x_2, x_3]^\top$ and four possible operations $\{R_1, S_{12}, S_{23}, S_{31}\}$ in the system. Then,

$$R_1 = \begin{bmatrix} \frac{1}{2} + \zeta_{11}^{R_1} & 0 & 0 \\ 0 & 1 + \zeta_{22}^{R_1} & 0 \\ 0 & 0 & 1 + \zeta_{33}^{R_1} \end{bmatrix} \quad (4.6)$$

$$S_{12} = \begin{bmatrix} \frac{2}{3} + \zeta_{11}^{S_{12}} & \frac{1}{3} + \zeta_{12}^{S_{12}} & 0 \\ \frac{1}{3} + \zeta_{21}^{S_{12}} & \frac{2}{3} + \zeta_{22}^{S_{12}} & 0 \\ 0 & 0 & 1 + \zeta_{33}^{S_{12}} \end{bmatrix} \quad (4.7)$$

$$S_{23} = \begin{bmatrix} 1 + \zeta_{11}^{S_{23}} & 0 & 0 \\ 0 & \frac{2}{3} + \zeta_{22}^{S_{23}} & \frac{1}{3} + \zeta_{23}^{S_{23}} \\ 0 & \frac{1}{3} + \zeta_{32}^{S_{23}} & \frac{2}{3} + \zeta_{33}^{S_{23}} \end{bmatrix} \quad (4.8)$$

$$S_{31} = \begin{bmatrix} \frac{2}{3} + \zeta_{11}^{S_{31}} & 0 & \frac{1}{3} + \zeta_{13}^{S_{31}} \\ 0 & 1 + \zeta_{22}^{S_{31}} & 0 \\ \frac{1}{3} + \zeta_{31}^{S_{31}} & 0 & \frac{2}{3} + \zeta_{33}^{S_{31}} \end{bmatrix} \quad (4.9)$$

where $\{\zeta_{pq}^k \mid p, q \in [3], k \in \{R_1, S_{12}, S_{23}, S_{31}\}\}$ are constants chosen from some neighborhood of 0.

Let $\zeta_{ij}^k = 0$ for $i, j \in [3], k \in \{R_1, S_{12}, S_{23}, S_{31}\}$. Then the matrices $R_1, S_{12}, S_{23}, S_{31}$ are invertible and symmetric. Therefore, their singular values are positive and coincide with eigenvalues. The system is LS, since $\rho(R_1) = \rho(S_{12}) = \rho(S_{23}) = \rho(S_{31}) = 1$. The system is also SAS, there is no common eigenvector with eigenvalue 1 for all $R_1, S_{12}, S_{23}, S_{31}$. This is in agreement with Theorem 21.

However, the system is not AAS, since $\lim_{n \rightarrow \infty} R_1^n \neq 0$. Consequently, there exists no $\epsilon > 0$ such that the system is ϵ -rAAS. Finally, by Theorem 23, there exists no $\epsilon > 0$ such that the system is ϵ -rSAS.

4.2 Regular Stability

In this chapter, we provide a method of getting rid of the restrictions on switching sequences imposed by the Muller automaton using the Kronecker product and converting regular asymptotic stability problems into several simpler absolute asymptotic stability problems or shuffle asymptotic stability problems. The lemmas and theorems presented in this chapter can be viewed as extensions to the lemmas and theorems given in Chapter 4.1.

Consider a discrete-time linear autonomous system under regular switching sequencessystem (\mathbf{X}, A) , where $\mathbf{X} \in \mathbb{M}_k^n$ and $A = (\mathbf{S}, \Sigma, \mathbf{T}, \mathbf{s}^{\text{init}}, \mathbf{F})$. Let $\mathbf{S} = \{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_p\}$, $\Sigma = [n]$ and $\mathbf{T} = \{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_q\}$. For simplicity, we define a map $\text{Ind} : \mathbf{T} \mapsto [q]$ by $\text{Ind}(\mathbf{t}_i) = i$. Recall that each transition is also labeled by another number $\text{Lbl}(\mathbf{t}_j) = i \in [n]$. For a sequence of transitions $\mathbf{t} = \mathbf{t}_1 \mathbf{t}_2 \dots$ of \mathbf{T} , define $\text{Ind}(\mathbf{t}) = \text{Ind}(\mathbf{t}_1) \text{Ind}(\mathbf{t}_2) \dots$

For each transition $\mathbf{t}_i \in \mathbf{T}$ from $\text{Src}(\mathbf{t}_i) = s_u$ to $\text{Dst}(\mathbf{t}_i) = s_v$, define the *transition matrix* as

$$Y_{\text{Ind}(\mathbf{t}_i)} = E_{\text{Ind}(\mathbf{t}_i)} \otimes X_{\text{Lbl}(\mathbf{t}_i)}, \quad (4.10)$$

where

$$(E_{\text{Ind}(\mathbf{t}_i)})_{pq} = \begin{cases} 1, & p = v, q = u \\ 0, & \text{otherwise.} \end{cases} \quad (4.11)$$

The set of transition matrices ordered by $\text{Ind}(t_i)$ forms a tuple, which is denoted by \mathbf{Y} in the rest of the chapter. For each $\mathbf{f} \in \mathbf{F}$, denote the matrix tuple $\{Y_{\text{Ind}(\mathbf{t}_i)} \mid \mathbf{t}_i \in \mathbf{f}\}$ by $\mathbf{Y}_{\mathbf{f}}$. By definition, the following lemma holds.

Lemma 26. *For a finite sequence of transitions \mathbf{t} , if it is a fragment of computation, then*

$$(E_{\text{Ind}(\mathbf{t})})_{pq} = \begin{cases} 1, & p = \text{Dst}(\mathbf{t}), q = \text{Src}(\mathbf{t}) \\ 0, & \text{otherwise.} \end{cases} \quad (4.12)$$

Otherwise, there exists $i \in \mathbb{Z}^+$ such that $E_{\text{Ind}(\mathbf{t}_{[1,i]})} = 0$.

For an set of invertible matrices, with the help of the Kronecker product, we can convert a regular asymptotic stability problem into several shuffle asymptotic stability problems.

Theorem 27. *For $\mathbf{X} \in \mathbb{I}_k^n$, the system (\mathbf{X}, A) is regularly asymptotically stable iff for any $\mathbf{f} \in \mathbf{F}$, the system $\mathbf{Y}_{\mathbf{f}}$ is shuffle asymptotically stable.*

Proof. Sufficiency. As noted in Section 2.1, given a word $\sigma \in \text{Lang}(A)$, there is an accepting computation $\mathfrak{t} \in T^\omega$ satisfying $\sigma = \text{Lbl}(\mathfrak{t})$. In addition, there exists some $i \in \mathbb{Z}^+$ such that $\mathfrak{t}_{[i, \infty]}$ is a shuffle sequence of some $\mathfrak{f} \in F$. Since $\mathbf{Y}_{\mathfrak{f}}$ is SAS, $Y_{\text{Ind}(\mathfrak{t}_{[i, \infty]})} = 0$. By Lemma 26, we have $X_{\text{Lbl}(\mathfrak{t}_{[i, \infty]})} = 0$. Therefore, the system (\mathbf{X}, A) is RAS.

Necessity. For $\mathfrak{f} \in F$, let $\gamma \in \mathfrak{f}^\omega$. There are two cases for γ .

(1) If $\gamma \in \text{Frag}(\mathfrak{f})$, then there exists $i \in \mathbb{Z}^+$ and a computation \mathfrak{t} such that $\gamma = \mathfrak{t}_{[i, \infty]}$. By definition, \mathfrak{t} is also an accepting computation, hence $X_{\text{Lbl}(\mathfrak{t})} = 0$. Noting that $\mathbf{X} \in \mathbb{I}_k^n$, $X_{\text{Lbl}(\gamma)} = 0$. By Lemma 26, we have $Y_{\text{Ind}(\gamma)} = 0$.

(2) Otherwise, $\gamma \notin \text{Frag}(\mathfrak{f})$. By Lemma 26, for some $i \in \mathbb{Z}^+$, $E_{\text{Ind}(\gamma_{[1, i]})} = 0$. Thus, $Y_{\text{Ind}(\gamma)} = Y_{\text{Ind}(\gamma_{[1, i]})} = 0$. In sum, $\mathbf{Y}_{\mathfrak{f}}$ is SAS. \square

By the spirit of Theorem 21 which states that shuffle asymptotic stability implies Lyapunov stability if the matrices in \mathbf{X} are invertible, we prove the following statement.

Lemma 28. *For $\mathbf{X} \in \mathbb{I}_k^n$, if the system (\mathbf{X}, A) is regularly asymptotically stable, then for any $\mathfrak{f} \in F$, the system $\mathbf{Y}_{\mathfrak{f}}$ is Lyapunov stable.*

Proof. For $\mathfrak{f} \in F$, denote the set of nodes in $\mathcal{G}_{\mathfrak{f}}$ by $\text{State}(\mathfrak{f})$. For each $s \in \text{State}(\mathfrak{f})$, denote the set of all finite fragments of computation starting from s by

$$\text{Frag}_s(\mathfrak{f}) = \{\mathfrak{t} \in \mathfrak{f}^* \mid \text{Src}(\mathfrak{t}) = s, \mathfrak{t} \text{ is a finite fragments of computation}\}. \quad (4.13)$$

Let $\text{Frag}(\mathfrak{f}) = \cup_{s \in \text{State}(\mathfrak{f})} \text{Frag}_s(\mathfrak{f})$. Recalling the definition of \mathbf{Y} , it suffices to prove that $\{Y_{\text{Ind}(\mathfrak{t})} \mid \mathfrak{t} \in \text{Frag}(\mathfrak{f})\}$ is bounded.

Suppose that $\{Y_{\text{Ind}(\mathfrak{t})} \mid \mathfrak{t} \in \text{Frag}(\mathfrak{f})\}$ is unbounded. Then for some $s \in \text{State}(\mathfrak{f})$, $\{Y_{\text{Ind}(\mathfrak{t})} \mid \mathfrak{t} \in \text{Frag}_s(\mathfrak{f})\}$ is unbounded. By Lemma 26,

$$\mathbf{W}_0 = \{X_{\text{Lbl}(\mathfrak{t})} \mid \mathfrak{t} \in \text{Frag}_s(\mathfrak{f})\} \quad (4.14)$$

is also unbounded. Pick $\tau_1 \in \text{Frag}_s(\mathfrak{f})$ such that $\|X_{\text{Lbl}(\tau_1)}\| > 1$. Since the sub-graph formed by $\text{State}(\mathfrak{f})$ is strongly connected, we can find a fragment of computation $\gamma_1 \in \text{Frag}(\mathfrak{f})$ from $\text{Dst}(\tau_1)$ back to s . Let

$$\mathbf{W}_1 = \{X_{\text{Lbl}(\mathfrak{t})} X_{\text{Lbl}(\gamma_1)} X_{\text{Lbl}(\tau_1)} \mid \mathfrak{t} \in \text{Frag}_s(\mathfrak{f})\} \subset \mathbf{W}_0 \quad (4.15)$$

If \mathbf{W}_1 is bounded, by invertibility of $X_{\text{Lb1}(\gamma_1)}X_{\text{Lb1}(\tau_1)}$, \mathbf{W}_0 will be bounded. Otherwise, pick $\tau_2 \in \text{Frag}_s(\mathbf{f})$ such that $\|X_{\text{Lb1}(\tau_2)}X_{\text{Lb1}(\gamma_1)}X_{\text{Lb1}(\tau_1)}\| > 2$. Again we can find a fragment of computation $\gamma_2 \in \text{Frag}(\mathbf{f})$ from $\text{Dst}(\tau_2)$ back to s . Let

$$\mathbf{W}_2 = \{X_{\text{Lb1}(\mathbf{t})}X_{\text{Lb1}(\gamma_2)}X_{\text{Lb1}(\tau_2)}X_{\text{Lb1}(\gamma_1)}X_{\text{Lb1}(\tau_1)} \mid \mathbf{t} \in \text{Frag}_s(\mathbf{f})\} \subset \mathbf{W}_1 \quad (4.16)$$

By invertibility of $X_{\text{Lb1}(\gamma_2)}X_{\text{Lb1}(\tau_2)}X_{\text{Lb1}(\gamma_1)}X_{\text{Lb1}(\tau_1)}$, if \mathbf{W}_2 is bounded, then \mathbf{W}_0 will be bounded. Otherwise, pick $\tau_3 \in \text{Frag}_s(\mathbf{f})$ such that

$$\|X_{\text{Lb1}(\tau_3)}X_{\text{Lb1}(\gamma_2)}X_{\text{Lb1}(\tau_2)}X_{\text{Lb1}(\gamma_1)}X_{\text{Lb1}(\tau_1)}\| > 2. \quad (4.17)$$

By repeating the above procedure, if for some $p \in \mathbb{Z}^+$,

$$\mathbf{W}_p = \{X_{\text{Lb1}(\mathbf{t})}X_{\text{Lb1}(\gamma_p)}X_{\text{Lb1}(\tau_p)} \cdots X_{\text{Lb1}(\gamma_1)}X_{\text{Lb1}(\tau_1)} \mid \mathbf{t} \in \text{Frag}_s(\mathbf{f})\} \subset \mathbf{W}_{p-1} \quad (4.18)$$

is bounded, since $X_{\text{Lb1}(\gamma_p)}X_{\text{Lb1}(\tau_p)} \cdots X_{\text{Lb1}(\gamma_1)}X_{\text{Lb1}(\tau_1)}$ is invertible, \mathbf{W}_0 is bounded. Otherwise, we will derive an accepting computation $\tau = \tau_1\gamma_1 \dots \tau_p\gamma_p \dots$, such that $X_{\text{Lb1}(\tau)}$ does not converge to 0. This is in contradiction with the assumption that (\mathbf{X}, A) is RAS. \square

Remark 29. *From the proof, we know that, for $\mathbf{X} \in \mathbb{I}_k^n$, if (\mathbf{X}, A) is RAS, then for any $\mathbf{f} \in \mathbf{F}$, the set $\{X_{\text{Lb1}(\mathbf{t})} \mid \mathbf{t} \in \text{Frag}(\mathbf{f})\}$ is norm bounded.*

By the above lemma, when $\mathbf{X} \in \mathbb{I}_k^n$, a regular asymptotic stability problem can be converted to many Lyapunov stability problems. Furthermore, we show that a robust regular asymptotic stability problem can be converted to many absolute asymptotic stability problems. Again, due to the denseness of invertible matrices, the requirement of invertibility can be removed.

Lemma 30. *If a system (\mathbf{X}, A) is robustly regularly asymptotically stable then for any $\mathbf{f} \in \mathbf{F}$, $\mathbf{Y}_{\mathbf{f}}$ is absolutely asymptotically stable.*

Proof. Let $\mathbf{X}' = \mathbf{X} + \delta I$. By the proof of Theorem 23, for small enough $\delta > 0$, $\mathbf{X}' \in \mathbb{I}_k^n$ and $\mathbf{X}' \in \mathcal{B}_{\|\cdot\|}(\mathbf{X}, \epsilon)$. By Remark 29, since (\mathbf{X}', A) is RAS, for any $\mathbf{f} \in \mathbf{F}$, $\{X'_{\text{Lb1}(\mathbf{t})} \mid \mathbf{t} \in \text{Frag}(\mathbf{f})\}$ is norm bounded by some constant $K > 0$. Let

$K > \|I\|$. For any $\mathfrak{t} \in \text{Frag}(\mathfrak{f})$, letting $\sigma = \text{Lb1}(\mathfrak{t})$ and $m = |\sigma|$, we have

$$\begin{aligned}
\|X_\sigma\| &= \|(X'_{\sigma_m} - \delta I)(X'_{\sigma_{m-1}} - \delta I) \cdots (X'_{\sigma_1} - \delta I)\| \\
&= \|X'_\sigma - \delta \sum_{1 \leq i \leq m} X'_{\sigma_{[i+1, m]}} X'_{\sigma_{[1, i-1]}} \\
&\quad + \delta^2 \sum_{1 \leq i < j \leq m} X'_{\sigma_{[j+1, m]}} X'_{\sigma_{[i+1, j-1]}} X'_{\sigma_{[1, i-1]}} + \cdots + (\delta)^m I\| \\
&\leq \|X'_{\sigma_{[1, m]}}\| + \delta \sum_{1 \leq i \leq m} \|X'_{\sigma_{[i+1, m]}}\| \|X'_{\sigma_{[1, i-1]}}\| \\
&\quad + \delta^2 \sum_{1 \leq i < j \leq m} \|X'_{\sigma_{[j+1, m]}}\| \|X'_{\sigma_{[i+1, j-1]}}\| \|X'_{\sigma_{[1, i-1]}}\| + \cdots + \delta^m \\
&\leq K + K^2 m \delta + K^3 \frac{m(m-1)}{2} \delta^2 + \cdots + K^{m+1} \delta^m \\
&= K(1 + K\delta)^m.
\end{aligned} \tag{4.19}$$

By Lemma 26, the joint spectral radius

$$\rho(\mathbf{Y}_\mathfrak{f}) = \lim_{m \rightarrow \infty} \left(\sup_{\mathfrak{t} \in \text{Frag}(\mathfrak{f}), |\mathfrak{t}|=m} \|X_{\text{Lb1}(\mathfrak{t})} \otimes E_{\text{Ind}(\mathfrak{t})}\|_* \right) \tag{4.20}$$

where $\|\cdot\|_*$ is a norm for $\mathbf{Y}_\mathfrak{f}$. Without loss of generality, let $\|\cdot\|$ and $\|\cdot\|_*$ be spectral norms. Then, we have $\rho(\mathbf{Y}_\mathfrak{f}) \leq 1 + K\delta$. As $\delta \searrow 0$, we have $\rho(\mathbf{Y}_\mathfrak{f}) \leq 1$.

We have shown that if (\mathbf{X}, A) is ϵ -rRAS for some $\epsilon > 0$, then $\rho(\mathbf{Y}_\mathfrak{f}) \leq 1$. Now let $c = 1 + \frac{\epsilon}{2\|\mathbf{X}\|}$ and consider $\mathbf{X}' = c\mathbf{X}$. Since $\mathcal{B}_{\|\cdot\|}(\mathbf{X}', \frac{1}{2}\epsilon) \subset \mathcal{B}_{\|\cdot\|}(\mathbf{X}, \epsilon)$, we know that (\mathbf{X}', A) is $\frac{\epsilon}{2}$ -rSAS. For the set of transition matrices generated by \mathbf{X}' , which we denote by $\mathbf{Y}'_\mathfrak{f}$, we have $\rho(\mathbf{Y}'_\mathfrak{f}) \leq 1$. Consequently, $\rho(\mathbf{Y}_\mathfrak{f}) \leq \frac{1}{c} < 1$. Recalling Lemma 19, we prove that the system $\mathbf{Y}_\mathfrak{f}$ is AAS. \square

Now combining lemma 28 and lemma 30, we show that a robust regular asymptotic stability problem is equivalent to many robust absolute asymptotic stability problems.

Theorem 31. *The followings are equivalent*

1. *the system (\mathbf{X}, A) is robustly regularly asymptotically stable*
2. *for all $\mathfrak{f} \in \mathbb{F}$, $\mathbf{Y}_\mathfrak{f}$ is absolutely asymptotically stable*
3. *for all $\mathfrak{f} \in \mathbb{F}$, $\mathbf{Y}_\mathfrak{f}$ is exponentially stable*

4. for all $\mathbf{f} \in \mathbf{F}$, $\mathbf{Y}_{\mathbf{f}}$ is robustly absolutely asymptotically stable.

Proof. First we note that (1) \Rightarrow (2) derives from Lemma 30 and the equivalence of (2)(3)(4) derives from Corollary 24.

(4) \Rightarrow (1). Let $\|\cdot\|$ and $\|\cdot\|_*$ be norms for \mathbf{X} and $\mathbf{Y}_{\mathbf{f}}$ respectively. For $\epsilon' > 0$, there exists $\epsilon > 0$ such that for any $\mathbf{X}' \in \mathcal{B}_{\|\cdot\|}(\mathbf{X}, \epsilon)$, $\mathbf{Y}'_{\mathbf{f}} \in \mathcal{B}_{\|\cdot\|_*}(\mathbf{Y}_{\mathbf{f}}, \epsilon)$ where $\mathbf{Y}'_{\mathbf{f}}$ are generated by \mathbf{X}' . For any $\mathbf{f} \in \mathbf{F}$ and an accepting computation \mathfrak{t} , there exist $i \in \mathbb{Z}^+$ such that $\mathfrak{t}_{[i, \infty]} \in \mathbf{f}^\omega$. Since $Y'_{\text{Ind}(\mathfrak{t}_{[i, \infty]})} = 0$, by Lemma 26, $X'_{\text{Lb1}(\mathfrak{t})} = X'_{\text{Lb1}(\mathfrak{t}_{[i, \infty]})} = 0$. Therefore, the system (\mathbf{X}, A) is ϵ -robustly regularly asymptotically stable. \square

Remark 32. For $\mathbf{C} \in \mathbb{K}_k^n$ satisfying that $\mathbf{C} \geq \mathbf{C}(\mathbf{X})$, Theorem 27, Lemma 28, Lemma 30 and Theorem 31 still hold after replacing $\mathbf{C}(\mathbf{X})$ with \mathbf{C} .

Example 4. Suppose that we design the following algorithm for generating switching sequences in Example 3, in an attempt to achieve fast convergence.

- red_1 is followed by $\text{cons}_{1,2}$ or $\text{cons}_{2,3}$
- $\text{cons}_{1,2}$ and $\text{cons}_{2,3}$ is followed by $\text{cons}_{3,1}$
- $\text{cons}_{3,1}$ is followed by red_1 .
- $\{\text{red}_1, \text{cons}_{1,2}, \text{cons}_{3,1}\}$ appears infinitely often

The above constraints on actions can be represented by the automaton B given in Example 2 with $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ standing for $\text{red}_1, \text{cons}_{1,2}, \text{cons}_{3,1}$ and $\text{cons}_{2,3}$ and $F = \{f\} = \{(\mathfrak{s}_1, \text{red}_1, \mathfrak{s}_2), (\mathfrak{s}_2, \text{cons}_{1,2}, \mathfrak{s}_3), (\mathfrak{s}_3, \text{cons}_{3,1}, \mathfrak{s}_1)\}$. Then the problem is formulated by the discrete-time linear autonomous system under regular switching sequences (\mathbf{X}, B) where $\mathbf{X} = \{R_1, S_{12}, S_{31}, S_{23}\}$ is given in Equation 4.6-4.9 with $\zeta_{ij}^k = 0$ for $i, j \in [3], k \in \{R_1, S_{12}, S_{23}, S_{31}\}$. The convergence of this algorithm is captured by the regular asymptotic stability of the system.

By Example 2, the sequences accepted by the automaton B has the form

$$\text{Lang}(B) = ((R_1 S_{31} S_{23})^* R_1 S_{12} S_{23})^* (R_1 S_{12} S_{23})^\omega. \quad (4.21)$$

Since the spectral radius $\rho(R_1 S_{31} S_{23}), \rho(R_1 S_{31} S_{23}) < 1$, we learn that the system is RAS. Furthermore, there exists $\epsilon > 0$ such that the system is ϵ -RAS.

On the other hand, we construct the transition matrices $\mathbf{Y} = \{R'_1, S'_{12}, S'_{23}, S'_{31}\}$ corresponding to the transitions in automaton B using the Kronecker product,

where

$$\begin{aligned}
R'_1 &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ R_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} & S'_{12} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & S_{12} & \mathbf{0} \end{bmatrix} \\
S'_{31} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & S_{31} & \mathbf{0} \end{bmatrix} & S'_{23} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & S_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
\end{aligned} \tag{4.22}$$

where $\mathbf{0}$ is a 3×3 matrix with all elements equal to 0.

We can prove that $\mathbf{Y}_f = \{R'_1, S'_{12}, S'_{23}\}$ is AAS, hence SAS, by considering the Lyapunov function $V(x) = x^\top Px$, where

$$P = \begin{bmatrix} P_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & P_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & P_3 \end{bmatrix}, \tag{4.23}$$

$$P_1 = \begin{bmatrix} 0.9383 & -0.0048 & -0.0088 \\ -0.0048 & 1.1913 & -0.1243 \\ -0.0088 & -0.1243 & 1.2021 \end{bmatrix}, \tag{4.24}$$

$$P_2 = \begin{bmatrix} 1.1910 & 0.1045 & 0.0051 \\ 0.1045 & 0.9925 & -0.0121 \\ 0.0051 & -0.0121 & 1.0762 \end{bmatrix}, \tag{4.25}$$

$$P_3 = \begin{bmatrix} 1.1926 & -0.1115 & -0.0117 \\ -0.1115 & 1.0961 & 0.1013 \\ -0.0117 & 0.1013 & 0.9778 \end{bmatrix}. \tag{4.26}$$

For any $x \in \mathbb{R}^9$, $V(x) > 0$, $V(R'_1 x) < V(x)$, $V(S'_{12} x) < V(x)$ and $V(S'_{23} x) < V(x)$. Therefore, by Theorem 27, (\mathbf{X}, B) is RAS. In addition, by Theorem 31, there exists $\epsilon > 0$ such that (\mathbf{X}, B) is ϵ -rRAS. This result agrees with the conclusions drawn from the direct analysis above.

CHAPTER 5

CONCLUSION

In this work, we introduced a discrete-time linear autonomous system under regular switching sequences and the definition of regular asymptotic stability, which generalized the two well-known notions, absolute asymptotic stability and shuffle asymptotic stability. The Kronecker product proved to be a central tool for studying this problem. By comparing different definitions of stability and the corresponding robust versions, we proved that absolute asymptotic stability, robust absolute asymptotic stability and robust shuffle asymptotic stability are equivalent to uniform exponential stability. In addition, we showed how to convert a regular stability problem into the conjunction of some shuffle asymptotic stability problems and proved that a robust regular stability problem is equivalent to the conjunction of several robust absolute asymptotic stability problems.

REFERENCES

- [1] D. Liberzon, *Switching in systems and control*. Springer, 2003.
- [2] P. Prabhakar, S. Mitra, and M. Viswanathan, “On convergence of concurrent systems under regular interactions,” in *CONCUR 2009-Concurrency Theory*. Springer, 2009, pp. 527–541.
- [3] J.-W. Lee and G. E. Dullerud, “Uniform stabilization of discrete-time switched and markovian jump linear systems,” *Automatica*, vol. 42, no. 2, pp. 205 – 218, 2006.
- [4] L. Gurvits, “Stability of discrete linear inclusion,” *Linear Algebra and its Applications*, vol. 231, pp. 47–85, 1995.
- [5] R. Jungers, *The joint spectral radius: theory and applications*. Springer, 2009, vol. 385.
- [6] A. A. Ahmadi, R. Jungers, P. A. Parrilo, and M. Roozbehani, “Analysis of the joint spectral radius via lyapunov functions on path-complete graphs,” in *Proceedings of the 14th international conference on Hybrid systems: computation and control*. ACM, 2011, pp. 13–22.
- [7] J. C. Lagarias and Y. Wang, “The finiteness conjecture for the generalized spectral radius of a set of matrices,” *Linear Algebra and its Applications*, vol. 214, no. 0, pp. 17 – 42, 1995.
- [8] V. D. Blondel, J. Theys, and A. A. Vladimirov, “An elementary counterexample to the finiteness conjecture,” *SIAM Journal on Matrix Analysis and Applications*, vol. 24, no. 4, pp. 963–970, Jan. 2003.
- [9] V. Kozyakin, “A dynamical systems construction of a counterexample to the finiteness conjecture,” in *44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05*, 2005, pp. 2338–2343.
- [10] J. N. Tsitsiklis and V. D. Blondel, “The lyapunov exponent and joint spectral radius of pairs of matrices are hard-when not impossible-to compute and to approximate,” *Mathematics of Control, Signals and Systems*, vol. 10, no. 1, pp. 31–40, Mar. 1997.

- [11] V. D. Blondel and J. N. Tsitsiklis, “The boundedness of all products of a pair of matrices is undecidable,” *Systems & Control Letters*, vol. 41, no. 2, pp. 135–140, 2000.
- [12] J.-W. Lee and G. E. Dullerud, “Uniformly stabilizing sets of switching sequences for switched linear systems,” *Automatic Control, IEEE Transactions on*, vol. 52, no. 5, pp. 868–874, 2007.
- [13] D. P. Stanford, “Stability for a multi-rate sampled-data system,” *SIAM Journal on Control and Optimization*, vol. 17, no. 3, pp. 390–399, May 1979.
- [14] D. P. Stanford and J. M. Urbano, “Some convergence properties of matrix sets,” *SIAM Journal on Matrix Analysis and Applications*, vol. 15, no. 4, pp. 1132–1140, 1994.
- [15] G. E. Dullerud and F. Paganini, *A course in robust control theory*. Springer New York, 2000.
- [16] W. Thomas, “Handbook of theoretical computer science (vol. b),” J. van Leeuwen, Ed. Cambridge, MA, USA: MIT Press, 1990, ch. Automata on infinite objects, pp. 133–191.
- [17] M. Mukund, “Finite-state automata on infinite inputs,” *Theoretical Computer Science*, vol. 96, p. 2, 1996.
- [18] G.-C. Rota and W. Strang, “A note on the joint spectral radius,” *Indagationes Mathematicae*, vol. 22, pp. 379–381, 1960.
- [19] K. G. Hare, I. D. Morris, N. Sidorov, and J. Theys, “An explicit counterexample to the lagarias–wang finiteness conjecture,” *Advances in Mathematics*, vol. 226, no. 6, pp. 4667–4701, 2011.
- [20] I. Daubechies and J. C. Lagarias, “Sets of matrices all infinite products of which converge,” *Linear Algebra and its Applications*, vol. 161, pp. 227–263, Jan. 1992.