J - HOLOMORPHIC CURVES AND THEIR APPLICATIONS

BY

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DISSERTATION

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ABSTRACT

This thesis covers four results:

1. We prove an analog of Whitney's embedding theorem for J-holomorphic discs.

2. For $z_j = x_j + iy_j \in \mathbb{C}$ and let $\mathbb{D}^2_{\mathbb{R}} = \{(z_1, z_2) \in \mathbb{C}^2 : x_1^2 + x_2^2 < 1, y_1^2 + y_2^2 < 1\}$ be the real bi-disc in $\mathbb{C}^2$. We find the sharp lower bound for R such that $\mathbb{D}^2_{\mathbb{R}}$ admits a symplectic embedding into $\mathbb{D}(R) \times \mathbb{C}$, the complex cylinder with base radius R. The sharp lower bound for R is shown to be $\frac{2}{\sqrt{\pi}}$. As a consequence, we know that $\mathbb{D}^2_{\mathbb{R}}$ and $\mathbb{D}^2$ are not symplectomorphic.

3. We extend the second result by showing that if T is an orthogonal matrix on $\mathbb{R}^4 \simeq \mathbb{C}^2$, then $T\mathbb{D}^2$ is symplectomorphic to $\mathbb{D}^2$ if and only if T is unitary or conjugate to unitary.

4. A high dimensional case of the second result: for $r \geq 1$ and $n \geq 2$, we show that $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}(r)$ and $\mathbb{D}^2 \times \mathbb{D}^{n-2}(r)$ are not symplectomorphic.

To my parents and my girlfriend, for their love and support.

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

In this thesis, we will establish a perturbation property of J-holomorphic curves and apply J-holomorphic curves to solve symplectic rigidity problems.

In Chapter 2, we will review the basic notions of symplectic manifold and J-holomorphic curve.

In Chapter 3, we will show that given a J-holomorphic disc in an almost complex manifold M, we can perturb it to obtain an embedded J-holomorphic disc that is smooth up to the boundary, provided that $\dim_{\mathbb{C}} M > 2$. This work extends the result of A. Sukhov and A. Tumanov [6] stating that a given J-holomorphic disc in M can be perturbed to be immersed provided that $\dim_{\mathbb{C}} M \geq 2$.

In Chapter 4, we will study under what condition (on the radius R) the real bi-disc $\mathbb{D}^2_{\mathbb{R}}$ can be symplectically embedded into the complex cylinder $\mathbb{D}(R) \times \mathbb{C}$ with radius R. A. Sukhov and A. Tumanov showed in [7] that the real bi-disc $\mathbb{D}^2_{\mathbb{R}}$ and the complex bi-disc $\mathbb{D}^2$ are not symplectomorphic. Moreover they showed that if the real bi-disc can be symplectically embedded into a complex cylinder, then the cylinder must have radius $R > 1$; the sharp lower bound for R was not known from their result. Our main result in this chapter is to provide a sharp lower bound for R.

If we consider the real bi-disc as obtained by performing an orthogonal transformation on the complex bi-disc, then our result in Chapter 4 gives an example that the orthogonally transformed complex bi-disc is not symplectomorphic to the complex bi-disc itself. Our first result in Chapter 5 will give a full description of which orthogonal transformations of complex bi-disc are symplectomorphic to the complex bi-disc.

The second result in Chapter 5 extends the result in Chapter 4 to a high dimensional case. More precisely, we show, for any $n \geq 2$ and $r \geq 1$, that the domains $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}(r)$ and $\mathbb{D}^2 \times \mathbb{D}^{n-2}(r)$ in $\mathbb{C}^n$ equipped with the standard symplectic form on $\mathbb{C}^n$ are not symplectomorphic. The case $r > 1$ is easier than the case $r = 1$ because we can adapt the idea used in Chapter 4 to show that $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}(r)$ is symplectomorphic to a domain in $\mathbb{C}^n$ which contains a ball of radius strictly greater than 1. To prove the case $r = 1$ we need to make use of the number of minimal surfaces passing through the origin in each domain.

CHAPTER 2

PRELIMINARIES

In this chapter we will recall basic notions concerning J-holomorphic discs, symplectic manifolds, Fredholm operators, and the Cauchy-Green operator.

2.1 J-holomorphic discs and symplectic manifolds

Definition 2.1 *A smooth map $\phi : (M, J) \to (M', J')$ from one almost complex manifold to another is said to be (J, J')-holomorphic if its derivative $d\phi$ is complex linear, that is*

$$d\phi \circ J = J' \circ d\phi. \tag{2.1}$$

Denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$ and by J_{st} the standard complex structure of $\mathbb{C}^n$. A J-holomorphic disc or pseudo-holomorphic disc is a (J_{st}, J)-holomorphic map

$$u : \mathbb{D} \to M$$

from $\mathbb{D}$ to an almost complex manifold (M, J).

In local coordinates $z \in \mathbb{C}^n$, an almost complex structure J is represented by a $\mathbb{R}$-linear operator $J(z) : \mathbb{C}^n \to \mathbb{C}^n$ such that $J(z)^2 = -I$, where I is the identity map. Now the Cauchy-Riemann equations (2.1) for a J-holomorphic disc $z : \mathbb{D} \to \mathbb{C}^n$ can be written in the form

$$z_\eta = J(z)z_\xi, \zeta = \xi + i\eta \in \mathbb{D}.$$

We represent J by a complex $n \times n$ matrix function $A = A(z)$ and obtain the equivalent equations

$$z_{\overline{\zeta}} = A(z)\overline{z}_{\overline{\zeta}}, \zeta \in \mathbb{D}. \tag{2.2}$$

We recall the relation between J and A for fixed z. let $J : \mathbb{C}^n \to \mathbb{C}^n$ be a $\mathbb{R}$-linear map so that $\det(J_{\text{st}} + J) \neq 0$, where $J_{\text{st}}v = iv$. Set

$$Q = (J_{\text{st}} + J)^{-1}(J_{\text{st}} - J).$$

Lemma 2.2 *$J^2 = -I$ if and only if $QJ_{st} + J_{st}Q = 0$.*

Proof. Since $J_{\text{st}}^2 = -I$, we have

$$\begin{aligned} Q &= (J_{\text{st}} + J)^{-1}(J_{\text{st}} - J) \\ &= (J_{\text{st}} + J)^{-1}J_{\text{st}}^{-1}J_{\text{st}}(J_{\text{st}} - J) \\ &= (-I + J_{\text{st}}J)^{-1}(-I - J_{\text{st}}J) \\ &= (I - K)^{-1}(I + K), \text{ where } K = J_{\text{st}}J \end{aligned}$$

Notice that $(I - K)^{-1}$ and $(I + K)$ commute. Hence

$$
\begin{aligned}
QJ_{\text{st}} + J_{\text{st}}Q = 0 &\iff (I-K)^{-1}(I+K)J_{\text{st}} + J_{\text{st}}(I+K)(I-K)^{-1} = 0 \\
&\iff (I+K)J_{\text{st}}(I-K) + (I-K)J_{\text{st}}(I+K) = 0 \\
&\iff (I+K)(J_{\text{st}}+J) + (I-K)(J_{\text{st}}-J) = 0 \\
&\iff 2J_{\text{st}} + 2KJ = 0 \\
&\iff J^2 = -I
\end{aligned}
$$

□

Notice that $QJ_{\text{st}} + J_{\text{st}}Q = 0$ is equivalent to Q being a complex anti-linear operator. Therefore Lemma 2.2 implies that there is a unique matrix $A \in$Mat$(n, \mathbb{C})$ such that

$$Av = Q\overline{v}, v \in \mathbb{C}^n.$$

We introduce

$$\mathcal{J} = \{J : \mathbb{C}^n \to \mathbb{C}^n : J \text{ is } \mathbb{R}\text{-linear}, J^2 = -I, \det(J_{\text{st}} + J) \neq 0\},$$

$$\mathcal{A} = \{A \in \text{Mat}(n, \mathbb{D}) : \det(I - A\overline{A}) \neq 0\}.$$

It is proved in [1] and [2] that the map $J \mapsto A$ is a birational homeomorphism $\mathcal{J} \to \mathcal{A}$.

Let J be an almost complex structure in a domain $\Omega \subset \mathbb{C}^n$. Suppose $J(z) \in \mathcal{J}, z \in \Omega$. Then J defines a unique complex matrix function A in Ω such that $A(z) \in \mathcal{A}, z \in \Omega$. We call A the complex matrix of J. The matrix A has the same regularity properties as J. A function $f : \Omega \to \mathbb{C}$ is

(J, J_{st})-holomorphic if and only if it satisfies the Cauchy-Riemann equations

$$f_{\bar{z}} + f_z A = 0, \tag{2.3}$$

where $f_{\bar{z}} = (f_{\overline{z_1}}, \ldots, f_{\overline{z_n}})$ and $f_z = (f_{z_1}, \ldots, f_{z_n})$ are considered as row vectors.

Let M be a smooth manifold of real dimension $2n$. A closed non-degenerate exterior 2-form ω on M is called a symplectic form on M. A couple (M, ω) is called a symplectic manifold. A basic example is $M = \mathbb{C}^n$ with the coordinates $z_j = x_j + iy_j, j = 1, \ldots, n$. The form $\omega_{\text{st}} = \sum_{j=1}^n dx_j \wedge dy_j = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\overline{z_j}$ is called the standard symplectic form on $\mathbb{C}^n$.

A symplectic form ω tames an almost complex structure J on M if $\omega(u, Ju) > 0$, for all $u \neq 0$. A basic example is $(M, \omega, J) = (\mathbb{C}^n, \omega_{\text{st}}, J_{\text{st}})$

Lemma 2.3 *Let J be an almost complex structure on $\mathbb{C}^n$, then J is tamed by ω_{st} if and only if the complex matrix A of J satisfies the condition*

$$\|A(z)\| < 1, \text{ for all } z \in \mathbb{C}^n. \tag{2.4}$$

Here the matrix norm is induced by the Euclidean inner product, that is, $\|A\| = \max_{0 \neq v \in \mathbb{R}^{2n}} |Av|_{\mathbb{R}^{2n}} / |v|_{\mathbb{R}^{2n}}$.

Proof. Recall the complex matrix A is defined by $Av = Q\overline{v}$ where $Q = (J_{\text{st}} + J)^{-1}(J_{\text{st}} - J)$ provided that $\det(J_{\text{st}} + J) \neq 0$. We need to first show that $\det(J_{\text{st}} + J) \neq 0$ if J is tamed by ω_{st}, hence Q is well-defined. Note that for any $0 \neq v \in \mathbb{R}^{2n}$,

$$\omega_{\text{st}}(v, (J + J_{\text{st}})v) = \omega_{\text{st}}(v, Jv) + \omega_{\text{st}}(v, J_{\text{st}}v) > 0,$$

which implies $\ker(J + J_{\text{st}}) = \{0\}$. Hence $J + J_{\text{st}}$ is invertible.

In the proof of Lemma 2.2 we showed that, for any $\mathbb{R}$-linear map J such that $\det(J_{\text{st}} + J) \neq 0$, $Q = (I - K)^{-1}(I + K) = (I + K)(I - K)^{-1}$ where $K = J_{\text{st}}J$. Denote by $|\cdot|$ the standard Euclidean norm on $\mathbb{R}^{2n}$, then we have

$$\begin{aligned}
\|Q\| < 1 &\Longleftrightarrow |Qx| < |x|, \forall 0 \neq x \in \mathbb{R}^{2n} \\
&\Longleftrightarrow |(I + K)(I - K)^{-1}x| < |x|, \forall x \neq 0 \\
&\Longleftrightarrow |(I + K)x| < |(I - K)x|, \forall x \neq 0 \\
&\Longleftrightarrow \|I + K\| < \|I - K\|
\end{aligned}$$

Now it remains to show that $\|I + K\| < \|I - K\|$ if and only if J is ω_{st} tamed. For each nonzero $v \in \mathbb{R}^{2n}$, we compute that

$$\begin{aligned}
|v - Kv|^2 - |v + Kv|^2 &= \omega_{\text{st}}(v - Kv, J_{\text{st}}(v - Kv)) - \omega_{\text{st}}(v + Kv, J_{\text{st}}(v + Kv)) \\
&= \omega_{\text{st}}(v - Kv, J_{\text{st}}v + Jv) - \omega_{\text{st}}(v + Kv, J_{\text{st}}v - Jv) \\
&= 2\omega_{\text{st}}(v, Jv) - 2\omega_{\text{st}}(Kv, J_{\text{st}}v) \\
&= 2\omega_{\text{st}}(v, Jv) - 2\omega_{\text{st}}(Jv, v) \\
&= 4\omega_{\text{st}}(v, Jv).
\end{aligned}$$

Therefore J is ω_{st} tamed if and only if $\omega_{\text{st}}(v, Jv) > 0$ for all nonzero $v \in \mathbb{R}^{2n}$, if and only if $|v - Kv| > |v + Kv|$ for all nonzero $v \in \mathbb{R}^{2n}$, if and only if $\|Q\| < 1$, if and only if $\|A\| < 1$. □

For a map $u : \mathbb{D} \to \mathbb{C}^n$, the (symplectic) area of u is given by

$$\text{Area}(u) = \int_{\mathbb{D}} u^*\omega_{\text{st}}. \tag{2.5}$$

In the case where u is a J-holomorphic disc and J is ω_{st} tamed, consider the canonical Riemannian metric $g(X,Y) = \frac{1}{2}(\omega_{\text{st}}(X,JY) + \omega_{\text{st}}(Y,JX))$ determined by J and ω_{st}. For $s+it \in \mathbb{D}$, we have $\partial_t u = J\partial_s u$. Therefore $\partial_s u, \partial_t u$ are g orthogonal vectors with equal length. Thus the geometric area of the parallelogram spanned by these two vectors is simply

$$|\partial_s u|_g \cdot |\partial_t u|_g = |\partial_s u|_g^2 = \omega_{\text{st}}(\partial_s u, J\partial_s u) = \omega_{\text{st}}(\partial_s u, \partial_t u),$$

hence the symplectic area of Z coincides with the area induced by the Riemannian metric canonically defined by J and ω_{st}; in particular, it coincides with the Euclidean area if $J = J_{\text{st}}$ (see [14] for more details). We also use (2.5) for the Euclidean area of complex analytic sets in $\mathbb{C}^n$.

2.2 Fredholm operators and the Cauchy-Green operator

We denote by $C^{k,\alpha}(\mathbb{D})$ the space of functions in $\mathbb{D}$ whose partial derivatives to order k satisfy a Hölder condition with exponent $0 < \alpha < 1$, that is

$$C^{k,\alpha}(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C}^n \colon \sup_{\substack{z,w\in\mathbb{D} \\ z\neq w}} \frac{|D^k f(z) - D^k f(w)|}{|z-w|^\alpha} < \infty \right\}.$$

We write $C^\alpha(\mathbb{D}) = C^{0,\alpha}(\mathbb{D})$. We denote by $W^{k,p}(\mathbb{D})$ the Sobolev space of functions with derivatives to order k in $L^p(\mathbb{D})$.

Theorem 2.4 *(Hölder's inequality) Let (S, Σ, μ) be a measure space and let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all measurable real or complex-valued* *functions f and g on S,*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Definition 2.5 *A linear operator $T : X \to Y$ between two Banach spaces is called a compact operator if the image under T of the unit ball in X has compact closure in Y.*

Definition 2.6 *Let $T : X \to Y$ be a bounded linear operator between two Banach spaces. T is said to be Fredholm if the following hold.*

1. $\ker T$ *is finite dimensional.*

2. $\mathrm{Coker} T$ *is finite dimensional.*

It is a standard result in operator theory that an operator $T : X \to Y$ is Fredholm if and only if it is invertible modulo compact operators, that is, if there exists a bounded linear operator $S : Y \to X$ such that

$$\mathrm{Id}_X - ST \text{ and } \mathrm{Id}_Y - TS$$

are compact operators on X and Y respectively.

The following result follows immediately from this alternative characterization of Fredholm operator.

Proposition 2.7 *If $K : X \to X$ is a compact operator, then $I + K$ is Fredholm.*

The main tool that we will use to study J-holomorphic discs is the Cauchy-Green operator (or Cauchy-Green integral)

$$Tu(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{u(\omega) d\omega \wedge d\overline{\omega}}{\omega - \zeta}.$$

The following regularity properties of the Cauchy-Green operator can be found in [3].

Proposition 2.8 *(i) Let $p > 2$ and $\alpha = (p-2)/p$. Then the linear operator $T : L^p(\mathbb{D}) \to W^{1,p}(\mathbb{D})$ is bounded. The inclusion $W^{1,p}(\mathbb{D}) \subset C^\alpha(\mathbb{D})$ is bounded, hence the operator $T : L^p(\mathbb{D}) \to C^\alpha(\mathbb{D})$ is bounded, and $T : L^p(\mathbb{D}) \to L^\infty(\mathbb{D})$ is compact. If $f \in L^p(\mathbb{D})$, then $\partial_{\bar{\zeta}} Tf = f$ for $\zeta \in \mathbb{D}$, in the sense of distribution.*

(ii) Let $1 < p < 2$ and $s = 2p(2-p)^{-1}$. Then $T : L^p(\mathbb{D}) \to L^s(\mathbb{D})$ is bounded.

(iii) Let $k \geq 0$ be an integer and let $0 < \alpha < 1$. Then $T : C^{k,\alpha}(\mathbb{D}) \to C^{k+1,\alpha}(\mathbb{D})$ is bounded.

CHAPTER 3

WHITNEY'S EMBEDDING THEOREM FOR J-HOLOMORPHIC DISCS

Thom's transversality theorem and Whitney's approximation theorem are useful tools in analysis and geometry. They are proved by local perturbations and use cut-off functions to obtain a global result. Since there are no holomorphic cut-off functions, we might need a global perturbation in order to prove the corresponding results. Kaliman and Zaidenberg [4] proved the jet transversality theorem for any holomorphic mapping from a Stein manifold to a complex manifold if the domain of the initial map is shrunk. Forstnerič [5] proved a similar result without shrinking the domain, but the target space is required to be either subelliptic or satisfy the Oka property. In the almost complex category, Sukhov and Tumanov [6] proved Thom's transversality theorem and Whitney's immersion theorem for J-holomorphic discs. In this chapter we will show that Whitney's embedding theorem holds for J-holomorphic discs.

Theorem 3.1 *Let (M, J) be a C^∞-smooth almost complex manifold with $dim_{\mathbb{C}} M > 2$, fix $m \geq 0$ and let $f_0 : \mathbb{D} \to M$ be a J-holomorphic disc of class $C^m(\overline{\mathbb{D}})$. Then there exists a J-holomorphic embedding $f : \mathbb{D} \to M$ arbitrarily close to f_0 in $C^m(\overline{\mathbb{D}})$.*

3.1 Fredholm theory and surjectivity of the jet map

Let B_1, B_2 be $n \times n$ matrix functions on $\mathbb{D}$ of class $L^p(\mathbb{D})$, $p > 2$. Solutions of the equation

$$u_{\bar{\zeta}} = B_1 u + B_2 \overline{u} \tag{3.1}$$

in the class $W^{1,p}(\mathbb{D})$ are called generalized holomorphic vectors.

In this section we will prove the existence of solution of equation (3.1) with prescribed value at two given points.

Fix $\tau \in \mathbb{D}$, we define an operator

$$T_\tau u = Tu - Tu(\tau), \text{ for } \tau \in \mathbb{D},$$

$$P_\tau u = u - T_\tau(B_1 u + B_2 \overline{u}).$$

Let $r > 2p(p-2)^{-1}$. Then $s = (1/p + 1/r)^{-1} > 2$. If $u \in L^r(\mathbb{D})$, then by Hölder's inequality $B_1 u, B_2 \overline{u} \in L^s(\mathbb{D})$. Then $T(B_1 u + B_2 \overline{u}) \in W^{1,s}(\mathbb{D})$ is continuous, hence the operator

$$P_\tau : L^r(\mathbb{D}) \to L^r(\mathbb{D})$$

is bounded. The equation (3.1) is equivalent to the equation

$$P_\tau u = \phi \tag{3.2}$$

where ϕ is a $\mathbb{C}^n$-valued holomorphic function on $\mathbb{D}$ with $\phi(\tau) = u(\tau)$. In the case $n = 1$, equation (3.2) admits a solution for every holomorhpic function ϕ. This is a fundamental result of the theory of generalized analytic functions [3]. If $n > 1$, then this equation does not necessarily give a one-to-one correspondence between generalized holomorphic vectors and holomorphic vectors.

By Proposition 2.8, the operator $u \mapsto T_\tau(B_1 u + B_2 \overline{u})$ is compact and hence Proposition 2.7 shows that P_τ is Fredholm. Therefore the kernel of P_τ is finite dimensional. We will show in the theorem below that we can modify P_τ by adding a small holomorphic term to obtain an operator with trivial kernel.

Theorem 3.2 *Let B_1, B_2 be $n \times n$ matrices in $L^p(\mathbb{D}), p > 2$ and $\tau \in \mathbb{D}$.*

(i) Let $w_1, \ldots, w_d$ form a basis of $\ker P_\tau$ over $\mathbb{R}$ and let $r > 2p(p-2)^{-1}$, then there exists holomorphic polynomial vectors $p_1, \ldots, p_d$ with $p_1(\tau) = \ldots = p_d(\tau) = 0$ such that the operator $\widetilde{P_\tau} : L^r(\mathbb{D}) \to L^r(\mathbb{D})$ defined by

$$\widetilde{P_\tau} u = P_\tau u + \sum_{j=1}^{d} (\mathrm{Re}(u, w_j)) p_j \tag{3.3}$$

has trivial kernel. The polynomials p_j can be chosen to be arbitrarily small.

(ii) If $B_1, B_2 \in C^{k,\alpha}(\mathbb{D})$, $0 < \alpha < 1$ (resp. $W^{k,p}(\mathbb{D})$) then $\widetilde{P_\tau}$ is an invertible bounded operator in $C^{k+1,\alpha}(\mathbb{D})$ (resp. $W^{k+1,p}(\mathbb{D})$). The function $\phi = \widetilde{P_\tau} u$ is holomorphic if and only if u satisfies equation (3.1). Furthermore, $\widetilde{P_\tau} u(\tau) = u(\tau)$.

Before proving Theorem 3.2, we first define the adjoint P_τ^* of the operator P_τ. For vector functions $u = (u_1, \cdots, u_n), v = (v_1, \cdots, v_n)$, denote by $(\cdot, \cdot)$ the usual inner product

$$(u, v) = \sum_{j=1}^{n} \frac{i}{2} \int_{\mathbb{D}} u_j \overline{v_j} d\zeta \wedge \overline{d\zeta}.$$

We consider the adjoint with respect to the real inner product $\mathrm{Re}(\cdot, \cdot)$. The operator P_τ^* is defined on $L^q(\mathbb{D})$ with $q = r(r-1)^{-1}$. Given a function χ, denote by S_χ the operator defined by

$$S_\chi u = \frac{1}{2\pi i}\int_{\mathbb{D}} \chi(\zeta)u(\zeta)d\zeta \wedge d\overline{\zeta}.$$

We write S_1 for this operator when $\chi = 1$.

Lemma 3.3 *(i) The adjoint of the operator of matrix multiplication $u \mapsto B_j u$ has the form $u \mapsto B_j^* u$, where B_j^* denotes the hermitian transpose;*

(ii) $T^* = -\overline{T}$, where $\overline{T}u := \overline{T(\overline{u})}$;

(iii) The conjugation operator $\sigma : u \mapsto \overline{u}$ is self-adjoint: $\sigma^* = \sigma$;

(iv) $S_\chi^* = \overline{\chi} S_1$;

(v) $T_\tau^* = -\overline{T} - (\overline{\zeta - \tau})^{-1} S_1$;

(vi) $P_\tau^* = I + (\overline{\zeta - \tau})^{-1} B_1^* \overline{T}(\overline{\zeta - \tau}) + (\zeta - \tau)^{-1} \overline{B}_2^* T(\zeta - \tau)\sigma$.

Proof. Parts (i) - (iv) follow from simple calculations, so their proofs will be omitted.

For part (v), we compute

$$
\begin{aligned}
\mathrm{Re}(T_\tau u, v) &= \mathrm{Re}\sum_{j=1}^{n}\frac{i}{2}\int_{\mathbb{D}}(Tu_j - Tu_j(\tau))\overline{v_j}d\zeta\wedge d\overline{\zeta} \\
&= \mathrm{Re}(Tu, v) - \mathrm{Re}\sum_{j=1}^{n}\frac{i}{2}\int_{\mathbb{D}} Tu_j(\tau)\overline{v_j}d\zeta\wedge d\overline{\zeta} \\
&= \mathrm{Re}(u, T^* v) - \mathrm{Re}\sum_{j=1}^{n}\frac{i}{2}\int_{\mathbb{D}}\frac{1}{2\pi i}\left(\int_{\mathbb{D}}\frac{u_j(w)}{w-\tau}dw\wedge d\overline{w}\right)\overline{v_j}(\zeta)d\zeta\wedge d\overline{\zeta} \\
&= \mathrm{Re}(u, T^* v) - \mathrm{Re}\sum_{j=1}^{n}\frac{i}{2}\int_{\mathbb{D}}\frac{1}{2\pi i}\left(\int_{\mathbb{D}}\overline{v_j}(\zeta)d\zeta\wedge d\overline{\zeta}\right)\frac{u_j(w)}{w-\tau}dw\wedge d\overline{w} \\
&= \mathrm{Re}(u, T^* v) - \mathrm{Re}\sum_{j=1}^{n}\frac{i}{2}\int_{\mathbb{D}}\overline{S_1 v_j}\frac{u_j(w)}{w-\tau}dw\wedge d\overline{w} \\
&= \mathrm{Re}(u, T^* v) - \mathrm{Re}\sum_{j=1}^{n}\frac{i}{2}\int_{\mathbb{D}}u_j(w)\overline{\frac{S_1 v_j}{\overline{w}-\overline{\tau}}}dw\wedge d\overline{w} \\
&= \mathrm{Re}(u, T^* v) - \mathrm{Re}\left(u(\zeta), \frac{S_1 v}{\overline{\zeta}-\overline{\tau}}\right).
\end{aligned}
$$

Hence $T_\tau^* = -\overline{T} - (\overline{\zeta-\tau})^{-1}S_1$.

For part (vi), notice that

$$
\begin{aligned}
P_\tau^* &= I - (B_1^* + \sigma B_2^*)T_\tau^* \\
&= I - (B_1^* + \sigma B_2^*)(-\overline{T} - (\overline{\zeta}-\overline{\tau})^{-1}S_1) \\
&= I + B_1^*(\overline{T} + (\overline{\zeta}-\overline{\tau})^{-1}S_1) + \overline{B}_2^*(T\sigma + (\zeta-\tau)^{-1}\sigma S_1)
\end{aligned}
$$

Therefore to prove Part (vi), it suffices to prove that $\overline{T} + (\overline{\zeta}-\overline{\tau})^{-1}S_1 = (\overline{\zeta-\tau})^{-1}\overline{T}(\overline{\zeta-\tau})$ and $T\sigma + (\zeta-\tau)^{-1}\sigma S_1 = (\zeta-\tau)^{-1}T(\zeta-\tau)\sigma$.

We will give a proof for $\overline{T} + (\overline{\zeta}-\overline{\tau})^{-1}S_1 = (\overline{\zeta-\tau})^{-1}\overline{T}(\overline{\zeta-\tau})$. The formula $T\sigma + (\zeta-\tau)^{-1}\sigma S_1 = (\zeta-\tau)^{-1}T(\zeta-\tau)\sigma$ can be proved similarly.

A direct computation shows that

$$\begin{aligned}\overline{T}u + (\overline{\zeta}-\overline{\tau})^{-1}S_1 u &= \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{u(w)}{\overline{w}-\overline{\zeta}} dw\wedge d\overline{w} + \frac{1}{\overline{\zeta}-\overline{\tau}}\frac{1}{2\pi i}\int_{\mathbb{D}} u(w) dw\wedge d\overline{w} \\ &= \frac{1}{2\pi i}\int_{\mathbb{D}} \frac{u(w)}{\overline{w}-\overline{\zeta}} + \frac{u(w)}{\overline{\zeta}-\overline{\tau}} dw\wedge d\overline{w} \\ &= \frac{1}{\overline{\zeta}-\overline{\tau}}\frac{1}{2\pi i}\int_{\mathbb{D}} \frac{(\overline{w}-\overline{\tau})u(w)}{\overline{w}-\overline{\zeta}} dw\wedge d\overline{w} \\ &= \frac{1}{\overline{\zeta}-\overline{\tau}}\overline{T}((\overline{\zeta}-\overline{\tau})u(\zeta)).\end{aligned}$$

□

The proof of the following lemma and corollary can be found in [6].

Lemma 3.4 *Let $u \in W^{1,p}(\mathbb{D})$ be a solution of equation (3.1), and let $\zeta_0 \in \partial\mathbb{D}$. Then there exists a neighborhood U of ζ_0 in $\mathbb{C}$, a continuous nonsingular matrix function S in $\overline{\mathbb{D}}\cap U$, and a vector function ϕ holomorphic in $\mathbb{D}\cap U$ and continuous in $\overline{\mathbb{D}}\cap U$, such that $u = S\phi$.*

Corollary 3.5 *Let $u \in W^{1,p}(\mathbb{D})$ be a solution of equation (3.1). Suppose $u = 0$ on an arc (or a set of positive length) $E \subset \partial\mathbb{D}$. Then $u \equiv 0$.*

Denote by H_τ the space of all holomorphic vector functions $h \in L^r(\mathbb{D})$ such that $h(\tau) = 0$, then H_τ is a closed subspace of $L^r(\mathbb{D})$.

Lemma 3.6 H_τ+Range$P_\tau = L^r(\mathbb{D})$.

Proof. Let $q = r(r-1)^{-1}$, hence $1 < q < 2$. Note that $(L^r(\mathbb{D}))^* = L^q(\mathbb{D})$. Suppose there exists $v \in L^q(\mathbb{D})$ such that v is orthogonal to both H_τ and RangeP_τ. We will show that $v = 0$.

Since $(\text{Range}P_\tau)^\perp = \ker P_\tau^*$, we have $P_\tau^* v = 0$, hence

$$(\overline{\zeta}-\overline{\tau})v = -B_1^*\overline{T((\zeta-\tau)\overline{v})} - \frac{\overline{\zeta}-\overline{\tau}}{\zeta-\tau}\overline{B_2}^* T((\zeta-\tau)\overline{v}). \tag{3.4}$$

We now apply bootstrapping to show that $(\zeta-\tau)\overline{v} \in L^p(\mathbb{D})$ (Recall $p > 2$ is a fixed constant). In fact, suppose $(\zeta-\tau)\overline{v} \in L^t(\mathbb{D})$ for $1 < t < 2$, then by Proposition 2.8 we know that $T((\zeta-\tau)\overline{v}) \in L^s(\mathbb{D})$ with $s = 2t(t-1)^{-1}$. Then by equation (3.4) and Hölder's inequality, we have $(\zeta-\tau)\overline{v} \in L^k(\mathbb{D})$ with

$$k = \frac{1}{p^{-1}+s^{-1}} = \frac{t}{1-\beta t} > \frac{t}{1-\beta}$$

where $\beta = \frac{1}{2} - \frac{1}{p}$ with $0 < \beta < \frac{1}{2}$. Now we are given that $(\zeta-\tau)\overline{v} \in L^q(\mathbb{D})$ where $1 < q < 2$, then we can apply the above argument finitely many times to get $(\zeta-\tau)\overline{v} \in L^p(\mathbb{D})$. Put $u = T((\zeta-\tau)\overline{v}) \in L^q(\mathbb{D})$, then by Proposition 2.8 we have $u \in W^{1,p}(\mathbb{D}) \subset C^\alpha(\overline{\mathbb{D}})$ with $\alpha = (p-2)p^{-1}$.

On the other hand, since v is orthogonal to H_τ, the function $u = T((\zeta-\tau)\overline{v})$ vanishes on $\mathbb{C}\setminus\overline{\mathbb{D}}$ and, in particular, on $\partial\mathbb{D}$. Moreover by equation (3.4), the function u on $\mathbb{D}$ satisfies

$$u_{\overline{\zeta}} = -\overline{B_1}^* u - \frac{\zeta-\tau}{\overline{\zeta}-\overline{\tau}} B_2^* \overline{u}.$$

Therefore we can apply Corollary 3.5 to the above equation and conclude that $u \equiv 0$. Hence $v \equiv 0$ on $\mathbb{D}$. This completes the proof of the lemma. □

Proof of Theorem 3.2. Since P_τ is Fredholm, it has finite dimensional cokernel, hence by Lemma 3.6 there exists $p_1, \cdots, p_d \in H_\tau$ such that

$$\text{Span}_{\mathbb{R}}(p_1, \ldots, p_d) \oplus \text{Range} P_\tau = L^r(\mathbb{D}) \tag{3.5}$$

By polynomial approximation, we can choose p_j to be polynomial. We now show that the operator $\widetilde{P_\tau}$ defined by equation (3.3) has trivial kernel. Let $\widetilde{P_\tau} u = 0$. Then by equation (3.5) we have $P_\tau u = 0$ and $\text{Re}(u, w_j) = 0$, for $j = 1, \ldots, d$. Since the functions $w_1, \ldots, w_d$ form a basis of $\ker P_\tau$ over

$\mathbb{R}$, we get $u = 0$. This proves Part (i). Part (ii) is immediate. The proof of Theorem 3.2 is completed. □

Theorem 3.7 *Let B_1, B_2 be $n \times n$ matrix functions on $\mathbb{D}$ of class $C^{k-1,\alpha}(\mathbb{D})$, $k \geq 1, 0 < \alpha < 1$. Then for all $\zeta_1, \zeta_2 \in \mathbb{D}$, $a_1, a_2 \in \mathbb{C}^n$, there exists a solution u of equation (3.1) such that $u \in C^{k,\alpha}(\mathbb{D})$ and*

$$u(\zeta_1) = a_1, u(\zeta_2) = a_2.$$

Proof. By Theorem 5.1 in [6], there exists $u_1 \in C^{k,\alpha}(\mathbb{D})$ such that $u_1(\zeta_1) = a_1$. Let $u(\zeta) = u_1(\zeta) + (\zeta - \zeta_1)w(\zeta)$ where w has to be determined. If we require $B_1 u + B_2 \overline{u} = u_{\overline{\zeta}}$ and $u(\zeta_2) = a_2$, then w has to satisfy

$$B_1(\zeta)w(\zeta) + B_2(\zeta)\frac{\overline{\zeta - \zeta_1}}{\zeta - \zeta_1}\overline{w(\zeta)} = w_{\overline{\zeta}}(\zeta).$$

and

$$w(\zeta_2) = \frac{a_2 - u_1(\zeta_2)}{\zeta_2 - \zeta_1} \triangleq \phi \text{ which is a constant vector.}$$

Note that $\widehat{B_2(\zeta)} = B_2(\zeta)\frac{\overline{\zeta - \zeta_1}}{\zeta - \zeta_1} \in L^p(\mathbb{D})$ for all $2 < p < \infty$, therefore we can apply Theorem 3.2 with $\tau = \zeta_2$ and $B_2 = \widehat{B_2}$. Set $w = \widetilde{P_{\zeta_2}}^{-1}(\phi)$, then we have $w \in W^{1,p}(\mathbb{D})$ for all $2 < p < \infty$, hence $w \in C^{\alpha}(\mathbb{D})$, therefore $\phi = \widetilde{P_{\zeta_2}} w(\zeta_2) = P_{\zeta_2} w(\zeta_2) = w(\zeta_2)$, and $\widetilde{P}_{\zeta_2} w = \phi$ implies $w_{\overline{\zeta}} = B_1 w + \widehat{B_2}\overline{w}$.

It remains to show that u is in the class $C^{k,\alpha}(\mathbb{D})$. We first choose $0 < r < 1$ so that $\mathbb{D}(r)$, the disc centered at the origin with radius r, satisfying $\mathbb{D}(r) \subset \mathbb{D}$ and $\zeta_1 \in \mathbb{D}(r)$. Note that the coefficient matrices $B_1, \widehat{B_2}$ are in $C^{k-1,\alpha}(\mathbb{D} \setminus \mathbb{D}(r))$ and it is proved that $w \in C^{\alpha}(\mathbb{D})$, so we can apply bootstrapping to the equation $\widetilde{P_{\zeta_2}} w = \phi$ to conclude that w is in $C^{k,\alpha}(\mathbb{D} \setminus \mathbb{D}(r))$.

Note that the function $v(\zeta) = (\zeta - \zeta_1)w(\zeta)$ satisfies equation (3.1), then we have

$$v(z) = T(B_1 v + B_2 \overline{v})(z) + \psi(z),$$

for some vector $\psi(z)$ holomorphic on $\mathbb{D}$ and in $C^\alpha(\mathbb{D})$. We claim that $\psi \in C^{k,\alpha}(\partial\mathbb{D})$: write $T(B_1 v + B_2 \overline{v})(z)$ as

$$\frac{1}{2\pi i}\int_{\mathbb{D}(r)} \frac{B_1 v + B_2 \overline{v}}{\zeta - z} d\zeta \wedge d\overline{\zeta} + \frac{1}{2\pi i}\int_{\mathbb{D}\setminus\mathbb{D}(r)} \frac{B_1 v + B_2 \overline{v}}{\zeta - z} d\zeta \wedge d\overline{\zeta},$$

for which we understand it as an integration applied to each entry of the vector function. Now the first term is holomorphic on $\partial\mathbb{D}$ since the function $\frac{1}{\zeta - z}$ is holomorphic in z on $\partial\mathbb{D}$ whenever ζ is in $\mathbb{D}(r)$, hence it is in $C^{k,\alpha}(\partial\mathbb{D})$. The second term is in $C^{k,\alpha}(\mathbb{D} \setminus \mathbb{D}(r))$ since $v \in C^{k,\alpha}(\mathbb{D} \setminus \mathbb{D}(r))$ and $B_1, B_2 \in C^{k-1,\alpha}(\mathbb{D} \setminus \mathbb{D}(r))$. Since also $v \in C^{k,\alpha}(\mathbb{D} \setminus \mathbb{D}(r))$, we have $\psi \in C^{k,\alpha}(\partial\mathbb{D})$.

Now ψ is holomorphic in $\mathbb{D}$ and in $C^{k,\alpha}(\partial\mathbb{D})$; hence by the regularity of the Laplace operator we have $\psi \in C^{k,\alpha}(\mathbb{D})$ (see, for example, [9]). By using $v \in C^\alpha(\mathbb{D})$, we can apply bootstrapping to the equation

$$v = T(B_1 v + B_2 \overline{v}) + \psi$$

to conclude that v, and hence u, is in fact in $C^{k,\alpha}(\mathbb{D})$. □

3.2 Approximation by embedded J-holomorphic discs

Given $f_0 : \mathbb{D} \to M$ a J-holomorphic disc of class $C^m(\overline{\mathbb{D}})$, we first use Theorem 1.1 in [6] to approximate f_0 by a $C^\infty(\overline{\mathbb{D}})$ J-holomorphic immersion $f_1 : \mathbb{D} \to M$, so that f_1 is arbitrarily close to f_0 in $C^m(\overline{\mathbb{D}})$.

For $0 < r < 1$, the function $z \mapsto f_{2,r}(z) = f_1(rz)$ is arbitrarily close to f_1 in $C^m(\overline{\mathbb{D}})$ as r approaches 1: By Nash's embedding theorem we can assume

$M = \mathbb{C}^N$ for some N sufficiently large, then the result can be easily proved by the fact that the i-derivative $f^{(i)}$ is uniformly continuous on $\overline{\mathbb{D}}$.

For all $\varepsilon > 0$, choose r_0 so that f_{2,r_0} is ε-close to f_1 in $C^m(\overline{\mathbb{D}})$, for simplicity denote f_{2,r_0} by f_2. Note that f_2 is an immersed J-holomorphic disc.

Lemma 3.8 *Let*

$$R = \inf\{|c_1 - c_2| : c_1, c_2 \in \overline{\mathbb{D}}, c_1 \neq c_2, f_2(c_1) = f_2(c_2)\},$$

then $R > 0$.

Proof. Suppose $R = 0$, then there exists two sequences c_1^n, c_2^n in $\overline{\mathbb{D}}$ such that $|c_1^n - c_2^n| \to 0$ as $n \to \infty$, which means $c_j^n \to c_0 \in \overline{\mathbb{D}}$ for each $j = 1, 2$. Note that $r_0 c_0 \in \mathbb{D}$ and f_1 is an immersion at $r_0 c_0$ implies f_1 is injective on some neighborhood $N \subset \mathbb{D}$ of $r_0 c_0$. However for n large enough, both $r_0 c_1^n, r_0 c_2^n$ belong to N and we have $f_1(r_0 c_1^n) = f_2(c_1^n) = f_2(c_2^n) = f_1(r_0 c_2^n)$, a contradiction. Therefore $R > 0$. $\square$

Define

$$U = \left\{(\zeta_1, \zeta_2) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}} : |\zeta_1 - \zeta_2| < \frac{R}{2}\right\}.$$

Let $\mathcal{V}^{k,\alpha}$ be the space of all solutions of equation (3.1) in the class $C^{k,\alpha}(\mathbb{D})$. By using the compactness argument as in Proposition 5.2 of [6], we have the following corollary.

Corollary 3.9 *For* $B_1, B_2 \in C^{k-1,\alpha}(\mathbb{D})$ $(k \geq 1, 0 < \alpha < 1)$*, there exists a subspace* $V \subset \mathcal{V}^{k,\alpha}$ *with* $\dim V < \infty$*, such that for all* $(\zeta_1, \zeta_2) \in (\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U$*, the mapping* $u \mapsto (u(\zeta_1), u(\zeta_2))$ *is surjective from* V *onto* $\mathbb{C}^{2n}$.

Assume for the moment that $f_2 : \mathbb{D} \to M$ is a J-holomorphic disc of class $C^{k,\alpha}(\mathbb{D})$ where k is large and $0 < \alpha < 1$, the particular values of k and α

are unimportant. We first recall the result in the proof of Theorem 1.3 of [6] about constructing a coordinate chart in a neighborhood of f_2.

Following [10], by replacing each disc in M by its graph in $\overline{\mathbb{D}} \times M$, we can assume f_2 is an embedding and contained in a single chart in $\mathbb{C} \times M$. Now we specify the choice of the chart.

For every point $p \in M$ there is a coordinate chart $\psi : U \subset M \to \mathbb{C}^n$ such that $p \in U, \psi(p) = 0$, and for the push-forward $\psi_* J = d\psi \circ J \circ d\psi^{-1}$ we have $\psi_* J(0) = J_{\mathrm{st}}$. We first show it is possible to choose such charts ψ^ζ for every point $p = f_2(\zeta)$ so that they $C^{k,\alpha}$-smoothly depend on $\zeta \in \overline{\mathbb{D}}$.

Let $G : \mathcal{U} \subset TM \to M$ be the exponential map for an arbitrary Riemannian metric on M defined in a neighborhood of the zero-section of $T(M)$ such that for $G_p := G|_{T_pM} : T_pM \to M$ we have $G_{p*}(0) =$id. The pull-back bundle $f_2^*(TM) \to \overline{\mathbb{D}}$ is trivial as a complex bundle over $\overline{\mathbb{D}}$. Then there exist J-complex linearly independent $X_1(\zeta), \dots, X_n(\zeta) \in T_pM, p = f_2(\zeta)$, which are $C^{k,\alpha}$ in $\zeta \in \overline{\mathbb{D}}$ and smooth on $\mathbb{D}$. Define $\psi^\zeta(q) = z = (z_1, \dots, z_n) \in \mathbb{C}^n$ so that $G_p^{-1}(q) = \sum_{j=1}^n z_j X_j(\zeta), p = f_2(\zeta)$. Here the product is computed using $J(p)$. Then $\psi_*^\zeta J(0) = J_{\mathrm{st}}$ because by construction ψ_*^ζ is (J, J_{st})-linear at $p = f_2(\zeta)$. The map ψ^ζ is $C^{k,\alpha}$ in $\zeta \in \overline{\mathbb{D}}$ and smooth on $\mathbb{D}$.

We can use the map $(\zeta, q) \mapsto (\zeta, \psi^\zeta(q))$ to define $C^{k,\alpha}$ coordinates in a neighborhood of the graph of f_2 in $\overline{\mathbb{D}} \times M$. By shrinking $\psi^\zeta(q)$ in ζ we obtain smooth coordinates.

Let $0 < r < 1$ and $H(\zeta, q) = (\zeta, \psi^{r\zeta}(q))$, where $\zeta \in \overline{\mathbb{D}}$ and $q \in M$ close to $f_2(\zeta)$. For r sufficiently close to 1, the smooth map

$$H : U \subset \overline{\mathbb{D}} \times M \to \overline{\mathbb{D}} \times \mathbb{C}^n$$

defines coordinates $(\zeta, z) \in \overline{\mathbb{D}} \times \mathbb{C}^n$ in a neighborhood U of the graph of f_2.

Consider the almost complex structure $J_{\text{st}} \otimes J$ on $\mathbb{C} \times M$. Then the push-forward $\tilde{J} = H_*(J_{\text{st}} \otimes J)$ is defined in a neighborhood of $\overline{\mathbb{D}} \times \{0\} \subset \overline{\mathbb{D}} \times \mathbb{C}^n$. By the definition of $\tilde{J}$, we have $\tilde{J}|_{\mathbb{D}\times\{0\}} = J_{\text{st}}$, hence $\tilde{J}$ has complex matrix $\tilde{A}$ such that $\tilde{A}(\zeta, 0) = 0$. Note that the projection $(\zeta, z) \mapsto \zeta$ is $(\tilde{J}, J_{\text{st}})$-holomorphic, hence, by equation (2.3), the matrix $\tilde{A}$ has the form

$$\tilde{A} = \begin{pmatrix} 0 & 0 \\ b & A \end{pmatrix},$$

where the matrix $A(\zeta, \cdot)$ is the complex matrix of the push-forward $\psi^{r\zeta}_* J$.

A map $f : \mathbb{D} \to M$ close to f_2 is J-holomorphic if and only if $\zeta \mapsto (\zeta, f(\zeta))$ is $J_{\text{st}} \otimes J$-holomorphic, which is equivalent to $\zeta \mapsto (\zeta, \psi^{r\zeta}(f(\zeta)))$ being $\tilde{J}$-holomorphic. Finally, it is equivalent to the map $\zeta \mapsto g(\zeta) := \psi^{r\zeta}(f(\zeta)) \in \mathbb{C}^n$ satisfying the equation

$$g_{\overline{\zeta}} = A(\zeta, g)\overline{g_{\zeta}} + b(\zeta, g). \tag{3.6}$$

Let $g_0(\zeta) = \psi^{r\zeta}(f_2(\zeta)) \in C^{k,\alpha}(\mathbb{D})$ and put $A_0(\zeta) = A(\zeta, g_0(\zeta))$. For any g, which is a solution of (3.6) in $C^{k,\alpha}(\mathbb{D})$, put $h = g - A_0\overline{g}$. By a direct computation equation (3.6) can be written as

$$h_{\overline{\zeta}} = K_0\overline{h}_{\overline{\zeta}} + K_1 h + K_2\overline{h} + q \tag{3.7}$$

for some matrix functions K_0, K_1, K_2, q. The following lemma shows that the correspondence $g \leftrightarrow h$ is one-to-one:

Lemma 3.10 *Let $H = \{h : \mathbb{D} \to \mathbb{C}^n \mid h$ satisfies (3.7)$\}$ and $G = \{g : \mathbb{D} \to \mathbb{C}^n \mid g$ satisfies (3.6)$\}$. The function $F : G \to H$ defined by $F(g) = g - A_0\overline{g}$ is bijective.*

Proof. The inverse of F is defined by $h \mapsto (I - A_0\overline{A_0})^{-1}(h + A_0\overline{h})$. □

Put $h_0 = g_0 - A_0\overline{g}_0$. We now conclude the discussion above:

Proposition 3.11 *There is a one-to-one correspondence between the set of J-holomorphic discs close to f_2 and the set of solutions of equation (3.7) close to h_0.*

Equation (3.7) is equivalent to

$$h = T_0(K_0\overline{h}_{\overline{\zeta}} + K_1 h + K_2\overline{h} + q) + \phi + \phi_0,$$

where ϕ is holomorphic, ϕ_0 is a fixed holomorphic function such that (3.7) holds with $h = h_0$ and $\phi = 0$.

Consider the C^∞ map

$$h \mapsto F_0(h) = \phi = h - T_0(K_0\overline{h}_{\overline{\zeta}} + K_1 h + K_2\overline{h} + q) - \phi_0,$$

and let $P = F_0'(h_0)$. By Theorem 3.2 we can modify F_0 to get a new function $F : h \mapsto F(h)$ where

$$F(h) = h - T_0(K_0\overline{h}_{\overline{\zeta}} + K_1 h + K_2\overline{h} + q) - \phi_0 + \sum_{j=1}^{d} \mathrm{Re}(w_j, h - h_0)p_j.$$

Note that the Fréchet derivative of F at h_0 has the form

$$F'(h_0)u = u - T_0(B_1 u + B_2\overline{u}) + \sum_{j=1}^{d} \mathrm{Re}(w_j, u)p_j, \text{ for } B_1, B_2 \in C^{k-1,\alpha}(\mathbb{D})$$

In fact $F'(h_0)$ is an isomorphism, hence by the inverse function theorem F^{-1} is well-defined and smooth in a neighborhood of zero in the space of all vector functions of class $C^{k,\alpha}(\mathbb{D})$. The map F gives a one-to-one correspondence between all solutions of (5) close to h_0 and all holomorphic functions ϕ close to 0 in $C^{k,\alpha}(\mathbb{D})$.

For any function $f: X \to Y$, let $E_{\zeta_1,\zeta_2} f = Ef(\zeta_1, \zeta_2)$ be the 0-jet of $f \times f$ at $\zeta_1, \zeta_2 \in X$ defined by

$$E_{\zeta_1,\zeta_2} f = (\zeta_1, f(\zeta_1), \zeta_2, f(\zeta_2)).$$

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let

$$J_1 = \left\{ \text{0-jet of } f \times f : (\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U \to M \times M \mid f \text{ is } J\text{-holomorphic} \right\},$$

$$J_2 = \left\{ \text{0-jet of } h \times h : (\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U \to \mathbb{C}^{2n} \mid h \text{ satisfies (3.7)} \right\}.$$

Proposition 3.11 gives rise to a diffeomorphism

$$\Psi : W \to \widetilde{W}$$

defined in the neighborhood $W \supset Ef_2((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U)$ in J_1, and we denote $\widetilde{W} = \Psi(W)$ in J_2.

Let

$$\Delta = \left\{ (\zeta_1, x, \zeta_2, x) \in \overline{\mathbb{D}} \times M \times \overline{\mathbb{D}} \times M \mid (\zeta_1, \zeta_2) \in (\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U, x \in M \right\}$$

be a subset of J_1, then Ef is transversal to Δ if and only if the corresponding Eh is transversal to $\widetilde{\Delta} = \Psi(\Delta)$. (Here we only consider the case $\Delta \cap W \neq \emptyset$ and denote $\Delta \cap W$ by Δ again, otherwise $Ef \pitchfork \Delta$ for small perturbation of f_2.)

Let $u_1, \ldots, u_N$ be the basis of $V \subset \mathcal{V}^{k,\alpha}$, the space of all solutions of

$u_{\overline{\zeta}} = B_1 u + B_2 \overline{u}$ in the class $C^{k,\alpha}(\mathbb{D})$. Let $\phi_j = F'(h_0)(u_j)$ and $\phi_s = \sum_l s_l \phi_l$ for $s = (s_1, \ldots, s_N) \in \mathbb{R}^N$. The map

$$\begin{aligned} \Phi : ((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U) \times \mathbb{R}^N &\to J_2 \\ (\zeta_1, \zeta_2, s) &\to EF^{-1}(\phi_s)(\zeta_1, \zeta_2) \end{aligned}$$

is defined for small $s \in \mathbb{R}^N$. By Corollary 3.9 the mapping is a submersion for $s = 0, (\zeta_1, \zeta_2) \in (\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U$, by shrinking the domain of s in $\mathbb{R}^N$, we see that Φ is a submersion. Hence by parametric transversality, there is $s \in \mathbb{R}^N$ arbitrarily close to 0 such that $EF^{-1}(\phi_s)$ is transversal to $\widetilde{\Delta}$, hence the corresponding Ef is transversal to Δ.

Now Ef is transversal to Δ implies $Ef((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U) \cap \Delta = \emptyset$. To see this, first observe that $\dim_{\mathbb{R}}((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U) = 4$, so

$$\dim_{\mathbb{R}} Ef((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U) = 4, \ \dim_{\mathbb{R}}(((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U) \times M) = 4 + 2n$$

and

$$\dim_{\mathbb{R}}(\overline{\mathbb{D}} \times M \times \overline{\mathbb{D}} \times M) = 4 + 4n.$$

Therefore if $Ef((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U) \cap \Delta$ is nonempty, it will imply $4+4+2n \geq 4+4n$ which gives $\dim_{\mathbb{C}} M = n \leq 2$, a contradiction.

Finally we need to check that f is injective on $\overline{\mathbb{D}}$. Suppose $f(\zeta_1) = f(\zeta_2)$ for ζ_1, ζ_2 in $\overline{\mathbb{D}}$. $Ef((\overline{\mathbb{D}} \times \overline{\mathbb{D}}) \setminus U) \cap \Delta = \emptyset$ implies $(\zeta_1, \zeta_2) \in U$, which means $|\zeta_1 - \zeta_2| < \frac{R}{2}$. By the definition of R, we have $\zeta_1 = \zeta_2$.

Hence the immersed J-holomorphic disc f is injective. Because $\overline{\mathbb{D}}$ is compact, f is in fact an embedding. Therefore we have found a J-holomorphic disc f in $C^\infty(\overline{\mathbb{D}})$ which is an embedding and arbitrarily close to f_2 and hence to f_0 in $C^m(\overline{\mathbb{D}})$. The proof of Theorem 3.1 is complete. $\square$

CHAPTER 4

SYMPLECTIC RIGIDITY OF REAL BI-DISC

Let $z_j = x_j + iy_j \in \mathbb{C}$ $(j = 1, 2)$ and let $\mathbb{D}^2_{\mathbb{R}} = \{(z_1, z_2) \in \mathbb{C}^2 : x_1^2 + x_2^2 < 1, y_1^2 + y_2^2 < 1\}$ be a real bi-disc in $\mathbb{C}^2$. Let $x_1, y_1, ..., x_n, y_n$ be the standard coordinates on the $2n$-dimensional Euclidean space, the standard symplectic form on the space is given by $dx_1 \wedge dy_1 + \cdots dx_n \wedge dy_n$. All the symplectic embeddings and symplectomorhpisms considered in this chapter will be with respect to the standard symplectic form on $\mathbb{R}^{2n}$ unless otherwise specified.

Let $\mathbb{D}(R) = \{z \in \mathbb{D} \colon |z| < R\}$. It is proved by A. Suhkov and A. Tumanov [7] that the real bi-disc $\mathbb{R}$ cannot be symplectically embedded into the complex cylinder $\mathbb{D}(R) \times \mathbb{C}$ when $R = 1$. On the other hand, it is clear that such a symplectic embedding exists for R sufficiently large. It is then natural to ask for a lower bound of R so that Ω admits such an embedding. In this chapter we will find the sharp lower bound for R.

Let $Q = \{z = x + iy \in \mathbb{C} \colon |x| < 1, |y| < 1\}$ be an open square in $\mathbb{C}$, we first consider the mapping $f_0 : \mathbb{C} \to \mathbb{C}$ defined by

$$f_0(z) = \frac{2}{\sqrt{\pi}} x e^{i\frac{\pi}{4}\frac{y}{x}}, \text{ if } -x < y \leq x \tag{4.1}$$

and f_0 maps the origin to itself. This mapping can be extended to $\mathbb{C}$ by reflecting about the lines $y = \pm x$. Note that f_0 is a symplectomorphism on $\mathbb{R}^2$ except on $y = \pm x$ and $f_0(\overline{Q}) = \overline{\mathbb{D}\left(\frac{2}{\sqrt{\pi}}\right)}$.

Let $Q^2 = \{(z_1, z_2) \in \mathbb{C}^2 \colon |x_j| < 1, |y_j| < 1, z_j = x_j + iy_j, j = 1, 2\}$ be a hypercube, $\mathbb{D}^2(R) = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_j| < R, j = 1, 2\}$ be a complex bi-disc of radius R and $F_0 = f_0 \times f_0$.

Lemma 4.1 *$B^4\left(\frac{2}{\sqrt{\pi}}\right) \subset F_0(\mathbb{D}^2_{\mathbb{R}})$, where $B^4(R) = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1|^2 + |z_2|^2 < R^2\}$*

Proof. It suffices to prove that $F_0^{-1}\left(B^4\left(\frac{2}{\sqrt{\pi}}\right)\right) \subset \mathbb{D}^2_{\mathbb{R}}$. For $z = x + iy$, denote $|z|_1 = \max\{|x|, |y|\}$. It is clear from the definition of f_0 that $|f_0(z)| = \frac{2}{\sqrt{\pi}}|z|_1$. Take $(z_1, z_2) \in F_0^{-1}\left(B^4\left(\frac{2}{\sqrt{\pi}}\right)\right)$ where $z_j = x_j + iy_j, j = 1, 2$, we have

$$
\begin{aligned}
&|f_0(z_1)|^2 + |f_0(z_2)|^2 < \frac{4}{\pi} \\
\Rightarrow &|z_1|_1^2 + |z_2|_1^2 < 1 \\
\Rightarrow &|x_1|^2 + |x_2|^2 < 1 \text{ and } |y_1|^2 + |y_2|^2 < 1 \\
\Rightarrow &(z_1, z_2) \in \mathbb{D}^2_{\mathbb{R}}
\end{aligned}
$$

□

Let Q_ε be an open domain in $\mathbb{C}$ with smooth boundary and area 4, assume also that $Q_\varepsilon \to Q$ as $\varepsilon \to 0$, then there exist symplectomorphisms $f_\varepsilon : Q_\varepsilon \to \mathbb{D}\left(\frac{2}{\sqrt{\pi}}\right)$ such that $f_\varepsilon \to f_0$ in C^0-topology as $\varepsilon \to 0$. Hence $f_\varepsilon : Q_\varepsilon \to \mathbb{D}\left(\frac{2}{\sqrt{\pi}}\right)$ is a smooth approximation of $f_0 : Q \to \mathbb{D}\left(\frac{2}{\sqrt{\pi}}\right)$. Finally set $\mathbb{D}^2_{\mathbb{R}}(\varepsilon) = \mathbb{D}^2_{\mathbb{R}} \cap (Q_\varepsilon \times Q_\varepsilon)$ then we have a symplectic embedding

$$F_\varepsilon = f_\varepsilon \times f_\varepsilon : \mathbb{D}^2_{\mathbb{R}}(\varepsilon) \to \mathbb{D}^2\left(\frac{2}{\sqrt{\pi}}\right).$$

Theorem 4.2 *Suppose $\mathbb{D}^2_{\mathbb{R}}$ admits a symplectic embedding into $\mathbb{D}(R) \times \mathbb{C}$, then $R \geq \frac{2}{\sqrt{\pi}}$.*

Proof. Suppose such a symplectic embedding exists. Because $\mathbb{D}^2_{\mathbb{R}}(\varepsilon)$ is symplectomorphic to $F_\varepsilon(\mathbb{D}^2_{\mathbb{R}}(\varepsilon))$, we have a symplectic embedding from $F_\varepsilon(\mathbb{D}^2_{\mathbb{R}}(\varepsilon))$ into $\mathbb{D}(R) \times \mathbb{C}$. By Lemma 4.1 and the fact that f_ε is a smooth approximation of f_0 in C^0-topology, there exist a ball $B^4(r_\varepsilon)$ of radius r_ε contained in $F_\varepsilon(\mathbb{D}^2_{\mathbb{R}}(\varepsilon))$ where $r_\varepsilon \to \frac{2}{\sqrt{\pi}}$ as $\varepsilon \to 0$. Therefore Gromov's non-squeezing theorem [11] implies that $R \geq r_\varepsilon$ for all $\varepsilon > 0$. By letting $\varepsilon \to 0$, we have $R \geq \frac{2}{\sqrt{\pi}}$. □

CHAPTER 5

OTHER RESULTS ON SYMPLECTIC RIGIDITY

We showed in Chapter 4 that a non-holomorphic change of coordinates

$$(x_1, y_1, x_2, y_2) \mapsto (x_1, x_2, y_1, y_2)$$

of $\mathbb{D}^2$ is not symplectomorphic to $\mathbb{D}^2$ itself.

In this chapter we will prove two theorems that generalize the result in Chapter 4. All the symplectic embeddings and symplectomorhpisms considered in this chapter will be with respect to the standard symplectic form on $\mathbb{R}^{2n}$ unless otherwise specified.

In Section 5.1 we will show that if $T \in O(4)$ is any orthogonal transformation on $\mathbb{R}^4 = \mathbb{C}^2$, then $T(\mathbb{D}^2)$ is symplectomorphic to $\mathbb{D}^2$ if and only if T is unitary or conjugate to unitary. We will give a more precise statement in Section 5.1.

In Section 5.2 we consider a high dimensional problem. We will show that for $r \geq 1$ and $n \geq 2$, $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}(r)$ is not symplectomorphic to $\mathbb{D}^2 \times \mathbb{D}^{n-2}(r)$.

Besides from the symplectic rigidity of real bi-disc, there are many results about symplectic rigidity of other domains such as ellipsoid or polydisc $\mathbb{D}(r_1) \times \cdots \times \mathbb{D}(r_n)$. For example, McDuff [12] studied when a 4-dimensional ellipsoid can be symplectically embedded in a ball; Guth [13] gave an asymptotic result on when a polydisc can be symplectically embedded into another.

There are a lot of open problems concerning symplectic rigidity, for instance it is not known that whether $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}(r)$ is symplectomorphic to $\mathbb{D}^2 \times \mathbb{D}(r)$ when $r < 1$. The results in this chapter only show that such symplectomorphism does not exist when $r \geq 1$. Another interesting open problem is to characterize when two given polydiscs are symplectomorphic.

5.1 Orthogonal transformation of complex bi-disc

Let $T \in O(4)$ be an orthogonal transformation on $\mathbb{R}^4 \simeq \mathbb{C}^2$, let $\mathbb{D}^2 = \{|z_j| < 1\colon j = 1, 2\}$ be the complex bi-disc. In this section we will give a necessary and sufficient condition for $T(\mathbb{D}^2)$ to be symplectomorphic to $\mathbb{D}^2$ with respect to the standard symplectic form on $\mathbb{C}^2$.

First of all, we define the notion of holomorphic radius and state a theorem proved by A. Sukhov and A. Tumanov [7] which provides a necessary condition on holomorphic radius for the existence of symplectic embedding.

Definition 5.1 *Let Ω be a complex manifold (e.g. a domain in $\mathbb{C}^n$). A set $A \subset \Omega$ is called a (complex) analytic set if it is, in a neighborhood of each of its points, the set of common zeros of a certain finite family of holomorphic functions.*

Definition 5.2 *A point p of an analytic set A in a complex manifold Ω is called regular if there is a neighborhood U in Ω containing p such that $A \cap U$ is a complex submanifold of Ω. The complex dimension of this submanifold is said to be the dimension of A at its regular point p, and is denoted by* $\dim_p A$. *The set of all regular points of A is denoted by* $\mathrm{reg} A$.

It is a fundamental result of complex analytic sets that the set of all regular points of an analytic set A is dense in A (see, for example, [15]).

Definition 5.3 *Let A be an analytic set in a complex manifold Ω. The dimension of A at an arbitrary point $p \in A$ is the number*

$$\dim_p A := \overline{\lim_{z \to p}} \dim_z A \text{ for } z \in \text{reg}A.$$

Then we define the dimension of A to be the maximum of its dimensions at points:

$$\dim A := \max_{z \in A} \dim_z A = \max_{z \in \text{reg}A} \dim_z A.$$

A purely m-dimensional analytic set is an analytic set of dimension m such that its dimension at all points coincide.

Definition 5.4 *Let G be a domain in $\mathbb{C}^n$ containing the origin. Denote by $\mathcal{O}_0^1(G)$ the set of closed complex purely one-dimensional analytic sets in G passing through the origin. Denote by $E(X)$ the Euclidean area of $X \in \mathcal{O}_0^1(G)$. The holomorphic radius $rh(G)$ of G is defined as*

$$\text{rh}(G) = \inf\{\lambda > 0 : \exists X \in \mathcal{O}_0^1(G), E(X) = \pi\lambda^2\}.$$

Theorem 5.5 *([7]) Let G_1 be a domain in $\mathbb{C}^2$ containing the origin and let G_2 be a domain in $\mathbb{D}(R) \times \mathbb{C}$ for some $R > 0$. Assume there exists a symplectomorphism $\phi : G_1 \to G_2$, then $rh(G_1) \leq R$.*

For $v = (v_1, \ldots, v_4), w = (w_1, \ldots, w_4) \in \mathbb{R}^4$, we denote the real inner product by $\langle v, w \rangle_{\mathbb{R}^4} = \sum_{j=1}^4 v_j w_j$. Similarly for $v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{C}^2$, we denote the complex inner product by $\langle v, w \rangle_{\mathbb{C}^2} = \sum_{j=1}^2 v_j \overline{w_j}$. Notice that $\langle v, w \rangle_{\mathbb{R}^4} = \text{Re}\langle v, w \rangle_{\mathbb{C}^2}$.

Lemma 5.6 *(i) Let $L \in \mathbb{C}^2$ be a real two dimensional plane. Denote by $L^{\perp_{\mathbb{R}^4}}$ the perpendicular complement of L with respect to $\langle \cdot, \cdot \rangle_{\mathbb{R}^4}$ and by*

$L^{\perp_{\mathbb{C}^2}}$ the perpendicular complement of L with respect to $\langle\cdot,\cdot\rangle_{\mathbb{C}^2}$. If L is a complex line, that is $v \in L$ if and only if $iv \in L$ for all $v \in \mathbb{C}^2$, then $L^{\perp_{\mathbb{R}^4}} = L^{\perp_{\mathbb{C}^2}}$.

(ii) If $L \in \mathbb{C}^2$ is a complex line, then $L^{\perp_{\mathbb{C}^2}}$ is also a complex line.

Proof. (i) Since $\mathrm{Re}\langle\cdot,\cdot\rangle_{\mathbb{C}^2} = \langle\cdot,\cdot\rangle_{\mathbb{R}^4}$, we have $L^{\perp_{\mathbb{C}^2}} \subset L^{\perp_{\mathbb{R}^4}}$. If $v \in L^{\perp_{\mathbb{R}^4}}$, we have $\mathrm{Re}\langle v,w\rangle_{\mathbb{C}^2} = 0$ for all $w \in L$. Since L is a complex line, we know that $w \in L$ if and only if $iw \in L$. Therefore $0 = \langle v, iw\rangle_{\mathbb{R}^4} = \mathrm{Im}\langle v,w\rangle_{\mathbb{C}^2}$ for all $w \in L$. Therefore $L^{\perp_{\mathbb{R}^4}} \subset L^{\perp_{\mathbb{C}^2}}$.

(ii) follows immediately from the linearity property of the inner product.

□

We denote by $\mathfrak{I}$ the set consisting of four diagonal matrices:

$$\mathfrak{I} = \left\{ \begin{pmatrix} 1 & & & \\ & a & & \\ & & 1 & \\ & & & b \end{pmatrix} \text{ where } a = \pm 1, b = \pm 1 \right\}$$

The following is the main theorem of this section. We used the canonical identification between complex matrices on $\mathbb{C}^2$ and real matrices on $\mathbb{R}^4$:

Theorem 5.7 *Let $T \in O(4)$ be an orthogonal transformation. $T\mathbb{D}^2$ is symplectomorphic to $\mathbb{D}^2$ with respect to the standard symplectic form on $\mathbb{R}^4$ if and only if there exists $U \in U(2)$ such that $UT \in \mathfrak{I}$.*

Proof. ($\Rightarrow$) Let (z_1, z_2) be the coordinate on $\mathbb{C}^2$. First of all, let $\partial\mathbb{D}^2 \cap \partial B^4(1) = S_1 \cup S_2$ where $S_1 = \{|z_1| = 1, z_2 = 0\}$ and $S_2 = \{z_1 = 0, |z_2| = 1\}$. Therefore S_1 and S_2 are contained in the complex line $H_1 = \{z_2 = 0\}$ and $H_2 = \{z_1 = 0\}$ respectively. For $i = 1, 2$, let $u_i, v_i \in \mathbb{C}^2$ be orthonormal basis of TH_i under the real inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}^4}$ on $\mathbb{R}^4$. Note that TS_i can be parameterized by

$$\frac{1}{2}\left(t + \frac{1}{t}\right) u_i + \frac{1}{2i}\left(t - \frac{1}{t}\right) v_i$$

for $|t| = 1$ in $\mathbb{C}$. The complexification of TS_i, denoted by $\widetilde{TS_i}$, is given by the same parametrization but allowing $t \in \mathbb{P}$. Hence $\widetilde{TS_i}$ is a complex algebraic curve in $\mathbb{P}^2$ parameterized by $t \in \mathbb{P}$.

Notice that for $i = 1, 2$, $\widetilde{TS_i}$ passes through the origin in $\mathbb{C}^2$ if and only if u_i and v_i are $\mathbb{C}$-dependent.

Suppose $T\mathbb{D}^2$ is symplectomorphic to $\mathbb{D}^2$, then Theorem 5.5 implies that $\mathrm{rh}(T\mathbb{D}^2) \leq 1$. By Bishop's convergence theorem there exists $X \in \mathcal{O}_0^1(T\mathbb{D}^2)$ such that $E(X) = \pi(\mathrm{rh}(T\mathbb{D}^2))^2$. Suppose there exists $p \in \partial X \cap \partial B^4(1)$ such that $p \in \mathrm{Int}(T\mathbb{D}^2)$; then X is not entirely contained in $B^4(1)$. Hence $E(X) > E(X \cap B^4(1)) \geq \pi$, which implies $\mathrm{rh}(T\mathbb{D}^2) > 1$, a contradiction. Therefore $\partial X \subset \partial B^4(1) \cap \partial T\mathbb{D}^2 = TS_1 \cup TS_2$ and X is a complex one dimensional analytic subset in $\mathbb{C}^2 \backslash (TS_1 \cup TS_2)$. Since $TS_1 \cup TS_2$ is a real one dimensional curve, it is totally real. Hence, by the reflection principle for analytic sets (see, for example, Section 20.5 of [15]), X extends as a complex one dimensional analytic set to a neighborhood of $TS_1 \cup TS_2$. By the uniqueness theorem X is contained in the complex algebraic curve $\widetilde{TS_1} \cup \widetilde{TS_2}$.

Since X contains the origin in $\mathbb{C}^2$, without loss of generality we can assume $\widetilde{TS_1}$ contains the origin. By the discussion above, we know that u_1 and v_1 are $\mathbb{C}$-dependent. Hence $TH_1 = \mathrm{span}_{\mathbb{R}}\{u_1, v_1\} = \mathrm{span}_{\mathbb{R}}\{u_1, iu_1\} = \mathrm{span}_{\mathbb{C}}\{u_1\} = \widetilde{TS_1}$. This shows that TH_1 is a complex line.

By Lemma 5.6, $H_2 = H_1^{\perp_{\mathbb{C}^2}} = H_1^{\perp_{\mathbb{R}^4}}$. Since T is an orthogonal matrix, we have $TH_2 = (TH_1)^{\perp_{\mathbb{R}^4}} = (TH_1)^{\perp_{\mathbb{C}^2}}$ where the last equality follows from Lemma 5.6 and the fact that $TH_1 = \widetilde{TS_1} = \mathrm{span}_{\mathbb{C}}\{u_1\}$ is a complex line.

Therefore Lemma 5.6 implies that TH_2 is a complex line.

We've showed that if T is orthogonal and $T\mathbb{D}^2$ is symplectomorphic to $\mathbb{D}^2$, then T maps the complex lines $H_1 = \{z_2 = 0\}, H_2 = \{z_1 = 0\}$ to complex lines TH_1, TH_2. Therefore there exists a unitary matrix $U \in U(2)$ such that $UT \in \mathfrak{I}$.

($\Leftarrow$) Suppose there exists an $U \in U(2)$ such that $UT \in \mathfrak{I}$, then we know that $UT\mathbb{D}^2 = \mathbb{D}^2$ as a set. Furthermore $U \in U(2)$ is a linear symplectomorphism on $\mathbb{C}^2$. Hence $T\mathbb{D}^2$ is symplectomorphic to $\mathbb{D}^2$. □

5.2 Symplectic rigidity in high dimensional case

Let $\mathbb{D}^m(r) = \{(z_1, \dots, z_m) \in \mathbb{C}^m : |z_j| < r \text{ for } j = 1, \dots, m\}$ by the m-th product of discs of radius r. The following is the main theorem in this section:

Theorem 5.8 *For $r \geq 1$ and $n \geq 2$, the domains $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}(r)$ and $\mathbb{D}^2 \times \mathbb{D}^{n-2}(r)$ in $\mathbb{C}^n$ equipped with the standard symplectic form are not symplectomorphic.*

We will first give the proof for the case $r > 1$ by adapting the idea in the proof of Theorem 4.2. Later we will develop a new method to prove Theorem 5.8 for the case $r = 1$.

5.2.1 The case $r > 1$

Theorem 5.9 *For $r > 1$ the domains $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}(r)$ and $\mathbb{D}^2 \times \mathbb{D}^{n-2}(r)$ in $\mathbb{C}^n$ are not symplectomorphic.*

Proof. It suffices to prove that if there exists a symplectic embedding of $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}(r)$ into $\mathbb{D}(R) \times \mathbb{C}^{n-1}$, then we must have $R > 1$. Indeed, assuming the previous statement is true and suppose $\psi : \mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}(r) \to \mathbb{D}^2 \times \mathbb{D}^{n-2}(r)$ is a symplectomorphism, then $\mathrm{Id} \circ \psi$ provides a symplectic embedding of $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}(r)$ into $\mathbb{D}(1) \times \mathbb{C}^{n-1}$, which gives a contradiction.

Recall the function given by equation (4.1) in Chapter 4

$$f_0(z) = \frac{2}{\sqrt{\pi}} x e^{i \frac{\pi}{4} \frac{y}{x}}, \text{ if } -x < y \leq x$$

extended to $\mathbb{C}$ by reflecting about the lines $y = \pm x$. Let $Q = \{z = x + iy \in \mathbb{C} : |x| < 1, |y| < 1\}$ be an open square in $\mathbb{C}$, then f_0 is a symplectomorphism on $\mathbb{R}^2$ except on $y = \pm x$ and $f_0(\overline{Q}) = \overline{\mathbb{D}(\frac{2}{\sqrt{\pi}})}$. By Lemma 4.1, we have $B^4(\frac{2}{\sqrt{\pi}}) \subset F_0(\mathbb{D}^2_{\mathbb{R}})$ where $F_0 = f_0 \times f_0$. As in the proof of Theorem 4.2, let Q_ε be an open domain in $\mathbb{C}$ with smooth boundary and area 4 such that $Q_\varepsilon \to Q$ as $\varepsilon \to 0$. We take the symplectomorphisms $f_\varepsilon : Q_\varepsilon \to \mathbb{D}(\frac{2}{\sqrt{\pi}})$ such that $f_\varepsilon \to f_0$ in C^0-topology as $\varepsilon \to 0$. Setting $\mathbb{D}^2_{\mathbb{R}}(\varepsilon) = \mathbb{D}^2_{\mathbb{R}} \cap (Q_\varepsilon \times Q_\varepsilon)$ then we have a symplectic embedding

$$F_\varepsilon = f_\varepsilon \times f_\varepsilon : \mathbb{D}^2_{\mathbb{R}}(\varepsilon) \to \mathbb{D}^2(\frac{2}{\sqrt{\pi}}).$$

Suppose there exists a symplectic embedding $\phi : \mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}(r) \to \mathbb{D}(R) \times \mathbb{C}^{n-1}$, then we need to prove that $R > 1$. For any $\varepsilon > 0$, we know that $\mathbb{D}^2_{\mathbb{R}}(\varepsilon) \times \mathbb{D}^{n-2}(r) \subset \mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}(r)$ and $\mathbb{D}^2_{\mathbb{R}}(\varepsilon) \times \mathbb{D}^{n-2}(r)$ is symplectomorphic to $F_\varepsilon(\mathbb{D}^2_{\mathbb{R}}(\varepsilon)) \times \mathbb{D}^{n-2}(r)$. Argue as in the proof of Theorem 4.2, there exists a ball $B^n(r_\varepsilon)$ contained in $F_\varepsilon(\mathbb{D}^2_{\mathbb{R}}(\varepsilon)) \times \mathbb{D}^{n-2}(r)$ where $r_\varepsilon \to \min(\frac{2}{\sqrt{\pi}}, r) > 1$ as $\varepsilon \to 0$. Therefore by Gromov's non-squeezing theorem [11] we must have $R \geq \min(\frac{2}{\sqrt{\pi}}, r) > 1$. The proof of the theorem is completed. □

5.2.2 The case $r = 1$

In order to prove the case $r = 1$, we will need to consider the limit of a sequence of analytic sets with bounded area, as well as the existence of J-holomorphic discs. We will start this subsection by stating some useful theorems.

The following theorem is known as Bishop's convergence theorem (see, for example, [15]):

Theorem 5.10 *Let $\{A_j\}$ be a sequence of purely p-dimensional analytic subsets in a complex manifold Ω with locally uniformly bounded volumes:*

$$\mathrm{Vol}_{2p}(A_j \cap K) \leq M_K < \infty$$

for any compact set $K \subset \Omega$. Here M_K is a constant depending only on K. Then we can extract a subsequence from $\{A_j\}$ converging in Ω (in Hausdorff sense) to a purely p-dimensional analytic subset or to the empty set.

The following theorem regarding the existence of J-holomorphic discs is due to A. Suhkov and A. Tumanov [8]. The original statement was about the triangular cylinder $\Delta \times \mathbb{C}^{n-1}$ where $\Delta = \{z \in \mathbb{C} : 0 < \mathrm{Im}z < 1 - |\mathrm{Re}z|\}$ instead of the circular cylinder $\mathbb{D} \times \mathbb{C}^{n-1}$. However the result still holds for the circular cylinder since $\mathbb{D}$ and Δ are biholomorphic.

Theorem 5.11 *(A. Suhkov and A. Tumanov [8]) Let A be a continuous $n \times n$ matrix function on $\mathbb{C}^n$ vanishing on $\mathbb{C}^n \setminus (\mathbb{D} \times \mathbb{C}^{n-1})$. Suppose there is a constant $0 < a < 1$ such that*

$$\|A(z)\| \leq a, \forall z \in \mathbb{D} \times \mathbb{C}^{n-1}. \tag{5.1}$$

Then there exists $p > 2$ such that for every point $x \in \mathbb{D} \times \mathbb{C}^{n-1}$ there is a solution $Z \in W^{1,p}(\mathbb{D})$ of equation (2.2)

$$Z_{\overline{\zeta}} = A(Z)\overline{Z_{\zeta}}$$

such that $Z(\overline{\mathbb{D}}) \subset \overline{\mathbb{D} \times \mathbb{C}^{n-1}}, x \in Z(\mathbb{D})$, $\text{Area}(Z) = \pi$ and

$$Z(\partial\mathbb{D}) \subset \partial(\mathbb{D} \times \mathbb{C}^{n-1}) = (\partial\mathbb{D}) \times \mathbb{C}^{n-1}.$$

Furthermore, if we denote the components of Z by $Z = (f_1, \ldots, f_n)$, then we have the following area property

$$\text{Area}(f_1) = \pi, \text{Area}(f_j) = 0, \text{ for } j = 2, \ldots, n.$$

For $1 \leq j \leq n$, let M_j be the holomorphic disc $M_j = (m_1, \ldots, m_n) : \mathbb{D} \to \mathbb{C}^n$ where $m_k(z) = 0$ if $k \neq j$ and $m_j(z) = z$. Notice that the minimal area of an analytic set passing through the origin in $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}$ is π and the minimal analytic set is given by an image of one of the $n-2$ distinct holomorphic discs $M_3, \ldots, M_n$. On the other hand, the minimal area of an analytic set passing through the origin in $\mathbb{D}^n$ is also π, but the minimal analytic set is given by an image of one of the n distinct holomorphic discs $M_1, \ldots, M_n$. The following lemma is an immediate consequence of the above observation:

Lemma 5.12 *Let A_j be a convergent sequence of analytic sets in $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}$ passing through the origin so that*

$$\lim_{j\to\infty} \text{Area}(A_j) = \pi.$$

Then the limiting analytic set A_∞ is an image of one of the $n-2$ distinct *holomorphic discs $M_3, \ldots, M_n$.*

We are now ready to prove the main theorem in this subsection.

Theorem 5.13 *The domains $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}$ and $\mathbb{D}^n$ equipped with the standard symplectic form on $\mathbb{C}^n$ are not symplectomorphic.*

Proof. Suppose on the contrary that $\psi : \mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2} \to \mathbb{D}^n$ is a symplectomorphism. By composing Möbius transformation on $\mathbb{D}$, we can assume that $\psi(0) = 0$.

Consider the standard almost complex structure J_{st} on $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}$ and let $J = \psi_* J_{\mathrm{st}}$ be the complex structure on $\mathbb{D}^n$ given by the push-forward of J_{st} by ψ. Since $\psi^* \omega_{\mathrm{st}} = \omega_{\mathrm{st}}$, then the almost complex structure J is tamed by ω_{st}. Then the complex matrix $\widetilde{A}$ of J satisfies $\|\widetilde{A}(z)\| < 1$ for $z \in \mathbb{D}^n$.

Let $\{K_l\}_{l=1}^{\infty}$ be a compact exhaustion of $\mathbb{D}^n$ so that $0 \in \mathrm{Int} K_l$, that is, $K_l \subset K_{l+1}$, K_l is a compact subset of $\mathbb{D}^n$ for all l and $\cup_{l=1}^{\infty} K_l = \mathbb{D}^n$. For each l, let χ_l be a smooth cut-off function on $\mathbb{C}^n$ with support in $\mathbb{D}^n$ and equal to 1 on K_l. Define $A_l = \chi_l \widetilde{A}$ to be a $n \times n$ matrix function on $\mathbb{C}^n$ such that $A_l = 0$ outside $\mathbb{D}^n$. Since $\|\widetilde{A}\| < 1$ on $\mathbb{D}^n$, there is a constant $0 < a < 1$ such that (5.1) holds for A_l. Let J_l be the almost complex structure on $\mathbb{C}^n$ corresponding to the complex matrix A_l.

By considering $\mathbb{D}^n$ as a subset of $\mathbb{D} \times \mathbb{C}^{n-1}$, we can apply Theorem 5.11 so that for each l, there exists a J_l-holomorphic disc $f_l : \mathbb{D} \to \mathbb{D} \times \mathbb{C}^{n-1}$ such that the image of f_l passes through the origin. Also if we write $f_l = (f_{l,1}, \ldots, f_{l,n})$, then we have $\mathrm{Area}(f_{l,j}) = \delta_{j1}\pi$ for all l, here δ_{j1} is the Kronecker delta.

Fix an integer N, for each $l \geq N$, $\psi^{-1}(f_l(\mathbb{D}) \cap K_N)$ is an analytic set in $\psi^{-1}(K_N) \subset \mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}$ passing through the origin. Since ψ is a symplectomorphism, we have

$$\text{Area}(\psi^{-1}(f_l(\mathbb{D}) \cap K_N)) \leq \text{Area}(f_l(\mathbb{D}) \cap K_N) \leq \pi.$$

Therefore by Theorem 5.10, after passing to a subsequence,

$$F_N = \lim_{l \to \infty} \psi^{-1}(f_l(\mathbb{D}) \cap K_N)$$

exists and $\text{Area}(F_N) \leq \pi$. Notice F_N is not an empty set for N sufficiently large. Indeed, choose b large so that the minimal area of analytic set in $\psi^{-1}(K_b)$ passing through 0 equals $B \neq 0$, then for all $N > b$, $\text{Area} F_N \geq B > 0$.

The above argument holds for all N, so we can apply Theorem 5.10 again to the sequence of analytic set F_N as $N \to \infty$. After passing to a subsequence, denote the limit of F_N by F. Now F is an analytic set in $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}$ passing through the origin with $\text{Area}(F) \leq \pi$ and $\partial F \subset \partial(\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2})$. Since the minimal area of analytic set in $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}$ through the origin is π, so we must have $\text{Area}(F) = \pi$. Therefore F is an image of one of the holomorphic discs M_j for $3 \leq j \leq n$.

Let $E = \psi(F)$. We now know that $\text{Area} f_l = \pi$ for all l and $f_l(\mathbb{D}) \cap \mathbb{D}^n \to E$ as $l \to \infty$, also we have $\text{Area}(E) = \pi$. We want to show that $f_l(\mathbb{D}) \to E$ as $l \to \infty$. Let $X_l = f_l(\mathbb{D}) \setminus \mathbb{D}^n$, the image of f_l that is not in $\mathbb{D}^n$. By the construction of A_l and J_l, we know that $J_l = J_{\text{st}}$ outside $\mathbb{D}^n$, hence X_l is an usual analytic set in $(\mathbb{D} \times \mathbb{C}^{n-1}) \setminus \mathbb{D}^n$. Since $\text{Area} X_l \leq \text{Area} f_l = \pi$ for all l, we can apply Theorem 5.10 to conclude that, after passing to a subsequence, X_l converges to an analytic set X. However $f_l(\mathbb{D}) \cap \mathbb{D}^n \to E$ as $l \to \infty$ and

$\text{Area}(E) = \pi$ implies that

$$\lim_{l \to \infty} \text{Area}(f_l(\mathbb{D}) \cap \mathbb{D}^n) = \pi,$$

and by construction $\text{Area}(f_l) = \pi$ for all l, hence we have $\text{Area}(X_l) \to 0$ as $l \to \infty$. Therefore X is an empty set and we can conclude that

$$\lim_{l \to \infty} f_l(\mathbb{D}) \subset \mathbb{D}^n,$$

and hence

$$\lim_{l \to \infty} f_l(\mathbb{D}) = E.$$

Since $\text{Area}(f_{l,j}) = \delta_{j1}\pi$, if we write $\omega_{\text{st}} = \omega_1 + \cdots + \omega_n$ where $\omega_j = dx_j \wedge dy_j$ for $j = 1, \ldots, n$, then we have

$$\int_E \omega_j = \delta_{j1}\pi.$$

Now for $1 \leq k \leq n$, by considering $\mathbb{D}^n$ as a subset of the cylinder $\mathbb{C}^{k-1} \times \mathbb{D} \times \mathbb{C}^{n-k} \cong \mathbb{D} \times \mathbb{C}^{n-1}$, we can apply the above argument to obtain a 2 real-dimensional set E_k in $\mathbb{D}^n$ passing through the origin, satisfying the following conditions:

1. $\int_{E_k} \omega_j = \delta_{jk}\pi$ for $j = 1, \ldots, n$, hence all E_k are distinct for $1 \leq k \leq n$.

2. The preimage $F_k = \psi^{-1}(E_k)$ is an analytic set in $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}$ passing through the origin.

3. F_k are distinct analytic sets for $1 \leq k \leq n$ since E_k's are distinct and ψ is a bijection.

4. $\text{Area}(F_k) = \pi$ for $1 \leq k \leq n$.

Hence for each $1 \leq k \leq n$, F_k must be an image of one of the holomorphic discs M_j for $3 \leq j \leq n$, but this is impossible since all F_k are distinct, so we arrived at a contradiction. Therefore $\mathbb{D}^2_{\mathbb{R}} \times \mathbb{D}^{n-2}$ and $\mathbb{D}^n$ equipped with the standard symplectic form on $\mathbb{C}^n$ are not symplectomorphic.

$\square$

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