Hildebrandt

Hyperbolic Functions

Mathematics
A. B.
1905
THE HYPERBOLIC FUNCTIONS

BY

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THESIS

FOR THE

DEGREE OF BACHELOR OF ARTS IN MATHEMATICS

IN THE

COLLEGE OF LITERATURE AND ARTS

OF THE

UNIVERSITY OF ILLINOIS

PRESENTED JUNE, 1905
UNIVERSITY OF ILLINOIS

May 25, 1905

THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Theophil H. Hildebrandt

ENTITLED The Hyperbolic Functions.

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE OF Bachelor of Arts.

HEAD OF DEPARTMENT OF Mathematics.
# Table of Contents

## Introduction

<table>
<thead>
<tr>
<th>Chapter I Definitions</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>§1. Analytical Definition</td>
<td>5</td>
</tr>
<tr>
<td>§2. Analogy and Relation to the Circular Functions</td>
<td>6</td>
</tr>
<tr>
<td>§3. Geometrical Analogy leading to a Geometrical Definition</td>
<td>7</td>
</tr>
<tr>
<td>§4. Proof that these two Definitions are identical</td>
<td>10</td>
</tr>
<tr>
<td>§5. Inverse Hyperbolic Functions</td>
<td>11</td>
</tr>
</tbody>
</table>

## Chapter II Elementary Relations between the Hyperbolic Functions of One Argument and of Two or More Arguments

| §6. Negative Arguments | 12 |
| §7. Relations between Functions of One Argument | 12 |
| §8. Addition Formulas | 13 |
| §9. Conversion Formulas | 15 |

## Chapter III Differentiation and Integration of the Hyperbolic Functions

| §10. Differentiation of Simple Functions | 17 |
| §11. Derivatives of the inverse Hyperbolic Functions | 18 |
| §12. Integration of the Hyperbolic Functions | 19 |
§ 13. Integration of other Expressions giving Inverse Hyperbolic Functions 20

Chapter IV Derives for the Hyperbolic Functions

Variations of the Functions with $x$; Graphs 22

§ 14. Derivatives for the Hyperbolic Functions 22

§ 15. Variations of the Hyperbolic Functions 23

§ 16. Graphs of the Functions 25

§ 17. Derivatives for the Inverse Hyperbolic Functions 27

Chapter V. The Indeterminant

§ 18. The Indeterminant and its Inverse 29

§ 19. Geometrical Consideration 32

§ 20. Differentiation of the Indeterminant Function 33

§ 21. Derivatives for the Indeterminant and the Inverse: Graphs 34

Chapter VI. Functions of Imaginary and Complex Arguments 36

§ 22. Pure Imaginary Arguments 36

§ 23. Complex Functions 37

§ 24. Periodicity 38

§ 25. Argand Diagrams 39

Chapter VII. Applications to Other Branches of Mathematics 41

§ 26. General 41

§ 27. Applications to the Integral Calculus and Differential Equations 42

§ 28. Applications to the Theory of Equations 43
§29 Some Problems simplified by the Hyperbolic Functions .............. 46

Chapter VIII Applications of the Hyperbolic Functions involving Mechanical Problems ....... 49

§30 The catenary ........................................ 49

§31 Center of Gravity and moment of inertia of the Arc of the Parabola $y^2 = 2px$ .... 52

§32 Applications to the Bending of Beams ........................................ 53

Historical ........................................ 57

References ........................................ 61
Introduction

Like many other branches of higher mathematics, the hyperbolic trigonometric functions have within recent years received much attention both for their own sake and for their many important applications. So rapid has been their development that some authors are beginning to introduce them into elementary text books on Integral Calculus and Differential Equations. Few text books on Trigonometry treat this topic, although it is so closely allied to that of the circular functions. This is due no doubt to the briefness of the usual school course rather than to the inherent difficulties of the subject. In view of this fact the primary object of the present writer was to provide an account and explanation of the hyperbolic functions which will render them intelligibly useful to the students of pure and applied mathematics.

Moreover the investigations in this and the allied branches of mathematics are so numerous and cover so wide a range that the literature of
the subject is distributed through numerous books and journals. Hence, the second object of this thesis is to combine in a single volume the results of all accessible investigations heretofore made in this subject, in order that the future investigator along this line may have a convenient and concise synopsis, containing all desirable, elementary information. Perchance, in making use of this thesis he may receive some suggestions to inspire him to carry on further research along some line closely connected with the hyperbolic function and thus enrich the theory of them. With these two objects in mind, this thesis has been written.

In planning the order of discussing the different topics the writer has attempted to treat first the most elementary subjects and later discuss those more difficult to understand. Finally he was guided more or less by the requirements of logical sequence and systematic arrangement. The thesis is divided into eight chapters and each of these into several sections differing in number and length according to the importance of the subject treated. The first chapter is concerned entirely with the matter of definition, the definition from the analytical standpoint being taken as the fundamental one. The geometric definition is introduced, since no treatise on the hyperbolic
functions is complete without it, and since it is from their relation to the hyperbola that they take their name. However, it is not made use of in the work, as all the properties can be explained much more easily from the analytical standpoint. The second chapter takes up the subject of relations between the functions and is quite similar to the corresponding discussion for the circular functions in our Trigonometry. The following chapter treats of the differentials and integrals of the various functions and inverse functions. In the fourth chapter the series for the various functions are developed, the method of determining the numerical values and the curves of the functions being treated incidentally. The fifth chapter takes up the relations existing between the hyperbolic and circular functions when the arguments are connected by the relation \( v = q w \). The sixth chapter treats of functions of imaginary and complex arguments, the importance of which lies in the fact that these functions can be expressed as pure imaginary or complex functions in the form \( a + ib \), through the introduction of the circular functions, and further that the hyperbolic functions have an imaginary period. In these six chapters
the most important points belonging to the theory of the hyperbolic functions are treated. The last two chapters contain a few examples of the uses which can be made of the hyperbolic functions in pure and applied mathematics respectively. The thesis is concluded by a short sketch of the development of these functions, and a short list of the references used.
Chapter I

Definitions

81. Analytical Definition. Let the exponential function \( a^x \) be of such a nature that it is the sum of two functions \( P \) and \( Q \), say, and its reciprocal \( a^{-x} \) or \( a^{-x} \) be the difference of the same two functions; that is

\[
P + Q = a^x \quad \text{and} \quad P - Q = a^{-x}.
\]

Solving for \( P \) and \( Q \) in terms of \( a^x \) and \( a^{-x} \) we have

\[
P = \frac{a^x + a^{-x}}{2}, \quad Q = \frac{a^x - a^{-x}}{2}.
\]

Let us call \( P \) the hyperbolic cosine of \( x \) with respect to the base \( a \), and \( Q \) the hyperbolic sine of \( x \) with respect to the base \( a \), or symbolically

\[
\cosh(x, a) = P \quad \text{and} \quad \sinh(x, a) = Q.
\]

The particular case for which \( a = e \) the base of the Napierian or natural system of logarithms gives

\[
\cosh(x, e) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x, e) = \frac{e^x - e^{-x}}{2}.
\]

For convenience, as in the case of logarithms the \( e \) is usually omitted in the symbols \( \cosh(x, e) \) etc., and they are written \( \cosh x \), \( \sinh x \), etc.

Analogously to the circular functions we obtain the following:

\[
\tanh(x, a) = \frac{\sinh(x, a)}{\cosh(x, a)} = \frac{a^x - a^{-x}}{a^x + a^{-x}}.
\]
\[ \text{cosh}(x,a) = \frac{e^x + e^{-x}}{2} \]
\[ \text{sinh}(x,a) = \frac{e^x - e^{-x}}{2} \]
\[ \text{sech}(x,a) = \frac{2}{e^x + e^{-x}} \]
\[ \text{cosech}(x,a) = \frac{2}{e^x - e^{-x}} \]

For the case \( a = \varepsilon \)

\[ \text{tanh}(x) = \frac{\text{sinh}(x)}{\text{cosh}(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \]
\[ \text{coth}(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} \]
\[ \text{sech}(x) = \frac{2}{e^x + e^{-x}} \]
\[ \text{cosech}(x) = \frac{2}{e^x - e^{-x}} \]

The hyperbolic functions with respect to the base \( e \) may however just as well be expressed as functions to the base \( \varepsilon \); for since \( a^x = \varepsilon^{x \log_a} \),

\[ \text{cosh}(x,a) = \frac{a^x + a^{-x}}{2} = \varepsilon^{x \log_a} + \varepsilon^{-x \log_a} = \text{cosh}(x \log_a) \]

and
\[ \text{sinh}(x,a) = \frac{a^x - a^{-x}}{2} = \varepsilon^{x \log_a} - \varepsilon^{-x \log_a} = \text{sinh}(x \log_a) \]

Since, therefore, the general hyperbolic function to the base \( a \) is easily expressed as a function to the base \( e \), we shall henceforth consider the term hyperbolic functions as designating functions referred to the base of the system of natural logarithms.

32. **Analogy and Relation to the Circular Functions.**—As already stated in the introduction, the hyperbolic functions are in many points analogous to the circular functions. The values of the circular functions in exponential form are:

\[ \sin x = \frac{e^{xi} - e^{-xi}}{2i} \]
\[ \cos x = \frac{e^{xi} + e^{-xi}}{2} \]

and so on for the others, where \( i = \sqrt{-1} \). The following relations between the two sets of
functions may be deduced by comparison:
\[ \cos x = \frac{e^{-ix} + e^{ix}}{2}, \quad \sin x = \frac{e^{-ix} - e^{ix}}{2i} \]

From this, it is apparent that if we have a circular function of an imaginary argument, it is easily expressed as a hyperbolic function of a real argument.

§ 2. Geometrical Analogy leading up to a Geometrical Definition of the Hyperbolic Functions.

The analogy between the hyperbolic and circular functions is quite as apparent geometrically, since the hyperbolic functions bear the same relation to the hyperbola as the circular functions bear to the ellipse or, in their generalized form, to the ellipse. It may be stated here, that the hyperbolic functions bear no analogy whatever to the elliptic functions, which have gone out of attempts to rectify the ellipse, a problem quite foreign to the present one.

To lead up more easily to the geometrical definition of the hyperbolic functions we shall first define the circular functions in a way differing somewhat from that usually given in trigonometry, but perfectly in harmony with them. With \( O \) as a center describe a circle of radius \( a \). Let \( \phi \) be the angle between two radii \( O\alpha \) and \( O\beta \) and let \( OP \)
be perpendicular to \( OP \). Draw lines \( PP', PP'' \), and \( P'P'' \), and let \( TT' \) be tangent to the circle at \( Q \). It can easily be shown that:

- area of sector \( OPP = \frac{1}{2} a^2 \phi \),
- area \( \Delta OPP = \frac{1}{2} a^2 \sin \phi \),
- area \( \Delta OPP' = \frac{1}{2} a^2 \cos \phi \),
- and area \( \Delta OPP'' = \frac{1}{2} a^2 \).

From this it follows that:

\[
\phi = \frac{\text{sector} \ OPP}{\Delta OPP}, \quad \sin \phi = \frac{\Delta OPP}{\Delta OPP'}, \quad \cos \phi = \frac{\Delta OPP'}{\Delta OPP}.
\]

In a similar manner since

\[
\Delta OTP = \frac{1}{2} a^2 \tan \phi, \quad \Delta OTP' = \frac{1}{2} a^2 \sec \phi, \quad \Delta OTP'' = \frac{1}{2} a^2 \cosec \phi,
\]

we get:

- \( \tan \phi = \frac{\Delta OTP}{\Delta OPP} \),
- \( \cot \phi = \frac{\Delta OTP'}{\Delta OPP'} \),
- \( \sec \phi = \frac{\Delta OTP'}{\Delta OPP} \),
- \( \cosec \phi = \frac{\Delta OTP}{\Delta OPP} \).

These formulas may be taken as definitions of the argument \( \phi \) and its various functions, the basis of comparison being the area of the right triangle \( OPP \), having two conjugate diameters as sides. They can be immediately extended to the ellipse, the only modifications necessary being that \( OP \) and \( OP' \) shall be conjugate semi-diameters. The area \( OPP' \) is then \( \frac{1}{2} ab \), and \( \phi \) is equal to the difference between the eccentric angles of \( P \) and \( Q \).
Passing to the hyperbola, since of two conjugate
diameters only one meets the hyperbola in real points,
the conjugate must be employed also. Let the diameter meet
the hyperbola in P, and its
conjugate, the conjugate hyper-
bola, in P'. Also draw line
Q, meeting the hyperbola in R.
Then we may define the hyperbolic
functions in strict analogy to the circular functions
defined above, as follows:

\[
\begin{align*}
\cosh u &= \frac{D_{QQ'}^{2}}{D_{PP'}}, \\
\sinh u &= \frac{D_{QQ'}^{2}}{D_{PP'}}, \\
\tanh u &= \frac{D_{QQ'}}{D_{PP'}}, \\
\coth u &= \frac{D_{QQ'}}{D_{PP'}}, \\
\text{sech} u &= \frac{D_{QQ'}}{D_{PP'}}, \\
\text{cosech} u &= \frac{D_{QQ'}}{D_{PP'}}.
\end{align*}
\]

Let us assume the hyperbola referred to its axes as
axes of coordinates, its equation being

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.
\]

Then, if \( P = (x, y) \) and \( Q = (x', y') \), it is easily shown

\[
\sinh u = \frac{x' - x}{y} ; \quad \cosh u = \frac{x'^2 - x^2}{y^2}.
\]

If the axes are the conjugate diameters chosen
for reference, the above formulas reduce to

\[
\begin{align*}
\sinh u &= \frac{D_{QQ'}^{2}}{D_{PP'}}, \\
\cosh u &= \frac{D_{QQ'}^{2}}{D_{PP'}}.
\end{align*}
\]

If \( u = \text{sector} \frac{PP'}{D_{PP'}} \).
\[
\sinh u = \frac{y}{a}, \quad \cosh u = \frac{x}{a}, \quad \tanh u = \frac{y}{x},
\]

Now we make these definitions still less general by taking the hyperbola as rectangular, it is easily shown that if \((x, y, z)\) be the coordinates of \(D\)

\[
\sinh u = \frac{y}{a}, \quad \cosh u = \frac{x}{a}, \quad \tanh u = \frac{y}{x},
\]

definitions quite similar to the ratio definitions of the circular functions.

84 To show that these two sets of definitions are identical. Let \(\omega = \text{sector } \frac{\text{opp}}{\text{opp}},\)

\[
\omega_2 = \text{sector } \frac{\text{opp}}{\text{opp}}, \quad \text{and } \omega = \text{sector } \frac{\text{opp}}{\text{opp}}.
\]

Then it is evident, since \(\Delta \text{opp} = \text{opp},\)

\[
\omega = \omega_1 - \omega_2.
\]

Now the area of the sector \(\text{opp}\) is easily shown by integral calculus to be

\[
\frac{\omega_1}{2} \log \left( \frac{x^2 + \sqrt{x^2 + a^2}}{a} \right)
\]

and the area of the sector \(\text{opp}\) is

\[
\frac{\omega_1}{2} \log \left( \frac{x^2 + \sqrt{x^2 + a^2}}{a} \right).
\]

Hence

\[
\omega_1 = \frac{\omega_1}{2} \log \left( \frac{x^2 + \sqrt{x^2 + a^2}}{a} \right) = \log \left( \frac{x^2 + \sqrt{x^2 + a^2}}{a} \right),
\]

and

\[
\omega_2 = \frac{\omega_1}{2} \log \left( \frac{x^2 + \sqrt{x^2 + a^2}}{a} \right) = \log \left( \frac{x^2 + \sqrt{x^2 + a^2}}{a} \right).
\]

Hence

\[
\omega = \log \left( \frac{x^2 + \sqrt{x^2 + a^2}}{a} \right).
\]

Or

\[
\frac{x^2 + y^2}{a^2 + b^2} = e^{-\omega}.
\]

Multiplying numerator and denominator by \(\frac{x^2 - y^2}{a^2 + b^2}\), and remembering that the point \((x, y)\) is on the hyperbola \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\),
\[
\left( \frac{x_1 - y_1}{a} \right) \left( \frac{x_2 + y_2}{a} \right) = e^w, \\
or \quad \frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} + \frac{x_1 y_2 - x_2 y_1}{ab} = e^w.
\]

But by our definition in \$3\$, \[
\frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} = \cosh \theta \quad \text{and} \quad \frac{x_1 y_2 - x_2 y_1}{ab} = \sinh \theta.
\]
Hence \( e^w = \cosh \theta + \sinh \theta \).

But this relation is also true by our analytical definition. Hence the two definitions are identical, and in what follows we can use either as a basis of proof for our theorems.

\$5\$. \textbf{The Inverse Hyperbolic Functions.} Suppose \( x = \cosh u, \ y = \sinh u \) and so on.

Then in a manner similar to that used for the circular functions, we may write these equations:

\[
\cosh^{-1} x = u, \quad \sinh^{-1} y = w, \quad \text{and so on.}
\]

Read "the anti-hyperbolic cosine of \( x \)," "the anti-hyperbolic sine of \( y \)," and so on.

If \( x = \cosh u \), \( u = \frac{e^u + e^{-u}}{2} \) by \$4\.

Multiplying by \( 2 e^u \) and transposing, we get \( e^{2u} - 2xe^u + 1 = 0 \).

Solving for \( e^u \):
\[
e^u = x \pm \sqrt{x^2 - 1} = x + \sqrt{x^2 - 1} \quad \text{or} \quad x - \sqrt{x^2 - 1}
\]

Hence \( u = \pm \log (x + \sqrt{x^2 - 1}) \).

The positive sign is always taken. Hence when \( x \) is real \( \cosh^{-1} x \) is a single valued function.

In a way similar to the above, we can show that
\[
\tanh^{-1} x = \frac{1}{2} \log \frac{x - 1}{x + 1}.
\]
Chapter II
Elementary Relations between the Hyperbolic Functions of One Argument and
of Two or More Arguments

86. Negative Arguments. From our definition in §1, we have
\[ \sinh u = \frac{e^u - e^{-u}}{2}, \quad \cosh u = \frac{e^u + e^{-u}}{2}. \]
If we substitute \(-u\) for \(u\) we see that
\[ \sinh(-u) = \frac{e^{-u} - e^u}{2} = -\sinh u, \]
\[ \cosh(-u) = \frac{e^{-u} + e^u}{2} = \cosh u. \]

Hence we have the following relations
\[ \sinh(-u) = -\sinh u, \quad \cosh(-u) = \cosh u, \]
\[ \tanh(-u) = -\tanh u, \quad \coth(-u) = -\coth u, \]
\[ \text{sech}(-u) = \text{sech} u, \quad \text{csch}(-u) = -\text{csch} u. \]

§7 Relations between Different Functions of the Same Argument. From our definition in §1
\[ \cosh u + \sinh u = e^u, \]
\[ \cosh u - \sinh u = e^{-u}. \]
Multiplying these two equations together, we get
\[ \cosh^2 u - \sinh^2 u = 1. \]  \(1\)
This relation between the \(\sinh\) and \(\cosh\), together with the four elementary relations given in the definitions in §1, gives us five independent relations.
between the six hyperbolic functions, such that each can be expressed in terms of the others five. By dividing this formula through by \( \cosh^2 u \) we have at once

\[
1 - \tanh^2 u = \text{sech}^2 u, \quad \ldots \quad (2)
\]

and dividing by \( \sinh^2 u \),

\[
\coth^2 u - 1 = \text{cosech}^2 u, \quad \ldots \quad (3)
\]

direct relations between the hyperbolic secant and tangent, and cosecant and cotangent. The values of each function in terms of the other functions along with the proper signs are given in the following table:

<table>
<thead>
<tr>
<th>( \sinh u )</th>
<th>( \cosh u )</th>
<th>( \tanh u )</th>
<th>( \text{sech} u )</th>
<th>( \text{cosech} u )</th>
<th>( \text{coth} u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \sqrt{t^2 - 1} )</td>
<td>( \sqrt{t^2 + 1} )</td>
<td>( \frac{1}{t} )</td>
<td>( \frac{1}{\sqrt{t^2 + 1}} )</td>
<td>( \frac{1}{\sqrt{t^2 - 1}} )</td>
<td>( \frac{t}{\sqrt{t^2 - 1}} )</td>
</tr>
<tr>
<td>( \frac{t}{\sqrt{t^2 - 1}} )</td>
<td>( \frac{1}{\sqrt{t^2 - 1}} )</td>
<td>( t )</td>
<td>( \sqrt{t^2 - 1} )</td>
<td>( \frac{1}{t} )</td>
<td>( \frac{1}{\sqrt{t^2 - 1}} )</td>
</tr>
<tr>
<td>( \frac{1}{t} )</td>
<td>( \frac{1 + t^2}{t} )</td>
<td>( \frac{1}{\sqrt{t^2 + 1}} )</td>
<td>( \frac{1}{\sqrt{t^2 + 1}} )</td>
<td>( \frac{1}{t} )</td>
<td>( \sqrt{t^2 + 1} )</td>
</tr>
<tr>
<td>( \pm \frac{1}{\sqrt{t^2 - 1}} )</td>
<td>( \pm \frac{t}{\sqrt{t^2 - 1}} )</td>
<td>( \frac{1}{t} )</td>
<td>( \pm \frac{1}{\sqrt{t^2 - 1}} )</td>
<td>( t )</td>
<td>( \sqrt{t^2 - 1} )</td>
</tr>
</tbody>
</table>

§ 8. **Addition Formulas.** Since by § 1

\[
e^u = \cosh u + \sinh u \quad \text{and} \quad e^v = \cosh v + \sinh v,
\]

\[
e^{u+v} = (\cosh u + \sinh u)(\cosh v + \sinh v) = \cosh u \cosh v + \cosh u \sinh v + \sinh u \cosh v + \sinh u \sinh v
\]

Similarly, since

\[
e^{-u} = \cosh u - \sinh u \quad \text{and} \quad e^{-v} = \cosh v - \sinh v,
\]
\[ e^{-u-v} = (\cosh u - \sinh u)(\cosh v - \sinh v) \]

Now, \( \sinh(u+v) = \frac{e^{u+v} - e^{-u-v}}{2} \)
and \( \cosh(u+v) = \frac{e^{u+v} + e^{-u-v}}{2} \).

Hence substituting the above values for \( e^{u+v} \) and \( e^{-u-v} \) we get
\[ \sinh(u+v) = \sinh u \cosh v + \cosh u \sinh v \] \hspace{1cm} (1)
and \[ \cosh(u+v) = \cosh u \cosh v + \sinh u \sinh v \] \hspace{1cm} (2)

These formulae have been proved independently of whether \( u \) and \( v \) are positive or negative. They therefore hold just as well if we replace \( u \) by \( -u \). Remembering that \( \cosh(-u) = \cosh u \) and \( \sinh(-u) = -\sinh u \), by \( \S 6 \), we obtain the following:
\[ \sinh(u-v) = \sinh u \cosh v - \cosh u \sinh v \] \hspace{1cm} (3)
\[ \cosh(u-v) = \cosh u \cosh v - \sinh u \sinh v \] \hspace{1cm} (4)

We may also let \( u = v \) in (1) and (2) and we get functions of twice an argument in terms of functions of the argument, viz:
\[ \sinh 2u = 2 \sinh u \cosh u \] \hspace{1cm} (5)
\[ \cosh 2u = \cosh^2 u + \sinh^2 u = 1 + 2 \sinh^2 u = 2 \cosh^2 u - 1 \] \hspace{1cm} (6)

If we let \( v = 2u \) we have
\[ \sinh 3u = \sinh u \cosh 2u + \cosh u \sinh 2u \]
\[ = \sinh u (1 + 2 \sinh^2 u) + 2 \cosh^2 u \sinh u \]
\[ = \sinh u + 2 \sinh^3 u + 2 \sinh u (1 + \sinh^2 u) \]
\[ = 3 \sinh u + 4 \sinh^3 u \] \hspace{1cm} (7)
\[
\cosh^3 u = \cosh u \cosh^2 u + \sinh u \sinh^2 u
= \cosh u (2 \cosh^2 u - 1) + 2 \sinh^2 u \cosh u
= 4 \cosh^3 u - 3 \cosh u \quad \cdots \cdots \quad (8)
\]

Since \( \tanh (u + v) = \frac{\sinh (u + v)}{\cosh (u + v)} \),

\[
\tanh (u + v) = \frac{\sinh u \cosh v + \cosh u \sinh v}{\cosh u \cosh v + \sinh u \sinh v}
\]

Dividing numerator and denominator by \( \cosh u \cosh v \), we get

\[
\tanh (u + v) = \frac{\frac{\sinh u \cosh v + \cosh u \sinh v}{\cosh u \cosh v}}{\frac{\cosh u \cosh v + \sinh u \sinh v}{\cosh u \cosh v}}
= \frac{\frac{\sinh u \cosh v + \cosh u \sinh v}{\cosh u \cosh v}}{\frac{\cosh u \cosh v + \sinh u \sinh v}{\cosh u \cosh v}}
= \frac{\tanh u \cosh v + \cosh u \sinh v}{\cosh u \cosh v + \sinh u \sinh v}
= \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v} \quad \cdots \cdots \quad (9)
\]

Replacing \( u \) by \( -u \) and remembering that \( \tanh (-u) = -\tanh u \),

\[
\tanh (u - v) = \frac{\tanh u - \tanh v}{1 - \tanh u \tanh v} \quad (10)
\]

Also if \( u = \frac{v}{2} \) then

\[
\tanh u = \frac{2 \tanh u}{1 + \tanh^2 u} \quad \cdots \cdots \quad (11)
\]

Similar formulae can easily be derived for

3.2. Conversion Formulas. Adding and subtracting formulae (11) and (3) and (2) and (4) we get

\[
\sinh (u + v) + \sinh (u - v) = 2 \sinh u \cosh v
\]

\[
\sinh (u + v) - \sinh (u - v) = 2 \cosh u \sinh v
\]

\[
\cosh (u + v) + \cosh (u - v) = 2 \cosh u \cosh v
\]

\[
\cosh (u + v) - \cosh (u - v) = 2 \sinh u \sinh v
\]
Now letting \( u + v = x \) and \( u - v = y \)
and hence \( u = \frac{x+y}{2} \) and \( v = \frac{x-y}{2} \), we have

\[
\sinh x + \sinh y = 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2}
\]

\[
\sinh x - \sinh y = 2 \cosh \frac{x+y}{2} \sinh \frac{x-y}{2}
\]

\[
\cosh x + \cosh y = 2 \cosh \frac{x+y}{2} \cosh \frac{x-y}{2}
\]

\[
\cosh x - \cosh y = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2}
\]

No doubt the reader has noticed throughout this chapter the similarity between the relations between the hyperbolic functions and the corresponding relations of the circular functions.

As a matter of fact, all these formulas could have been derived from those of the circular functions through the relations

\[
\sin x = i \sinh x
\]

and

\[
\cos x = \cosh x
\]

It may be stated further, that the fundamental relations as (1) of §7 and (10) of §8 could just as well have been derived from the geometrical definition given in §3.
Chapter III

Differentiation and Integration of the Hyperbolic Functions

§10. Derivatives of the Hyperbolic Functions.

As in the case of the circular functions, the derivatives of the hyperbolic functions are easily found. They can be derived independently by the method of increments, and also by making use of their analytical definitions. Of these two methods we shall use the latter, although the other would have been just as good.

Since \( \sinh u = \frac{e^u - e^{-u}}{2} \) and \( \cosh u = \frac{e^u + e^{-u}}{2} \)

\[
\frac{d}{du} (\sinh u) = \frac{e^u + e^{-u}}{2}
\]

and

\[
\frac{d}{du} (\cosh u) = \frac{e^u - e^{-u}}{2}
\]

These two derivatives are the fundamental ones and are sufficient to give us the derivatives of the other functions by ordinary differentiation.

That is,

\[
\frac{d}{du} (\tanh u) = \frac{d}{du} \left( \frac{\sinh u}{\cosh u} \right) = \frac{1}{\cosh^2 u} = \text{sech}^2 u \quad \ldots \quad (3)
\]
and \( \frac{d}{du} \cosh u = \frac{1}{2} \cosh u \sinh u \) 
\[ = \frac{\sinh u}{\sinh u} = \frac{1}{\sinh u} \]
\[ = \cosech u. \] (4)

Similarly \( \frac{d}{du} \sec u = \frac{1}{\sec u} \tan u \) 
\[ = -\tan u \sec u \] (5)

and \( \frac{d}{du} \cosech u = \frac{d}{du} (\sinh u) \) 
\[ = -\cosh u \cosech u \] (6)

It is evident from these formulas, that the hyperbolic sine and cosine reproduce themselves and their successive derivatives in a way similar to the circular sine and cosine, except that the signs do not change. This gives a ready solution to the problem: "What function repeats itself in its second derivative?"

The similarity of the derivative forms of the circular functions to the corresponding hyperbolic is obvious.

§ 11. Differentials of the Inverse Hyperbolic Functions. Let \( x = \sinh u. \)

Then \( \sinh x = u. \)

Differentiating \( \cosh x \, dx = du, \) or 
\[ \sqrt{1 + \sinh^2 x} \, dx = du. \]

Hence 
\[ dx = d(\sinh^{-1} u) = \frac{du}{\sqrt{1 + \sinh^2 x}} = \frac{du}{\sqrt{1 + u^2}}. \] (7)
Again let \( x = \cosh u \) or \( \cosh x = u \) 

Differentiating, \( \sinh x \, dx = du \), 
\[
\sqrt{\cosh^2 x - 1} \, dx = du, \\
ax = d(\cosh^{-1} u) = \frac{du}{\sqrt{u^2 - 1}} \quad \ldots \quad (2)
\]

Similarly if \( x = \tanh u \),
\[
du = \text{sech}^2 x \, dx, \\
ax = d(\tanh^{-1} u) = \frac{du}{1-u^2} \quad \ldots \quad (3)
\]

And in the manner
\[
d(\text{coth}^{-1} u) = \frac{-du}{u^2 - 1} \quad \ldots \quad (4)
\]

Of these two formulas \([1, 3, 4]\), the first holds only when \( u \) is less than one, and the second when \( u \) is greater than one. (The reason for this will be explained in \( \S 15 \).)

Again, let \( x = \text{sech} u \).

Then
\[
du = -\text{sech} u \tanh u \, dx, \\
du = -\text{sech} u \sqrt{1 - \text{sech}^2 x} \, dx,
\]

and hence
\[
ax = \frac{du}{\sqrt{1 - u^2}} \quad \ldots \quad (5)
\]

Similarly
\[
d(\text{csch}^{-1} u) = \frac{-du}{u \sqrt{u^2 + 1}} \quad \ldots \quad (6)
\]

\( \S 12 \) Integration of the Hyperbolic Functions

The following formulas follow immediately from \( \S 10 \) from the fact that integration is the inverse of differentiation,

\[
\begin{align*}
\sinh u \, du &= \cosh u \quad \ldots \quad (1) \\
\cosh u \, du &= \sinh u \quad \ldots \quad (2) \\
\text{sech} u \, du &= \tanh u \quad \ldots \quad (3) \\
\text{csch} u \, du &= -\coth u \quad \ldots \quad (4)
\end{align*}
\]
\[ \text{sech } u \, \text{and } \int \text{sech } u \, du = \text{log } \text{sech } u + \text{constant} \quad (5) \]

\[ \text{cosech } u \, \text{and } \int \text{cosech } u \, du = \text{log } \text{cosech } u + \text{constant} \quad (6) \]

Other formulas are easily derived as follows:

\[ \int \text{sech } u \, du = \int \frac{\text{sech } u \, \text{tanh } u}{\text{sech } u} \, du = \log \cosh u + \text{constant} \quad (17) \]

\[ \int \text{cosh } u \, du = \int \frac{\text{cosh } u}{\text{cosh } u} \, du = \log \cosh u + \text{constant} \quad (8) \]

\[ \int \text{sech } u \, du = \int \frac{\text{sech } u \, \text{tanh } u}{\text{sech } u} \, du = \log \text{cosech } u + \text{constant} \quad (9) \]

\[ \int \text{cosech } u \, du = \int \frac{\text{cosech } u \, \text{cosech } u}{\text{cosech } u \, \text{cosech } u} \, du = \int \frac{\text{cosech } u \, \text{cosech } u}{\cosh u \, \cosh u} \, du = \log \text{cosech } u \quad (10) \]

\[ \int \text{cosech } u \, du = \log \frac{1 - \text{coth } u}{\text{sinh } u} = \log \frac{2 \text{sinh } u}{2 \text{sinh } u + \text{cosh } u} \]

\[ = \log \tanh \frac{u}{2} \quad (10) \]

\[ \int \text{cosech } u \, du = \log \frac{1 - \text{coth } u}{\text{sinh } u} = \log \frac{2 \text{sinh } u}{2 \text{sinh } u + \text{cosh } u} \]

\[ = \log \tanh \frac{u}{2} \quad (10) \]

8.13. Integration of expressions resulting in the inverse hyperbolic functions. We easily obtain the following integrals from the differentiation formulas of § 11:

\[ \int \frac{du}{\sqrt{u^2 - a^2}} = \sinh^{-1} \frac{u}{a} + \text{constant} \quad (1) \]

\[ \int \frac{du}{\sqrt{a^2 - u^2}} = \cosh^{-1} \frac{u}{a} + \text{constant} \quad (2) \]

\[ \int \frac{du}{u^2 - 1} = \tanh^{-1} \frac{u}{a} + \text{constant} \quad (3) \]

\[ \int \frac{du}{u^2 - 1} = \tanh^{-1} \frac{u}{a} + \text{constant} \quad (3) \]

\[ \int \frac{du}{u^2 - a^2} = \frac{1}{2} \coth^{-1} \frac{u}{a} + \text{constant} \quad (4) \]

*Compare this result with that in § 20.*
From these fundamental integrals the following are easily derived:

\[
\int \frac{du}{u \sqrt{1 - u^2}} = \text{arc} \sinh u \quad \int \frac{du}{u \sqrt{a^2 - u^2}} = \frac{1}{a} \text{arc} \cosh \frac{u}{a}
\]

\[
\int \frac{du}{u \sqrt{u^2 + 1}} = \text{arc} \coth u \quad \int \frac{du}{u \sqrt{a^2 + u^2}} = \frac{1}{a} \text{arc} \coth \frac{u}{a}
\]

\[
\int \frac{dx}{ax^2 + bx + c} = \frac{1}{\sqrt{a}} \text{arc} \tanh \frac{ax + b}{\sqrt{a} \sqrt{b^2 - ac}}, \quad \text{a positive } ac > b^2,
\]

\[
= \frac{1}{\sqrt{a}} \text{arc} \coth \frac{ax + b}{\sqrt{a} \sqrt{b^2 - ac}}, \quad \text{a negative } ac < b^2.
\]

\[
\int \frac{dx}{ax^2 + bx + c} = \frac{1}{a} \tan^{-1} \frac{ax + b}{\sqrt{a} \sqrt{b^2 - ac}}, \quad \text{a negative } ac > b^2,
\]

\[
= \frac{1}{a} \cot^{-1} \frac{ax + b}{\sqrt{a} \sqrt{b^2 - ac}}, \quad \text{a positive } ac < b^2.
\]

\[
\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left| x + \sqrt{x^2 - a^2} \right|
\]

\[
\int x \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln \left| x + \sqrt{x^2 + a^2} \right|
\]

\[
\int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}
\]
Chapter IV

Series for the Hyperbolic Functions

Variations of the function with $x$;

Graphs of the Functions.

§ 14. Series for the Hyperbolic Functions

From 51, by definition, sinh $x = \frac{e^x - e^{-x}}{2}$.

But $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$.

and $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$.

Hence by substitution

\[ \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \quad (1) \]

Similarly, since $\cosh x = \frac{e^x + e^{-x}}{2}$,

\[ \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \quad (2) \]

We can also develop these formulae from Mcclain's Theorem.

\[ f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots \]

If $f(x) = \sinh x$, $f'(x) = \cosh x$, $f''(x) = \sinh x$,...

Then $f(0) = 0$, $f'(0) = 1$, $f''(0) = 1$, $f'''(0) = 0$.

Hence $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$.

Similarly, by differentiation

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$.

From the addition of these two we get

$\cosh x + \sinh x = e^x$.

* These values may be obtained by substituting into formulas $\frac{x^3}{3!} \pmb{2} \pmb{2}$.

(22)
This last result would seem to be the result of reasoning in a circle, at least on the surface. But however, we obtain the differentiation formulas independently of the analytical definition, as it is possible to do, and obtain the values for \( \sinh, \cosh, \tanh, \coth \), from the geometrical definition; the result shows again the agreement of the two definitions.

From series (1) and (2) the following may be obtained by division:

\[
\tanh \frac{\sinh}{\cosh} = 1 - \frac{1}{3} u^2 + \frac{2}{15} u^4 - \frac{17}{315} u^6 + \ldots \quad (3)
\]

\[
\sech \frac{\sinh}{\cosh} = 1 - \frac{1}{2} u^2 + \frac{5}{24} u^4 - \frac{61}{720} u^6 + \ldots \quad (4)
\]

\[
\coth \sinh = \frac{1 + \frac{1}{3} u^2 - \frac{4}{45} u^4 + \frac{2}{945} u^6 + \ldots}{\sinh} \quad (5)
\]

\[
\cosech \sinh = \frac{1 + \frac{1}{6} u^2 + \frac{7}{360} u^4 + \frac{31}{15720} u^6 + \ldots}{\sinh} \quad (6)
\]

These series are however seldom used, because there is no known law by which the coefficients progress. Moreover the \( \tanh \), \( \text{sech} \), \( \coth \), and \( \text{cosech} \) can be easily found if the values of \( \cosh \) and \( \sinh \) are known.

\( \text{§ 15. Variations of the Hyperbolic Functions} \)

The above series give us a set of criteria by which we can tell the values of the different hyperbolic functions as \( u \) varies.

If \( u \) is zero it is evident that

\[
\cosh 0 = 1, \sinh 0 = 0, \tanh 0 = 0, \coth 0 = \infty, \text{sech} 0 = \infty, \text{cosech} 0 = 0, \text{cosech} 0 = \infty.
\]
If \( u \) varies from zero to infinity, the values of \( \cosh u \) and \( \sinh u \) will increase indefinitely and become infinite when \( u \) is infinite. It is apparent from the series that they approach infinity much faster than \( u \) showing that \( \cosh u \) and \( \sinh u \) are infinites of a higher order than \( u \). Although the hyperbolic sine and cosine approach infinity simultaneously, they are never equal to each other while finite, as is evident from § 7 (1), viz.

\[
\cosh^2 u - \sinh^2 u = 1, \quad \cosh u \to \infty \quad \text{always being the greater.}
\]

\( \tanh u \) will also increase from zero upward as \( u \) increases, but it will never become greater than unity, as is evident from the fact that

\[
\tanh u = \frac{\sinh u}{\cosh u}, \quad \text{and that } \cosh u + \sinh u \to \infty \text{ as } u \to \infty.
\]

\( \cosh u \) will approach unity as its limit as \( u \) approaches infinity, decreasing as \( u \) increases. This is immediately evident from the fact that the cotangent is the reciprocal of the tangent.

\( \sech u \) approaches zero as \( u \) approaches infinity, decreasing from unity as \( u \) increases. The hyperbolic cosecant also approaches \( 0 \) as its limit, but decreases from infinity as \( u \) increases. Hence:

\[
\sinh \infty = \infty, \quad \cosh \infty = \infty, \quad \tanh \infty = 1, \quad \sech \infty = 0, \quad \cosech \infty = 0, \quad \coth \infty = 1.
\]
When \( r \) varies from zero negatively the values of the hyperbolic cosine and secant are the same as for the positive values, while the values of the other functions are the negative. This follows directly from \( \S 6 \).

Hence we have the following conclusions:
- \( \cosh \) is never less than one, \( \sinh \) can have all values, \( \tanh \) varies from positive unity to negative unity, \( \sech \) varies between zeros and unity, \( \cosech \) can have all values from positive infinity to negative infinity, \( \coth \) between infinity and unity, and \( \csch \) between infinity and negative infinity and negative one.

\( \S 16 \). Graphs of the Functions. By substituting for \( r \) some numerical value in the series of \( \S 14 \), we obtain the corresponding values of the functions. By repeating this process, making \( r \) vary at regular intervals, tables can be constructed, giving the values of the functions. Such tables have been worked out with great care by Budehmann and numerous other investigators. Tables sufficient for ordinary use are given in McMahon’s article on the hyperbolic functions. The tables above referred

to are accessible to most persons interested in the hyperbolic functions and are omitted here as being beyond the scope of the present thesis.

By laying off values of $u$ as abscissae and the values of the functions as ordinates, curves representing the variation of the hyperbolic functions can be plotted. We give the curves for the different functions below:

These curves illustrate clearly the conclusions of the preceding section.
§ 17. Series for the Inverse Hyperbolic Functions.

From (11) we have:
\[
\frac{d}{du} \sinh^{-1} u = \frac{1}{\sqrt{1+u^2}} = (1+u^2)^{-\frac{1}{2}} = (1+u^2)^{-\frac{1}{2}} = 1 - \frac{1}{2} u^2 + \frac{1}{2} \cdot \frac{3}{4} u^4 - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} u^6 + \ldots
\]

By the binomial theorem,

Integrating both sides of this equation,
\[
\sinh^{-1} u = u - \frac{1}{2} \frac{u^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \frac{u^5}{5} - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \frac{u^7}{7} + \ldots
\]

the integration constant being zero since \( \sinh^{-1} 0 = 0 \).

This series is convergent only when \( u \) is less than one, and hence cannot be used for values greater than one. Another series convergent when \( u^2 \) is obtained by writing \( \frac{1}{u} (1 + \frac{1}{u^2})^{-\frac{1}{2}} \) for \( (1+u^2)^{-\frac{1}{2}} \) and expanding it:
\[
\frac{d}{du} \sinh^{-1} u = \frac{1}{u} \left(1 + \frac{1}{u^2}\right)^{-\frac{1}{2}} = \frac{1}{u} \left(1 - \frac{1}{2} u^{-2} + \frac{1}{2} \cdot \frac{3}{4} u^{-4} - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} u^{-6} + \ldots\right)
\]

Integrating,
\[
\sinh^{-1} u = C + \log u + \frac{1}{2} \cdot \frac{3}{4} u^{-2} - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} u^{-4} + \ldots
\]

If \( u = \infty \) we get
\[
\lim_{u \to \infty} (\sinh^{-1} u - \log u) = 0
\]

But by § 15 \( \sinh^{-1} u = \log (u + \sqrt{u^2 + 1}) \)

Hence \( C = \lim_{u \to \infty} \log (u + \sqrt{u^2 + 1}) = \lim_{u \to \infty} \log (u + \sqrt{u + \frac{1}{u}}) = \log 2 \)

Hence \( \sinh^{-1} u = \log 2 + \frac{1}{2} \cdot \frac{3}{4} u^{-2} - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} u^{-4} + \ldots \)

when \( u \) is greater than one.

The \( \cosh^{-1} u \) can be developed in a similar way,
\[
\frac{d}{du} \cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}} = (u^2 - 1)^{-\frac{1}{2}} = \frac{1}{u} (1 + \frac{1}{2} u^{-2} + \frac{1}{2} \cdot \frac{3}{4} u^{-4} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} u^{-6} + \ldots)
\]
\[ \cosh u = e^u + e^{-u} = 2 + \frac{1}{2!} \cdot \frac{u^2}{2!} + \frac{1}{3!} \cdot \frac{u^4}{2!} - \frac{1}{4!} \cdot \frac{u^6}{2!} - \cdots \] (3)

The value of \( C \) is again found to be \( \log 2 \).

It is evident from \( \S 15 \) that when \( u \) is less than one, \( \cosh u \) is not real. Hence this series will not give real values when \( u \) is less than unity, that is, it is not convergent.

Again \( \tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}} = 1 - u^2 + u^4 - u^6 + \cdots \)

Hence \( \tanh u = u + \frac{u^3}{3} + \frac{u^5}{5} + \frac{u^7}{7} + \cdots \) \hspace{1cm} (4)

The integration constant being zero, since \( \tanh 0 = 0 \)

\[ \text{sech}^{-1} u = \log \frac{1}{2} - \frac{1.3 u^2}{2.2} - \frac{1.3 \cdot 5 u^4}{2.4 \cdot 4} - \cdots \] \hspace{1cm} (5)

\[ \text{cosech}^{-1} u = \frac{1}{u} - \frac{1}{2} \cdot \frac{3}{2} u^2 - \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} u^4 - \cdots \] \hspace{1cm} (6)

when \( u \) is greater than unity,

and \( \text{coth}^{-1} u = \frac{1}{u} - \frac{1.3 u^2}{2.2} - \frac{1.3 \cdot 5 u^4}{2.4 \cdot 4} - \cdots \) \hspace{1cm} (7)

when \( u \) is less than unity,

\[ \text{coth}^{-1} u = \tanh^{-1} \frac{1}{u} = \frac{1}{u} + \frac{1}{2} \cdot \frac{3}{2} u^2 + \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} u^4 + \cdots \] \hspace{1cm} (8)

These series are convergent for all values of \( u \) which make these functions real.
Chapter V

The Gudermannian

We have hitherto considered relations between the different hyperbolic functions and also the relations connecting the circular functions of imaginary argument with the hyperbolic functions of equal real arguments. The question naturally arises: Is there any relation between the hyperbolic and circular functions when they are functions of real arguments only? That there is will appear in the course of this chapter.

§ 18. The Gudermannian. If \( u \) varies from zero to infinity, by §15, which \( u \) will vary from zero to infinity, taking any and all values between zero and infinity, provided \( u \) be taken at intervals differing by sufficiently small quantities; in other words, \( \sinh u \) is a continuous function of \( u \) as \( u \) varies from zero to infinity. Again, in the circular functions \( \tan v \) varies from zero to infinity as \( v \) varies from zero to \( \pi/2 \), taking any and all values between zero and infinity if \( v \) is taken in sufficiently small intervals. Hence
we see that, if we find the value of $u$ corresponding to some value of $\sin u$, it is possible to find a value of $v$ such that

$$u = \tan v,$$

since $\tan v$ varies continuously between zero and infinity as $v$ varies between zero and $\frac{\pi}{2}$, and since $u$ varies from zero to infinity as $v$ varies from zero to infinity. Hence we see that, for every value of $u$ between zero and $\frac{\pi}{2}$, and hence $v$ is a function of $u$, or the reverse. This correspondence of functional relations is expressed by saying that $v$ is the gudernanius of $u$ and $u$ is the anti gudernanius of $v$, written,

$$v = gdw\quad\text{and}\quad u = gd^{-1}v \quad \ldots \quad (1)$$

If this relation connects $u$ and $v$, it is evident that

$$\sin hu = \tan vu \quad \ldots \quad (2)$$

Furthermore

$$\cosh hu = \sqrt{1 + \sin^2 hu} = \sqrt{1 + \tan^2 u} = \sec v \quad (3)$$

Dividing (2) by (3),

$$\tanh hu = \frac{\tan vu}{\sec v} = \sin v \quad \ldots \quad (4)$$

Similarly

$$\sinh u = \csc v\quad (5)$$

$$\cosh u = \sec v\quad (6)$$

$$\cosh hu = \cot v\quad (7)$$

These relations are easily remembered on account of their symmetry, viz.
\[ \sinh u = \tanh v, \quad \tanh u = \sin v, \]
\[ \cosh u = \sec v, \quad \sec u = \cosh v, \]
\[ \coth u = \cosec v, \quad \cosec u = \cot v. \]

Again by (3)
\[ \sinh u + \cosh u = e^u. \]

Hence
\[
\begin{align*}
\cosh^2 v &= 1 + \sinh^2 v \\
&= 1 + \tanh^2 v \\
&= 1 + \cosec^2 v (v + \pi/2) \\
&= \cosh (v + \pi/2)
\end{align*}
\]

Therefore
\[ u = g \log \cosh (v + \pi/2) \ldots (8) \]

It is apparent from this that if we had a table of natural logarithms of sines, it would not be difficult to find the value of \( u \) in terms of \( v \). Tables of this kind have been constructed, but are not in general use. It would, however, be possible to find \( u \) from a table of Briggsian logarithms by dividing by the modulus of the system.

Further
\[ \tan \frac{v}{2} = \frac{\sin v}{1 + \cos v} = \frac{\tanh v}{1 + \cosec v} \]
and
\[ \tanh \frac{v}{2} = \frac{\sinh u}{1 + \cosh u} \]

Hence, since from above \( \sinh u = \tanh v \) and \( \cosh u = \sec v \),
\[ \tan \frac{v}{2} = \tan \frac{\pi}{2} \ldots (9) \]

At first glance it would seem that, since \( \tan v = \sinh u \),
\[ \tan \frac{v}{2} = \sinh \frac{u}{2}, \]
and therefore
\[ \tanh \frac{v}{2} = \sinh \frac{u}{2}, \]
which is impossible. The fallacy lies in the fact that
\[ g \sinh \frac{u}{2} = \frac{1}{2} g \sinh u. \]
which equality would have to be assumed in the above reasoning. However, the equation, to be exact, might be written

\[ \tan \frac{1}{2} \theta \cot \phi = \sinh \theta \tan \frac{\phi}{2} \]

Similarly, \[ \tan \frac{1}{2} \phi \cot \theta = \sin \phi \tan \frac{\theta}{2} \]

giving easily a solution for

\[ \tan \theta = \sin \phi \]

and \[ \tan \phi = \sin \theta \]

319. Geometric Proof. These formulas can all be proved geometrically as follows:

Let \( P \) be a point on a rectangular hyperbola \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \). Let \( MT \) be a tangent to the circle of radius \( OA \) from the foot of the ordinate \( PM \).

Further let \( u = \frac{\text{sector } SOP}{\theta^2} \) (see 853)

and \( v = \angle A OP \).

Then \( \sec u = \frac{OP}{a} = \frac{OP}{a} \)

and \( \cosh v = \frac{OM}{a} \).

Hence \( \sec u = \cosh v \)

which agrees with formula 3 of 818.

The above proof is for the case when \( v \) is referred to the circle and \( u \) referred to the rectangular hyperbola. They might just as well have been referred to the general ellipse and hyperbola, in which case \( u \) and \( v \) would simply be the ratio of the sectors to the triangles of
reference; but the ratio ω would be the same as the radian measure of the angle ω referred to the circle. To prove the formulae, use would have to be made of the theory of correspondence of points on ellipse and hyperbola. In the first case we treat of the gudermannian angle and in the second of the gudermannian function only.

§ 70. Differentiation of the Gudermannian Function. Let \( ν = \text{gd} u \).

Then \( \tan ν = \text{sech} u \).

Differentiating \( \sec^2 ν \text{~} dν = \cosh \text{~} du \)

Belt \( \sec ν = \cosh u \).

Hence \( \text{~} du = \text{~} (\text{gd} ν)' = \sec ν \text{~} du \) \( \cdots \\cdots 11 \)

Again from 11 \( \text{~} du = \cos ν \text{~} du = \text{sech} \text{~} du \).

Hence \( d (\text{gd} u) = \text{sech} \text{~} du \) \( \cdots \\cdots 12 \).

We see therefore that the following integrals might be expressed:

\[
\sec ν \text{~} dv = \text{gd}^{-1} ν \quad \cdots \cdots 13
\]

and

\[
\text{sech} \text{~} du = \text{gd} u \quad \cdots \cdots 14
\]

These agree with the usual forms of these integrals since

\[
ν = \text{gd}^{-1} ν = \log \tan \left( \frac{u}{2} + \frac{π}{4} \right)
\]

and

\[
ν = \text{gd} u = \tan^{-1} \sinh u
\]

* Cf. § 12 formula 9
§ 21 Series for the Gudermannian and its Inverse. Graph. Since \( d(x) = \sec v \) and \( x = \arctan x \) \( \sec^2 x = 1 + \tan^2 x \), we obtain
\[
\frac{d}{dx} \tan x = \sec^2 x - 2 \sec x \tan x = \frac{2}{\sec x} \tan x + \ldots
\]

Similarly, by substituting in
\[
d(\tan^{-1} x) = \sec^2 x - 2 \sec x \tan x = \frac{2}{\sec x} \tan x + \ldots
\]
the series for \( \tan x \) and integrating,
\[
\tan^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots
\]

These series are however seldom used since there is no known law by which the coefficients progress. If it is necessary to find values, they are usually calculated from relations like
\[
gdn = \tan^{-1} x
\]
from tables of natural logarithmic and hyperbolic sines. As used is made of formula 8 of § 16,
\[
\mu = \log \tan \left( \frac{x}{2} + \frac{1}{2} \right),
\]
the use of which is discussed in that paragraph. For tables of \( gdn \) and \( gdn^{-1} \) the reader is referred to the excellent set given by Gudermann in parts 7, 8, 9, 10 of Journal der Mathematik.

As in the case of the hyperbolic functions, the gudermannian function can also be plotted; values of \( x \) being taken as abscissae and the values of \( gdn \) as ordinates. The
curve is given in the figure below.

\[ \begin{align*}
&\pi/2 \\
-3 &\quad -2 &\quad -1 \\
&\quad 1 &\quad 2 &\quad 3 \\
&-\pi/2
\end{align*} \]

It is clear that the curve is asymptotic to \( \pi/2 \) and \(-\pi/2\). Also that it is always less than \( \pi/2 \) and greater than \(-\pi/2\).
Chapter VI

Hyperbolic Functions of Imaginary and Complex Argument

Thus far we have treated only of the hyperbolic functions of real arguments. Imaginary and complex arguments have not yet been considered, and since no work on the hyperbolic functions is complete without a discussion of them, we shall close the consideration of the theory of these functions with a short discussion of these two kinds of arguments before turning our attention to the applications of the hyperbolic functions.

§ 22. Pure Imaginary Arguments.

We have shown in § 21 that the circular functions of imaginary arguments can be reduced to hyperbolic functions of equal real arguments. It would naturally seem possible to do the reverse, that is, express the hyperbolic functions of imaginary in terms of circular functions of reals. For if in the formulas of § 21 we substitute \( x = \text{ix} \), we obtain
\[
\cosh x + i = \cos(x + i) = \cos(x - i) = \cosh x
\]
\[
\sinh x + i = \sin(x + i) = \sin(x - i) = \sinh x
\]

Similarly, or by division,
\[
\tanh x + i = \frac{\cosh x}{\sinh x}
\]

and so on for the other functions. The same results may be obtained by replacing \( x \) by \( x + i \) in the definitions of § 7 of the hyperbolic cosine and sine, but this really amounts to the same thing as the above.

It is apparent therefore that the hyperbolic cosine of a pure imaginary argument is real while the sine and tangent are imaginary.

§ 13. Complex functions. Since the addition formulas of § 8 were proved independently of whether \( \mu \) and \( \nu \) were real or imaginary, they hold for all values of \( \mu \) and \( \nu \), and hence, for values of \( \mu \) real and \( \nu \) imaginary. Therefore we have
\[
\sinh(\mu + i \nu) = \sinh \mu \cosh i \nu + \cosh \mu \sinh i \nu.
\]

But \( \cosh i \nu = \cosh \mu \) and \( \sinh i \nu = i \sin \nu \).

Hence
\[
\sinh(\mu + i \nu) = \sinh \mu \cos \nu + i \cosh \mu \sin \nu.
\]

Also
\[
\cos(\mu + i \nu) = \cos \mu \cosh i \nu - \sin \mu \sinh i \nu.
\]

\[
= \cos \mu \cosh \nu + i \sin \mu \sinh \nu
\]
When $vi$ is negative, the sign before the second half of the second number of each identity is changed to minus.

The above formulae show us that the hyperbolic sine and cosine of a complex argument are reducible to complex functions of the form $a + ib$, $c + id$. If

\[ \sinh (u + vi) = a + ib \]

then

\[ a = \sinh \text{ even} \]
\[ b = \cosh \text{ even} \]

and if \( \cosh (u + vi) = c + id \)

\[ c = \cosh \text{ even} \]
\[ d = \sinh \text{ even} \]

This shows us that if we have a table of hyperbolic sines and cosines and also of the circular sines and cosines, we can easily calculate $a$, $b$, $c$, and $d$, and hence the hyperbolic sine and cosine of any complex number.

\section{Periodicity of the Hyperbolic Functions.}

If in the formulae of §23 we take the value $\frac{\pi}{2}$, we obtain:

\[ \sinh (u \pm \frac{\pi}{2} i) = \sinh u \cos \frac{\pi}{2} \pm i \cosh u \sin \frac{\pi}{2} = \pm i \cosh u \quad (1) \]

and \( \cosh (u \pm \frac{\pi}{2} i) = \cosh u \cos \frac{\pi}{2} \pm i \sinh u \sin \frac{\pi}{2} = ti \sinh u \quad (2) \)

Again if $\pi$ is $\pi$, we obtain:
\[
\sinh (u \pm \pi i) = \sinh u \cos i \pi \pm i \cosh u \sin i \pi
\]
\[
= -\sinh u \quad \ldots \quad (3)
\]

and
\[
\cosh (u \pm \pi i) = \cosh u \cos i \pi \pm i \sinh u \sin i \pi
\]
\[
= -\cosh u \quad \ldots \quad (4)
\]

If \( v = \frac{3\pi}{2} \),
\[
\sinh (u \pm \frac{3\pi}{2} i) = \sinh u \cos \frac{3\pi}{2} \pm i \cosh u \sin \frac{3\pi}{2}
\]
\[
= \mp \cosh u \quad \ldots \quad (5)
\]

and
\[
\cosh (u \pm \frac{3\pi}{2} i) = \mp \sinh u \quad \ldots \quad (6)
\]

If \( v \) is \( 2\pi \)
\[
\sinh (u \pm 2\pi i) = \sinh u \quad \ldots \quad (7)
\]
\[
\cosh (u \pm 2\pi i) = \cosh u \quad \ldots \quad (8)
\]

and in general it is easily proved that if \( n \) be any integer
\[
\sinh (u \pm 2n\pi i) = \sinh u \quad \ldots \quad (9)
\]
\[
\cosh (u \pm 2n\pi i) = \cosh u \quad \ldots \quad (10)
\]

This shows us that the hyperbolic functions, that is, the sinh and cosh, are periodic functions whose period being an imaginary \( \pi i \). This also appears from the relations between the circular and hyperbolic functions since the period of the circular functions is \( 2\pi \), a real one. It is easily shown that
\[
\tanh (u \pm 2n\pi i) = \tanh u
\]
from which it appears that the tangent has the period \( \pi i \).

S25. Let the cosh \( z = \xi \) where \( \xi \) is of the form \( x + iy \) and \( z \) of the form \( x + iy \). Further let the complex numbers \( z \) and \( z \) be represented by
Argand diagrams in the usual way by points whose coordinates are \( x, y, X, Y \).

If \( z \) traces the line \( x = m \) parallel to the \( y \)-axis, \( z \) will trace out an ellipse as is shown by eliminating \( y \) from \( x = \cos \theta m \csc \theta \), \( y = \sin \theta m \sec \theta \) with
\[
\cos^2 \theta + \sin^2 \theta = 1
\]
\[
\frac{x^2}{\csc^2 \theta} + \frac{y^2}{\sec^2 \theta} = 1
\]

This will represent a series of confocal ellipses as \( m \) varies with \( \csc \theta \), \( \sec \theta \).

Similarly if \( z \) traces out a line parallel to the \( x \)-axis say \( y = m \), \( z \) will trace out a hyperbola whose equation obtained by eliminating \( x \) from the equations
\[
X = \cos \theta m \csc \theta \quad Y = \sin \theta m \sec \theta
\]
is
\[
\frac{x^2}{\csc^2 \theta} - \frac{y^2}{\sec^2 \theta} = 1
\]

As \( m \) varies this also represents a set of confocal hyperbolas also confocal with the ellipses above. The intersection of the ellipse and hyperbola represented by the above equation will give the value of \( X + Y \) for \( x = m \), \( y = n \) and hence the value for each \( (m + in) \). Hence if accurate diagrams of ellipses and hyperbolas be drawn for different values of \( m \) and \( n \), the value of \( \csc \theta (m + in) \) can be read off at the intersection of the ellipse whose parameter is \( m \) with the hyperbola of parameter \( n \).

Similar diagrams can be constructed for the hyperbolic sine \( \sinh \) equations being
\[
\frac{x^2}{\sinh^2 m} + \frac{y^2}{\cosh^2 m} = 1
\]
and
\[
\frac{x^2}{\sin^2 m} - \frac{y^2}{\cosh^2 m} = 1
\]
a set of ellipses and hyperbolas the result passing through the ellipse and hyperbolas of constant \( \sinh m + \cosh m \) through a right angle, as is easily seen from the equations.
No doubt the reader has before this point asked the question "What use can be made of these functions and the theory of them?" Like the circular functions, they enjoy a wide range of applications, and no doubt they deserve as wide an introduction into the mathematical sciences. The applications of these functions fall under two heads, viz.: those to other branches of pure mathematics which we shall treat in the present chapter, and those to applied mathematics, mechanics and physics, mostly through the medium of other mathematics.

§26. The first use of importance that can be made of these functions is to replace the longer expressions, like
\[ \frac{x^3 + e^{-x}}{2} \log \left( x + \sqrt{x^2 - a^2} \right) \text{ etc.,} \]
by the shorter and simpler expressions:
\[ \cosh x, \cosh^2 x, \text{ etc.} \]
Since expressions of this nature often arise...
in mathematics, it is much more convenient and expedient to use the notation of the hyperbolic functions corresponding to these expressions.

327. **Applications to Integral Calculus and Differential Equations.** The most important application to the integral calculus arise from the analogy to the circular function. By this analogy the remembering of many complicated formulas of the integral calculus is greatly simplified. For example, we have from integral calculus,

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \sin^{-1} \frac{x}{a} = -\cos^{-1} \frac{x}{a}.$$

Further by \(3 \frac{13}{3}\),

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \frac{x}{a}$$

and

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a},$$

which carries us through all the fluctuations of signs possible between \(x^2\) and \(a^2\). Thus the student is led immediately to associate with the square root of the sum or difference of two squares, one being the square of the variable and the other a constant, in the denominator, the sine or cosine, the constant being determined from the special case under consideration. The same is true for all other integrals involving inverse functions.
No to the differential equations we have already referred to one case in § 11 in the solution of the problem: what function of \( x \) recurs in its second derivative? and \( \cos x \) and \( \sin x \); so that the general solution of
\[
\frac{d^2 y}{dx^2} - y = 0
\]

is
\[
y = A \cos x + B \sin x.
\]

Comparing this result with the solution of
\[
\frac{d^2 y}{dx^2} + y = 0,
\]

as
\[
y = A \cos x + B \sin x,
\]

the connection is immediately apparent. Thus we might continue for many other similar equations.

§ 18. Application to the Theory of Equations

With the aid of the hyperbolic functions it is possible to solve completely the general cubic equation. Every cubic equation can be reduced to the form
\[
x^3 = bx + c \quad \ldots \ldots \ldots \ldots (1)
\]
in which the roots are \( x, x', x'' \); to zero.

If we let \( x = v \cosh w \),
\[
v^3 \cosh^3 w = bv \cosh w + c, \quad \ldots \ldots (2)
\]

or
\[
\cosh^3 w = \frac{b}{v^2} \cosh w + \frac{c}{v^2} \quad \ldots \ldots (2)
\]

But by formula (8) of § 8,
\[
\cosh^3 w = \frac{3}{4} \cosh w + \frac{1}{4} \cosh 3w \ldots (3)
\]

Equating coefficients we obtain:
\[
\frac{b}{v^2} = \frac{3}{4} \quad \ldots \ldots (4)
\]
and \( \frac{a}{b^3} = \frac{1}{4} \cosh 3w \); \( \frac{c^{3/2} \sqrt{26w}}{26w} = \cosh 3w \) .... (15)

This makes equation (11) an identity and hence \( x = v \cosh \theta \) is a root of the equation. If we replace \( 3u \) by \( 4 \), we obtain the three roots of equation (11)

\[
x = \sqrt[4]{\frac{3}{6}} \cosh \frac{\theta}{3},
\]

\[
x' = \sqrt[4]{\frac{3}{6}} \cosh \left( \frac{\theta}{3} + \frac{2}{3} \pi \sqrt{2} \right),
\]

\[
x'' = \sqrt[4]{\frac{3}{6}} \cosh \left( \frac{\theta}{3} + \frac{2}{3} \pi \sqrt{2} \right)
\]

in which the value of \( \theta \) is obtained from equation (8). These three values of \( x \) give us the three roots of the above cubic for all values of \( b \). Hence it is necessary to distinguish between several cases:

I. When \( b \) and \( c \) are positive, and \( \frac{c^{3/2} \sqrt{26w}}{26w} \) is real and \( \frac{c^{3/2} \sqrt{26w}}{26w} \) is real and \( \frac{c^{3/2} \sqrt{26w}}{26w} \) is real.

II. When \( b \) is positive and \( c \) is negative, but \( \frac{c^{3/2} \sqrt{26w}}{26w} \) is real and \( \frac{c^{3/2} \sqrt{26w}}{26w} \) is real.

III. When \( b \) is negative.

IV. When \( b \) is positive, but \( \frac{c^{3/2} \sqrt{26w}}{26w} \) is real and \( \frac{c^{3/2} \sqrt{26w}}{26w} \) is real.

Case I. When \( b \) and \( c \) are positive and \( \frac{c^{3/2} \sqrt{26w}}{26w} \) is real and \( \frac{c^{3/2} \sqrt{26w}}{26w} \) is real.

In this case \( \theta \) is real, and the value of the roots can be obtained from the above formulae. These can be reduced as follows:

\[
x = \sqrt[4]{\frac{3}{6}} \cosh \frac{\theta}{3}
\]

\[
x' = \sqrt[4]{\frac{3}{6}} \cosh \left( \frac{\theta}{3} + \frac{2}{3} \pi \sqrt{2} \right)
\]

\[
x'' = \sqrt[4]{\frac{3}{6}} \cosh \left( \frac{\theta}{3} + \frac{2}{3} \pi \sqrt{2} \right)
\]

\[
x = -\frac{\theta}{2} + i \sqrt{w} \sinh \frac{\theta}{3}
\]
\[ x'' = \sqrt{\frac{4}{3}} v \cosh \left( \frac{\theta}{3} + \frac{2}{3} \pi i \right) \]
\[ = -\frac{x}{2} - i\sqrt{6} \sinh \frac{\theta}{3} \]

**Case II.** When \( b \) is positive, \( c \) is negative, but \( \frac{\sqrt{3} b}{2 \sqrt{6} v} > 1 \).

In this case \( \theta \) is imaginary because \( \cosh \) is positive for all values of \( \theta \). However by Cauchy,

\[ \cosh(u + i\pi) = -\cosh u. \]

Hence

\[ x = \sqrt{\frac{4}{3}} v \cos \left( \frac{\theta}{3} + \frac{2}{3} \pi i \right), \]
\[ x' = \sqrt{\frac{4}{3}} v \cosh \left( \frac{\theta}{3} + \pi i \right) \cos \frac{\theta}{3}, \]
\[ x'' = \sqrt{\frac{4}{3}} v \cosh \left( \frac{\theta}{3} + \frac{5}{3} \pi i \right) \sin \frac{\theta}{3}. \]

It is easily shown that

\[ x = -\frac{x'}{2} + i\sqrt{6} \sinh \frac{\theta}{3}, \]
\[ x' = -\sqrt{\frac{4}{3}} v \cosh \frac{\theta}{3}, \]
\[ x'' = -\frac{x'}{2} - i\sqrt{6} \sin \frac{\theta}{3}. \]

**Case III.** When \( b \) is negative.

Since \( \cosh \left( u + \frac{3}{2} \pi i \right) = \sinh u \), substitute \( \theta + \frac{3}{2} \pi i \) for \( \theta \) and we get:

\[ x = -\sqrt{\frac{4}{3}} v \sinh \frac{\theta}{3}, \]
\[ x' = -\frac{x'}{2} + i\sqrt{6} \cosh \frac{\theta}{3}, \]
\[ x'' = -\frac{x'}{2} - i\sqrt{6} \sinh \frac{\theta}{3}. \]

**Case IV.** When \( b \) is positive, but \( \frac{\sqrt{3} b}{2 \sqrt{6} v} < 1 \).

Substitute for \( \theta \), \( i \theta \) and pass to the circular functions. This gives

\[ x = \sqrt{\frac{4}{3}} v \cos \theta, \]
\[ x' = \sqrt{\frac{4}{3}} v \cos \left( \frac{\theta}{3} + \frac{2}{3} \pi \right) = -\frac{x}{6} + \sqrt{6} \sin \frac{\theta}{3}, \]
\[ x'' = \sqrt{\frac{4}{3}} v \cos \left( \frac{\theta}{3} - \frac{2}{3} \pi \right) = -\frac{x}{6} - \sqrt{6} \sin \frac{\theta}{3}. \]

This gives us the values of the roots of the cubic for all possible values of \( b \) and \( c \), and hence we arrive at the complete solution of the cubic.
To use the formulas the only thing necessary is a set of tables of the hyperbolic sine and cosine.

379 Some Problems the Solutions of which are simplified by the Introduction of the Hyperbolic Functions

I To find the arc of the logarithmic spiral

\[ y = e^x. \]

The arc of any curve is given by

\[ s = \int \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx. \]

In this case \( \frac{dy}{dx} = \frac{x}{\text{log} e} = \frac{y}{\text{log} e}. \)

Let \( \log e = M \) and \( \frac{y}{M} = \sinh u. \)

Then \( dy = M \cosh u \, du. \)

Hence

\[ s = M \int \frac{\cosh u}{\sinh u} \, du = M \int \cosh u \, du = M \left[ \sinh u + \frac{1}{2} \cosh u \right]. \]

II To find the arc of the spiro of Archimedes:

\[ r = a \theta. \]

The arc of any curve in polar coordinates is

\[ s = \int \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta. \]

Hence

\[ \frac{dr}{d\theta} = a. \]

Hence

\[ r = \int \sqrt{a^2 + a^2} \, d\theta. \]

Let \( \theta = \sinh u, \, d\theta = \cosh u \, du, \, r = a \theta = a \sinh u. \)

Then

\[ s = \int \sqrt{a^2 \sinh^2 u + a^2} \, \cosh u \, du = a \int \cosh^2 u \, du, \]
or since \( \cosh^2 u = \frac{1}{2}(\cosh 2u + 1) \),
\[ s = \frac{\alpha \sqrt{b} (\cosh 2u + 1)}{\sqrt{y}} \]
\[ = \alpha \left( \frac{\cosh b u + 2u}{\sqrt{y}} \right) \]
\[ = \frac{\alpha}{2} \left( \theta + \theta^2 + 2 \cosh^{-1} \theta \right). \]

III. To find the area of a zone of an oblate spheroid generated by revolving the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), or \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), about the y-axis.

Swing any surface of revolution, the surface is
\[ \phi = 2\pi \int_0^a \sqrt{1 \left( \frac{dy}{dx} \right)^2} \, dy. \]
Here \( \frac{dy}{dx} = \frac{x}{\sqrt{1 - x^2}} \).
Hence
\[ \phi = 2\pi \int_0^a \sqrt{1 \left( \frac{x}{\sqrt{1 - x^2}} \right)^2} \, dy \]
\[ = \frac{2\pi}{\sqrt{1 - x^2}} \int \sqrt{x^2(1 - x^2) + y^2} \, dy, \]
or since
\[ x^2 = \frac{a^2 - y^2}{1 - x^2}, \]
\[ \phi = \frac{2\pi}{\sqrt{1 - x^2}} \int \sqrt{y^2(1 - x^2) + \frac{a^2 - y^2}{1 - x^2} \, dy.} \]

Let \( \frac{a^2 - y^2}{b^2} = \sinh u \), \( dy = \frac{a^2 - y^2}{b^2} \cosh u \, du \).
Then
\[ \phi = \frac{2\pi}{b^2} \int \sqrt{1 + \sinh^2 u} \cosh^2 u \, du \]
\[ = \frac{2\pi}{b^2} \int \cosh^2 u \, du \]
\[ = \frac{\pi b^2}{2} (\sinh u + 2u) \]
\[ = \frac{\pi b^2}{2} \left( \frac{a^2}{b^2} \sqrt{y^2 + b^2(1 - x^2)} + \sinh^{-1} \frac{y}{b} \right) \]

IV. To find the radius of curvature of the hyperbola \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \).

In general
\[ R = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2} \]
\[ \frac{1}{\sqrt{y}} \frac{dy}{dx} = -\frac{a^2}{b^2}, \quad \frac{1}{\sqrt{y}} \frac{dy}{dx} = -\frac{b^2}{a^2}. \]
Then \[ R = \frac{\left[ 1 + \frac{b^2 \tan^2 \phi}{a^2} \right]^{3/2}}{a^2} \]

Let \( \frac{b}{a} = \sin \alpha \) and \( \frac{a}{b} = \cos \alpha \).

Then \[ R = \frac{\left( a \sin \alpha \cos \alpha + b \cos \alpha \sin \alpha \right)^{3/2}}{ab} \]

To find the equation of the curve, the length of whose tangent from the point of contact to the \( x \)-axis is constant.

Let \( PT \) be the tangent at any point \( P \).

Let it make an angle \( \phi \) with the \( y \)-axis and let its tangent be \( a \). Then from the figure it is apparent that \( y = a \cos \phi \).

Now \( \phi = \tan^{-1} \frac{dy}{dx} \) from differential calculus.

Hence \( y = a \cos \tan^{-1} \frac{dx}{dy} \).

Reversely we get \[ \frac{dx}{dy} = \tan \phi = \frac{y}{a} \cos \phi \] and \( \frac{dy}{dx} = \frac{a^2 - y^2}{y} \).

Integrating, \( x = a \sec \phi \). Then \( \frac{dy}{dx} = \sqrt{a^2 - y^2} + c \) \( x = a \sec \phi \) when \( x = 0 \). Then \( \phi = 0 \).

Now \( \frac{dy}{dx} = \cos \phi \).

Hence \( x = a \sec \phi \cos \phi - a \sin \phi = a \left( \sec \phi - \tan \phi \right) \).

This equation, together with \( y = a \cos \phi \), gives a simple single-parameter form from which the curve can be easily found.

Let \( \alpha = \phi \). Then \( \sec \phi = \frac{1}{\alpha} \).

If \( t = a \), the above equation becomes
\[ x = a \sec t \tan \theta, \quad y = a \sec \theta \]

This curve is commonly known as the tractrix.
Chapter VIII

Applications of the Hyperbolic Functions

involving Mechanical Problems.

§ 30. The catenary. If a perfectly flexible uniform string be suspended by its ends what will be the equation of the curves so formed?

Let \( w \) be the weight of a unit length and let \( p \) be the length of the portion \( AP \). Then the weight of \( AP \) is \( pw \). Since the string is in equilibrium, the force acting on it must be balanced. Other forces acting are \( H \), the horizontal tension along the tangent at \( A \), and \( T \), the terminal tension along the tangent at \( P \). Hence from \( \Delta PLH \), where \( PL = T \), \( LN = H \), \( PN = wP \), if \( \phi \) be the angle made by the tangent with the \( x \)-axis, it is evident that

\[ \tan \phi = \frac{H}{w} = \frac{a}{w} \]

Hence \( \tan \phi = \frac{dy}{dx} \).

But \( \tan \phi = \frac{dy}{dx} \) \( \therefore \frac{dy}{dx} = \frac{a}{w} \).

Then \( \frac{dy}{dx} = \frac{a}{w} \) \( \therefore \frac{dy}{dx} = \frac{a}{w} \).

or

\[ \frac{dy}{dx} = \frac{a}{w} \]
Integrating, \( \frac{\theta}{\alpha} = \sinh \frac{\alpha}{\alpha} \)

and \( \frac{\theta}{\alpha} = \sinh \frac{\alpha}{\alpha} \) .................. (1)

But \( \frac{\theta}{\alpha} = \frac{\alpha}{\alpha} = \sinh \frac{\alpha}{\alpha} \).

Hence \( \frac{\theta}{\alpha} = \cosh \frac{\alpha}{\alpha} \) .................. (2)

which is the required equation of the catenary.

From above, \( \tan \phi = \sinh \frac{\alpha}{\alpha} \).

Hence by \( \tan \theta = a \gamma \phi \), and hence \( \gamma = a \sec \phi \),

the equation of the catenary expressed in terms of the angle which the tangent makes with the x-axis.

It is also easily shown that

\[ \gamma^2 = \alpha^2 + a^2 \] .................. (3)

The catenary is one of the most interesting curves in mathematics. Among its many properties, one may be mentioned that the length of the perpendicular upon the tangent from the foot of the ordinate of the point of contact is constant and equal to the parameter \( a \). Hence as seen from equation (4),

\[ \gamma^2 = \alpha^2 + a^2 \]

it is evident that the length of the tangent between the point of contact and the foot of this perpendicular is equal to the length of the curve from the lowest point to the point of contact. It is further evident that if a right triangle, one leg of constant length, be moved so that the hypotenuse is always perpendicular to a given line and the variable leg equal to the length of
the curve traced by the point of intersection of the leg with the hypotenuse, the curve will be the catenary. It would seem not difficult to devise a mechanism fulfilling these conditions.

B. To find the center of gravity of an arc. The arc of the catenary is by equation (1),

\[ s = a \sinh \frac{x}{a} = a \sinh u \quad (u = \frac{x}{a}). \]

From integral calculus,

\[ \bar{x} = \frac{\int_{0}^{s} x ds}{\int_{0}^{s} ds} = \frac{a^{2} \int_{0}^{u} \sinh u \ du}{a^{2} \int_{0}^{u} \cosh u \ du}. \]

Hence

\[ \bar{x} = a^{2} \left[ \sinh u - \cosh u \right]_{0}^{u} = a^{2} \left[ \sinh u - \cosh u + 1 \right]. \]

\[ \bar{y} = \frac{\int_{0}^{s} y ds}{\int_{0}^{s} ds} = a^{2} \int_{0}^{u} \cosh u \ du. \]

Hence

\[ x \bar{y} = \frac{a^{2}}{4} \left( \cosh 2u \right) \]

and

\[ 4xy = a^{2} \left( \sinh 2u + 2u \right). \]

To find the moment of inertia of this arc about the terminal abscissa, QP.

In any case by calculus,

\[ I = \int (y - y')^{2} dm. \]

Here \( r = (y, -y) \) \( dm = \mu ds \), where \( \mu \) is mass per unit length.

Hence

\[ I = \mu \int (y - y')^{2} ds \]

\[ = \mu \left[ \int y^{2} ds - 2y'y ds + y'^{2} ds \right] \]

\[ = \mu \left[ y^{2} ds - 2y'y ds + y'^{2} ds \right]. \]

Now

\[ y = a \cosh u \]

\[ ds = a \cosh u \ du \]

\[ \therefore \quad I = \mu \left[ y^{2} ds - 2y'y ds + a^{3} \int \cosh^{3} u \ du \right] \]

\[ = \mu \left[ y^{2} ds - 2y'y ds + a^{3} \left( \frac{1}{2} \sinh^{2} u + \frac{3}{4} \sinh u \right) \right]. \]
Since \( y = a \cosh \theta \), \( a = \frac{y}{\cosh \theta} \), we obtain

\[
I = \mu \left[ a^3 \cosh^2 \theta \sinh \theta - a^3 \cosh^2 \theta \sinh \theta - \frac{a^3}{2} \sinh^2 \theta \right] + \frac{a^3}{12} \sinh^3 \theta + 3 \frac{a^3}{4} \sinh \theta \]

\[= \mu a^3 \left[ \frac{1}{2} \sinh^3 \theta + \frac{3}{4} \sinh \theta - \frac{a^3}{4} \cosh^3 \theta \right]. \]

**3.1 Center of Gravity and Moment of Inertia of the arc of the parabola \( y^2 = 2px \).**

Before finding the center of gravity, i.e., it might be well to find the length of the arc with the help of the hyperbolic functions.

For any arc \( s = \int \sqrt{1 + (\frac{dy}{dx})^2} \, dy \).

Here \( \frac{dy}{dx} = \frac{\cosh \theta}{a} \).

Therefore \( s = a \int \sqrt{1 + (\frac{\cosh \theta}{a})^2} \, dy \).

Let \( \frac{1}{a} = \sinh \alpha \). Then \( dy = a \cosh \alpha \, d\alpha \).

Hence

\[
s = a \int \sqrt{1 + \sinh^2 \alpha} \cosh \alpha \, d\alpha = a \int \cosh \alpha \cosh \alpha \, d\alpha = a \cosh \alpha \left( \frac{1}{2} \sqrt{y^2 + p^2} + \sinh^{-1} \frac{y}{p} \right).
\]

The center of gravity is given by

\[
x = \frac{\int x \, dy}{\int dy} \quad \text{and} \quad y = \frac{\int y \, dy}{\int dy}.
\]

Hence

\[
x = \frac{\int x \, dy}{\int dy} = \frac{\int x \, \cosh \alpha \, d\alpha}{\int \cosh \alpha \, d\alpha} = \frac{p^2}{2} \sinh \alpha \cosh \alpha \left( \frac{1}{2} \sqrt{y^2 + p^2} + \sinh^{-1} \frac{y}{p} \right).
\]

The center of gravity is given by

\[
x = \frac{\int x \, dy}{\int dy} = \frac{p^2}{2} \frac{y + \sqrt{y^2 + p^2}}{a}.
\]

Hence

\[
\bar{x} = \frac{\int x \, dy}{\int dy} = \frac{p^2}{2} \frac{\sqrt{y^2 + p^2}}{a}.
\]

**For a discussion of the elasticity of uniform strength and the elastic energy see Minnaert and Lordewerd Higher Math. Pp 147-9**
\[ \phi \frac{p}{2} (\sinh u + u - u) \]
and \[ 6\phi \frac{p}{2} = p^2 (\sinh u + u - u) \cdot \]

Similarly,
\[ \phi \frac{j}{2} = \int_0^\infty y dy \]
\[ = \frac{p^2}{2} \int_0^\infty \sinh u \cosh u du \]
\[ = \frac{p^2}{3} (\cosh^3 u) = \frac{p^2}{3} (\cosh^3 u - 1), \]
or \[ 3\phi j = \cosh^3 u - 1. \]

The moment of inertia of this arc about its

Turning axis is
\[ I = \phi \frac{p}{2} (x - x)^2 dx \]
\[ = \mu \left[ \frac{1}{2} \left( x^2 + x \right) dx + \frac{p^2}{2} \right] \]
\[ = \mu \left( x^2 + 2x + \frac{p^2}{4} \right) \sinh^2 \cosh^2 u. \]

Now
\[ \int \sinh^2 u \cosh^2 u = \frac{1}{8} \int \cosh 2u + 1 \left( \cosh 2u - 1 \right) \]
\[ = \frac{1}{8} \int \left( \cosh^2 2u - \cosh 2u \cosh 2u \right) \cosh^2 2u = \frac{1}{4} \cosh (u + \frac{3}{2}) \cosh 2u \text{ and} \]
\[ \cosh 2u = \frac{1}{2} \cosh (u + \frac{3}{2}). \]

Hence
\[ \int \sinh^2 u \cosh^2 u = \frac{1}{8} \int \frac{1}{4} \cosh 6u - \frac{1}{4} \cosh 4u - \frac{3}{4} \cosh 2u \]
\[ = \frac{1}{7} \sinh 6u - \frac{1}{4} \sinh 4u - \frac{1}{2} \sinh 2u - \frac{1}{2}. \]

Hence
\[ I = \mu \left( x_0 \left( x - 2x \right) + \frac{1}{2} p^3 N \right), \text{ where} \]
\[ N = \frac{1}{2} \sinh 6u - \frac{1}{4} \sinh 4u - \frac{1}{2} \sinh 2u + u. \]

3.32 Applications to the Bending of Beams.

If a beam is built in at one end and a load \( P \) is applied at the other, and also a horizontal tensile force \( L \) is applied at the same point, to find the equation of the curve assumed by the neutral
surface. If \((a, y)\) be any point on the surface such as free end or origin, the bending moment of this point is \(2y - Px\). Hence with the usual notation of the theory of flexure,

\[
EI \frac{d^2y}{dx^2} = 2y - Px.
\]

Let \(\frac{Q}{EI} = n^2\) and \(\frac{P}{EI} = mn^2\).

Then \(\frac{d^2y}{dx^2} = n^2(y - mn)\).

If \(y - mn = u\), \(\frac{d^2u}{dx^2} = 0\),

and the equation assumes the form

\[
\frac{d^2u}{dx^2} = n^2 u,
\]

the solution of which is

\[
u = A \cosh nx + B \sinh nx,
\]

or

\[
y = xee^{nx} + B \sinh nx + mx.
\]

The arbitrary constants \(A\) and \(B\) can be determined from the conditions of the problem. Hence at the free end \(x = 0, y = 0\), \(A\) must be 0, and hence

\[
y = B \sinh nx + mx,
\]

and

\[
\frac{dy}{dx} = nB \cosh nx + m.
\]

At the fixed end \(dy/dx = 0\), and \(x = l\),

\[
\therefore B = \frac{\text{const}}{\text{const}}
\]

Hence the required equation is

\[
y = mx - \frac{\text{const} \times nB \cosh nx + m}{\text{const} \times nB \cosh nx + m}
\]

B. If the load is uniformly distributed over the beam \(w\) per unit area, the equation assumes the form
\[ EI \frac{d^2 y}{dx^2} = 2y - \frac{1}{2} \omega x^2, \]

so, if \( \frac{d^2 y}{dx^2} = \frac{\omega}{m}, \) and \( \frac{d y}{dx} = \frac{\omega}{m} x, \)
\[ \frac{d y}{dx} = \omega \left( \frac{n}{m} x^2 \right) \]

Let \( y - nx^2 = u \) and \( \frac{d u}{dx} = \frac{m}{n} u + \omega x. \)

Hence \( \frac{d u}{dx} - \frac{m}{n} u = -2 \omega x, \)
the solution of which is
\[ u = A \cosh nx + B \sinh nx + \frac{2m}{n} x, \]
so
\[ y = n \cosh nx + B \sinh nx + \frac{2m}{n} x. \]

Since at the origin \( y = 0, x = 0, A = -\frac{2m}{n}. \)

\[ y = \frac{2m}{n} \cosh nx + B \sinh nx + \frac{2m}{n} x, \]
\[ \frac{d y}{dx} = \frac{m}{n} \sinh nx + B \cosh nx + \frac{2m}{n} x. \]

At the fixed end \( \frac{d y}{dx} = 0, x = l. \)

Hence \( B = -\frac{2ml}{n} \frac{\sinh nl}{\cosh nl}. \)

Hence
\[ m = \frac{2m}{n} \cosh nl \frac{\sinh nl}{\cosh nl} + \frac{2m}{n} x^2. \]

Many other problems of a nature similar to the above might be mentioned.

Hyperbolic functions are used also for alternating current, which fall back on the solution of a differential equation of the form
\[ \frac{d^2 y}{dx^2} - a^2 y = f(x), \]
already discussed in §27, and hence we need not go into any detail here.
Applications to the field of potential along a wire are found in Byerly's Fourier Series, p. 99 ff. In the same volume are given many other applications of the hyperbolic functions to problems in sound, heat, and electricity.
Historical.

The hyperbolic functions date their origin back to the eighteenth century. The foundations for these functions were laid long before their theory was actually developed. Unconsciously, Gregory and St. Vincentius (1554-1667) David Gregory (1667-1703) and others contributed to the early development by finding the area of the rectangular hyperbola. Newton further helped by drawing comparisons between the circle and the rectangular hyperbola, while Weierstrass found that in the case of imaginary arguments the functions of the circle could be changed into similar real functions of the hyperbola. The first one who actually founded a theory of the hyperbolic functions was Enrico Riccati (1707-1775) who worked out the most important relations connected with them, from geometrical considerations. Lambert made use of this theory and extended it considerably, applying it in various ways, and especially to the solution of some trigonometric problems. For the next half century the advance made was comparatively slow. The next work of importance along this line was done by Euler and Legendre in his article on the Potential Functions. The chief
addition which he made was the so-called
quadrascaranian function, connecting the hyper-
belic and circular functions. He further worked
out extensive tables for the hyperbelic functions
and the quadrascaranian, and this is still one of
the most complete set of tables of the functions
published. Other investigators who have published
tables of these functions are: Deonan (1860), Angelus
Forti (1862), Donel (1864), Nasaal (1872), and
Ligowski (1873) and (1889).

Since the publication of Deonan's article, the
theory has been extended considerably, especially
in the latter half of the nineteenth century. The
chief extensions consist in various attempts at
generalizing these functions. Perhaps the most
interesting of these is set forth in Liouville's article
'Essai sur les fonctions hyperboliques,' (Paris, 1879),
in which he defined the functions with respect to
the general hyperbola instead of using the
rectangular hyperbola, as had previously been
done. In a similar way the circular func-
tions have been extended to the ellipse. Another
generalization was attempted by Dühnhe (1858),
by making them functions of the angle which

*This function was called by him the "longitudinal," expressed
\( \ell \), and the inverse, the "längzahel" (LW). The name "quadrascaranian"
was first applied by Cayley: Elliptic Functions, 1874.
the radius vector of the curve \( x^m - y^m = 1 \) makes with the \( x \)-axis, the hyperbolic functions being the result of the special case in which \( m = 2 \). The corresponding extensions of the circular functions is with reference to the curve \( x^m + y^m = 1 \), the circular functions being as above, the case where \( m = 2 \). A third generalization which might be mentioned was made by dropping out at regular intervals certain terms in the expansion

\[
  x^2 = 1 + x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \frac{x^5}{5^2} + \ldots
\]

Thus

\[
  f_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \ldots
\]

\[
  f_1(x) = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \ldots
\]

\[
  f_2(x) = \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \ldots
\]

\[
  f_n(x) = \frac{x^n}{n!} + \frac{x^{n+2}}{(n+1)!} + \frac{x^{n+4}}{(n+2)!} + \ldots
\]

The hyperbolic sine and cosine result when \( n = 2 \) for there we have

\[
  f_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{4^2} + \frac{x^6}{6^2} + \frac{x^8}{8^2} + \ldots
\]

\[
  f_1(x) = x + \frac{x^3}{3^2} + \frac{x^5}{5^2} + \frac{x^7}{7^2} + \ldots
\]

which are clearly the series for the sine and cosine. These series had already suggested themselves to Riccati. They are discussed to a considerable extent in Euler's "On a set of functions analogous to the circular functions." * Kühn has also treated them in his book on the hyperbolic functions.

* * *
An attempt in an altogether different direction was made by James Booth (1862) in the creation of a parabolic trigonometry, which is rather an extension of the circular trigonometry, the functions being referred to the arc of a parabola. This trigonometry has not gained a very wide introduction into the mathematical sciences.

What the future will bring us in the work on these functions is difficult to say, but it is hoped that by degrees the importance of the hyperbolic trigonometry will be recognized, and that it will be placed on an equal footing with the now so common circular trigonometry.
References.

4. Dürer; Die Lehre von den gewöhnlichen und verallgemeinerten hyperbel funktionen, Halle, 1851.
5. Plücker; On a set of functions analogous to the circular functions. Read, Proc. of Math v. 7.
7. Rayleigh; Fourier. I, art. 176, 477.

Good elementary discussions are found in Loney’s Trigonometry
Levi & Davison’s Trigonometry.

For Historical notes see