An Investigation of Green's Theorem for Discontinuous Functions

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

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* with the assention of Prof. E. Townsend.
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**Proper and Improper Integrals**

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Introduction

Green's Theorem first appeared in "An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism," which was published by subscription at Nottingham in 1828. The essay may be found in the "Mathematical Papers of the Late George Green" edited by W. M. Ferrier. The theorem has important applications in the theories of Electricity and Magnetism, Potential, and Hydrodynamics. The theorem expresses the surface integral of a function of two functions \( W \) and \( V \) and their partial derivatives with respect to the normal in terms of the volume integral of a function of the same two

1) Maxwell, Electricity, and Magnetism, P. 111 et seq.
3) Lamb, Hydrodynamics, P. 47 et seq.
functions \( W \) and \( V \) and their second

differential parameters.

Suppose \( W \) and \( V \) and their first

partial derivatives in any direction

are uniform, continuous functions of

the coordinates \( x, y, \) and \( z \) within a

closed surface \( S \) and on the bound-

ary. Also, let \( D_m \) denote the partial
derivative with respect to the exterior

normal and let \( \Delta \) denote the second
differential parameter, namely,

\[
\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2} . \]

Then

\[
\iint (W D_m V + V D_m W) \, dS = \iiint (W \Delta V - V \Delta W) \, dx \, dy \, dz , \quad (1)
\]

where the triple integrals are to be

extended to all points within \( S \), and

the surface integrals to all points on \( S \).

For line and surface integrals

this becomes

\[
\int (W D_m V + V D_m W) \, ds = \iiint (W \Delta V - V \Delta W) \, dx \, dy \quad (2)
\]
where $W$ and $V$ and their derivatives are uniform continuous functions of $x$ and $y$ within a closed curve $C$, the line integrals extending to all points on $C$ and the surface integrals to all points within $C$. \( \Delta = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}. \)

Equations (1) and (2) give Green’s Theorem in the second form, in which it is the most often used.

In the applications of the theorem, the integrands are assumed to be continuous. For example, if $W$ or $V$ represents the potential function due to a continuous distribution of matter, this function is everywhere finite.

Discontinuities in the function would have no physical application. Maxwell has discussed the case where the function

---

1) Peirce, Newtonian Potential Function;
2) Maxwell, Electricity and Magnetism, p. 2.
\( W \) or \( V \) may not be uniform. It is the purpose of this thesis to investigate under what conditions different forms of this theorem hold when \( W \) or \( V \) becomes discontinuous at a set of points in the region considered.
Grice Integrals

1. Definition of Grice Integral — If a given line is divided in any way into infinitesimal elements, and the length of each element is multiplied by the value which a given point function, continuous along the line, has at some point within the element, the limit approached by the sum of these products as each element is indefinitely decreased, is called the line integral of the given function along the line in question.

2. Complex Variable — Cauchy’s Theorem — In the theory of functions of a complex variable, it is often necessary to consider the line integral of a function around a point in the complex plane at which the function has a singularity. Consider, for example, the
function \( \frac{1}{z-a} \), which becomes infinite for 
\( z = a \), i.e. has a pole at \( a \). Integrating
the function around a closed curve \( C \), enc
closing the point \( a \), we have

\[ \oint_{C} \frac{dz}{z-a} = 2\pi i \]

(3)

If \( f(z) \) is a function of the complex
variable \( z \), which is regular within a
region bounded by the closed curve \( C \), and
if \( a \) is a point within this region, then
we have the following more general theorem,
the well-known theorem of Cauchy:

\[ \oint_{C} \frac{f(z) dz}{z-a} = 2\pi i f(a) \]

(4)

The coefficient of the \((-1)st\) power of \( z-a \)
in the development of a function for the
neighborhood of a pole \( z = a \) is called the
residue of the function for this pole. If
\( f(z) \) is expressed in the form:

1) Buschhardt, Funktionentheorie, V.I. p.114.
2) Ibid. p.115.
\[ f(z) = \frac{A_{-m}}{(z-a)^m} + \frac{A_{-m+1}}{(z-a)^{m+1}} + \cdots + \frac{A_{-1}}{z-a} + A_0 + A_1(z-a) + A_2(z-a)^2 + \cdots \] (5)

then

\[ \int \frac{f(z)}{z-a} \, dz = 2\pi i \cdot A_{-1} \] (6)

\( A_{-1} \), being the residue of the function \( f(z) \) relative to the pole \( a \). This is known as Cauchy's Theorem of Residues.

The form (4) is a special case of (5) for, if \( f(z) \) is developed into a series of the form

\[ f(z) = A_{-1} + A_0(z-a) + A_1(z-a)^2 + \cdots \] (7)

then

\[ \frac{f(z)}{z-a} = \frac{A_{-1}}{z-a} + A_0 + A_1(z-a) + \cdots \] (8)

and by (6) the integral of a function \( f(z) \) in this form gives

\[ \int \frac{f(z)}{z-a} \, dz = 2\pi i \cdot A_{-1} \] (9)

But, by (7)

\[ f(a) = A_{-1} \] (10)

that is to say, the residue in this case is the value of the function at the point, and equation (9) is the same as equation (4).

3. Real Variable — Condition that Given Integral is Independent of Path — For equation (1) put \( W = 1 \), then
\[ \oint_{C} D_{n}\mathbf{v} \cdot ds = \iint_{A} \mathbf{\nabla} \mathbf{v} \cdot dA \]  

(11)

By this relation we will show that the condition that the line integral of \( \mathbf{\nabla} \mathbf{v} \) be independent of the path of integration \( C \), is equivalent to the condition that \( \mathbf{v} \) be a harmonic function everywhere within \( C \), i.e., that \( \mathbf{v} \) be uniform, continuous and satisfy the condition \( \mathbf{\nabla}^{2} \mathbf{v} = 0 \). For, if \( \mathbf{v} \) is harmonic, we have

\[ \mathbf{\nabla}^{2} \mathbf{v} = 0 \]  

(12)

and, substituting this in the above equation, it reduces to

\[ \oint_{C} D_{n}\mathbf{v} \cdot ds = 0 \]  

(13)

where \( \mathbf{v} \) is harmonic within \( C \).

In the figure, the line integral of \( D_{n}\mathbf{v} \cdot ds \) around the shaded region following the arrows will be zero if \( \mathbf{v} \) is harmonic between the curves \( C_{a} \) and \( C_{b} \). For, by (13),
\[ \oint_{C} \mathbf{D} \cdot \mathbf{V} \, ds + \int_{a}^{b} \mathbf{D} \cdot \mathbf{v} \, ds + \oint_{\Delta} \mathbf{D} \cdot \mathbf{V} \, ds = 0 \]  

But the integrals along \(ab\) in opposite directions cancel, and, reversing the direction of integration along \(c\), we have

\[ \oint_{C} \mathbf{D} \cdot \mathbf{V} \, ds = \int_{\Delta} \mathbf{D} \cdot \mathbf{V} \, ds \]  

i.e. the integral is independent of the path.

4. **Cauchy's Theorem for Real Variables**

Consider the line integral of the function \[ \frac{1}{\sqrt{(x-a)^2 + (y-b)^2}} \] around a circle \(K\) with center \((a,b)\). We have

\[ \oint_{K} \frac{ds}{\sqrt{(x-a)^2 + (y-b)^2}} = \int_{0}^{2\pi} \frac{rd\theta}{r} = 2\pi \]  

The value of the integral is independent of the size of the circle. Also, since the function \(\frac{1}{r}\) is harmonic, by \((15)\) the integral will be the same around any closed curve enclosing the point \((a,b)\).

So we may write:
\[ \int_C \frac{ds}{\sqrt{(x-a)^2 + (y-b)^2}} = 2\pi \quad (17) \]

where \( C \) is any closed curve surrounding the point \((a, b)\).

Consider the line integral of the function

\[ \frac{f(x, y)}{\sqrt{(x-a)^2 + (y-b)^2}} \quad (18) \]

around any closed curve \( C \), enclosing the point \((a, b)\), where \( f(x, y) \) is continuous everywhere within \( C \). The function (18) becomes infinite at one point \((a, b)\) within the contour \( C \). Surround the point \((a, b)\) by a small circle \( K \) with arbitrary small radius \( r_K \). Then we may apply (15) provided the function (18) is of the form \( D_{r_K} \), i.e. the further condition must be imposed on the function \( f(x, y) \) that it is of the form \( r D_{r_k} \), where \( V \) is a harmonic function in the region between \( C \) and \( K \). Then

\[ \int_C \frac{f(x, y) \, ds}{\sqrt{(x-a)^2 + (y-b)^2}} = \int_K \frac{f(x, y) \, ds}{\sqrt{(x-a)^2 + (y-b)^2}} \quad (19) \]
This second integral may be written
\[
\int \frac{f(x,y) \, ds}{\sqrt{(x-a)^2 + (y-b)^2}} = \int \frac{f(a,b) \, ds}{\sqrt{(x-a)^2 + (y-b)^2}} + \int \frac{f(x,y) - f(a,b) \, ds}{\sqrt{(x-a)^2 + (y-b)^2}}
\]

By (16),
\[
\int \frac{f(a,b) \, ds}{\sqrt{(x-a)^2 + (y-b)^2}} = f(a,b) \int \frac{ds}{\sqrt{(x-a)^2 + (y-b)^2}} = 2\pi f(a,b).
\]

Also, by virtue of the continuity of \( f(x,y) \) for an arbitrarily small \( \varepsilon \) we can find a \( \delta \) such that
\[
|f(x,y) - f(a,b)| < \varepsilon \quad (22)
\]
when \(|x-a| < \delta\), \(|y-b| < \delta\), i.e. \( rK < \delta v \).

The third integral may be written
\[
\int \frac{[f(x,y) - f(a,b)] \, rK \, ds}{rK} \quad (23)
\]

The \( rK \)'s will always cancel, and, by making \( rK \) small enough, we can make
\[
|f(x,y) - f(a,b)| < \varepsilon \quad (24)
\]

Hence this integral vanishes.

We then have
\[
\int \frac{f(x,y) \, ds}{\sqrt{(x-a)^2 + (y-b)^2}} = 2\pi f(a,b) \quad (24)
\]

This theorem is analogous to Cauchy's Theorem for a complex variable, (4). The
extension to the case where there is a set of poles within the contour will be made later.

5. Residue Theorem for a Single Pole —

We will now develop a theorem similar to (24) in another form, using the theory by which the theorems in the next section, for three dimensions, are deduced. This theory for three variables has been developed by Apple.

Let \( V_u (x, y) \) denote the most general homogeneous polynomial of degree \( n \) in \( x \) and \( y \) satisfying the equation

\[
\Delta V_u = \frac{\partial^2 V_u}{\partial x^2} + \frac{\partial^2 V_u}{\partial y^2} = 0
\]

(25)

Consider a function \( F(x, y) \), uniform in all space and regular at all points except certain points which we call singular points. If \( P(a, b) \) is one of these points, we say that this point is a pole or non-essential singular point of
degree \( n \) of the function \( F \) if there exists a function \( \Phi \) of the form

\[
\Phi(x-a, y-b) = V_1(x-a, y-b) + V_2(x-a, y-b) + \ldots + V_n(x-a, y-b)
\]  

such that

\[
F(x, y) - \Phi(x-a, y-b)
\]

is regular at the point \( P \). The first term of \( \Phi \) is of the form

\[
V_1(x-a, y-b) = \frac{\lambda}{\sqrt{(x-a)^2 + (y-b)^2}}
\]

and the coefficient \( \lambda \) is called the residue of the function \( F \) relative to the pole \( P \).

If no such function \( \Phi \) exists, \( P \) is called an essential singular point.

A function \( F(x, y) \), uniform, continuous and having partial first and second derivatives and satisfying \( \Delta F = 0 \) at all points within the space between two circles with the same center \((a, b)\) and radii \( R, R' \), \( R > R' \), is developable into a series of the form
\[ F(x, y) = \sum_{V = -\infty}^{V = +\infty} V_v (x-a, y-b) \quad (29) \]

uniformly convergent at all points between the two circles. This is analogous to Gauss's theorem for complex variables.

We will now prove the following theorem:

**Theorem** - If a function \( F(x, y) \), satisfying the equation \( \Delta F = 0 \), has a pole or essential singular point \( P \), the line integral

\[ \oint_C \nabla F \, ds \]

around any closed contour \( C \), enclosing the point \( P \), and containing no other singular points, is equal to \(-2\pi i\) times the residue of \( F \) relative to the point \( P \).

Let \( K \) be any circle with center \((a, b)\) and radius \( r \) lying wholly within \( C \); then, since the function \( F \) is harmonic in the region between \( C \) and \( K \), we have, by applying (15),

\[ \oint_C \nabla F \, ds = \oint_K \nabla F \, ds \quad (30) \]
that is to say, the determination of the integral around the curve C may be replaced by that around a circle K. Between this circle K and an arbitrarily small concentric circle within K, the function \( F \) can be developed by (29) in the form

\[
F(x, y) = \sum_{v=-\infty}^{v=\infty} V_v (x-a, y-b)
\]  

(31)

On the circle K,

\[
x - a = r \cos \theta \\
y - b = r \sin \theta \\
0 \leq \theta \leq 2\pi
\]

Substituting in (31)

\[
F(x, y) = \sum_{v=-\infty}^{v=+\infty} V_v [\rho \cos \theta, \rho \sin \theta]
\]

But \( V_v (\rho \cos \theta, \rho \sin \theta) = r^v \mathcal{I}_v (\rho) \), where \( \mathcal{I}_v \) denotes a function of \( \rho \). Then

\[
F(x, y) = \sum_{v=1}^{v=\infty} r^v \mathcal{I}_v (\rho) + \mathcal{I}_0 (\rho) + \sum_{v=1}^{v=\infty} r^{-v} \mathcal{I}_v'' (\rho)
\]

(32)

where \( \mathcal{I}_0 (\rho) \) is a constant and \( \mathcal{I}_v'' \) denotes another \( \mathcal{I}_v \) function with different constants.
Then:

$$D_n F = \sum_{v=1}^{\infty} v^{v-1} Y_v'(0) - \sum_{v=1}^{\infty} v^{v-1} Y_v''(0)$$  \hspace{1cm} (33)$$

Now substitute in \( \int_K D_n F \, ds \) the value of \( DF \) in (33) and also \( ds = r \, d \theta \). Also note that for any function of \( G \)-place, \( Y_v'' \),

$$\int_0^{2\pi} Y_v'(0) \, d \theta = 0 \quad \text{for} \ v \neq 0$$  \hspace{1cm} (34)$$

Since \( Y_v(0) \) is homogeneous in \( \sin \theta \) and \( \cos \theta \).

We then have

$$\int_K D_n F \, ds = -\int_0^{2\pi} Y_v''(0) \, d \theta = -2\pi Y''(0),$$  \hspace{1cm} (35)$$

\( Y''(0) \) being the residue of \( F \) relative to the point \( P \).

6. Extension of Residue Theorem to \( \infty \) Poles.

Suppose now that the function \( F(x, y) \) has any finite number of poles or essential singular points within the contour \( C \). We will demonstrate for this case the following:

**Theorem:** If a function \( F(x, y) \), satisfying the equation \( \Delta F = 0 \), is uniform
and regular at all points within and on a contour $C$, except at certain interior points $P, P_2, \ldots, P_p$, the line integral

$$\int_C D \frac{\partial F}{\partial s} ds$$

taken around the contour $C$, is equal to $-2\pi$ times the sum of the residues of the function $F$ relative to these points $P, P_2, \ldots, P_p$.

Surround each point $P_i$ by a circle $K_i$ situated within $C$ and having for its center the point $P_i$ and a radius small enough so that it will not cut any of the other circles $K_j$ having for centers the points $P_j$. Representing by $A$ the area interior to $C$ and extended to the circles $K, K_2, \ldots, K_p$, the function $F$ is regular throughout this area and, by (13), the line integral

$$\int_C D \frac{\partial F}{\partial s} ds = 0$$

taken along the contour limiting the area $A$. But this integral is the sum of
\[ p+1 \text{ integrals taken along the contours } C, \, K, \, K_1, \ldots, \, K_p, \text{ so that} \]
\[ \int \nabla F \, ds + \sum_{k=1}^{p} \int_{K_k} \nabla^2 F \, ds = 0 \quad (36) \]

By the preceding theorem
\[ \int_{K_k} \nabla^2 F \, ds = 2\pi R_k \quad (37) \]

\( R_k \) denoting the residue of \( F \) relative to the point \( P_k \), the + sign being used since the derivative is taken with respect to the exterior normal to the contour enclosing the area \( A \), i.e. the interior normal to the circle \( K_k \). Then, from (36)
\[ \int_{C} \nabla^2 F \, ds = -2\pi \left( R_1 + R_2 + \cdots + R_p \right) \quad (38) \]

which proves the theorem.
7. Definition of Surface Integral — If a given surface is divided in any way into infinitesimal elements such that the distance between the two most widely separated points within each element is infinitesimal and the area of each element is multiplied by the value which a given point function, continuous over the surface, has at some point within the element, the limit approached by the sum of these products as each element is indefinitely decreased, is called the surface integral of the given function over the surface in question.

8. Case Where the Function becomes Infinite or Indeterminate — Under what condition does \( \int \int f(x,y) \, dx \, dy \) have a meaning if \( f(x,y) \) becomes infinite or indeterminate within the

1) Picard, Traité d'Analyse. V. I. Chap. V."
field of integration?

Suppose \((a, b)\) to be the singular point. Trace around the point a small closed curve \(\gamma\) and extend the integral to that part of the domain without \(\gamma\). If this integral approaches a limit always the same as \(\gamma\) approaches indefinitely near the point \((a, b)\) by any law whatever, then that limit represents the value of the integral. Consider the integral

\[
\iiint_{\gamma} \frac{\partial^2 V}{\partial x \partial y} \, dx \, dy
\]

over the rectangle formed by the coordinate axes and the lines \(x = a, \ y = a'\). First suppose \(V\) continuous within and on the rectangle of integration. Then, integrating with respect to \(x\), the integral \((39)\) becomes

\[
\int_0^{a'} \frac{d}{dy} \left[ V(a, y) - V(0, y) \right] \, dy
\]

\[
= \ V(a, a') - V(a, 0) - V(0, a') + V(0, 0)
\]

Now suppose that \(V(x, y)\) is infinite or indeterminate for \(x = 0, \ y = 0\). Take for \(\gamma\) a
Small rectangle with sides 
$\xi'$ parallel to the axes.

Denoting the rectangles so
formed by I, II, III, we have

\[ \iint \frac{\partial^2 V}{\partial x \partial y} \, dx \, dy = V(a, \xi') - V(a, 0) - V(0, \xi') + V(0, 0) \]  \tag{42} 

\[ \iint \frac{\partial^2 V}{\partial x \partial y} \, dx \, dy = V(0, a') - V(\xi, \xi') - V(0, a') + V(0, 0) \]  \tag{43} 

\[ \iint \frac{\partial^2 V}{\partial x \partial y} \, dx \, dy = V(a, a') - V(0, a') - V(0, \xi') + V(0, 0) \]  \tag{44} 

Subtracting (44) from the sum of (42) and (43), we have

\[ \iint \frac{\partial^2 V}{\partial x \partial y} \, dx \, dy = V(a, a') - V(a, 0) - V(0, a') + V(0, 0) - V(0, \xi') \]  \tag{45} 

If (45) has a definite value, independent of
the limit of the ratio $\frac{\xi}{\xi'}$, it defines the integral.
The last three terms of (45) may, however, be-
come infinite, or have a limit depending on
the limit of the ratio $\frac{\xi}{\xi'}$, in which case
the integral has no meaning.

Example:

\[ V = \tan^{-1} \frac{y}{x} \]

\[ \frac{\partial^2 V}{\partial x \partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \, dx \, dy \]
\[ \iint \frac{y^2 - x^2}{(y^2 + x^2)^2} \, dx \, dy = \tan^{-1} \frac{a'}{a} - \tan^{-1} \frac{\epsilon}{a} - \tan^{-1} \frac{a'}{\epsilon} + \tan^{-1} \frac{\epsilon}{\epsilon} \]

Since this latter expression depends on the limit of \( \frac{\epsilon}{\epsilon} \), the integral has no meaning extended to the rectangle \((a, a')\).

An analogous method may be employed in the case of triple integrals. Surround the singular point by a sphere and study the limit of the integral as this sphere approaches zero. If the function becomes infinite at a set of isolated points, we may surround each point by a sphere and study the limit of the integral as these spheres together approach zero.

We thus have a criterion for determining whether the integrals have a meaning when singular points occur in the field of integration.

9. **Theorems on Harmonic Functions of Three Real Variables** - If, in the second form of Green's Theorem, for surface and
volume integrals, equation (11), we put $V = 1$
and $W = F$ where $\Delta F = 0$; we have

$$
\iiint D F \, ds = 0
$$

and hence we may state the theorem analogous to (13):

**Theorem I** — If a function $F(x, y, z)$, satisfying the equation $\Delta F = 0$, is uniform and regular at all points within and on a surface $S$, then

$$
\iiint_{S} D F \, ds = 0
$$

appears that proofs the following theorems for

three variables by a method analogous to

that used in Art. 5 for functions of two

variables.

**Theorem II**

If a function $F(x, y, z)$, satisfying

1) P. Appell, "Sur les Fonctions de Trois Variables Régulières Satisfaisant à l'Equation Différentielle $\Delta F = 0"$ — Acta Math. V III.

2) Since the integral of a function $F$ over any surface is the same as the integral over a sphere, if the function satisfies the equation $\Delta F = 0$ between the surface and the sphere, this theorem will hold when the integral is extended to any surface enclosing the point $F$. 
the equation $\Delta F = 0$, has a pole or essential singular point $P$, the integral

$$\int\int_D \Delta F \, ds$$

extended to the surface of a sphere with center $P$ and containing no other singular points, is equal to $-4\pi$ times the residue of $F$ relative to the point $P$.

**Theorem III** — If a function $F(x, y, z)$, satisfying the equation $\Delta F = 0$, is uniform within a surface $S$ and regular at all points within and on $S$ except at certain interior points $P, P_1, \ldots, P_p$, the integral

$$\int\int_D \Delta F \, ds$$

extended to the surface $S$ is equal to $-4\pi$ times the sum of the residues relative to the points $P, P_1, \ldots, P_p$.

This theorem is proved by enclosing each point $P_k$ by a sphere $S_k$ and proceeding by a method similar to that of Art. 5.
Theorem IV — If \( F(x, y, z) \) is a function satisfying the equation \( \Delta F = 0 \), uniform within a closed surface \( S \) and regular at all points within and on \( S \), then

\[
F(x, y, z) = \frac{1}{4\pi} \int_S (F \cdot \mathbf{n}) \, dS
\]

where \( (x, y, z) \) is a fixed point within \( S \) and

\[
T = \frac{1}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}
\]

in referring to the exterior normal. For the proof of this theorem see Heine, Kugelfunctionen, Vol. 2, P. 93. As an example, let \( F \) be the Newtonian Potential Function \( V \), due to any distribution of matter, and \( T = \frac{1}{r} \), where \( r \) is the distance from a point \( P \) within the space included by \( S \). Then we will have

\[
V_P = \frac{1}{4\pi} \int_S (V \cdot \mathbf{n}) \, dS - \frac{1}{r} \int_S D_n V \, dS
\]

where \( V_P \) is the value of the potential at the point \( P \), according to J. Webster, "Electrostatics and Magnetism," P. 64.
point $P$, i.e., the value of a harmonic function at any point is determined if the value of the function together with that of its normal component is given at all points of a surface $S$.

We have similarly, if $V$ represents the logarithmic potential function

$$V_P = \frac{1}{2\pi i} \int_\gamma (\log z \, dV - V \, d\log z) \, ds \quad (48)$$

We thus see the analogy between these theorems and Cauchy's theorem for complex variables, viz.

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-s} \, ds \quad (49)$$

which gives the value of a regular function of a complex variable $z$ at any points $s$ within a closed curve $\Gamma$, in terms of the value of the same function on the boundary $\Gamma$.

10. Definition of Curvilinear Integral

Let \( P(x, y) \) be a point function of \( x \) and \( y \), and \( C \) a curve in the \( xy \)-plane joining the points \( A \) and \( B \), with coordinates \((a, A)\) and \((b, B)\).

Divide the curve \( C \) into any number of intervals by the points whose coordinates are \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\).

Then form the sum:

\[
P(a, A)(x_1 - a) + P(x_1, y_1)(x_2 - x_1) + \ldots + P(x_{n-1}, y_{n-1})(b - x_n)
\]

which differs from that of the ordinary definite integral only in that the function \( P \) depends on \( y \) as well as on \( x \).

We then take the limit of this sum as the intervals \((x_i, x_{i+1})\) approach zero, their number increasing indefinitely.

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1. Picard, Traité d'Analyse, VI, P70-71.
Let \( y = \phi(x) \) be the equation of the curve \( \beta \), where \( \phi(x) \) is single-valued and continuous from \( a \) to \( b \). If we put \( P(x, \phi(x)) = \Phi(x) \), the sum (5.0) takes the form

\[
P'(a)(x-a) + P'(x)(x-x_1) + \cdots + P'(x_n)(x-x_{n-1}),
\]

the limit of which is the definite integral

\[
\int_a^b P(x, \phi(x)) \, dx
\]

This integral, represented by the symbol

\[
C \int P(x,y) \, dx
\]

we call the curvilinear integral of \( P(x,y) \) along the curve \( C \) from \( a \) to \( b \). Its value depends on the path from \( a \) to \( b \) as well as on the end points.

We define likewise the curvilinear integral

\[
C \int Q(x,y) \, dy
\]

as the limit of the sum

\[
Q(\alpha, \lambda)(y_1 - y) + Q(x, y_1)(y_2 - y) + \cdots + Q(x_n, y_{n-1})(y_n - y),
\]

the summation being taken in this case with respect to \( y \).
11. Green's Theorem — In equations (1) and (2), Green's Theorem in its second form was stated for two and three dimensional regions. A special case of (2), when $w = 1$ and $v$ harmonic, was deduced in Art. 3. In Art. 9, a special case of (1) was given, where $V = 1$ and $W = F$, a harmonic function.

At this example, if (1) and (2), when one function is harmonic and the other the Newtonian or logarithmic potential function, were given in Art. 9, we will now develop Green's Theorem for curvilinear integrals.

Suppose $C$ is a closed curve enclosing an area $A$, and $P(x,y)$ and $Q(x,y)$ are given point functions of $x$ and $y$. We will consider the

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Maschke,GREEN'S THEOREM ON FUNCTION THEORY,
University of Chicago 1905.
curvilinear integral

\[ \oint_C P\,dx + Q\,dy \quad (55) \]

obtained by combining the integrals (52) and (53).

Get \( P \) represent the function \( P(x,y) \) along the lower part of the curve \( C \) from \( B \) to \( A \) i.e. \( P_1 = P(x,y) \) and let \( P_2 \) represent the function \( P(x,y) \) along the upper part of the curve \( C \) from \( A \) to \( B \) i.e. \( P_2 = P(x,y) \), \( y_1 \) and \( y_2 \) being the ordinates of the lower and upper points of the curve respectively. We will integrate around \( C \) in the negative direction i.e. clockwise.

\[ \oint_C P\,dx = \oint_a^b P_2\,dx + \oint_a^b P_1\,dx \]

\[ = \oint_a^b P_2\,dx - \oint_a^b P_1\,dx \]

\[ = \oint_a^b (P_2 - P_1)\,dx \quad (56) \]

We will find the surface integral \( \iint\sigma P\,dxdy \) by first integrating along a strip parallel to the y-axis, \( y \) ranging from \( y_1 \) to \( y_2 \) and then summing these strips.

\[ \iint_a^b D_y P\,dxdy = \oint_a^b \left[ \int_{y_1}^{y_2} D_y P\,dy \right]dx \quad (57) \]
Then, if \( P \) is continuous from \( y_1 \) to \( y_2 \), i.e. continuous everywhere within \( C \),

\[
\int_{y_1}^{y_2} \frac{dP}{dy} dy = P[y_2] - P[y_1]
\]

Hence

\[
\int \int \frac{dP}{dy} dx dy = \int \int (P[x] - P[y]) dx
\]

and so, by equation (56), we have

\[
\int \int P dx = \int \int \frac{dP}{dy} dx dy
\]

Consider now \( \int \int dq dy \).

Get \( Q \), represent the function \( Q(x,y) \) along the left part of the curve \( C \) from \( D \) to \( E \), i.e. \( Q = Q(x,y) \), and let \( Q_2 \) represent the function \( Q(x,y) \) along the right part of the curve \( C \) from \( E \) to \( D \), i.e. \( Q_2 = Q(x,y) \), \( x \), and \( x_2 \) being the abscissas of the left and right points of the curve respectively. Integrating around \( C \) in the negative direction, we have

\[
\int \int Q dy = \int \int Q_1 dy + \int \int Q_2 dy
\]

\[
= \int \int (Q_1 - Q_2) dy
\]
we will find the surface integral
\[
\iint_D x \, dx \, dy = \frac{1}{a} \int \left[ \int_{x=1}^{x=2} q \, dx \right] dy
\]
by first integrating along a strip parallel to the x-axis, x varying from x, to x, and then summing these strips.

We thus have
\[
\iint_D x \, dx \, dy = \frac{1}{a} \int \left[ \int_{x=1}^{x=2} q \, dx \right] dy
\]

But, if q is not discontinuous within C,
\[
\int_{x=1}^{x=2} q \, dx = \left[ q \right]_{x=1}^{x=2} = q_2 - q_1.
\]

Hence,
\[
\iint_D q \, dx \, dy = \frac{1}{a} \int (q_2 - q_1) \, dy
\]

and so, by equation (62), we have
\[
\oint_C P \, dx + Q \, dy = -\iint_D q \, dx \, dy
\]

Combining (61) and (62) we have Green's Theorem; viz,
\[
\oint_C P \, dx + Q \, dy = \iint_D (Q_x - Q_y) \, dx \, dy
\]
in which the curvilinear integral on the left is taken in the negative direction and P and Q are point functions of x and y.
which are continuous throughout the region.

12. Condition that Integral shall be Independent of Path — The necessary and sufficient condition that \( \int Pdx + Qdy \) be independent of the path \( C \) is deduced in Picard, Tracté d'Analyse, V.1, p. 73-79, and is shown to be

\[
D_yP = D_xQ
\]

If in equation (64) we put \( D_yP = D_xQ \) we have

\[
\int_C Pdx + Qdy = 0
\]

which agrees with the above result.

We have shown in Art. 3 that if \( V \) is harmonic within \( C \) we have

\[
\int_C \Delta V \, ds = 0
\]

We will now show that the relation

\[
\int_C Pdx + Qdy = 0 \quad , \quad D_yP = D_xQ \quad , \quad (65)
\]

for curvilinear integrals, is equivalent to the relation

\[
\int_C D_xV \, ds = 0 \quad , \quad \Delta V = 0 \quad , \quad (66)
\]

for line integrals.
Since \[ D_x V = D_x D_x x + D_y V D_y y \]
we have
\[ D_x V ds = D_x V \cos(xm) ds + D_y V \cos(ym) ds \]
where \( \cos(xm) \) denotes the \( \cos \) of the angle between \( x \) and \( m \), etc., Fig. 6.

But \( \cos(xm) = \cos(yt) \)
and \( \cos(ym) = \cos(xt) \)

Then \( \cos(xm) ds = \cos(yt) ds = dy \)
and \( \cos(ym) ds = \cos(xt) ds = -dx \)

Hence \[ D_x V ds = D_x V dy - D_y V dx \]

Putting \( P = -D_y V \) and \( Q = D_x V \), this becomes \[ D_x V ds = P dx + Q dy \]

Then (65) is equivalent to (66), the condition \[ D_y P = D_x Q \]
becomes \[ -D_y Q = D_x P \]
or \[ \Delta V = 0 \]

As was stated before, equation (64), and hence condition (65) does not hold if \( P \) or \( Q \) becomes infinite within the contour \( C \).
As an example we will take

\[ P = -\frac{y}{x^2 + y^2} \quad \text{and} \quad Q = \frac{x}{x^2 + y^2} \]

Here \( P \) and \( Q \) satisfy the condition \( D_y P = D_x Q \) for

\[ D_y P = \frac{y^2 - x^2}{(x^2 + y^2)^2} = D_x Q \quad ; \quad x = 0; y = 0. \]

But \( P \) and \( Q \) both become infinite at the point \((0,0)\). For by substituting

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

we have \( P = -\frac{\sin \theta}{r} \) and \( Q = \frac{\cos \theta}{r} \)

both of which become infinite for \( r = 0, \theta = n \pi \)

In this case

\[ \int P \, dx + Q \, dy \]

taken around a circle with center at the origin, becomes

\[
\int_{C} \frac{x \, dy - y \, dx}{x^2 + y^2} = \int_{C} \frac{r^2 \, d\theta}{r^2} = \int_{0}^{2\pi} d\theta = 2\pi \quad (65)
\]
13. Definition of Proper Integral: We shall denote by an ordinary line any closed or open line which has in common with parallels to the x and y axes a finite number of points. Let $R$ be a region in the $xy$ plane bounded by ordinary lines and let $f(x,y)$ be uniquely defined for every point within this region and on the boundary. Get this region $R$ be divided into polygons by a network of ordinary lines. For convenience let the lines be chosen parallel to the $x$ and $y$ axes, in which case $R$ will be divided into rectangles. In, some of them reaching over the boundary of $R$. In each rectangle there must be points of $R$. and each

point of \( R \) must belong to some rectangle \( T_n \). The number of rectangles, \( n \), is arbitrary. Let \( g_{n} \) and \( k_{n} \) be the upper and lower limits respectively of \( f(x,y) \) over the rectangle \( T_n \). Then,

\[
\lim_{T_n \to 0} \sum_{i=1}^{m} g_{n} T_n \Rightarrow G
\]

is called the upper double integral of

the function \( f(x,y) \) over the region \( R \),

and is denoted by Jordan as

\[
\int_{R} f(x,y) \, dA
\]

\( dA \) representing the area of the arbitrarily small rectangle \( T_n \). Also

\[
\lim_{T_n \to 0} \sum_{i=1}^{m} k_{n} T_n \Rightarrow K
\]

is called the lower double integral of

the function \( f(x,y) \) over the region \( R \),

and is denoted by

\[
\int_{R}^{2} f(x,y) \, dA
\]

where \( S \) is a symbol for the summation.
extended to the area $R$ for an arbitrary mode of subdivision.

If $G = K$, then their common value is called the Proper Double Integral of the function $f(x, y)$ over the region $R$ and is denoted by

$$\int_{R} f(x, y) \, dA.$$  \hfill (70)

or, in the case where the polygons in are rectangles, $dA$ becomes $dx dy$ and the above expression may be written

$$\int_{R} f(x, y) \, dx dy.$$  \hfill (71)

If we would get the same limits $G$ and $K$ by using only those rectangles which lie wholly within $R$ or reach to the boundary of $R$. We will make use of the following theorems:

Theorem I: A real function $f(x, y)$ which at every point of a region $R$, the

boundary included, is continuous with respect to both variables \( x \) and \( y \), has a proper double integral over this region.

**Theorem II** — A function \( f(x, y) \) which is finite throughout a region \( D \) and continuous with respect to \( x \) and \( y \) at all points of \( D \), with the exception of a finite number of isolated points and points on a finite number of ordinary lines, has a proper integral over this region. The value of the integral is independent of the values of the function at these points of discontinuity, but the discontinuities must be finite.

14. **Green's Theorem for Proper Integrales**

The previous statements of this theorem have been for a single closed boundary. We may generalize the theorem by extending it to a boundary of an inner

\( \text{Get } P \) be a region bounded by an outer
rim \( r \) and \( n \) inner rims, \( r, r_2, \ldots, r_m \). The special case for one rim is obtained by putting \( m = 0 \). Suppose that these rims are traced in a positive sense by a point whose coordinates \( x, y \) are expressed as functions \( f \) of an increasing parameter \( \tau \). Denote the coordinates of this point for the outer rim by

\[ x_\tau = \phi(\tau), \quad y_\tau = \psi(\tau), \quad 0 \leq \tau \leq \beta \]

and for the inner rim by

\[ x_{\tau r} = \phi_r(\tau), \quad y_{\tau r} = \psi_r(\tau), \quad 2\pi \leq \tau \leq 3\pi, \quad r = 1, 2, \ldots \]

The intervals \( (\alpha, \beta) (2\pi, 3\pi) \) may be either finite or infinite. The functions \( \phi(\tau), \psi(\tau), \phi_r(\tau), \psi_r(\tau) \) are continuous for every value of \( \tau \) in the intervals, end points included, and have continuous differential quotients at all points, except, perhaps, a finite number. The function \( f(x,y) \) has a proper double integral by Theorem III, hence we write

\[ \int \int_{S} f(x,y) \, dx \, dy = \mathcal{J} \]  

(72)
Theorem III \[ i \quad \text{Get } F(x,y) \text{ be a continuous function at all points of the domain } \mathcal{D}, \text{ whose partial derivative with respect to } x \text{ is } D_x F, \text{ at least at all points of } \mathcal{D} \text{ where } D_x F \text{ is continuous, then,} \]

\[
\int \int_{\mathcal{D}} D_x F \, dx \, dy = \int_{\mathcal{D}} F(x,y) \, \frac{dy}{dt} \, dt - \sum_{r=1}^{m} \int F(x_{r},y_{r},t_{r}) \, \frac{dy_{r}}{dt} \, dt (73)
\]

Theorem IV \[ i \quad \text{Get } G(x,y) \text{ be a continuous function at all points of the domain } \mathcal{D}, \text{ whose partial derivative with respect to } y \text{ is } D_y G \text{ at least at all points of } \mathcal{D} \text{ where } D_y G \text{ is continuous, then} \]

\[
\int \int_{\mathcal{D}} D_y G \, dx \, dy = \int_{\mathcal{D}} G(x,y) \, \frac{dx}{dt} \, dt + \sum_{r=1}^{m} \int G(x_{r},y_{r},t_{r}) \, \frac{dx_{r}}{dt} \, dt (74)
\]

Combining equations (73) and (74), we get

Theorem V

\[
\int \int_{\mathcal{D}} (D_x F - D_y G) \, dx \, dy = \int_{\mathcal{D}} F(x,y) \, \frac{dy}{dt} \, dt + G(x,y) \, \frac{dx}{dt} \, dt
\]

\-

\[
- \sum_{r=1}^{m} \int F(x_{r},y_{r},t_{r}) \, \frac{dy_{r}}{dt} \, dt + G(x_{r},y_{r},t_{r}) \, \frac{dx_{r}}{dt} \, dt (75)
\]

\[1 \text{ For proof of Theorems III and IV, see Stilj, V.3, P.97 et seq.}\]
which is the second form of Green's theorem for proper integrals for \( m+1 \) rims. This is an extension of (64) and reduces to that form if \( m = 0 \) and we replace the \( F \) and \( G \) functions by \( Q \) and \( P \) respectively. The sign, however, will be changed, since in (75-1) the rims are traced in the positive direction. We note further that if \( D_x F = D_y G \) in (75), which is the condition that the integral be independent of the path, we will have the curvilinear integral around the \( m+1 \) rims equal to zero or the integral around the outer rim, \( r_2 \), equal to that around the \( m \) inner rims, \( r_1, r_2, \ldots, r_m \).

15. Definition of Improper Integral

Suppose that the function \( f(x, y) \) becomes infinite at one or more points of a finite domain \( RF \), limited by ordinary lines.
However, let \( f(x, y) \) be finite at least within every region \( G \) which lies wholly within \( P \), i.e., has no points in common with the boundary of \( P \). If a region should contain points in whose neighborhood the function is infinite, we may draw lines through these points and thus divide the region into smaller regions having the property of \( P \).

Under these conditions the function \( f(x, y) \) does not possess a proper integral over the region \( P \), but it will possess a proper integral over any region \( G \) having an ordinary boundary and lying wholly within \( P \). Denote it by

\[
\int_G f(x, y) \, dA
\]

If this integral has a limit always the same as \( G \) approaches \( P \) in any way, we denote this limit by
\[ J = \int f(x,y) \, dA \]  

(76)

and call it the improper double integral of \( f(x,y) \) over the region \( R \).

That is, for an arbitrarily small \( \epsilon > 0 \) a \( \delta > 0 \) exists such that

\[ |J - \int \int f(x,y) \, dA| < \epsilon \]

for

\[ 0 < \delta - \xi < \delta \]

get

\( x = \phi(\tau, \sigma) \), \( y = \psi(\tau, \sigma) \),

represent the coordinates of a point tracing the curve or curves which approach the boundary of \( R \) as the parameter \( \tau \) approaches 0. Denoting by \( G(\sigma) \) the region bounded by these variable curves, we have

\[ \lim_{\tau \to 0} G(\sigma) = R \]

We may then state the following

**Theorem** — If the function \( f(x,y) \) has, over the region \( R \), an improper
integral

\[ J = \int_B f(x,y) \, dA \]

then \( J \) is the limit of the proper integral of \( f(x,y) \) over the variable region \( B(\sigma) \) which approaches \( B \) indefinitely, i.e.

\[ J = \lim_{\sigma \to +0} \int_{B(\sigma)} f(x,y) \, dA \]  \hspace{1cm} (77)

16. **Green's Theorem for Improper Integrals**

Suppose that the following conditions hold: (1) The boundary of a region \( B \) is composed of regular lines, i.e. if

\[ x = \phi(t), \quad y = \psi(t), \quad \alpha \leq t \leq \beta \]  \hspace{1cm} (78)

are the equations of the outer rim \( r \), then the functions \( \phi(t) \) and \( \psi(t) \) are finite and have continuous derivatives with respect to \( t \) for each \( t \) in the interval \((\alpha, \beta)\) or \( r \) is divided into a finite number of parts in each of which \( \phi(t) \) and \( \psi(t) \) possess this property.
(2) The functions \( D_x F \) and \( D_y G \) satisfy the conditions of Art. 14 within a region which lies wholly within \( R \). (3) The function \( F \) and \( G \) are defined as in Theorems III and IV, Art. 14. (4) The functions \( D_x F \) and \( D_y G \) possess an improper integral over the region \( R \).

Take a system of regular curves \( r_0 \) whose equations are

\[
X = X_{r_0} = \phi_1 (r, \sigma), \quad Y = Y_{r_0} = \psi_1 (r, \sigma), \quad (2 \leq r \leq 3) \quad (79)
\]

and which approach the outer rim \( r \) as \( \sigma \) approaches \( 0 \). Also take a system of regular curves \( r_0 \), whose equations are

\[
X = X_{1, r_0} = \phi_2 (r, \sigma), \quad Y = Y_{1, r_0} = \psi_2 (r, \sigma), \quad (2 \leq r \leq 3) \quad (80)
\]

and which approach the inner rim \( r \), as \( \sigma \) approaches 0. The intervals 

\((2, 3) \quad (2, 3)\)

are finite. We may take
so small that the curves (79) and (80) do not cut, in which case they enclose a region which we will call \( G(\sigma) \). Then, in this region, the functions \( D_x F \) and \( D_y G \) are properly integrable and we have by equations (73) and (74)

\[
\int_{G(\sigma)} D_x F \, dx \, dy = \int_{x=\alpha}^{x=\beta} F(x, y, \sigma) \frac{dy}{dt} \, dt - \int_{y=\gamma}^{y=\delta} F(x, \gamma, \sigma) \frac{dx}{dt} \, dt \quad (81)
\]

and

\[
\int_{G(\sigma)} D_y G \, dx \, dy = -\int_{x=\alpha}^{x=\beta} G(x, y, \sigma) \frac{dx}{dt} \, dt + \int_{y=\gamma}^{y=\delta} G(\beta, y, \sigma) \frac{dx}{dt} \, dt \quad (82)
\]

Now, since \( D_x F \) and \( D_y G \) are improperly integrable over \( F \), then, by (77),

\[
J_1 = \lim_{\sigma \to +0} \int_{G(\sigma)} D_x F \, dx \, dy
\]

and

\[
J_2 = \lim_{\sigma \to +0} \int_{G(\sigma)} D_y G \, dx \, dy
\]

Taking the limits of both members of equations (81) and (82) as \( \sigma \) approaches 0, we get
\[ J_1 = \iint_{\mathbb{R}} D_x F \, dx \, dy = \int_\mathcal{R} F(x, y) \frac{dy}{dy} \, dx - \int_\mathcal{R} F(x, y) \frac{dy}{dy} \, dx \quad (83) \]

and

\[ J_2 = \iint_{\mathbb{R}} D_y G \, dx \, dy = -\int_\mathcal{R} G(x, y) \frac{dx}{dx} \, dx + \int_\mathcal{R} G(x, y) \frac{dx}{dx} \, dx \quad (84) \]

Combining equations (83) and (84),

we get

\[ \iint_{\mathcal{R}} (D_x F - D_y G) \, dx \, dy = \int_\mathcal{R} F(x, y) \frac{dy}{dy} \, dx + G(x, y) \frac{dx}{dx} \, dx \]

\[ -\int_\mathcal{R} F(x, y) \frac{dy}{dy} \, dx + G(x, y) \frac{dx}{dx} \, dx \quad (85) \]

which is the second form of Green's theorem for improper integrals.

In equation (75) the functions $D_x F$ and $D_y G$ may have at most only finite discontinuities at a finite number of isolated points or at points on a finite number of ordinary lines. In the latter
form of the theorem (85), the functions $D_x F$ and $D_y G$ may have infinite discontinuities at points along a finite number of ordinary lines. The curvilinear integrals in the right members of these equations (75) and (85) must, however, exist if the theorem holds. Hence the set of points of discontinuity which lies on the lines must be such that the curvilinear integrals along the lines exist.
17. Special Case of the Proceeding theorem.

The following theorem was proved by

Get $R$ be a region bounded by an ordinary closed curve $C$ and lying between
the lines $y = a$ and $y = b$. If $F(x,y)$ and
$D_xF$ be defined for all those points within
$R$ lying on the lines $y = y_k$, where the
points $y_k$ are everywhere dense in $(a,b)$,
and if, $F(x,y)$ being defined for all
points of $C$, the integrals $\int_C F(x,y) \, dy$ and
$\int_R D_xF(x,y) \, dx \, dy$ exist in Riemann's sense, then

$$\int_C F(x,y) \, dy = \int_R D_xF \, dx \, dy$$ (86)

We may make this extension: Get the
region $R$ also lie between the lines $x = c$
and $x = d$; then, if $F(x,y)$ and $D_xF$ be defined

1) i.e. The functions $F(x,y)$ and $D_xF$ are either properly or
improperly integrable.
for all those points within $R$ lying on the lines $x = x_k$, where the points $x_k$ are everywhere dense in $(c, d)$, and if, $G(x, y)$ being defined for all points of $C$, the integrals

$$
- \int_C G(x, y) \, dx = \iint_R D_{yx} \, dxdy
$$

exist in Riemann's sense, then

$$
\text{Fig. 8.}
$$

We may show this as follows. Turn the axes to the right through an angle $=90^\circ$. Denote by $x'$ the direction opposite to $x$. Then, the old $x$ axis is replaced by the $y'$ axis, and the old $y$ axis by the $x'$ axis. Now, forming an integral similar to (86) for a $G$ function we have

$$
\int_C G(y, -x') \, dx' = \iint_R D_{yx'} \, dx'dy = J
$$

Since the element of the axis is changed in sign while the element of area is always positive, i.e.
The improper integral \( J \) will not change, but the convoluting integral changes sign, we obtain (8.7) by substituting in (8.8)

\[
x' = -x  \\
dx' = -dx  \\
dx'dy = dxdy
\]

Combining equations (8.6) and (8.7), we get

\[
\int_{\mathbb{R}} (D_x F - D_y g) \, dx \, dy = \int_{\mathbb{R}} F(x,y) \, dy + g(x,y) \, dx
\]

which shows that Green's Theorem holds under the given conditions, where the functions \( D_x F \) and \( D_y g \) may become infinite at point sets lying on lines \( y = \tilde{y}_k \) and \( x = \tilde{x}_k \) respectively, where \( \tilde{y}_k \) and \( \tilde{x}_k \) represent the sets complementary to \( Y_k \) and \( X_k \) respectively. The point sets over which these functions become infinite are of content zero.
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