The Projective and Kinematic Geometry of Closed Polygons

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THE PROJECTIVE AND KINEMATIC GEOMETRY OF CLOSED POLYGONS.

I. Introduction.

Professor Emch has shown that in any closed convex curve at least one square may be inscribed* and that about such a curve may be circumscribed at least one square. He has also shown that the results may be extended to convex polygons formed by analytic arcs by making use of projective pencils of medians.

This investigation leads to methods for actually constructing the inscribed squares of all regular polygons and of any quadrilateral, and the circumscribed square of any quadrangle. The theorems concerning the quadrilateral and the quadrangle are the basis of some kinematic propositions which are illustrated by mechanisms.

II. Definitions.

I. Minkowski* gives the following definition for a continuous domain of points enclosed by a convex boundary in a plane:

(a) The domain contains with any two points the entire segment between the two points.

(b) The domain is finite.

(c) The domain is closed.

II. By inscribed square is meant a square whose vertices lie on the boundary lines of the figure.

III. By circumscribed square of a quadrangle is meant a square each of whose sides passes through one of the vertices of the quadrangle.


*Theorie der konvexen Körper, Gesammelte Abhandlungen, V.II, P154
III. Preliminary Theorems on Symmetrical Figures.

Theorem I. If a closed curve is symmetric with respect to one axis, at least one square may be inscribed in the curve in such a way that the center of the square is on the axis of symmetry and two sides are parallel to this axis.

If a continuous curve is symmetrical with respect to the line AB, Fig. 1, any pair of lines, EF and GH, equidistant from and parallel to AB, will cut the curve in four vertices of at least one rectangle, real or imaginary.

The real rectangle varies in form and size from the segment AB to the segments CD, C'D', etc., perpendicular to AB.

Hence, since the rectangle varies continuously from the line AB to the segments CD, C'D', etc., there is at least one position in which it is a square.

Theorem II. If a closed curve is symmetric with respect to two perpendicular axes, there is at least one inscribed square having its center at the center of symmetry and its sides parallel by twos to the axes of symmetry.

This is evident, since the bisectors of the angles between the axes of symmetry cut such a curve in the four vertices of a square.

Theorem III. If a curve has a center of symmetry, the median curves all pass through this center and consequently there is at least one inscribed square whose center is the center of symmetry.

From any pair of perpendicular lines through the center of symmetry will be cut out segments which are bisected by the center. If these are rotated through $90^\circ$ from a given position, the segments will have exchanged places, i.e., since the change in length is continuous, there is some position in between the ini-
tial position and the rotated position in which the segments are equal.

IV. Squares inscribed in Regular Polygons.

Pentagon. Since the method used* will apply to any regular polygon, it will suffice to apply it here to the pentagon, Fig. 2.

By Theorem I, the center of an inscribed square lies on AB and two of its sides are parallel to AB.

Draw any line DF, parallel to AB and cutting the sides BE and EG in D and F, respectively. Draw DC perpendicular to DF and cutting BK in C. Draw DL perpendicular to BE and on DL lay off DM = DC and DN = DF.

From O, the sect-point of BM and EN, draw OP perpendicular to BE. Then P is a vertex of the required square and PT and PQ are sides parallel to CD and DF, respectively.

That PT = PO is evident from the pairs of similar triangles, BDM, BPO and BCD, BTP.

Any regular polygon of 2n sides comes under Theorem II, and the vertices of a required square can be determined by drawing the bisectors of the angles between the axes of symmetry. The sect-points of these bisectors and the sides of the polygon are the required vertices.

In a regular polygon of 4n sides may be inscribed an infinite number of squares, since pairs of opposite sides are parallel and perpendicular and any pair of perpendicular lines through the center cut the polygon in four points which are equidistant.

*This method is due to Professor Emch.
from the center and therefore form a square. These four points lie on four sides of the polygon which, if produced, would form a square.

V. Squares circumscribed to a quadrangle* and their Kinematic Properties.

Given three collinear or non-collinear points, O(o,o), E(c,o), F(a,b).

To find the locus enveloped by the fourth side of the square, three of whose sides pass through these three points, Fig. 3.

Through (o,o) draw the line \( y = nx \) and through (a,b) pass the line \( ny + x = a + bn \), perpendicular to it. Through (c,o) pass a line parallel to \( y = nx \).

The side \( AB \) of our square = \( OE \cdot \sin(\tan^{-1} n) = \frac{cn}{\sqrt{1 + n^2}} \)

The fourth side of the square is a line parallel to \( AB \) and at the distance \( AB \) from it. Then its equation is \( n(y + c - b)x - a = 0 \), or

\[
\frac{y - (b + c)}{x - a} = -\frac{1}{n}.
\]

The form of the equation shows that this line always passes through the point \( (a, b \pm c) \). The join of this point and the point \( (a, b) \) is perpendicular and equal to \( OE \). The locus enveloped by the fourth side is therefore the point \( (a, b \pm c) \).

Suppose now that an arbitrary quadrangle OFED be given. Then an infinite number of squares may be circumscribed about O, F, E, in the manner indicated. The fourth sides all pass through a fixed point K. Consequently there can be an infinite number of circum-

* See Journal für die reine und angewandte Mathematik, Vol. XXXIV, p. 281, where this problem is solved, but no mention is made of the kinematic properties.
scribed squares about OFED, only when D coincides with K. Hence we have

Theorem IV. The necessary and sufficient condition that an infinite number of squares may be circumscribed about a given quadrangle is that its diagonals be perpendicular and equal.

Such a quadrangle I call an "S-quadrangle".

The construction for the square about any quadrangle follows directly from this theorem, and is as follows.

Let HEFO in Fig. 3 be any quadrangle.

Draw FK perpendicular and equal to OE.

Then KH is a side of the required square. Through F pass a line parallel to KH and through O and E draw perpendiculars to KH, and the construction is complete.

It is at once evident from the figure that the four vertices of the variable square about OFEK describe four circles with OF, FE, EK, and KO for their respective diameters and that these circles are concurrent at P, the sect-point of the diagonals. Since the median line of the square parallel to OA always passes through R, the mid-point of OE, and the other median line passes through Q, the mid-point of KF, and these two are perpendicular, the locus of their intersection (center of the square) is the circle on QR as diameter. From this it follows that the center of the square with sides parallel and perpendicular to OE and KF is at the opposite extremity of the diameter of this circle of which P is one extremity. The coordinates of the center of this circle are \( \left( \frac{c^2 + 2a}{4}, \frac{2b^2 + c}{4} \right) \).

The mid-points of OF, FE, EK, and KO, respectively, are
determined by \((a^2, b^2), (a+c, b^2), \frac{a+c}{2}, \frac{b+c}{2}, \frac{a}{2}, \frac{b+c}{2}\). Hence they form a square. Moreover, the center of this square is the point \((\frac{2a+c}{4}, \frac{2b+c}{4})\) which, as we have seen, is the center of the circle on which the center of the circum-square moves.

Kinematically, the theorem and deductions of this article may be stated thus:

Theorem V. If one side of a variable square turns about a fixed point and the extremities of this side move with constant velocities on two circles which have the fixed point for one of their sect-points, then the other sides also turn about fixed points and the two remaining vertices of the square move with constant velocities on two circles which intersect each other and the first two circles in the remaining fixed points. The centers of the four circles form a square and the four circles have one point in common. The center of the variable square moves on a circle which passes through the sect-point of the lines joining opposite fixed points and through the mid-points of these lines. Its center is the center of the square formed by the centers of the circles on which the vertices move. The velocities of the vertices and center of the variable square are proportional to the radii of the corresponding circles, i.e., the angular velocities of vertices and center are constant.

See Plate I for a Mechanism illustrating this Theorem.

VI. Inscribed Square of any Quadrilateral* and Kinematic Properties.

In Fig. 4, let \(\delta\) be the angle made by a side of the inscribed.

* See foot-note on next page.
square with the side d of the quadrilateral. Let the other parts be lettered as shown. Then by the Law of Sines we have:

\begin{align}
(1) \quad \frac{x}{\sin \alpha} &= \frac{v}{\sin(180-(\alpha+\delta))} \quad & (2) \quad \frac{d-v}{\sin(90-(\beta-\delta))} &= \frac{y}{\sin \beta} \\
\text{or,} \quad v &= \frac{x \sin(\alpha+\delta)}{\sin \alpha} \quad & \text{or,} \quad v &= d - \frac{y \cos(\beta-\delta)}{\sin \beta}
\end{align}

Equating the values of v and observing that x=y, if the inscribed figure is a square, we have after some trigonometric reduction,

\begin{align}
(3) \quad x &= \frac{d}{\cos \delta + \cot \alpha \sin \delta + \cot \beta \cos \delta + \sin \delta} \\

\text{Similarly,} \\
(4) \quad w &= \frac{y \cos \delta}{\sin \beta} = a - \frac{t \sin(\gamma+\beta-\delta)}{\sin \gamma}.
\end{align}

But y=t=x, since they are sides of a square, i.e., from (4),

\begin{align}
(5) \quad x &= \frac{a}{\cos \delta \csc \beta + \cos \delta \cot \gamma \sin \beta + \sin \delta \sin(\sin \beta - \cos \beta \cot \gamma)}.
\end{align}

Equating (3) and (5) and solving for tan\(\delta\), we get

\begin{align}
(6) \quad \tan \delta &= \frac{d(\csc \beta + \cos \beta \cot \gamma \sin \beta) - a(1+\cot \beta)}{d(\cot \gamma \cos \beta - \sin \beta) + a(1+\cot \alpha)}.
\end{align}

Since (6) is a linear equation in tan\(\delta\), there is only one solution or an infinite number of solutions according to the

*Note for page 6.


The solutions of the special problems to merely inscribe and circumscribe squares to given quadrilaterals and quadrangles are original and were made in absolute independence of the previous work. In these earlier investigations no attempt was made to apply the methods of Kinematic Geometry to these problems, as is done in this Thesis.

It should be noticed that the methods worked out for the quadrilateral will apply to any convex polygon, since the vertices of the square lie on four sides of a quadrilateral formed by four definite sides of the polygon. The difficulty for a general convex polygon lies in the determination of those four sides.
following theorem stated by A. Hurwitz (Math. Annalen, Vol. XV. p. 8.)

"If it is possible in special cases to get more than $n$ solutions for a problem which in general has $n$ solutions determined by the roots of an equation of the $n$-th degree, then in this special case there are an infinite number of solutions."

Moreover, since $v$ and $w$ are linear in $\delta$, every point of the side of a quadrilateral in which an infinite number of squares can be inscribed is the vertex of one and only one inscribed square. By differentiating the expressions for $v$ and $w$ obtained from (1) and (4) by substituting $x = y$ from (5), and dividing $\frac{dv}{d\delta}$ by $\frac{dw}{d\delta}$, we get $\frac{dv}{dw} = -\frac{\cos(\alpha+\beta)}{\sin \alpha \cos \alpha}$, i.e., $= a$ constant.

The results thus far obtained may be summed up in the following theorems:

Theorem VI. In every quadrilateral may be inscribed at least one square having a vertex on each of the four sides. If there is more than one such square having its vertices on the four sides taken in a given order*, there is an infinite number.

In the latter case, I call it an "S-quadrilateral".

Theorem VII. If three vertices of a variable square move along three fixed straight lines, the fourth vertex moves along a fixed straight line and the centers of all the squares are collinear.

For, if each of two squares has three of its vertices on the three fixed lines, the line joining the fourth vertices will form with the three fixed lines a quadrilateral in which two, and hence

*There are six possible orders but only one definite order for the convex quadrilateral. See the article by Clausen, l. c.
an infinite number of squares are inscribed. Then the fourth vertex of the variable square moves along this line so determined.

Let ABC and A'B'C', Fig. 5, be three positions of opposite vertices. Since \( \frac{dv}{dw} \) is a constant and so for the remaining vertices, if AB=BC, then A'B'=B'C'.

Let F, G, H be the mid-points of AA', BB', and CC', respectively. Then FBHB' is a parallelogram and since the diagonal BB' bisects the diagonal FH, the points F, G, and H are collinear, i.e., the centers are collinear.

Also, in this case, FG=GH.

Kinematically, these results may be expressed in

Theorem VIII. If two vertices of a variable square move with constant velocities along two fixed straight lines, the other two vertices move with constant velocities along two fixed straight lines and the center moves with constant velocity along a fixed straight line.

A Mechanism to illustrate this Theorem is shown in Plate II.

Some special cases of this theorem have been stated by E. Study, who proved them by means of the "real pictures" of imaginary points*. The theorems stated by Study are unnecessarily restricted.

By means of these theorems we can easily prove the correctness of the following construction for the square inscribed in a given quadrilateral. This is a modification of a method due to Professor

Emch and those found in the works already referred to.

Given a quadrilateral ABCD, Fig. 6.

Draw AE making angle EAD = 45°, cutting DC at E.

Draw DF making angle FDA = 45°, cutting AB at F.

The line PQ, joining the free vertices of the squares APEG and FHDQ having AE and DF for diagonals, is the locus of the fourth vertex of a variable square, three of whose vertices move along AB, AD, and DC, according to Theorem VII. Therefore R, the sect-point of PQ and the side BC, is a vertex of the required square. The other three vertices may easily be determined, since by Theorem VII, \[
\frac{PR}{RQ} = \frac{PR'}{R'D} = \frac{GR''}{R''H} = \frac{AR''}{R''F}.
\]

If the segment AT, cut off from the perpendicular AP by the side BC, were exactly equal to EG, then APEG would be the required square. If AT is shorter than EG, a square may be inscribed with two vertices on AD, by the method used for inscribing squares in triangles. The same reasoning applies to corresponding lines drawn from B, C, or D. If neither of these two special cases arises, we must apply the method given above or one similar to it.

VII. Properties of the Parabolas enveloped by the sides of Squares in Theorem VII.

If one of the vertices of the variable square describes equal intervals along a side of the quadrilateral, the other vertices also describe equal intervals on the other three sides, so that the four sides of the variable square envelope four parabolas.

We shall show (1) that these four parabolas are confocal and (2) that their directrices form a square.
(1) Parabolas are confocal.

The focus $F$ of the parabola enveloped by the side $AB$, Fig. 7, is at the sect-point of the circles through $ABR$ and $A'B'R$, since "the circle determined by the sect-points of three tangents to a parabola passes through the focus" and we have here two such circles. Moreover, "the angle under which the variable segment $AB$ is seen from the focus is constant and equal to $180^\circ - R$ and the triangles $ABF$ and $A'B'F$ are similar", i.e., $\frac{FA}{FA'} = \frac{AB}{A'B'}$. In the triangles $FAC$ and $FA'C'$, angle $FAC = \angle FA'C' (= 90^\circ - FAB)$, and $\frac{FA}{FA'} = \frac{AC}{A'C'}$ (since $AC = AB$ and $A'C' = A'B'$).

Therefore, FAC and $FA'C'$ are similar, i.e., angle $AFC$ is constant. Hence $F$ is the focus of the parabola enveloped by the variable side $AC$. Similar proofs hold for the parabolas enveloped by $CD$ and $DB$.

(2) Directrices form a square.

Perpendiculars from the focus $F$ to the tangents $QR$ and $RS$ cut these tangents in points lying on the tangent at the vertex, i.e., $A'B'$ is the vertical tangent of the parabola enveloped by the side $AB$. Since every side $AB$ of the variable inscribed square is tangent to this parabola and every segment of a tangent cut out by $QR$ and $RS$ is a side $AB$, then $A'B'$ is a side of an inscribed square.

$A'C'$ is tangent to the parabola enveloped by $AC$ at its vertex because it passes through the foot of the perpendicular from the focus $F$ of that parabola to the tangent $QR$.

In like manner, $C'D'$ and $D'B'$ are the vertical tangents to the parabolas enveloped by $CD$ and $DB$.

Hence the four tangents and, therefore, the four directrices of these four parabolas form a square.
VIII. Properties of four confocal Parabolas whose vertical Tangents form a Square.

The equations of four confocal parabolas, which we call G-parabolas, whose vertical tangents form a square, may be written in the form

\[(G_1) \quad y^2 = 4px + 4p^2. \]
\[(G_2) \quad y^2 = 4(p-c)x + 4(p-c)^2. \]
\[(G_3) \quad x^2 = 4q^2 - 4qy. \]
\[(G_4) \quad x^2 = 4(c-q)y + 4(c-q)^2. \]

where \(c\) is the side of the square formed by the vertical tangents and the common focus is at the origin \(F\), Fig. 8.

The equations of tangents to \(G_1\) and \(G_2\) having the same slope \(m\) are

\[(t_1) \quad y = mx + \frac{m^2 + 1}{m} p, \quad (t_2) \quad y = mx + \frac{m^2 + 1}{m} (p-c). \]

The equations of tangents to \(G_3\) and \(G_4\) which are perpendicular to \(t_1\) and \(t_2\) are

\[(t_3) \quad y = \frac{1}{m}x + q \frac{m^2 + 1}{m^2}, \quad (t_4) \quad y = \frac{1}{m}x + \frac{m^2 + 1}{m^2} (q-c). \]

Inspection of equations \(t_1\) and \(t_2\) shows that the distance between them is equal to the distance between the lines \(t_3\) and \(t_4\). Hence we have

Theorem IX. Any four rectangular tangents to this system of parabolas form a square.

The sect-point of \(t_1\) and \(t_3\) is the point \((\frac{q}{m} - p, \frac{p}{m} + q)\), which lies on the line \(p(p+x) = q(y-q)\), designated by \(L_1\).

The loci of the sect-points of \(t_2\) and \(t_3\), \(t_2\) and \(t_4\), \(t_1\) and \(t_4\), respectively, are

\[(L_2) \quad (p-c)(x+p-c) = q(y-q), \quad (L_3) \quad (p-c)(x+p-c) = (q-c)(y-(q-c)), \]
\[(L_4) \quad p(x+p) = (q-c)(y-(q-c)). \]

These four lines are tangent to \(G_1\), \(G_2\), \(G_3\), \(G_4\), in pairs, i.e., each of the lines \(L_1\), \(L_2\), \(L_3\), \(L_4\), is tangent to two of the parab-
13.

The coordinates of the center of the square formed by the four tangents are \( \frac{2q-(m-1)c}{2m} \), \( q-(m+1)c-2p \) and the locus of the center is therefore the line

\[
(2y-(2q-c))(2q-c)=(2p-c)(2x+2p-c).
\]

Theorem X. The vertices of the variable square formed by the rectangular tangents of Theorem IX always lie on straight lines which are tangent to the parabolas in pairs. The center of the variable square lies on a fixed straight line.

The results thus far obtained in this section, if considered in connection with the results of the preceding section, show that there is a (1,1) correspondence between points of the parabolas and points of the sides of the quadrilateral whose inscribed squares envelope these parabolas. For, in the preceding section we saw that the S-quadrilateral uniquely determined the four confocal parabolas while in this section we have shown that the four confocal parabolas uniquely determine an S-quadrilateral. To every point on a side of the S-quadrilateral corresponds a side of an inscribed square and thence a point of one of the parabolas, and conversely. We shall use this property instead of carrying out some extended analytic processes in what follows.

Any line \( y=nx \), through the focus, cuts the parabolas \( G_1 \) and \( G_2 \) in the points

\[
A' \left( \frac{2p+2p\sqrt{1+n^2}}{n^2}, \frac{2p+2p\sqrt{1+n^2}}{n} \right) \quad \text{and} \\
C' \left( \frac{2(p-c)+2(p-c)\sqrt{1+n^2}}{n^2}, \frac{2(p-c)+2(p-c)\sqrt{1+n^2}}{n} \right),
\]

respectively.
The length of the segment cut out of the line by these two points is 
\[ d_1 = \frac{2c(1+\sqrt{1+n^2})}{n^2} \].

The line \( y = \frac{-1}{n}x \), perpendicular to \( y = nx \), cuts \( C_3 \) and \( C_4 \) in the points 
\[ B' \left( \frac{2q+2g\sqrt{1+n^2}}{n}, -\frac{2q+2g\sqrt{1+n^2}}{n^2} \right) \] and 
\[ D' \left( \frac{2(q-c)+2(q-c)\sqrt{1+n^2}}{n}, -\frac{2(q-c)+2(q-c)\sqrt{1+n^2}}{n^2} \right) \], respectively, and the length of their join is 
\[ d_2 = d_1 \], above.

From this and Theorem IV follows

Theorem XI. Any two perpendicular lines through the focus of this system of parabolas cuts the two opposite pairs of this system in four points forming an S-quadrangle, i.e., a quadrangle about which may be circumscribed an infinite number of squares.

Since the coordinates of these four points in the same order satisfy the four equations \( t_1, t_2, t_3, t_4 \), simultaneously, they must be the points of contact of a set of rectangular tangents and \( n \) is a function of \( m \). \( n(1-m^2) = 2m \).

Now for a given value of \( n \), the vertices of the infinite set of squares belonging to the S-quadrangle so determined lie on four fixed circles, \( C_1, C_2, C_3, C_4 \). These four circles must be tangent to the four lines \( L_1, L_2, L_3, L_4 \), on which the vertices of the squares circumscribed to the four \( G \)-parabolas lie. For, if the circles cut the lines, there would be two squares having the same points of contact with the \( G \)-parabolas and having their vertices on the four fixed lines. This is impossible since there is a \((1,1)\) correspondence between the points of these four lines and the four parabolas.

The same reasoning applies to the locus of centers of squares corresponding to a given value of \( n \), i.e., the circle \( K \) which is the locus of the centers of squares passing through these four
points is tangent to the line $M$ on which lie the centers of all
the squares tangent to the system of parabolas. Analytically, the
proof is as follows:

The circle $K$ is determined by the focus $F$ and the mid-points
of the segments cut out of $y=nx$ and $y=-\frac{1}{n}x$ by the parabolas, i.e.,
the mid-points of $A'C'$ and $B'D'$. The equation of this circle $K$ is

$$x^2-\frac{(2p+2nq-c-cn)(1\pm\sqrt{1+n^2})}{n^2}x+y^2-\frac{(c-2q+2pn-cn)(1\pm\sqrt{1+n^2})}{n^2}y = 0.$$ 

The equation of $M$, which we have determined, may be written

$$y = \frac{(4p-2c)x+(2p-c)^2+(2q-c)^2}{4q-2c}.$$ 

The sect-point of $M$ and $K$ has an abscissa given by the value
of $x$ in the equation, determined by substituting this value of $y$
in the equation of $K$ and reducing,

$$4n^2x^2+4n[n(2p-c)-(1\pm/1+n^2)(2q-c)]x+[n(2p-c)-(1\pm/1+n^2)(2q-c)]^2=0.$$ 

Since this expression is a perfect square and $M$ is not parallel
to the $y$-axis, $M$ is tangent to the circle $K$.

Hence we have five infinite sets of circles $C_1$, $C_2$, $C_3$, $C_4$, $K$,
tangent to five fixed lines, $L_1$, $L_2$, $L_3$, $L_4$, $M$. Since the circles,
whatever the value of $n$, pass through the focus $F$ and are always
tangent to these five fixed lines, their centers describe five
parabolas, $P_1$, $P_2$, $P_3$, $P_4$, $Q$, whose vertices are at the mid-points
of the perpendiculars from the focus to the five fixed lines. We
have seen in the preceding section that the feet of four of these
perpendiculars are the vertices of the square formed by the four
vertical tangents to the fixed parabolas. By Section V, the cen-
ter of this square is at the opposite extremity of the diameter
of the circle $K$ which has one extremity at the focus. Moreover,
this center is on the line $M$ and therefore the line $M$ is perpen-
dicular to the diameter mentioned, i.e., the center of the square
formed by the vertical tangents is at the foot of the perpendicular
from the focus F to the line M. Hence the vertex of the parabola Q,
which is the mid-point of this perpendicular, is the center of the
square formed by the vertices of the four P-parabolas. This
would follow directly from the fact that the S-quadrangle in this
case is formed by the four vertices of the G-parabolas and the
vertices of the P-parabolas and the Q-parabola are the centers of
the circles belonging to this S-quadrangle.

The results of this section may be summed up as follows:
Given four confocal parabolas whose vertical tangents form
a square;

Theorem XII. Any four rectangular tangents to these four parab-
olas form a square whose vertices lie on four fixed lines and
whose center lies on a fifth fixed line.

Theorem XIII. Any two perpendicular lines through the common
focus cut the four parabolas in four points which form an S-quad-
rangle, about which an infinite number of squares may be circum-
scribed whose vertices lie on four circles tangent to the four
fixed lines of Theorem XII and whose centers lie on a fifth circle
tangent to the fifth fixed line of Theorem XII.

Theorem XIV. The centers of the four circles in Theorem XIII
lie on four parabolas confocal with the given parabolas, whose
vertices are the mid-points of the sides of the quadrilateral
formed by the vertices of the given parabolas, and therefore
form a square. The center of this square is the vertex of a parab-
ola, confocal with the others of the set, which is the locus of
the center of the fifth circle in Theorem XIII.
Hence we have nine confocal parabolas, five fixed lines, and five infinite sets of circles tangent to the five fixed lines and concurrent at the common focus of the parabolas. These seem to explain the apparent (though only apparent) dualism of the theorems relating to quadrilaterals having an infinite number of inscribed squares and quadrangles having an infinite number of circumscribed squares.*

*For mention of this apparent dualism and doubt as to its nature, see the article by Thomas Clausen, l. c.
The slotted pieces $AB$, $BD$, $DC$, $CA$, turn about the fixed points $F$, $E$, $K$, $O$, respectively. The pins $A$, $B$, $C$, $D$, are the four vertices of a variable square and move on the circles with centers $C_1$, $C_2$, $C_3$, $C_4$. The center of the variable square describes the circle with center at the point $C_5$. 
MECHANISM TO ILLUSTRATE THEOREM VIII.

The pins A, E, C, D, E move in the slotted pieces a, b, c, d, e. In every position, ABCD is a square.