A PROPOSED COURSE IN ADVANCED KINEMATICS

...BY...

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A Proposed Course
in
Advanced Kinematics.

I. Velocity Constructions for a moving plane system.

1. The velocity of interaction of moving curves.

(a) The construction for the velocity of interaction of a moving with a fixed curve.

The velocity with which the intersection of a moving curve with a fixed curve, moves along its path, can be easily determined by aid application of the velocity diagram. We will assume for this case that the curve m in Fig. 1. moves and constantly intersects the fixed curve n, that AA, is the velocity of the point of the curve m coinciding with A, the point of intersection, and that the tangents to the curves at the
intersection $A$ are $m_f$ and $n_f$. Then drawing $A_r A_r'$ parallel to $m_f$ until it intersects $n_f$ in $A_r''$, we obtain in $A A_r''$ the velocity of the intersection $A$ along the fixed curve $n$. If further we draw $A_r'' A_r'$ parallel to $AA_r$ until it intersects $m_f$ in $A_r''$, we obtain in $AA_r''$ the velocity of the intersection $A$ along the moving curve $m$.

If further we resolved the velocity $AA_r$ along the normal and tangent to the moving curve $m$ as $AA_r$, and $AA_r'$ respectively, we find that the velocity of the point of intersection along the rigid curve $n$ is independent of the tangential component, the normal component being the determining factor.

A more general case is the following: The moving curve $m$ which cuts the rigid curve $n$ in Fig. 2 in $A$, belongs to a rigid system $S_2$, which system has motion relative to a system $S_3$, and as point $B$, which lies on the normal to the curve $m$, has the
velocity \( BB_v \). Knowing the velocity of the point \( B \) we resolve it along and perpendicular to the normal \( m_n \), obtaining respectively \( BB_n \) and \( B_n B_v \). As the points \( A \) and \( B \) are points of the rigid system, \( S_2 \) and both lie on the normal to the curve \( m \), the normal components of their velocities must be equal, and as we have found in the first case that the normal component is the determining factor for the velocity of the point \( A \) on the fixed curve \( n \) we obtain this velocity by making \( A A_n \) equal to \( BB_n \) and drawing \( A_n A_r \) perpendicular to \( m_n \) until it intersects \( n_r \) in \( A_v \). \( A A_v \) is the required velocity of the point \( A \) along the rigid curve \( n \).
If we resolve the velocity of \( B \) through \( 90^\circ \) into \( BB' \), our construction is simplified in that by drawing \( B'B' \) parallel to \( mm \) until it intersects the direction of the resolved velocity of \( A \) in \( A' \), we obtain immediately in \( AA' \) the magnitude of the velocity of \( A \) on the curve \( n \). This construction follows directly from the image method of obtaining velocities, which is assumed known.

(b) Construction for the velocity of intersection of two moving curves.

This is a more general case than the foregoing, but the construction is very similar. We will again assume the curves \( m \) and \( n \), Fig. 3, to intersect in \( A \), the two belonging respectively to the systems \( s_2 \) and \( s_3 \) which are moving relative to a fixed system \( s_1 \). The velocity \( BB' \) of a point \( B \)
in the system $S_z$, on the normal to the curve $y$, and the velocity $v_A$ at a point $C$ in the system $S_y$, on the normal to the curve $n$, are given. Then to find the velocity $AA'$ of the point $A$ in its path, we know that the end of its revolved velocity must lie on a line drawn through the end of the revolved velocity of $B$ and parallel to $m$, and that it must also lie on the line drawn through the end of the revolved velocity of $C$ parallel to $n$, therefore it must lie at their intersection at $A'$. Revolving through $90^\circ$ we obtain $AA'$, the velocity of the point $A$ in its path. $AA'$ being the velocity of the point in the path will therefore be the tangent to the path.

If the points $B$ and $C$ should coincide with $A$, we would use their revolved velocities, drawing them the end of these velocities parallel to the normal of the curves to which the points belong. The intersection of the two parallels would give the end of the revolved velocity, as in the foregoing.
The actual motion of the systems \( S_2 \) and \( S_3 \) in the preceding case is however not fully determined unless the two instantaneous centers \( P_{12} \) and \( P_{13} \) are given. Here being given as in Fig. 4, we can find the velocity of the points of the systems \( S_2 \) and \( S_3 \) coincident with \( A \), relative to the system \( S_1 \). The ends of these velocities when revolved must lie at the intersections of \( P_{12} A \) and \( P_{13} A \) with \( B' A' \) and \( C' A' \) respectively, that is at \( A_{v}^{2} \) and \( A_{v}^{3} \). \( A A_{v}^{2} \) will then be the revolved velocity of the point of the system \( S_2 \) coincident with \( A \) relative to the system \( S_1 \), and \( A A_{v}^{3} \) will be the revolved velocity of the point of the system \( S_3 \) coinciding with \( A \), relative to the system \( S_1 \). Further, \( A_{v}^{2} A' \) is the revolved velocity with which the point of intersection \( A \) moves along the curve \( m \) and \( A_{v}^{3} A' \) the revolved velocity of \( A \) along the curve \( n \).
As a special case we will assume the velocity $A'v_1$ of the point $A$ on the curve $m$ to be zero, then the point $A$ may be considered a point of the rigid system $s_2$. The problem then resolves itself into the following: The point $B$ of the system $s_2$, Fig. 5, moves along the curve $\ell$ with the velocity $BB_v$, and another point of the system $s_2$ as $A$ moves on a curve $\eta$, which latter is part of a moving system $s_3$, and has on it normal $C$ whose resolved velocity $CC_v$ is given. Hence by drawing $Cv', A_v'$ and $B_v' A_v'$ parallel respectively to $CA$ and $BA$ until they intersect in $A'$, we obtain in $AA_v'$ the resolved velocity of the point $A$ in its path $p$, and in the intersection $P_2$ of $AA_v'$ and $BB_v'$ the instantaneous center of the relative motion of $s_2$ and $s_3$. Should the point $C$ coincide with $A$, and the
revolved velocity of the coincident joint be given, then the revolved velocity of \( A \) in the path \( p \) can be determined as before, and also the pole \( P_{12} \). If the velocity of \( A \) on both the curves \( m \) and \( n \) had been zero, we could consider \( A \) a point of either the system \( s_2 \) or \( s_3 \), and could therefore consider it a point of the links \( AB \) and \( AC \). The construction of \( A \)'s velocity would not be altered from the one stated. If the velocity of \( B \) and \( C \) were given the poles would be found as before by the intersection of the revolved velocities \( A \) and \( B \) and \( A \) and \( C \).

1. (c) Construction for the velocity of the intersection of two moving curves by means of the poles.

The motion of the systems \( s_2 \) and \( s_3 \), which contain respectively the curves \( m \) and \( n \), Fig. 6, is wholly determined if the poles \( P_{12}, P_{13} \) and \( P_{23} \), and the revolved velocity of \( P_{23} \) are given. Let \( P_{23}, P_{23}', \) be the revolved velocity of \( P_{23} \), and let us draw \( P_{23} BC \) any straight
line through \( P_{23} \) intersecting the normals to the curves \( m \) and \( n \) in \( B \) and \( C \). Thus we know that \( B' \) and \( C' \) revolved velocity lies on \( P_{23} \) \( B \) and \( C \).

\( C' \) revolved velocity lies on \( P_{23} \) \( C \). If then we draw \( P_{23} \) \( C' \) parallel to \( P_{23} \) \( C \) we obtain in the intersections \( B' \) and \( C' \) of the line \( P_{23} \) \( C' \) with \( P_{23} \) \( B \) and \( P_{23} \) \( C \) respectively, the revolved velocities \( BB' \) and \( CC' \) of the points \( B \) and \( C \). Then drawing \( B' A' \) and \( C' A' \) parallel to \( BA \) and \( CA \) respectively we obtain in the intersection \( A' \) the revolved velocity \( AA' \) of the point \( A \) in its path \( E \). If the line \( P_{23} \) \( C' \) passed through the point \( D \), we would have \( A' \) coinciding with \( D \), thus simplifying the construction.

We might have the special case in which the line \( P_{23} \) \( B \) in passing through \( A \), in which case the points \( A \), \( B \) and \( C \) would coincide. However we
can solve this problem in the same
manner as the one above. We will
again assume $F_{23}, F'_{23}, v$ the revolved ve-
cocity of $F_{23}$, Fig. 7. Then drawing $F_{23}, v A'_{m}, v$
parallel to $F_{23}, A$, we have in the inter-
sections $A_{m}, v$ and $A'_{n}, v$ the ends of the re-
volved velocity of the coincident joint
at A of the curves $m$ and $n$. Then
drawing $A'_{n}, v A'_{v}$
perpendicular to
the tangent to the
curve $n$ at $A$,
and $A_{m}, v A'_{v}$ per-
pendicular to the
tangent to the curve $m$ at $A$, we obtain in
their intersection $A'_{v}$, the end of the revolved
velocity of $A$ along its path in the fixed
system $s$. Revolving $A A'_{v}$, then $90^\circ$ we
obtain $A A_{v}$, the velocity of the point $A$ a-
long this path, and therefore the tangent
to the path.

As a problem we will apply
this construction to a four linked
mechanism, in order to find the
tangent to the centroids. Let us take the
quadric chain as shown in Fig. 8,
P_{23} P_{23}^v$ being the assumed revolved velocity
of $P_{23}$. Drawing $P_{23}^v P_{23}^v$ parallel to $P_{23} P$ we
obtain in $P_{23}^v$
and $P_{3}^v$ the ends
of the revolved
velocities of the
coincident joints
at $P$ due to the
rotation of
links 2 and 3.

Then drawing $P_{23}^v P_{2}^v$ and $P_{3}^v P_{3}^v$ perpendicularly
respectively to $P_{23} P$ and $P_{3} P$ we obtain in
their intersection $P_{2}^v$, the end of the revolved
velocity of $P$, and revolving this thru $90^\circ$
we obtain the actual velocity of the joint
$P$ in the system represented by the link
1, and its direction will be tangent to
the centroids, for by definition, the rate
of $P$ in the centroids.

2. (a) Boillier's Construction.

We are now prepared to derive a
construction for the tangent to the cen-
triodes known as Boillier's Constr-
The construction in Fig. 9 is the same as in Fig. 8, $P_{23} B$ being drawn parallel to $P_{23} P$ and at the intersections $A$ and $B$, the perpendiculars $AC$ and $BC$ are drawn to the lines $AP$ and $BP$, giving us $C$ the end of the revolved velocity of $P$. Rotating thru $90^\circ$ we obtain the tangent $t$. Now Robilard's construction states...
that the angle $\phi$ made by the tangent $t$, with the side $PB$ is equal to the angle $\alpha$ made by the collineation axis $PP_3$ with the opposite side $PA$. The proof referring to Fig. 9 is as follows:

Angle $\alpha = $ Angle $\beta$, being opposite interior angles.

" $\phi = $ $\theta$, sides mutually perpendicular.

" $\beta = $ $\phi$, have common intercept $BP$.

Therefore, angle $\alpha = $ angle $\theta$ and angle $\alpha = $ angle $\phi$ Q.E.D.
II. Some principles of acceleration in a moving plane system.


Having discussed to some extent the velocity of points in a moving plane system, and the construction of normals and tangents to their paths, we will now take up some of the principles of acceleration.

We know that the movement of any plane system is determined by means of the motion of any three of its points, and if further the velocities of these points are given the velocity of all other points of the system can be determined. In a similar manner the acceleration of all points of the system are dependent on the accelerations of the three points. The investigation of this principle will be the object in the following discussion.

We will investigate the acceleration of points $P$ of a rigid system, which at the time $t$ has a translation or rotation.
In case of a translation all points of the system will have the same velocity at the time \( t \), and will describe, in the next following time element \( d\,t \), geometrically equal elementary distances \( d\,s \), such that \( \mathbf{v} \left( = \frac{d\,s}{d\,t} \right) \) represents the common velocity. If in the following time element the translation of the system is continued, all points of the system will during this time element describe an equal elementary distance in the same direction or slightly deviating from it, and will have the common velocity \( \mathbf{v}' \), into which \( \mathbf{v} \) has passed. Therefore all points of the system have equal elementary accelerations, which change the velocity \( \mathbf{v} \) in magnitude and direction, and in consequence thereof have the same acceleration \( \mathbf{a} \) with its tangential and normal components \( \mathbf{a}_t = \frac{d\,\mathbf{v}}{d\,t} \) and \( \mathbf{a}_n = \frac{\mathbf{v} \cdot \mathbf{n}}{r} \). The acceleration \( \mathbf{a} \) of one point is sufficient to determine the acceleration of all points of the system possessed a translation.
not only at the time \( t \), but during a definite length \( \Delta t \) time, then the above holds for every moment of the time. The acceleration of all points are in every moment geometrically equal, but may change from moment to moment their common amount and direction.

If the system possessed a rotation at the time \( t \), then the velocity \( \omega \) of the points of the system are proportional to \( r \), their distance from the axis of rotation, and if \( w \) denote the angular velocity, the velocity of the points equal \( \omega r \). Because \( \omega \) this velocity the points of the system describe in the next time-element \( \Delta t \), distances \( dr = \omega \Delta t r \), which distances are arcs of circles whose centers lie on the axis of rotation, and whose radii are the distances of the points from the axis. If in the following time-element the rotation continues about the same axis, then the radii are the radii of cur-
nature of the paths of the points of the system, and as they are equal in both time
ments, after \( t \), we obtain

\[
P_t = \frac{d \mathbf{v}}{dt} = \mathbf{r} \frac{d \omega}{dt} = \mathbf{r} \omega', \quad P_n = \frac{\mathbf{r} \cdot \mathbf{v}}{\mathbf{r}} = \mathbf{r} \omega^2
\]
as the tangential and normal components of acceleration, where \( \omega' = \frac{d \omega}{dt} \). The total
acceleration, \( \mathbf{a} = \sqrt{\omega'^2 + \omega^2} = \sqrt{\omega^2 + \omega^2} \),.

The tangential component \( \omega' \) is the angular acceleration of the system.

If the system possesses a rotation about the same axis during a definite period, the former statement
will hold for every moment of the period. If the rotation is uniform
i.e. \( \omega \) is constant, then \( P_t = 0 \), \( P_n = \mathbf{r} \omega^2 = \mathbf{a} \).
The total acceleration is reduced to the normal acceleration and is perpendicular
to the axis of rotation. From the acceleration of one point of a rotating
system, not lying on the axis, the acceleration of all other joints can be found.
2. Acceleration of plane motion.

(a) We will pass from the acceleration of translation and rotation to that of the more general, acceleration of plane motion.

The momentary condition of velocity of a rigid plane system moving in a plane at the time \( t \) is wholly determined by the virtual center \( P \) and the magnitude and sense of the angular velocity. The velocity \( \mathbf{v} = \mathbf{r} \times \mathbf{\omega} \) at point \( A \) on concentric circles about \( P \) at a distance \( PA = r \) from \( P \) is constant in magnitude and constant direction, on the same radii of the point, but of opposite sense for points on opposite sides of \( P \). The point \( A \) describe in an interval \( \Delta t \) after \( t \), are arc \( ds = r \mathbf{\omega} \Delta t \), perpendicular to \( PA \) and in sense corresponding with \( \mathbf{\omega} \). The virtual center whose velocity momentarily is zero, changes in general its position in the system as well as in the plane in which the movement is taking...
place. The locus of the positions of the virtual center \( P \) in the plane of movement is called the moving center, while the locus of \( P \) in the fixed system is called the fixed center, and in the process of motion these centers roll one on one another without sliding, such that the point of contact for any position in the virtual center of the system for that position. The position of the system is characterized by the position of the centers.

A similar condition is true of the acceleration of a system at the time \( t \), which at this time is added to the velocity of the system to change it to the velocity for the next following period \( t + dt \), that is, to change the angular velocity \( \omega \) about \( P \) into the angular velocity \( \omega + d\omega \) about the following center \( P' \).

Let \( \omega \) and \( \omega + d\omega \) be the angular velocities of the system at the times \( t \) and \( t + dt \) respectively, such that the system rotates about \( P \) with the
angular velocity \( \omega \) during the first time element, and about \( P' \) with the angular velocity \( \omega + d\omega \) during the second time element. The angular velocity \( \omega + d\omega \) is then equivalent to the angular velocity \( \omega \) about \( P \) plus the infinitesimal angular velocity \( d\omega \) about a point \( C \) lying on the line joining \( P \) and \( P' \). By then replacing \( \omega + d\omega \) by its components, we add the original motion the resultant \( \omega \) an angular velocity about \( P \) equal to \( \omega \) during both time elements and a rotation having an angular velocity \( d\omega \) about \( C \), Fig. 10, during the second time element. Due to the rotation about \( P \) the points at unit distance have a constant angular velocity \( \omega \) during both time elements, and hence their tangential acceleration will be zero, and their total acceleration will be centripetal and equal to \( \omega^2 \) acting towards \( C \). Due to the
rotation about \( C \) the point at unit distance from \( C \) obtain an increment in velocity equal to \( dw \) in a direction at right angles to the radius upon which they lie. This infinitesimal change of velocity \( dw \) is called the instantaneous angular acceleration of the system at the time \( t \), and \( C \) is its center. The same differential \( dw \) when taken with respect to the time \( dt \), that is, \( \frac{dw}{dt} = \alpha = \omega' \), is the angular acceleration of the system, about \( C \). The angular acceleration is positive or negative, according as \( dw \) is positive or negative, that is, according as the angular velocity increases or decreases.

The center of angular acceleration lies on the common tangent to the centrodics and at a distance \( PC \), such that \( \frac{PP'}{dw} = \frac{P'C}{\omega} = \frac{PC}{\omega + dw} \), and is measured in the sense \( PP' \) when \( dw \) is positive and in the sense \( PP' \) when \( dw \) is negative.
If $\mathbf{v}$ is the velocity with which the virtual center $P$ changes, that is $v = \frac{PP'}{dt}$, then eliminating $PP'$ and substituting in the preceding equation we obtain $PC = c = \frac{\omega \mathbf{v}}{\alpha}$, where $\alpha = \frac{d\omega}{dt}$.

This development leads to the following statement: The acceleration of a moving plane system, which changes its velocity at the time $t$, can be replaced by two components, the centripetal and the angular accelerations. The first is directed towards the virtual center $P$ and at unit distance has the magnitude $\omega^2$; the second is at right angles to the line passing through the center $C$, which center lies on the common tangent to the centroids and is at a distance $c = \omega \mathbf{v}$ from $P$. Points at unit distance from both $P$ and $C$ obtain due to this acceleration components the elementary velocity changes $\omega^2 dt$ and $\alpha dt = \frac{d\omega}{dt}$, in the direction of the component.

If the virtual center is fixed that
is if \( \omega = 0 \), then \( C \) will coincide with it. If the angular velocity \( \omega \) is constant, then \( C \) passes to infinity. If \( \omega \) becomes zero at the time \( t \), without \( \omega \) becoming infinitely large, then \( P \) and \( C \) will again fall together.

From the foregoing statement regarding the division of the movement of a system into a rotation about \( P \) with the angular velocity \( \omega \) during two time-elements, and a rotation of angular velocity \( \omega \) about \( C \) during the second time-element, we obtain the following: The acceleration of a point \( M \) whose distance from the center \( C \) velocity \( P \) is \( r \) and from the center \( C \) angular acceleration \( \omega \) is \( r' \) has two components, Fig. 11, the centrifugal acceleration \( \omega^2 r \) towards \( P \) and the acceleration \( \alpha r' \), due to the angular acceleration, in a direction perpendicular to \( r' \) and in sense so as to harmonize with

![Fig. 11](image-url)
not to harmonize with $w$, accordingly as $\alpha$ is positive or negative.

Another procedure fruitful of results is the following: To the system, during the second time-element impart two infinitesimal angular velocities, $d\alpha w$, with opposite senses, thus not changing the motion of the system, and if these two combining the one having the same sense as the elementary acceleration with the angular velocity $w$, thus changing it to $w + d\alpha w$, and combining the second with the elementary angular acceleration as about $C$, thus forming a rotor couple, $d\alpha w$ and $-d\alpha w$, which is equivalent to the infinitesimal translation $PC d\alpha w = \varepsilon t d\alpha w = \varepsilon x d\alpha t = W.M. dt$. The direction of this translation is parallel to the normal to the centrodie at $P$, and towards the opposite side of the tangent, from which the angular velocity $\omega$ rotates the system; the system, therefore, rotates along the first time-element with
angular velocity \( \omega \) about \( P \), Fig. 12, during the second time element with

\[
\begin{align*}
&\omega + d\omega \\
&\frac{d\omega}{C} \\
&P \\
&\frac{d\omega}{C'} \\
&\frac{dw}{C}
\end{align*}
\]

the angular velocity \( \omega + d\omega \) line-wise about \( C \), and
goes as in this
time element also the infinitesimal translation velocity \( \omega d t \).

The acceleration of a point of the system such as \( M \) is then the resultant of the acceleration due to rotation, which can be separated into the centripetal acceleration \( \omega^2 r \) and the tangential acceleration \( r \dot{\omega} = \omega \dot{r} \), and the acceleration \( \frac{d\omega}{dt} \) which is the same for all points of the system and is caused by the rotor couple. The latter component is due only to the change of the virtual center, and disappears if the system rotates only about \( P \). We can think of it as the accelerating couple \((\alpha, -\alpha)\) having the moment \( \alpha \rho = \omega \rho 
\)

We have therefore the following:
The acceleration of a point \( M \) is
system may be represented by two
components, of which one is the
acceleration, the joint would have if
the center of velocity were stationary,
and the other is due to the velo-
city to the virtual center. The
first may be replaced by the cen-
trifugal component \( \omega^2 r \), which is
directed towards the center \( P \), and the
tangential component \( \omega r \), which is
perpendicular to the line joining the
joint \( M \) with \( P \), and makes with
the normal \( PM \) of the path of the point
the constant angle \( \lambda \), for which
\( \tan \lambda = \frac{\omega}{c} \). The second is indepen-
dent of the position of the joint \( M 

in the system, is perpendicular to
the tangent \( M \) the fixed centroid, and
is directed away from that side of
the tangent, towards which the an-
gular velocity \( \omega \) turns the system. It
is expressed by the moment \( M \omega \) \( M 

of the angular accelerating couple.

We see here from that all points
of the system have one common
acceleration component, the acceleration of the virtual center $P$. For the point $P$, the acceleration component due to the rotation about $P$ is zero, $a_r$ being zero, and the acceleration due to the angular acceleration $a_e$ about $C$ is $a_e$ perpendicular to $C$, $a_e$ being equal to $\omega^2$. 

2. (b) Points of Zero Acceleration.

It may be asked whether there are points of the system whose acceleration is zero. From the foregoing discussion we see that for this to be possible we must find a point such that, let the components $w^2r$ and $\alpha r'$, of which the first is along $r$ and the second perpendicular to $r'$ coincide, and the same components must have opposite sense; and so their magnitude must be equal. Because of the first requirement we see that the points must lie on a circle, whose diameter is the line joining the center of angular acceleration with the center of velocity. However, if we follow
up the points on this circle we find that only the points on one half will fulfill the second condition. For positive and negative values of \( x \), the points having acceleration components in opposite directions are only those lying on that side of \( PC \) towards which the notation does not apply, the other side having acceleration components \( w^2 x \) and \( \dot{w} x \), both of which are directed towards \( P \). Now the first half the acceleration is \( w^2 x - x \dot{x} \) and on the second \( w^2 x + x \dot{x} \). The \( 3^\text{rd} \) condition provided that \( w^2 x - x \dot{x} \equiv 0 \), that is, 
\[
\frac{x}{\dot{x}} = \frac{w^2}{w} = \tan \lambda.
\] 
The region which satisfies this condition is a circle whose center lies on \( PC \) and which divides the distance \( PC \) harmonically in the proportion \( a : w^2 \). This circle intersects the foregoing semicircle in the only point whose acceleration is zero. The line joining this point \( P \) and this point, makes with the normal to the fixed centroid the angle \( \lambda \). From which we obtain the for-
lowing: Here is in a moving system only one point for each time element whose acceleration is zero, whose velocity in this moment, though in magnitude and direction is constant. We will call this point the center of acceleration, and will designate it by the letter $G$.

The center of acceleration $G$ lies on the side of the common tangent to the centrodics, opposite to that towards which the rotation of the system about $P$ occurs, and on that side of the normal to the centrodics on which the center $C$ of the angular acceleration lies. The distances $r_0$ and $r_0'$, Fig. 13, of the joint $G$ from $P$ and $C$ are obtained with the help of the following equations:

$$\omega^2 r_0 - \dot{\alpha} r'_0 = 0,$$

$$r_0 + \dot{r}_0 = \dot{\alpha}.$$

From which

$$r_0 = \frac{\kappa c}{\sqrt{\dot{\alpha}^2 + \alpha^2}},$$

and

$$r'_0 = \frac{\omega^2 c}{\sqrt{\omega^2 + \alpha^2}}.$$
A line drawn through a parallel to the normal \( n \) of the fixed centrifuge divides \( PC \) in the proportion, \( r_0^2 : r_0'^2 : a^2 : w^4 \).

If the angular velocity \( w \) is constant, \( a = 0 \), \( e = \frac{w^2}{\kappa} = \infty \), and \( r_0' = 0 \), but \( r_0 = \frac{w^2}{\kappa} \). The center of acceleration then falls on the normal to the fixed centrifuge. Its position in this case is a point \( J \), Fig. 13, which will be mentioned later and is called the Inertia Pole. As \( r_0 = \frac{\kappa}{w^2} = \frac{w^4}{\kappa} \), \( \frac{\kappa}{w^2} = e \tan \kappa \), therefore \( J \) is the intersection of \( CA \) with the normal \( n \) to the fixed centrifuge.

If the virtual center \( P \) passes to infinity, then \( W = 0 \), but in general the product \( rw \) is finite and for all points \( w \) constant and equal to \( e \). In this case the system has a motion of translation, and the points \( P, C \) and \( G \) will coincide at infinity. The parallel directions of the normals to the paths of all points of the system pass through \( G \), and the accelerations of all points are geometrically equal.

In Figs. 14 and 15, \( P, C \) and \( G \) are...
The center of velocity, angular acceleration, and acceleration, respectively, are:

\[ PC = C, \quad PG = r_0, \quad CG = r' \]  their distances from one another. Any point \( P \) of the system can as \( M \) situated so that \( PM = \lambda \), \( CM = r' \) and \( GM = p \), produce two acceleration components, \( w^2 r \) along \( MP \) towards \( P \), and \( \lambda \nabla \) perpendicular to \( CM \) and moving in a plane parallel to \( PG \). We resolved the centripetal component \( w^2 r \) along \( p \) and parallel to \( PG \). From the similarity of triangles in the figures we obtain in place of \( w^2 r \) the component \( w^2 p \) towards \( G \) and \( \lambda r \) parallel to \( GP \). The latter is independent of the position of \( M \) in the system, as all points coincide in magnitude, direction, and sense, and represents the centripetal acceleration of \( G \).

The component \( \lambda r' \), due to the centri-
law acceleration \( \mathbf{a} \) in circular motion into two components. With this in view we will insert two equal and opposite rotations about \( \mathbf{G} \), equal to \( \omega \), the elementary angular acceleration about \( \mathbf{G} \), and obtain thereafter in place of \( \omega \) about \( \mathbf{C} \), the elementary angular acceleration \( \omega \) about \( \mathbf{G} \) in combination with the rotor couple \( (\omega, -\omega) \) whose moment \( \mathbf{G} \times \omega \) is an infinitesimal translation velocity perpendicular to \( \mathbf{G} \) and in a sense harmonizing with \( \omega \) about \( \mathbf{C} \). If we think of the above as divided by the time element \( dt \), then in place of the angular acceleration \( \mathbf{a} \) about \( \mathbf{C} \) we have the same angular acceleration \( \mathbf{a}^\prime \) about \( \mathbf{G} \), in combination with the acceleration \( \mathbf{a}^\prime_0 \) perpendicular to \( \mathbf{G} \). The angular acceleration about \( \mathbf{G} \), causes the joint \( \mathbf{M} \) to have the acceleration components \( \mathbf{a}_p \) perpendicular to \( \mathbf{G} \mathbf{M} \) and in a sense harmonizing with \( \mathbf{a} \), which in combination with the acceleration \( \mathbf{a}^\prime_0 \).
can replace the acceleration component \( a_r' \) due to the angular acceleration \( \alpha \).

The component \( a_{\theta}' \) is like \( w^2 r \) independent of the position of the point \( M \) in the system, and represents the acceleration of the point \( G \) due to the angular acceleration. The two last named components therefore destroy one another at every position of \( M \), as they do at the center of acceleration. Therefore we have remaining at the point \( M \) the components \( w^2 p \) and \( a_p \) and we arrive at the following statement: The acceleration \( q \) of a point \( M \) in a system at a distance \( p \) from the center of acceleration \( a \), can be represented by two components; the centrifugal acceleration \( w^2 p \) directed towards the point \( G \) and proportional to the distance from the point, and the acceleration due to the angular acceleration \( a \) perpendicular to the line \( GM \), in sense harmonizing with \( a \) and likewise proportional to the distance \( p \). From these right inited components \( w^2 p \) and \( a_p \) we obtain...
the acceleration $\mathbf{a}$ and the angle $\lambda$ which it makes with the radius to the center of acceleration, namely:

$$\mathbf{a} = \mathbf{c} \sqrt{\omega^2 + \mathbf{r}^2} \quad \tan \lambda = \frac{\mathbf{r}}{\omega^2}.$$  

We see therefore that the accelerations of all points of a system are proportional to their distances from the center of acceleration, and is constant for points on concentric circles about this point; they all make the constant angle $\lambda$ with their radii, and along any radius they have the same sense. However on the central diameter, but on opposite sides of the center of acceleration, the senses are opposed.

The existence of the centrifugal acceleration component requires that $\lambda$ the angle which $\mathbf{a}$ makes with the radius shall never be greater than 90°.

The foregoing reduction of the acceleration is applicable not only to the point $\mathbf{G}$, but to any other point $\mathbf{B}$, only for such points the components $\mathbf{c}$ and $\mathbf{r}$ do not
destroy one another. Hence, however, in
place of having two components \( \omega \cdot \mathbf{BM} \) and \( \mathbf{a} \cdot \mathbf{BM} \), the first directed toward \( \mathbf{B} \), the second perpendicular to \( \mathbf{B} \), we have two additional components \( \omega^2 \times \mathbf{BP} \) parallel to \( \mathbf{BP} \), and \( \mathbf{a} \times \mathbf{BC} \) perpendicular to \( \mathbf{BC} \).

It is perhaps interesting to notice that, the system of centripetal accelerations, such as \( \omega^2 \mathbf{r} \), may be moved from one center to another if an acceleration be added to all points of the system, equal to the product of the square of the angular velocity and the distance between centers, in direction parallel to the joining line and in a sense from the second to the first center. Likewise we can change the center of any system of angular accelerations, if and accelerations of translation equal to the moment of the couple of angular acceleration formed by the angular acceleration about the first and second centers, be added.
If in the foregoing case we had transferred the center to \( P \), then we would have had \( w^2 r \) and \( a r \), in connection with \( w^2 r \) parallel to \( PG \) and \( a r \) perpendicular to \( CG \), which when added form the concomitant \( c^2 w^2 + a^2 = c w = \mathbf{A} \), which is the acceleration of the point \( P \). We recognize herein the well-known statement that the center of acceleration of any system can be changed from \( G \) to any point \( B \) by adding to the system the acceleration of the point \( B \).

The virtual center \( P \) has no centripetal acceleration, its acceleration being only that due to the angular acceleration \( a \) about \( C \), which remains as \( a \mathbf{C} = a \mathbf{A} \) parallel to the normal to the centrodics. In the acceleration of all points on the same radius from \( G \) and \( C \) are parallel, therefore the line \( PG \) is the locus of all points whose acceleration is parallel to the normal to the centrodics.

The center of angular acceleration \( C \) possesses only centripetal acceleration, therefore, the line \( GC \) is the locus of all
points whose acceleration is parallel to the tangent to the centripetal.

Of the two above mentioned lines $PG$ and $GC$, the first makes an angle with the normal $\theta$ second with the tangent equal to $\lambda$ where $\tan \lambda = \frac{g}{v^2}$ as they intersect in $G$, and $G$ is on $P$ and $C$. Therefore they are perpendicular to one another and may be used in finding the center of acceleration.

2. (c) Points of Zero Tangential Acceleration.

For points without tangential acceleration, the acceleration component was directed towards $P$ and the acceleration perpendicular to $MC$ due to the angular acceleration about $C$ must fall on the normal $PM$ to the path of the point $M$. For this case then $PM$ and $MC$ must become perpendicular to one another. Therefore, the circle drawn on the segment between the centers $P$ and $C$ of velocity and acceleration, respectively, as a diameter in the locus of all points having zero tangential acceleration.
This circle is entirely normal, and is expressed by \( g = \omega^2 r \pm ax \), where the minus sign refers to points on that side of \( PC \) on which the center of acceleration lies, the positive sign referring to points on the other side, and the direction is positive towards \( P \).

As the tangential acceleration, \( \alpha \), points on the circle are zero, therefore their change of velocity at the time \( t \) is zero. In general, a maximum or a minimum with respect to the time.

If we designate by \( \theta \) the angle \( CMP \) which the lines joining any point of the system \( M \) with the centers \( P \) and \( C \) makes with each other, positive in the sense of the angular velocity \( \omega \), then the acceleration (tangential) is expressed by the equation \( \alpha = \omega^2 \cos \theta \) and is positive for points outside \( P \) the circle drawn on \( PC \) as a diameter, and negative for points inside the same. We therefore have the following statement: - The circle which is the locus of points of zero tangential acceleration...
separates the points whose velocities at the time \( t \) are increasing from those points whose velocities are decreasing. The first group lying outside, the second inside the circle.

The points of equal tangential acceleration, \( \alpha \), must satisfy the equation \( \dot{r} = \alpha \), \( r' \cos \theta = \frac{\alpha}{\dot{r}} \). If \( \beta \) designates the angle which the radius vector \( \overline{PM} = r \), Fig. 16, drawn from the center \( P \) to \( M \) makes with \( PC \), then the chord which it subtends in the circle of zero tangential acceleration determines that \( \beta \cos \beta = r - r' \cos \beta \). We obtain tangents from \( P \) to the circle \( M \) the equation

\[ r = \alpha \cos \theta + \frac{\alpha}{\dot{r}} \]

is the locus of such points and therefore the distance \( \overline{DM} = \frac{\alpha}{\dot{r}} \) to the end of \( PD \) (the distance intersected by any line drawn from \( P \) by the circle of zero tangential acceleration) the distance \( \overline{DM} = \frac{\alpha}{\dot{r}} \); the end of constant tangential accela-
motion is therefore the circumference of 
Cassie.

For points of this curve outside of 
the circle the tangential acceleration is 
positive and equal to \( a = a \cdot \dot{q} \cdot \mathbf{N} \), for 
points inside negative. For the double 
point \( P \) the equation does not hold, as 
its acceleration is tangential and \( a = a \mathbf{c} \).

For \( a = a \mathbf{c} \), the curve is a cardioid, 
and for \( a = 0 \), a circle and \( PC \) and 
diameter.

2. (d) - Points of Zero Normal Acceleration.

The normal acceleration \( q_n \) of the 
point \( M \) consists of the centripetal com-
ponent \( q/c \), and the component \( q' \) arising 
due to the angular acceleration \( a \) about 
\( C \). We have shown that \( q_n = \omega \cdot \mathbf{r} = \mathbf{q}' \cdot \mathbf{r} \), 
where the curve \( \mathbf{q} \) is given in a for-
mal statement, is adhered to. The 
points \( M \) whose normal acceleration 
disappear must therefore satisfy the 
equation, \( \omega \cdot \mathbf{r} = a \cdot \mathbf{r} \cdot \sin \mathbf{S} = 0 \).

If by \( \theta \) we designate the angle 
which the normal to the centripetal 
makes with \( \mathbf{r} \), Fig. 17, then \( \mathbf{S} = \frac{\mathbf{a} \cdot \mathbf{r}}{\cos \theta} \).
or \( \cos \theta = \frac{r'}{r} \sin \theta \). Therefore substituting in the above equation we obtain

\[
r = \frac{a \sin \theta}{\omega^2} \cos \theta \,'
\]

as \( a \sin \theta = \omega^2 r \).

(r) If on the normal to the centrodle we lay off the distance \( PJ = \frac{a}{\omega} \), then the point \( M \) without normal acceleration are the projections of the points \( J \) on the radius joining the points with the pole \( P \).

We have therefore the statement: the locus of all points with zero normal acceleration is a circle whose diameter is \( a / \omega \), which touches the centrodle in the virtual center, and which lies on the same side of the tangent to the curve as the center of acceleration. This circle is called the inflection circle and it contains the points of the system which are getting into the inflection points to their paths. The magnitude \( a / \omega \), which determines the
position \( Q \), the inflection point, has a geometric meaning which will be found later.

From formal equations we find that the normal acceleration is obtained from the following equation:

\[
\gamma_n = \omega^2 (r - \frac{\omega^2}{6} \cos \theta) = \omega^2 (r - \frac{\alpha}{6} \cos \theta)
\]

or \( \gamma_n = \omega^2 (r - r_n) \) where \( r_n \) is the radius vector of the inflection circle with respect to \( P \). For points within the circle, therefore, it will be seen the normal acceleration is negative and for points outside positive. The acceleration of points outside is thus one towards the virtual center \( P \) and of points inside away from it.

As the normal acceleration of a point is directed towards the center of curvature of its path, it follows that the center of curvature of each point lying on the inside of the inflection circle falls on that circle. The virtual center \( P \) on which the circle lies, and \( P \) points outside, are the opposite side of this circle.
of the inner points therefore have their convex side, and the outer points their concave side towards the virtual center \( P \). The path of points of the circle forms the transition, and these points are points of inflection, which latter property probably determined the name, the inflection circle.

For points whose normal acceleration is constant and equal to \( b \), the following equation must hold,

\[ \omega^2 r - \alpha r^2 \sin \theta = b, \]

but as \( \alpha \sin \theta = \omega \cos \theta \),

\[ + \frac{b}{\omega^2} \quad \text{or} \quad r = \frac{\omega^2}{b} \cos \theta + \frac{b}{\omega^2}. \]

The magnitude \( \frac{\omega^2}{b} \cos \theta \) is the chord, which is determined by the radius \( r \) in the inflection circle. Therefore, the locus of the points of equal normal acceleration is a Cassini's limacon with the center of velocity \( P \) as a double point, the normal to the cardioid as axis of symmetry and its inflection circle as base.

2-18 Some Relations between Velocity and Acceleration in a Moving Plane System.
In Fig. 18 the centres of velocity and acceleration \( P \) and \( G \) are given. The velocity and acceleration \( AA_v \) and \( AA_j \) respectively of the point \( A \), from which data we can find the velocity and acceleration \( J \) any other point \( J \) of the system. Let us designate by \( \theta \) the angle between the velocity and acceleration of any point, by \( \lambda \) the angle made by the acceleration with the radius vector to the centre \( J \) of acceleration, and the angle \( \theta \) \( PAG \), made by the lines joining \( A \) with the centres of velocity and acceleration with one another, by \( \phi \); then it follows that \( \lambda = 90^\circ - \theta - \phi \).
As a consequence, the angle $\theta$ is constant for all points $F$ in any system which lie on a circle $K$, passing through the pole $P$ and the center $G$ of acceleration $G$, and which has the angle $\theta$ subtended by the chord $PG$ as a peripheral angle. We therefore have the statement: the geometrical locus of all points of a plane system whose velocity and acceleration make the constant angle $\theta$ with one another, is a circle passing through the virtual center $P$ and the center $G$ of acceleration $G$.

Every angle $\theta$ determines such a circle, and the whole family of such circles forms a family whose common points are $P$ and $G$. Any two of these circles whose angles are $\theta'$ and $\theta''$ intersect one another at an angle equal to $\theta' - \theta''$.

For $\theta = 0$ or $180^\circ$ the velocities and acceleration $G$ every point of the circle determined by the same must coincide, the normal acceleration must therefore be zero, that is, the circle obtained in the intersection circle,
which was mentioned in section 2-13), and is the circle drawn with $PF$ as a diameter, Fig. 18.

For $\theta = 90^\circ$, the points on the circle have their velocity and acceleration at right angles. For this case however, the virtual center $P$ cannot be considered as point of the system as its velocity is zero, while its acceleration is $PR$ in the direction of motion of $P$ for the following time-element. This circle contains all points whose tangential acceleration for the instant considered is zero and is the same as was found in section 2-6. All points of this circle move in equal elementary distances in any two consecutive time-elements. It intersects the reflection circle at right angles. The acceleration of all points of this circle must then the virtual center $P$ as can be seen from the figure.
III. A discussion of Curvature and the Determination of its Center in a Moving Plane System.

(1) - We have found in Section II, article 2-(a), the (the) motion of a system is dependent to some extent upon the velocity $v$ of the virtual center $P$. In this section, we shall deduce some of the relations existing between this velocity, the angular velocity about the center $P$, and the curvature of the curvatures at $P$.

We know that $u = \frac{dx}{d\theta}$, where $d\theta$ is the element of arc $\theta$ about the center $P$ passed over in the time $d\theta$. If now by $dE$ and $dE'$ we designate the contiguous angles of the curvatures, fixed and moving respectively, and we next assume that the osculatory circles of both curvatures, which may be substituted for the infinitesimal movement of the curves themselves, as they have two consecutive elements that are in common, let us write the same side of the common tangent, Fig. 19, then $dE - dE'$ represents the element
amplitude $d\omega$ of the rotation about the virtual center, when the positive sense of rotation is taken as a rotation toward that side of the common tangent on which the oscillatory circles lie.

As $\omega = \frac{dx}{dt}$, therefore from the former equation in $x$ we obtain $\frac{d\omega}{\omega} = \frac{dx}{x}$. This can be called the relative curvature of the centrodal angle of the centrodal and the quotient of the relative curvature of the centrodal.

We have therefore the statement: the quotient of the angular velocity about the virtual center, and the velocity of the latter is equal to the relative curvature of the centrodal. The reciprocal relation $\frac{d\omega}{\omega}$, also known as the diameter of the inflection circle, may be called the ratio of the radius of the relative curvature, then following out the analogy between the curvature and relative curvature. From the above developed
dependence of $\omega$ and $\omega$ we see that both are either constant or variable, according as the relative curvature is constant or variable, and that the one follows from the other, as soon as the centrodes and a pair of homologous points of the same, besides the sense in which the centrodes roll one upon another, or the sense of the change of the virtual center, are given. The relative curvature can be positive, negative, or zero, the several signs demanding a positive or negative sense of the angular velocity $\omega$; the change from zero demanding a momentary disappearance and reversal of the sense of the latter, and now the radii of curvature $\rho$ the fixed and moving centrodes are respectively $\rho$ and $\rho'$; then the curvature $\frac{1}{\rho}$ of these centrodes are

$$\frac{1}{\rho} = \frac{d \xi}{d \xi} \quad \text{and} \quad \frac{1}{\rho'} = \frac{d \xi'}{d \xi'},$$

and consequently we have the relative curvature which is the difference between these curvatures, namely $\frac{d \xi}{d \xi} = \frac{1}{\rho} - \frac{1}{\rho'}$. 
positive, zero, or negative, accordingly as the radius of curvature of the moving centrodle is greater, equal to, or less than the radius of curvature of the fixed centrodle. As long as the center of curvature of the moving centrodle lies outside of the center of curvature of the fixed centrodle, the rotation will have a positive angular velocity, as soon as it lies within, a negative angular velocity, and the passing of the two marks the change in sense.

If the osculatory circle of the moving centrodle falls on the opposite side of the common tangent to the centrodles, then $d\theta + d\theta'$, Fig. 20, represents the elementary amplitude $d\theta$ and we need only to change the sign of $d\theta'$ and therefore of $P'$ to have the above formula apply. In this case however
The center of curvature of the moving centrodode is always outside of that of the fixed centrodode and therefore we can not change the sign as before. Even should $p'$ pass with a change of sign through infinity, we would nevertheless not change its sign.

The combination of the development formulae forms the equation,
\[ \frac{1}{w} = \frac{1}{p} - \frac{1}{p'} \]
which always applies if $p'$ is looked upon as having the same or opposite sign to $p$, according as the center of curvature of the centrodode falls on the same or opposite sides of the common tangent.

We found in Section II, Art. 17, that $\frac{1}{w}$ is the diameter of the inflection circle. Therefore the statement:

The diameter $\frac{1}{w}$ of the inflection circle is the reciprocal of the sum of the curvatures of the fixed and moving centrododes.

The inflection circle has the property that the radius of curvature
of the paths & its points are infinitely large. This can be shown geometrically as follows: Through the virtual center P draw any straight line PM, Fig. 21, and then the following virtual center P', the end of the common element P' are PP' of the centrodies, a parallel P'M'. Another line through P making with the first, the elementary angle d\(\theta\), cuts the parallel P'M' in M', which point is the new position of the point M, at a distance PM = PM', due to the rotation of the system. Then MP is the normal to the path P M for its first position, and M'P' for its second position. The intersection of these normals is the center of curvature of the path P M, and as they are parallel, must lie at infinity. From the foregoing, therefore, we see that out every straight line through P, that is a point M, the radius
of curvature if whose path is infinite has the angle \( d\varphi \) is the same for all lines, the points \( M \) must lie on a circle about \( PP' \), which has the angle \( PMP' = d\varphi \) as tangential angle. The diameter of this circle is therefore, \( \frac{d\varphi}{dt} = \frac{d\varphi}{c} \). As the limit this circle is the locus of the points \( M \). The points such as \( M' \) lie only on that side of \( PP' \) from which the rotation \( d\varphi \) proceeds. The locus in question is therefore the inflection circle. If we invert the motion such that the fixed centrod becomes the moving and the moving becomes the fixed, then the inflection circle will fall on the opposite side of the common tangent to the centrod; it has received the same diameter, \( \frac{d\varphi}{c} \).

In Fig. 22 let \( MM' \) be the element of the path of any point \( M, J, J' \), the element of the path of the point \( J \), in which the straight line \( PM \) cuts the inflection circle, such that the angle \( PJ', P' = d\varphi \). If now \( M \) lies outside
of the incircle circle, but one that
side of the tangent to the fixed cur-
trode on which this circle lies, thus
the angle $PM'P'$,
which we designate
by $d\alpha$, is smaller
than $d\beta$, as $d\beta$
in the geometrical
angle of this cir-

icle. Hence the
normals, $MP$ and
$M'P'$. If the path $i$
$M$ intersects on the
side $J$ of $P$ opposite
$M$ in some point
$K$, and makes with one another the
angle $d\tau$, for which $d\tau = d\nu - d\mu$.
For points within the circle, $d\theta$ is
greater than $d\tau$, and the intersection
$K$ of the normals lies on that side
of the circle for which $d\tau = d\nu - d\mu$.
If however $M$ lies on the opposite
side of the tangent to the fixed cur-
trode from which the circle lies,
then the intersection $J$ of the normals will
between $P$ and $M$ and $d\gamma = d\nu + d\mu$.

We recognize therefore that the inflection circle must lie on that side of the tangent for which $d\gamma = \pm (d\nu - d\mu)$, because for these points only can $d\gamma$ become zero and the centre of curvature pass to infinity.

To determine the radius of curvature $MK$, Fig. 22, let $i$ be the angle which $PM$ makes with the normal to the entrode and let $PQ$ be the infinitesimal arc, drawn with $KP$ as a radius. Then we have for all cases,

$$PK: \frac{d\gamma}{d\nu} = PP'/\cos i, \quad PM: \frac{d\nu}{d\mu} = PP'/\cos i, \quad \frac{d\gamma}{d\nu} = \frac{d\nu}{d\mu}$$

where $PJ$ is the diameter of the inflection circle. If we substitute the values of $d\gamma$, $d\nu$ and $d\mu$, from the above, in the equations,

$$d\gamma = d\nu - d\mu, \quad d\gamma = d\mu + d\nu \quad \text{and} \quad d\gamma = d\nu + d\mu$$

which correspond to the various positions of $M$ and for $PM'$, but $PM$, then for these cases we have respectively:

$$\frac{d\gamma}{d\nu} = \left(\frac{1}{PM} + \frac{1}{PK}\right) \cos i, \quad \frac{d\gamma}{d\mu} = \left(\frac{1}{PM} - \frac{1}{PK}\right) \cos i$$

and $\frac{d\gamma}{d\nu} = \left(\frac{1}{PK} - \frac{1}{PM}\right) \cos i$. We may
However by replacing $PM$ by $z$ and $PK$ by $z'$ combining these three equations into one. In this case we must take account of the signs of $x$ and $x'$. Thus $z$ is positive for points on the same side of the tangent to the electrodes on which the inflection circle lies and negative for points on the opposite side. While $z'$ is negative when $X$ lies on the same side of the tangent as the inflection circle and positive when on the other side. Our equation will then be, $\frac{w}{w} = \frac{1}{z} + \frac{1}{z'} \cos i$, in which $PJ$ has been replaced by $z$.

As long as $i$ has the same value, $PJ$ will also. Therefore the sum of the reciprocals of the distances of the point $M$ and the center of curvature $P$ from the virtual center $P_0$ for points on the same straight line through $P$, is constant.

We see from Fig. 22, that $PJ$, is the projection of the diameter of the inflection circle on the line $PM$ and is equal to $PJ \cos i$. Thus it follows
that the constant equation of above is the reciprocal of the projection of the diameter (\( \omega \)) of the inscribed circle on the straight line \( PM \), or \( P_{ij} = \frac{r}{2} + \frac{1}{r} \).

If we designate by \( P \) the radius of curvature of the path \( PM \) and note that in the above three cases,

\[ r + r' = P, \quad r - P = r, \quad r' = r - P \quad \text{or} \]

\[ PJ_1 + MJ_1 = r, \quad PJ_1 = r + MJ_1, \quad PJ_1 = MJ_1 - r; \]

then by the elimination of \( PJ_1 \) it follows that \( r^2 = MJ_1 \cdot P \) or \( r^2 = \frac{\omega}{\omega - P \cdot \cos \beta} \).

We have therefore that the radius of curvature of the path \( PM \) is the third proportional to its distance from the virtual center \( P \), and the distance of the projection \( J_1 \) of the inscribed circle \( J_1 \) on the normal \( MP \) to the path.

If we find the point \( P \) on the normal \( PM \), which lies symmetrical to \( P \) with respect to \( M \), we find that the four points \( P, P, J_1, \) and \( K \) are harmonic, the center of curvature \( K \) being conjugate to the projection \( J_1 \) of the inscribed circle. We recognize this from the above equation.
in connection with the above consideration about the position of \( K \), from which it is found that the points \( P \) and \( P' \) are always separated by the points \( J \) and \( K \).

If on the common normal to the centrodes we take any point \( A \), Fig. 23, for whose path we wish to find the center of curvature, then the four points, \( P, P_0, J \) and \( K \) are harmonic, \( P_0 \) being a point lying symmetrical to \( P \) with respect to the point \( A \). If from \( P \) we draw any straight line and project the given points, \( A, P, P_0, J \) and \( K \) on it, then by the projection, \( A', P, P', J' \) and \( K' \) are harmonic, and \( A' \) lies in the middle of the segment \( PP' \). Therefore \( K' \) is the
center of curvature of the path of the joint A'.

The center of curvature of the path of any point of a system is the projection of the center of curvature of the point on the common normal to the central, whose projection is the point of the system. Let A' be the joint A' all lie on a circle about PA, and the joint K' all lie on a circle about PK as a diameter. The center of curvature of all points of a circle tangent to the fixed central at the virtual center P, lie on a second circle also tangent to the centrodle at P. The position and magnitude of the second circle is determined in that it contains the center of curvature K of the point A' of the first circle, or in diametrically opposite to the virtual center.

And the foregoing statements we can have a very simple construction for the center of curvature of the path of points of a sys-
tem. To a given one that two given points, A and B, of the virtual center, P and the insufficing pole, J, Fig. 24, be known. If to PJ we add JJ' equal to P, and in the same sense, and con-
nnect the system, joint M, the center of curvature J with the joint J', and then J' draw a parallel to J, then we obtain on the line PM the point P', which with reference to the joint M lies asymmetrical to P. If from the insufficing pole J we draw the straight line J, which projects J on PM at right angles, then the interaction of J, J, and J' P, is a joint G. Now as J in midway between P and J', therefore P, J, and P, are one pair and J and the joint at infinity are PJ another pair. By four harmonic points, and therefore the four
straight lines $GP, GP, GJ$ and the straight line $GL$ parallel to $PJ$ are harmonic, $GP, GP$ being one pair and $GJ, GL$ the other pair of conjugate lines. Therefore, it further follows that $P, P, J,$ and the intersection, $K$ of $GL$ and $PM$ are harmonic, and therefore, according to a preceding discussion, $K$ is the center of curvature for the path of the joint $M$. As the triangles $PLG$ and $JS'G$ are similar and two sides are parallel, therefore the third sides $PL$ and $JS$, are parallel and we have the following statement: To find the center of curvature of the path of any joint, when the virtual center $P$ and the inflection pole $J$ are known, connect the joint $M$ with $J$ by a straight line and at the virtual center $P$, erect a perpendicular, $PL$, to the normal $PM$ of the joint $M$. This perpendicular intersects the line $MJ$ in $L$; drawing $LK$ parallel to $PM$
PJ until it intersects PM in K, and we obtain in K, the desired center of curvature of the path of the point M. Fig. 25.
IV. Applications of the principles derived, to higher pairing.

(a). Velocity and acceleration in rotating cam trains.

In Fig. 26 is shown a simple three linked cam train, with lower pairs at $A$ and $B$ and an upper pair at $E$, represented by the contact of the curves $m$ and $p$. We will consider the link $AB$ fixed, and impart an angular velocity to the system represented by the curve $m$. It will because of the constraint due to higher pairing, impart to the system represented by the curve $p$, an angular velocity about the center of rotation $B$. We will now see if, by a legitimate substitution, for the above mechanism, we can obtain the displacement, velocity and
acceleration of all or any of its points.

As acceleration is the rate of change of velocity, we must compare velocities during two successive infinitesimal intervals to ascertain the change of velocity. The consideration of these two successive intervals of a body's motion means the contemplation of three consecutive positions, any contrivance, therefore, which will cause the body to assume these three positions is the kinematic equivalent of the actual motion, whether it be displacement, velocity, or acceleration, that we are considering.

Let $c$ and $o$ be the centers of curvature of the curves $m$ and $p$ respectively, at the point of contact $E$. Since from the mathematical definition of curvature, we know that each curve will have two consecutive elements in common with its corresponding osculatory circle, and as the systems represented by the curves $m$ and $p$ pass to the following two consecutive positions, this necessi-
tating the consideration of two consecutive elements of the curve \( y \) and \( p \), we therefore can replace the curve by their corresponding osculatory circles, whose centers are at the centers of curvature \( C \) and \( D \), and we may without changing the kinematic properties, for the instant considered, replace the mechanism of Fig. 26 by the quadric chain \( ABCD \).

Having thus reduced the original mechanism to a quadric chain, we can proceed to find the velocities. Knowing the angular velocity \( \omega \) of the system represented by \( m \), call it \( \omega_0 \), we can find the velocity \( CC_0 \) of the point \( C \) in that system, it being equal to \( AC \times \omega_0 \). Revolving \( CC_0 \) through \( 90^\circ \) we obtain \( CC_1 \), the revolved velocity of \( C \). Then drawing \( CD \parallel \) parallel to \( CD \) until it intersects the line \( BD \), we obtain in \( DD_1 \), the revolved velocity of \( D \), and revolving \( DD_1 \) back from \( 90^\circ \), we obtain \( DD_2 \), the velocity of the point \( D \). We can then obtain the angular velocity of the system represented by the curve \( p \), by dividing
the velocity DD, Z D by the radius BD. Having obtained the velocity of a point in
both systems, we can pass to the determination of the acceleration.

In the study of the acceleration of the systems represented by the curves m
and p we might assume the angular velocity of the system m about A as constant,
but to make the case general we will take it variable. Thus in

Fig. 27 let \( \omega \) be the angular velocity of

![Diagram](image-url)
The system \( m \), then the acceleration \( \gamma_0 \) of the point \( c \) is made up of the components, \( AC \times \omega^2 \) along \( CA \), and \( AC \times \gamma \) perpendicular to \( CA \). We will assume that \( \gamma \) known, and find the acceleration of the point \( D \) in the system, represented by the curve \( p \). Having the acceleration of one point of the system \( m \), we can obtain the acceleration of any other point of the system, such as \( P \), the instantaneous center of relative motion of \( BD \) and \( AC \), by constructing the triangle \( APP'_m \) similar to \( ACC'_0 \), \( PP'_m \) being the required acceleration of \( P \).

Since, according to the definition of instantaneous center of relative motion, the point \( P \) of the system \( p \) rests for an instant in coincidence with the point \( P \) of the system \( m \), its velocity relative to the latter system will for the instant considered be zero.

In section II article 2-(a) we found that the acceleration of any point of a system could be resolved into two components. One due to the angular
velocity about the instantaneous center, the other due to the angular acceleration, \( \alpha \), about a point \( C \), on the tangent to the centrodix. If we apply this to the case in hand we find that \( P \) will have zero acceleration due to the rotation about itself, but due to the angular acceleration about the point \( C \) on the tangent to the centrodix, it will have an acceleration along the normal, \( n \), to the centrodix at \( P \), which normal is found by the use of Biot-Savart's construction. The angle \( \phi \) is made equal to \( \theta \), which gives the tangent \( t \), with which center normal makes an angle \( 90^\circ \).

The point \( P \) of the centrodix attached to the system \( p \) is moving along the centrodix attached to the system \( m \), which centrodix is revolving about the center \( A \). We must therefore apply Coriolis' law to obtain the resultant total acceleration of the point \( P \) of the system \( p \). The velocity of \( P \) in the centrodix attached to the system \( m \) is
zeros, then one component a.w is zero, and the acceleration \( J_P \) in the system \( P \) will be the resultant of the acceleration \( PP_{f}^{m} \) of the point \( P \) in the system \( m \), and the relative acceleration along the normal to the centrodus. The point \( P \) has the velocity \( PM \), whether considered as point of the system \( m \) or \( P \). Therefore the normal accelerations of the point in the two systems are respectively:

\[
P_g = \frac{PM}{AP} \quad \text{and} \quad Ph = \frac{PM}{BP}
\]

therefore \( P_g : Ph = BP : AP \).

From the latter equation we can easily derive the following construction for the acceleration component \( Ph \).

Join \( B \) and \( P_{f}^{m} \) and then thru \( A \) draw \( AH \) parallel to \( BP_{f}^{m} \) until it intersects \( PP_{f}^{m} \) in \( H \), then thru \( H \) draw \( AK \) perpendicular to \( BP \), intersecting it in \( K \); then \( Ph \) will be the required normal acceleration \( J_P \) in the system \( P \). Thus having the normal component \( Ph \) of the acceleration \( J_P \), we know that the end
of the total acceleration must lie on \( hH \), the perpendicular, at the end of the normal component. If therefore we draw \( F_2^p \) parallel to the relative acceleration, which lies on the normal \( n \), until it intersects the perpendicular \( hH \) produced in \( F_2^p \), we obtain in \( PP^p \) the total acceleration of \( P \) considered as point of the system \( p \). Drawing \( F_2^p \parallel PP^p \) until it intersects the normal \( n \), we obtain in \( PP^p \), the relative acceleration component of \( P \). Having obtained the acceleration of one point of the rotation system \( p \), we can obtain the acceleration of any other point of the system, such as \( D \), by constructing the triangle \( BDD \); similar to \( BPP^p \), \( DD \), \( DD \) being the required acceleration of the point \( D \). We have thus shown that by means of a kinematical equivalent of the original mechanism, we can obtain the velocity and acceleration of all points, \( B \) as train of rotating curves. Had we turned
the angular velocity \( \omega \) constant, the construction would have been simplified in that we could have obtained immediately the acceleration of the point \( P \), it being equal to \( AP \cdot \omega^2 \) along \( PA \).

(b) - Velocity and acceleration in a mechanism, having a rotating cam and sliding follower.

In the mechanism shown in Fig. 28, we have a sliding \( B \) on the link \( B \) in the guides \( G \), the rod \( f \) which is fastened to the slide \( B \), being set in motion by the contact with the curve \( m \), which
latter rotates about the center $A$. We will again assume the curve $m$ to have an angular velocity $w$. To find the velocity $v$ of the slide $B$, it will not be necessary to obtain a kinematic equivalent of the mechanism as in the foregoing case, for we can proceed directly as follows: The velocity $v$ of the point of contact $C$ of the system $m$ will be $AC \times w$ at right angles to the line $AC$, let it be represented to some scale by $CC^\prime$. This velocity may be resolved into components $CC^\prime$ and $C'C^\prime$, parallel and perpendicular respectively to the direction of motion $v$ of the slide $B$. The component parallel to the motion of the slide, will then be the velocity of the slide, for if it were not the curve and rod would either separate or cut into one another, which is impossible under the conditions of the problem. The component perpendicular to the motion of the slide is the velocity of sliding of the curve $m$ on the rod $f$. 
Before passing to the determination of the acceleration we will find the instantaneous centers of relative motion, as they will be used in the determination of the acceleration. We may consider the mechanism a quadric chain as was done in the foregoing case, the center of curvature of \( f \) being at infinity horizontally and the center of curvature of the guide \( G \) being at infinity vertically. The instantaneous centers will then be at \( P \) the intersection of a horizontal line \( F \), the center of curvature of the curve at \( C \), and the vertical through \( A \), and at \( P' \), the intersection of the line \( A F \) produced with the vertical drawn through the center of curvature \( G \) the rod \( f \), which center is at infinity. The configuration axis in this case will be the line drawn thru \( P \) parallel to \( A F \). The tangent to the centroids at \( P \) being obtained as before by Bosillini's Construction, making the angle \( \theta \) equal to \( \phi \). Having found the instantaneous centers and the common tangent to the centroids, we can proceed to find the acceleration.
We will again take the angular velocity \( \omega \) as variable; then the acceleration of any point of the system \( m \) such as \( F \) will have the acceleration \( F_{F}^{m} \) as a resultant of the components, \( AF \times \omega^{2} \) along \( AF \), and \( AF \times \omega \) at right angles to \( AF \). We will again assume this known. Passing to the instantaneous center of relative motion, \( P \) we obtain its acceleration \( P_{F}^{m} \), considered at point \( F \) the acceleration \( m \) by constructing the triangle \( APP_{F}^{m} \) similar to \( AFF_{F}^{m} \). We must now apply Coriolis' law, for we have two centroids one moving upon the other and both rotating about fixed centers. The component \( 2 \omega \) will again drop out. The relative velocity of the centroids at the instantaneous center \( P \) being zero, according to the definition of the same. The acceleration \( 2P \) will therefore be the resultant of the relative acceleration which is normal to the centroids, and the acceleration \( 2P \) due to its rotation about \( A \), which latter is known. As the
direction of P's acceleration must be parallel to the line of motion of the slide, we have the following con-
struction: Draw the normal N, Fig. 1, to the centrodies at P, then thru P', the end of the acceleration by P in the system m draw P''P' parallel to the normal N until it intersects PP'' the line of section of P's acceleration in the system B, in the point P', PP'' will be the required acceleration of P and therefore also of the slide B. Drawing P''P' parallel to PP'' un-
til it intersects the normal to the centrodies in P', we obtain PP'' the relative acceleration component. Had the angular velocity been constant the prob-
lem of finding the acceleration would have been simplified in that we could have passed immediately to the point P, its acceleration being then AP\times w² along AP. The radius AP could represent to some scale for the acceleration, then drawing thru A a line parallel to the motion of the slide B, we would obtain the acceleration of
$y_P$ is the distance from $P$ to the intersection, to the same scale that $AP$ represents the acceleration of $P$ in the system $m$. This would also be the acceleration of the slide.

**Bibliography.**


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