Some Applications of Analytic Functions

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CLYDE WILBUR ÉMMONS

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CHAPTER I.

INTRODUCTION.

Sec. 1. Goursat's Theorem. It was shown by Goursat that the existence of a derivative $f'(z)$ at every point in a region $T$ is a sufficient condition for a function $f(z)$ to be analytic in the Cauchy-Riemann sense. His theorem and essentially his method of proof follow.

Theorem. Let $f(z)$ be defined and single valued for the closed region $T$ and let $f'(z)$ exist for all points in the same region. Then to a number $\varepsilon$, arbitrarily small, there corresponds a division of $T$ into sub-regions $\mathcal{C}$ in each of which there is a point $z_i$ such that, when $z_j$ and $z$ are any other points of the same closed sub-region $\mathcal{C}$, the following relation holds:

$$\left| \frac{f(z_i) - f(z_j)}{z_i - z} - f'(z_i) \right| < \varepsilon \quad \text{............................. (1)}.$$

Proof. Let us suppose that after the process of division has reached a certain stage relation (1) does not hold for division $\mathcal{C}'$, of $T$. Then let $\mathcal{C}'$ be divided into squares. Again it may happen that relation (1) does not hold for $\mathcal{C}''$, one of the sub-squares of $\mathcal{C}'$. Let $\mathcal{C}''$ be subdivided and the process continued.


Two cases may arise:

(1) After a finite number of divisions, relation (I) may be found to hold for every sub-square.

(2) After no finite number of divisions will the relation (I) be found to hold for every sub-square.

Case (1) presents nothing discordant with the theorem and so we pass at once to the consideration of the second case. Let it be understood that the subdivisions are so preformed that, at every stage, the square, under consideration in reference to relation (I), contains all of the set of squares that follows. This particular set of squares then forms a regular sequence and defines in its limit some point, say $z_i$, of the region $T$.

Case (2) says that in every one of these exceptional squares $\mathcal{C}', \mathcal{C}''$, $\mathcal{C}'''$, $\ldots$, $\mathcal{C}^{(n)}$ there can be found at least two points $z_\alpha$ and $z$ such that

$$\left| \frac{f(z_\alpha) - f(z)}{z_\alpha - z} - f'(z) \right| \geq \epsilon.$$

But as the process of subdivision proceeds it is necessary that these two points, $z_\alpha$ and $z$, shall approach closer and closer to $Z_i$. From the continuity of $f(z)$, because of its possession of a derived function $f'(z)$, it follows that

$$\lim_{z_\alpha \to z} \left| \frac{f(z_\alpha) - f(z)}{z_\alpha - z} - f'(z) \right| = \left| \frac{f(z) - f(z_i)}{z_i - z} - f'(z_i) \right|$$

and

$$\lim_{z \to z_i} \left| \frac{f(z) - f(z_i)}{z_i - z} - f'(z_i) \right| = \left| f(z_i) - f'(z_i) \right| \neq 0.$$
which last result follows from (2). But this contradicts the hypothesis that there should be at every point in the region $T$ a definite derivative $f'(z)$. Consequently case (2) cannot present itself and the theorem is established. Relation (I) is precisely the condition for the continuity of $f'(z)$, so that the theorem leads to the very important conclusion that the continuity of $f'(z)$ need not be stipulated in the definition of an analytic function, since it follows incidentally.

Sec. 2. Cauchy's Integral Formula. It is a well known fact that $\int f(z)dz = 0$, where the integration is performed in the postive direction along the whole boundary of the finite region in which $f(z)$ is analytic. For example, let us suppose that $f(z)$ is analytic in the region $S$ between the two curves ABC and DEF. The complete boundary of $S$ consists then of the two curves ABC and DEF. We have then

$$\int_{DEF} f(z)\,dz = \int_{ABC} f(z)\,dz = 0$$

or

$$\int_{DEF} f(z)\,dz = \int_{ABC} f(z)\,dz.$$ 

that is, we may deform the path of integration in any manner so long as we do not, in the process of deformation, pass over a singular point.

Theorem: Let $f(z)$ denote a function which is analytic in the region $T$, and let $\alpha$ denote any point within the region $T$, such that $f(\alpha) \neq 0$, then

$$f(\alpha) = \frac{1}{2\pi i} \int \frac{f(z)\,dz}{z - \alpha}.$$
the integral being taken positively around the whole boundary of the region.

Proof. Let \( f(z) \) be a function which is analytic in the region \( T \). Let \( \alpha \) be any point within the region such that \( f(\alpha) \neq 0 \). Consider the function \( \frac{f(z)}{z-\alpha} \) for \( z \neq \alpha \). If \( f(z) \) is analytic for the whole region enclosed by \( T \), then \( \frac{f(z)}{z-\alpha} \) is analytic for all points except \( \alpha \).

Construct the circle \( C \) of radius \( \rho \), center at \( \alpha \). \( \rho \) may be so chosen that the circle \( C \) shall lie wholly within the region \( T \). Then \( \frac{f(z)}{z-\alpha} \) is finite, continuous, and analytic for all values of \( z \) between the two curves \( T \) and \( C \).

Hence
\[
\int_{T} \frac{f(z)}{z-\alpha} \, dz = \int_{C} \frac{f(z)}{z-\alpha} \, dz \quad \ldots \quad \ldots \quad \ldots \quad (1)
\]

Let us now consider the second member of this equation and find its limit as \( \rho \) approaches zero. Points on the circle \( C \) satisfy the equation
\[
z-\alpha = \rho e^{i\theta}, \quad \ldots \quad \ldots \quad \ldots \quad (2)
\]
where \( \theta \) is a real variable which has \( 0 \) to \( 2\pi \) for its range.

Whence also along the curve \( C \)
\[
dz = i\rho e^{i\theta} \, d\theta
\]
and, making use of (2),
\[
\frac{dz}{z-\alpha} = i \, d\theta \quad \ldots \quad \ldots \quad \ldots \quad (3)
\]
Again, from (2), and the continuity of \( f(z) \),
\[
f(z) = f(\alpha + \rho e^{i\theta}) = f(\alpha) + \epsilon(\rho) \quad -----------(4)
\]
where \( \epsilon(\rho) \) is a continuous function of \( \rho \) and such that
\[
\lim_{\rho \to 0} \epsilon(\rho) = 0.
\]
So by means of (3) and (4) it follows that
\[
\int_C \frac{f(z) \, dz}{z - \alpha} = \int_0^{2\pi} [f(\alpha) + \epsilon(\rho)] \, i \, d\theta
\]
\[
= 2\pi i \, f(\alpha) + i \int_0^{2\pi} \epsilon(\rho) \, d\theta
\]
Let \( E \) be the value of the last integral and \( \eta \) the greatest value of the modulus of \( \epsilon(\rho) \) for points along the circle, then
\[
|E| \leq \int_0^{2\pi} |\epsilon(\rho)| \, d\theta \leq \int_0^{2\pi} \eta \, d\theta,
\]
or
\[
|E| \leq 2\pi \eta.
\]
Now let the radius of the circle \( C \) diminish and approach zero as its limit and \( \eta \) will approach the limit zero also. Hence
\[
\lim_{\rho \to 0} E = 0.
\]
Consequently
\[
\lim_{\rho \to 0} \int_C \frac{f(\alpha) \, dz}{z - \alpha} = 2\pi i \, f(\alpha),
\]
and it follows from (1) that
\[
2\pi i \, f(\alpha) = \int_{\Gamma} \frac{f(z) \, dz}{z - \alpha},
\]
or
\[
f(\alpha) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z) \, dz}{z - \alpha}.
\]
This last result is known as Cauchy's Integral Formula since by means of it the value of the function \( f(z) \), at any point \( \alpha \) within the region may be computed.

Sec. 3. On Differentiation under the Integral Sign.

Theorem. Given an analytic function of the parameter \( \alpha \) defined by an integral as follows,

\[
\phi(\alpha) = \int_{z_1}^{z} f(z, \alpha) \, dz.
\]

If \( f(z, \alpha) \) is continuous in the two variables \( z \) and \( \alpha \) taken together, and possesses a continuous partial derivative with respect to \( \alpha \) then

\[
\frac{\partial \phi(\alpha)}{\partial \alpha} = \int_{z_1}^{z} f_z'(z, \alpha) \, dz.
\]

References.

Forsyth, Theory of Functions. P. 27.

Osgood, Lebrbuch der Funktionen Theorie P. 251.

Whittaker, Modern Analysis P. 50.

Harkness and Morley (Treatise) P. 173.

The original proof by Cauchy appeared in Turiner Abhandlung, Von Jahre, 1831.

also in

Exercices d'Analyse Bd. 2. (1841). P. 52.
Proof. Since $\phi(\alpha)$ is analytic, it is a continuous function of $\alpha$.

Then

$$\phi(\alpha) = \int_{z_o}^{z_1} f(z, \alpha) \, dz$$

and

$$\phi(\alpha + \Delta \alpha) = \int_{z_o}^{z_1} f(z, \alpha + \Delta \alpha) \, dz,$$

from which it follows that

$$\phi(\alpha + \Delta \alpha) - \phi(\alpha) = \int_{z_o}^{z_1} [f(z, \alpha + \Delta \alpha) - f(z, \alpha)] \, dz. \tag{1}$$

Consider the integrand $f(z, \alpha + \Delta \alpha) - f(z, \alpha)$

By the law of the mean

$$f(z, \alpha + \Delta \alpha) - f(z, \alpha) = \Delta \alpha \cdot f'_\alpha(z, \alpha + \Theta \Delta \alpha), \tag{2}$$

where $0 < \Theta < 1$.

By reason of the assumed continuity of $f'_\alpha(z, \alpha)$, the right hand member of (2) may be written

$$\Delta \alpha \left[ f'_\alpha(z, \alpha) + \xi \right], \tag{3}$$

where $|z| < \epsilon \quad \text{for} \quad |\Delta \alpha| < \delta$.

By the aid of (2) and (3) equation (1) becomes

$$\phi(\alpha + \Delta \alpha) - \phi(\alpha) = \int_{z_o}^{z_1} \Delta \alpha \left[ f'_\alpha(z, \alpha) + \xi \right] \, dz \tag{4}$$

or

$$\phi(\alpha + \Delta \alpha) - \phi(\alpha) = \Delta \alpha \int_{z_o}^{z_1} f'_\alpha(z, \alpha) \, dz \tag{5}$$

since $\Delta \alpha$ is independent of $z$. Dividing through by $\Delta \alpha$, we have

$$\frac{\phi(\alpha + \Delta \alpha) - \phi(\alpha)}{\Delta \alpha} = \int_{z_o}^{z_1} f'_\alpha(z, \alpha) + \int_{z_o}^{z_1} \xi \, dz. \tag{6}$$

Consider the last integral; viz.,

$$\int_{z_o}^{z_1} \xi \, dz.$$
Since $\xi$ is independent of $z$ and arbitrarily small, we have
\[ \int_{z_0}^{z} \xi \, dz = \xi \int_{z_0}^{z} \, dz = \xi (z, -z_0). \]

And, since $\xi$ is arbitrarily small, and approaches zero with $\Delta \alpha$, we have that
\[ \lim_{\Delta \alpha \to 0} \frac{\phi(z_0 + \Delta \alpha) - \phi(z_0)}{\Delta \alpha} = \int_{z_0}^{z} f'(z, \alpha_0), \]
and, since $\alpha_0$ is any point which satisfies the conditions of the theorem, we have, in general
\[ \frac{d \phi(a)}{d \alpha} = \int_{z_0}^{z} \frac{\partial f(z, \alpha)}{\partial \alpha} \, dz. \]

Sec (4). The Successive Derivatives of $f(z)$.

In section (2) the conclusion was reached that
\[ f(a) = \frac{1}{2\pi i} \int_{T} \frac{f(z) \, dz}{z - a}, \]
where $z$ is allowed to assume all values along $T$, the boundary of the region in which the function $f(z)$ is analytic and $a$ is

References. The corresponding theorem for real variables is considered by Osgood in his

Lehrbuch der Funktionen Theorie P. 87.
any point within the region bounded by $T$. Now the function

$$\frac{f(z)}{z-a}$$

has a continuous derivative with respect to $\alpha$ for all values of $z$ and $\alpha$ which they are allowed to have in Cauchy's Integral Formula. The conditions which we have found to be sufficient for the differentiation under the integral sign with respect to the parameter $\alpha$ are fulfilled, so that

$$f'(\alpha) = \frac{1}{2\pi i} \int_T \frac{f(z)\,dz}{(z-a)^2}$$

$$--------------------------------- (1)$$

$$f''(\alpha) = \frac{2!}{2\pi i} \int_T \frac{f(z)\,dz}{(z-a)^3}$$

$$------------------------ (2)$$

$$f^n(\alpha) = \frac{n!}{2\pi i} \int_T \frac{f(z)\,dz}{(z-a)^{n+1}},$$

which is finite for $z \neq a$. Now since $\alpha$ is any point within the boundary $T$, it follows that within that region in which $f(z)$ is analytic the successive derivatives $f'(\alpha)$, $f''(\alpha)$, $f'''(\alpha)$,----- exist, and by Section 1, they are continuous. Hence the derivative of an analytic function is also analytic.

If

$$f(z) = u + i v,$$

then

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = u_x + i v_x,$$

$$f''(z) = \frac{\partial u_x}{\partial x} + i \frac{\partial v_x}{\partial x} = u_{xx} + i v_{xx},$$

$$f'''(z) = \frac{\partial u_{xx}}{\partial x} + i \frac{\partial v_{xx}}{\partial x} = u_{xxx} + i v_{xxx},$$

$$-------------------------------------------$$

$$f^n(z) = \frac{\partial u_x^n}{\partial x} + i \frac{\partial v_x^n}{\partial x} = \frac{\partial^n u}{\partial x^n} + i \frac{\partial^n v}{\partial x^n}.$$
From the existence and continuity of \( f(z), f'(z), f''(z), \ldots \)
follow the existence and continuity of

\[
\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial x^2}, \frac{\partial^2 v}{\partial x^2},
\]

and

\[
\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y^2}, \frac{\partial^2 v}{\partial y^2},
\]

So that conditions sufficient for the interchange of the order
of differentiation are fulfilled and therefore

\[
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}.
\]

Sec. 5. Laplace's Equation. If \( f(z) = u + iv \) is an
analytic function of \( z, u \) and \( v \) must be functions of \( x \) and \( y \)
which satisfy the Cauchy-Riemann differential equations

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
\]

In the previous section it was shown that the successive
partial derivatives of \( u \) and \( v \) exist and are continuous.

Hence we have

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial x \partial y},
\]

and

\[
\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x \partial y}.
\]

Adding these equations, member by member,
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

and similarly \[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \]

These are special forms of what is known as Laplace's equation. The operator \( \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \) is known as Laplace's operator and is quite commonly denoted by the symbols \( \nabla \) or \( \Delta \).

Sec. 6. Relation of Laplace's Equation to Analytic Functions.

It has been shown by Cauchy that the necessary and sufficient conditions that \( w(z) = u + iv \) be an analytic function of \( z(x + iy) \) are that the so-called conjugate functions \( u(xy) \) and \( v(xy) \) shall satisfy the following differential equations:

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \]

From the considerations presented in the previous sections, the conclusion has been reached that whenever \( w = u + iv \) is an analytic function of \( z \), \( u \) and \( v \) will be functions of \( x \) and \( y \) which will satisfy Laplace's Equation. But the converse does not hold in general, otherwise any two functions \( u \) and \( v \) which satisfy Laplace's Equation would form an analytic function when used in the combination \( u + iv \). That
this is not the case is made clear by a simple example. Let $u + iv = (x^2 - y^2) + i (e^x \cos y)$. Now, though the separate functions, $x^2 - y^2$ and $e^x \cos y$ satisfy Laplace's equation, they do not satisfy the Cauchy differential equations and hence the function is not analytic. In fact it is not even a function of $z$. The general solution $u(x,y) = c$, of Laplace's equation is a family of surfaces. Corresponding to the family of surfaces there is a family of surfaces $V(xy) = c_z$ which cut the first orthogonally. $u$ and $V$ satisfy the condition of orthogonality; viz,

$$\frac{\partial u}{\partial x} \cdot \frac{\partial V}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial V}{\partial y} = 0 \quad \text{------------------------(1)}$$

Now this condition may be written in the form

$$\frac{\partial u}{\partial x} \cdot \frac{\partial V}{\partial y} = - \frac{\partial u}{\partial y} \cdot \frac{\partial V}{\partial x} = \kappa$$

or

$$\frac{\partial u}{\partial x} \neq - \frac{\partial u}{\partial y} = 1 \quad \text{where } \kappa = KV. \quad \text{--------(2)}$$

This gives the two equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}$$

which are precisely the equations which define an analytic function $u + iv$. So that it is plain that to every solution $u$ of Laplace's equation there corresponds an orthogonal solution $V$, and when this solution is multiplied by the proper constant
so that equations (2) are satisfied, the functions \( u \) and \( v \) become the conjugate functions of the analytic function \( u + iv \).
RELATION OF LAPLACE'S EQUATION TO POTENTIAL THEORY.

Sec. 1. Newtonian Potential Function. Let any mass $M$ whose density is continuous be divided into $n$ elements $\Delta m$. Let $r_i$ be the distance from any point $(x, y, z)$ to the mass center $(x_i', y_i', z_i')$ of $\Delta_i m$. If $\rho_i$ is the mean density of the mass element, then

$$\Delta_i m = \rho_i \Delta_i v$$

where $\Delta v$ is the volume of the mass element. Then

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\rho_i \Delta_i v}{v_i}$$

is defined as the Potential Function for the mass $M$. It is commonly represented by the symbol $V$ and is equivalent to

$$\int \int \int \frac{\rho \, dx' \, dy' \, dz'}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{3/2}} \quad \text{(1)}$$

where the limits of integration are so taken as to include the whole mass $M$. The density factor is a function of the co-ordinates $(x', y', z')$ and is necessarily finite. Then the integrand is finite and continuous, if $(x, y, z)$ is not a point of mass $M$. That $V$ remains finite, even when the point $(x, y, z,)$ is a point of the mass $M$, is obvious if the expression for $V$ is transformed into polar co-ordinates $(r, \theta, \phi)$.
with the point \( P \equiv (x, y, z) \) as the pole. For then we have

\[
\nabla = \iiint \frac{\rho r^2 \sin \theta}{r} \, dr \, d\theta \, d\phi
\]

or

\[
\nabla = \iiint \rho r \sin \theta \, dr \, d\theta \, d\phi
\]

which is finite for finite limits of integration. On account of the continuity of the integrand of (1) for all points not included in \( M \), the partial derivatives of \( V \); viz, \( D_x V \), \( D_y V \), \( D_z V \) may be obtained by differentiation under the integral sign with respect to the parameters, \( x, y, \) and \( z \) respectively. The results of the first differentiation are

\[
\frac{\partial V}{\partial x} = \iiint \frac{(x' - x) \rho'}{(x' - x)^2 + (y' - y)^2 + (z' - z)^2} dx' \, dy' \, dz'
\]

\[
\frac{\partial V}{\partial y} = \iiint \frac{(y' - y) \rho'}{(x' - x)^2 + (y' - y)^2 + (z' - z)^2} dx' \, dy' \, dz'
\]

\[
\frac{\partial V}{\partial z} = \iiint \frac{(z' - z) \rho'}{(x' - x)^2 + (y' - y)^2 + (z' - z)^2} dx' \, dy' \, dz'
\]

These are, in fact, the expressions for \( X, Y, \) and \( Z \), the components of attraction at \( P \equiv (x, y, z) \) due to the mass \( M \).

References.

* Osgood: Lehrbuch der Funktionentheorie

\( \text{P. 87.} \)

* Kirchhoff: Vorlesungen über Mathematische Physik

\( \text{Vol. I P. 163.} \)
Differentiating again,-

\[
\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial X}{\partial x} = \iint \frac{3(x'-x)^2 - \nu^2}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^2} \cdot \rho' \, dx' \, dy' \, dz',
\]

\[
\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial Y}{\partial y} = \iint \frac{3(y'-y)^2 - \nu^2}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^2} \cdot \rho' \, dx' \, dy' \, dz',
\]

and

\[
\frac{\partial^2 \psi}{\partial z^2} = \frac{\partial Z}{\partial z} = \iint \frac{3(z'-z)^2 - \nu^2}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^2} \cdot \rho' \, dx' \, dy' \, dz'.
\]

Whence, by addition of these results

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0,
\]

which is Laplace's equation for three dimensions and defines what is known as the Newtonian Potential Function. An extension of Laplace's equation was made by Poisson who obtained as the general equation for the potential function

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -4\pi \rho,
\]

where \( \rho \) is the density at the point considered. This equation clearly reduces to Laplace's form when the point considered is not a point of the given mass.

Sec. 2. Logarithmic Potential. It is plain that in Laplace's equation none of the terms \( \frac{\partial^2 \psi}{\partial x^2} \), \( \frac{\partial^2 \psi}{\partial y^2} \), \( \frac{\partial^2 \psi}{\partial z^2} \) is identically zero for the general distribution of the

* Bulletin de la Societe Philomatique, 1813.
density of the mass.

But suppose \( \frac{\partial^2 V}{\partial z^2} = 0 \)

Then, since

\[
\frac{\partial^2 V}{\partial z^2} = \frac{\partial Z}{\partial z}
\]

the matter must be so distributed that the component of attraction parallel to the z-axis shall always be zero. Such a distribution can occur, at least in the ideal, if the mass be constructed of filaments of constant density and infinite extent in both directions, parallel to the z-axis. For a body so composed the potential function would be independent of z and would satisfy the equation

\[ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \]

Then \( V = f(x, y) + C \) expresses the general solution for the present case, so that the locus of the points having a given fixed value for potential will be a cylinder whose elements are parallel to the z-axis.

The attraction of a filament AB whose length is 1, for a unit mass concentrated at a point r units from the filament is given by the expression \( \frac{2\mu}{\lambda} \sin \frac{\Theta}{\lambda} \), where \( \mu \) is the
mass of a unit length of the filament and \( \Theta = \text{APB} \). Now as AB increases indefinitely in length the angle \( \Theta \) approaches \( \pi \) as its limit and the above expression approaches \( \frac{2\mu}{r} \) as its limit.

Considering this filament to have a rectangular cross-section whose dimensions are \( \Delta x' \) and \( \Delta y' \) then \( \mu = \rho A x' \cdot \Delta y' \) and the components of attraction are given as follows:

\[
\vec{X} = \frac{\partial V}{\partial x} = \iint \frac{2\rho (x'-x) \, dx' \, dy'}{(x'-x)^2 + (y'-y)^2}
\]

and

\[
\vec{Y} = \frac{\partial V}{\partial y} = \iint \frac{2\rho (y'-y) \, dx' \, dy'}{(x'-x)^2 + (y'-y)^2}
\]

A solution for \( V \) manifestly is

\[
V = \iint \rho \log \left[ (x'-x)^2 + (y'-y)^2 + (z'-z)^2 \right] \, dx' \, dy'
\]

\[
= \iint 2\rho \log r \, dx' \, dy' = \iint \rho \log r^2 \, dx' \, dy',
\]

where the integrals are so taken as to include the whole boundary of the \( x-y \) section of the body. From the character of the functions entering the result just obtained this is called Logarithmic Potential. Differentiating (2) and (3), with respect to \( x \) and \( y \) respectively,

\[
\frac{\partial X}{\partial x} = \frac{\partial V}{\partial x} = \iint 2\rho \left[ \frac{2(x'-x)^2 - r^2}{r^4} \right] \, dx' \, dy'
\]

and

\[
\frac{\partial Y}{\partial y} = \frac{\partial V}{\partial y} = \iint 2\rho \left[ \frac{2(y'-y)^2 - r^2}{r^4} \right] \, dx' \, dy'
\]
from which it follows, by addition, that

\[ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \]

As illustrations of this kind of potential, consider the following:

1. The simplest case is perhaps that of a long straight wire of uniform structure placed perpendicular to the xy plane. It is surrounded by a field of gravitational force. For the immediate vicinity of the wire, i.e., where the length of the wire is very large compared to the distance from the wire, the equipotential surfaces are cylinders whose axes coincide with the axis of the wire. The lines of force form an infinite set of planes all intersecting each other in the axis of the wire and cutting the system of cylinders orthogonally.

2. Suppose an electric current is sent through the long wire considered in the previous example. Then it is at once surrounded by a field of magnetic force.
In the $x$-$y$ plane the lines of magnetic force will be circles whose centers are the center of the wire.

Sec. 3. Laplace's Equation for two variables. Since one of the principal objects of this investigation is to correlate analytic functions of the complex variable $(x + iy)$ with physical phenomena through the medium of Laplace's equation which has now been found essential to both fields, it is proper to fix attention to that form of the equation which is common to the two fields; viz,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

With this restriction $V$ becomes a function of two variables and a noteworthy advantage is the possibility of geometrical representation of the functions which may present themselves. For example, if $V = x^2 - y^2$, the value of $V$ may be used as a co-ordinate measured perpendicular to the $xy$ plane as $z$ is measured and the resulting surface is the common hyperbolic paraboloid. The projection upon the $xy$ plane of those points for which $V$ has a given fixed value will be equilateral hyperbolas and may be called equipotential curves.
Sec. 4. Conjugate Functions. One of the simplest, yet most important of the correlations, presents itself in the conjugate functions. If $U$ is the conjugate of $V$, then it must satisfy these partial differential equations

$$\frac{\partial V}{\partial x} = \frac{\partial U}{\partial y} \quad \text{and} \quad \frac{\partial V}{\partial y} = -\frac{\partial U}{\partial x} \quad (1)$$

It should be noticed that when $U$ is the conjugate of $V$ then $-V$ (not $+V$) must be the conjugate of $U$. Equations (1) are precisely the conditions which $U$ and $V$ must satisfy if $V+iu$ is to be an analytic function of $x+iy$ in the Cauchy-Riemann sense. Taken together they are equivalent to the condition necessary for orthogonal systems of curves, as already stated in Chapter I, Sec. 6. Whence it follows that when

$$w = u + iv = f(x + iy)$$

the conjugate functions $u(x,y) = c_1$ and $v(x,y) = c_2$ give orthogonal systems of curves in the $xy$-plane. Now if $v(xy)$ gives the potential function, $v(x,y) = k$, where $k$ is an arbitrary parameter, defines the equipotential curves, then will $u(x,y) = k$ define the lines of flow, or lines of force. So that if $V$ is the conjugate function of $U$ then $w = u + iv$ is an analytic function while $u$ and $v$ characterize a state of streaming. Klein has made use of these facts to derive from physical phenomena theorems in the theory of functions of a complex
variable. An example has also been shown by Slichter in the case of water flowing through a thin layer of porous soil underlaid by impervious rock. If the pressure $p$ of the water on the horizontal plane is a function of $x$ and $y$, then the rates at which a particle of water is changing its position in the directions of the co-ordinate axes are given as follows:

$$ v_x = \frac{dx}{dt} = K \frac{\partial p}{\partial x} \quad \text{and} \quad v_y = \frac{dy}{dt} = K \frac{\partial p}{\partial y} $$

---------(2)

The equation of continuity for motion in a single plane is

$$ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 $$

---------(3)

This is the condition under which the quantity $Q$ of water, which enters a given rectangle in the $xy$ plane shall continually equal the quantity which leaves the rectangle. Applying condition (3) to equations (2) we obtain at once Laplace's equation

$$ \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 $$

---------(4)

*See Klein: On Riemann's Theory of Algebraic Functions.*

Equation (3) gives the necessary and sufficient condition that
\[ v_x \frac{dy}{ds} - v_y \frac{dx}{ds} = d\psi \]  
\[ \text{-----}(5) \]
shall be an exact differential.*

Hence
\[ v_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v_y = -\frac{\partial \psi}{\partial x} \]  
\[ \text{-----}(6) \]

We may write equation (5) in the form
\[ \frac{d\psi}{ds} = v_x \frac{dy}{ds} - v_y \frac{dx}{ds} \]  
\[ \text{-----}(7) \]
in which ds is the element of any path in the xy plane. Now
\[ \frac{dy}{ds} \quad \text{and} \quad \frac{dx}{ds} \]
are the direction cosines l, m of the inwardly
\[ \text{drawn normal to ds. Whence} \]
\[ \psi = \int_A^P (l v_x + m v_y) \, ds \]  
\[ \text{-----}(8) \]
where A and P are any two points on the path s. But the right
side of (8) gives the quantity \( Q \) which flows across the path AP.

If P moves along the plane in such a way that \( \psi \) maintains a
constant value it will trace a line of flow. Thus \( \psi = c \) is a
stream line and the function \( \psi \) may be called the current
function. Since \( p \) is the pressure and \( \psi \) is the current
function, we have
\[ \frac{\partial p}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \text{and} \quad \frac{\partial p}{\partial y} = -\frac{\partial \psi}{\partial x} \]

Thus the pressure function is conjugate to the current function.

* Townsend and Goodenough, Calculus. P. 263.
Sec. 5. The Potential Function for Two Variables. We shall now investigate the form which a function of \( x \) and \( y \) may take if it is to represent a potential function. Suppose that \( t \) is a function of the variables \( x \) and \( y \). This function \( t(x,y) \) may or may not be a solution of Laplace's equation. But let

\[
u = f(t)
\]

be a solution of Laplace's equation. Then we have

\[
\frac{\partial^2 u}{\partial x^2} = f(t) \frac{\partial^2 t}{\partial x^2} , \quad \frac{\partial^2 u}{\partial x \partial y} = f'(t) \frac{\partial^2 t}{\partial x \partial y} + f(t) \frac{\partial^2 t}{\partial x^2} \quad (2)
\]

and

\[
\frac{\partial^2 u}{\partial y^2} = f''(t) \frac{\partial^2 t}{\partial y^2} \quad \frac{\partial^2 u}{\partial y^2} = f''(t) \frac{\partial^2 t}{\partial y^2} + f'(t) \frac{\partial^2 t}{\partial y^2} . \quad (3)
\]

Therefore

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left( \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} \right) f''(t) + \left( \frac{\partial^2 t}{\partial x \partial y} + \frac{\partial^2 t}{\partial y \partial x} \right) f'(t) \quad (4)
\]

Since \( u \) satisfies the left hand side of (4) we have

\[
\left[ \left( \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} \right) f''(t) + \left( \frac{\partial^2 t}{\partial x \partial y} + \frac{\partial^2 t}{\partial y \partial x} \right) f'(t) \right] = 0 \quad (5)
\]

or

\[
\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = - \frac{f''(t)}{f'(t)} \quad (6)
\]

The right member of (6) depends upon \( t \) alone. So if the function \( t(x,y) \) has been selected, then the form of the function \( u = f(t) \) must be such as to meet the requirements of equation (6).
That equation (6) gives a sufficient condition for \( u = f(t) \) to be a solution of Laplace's equation may be shown as follows:

From equation (6) we have that

\[
\left[ \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} \right] f(t) + \left[ \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \right] f''(t) = 0
\]

or

\[
\left[ f'(t) \frac{\partial^2 t}{\partial x^2} + f''(t) \left( \frac{\partial f}{\partial x} \right)^2 \right] + \left[ f(t) \frac{\partial^2 t}{\partial y^2} + f''(t) \left( \frac{\partial f}{\partial y} \right)^2 \right] = 0
\]

Hence by (2) and (3)

\[
\frac{\partial^2 f(t)}{\partial x^2} + \frac{\partial^2 f(t)}{\partial y^2} = 0
\]

Such functions \( f(t) \) as satisfy Laplace's equation are called Harmonic Functions. We may, therefore, state the following:

**Theorem.** The necessary and sufficient conditions that \( u = f(t) \) shall be a harmonic function where \( t = \phi(x, y) \) and where \( t \) may, or may not, be a harmonic function, is that

\[
\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} \leq \frac{f''(t)}{f'(t)} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}
\]

**Examples.** 1. Consider the function \( t = (x^3 + y^3) \).

Here

\[
\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = \frac{6(x + y)}{9(x^4 + y^4)}
\]

Since the right member of (9) is no function of the argument \( t = (x^3 + y^3) \) alone, it is impossible to express it in the
form \(-f''(t)\). Consequently, there is no harmonic function 
\(f''(t)\)  
whose argument is \(x^3 + y^3\) alone.

2. Consider the function \(t = \frac{\sqrt{y}}{x}\)

In this case
\[
\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = \frac{2 \frac{y}{x^3}}{\left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial y}\right)^2} = \frac{2 \frac{y}{x}}{\frac{y^2}{x^4} + \frac{1}{x^2}} = \frac{2 \frac{t}{t^2 + 1}}{t^2 + 1} \quad \text{---(10)}
\]

Now let
\[\frac{2 \frac{t}{t^2 + 1}}{t^2 + 1} = -\frac{f''(t)}{f'(t)},\]
as in (6), and we get by integration,

\[\log f'(t) = -\log \left(t^2 + 1\right),\]
or
\[f'(t) = \frac{1}{t^2 + 1}\]

and by a second integration,

\[f(t) = \arctan t,\]
i. e.- \(u = \arctan \frac{t}{x}\) is a harmonic function. This is of interest since \(\arctan \frac{t}{x}\) is the amplitude of the complex variable.

3. Let \(t = \sqrt{x^2 + y^2}\)

The equation (6) becomes for this problem
\[
\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} \quad \text{---(10)}
\]

\[\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = \frac{1}{t} = -\frac{f''(t)}{f'(t)}\]
or\[-\frac{f''(t)}{f'(t)} = -\frac{1}{t}.\] ------(11)

By an integration we obtain
\[\log f'(t) = \log k - \log t = \log \frac{k}{t}\]

or
\[f'(t) = \frac{k}{t}\]

Another integration gives
\[f(t) = k \log t + c\]

where \(k\) is an arbitrary constant of integration. We are thus led to the conclusion that when
\[t = \sqrt{x^2 + y^2}\]

representing the distance from \(P = (x,y)\) to some point as origin, is the argument considered, then the only harmonic function with \(t\) as its argument is
\[k \log t,\]

except for an additive constant.

This example is of importance because it enables us to decide what functions of the argument
\[\gamma = \sqrt{(x-a)^2 + (y-b)^2}\]

are harmonic functions and what functions are not of that class. From the definition of potential we have
\[V = \frac{m}{\gamma};\]
But in two dimensions the form of the potential function in terms of $\nu$ is

$$V = k \log \nu.$$  

It has been shown then that: (1) The form of a harmonic function depends upon the character of the argument $t(x,y)$ and, (2) When the argument is the distance of the point $(x,y)$ from some fixed point the harmonic function is no less than $k \log \nu$.

Sec. 6. Applications of Logarithmic Potential. In this section we shall discuss some of the more general applications of the Logarithmic functions of a complex variable.

1. The Function $w = \log \frac{z - \alpha}{z - \beta}$

Consider the function $w = u + iv = \log \frac{z - \alpha}{z - \beta}$, where $\alpha$, $\beta$ and $z$ are the complex numbers $a + ib$, $c + id$, and $x + iy$ respectively. We have that $z - \alpha = (x + iy) - (a + ib) = (x-a) + iy - b$)

$$= [(x-a)^2 + (y-b)^2]^{1/2} \left\{ \frac{(x-a) + i (y-b)}{[(x-a)^2 + (y-b)^2]^{1/2}} \right\}$$

or

$$z - \alpha = \rho_1 (\cos \theta_1 + i \sin \theta_1) = \rho_1 e^{i \theta_1}, \quad \text{------(1)}$$

where

$$\rho_1 = \left[ (x-a)^2 + (y-b)^2 \right]^{1/2}$$

and

$$\theta_1 = \arctan \frac{y-b}{x-a}$$

Likewise $\overline{z} - \beta = \rho_2 (\cos \theta_2 + i \sin \theta_2) = \rho_2 e^{i \theta_2}, \quad \text{------(2)}$
where
\[ \rho_2 = \sqrt{(x-a)^2 + (y-b)^2} \]
and
\[ \Theta_2 = \arctan \frac{y-b}{x-a} \]

Using (1) and (2) we obtain
\[ u + iv = \log \frac{\rho_1 e^{i \Theta_1}}{\rho_2 e^{i \Theta_2}} = \log \frac{\rho_1}{\rho_2} + i(\Theta_1 - \Theta_2) \]

Equating real and imaginary parts,
\[ u = \log \frac{\rho_1}{\rho_2} \]
and
\[ v = \Theta_1 - \Theta_2 \]

\( v \) is the conjugate function of \( u \).

If \( u \) is a potential function then \( v = \Theta_1 - \Theta_2 \) characterizes the streaming. A system of equipotential curves may be obtained by considering \( u \) as having some constant value, say \( k \). Then equation (4) becomes
\[ \log \frac{\rho_1}{\rho_2} = k \]

or
\[ \frac{\rho_1}{\rho_2} = e^k = m, \]

where \( m \) is understood to represent \( e^k \) and has any arbitrary value from 0 to \(+\infty\).

But
\[ \frac{\rho_1}{\rho_2} = \frac{\sqrt{(x-a)^2 + (y-b)^2}}{\sqrt{(x-c)^2 + (y-d)^2}} = m. \]
Squaring, \( (x - a)^2 + (y - b)^2 = m^2 \)

Clearing of fractions and reducing

\[
(x^2 + y^2 - 2\left(\frac{a - m^2c}{1 - m^2}\right)x - 2\left(\frac{b - m^2d}{1 - m^2}\right)y = \frac{m^2c^2 + m^2d^2 - a^2 - b^2}{1 - m^2},
\]

or

\[
\left(\frac{x - \frac{a - m^2c}{1 - m^2}}{1 - m^2}\right)^2 + \left(\frac{y - \frac{b - m^2d}{1 - m^2}}{1 - m^2}\right)^2 = \left[\frac{m}{1 - m^2}\sqrt{(a - c)^2 + (b - d)^2}\right]^2
\]

Thus the lines of equipotential are circles. These circles have their centers on the line through the points \( \alpha = (a, b) \) and \( \beta = (c, d) \). It is evident that whatever the value of \( m^2 \) the center of the circle lies outside the line segment \( \alpha \parallel \beta \). It is nearer to \( \beta \) than to \( \alpha \) if \( m^2 > 1 \); nearer to \( \alpha \) than to \( \beta \) if \( m^2 < 1 \). That the circle cuts the line \( \alpha \parallel \beta \) some where between \( \alpha \) and \( \beta \) is shown in the following manner. If \( m^2 > 1 \) or \( m > 1 \), then the center of the circle is nearer to \( \beta = (c, d) \) than to \( \alpha = (a, b) \). The distance from the center

\[
\left(\frac{a - m^2c}{1 - m^2}, \frac{b - m^2d}{1 - m^2}\right)
\]

to the point \( (c, d) \) is given by the expression

\[
\frac{1}{1 - m^2} \sqrt{(a - c)^2 + (b - d)^2}
\]

\[\text{-----(8)}\]

\* It should be noticed that in clearing of fractions the discontinuity at the point \( \beta = (c, d) \) is removed.
while the radius of the circle is
\[ \frac{m}{1 - m^2} \sqrt{(a - c)^2 + (b - d)^2} \quad \text{-----(9)} \]

But the distance from the center \((a \cdot \frac{m \cdot c}{1 - m^2}, \frac{b - m \cdot d}{1 - m^2})\)
to the point \(\alpha = (a, b)\) is
\[ \frac{m^2}{1 - m^2} \sqrt{(a - c)^2 + (b - d)^2} \quad \text{-----(10)} \]

For \(m > 1\), we have the relation
\[ m^2 > m > 1 \]

Hence
\[ \frac{m^2}{1 - m^2} \sqrt{(a - c)^2 + (b - d)^2} > \frac{m}{1 - m^2} \sqrt{(a - c)^2 + (b - d)^2} \]

This means that the circle (7) cuts the line \(\alpha \beta\) at some point between \(\alpha\) and \(\beta\). That this point of intersection is closer to \(\beta\) than to \(\alpha\), for \(m > 1\), may be seen by getting the difference between the two distances (8) and (9). That is
\[ \frac{m - 1}{1 - m^2} \sqrt{(a - c)^2 + (b - d)^2} = \frac{m - 1}{1 - m^2} \sqrt{(a - c)^2 + (b - d)^2} \quad \text{-----(11)} \]

which for \(m > 1\), gives something less than
\[ \frac{1}{2} \left| \sqrt{(a - c)^2 + (b - d)^2} \right| \]

which is one half the length of the line \(\alpha \beta\). When \(m > 1\), a set of results corresponding to those just obtained will be found; viz, the center of the circle is nearer to \(\alpha\) than to \(\beta\).
The circle intersects the straight line \( \alpha \beta \) at a point between \( \alpha \) and \( \beta \). This point of intersection is between \( \alpha \) and the mid point of the line \( \alpha \beta \).

The figure shows the set of circles (7) which represent the lines of equi-potential.

The conjugate set of curves. When \( v = \theta_1 - \theta_2 = k \), where \( k \) may have any assigned values, positive or negative, we have a set of circles again. Since \( \theta_1 = \arctan \frac{y-b}{x-a} \) and \( \theta_2 = \arctan \frac{y-d}{x-c} \). Then \( k = \arctan \frac{y-b}{x-a} - \arctan \frac{y-d}{x-c} \).

Taking the tangents of both members,

\[
\tan k = \frac{\frac{y-b}{x-a} - \frac{y-d}{x-c}}{1 + \frac{y-b}{x-a} \cdot \frac{y-d}{x-c}}
\]

which reduces to the equation of a circle; viz,

\[
\chi^2 + y^2 = \frac{(a+c)\tan k + (b-d)}{\tan k} \chi - \frac{(b+d)\tan k + (c-d)}{\tan k} k = \frac{bc-ad-(ac+bd)\tan k}{\tan k}
\]

It is readily shown by direct substitution that this circle
passes through the points $\alpha$ and $\beta$. In fact for a succession of values of $k$ there is obtained a family of circles, every one of which passes through the points $\alpha$ and $\beta$. This system of circles (12) cuts the system (7) orthogonally.

An experiment which may be made with simple physical apparatus will show one of these systems of circles. Let two long parallel wires pass through a horizontal plate of glass at the points A and B. Through each wire pass a current of a given strength. Let the currents flow in opposite directions through the wires. Spread a thin layer of iron filings over the glass and tap the glass lightly. The filings arrange themselves longitudinally in the lines of magnetic force surrounding each wire. They show very plainly the system of circles corresponding to equation (7). Again suppose that the plane plate is a good conductor and unlimited in extent in all directions. Let a positive electrode touch the plane at the point A and the corresponding negative electrode touch at B. Then, the circuit being closed, the current will flow out into the plane at A and leave the plane at B. As soon as the flow becomes steady throughout the plane the lines of flow will be the circles (12), which pass through A and B. In this case the lines of equipotential are given by the circles (7).

Let water be supplied to a horizontal plane at the point A and let it escape through a sink at the point B. Now if the
flow be steady throughout the plane the lines of flow will be the circles (12) again while the curves of equal pressure will be circles (7).

Another method of obtaining arcs of circles as stream lines is to place on the plane an impervious water-divide MN along one side of which there is a narrow slit in the plane which may be used for a sink. Now allow a continuous and uniform sheet of water to fall upon the plane along the side of MN opposite to the sink PQ. If the source and sink have equal capacity the lines of flow will be given by equation (7). Obviously the sink can be distorted into any one of the equipotential curves, say PRQ, without affecting the lines of flow. In the case of electric flow corresponding to the one just given for liquids the only change which must be made is in the apparatus. Let the barrier MN be a dielectric and let it pass completely through the plane so that there can be no passage of electricity from one side of the barrier to the other,
except around the points \( M \) and \( N \), and on the plane. Then let two linear electrodes, joined to a battery, touch the plane on either side of the barrier. Let the electrodes be coextensive with the barrier. When the flow of electricity becomes steady through the plane, the lines of flow are the circles given by equation (7). These illustrations are sufficient to show the significance of the conjugate functions involved in the function

\[
w = \log \frac{z-a}{z-b},
\]

which is under consideration.

2. **Consider The Function.** \( w = \log \left[ (z-\alpha)(z-\beta) \right] \)

Let \( w = u + iv = \log \left[ (z-\alpha)(z-\beta) \right] \)

where \( z, \alpha \), and \( \beta \) have the same values as in previous examples. The real and imaginary parts can be separated with little difficulty, giving

\[
u = \log \left( \rho_1 \rho_2 \right)
\]

where

\[
\rho_1 = \left[ (x-a)^2 + (y-b)^2 \right]^{1/2}
\]

and

\[
\rho_2 = \left[ (x-c)^2 + (y-d)^2 \right]^{1/2}
\]

and

\[
u = \Theta_1 + \Theta_2
\]

where

\[\Theta_1 = \arctan \frac{y-b}{x-a}\]
First let us consider the potential function defined in equation (1). The curves of equal potential are given by

\[ u = \log(p^1 \cdot p^2) = \kappa \]

or

\[ p^1 \cdot p^2 = e^\kappa \]

This gives a system of Cassinian ovals when a succession of values are given to the parameter \( k \). The foci of these ovals are the points \( (a, b) \) and \( (c, d) \) in the \( z \)-plane. This set of curves may be obtained experimentally by means of the two long wires, glass plate, iron filings, etc., arranged as before. In this case let the current flow in the same direction through both wires and let the strength of current be the same in both wires. A field of force surrounds each wire and the resultant lines of force are marked by the filings. The lines of force are Cassinian ovals, one of which is the lemniscate.

Again suppose that a portion of a plane is bounded by the oval LMNR having its foci at A and B. Suppose that at A and B there are two artesian wells of equal strength which supply

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Russell's Treatise on the Theory of Alternating Currents,
Page 310.
water to the surface of this impervious plane. Let the water escape at the boundary LMNR. As soon as the interior of the oval is covered with a film of water and the streaming becomes steady in every portion, the system of Cassinian ovals becomes the system of equipotential lines. The same thing happens if the plane is made of a good conductor of electricity and is grounded uniformly along the boundary LMNR. If equal positive electrodes are placed at A and B the current will flow through the plane conductor in such a way that the Cassinian ovals are equipotential lines.

The lines of flow will be given by the equation

\[ \psi = \theta_1 + \theta_2 = K \]  \( \text{(4)} \)

on account of the conjugate relation shown in equations (1) and (2). The nature of these curves (4) is best seen from the equation in rectangular co-ordinates.

We have from (4)

\[ \tan(\theta_1 + \theta_2) = \tan k. \]

it is convenient to make \( \tan k = m \), and, remembering the
values of $\theta$, and $\theta$, we obtain

$$\frac{y-b}{x-a} + \frac{y-d}{x-c} = m.$$  

This equation may be reduced to the form

$$m x^2 - 2 x y - m y^2 + (b+d-am-cm)x + (a+c+bm+dn)y + (ac-bd)m - (bc+ad) = 0$$  

But equation (5) represents, for every finite value of $m$, an equilateral hyperbola, the inclination of whose axis to the $x$-axis is

$$\frac{1}{2} \arccos \left( \frac{m}{m} \right).$$

To each branch of a hyperbola passing through $\alpha$ there corresponds one passing through $\beta$ and these together constitute one of the equilateral hyperbolas (5).

So there are two clusters of curves, one radiating from $\alpha$ and the other from $\beta$ and these are the stream lines when equal sources are located at $\alpha$ and $\beta$ respectively.
Sec. 7. Potential Function for a set of long parallel wires. The attraction at the point P due to a fine straight wire of infinite extent, where the mass of a unit length of the wire is $m$ and the distance of P from the wire is $\lambda$, is given by the expression

$$A = \frac{2m}{\lambda}$$  \hspace{1cm} (1)

This is proportional to the component of potential $V$ along the radius $r$, i.e.

$$A = k \frac{dV}{dr} = \frac{2m}{r},$$  \hspace{1cm} (2)

whence

$$V = \frac{1}{k} \int \frac{2m}{r} dr$$

or

$$V = \frac{1}{k} \cdot 2m \log r$$  \hspace{1cm} (3)

For purposes of convenience let the units be so chosen that the proportionality factor equals one. Then we have

$$V = 2m \log r.$$  \hspace{1cm} (4)

The potential due to $n$ equal wires, all parallel and isolated, would be the algebraic sum of the potentials due to each separately.

Thus

$$V = 2m \log r + 2m \log r_1 + \ldots + 2m \log r_n.$$  \hspace{1cm} (5)

from which it follows that
\[ V = 2m \log (r_1 \cdot r_2 \cdot r_3 \cdots \cdot r_n) \] \hspace{1cm} (6)

where the subscripts refer to the wires considered. If two of the wires, say the \((n-1)\)th and the \(n\)th, coalesce, or, what is the same in effect, if the density of the \((n-1)\)th wire is doubled and the \(n\)th wire removed from consideration, we have

\[ V = 2m \log r_i + 2m \log r_2 + \cdots + 2m \log r_{n-1}, \]

or

\[ V = 2m \log \rho_i + 2m \log \rho_2 + \cdots + 2m \log \rho_{n-1}, \] \hspace{1cm} (7)

whence

\[ V = 2m \log (\rho_i \cdot \rho_2 \cdot \rho_3 \cdots \cdot \rho_{n-1}) \] \hspace{1cm} (8)

In general if the masses \(m\) per unit length are \(v_1, \ldots, v_n\) for the wires respectively

\[ V = 2 \log (v_i \cdot v_2 \cdot v_3 \cdots \cdot v_n) \] \hspace{1cm} (9)

In case there should be besides the \(n\) attracting wires, \(m\) wires of matter which repel with a force that varies inversely as the square of the distance, the expression for potential would be

\[ V = 2 \log (r_i \cdot r_2 \cdots \cdot r_n) - 2 \log (r_{n+1} \cdot r_{n+2} \cdots \cdot r_{n+m}) \]

or

\[ V = 2 \log \left( \frac{r_i \cdot r_2 \cdots \cdot r_n}{r_{n+1} \cdot r_{n+2} \cdots \cdot r_{n+m}} \right). \] \hspace{1cm} (10)

The conjugate potential function is found easily when one goes back to the simple case represented in equation (4) and remembers that \(\theta\) is the function conjugate to \(\log r\).
\[ w = \log r + i \theta \]
is a function of the complex variable \( z = x + iy \). Thus the conjugate of
\[
\log \left( r_1, r_2 \right) = \log r_1 + \log r_2
\]
is the function
\[
\theta_1 + \theta_2
\]  
and to \( \log \frac{r_1}{r_2} = \log r_1 - \log r_2 \)
corresponds the conjugate function
\[
\theta_1 - \theta_2
\]  
The conjugate of the function given in (7) is
\[
2m(\theta_1 + \theta_2 + \theta_3 + \cdots + 2\theta_{n+1})
\]  
And finally,
\[
U = 2(\nu_1\theta_1 + \nu_2\theta_2 + \cdots + \nu_m\theta_m + \mu_1\theta_{n+1} - \mu_{n+1}\theta_m) \quad \cdots (14)
\]
is conjugate to the function expressed in (10). It is possible
to combine these conjugate functions so as to form a function
of the complex variable \( z \). From (10) and (14)
\[
V + i U = 2 \sum_{K=1}^{m} \nu_K \alpha_K - 2 \sum_{E=M+1}^{2m+4} \mu_E \alpha_E + 2 \sum_{K=1}^{n+1} i \nu_K \alpha_K - 2 \sum_{E=M+1}^{2m+4} i \mu_E \alpha_E
\]
\[
= 2 \sum_{K=1}^{m} \nu_K (\log r_K + i \theta_K) - 2 \sum_{E=M+1}^{2m+4} \mu_E (\log r_K + i \theta_E)
\]
Hence to every arrangement of parallel wires each of which is homogeneous, either attractive or repellant, and of commensurable density, there corresponds an analytic function \( f(z) \) whose real part and imaginary part when set equal to arbitrary constants define respectively the curves of equal potential and the lines of force. For example consider the field of gravitational force about three wires of unit mass per unit length. Here all the wires are attractive and it is possible to write down at once the function

\[
f(z) = 2 \log \frac{(z-a_1)(z-\beta_1)^\mu_1 \cdots (z-\beta_n)^\mu_n}{(z-\beta_{n+1})^\mu_{n+1}} = f(x + iy)
\]  

Hence

\[
V + iU = \sum_{k=1}^{n} \lambda_k \log (z - a_k) - \sum_{\ell = n+1}^{m} \mu_\ell \log (z - \beta_\ell)
\]

\[
= \lambda_1 \log \frac{(z-a_1)^\mu_1\cdots (z-\beta_n)^\mu_n}{(z-\beta_{n+1})^\mu_{n+1}} = f(x + iy)
\]  

\[
\text{(15)}
\]

It is evident that the equation just obtained is of the sixth degree in the co-ordinates \( x \) and \( y \) and is not to be plotted.
Equipotential Lines for Three Equal Sources.
easily. The locus is known as the Cassinian oval of the third order. Likewise the current function has the form

\[ \mathcal{U} = \theta_1 + \theta_2 + \theta_3 \]

and the lines along which the force acts are given by

\[ \theta_1 + \theta_2 + \theta_3 = K \]

or

\[ \arctan \frac{y-b}{x-a} + \arctan \frac{y-d}{x-c} + \arctan \frac{y-f}{x-e} = K \quad \text{-----(17)} \]

Taking the tangent of both members, an equation is obtained which, when reduced, is found to be of the third degree in the variables \( x \) and \( y \). This locus is known as the hyperbola of the third order.

Examples. 1. Consider the field of force surrounding three long parallel conductors which pierce the \( z \)-plane at the three points \( A \equiv (1,0) \), \( B \equiv (0,1) \) and \( C \equiv (0,-1) \), when the current is the same, both in intensity and in direction, in all three wires. The adjoining figure shows a few of the lines of magnetic force. The analytic function associated with this example is

\[ w = \log (r_1, r_2, r_3) + i (\theta_1 + \theta_2 + \theta_3) \]

where \( r_1, r_2 \) and \( r_3 \) represent the distances from \( P \equiv (x,y) \) to \( A, B \) and \( C \) respectively and \( \theta_1, \theta_2 \) and \( \theta_3 \) represent the differences in amplitude between \( P \) and the points \( A, B \) and \( C \). Here the lines of force are given by the equation
\[ v_1 \cdot v_2 \cdot v_3 = c \]

where \( c \) is arbitrary. The curves are known as Cassinian ovals of the third order. The orthogonal set of curves, not shown in the figure, constitute what are known as hyperbolas of the third order. A bundle of these hyperbolic curves passes through each of the three points A, B and C.

2. If three springs of equal strength furnish water to a thin horizontal stratum of porous soil underlaid by an impervious stratum, and if the streaming through the soil is steady, the curves shown in the figure represent equipressural curves, while the lines of flow would be the hyperbolic curves mentioned in example 1.

3. Consider the field of force about two long parallel conductors when the current is twice as strong in one as in the other, the directions being opposite. Suppose that the wires pierce the \( (z) \) plane at the points \( A = +1 \) and \( B = -1 \) and let them be referred to as wire 1 and wire 2 respectively. Let the current in wire 2 be of unit strength and that in wire 1 be of two units strength. Then, as we have shown, the function which involves the potential function and the stream lines as conjugate parts may be written down at once as follows:

\[ w = \log \frac{(z - 1)^k}{(z + i)} \]

\[ \text{(1)} \]
Field of Force. Source A twice as strong as Sink B.
If the real and imaginary parts be separated, we have the relations

\[ U = \log \frac{r_i}{r_2} \]  

and

\[ V = 2 \theta_i - \theta_2. \]  

(2) gives the value of the potential at any point outside of conductors. Equation (2) may be written in the form

\[ \frac{r_i^2}{r_2} = e^U \]  

and when \( U \) is given a succession of values we obtain a set of equipotential curves in the \((x, y)\) plane; viz,

\[ \frac{r_i^2}{r_2} = m \]  

where \( m \) is an arbitrary constant. Equation (5) represents a family of Cassinian ovals of fractional order. The rectangular equation for (5) is

\[ \frac{(x - 1)^2 + y^2}{\sqrt{(x + 1)^2 + y^2}} = m. \]  

In the adjoining figure the point \( A \) is located in the Argand plane at the point \( z = 1 \) and the point \( B \) is located at \( z = -1 \). The current of double strength passes through \( A \) and the current of single strength through \( B \). The black curves surrounding the
points A and B represent a few of these ovals. The red curves represent some of the orthogonal set of curves whose equation is obtained from (3) and is

\[ 2\Theta_1 - \Theta_2 = \kappa. \]

The dark lines are lines of force and the red lines are lines of equipotential. At the point C, which is that point in the plane satisfying the relation

\[ \frac{2}{\text{CA}} - \frac{1}{\text{CB}} = 0, \]

there is situated what is known as a point of equilibrium. Here one of the lines of force crosses itself and the potential function becomes indeterminate.

4. An interesting case of streaming occurs when water is supplied to a horizontal plane at A at a rate twice as great as the rate at which it passes through a sink at B. Suppose that the plane extends to a considerable distance in all directions from A and B and that whatever water reaches the boundary of the plane runs off the plane and soaks away into porous soil. Then the red lines in the figure show lines of flow and the black lines are equipressure lines. At C there is no flow since here the equipressure curve has a double point and at this point the direction of the normal to the equipressure curve is indeterminate. Part of the lines of flow, which originate at A pass through B, the others extend outward indefinitely.
Sec. 8. The motion of Ground-water. In discussing the application of analytic functions to the motion of water, it is necessary to restrict our field of investigation as follows:

1. Those motions only can be studied which occur in a two-dimensional space. This is due to the two-dimensional nature of the variable $z$ which is the argument of the analytic function.

2. When the motion takes place through some medium such as soil we must require that the medium have uniform conductivity in all directions.

3. We must disregard the motions due to internal attractions of capillary nature.

Suppose that a horizontal impervious stratum is covered with a layer of homogeneous soil. Consider the horizontal stratum as the $xy$ plane. If at every point $(x, y)$ the depth of water ($h$) has a value which depends upon $x$ and $y$, the pressure of the water at the point $x, y$ will be proportional to $h$. Thus we may write $u = Kh = f(x, y)$, where $u$ denotes the pressure.

References.  
Lamb's Hydrodynamics Chap.IV. 
Basset's Treatise on Hydrodynamics Vol.1, Chap. V. 
Holzmüller in Zeit schrift für Math. und Physik  
Slichter Annual Report of the U. S. Geol. Survey (1899)  
Part II, P. 295.
If \( h \) is constant then the pressure is constant for every value of \( x \) and \( y \) and there is no tendency toward motion in the \( x-y \) plane. But if \( h \) varies continuously from point to point in the \( xy \) plane, we have the continuous function

\[
u = f(x, y)
\]

and there will be a tendency toward motion of water from points of higher pressure to points of lower pressure. Let us consider a simple case of streaming and seek the pressure function. Suppose at some point \( A \) on the surface of the water contained in a layer of soil the normal \( AL \) is erected. Pass a vertical plane through \( AL \) cutting the water surface in the curve \( ACB \). Suppose that the function \( u \) decreases in this section in the direction \( AB \), then the tendency toward motion will vary directly as the fall of pressure per unit distance. If \( AD = \Delta s \) and \( CD = \Delta u \) then the instantaneous velocity at \( A \) is given by

\[
u = \kappa \frac{1}{\Delta s} - \Delta u \Delta s = - \kappa \frac{\partial u}{\partial s}
\]

\[----(1)\]
The quantity $Q$ of water which would pass through a rectangle whose plane is perpendicular to the normal plane, whose altitude is $h$ and whose breadth is $b$ is given by

$$Q = -bk u \frac{du}{ds} \quad ------(2)$$

In particular, if the function $u$ is independent of $y$ so that all normal sections are alike, then by taking the axis OS as the $x$-axis we may write the equation (1) in the form

$$v_x = -k \frac{du}{dx} \quad ------(3)$$

and (2) becomes

$$Q = -bk u \frac{du}{dx} \quad ------(4)$$

If the conditions of source and sink are such as to maintain $Q$ a constant we have from (4) by integration

$$u^t = -2 \frac{Q}{bk} x + c \quad ------(5)$$

In the case of streaming electricity in parallel lines in a plane the expression for $Q$ would be

$$Q = -bk \frac{du}{dx} \quad ------(6)$$

the factor $u$ not appearing because we consider the quantity of electricity crossing a line having the dimension $b$. Thus for electricity equation (5) would take the form

$$u = -\frac{Q}{bk} x + c. \quad ------(7)$$
Examples. 1. Suppose that two pools of water are separated by a dam of porous structure and parallel vertical sides. Let water be pumped from pool II and discharged into pool I. The water levels will become different in the two pools and there will be a streaming of water through the porous dam. When a difference of head in the two pools is reached at which the water soaks through the dam at the same rate at which it is being pumped from II into I then the pressure function will have the form given in equation (5). Thus the curve AB will be a segment of a parabola.

2. Given a circular island having a horizontal impervious stratum below sea level. Let a circular well be situated in the center of the island and extend to the impervious stratum. If a pump takes water from the well and discharges it into the sea, the surface levels of the well and sea become changed and a streaming starts up in the direction of the well. By example 1, we have that the quantity $Q$ of water which soaks through the walls of a cylinder co-axial with the well and having the radius $r$ is

$$Q = -2\pi r ku \frac{du}{dr}$$  \hspace{1cm} (8)
when a state of steady streaming is established $Q$ will be constant $Q$, and the same for all cylinders. Upon integration (8)
gives

$$u^2 = -\frac{Q}{\pi} \log r + C$$

-------(9)

The corresponding streaming in electricity would be that which takes place when the positive electrode is a circle in the plane conductor and the negative electrode is a very small circle situated at the center of the former. In that case the potential function is simply

$$u = -\frac{Q}{\pi} \log r + C$$

-------(10)

Moreover it can be shown that to every example of streaming electricity in a plane with point electrodes as sources or sinks there corresponds one of under-ground streaming where wells may be considered as sources and sinks. The potential function in the case of electricity must be squared to become the pressure function for the case of water. Thus if a certain distribution of positive and negative electrodes in a plane conductor gives rise to the potential function

$$u = \nu_1 \log \nu_1 + \nu_2 \log \nu_2 + \cdots + \nu_n \log \nu_n$$

the same arrangement of wells of relative strength corresponds.

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ing to the strength of the electrodes will lead to the pressure function

\[ u^2 = \nu_1 \log \rho_1 + \nu_2 \log \rho_2 + \ldots + \nu_n \log \rho_n. \]

It is evident that the same system of curves may be used in both cases to represent the equipotential or equipressure curves; but the curve for which potential is \( c^2 \) would correspond to the curve for which pressure is \( c \).

We will next consider some analytic functions and their applications to the problem of water flowing through a thin layer of porous soil which is underlaid by an impervious stratum.

Examples. 1. Consider the transformation

\[ w = \cos h^{-1} \frac{z}{c}. \]  

This equation may be written in the form

\[ z = c \cosh w. \]  

Or

\[ x + iy = c \cosh (u + iv). \]

But

\[ \cosh u + iv = \frac{1}{2} \left( e^{u+i\nu} + e^{-u-i\nu} \right) \]

\[ = \frac{1}{2} \left( e^u \cos \nu + i \sin \nu + e^{-u} \cos \nu - i \sin \nu \right) \]

\[ = \frac{1}{2} (e^u + e^{-u}) \cos \nu + i \frac{1}{2} (e^u - e^{-u}) \sin \nu \]

\[ = \cosh u \cos \nu + i \sinh u \sin \nu. \]
Field of Force Defined by

\[ w = \cos \left( \frac{z}{\xi} \right) \]
Therefore \( x + iy = c \cosh u \cos v + i c \sinh u \sin v \) \( \ldots (3) \)

or \( x = c \cosh u \cos v \)

and \( y = c \sinh u \sin v \) \( \ldots (4) \)

Upon eliminating \( u \) we get

\[
\frac{x^2}{c^2 \cosh^2 u} - \frac{y^2}{c^2 \sinh^2 u} = 1 \quad \ldots (5)
\]

and eliminating \( v \), we get

\[
\frac{x^2}{c^2 \cos^2 u} + \frac{y^2}{c^2 \sinh^2 u} = 1 \quad \ldots (6)
\]

Equation (5) gives for constant values of \( v \) a set of confocal hyperbolas with foci at a distance \( C \) from the origin. The other equation (6) represents for constant values of \( u \), a system of ellipses having foci in common with the set of hyperbolas. Either set of curves may be considered equipotential curves while the other set is considered as the stream lines. For example, (1), if the line from \( A \) and extending to the right indefinitely is made a source of constant strength at every point as may be done by making it a long ditch in porous soil and keeping the water level constantly high; and if on the other hand the line from \( B \) to the left is made a constant sink by having it become a long ditch from which the water is continually drained, as by a pump, then the stream lines are arcs of the
confocal ellipses which start from points on the line $A^{\infty}$ and terminate in the line $B^{-\infty}$. In this case the hyperbolas are equipotential curves.

(2) Again if both $A^{\infty}$ and $B^{-\infty}$ are made to be ditches in which water is kept at high level while along the line $FG$ there is a ditch from which the water is drained so as to produce a sink, the stream lines are quadrant arcs of the confocal ellipses and the hyperbolas represent the equipotentials.

3. If a short ditch extends from $A$ to $B$ in a field of porous soil and the water is kept at a given height from an outside source, the water will soak into the soil surrounding it. The stream lines become quadrant arcs of the confocal hyperbolas. The equipotential curves now are the orthogonal set of confocal ellipses.

4. If the lines $A^{\infty}$ and $B^{-\infty}$ represent barriers which are impervious to water and which extend from the surface of the ground down to the horizontal impervious plane, and if a semi-ellipse such as $DGC$ is made a constant source by the high water ditch method while some semi-ellipse on the other side of the barriers is made a constant sink by the low water ditch method, a streaming through the soil will take place along the confocal hyperbolas.
Example 2. Consider the transformation
\[ z = w + e^{w} \]
or
\[ x + iy = u + iv + e^{u}(\cos v + i \sin v). \tag{1} \]
Here we obtain the two equations
\[ x = u + e^{u} \cos v \tag{2} \]
and
\[ y = v + e^{u} \sin v \tag{3} \]
Eliminating \( v \), we have
\[ y = \arccos \frac{x-u}{e^{u}} + \sqrt[2]{e^{2u} - (x-u)^2} \tag{4} \]
The elimination of \( v \) is somewhat more difficult and the work appears as follows:
From (2) and (3),
\[ e^{u} = \frac{x-u}{\cos v} = \frac{y-v}{\sin v} \tag{5} \]
Therefore
\[ \tan \nu \cdot (x-u) = y-v \]
or
\[ u \tan \nu - x \tan v = v-y. \]
Then
\[ u = \frac{x \tan \nu + v - y}{\tan \nu} \tag{6} \]
and substituting (6) in (2)
\[ x = \frac{v + x \tan \nu - y}{\tan \nu} + e^{u} \frac{v + \tan \nu - y}{\tan \nu} \cdot \cos \nu \tag{7} \]


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Field of Force Defined by

\[ Z = \omega + e^{\omega} \]
The figure on the opposite page represents the two sets of curves given by equations (4) and (7) when \( u \) and \( v \) respectively are given constant values. The oval shaped curves correspond to equation (4) and the radiating lines correspond to equation (7). Thus the stream lines and the equipressural lines for the case of a canal with long straight parallel banks which are impervious to water: If at the end of the canal as indicated by the points \( X \) and \( G \) the contents of the canal soak into porous soil underlaid by an impervious stratum, the stream lines will be of the type shown in ABC and DEF. If, however, the line LT is made a constant source by making a narrow ditch in which the water is kept at high pressure, and, if MK is made a narrow ditch from which the water is continually pumped, thus producing a sink along the line MK, then the nature of the streaming is different from that first considered in this example. Here the stream lines start from points on the line LG and terminate at some point on MK. In this case it is necessary to consider the lines MK and LG as being very long in comparison to the distance between them. Other illustrations of equipressure curves and stream lines for the case of water have been considered in connection with examples used to illustrate potential functions.
CHAPTER III.

THE LINES OF EQUAL CURRENT-STRENGTH AND CURRENT-DIRECTION.

Sec. 1. The Derived Function $f'(z)$. In an article entitled "Über einen Satz der Funktionen theorie und seine Anwendung auf isothermische Kurven systeme und auf einige Theorien der Mathematischen Physik." Holzmüller has made important extensions of the use of analytic functions in the field of Potential Theory. This chapter will be devoted to a discussion of Holzmüller's theory and the results obtained.

It will be necessary first to develop and discuss certain functional relations. The geometric significance of these relations will be investigated in order that they may be more intelligently applied to the physical problems.

Let $w = u + iv = f(z) = f(x+iy)$

then
$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

or
$$\frac{dw}{dz} = R \left( \cos \phi + i \sin \phi \right)$$

where
$$R = \sqrt{(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2} = \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}$$

Zeit schrift für Mathematik und Physik, 42: 217. (1897)
and \[ \Phi = \arctan \frac{\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial x}} = \frac{\pi}{2} - \arctan \frac{\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial x}} \]  \hspace{1cm} \text{(4)}

It is well known that if \( w \) is an analytic function of \( z \), \( \frac{dw}{dz} \) will also be an analytic function. Therefore \( R \cos \phi \) and \( R \sin \phi \) are both solutions of Laplace's equation. It will be convenient to have an analytic function which will involve \( R \) and \( \phi \) in the conjugate functions, but not both in the same function. A simple function which answers this requirement is

\[ \log \frac{dw}{dz} = \log R + i \phi \]

That \( R \) has a metrical significance is made evident as follows: since \( u \) and \( v \) are functions of \( x \) and \( y \), we have

\[
\begin{align*}
dw &= \frac{\partial u}{\partial x} \, dx + \frac{\partial u}{\partial y} \, dy \\
dv &= \frac{\partial v}{\partial x} \, dx + \frac{\partial v}{\partial y} \, dy
\end{align*}
\]

Squaring and adding,

\[
dw^2 + dv^2 = \left( \frac{\partial u}{\partial x} \right)^2 dx^2 + \left( \frac{\partial u}{\partial y} \right)^2 dy^2 + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} dx \, dy \\
&\hspace{1cm} + \left( \frac{\partial v}{\partial x} \right)^2 dx^2 + \left( \frac{\partial v}{\partial y} \right)^2 dy^2 + 2 \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} dx \, dy
\]

Making use of the Cauchy-Riemann relations,
\[ du^2 + dv^2 = \left[ \frac{\partial v}{\partial x} \right]^2 + \left( \frac{\partial v}{\partial y} \right)^2 dx^2 + \left[ \frac{\partial u}{\partial x} \right]^2 + \left( \frac{\partial u}{\partial y} \right)^2 dy^2 \]

But \( du^2 + dv^2 = ds^2 \) and \( dx^2 + dy^2 = ds^2 \), where \( ds \) and \( ds \) represent corresponding length elements in the \( w \) and \( z \) planes respectively.

Hence

\[ ds = R \cdot ds \]

Thus when a given curve \( c \) in the \( z \) plane is mapped in the \( w \)-plane into the curve \( C \) according to some functional relation \( w = f(z) \), the length of an element of the curve \( c \) is multiplied by the factor \( R \). Since this factor \( R \) is a function of \( x \) and \( y \) except in the trivial case of a linear function of \( z \) the amount of stretching or contraction which is necessary at the point \( z \), on curve \( c \) in order to deform \( c \) into \( C \) depends not only upon the form of the transforming function \( f(z) \) but also upon the position of the point \( z \). Since \( R \) is a function of position only and does not involve in any way the direction of the curve element, the relation (7) holds for all the curves which pass through \( z \). The angle between any two of these curves will be preserved by the
mapping. Therefore the triangle \( w, w_2 w_3 \) corresponding to the small triangle \( z, z_1 z_3 \) will be similar to it and the area of the former will be to the area of the latter in the ratio \( R^2:1 \).

The meaning of the term \( \phi \) can be seen in the following way. If the second of equations (5) be divided by the first the resulting relation is

\[
\frac{du}{dv} = \frac{\frac{2u}{dx} \frac{dx}{v} + \frac{2v}{dy} dy}{\frac{2u}{dx} \frac{dx}{x} + \frac{2v}{dy} \frac{dy}{x}} = \frac{\frac{2u}{dx} + \frac{2v}{dy} \frac{dy}{dx}}{\frac{2u}{dx} + \frac{2v}{dy} \frac{dy}{dx}}
\]

Then suppose that the slope of a given curve at \( z_1 \) in the \( z \)-plane equals zero, i.e. \( \frac{dy}{dx} = 0 \), and equation (8) reduces to

\[
\frac{du}{dv} = \frac{\frac{2u}{dx}}{2u} = \frac{2u}{dx}
\]

whence

\[
\arctan \frac{du}{dv} = \arctan \frac{\frac{2u}{dx}}{2u} = \phi
\]

We then reach the conclusion that \( \phi \) is the angle through which horizontal line elements in the \( z \)-plane are turned on passing into the \( w \)-plane. In like manner it may be shown that if the line element in the \( z \)-plane is vertical
so that

\[ \frac{dx}{dy} = 0, \]

\[ \frac{dw}{du} = -\frac{\frac{\partial w}{\partial x}}{\frac{\partial w}{\partial x}} \]

or

\[ \arctan \frac{dw}{du} = -\arctan \frac{\frac{\partial w}{\partial x}}{\frac{\partial w}{\partial x}} = \frac{\pi}{2} + \arctan \frac{\frac{\partial w}{\partial x}}{\frac{\partial w}{\partial x}} = \frac{\pi}{2} + \phi \]

When \( \phi \) is given some constant value as

\[ \arctan \frac{\frac{\partial w}{\partial x}}{\frac{\partial w}{\partial x}} = c \]

we have a functional relation which is satisfied by the co-

ordinates of every point in the z-plane at which, when the

z-plane is mapped into the w-plane, horizontal line ele-

ments are turned through the angle \( \phi \). But since the mapping

is conform all the line elements passing through those points

which satisfy (10) will be turned through the same angle \( \phi \).

There are then three types of changes which take place

when points in the z plane are mapped over into the w- plane:

1. The simple change of position according to the equa-
tion \( w = f(z) \).

2. The magnification which takes place at every point
(providing \( R \geq 1 \)).

3. The rotation of the system of radiating lines through
each point, through the angle \( \phi \).
Sec. 2. The Physical Significance of the Functions $R$ and $\phi$. Consider the particles of a thin film of matter spread out over a portion of a plane. Let them be considered as points in the $z$-plane. Then let the film be so distorted that the particles will be moved to their appropriate positions in the $w$-plane according to the relation $w = f(z)$.

In some parts of the film the particles will be crowded closer together while perhaps at other points the particles will be separated. If the surface density throughout the film was unity before the transformation, it will be represented at any given particle after the transformation by the reciprocal of the function $R$. Those points at which the surface density has some constant value satisfy an equation

$$\sqrt{(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2} = c,$$

where $c$ is some arbitrary constant. Now if the restraint is released and the particles are free to move, the tendency will be for the density to become uniform again. A flow of the particles will occur. The lines of flow will be $u = c$, while the equipressural lines will be $u = c$, $c$, being arbitrary. Suppose a case of stationary streaming is set up by making the points of greatest surface density constant sources, and points of least surface density constant sinks.
The lines \( v = c \) will continue to be lines of flow and along these lines there will be a decrease in density and a fall of potential. But the rate of the fall of potential along the line of flow will not be uniform in general, when \( dx \) and \( dy \) are both constant.

We should have the equation

\[
\frac{dv}{ds} = K
\]

or

\[
dv = Kds = KR dv
\]

for the determination of the locus of points at which the fall of potential is going on at the same rate. That is, when \( dx \) and \( dy \) are constant rates and \( dv \) stands for a constant rate of potential decrease, equation (1) can hold only when \( R \) has a constant value. It has been shown in the previous chapter that the derivative of the potential function with respect to a variable is the component of force exerted in a direction parallel to the axis of that variable. At all points represented by the equation

\[
R = \sqrt{\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2} = e^c
\]

the fall of potential is going on at the same rate as long as the conditions of the streaming remain unchanged. Hence we
have the following:

Theorem. Given the function \( u + iv = f(x + iy) \). If \( u = c \) and \( v = c \) represent respectively equipotential and stream lines, then the locus of the points at which the strength of flow is equal, is represented by the equation

\[
\mathcal{R} = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2} = c
\]

or, briefly, \( \log \mathcal{R} = c \), where \( c \) is arbitrary.

We may see the nature of \( \phi \) in the following way. Let \( u = c \) be the equation of a line of equipotential. Since \( u \) is a function of \( x \) and \( y \) we may write down the relation

\[
\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{2u}{}\frac{dy}{dx} = 0
\]

So that

\[
\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} = \cot \phi = \tan \left( \frac{\pi}{2} - \phi \right)
\]

But \( \frac{dy}{dx} \) represents the slope of a tangent to one of the system of curves \( u = c \). Hence for all those points in the curves \( u = c \) at which the slope of the tangent has a given constant value, the expression

\[
\phi = \arctan \frac{\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial x}}
\]
must be constant. We have then the following:

Theorem. Given \( u + iv = f(x + iy) \), where \( u = c \), and \( v = c \) represent equipotentials and lines of flow respectively. The locus of all points at which the direction of flow (perpendicular to lines of equipotential) is the same is given by the equation

\[
\phi = \text{const} \tan \left( \frac{\partial v}{\partial x} \right).
\]

A concrete example of this last function and its appearance in physics may serve to clarify our ideas. Suppose a large glass plate, perforated for the passage of a long straight electric conductor, to be strewn with a large number of compasses whose needles are short. When no current flows in the conductor it will be found that all the compass needles point northward. The magnetic field is uniform with the distant earth pole as the attracting pole. Now send a strong current through the conductor and in the plane of the glass there is produced a new field of force of such intensity that the previous field furnishes a negligible component. The compass needles have new directions. These directions are actually the tangents to the lines of force passing through the pivot points of the compasses. It has required a certain torsional force to turn each of these compasses through the
angle which its direction now makes with the north and south line. So if the analytic function

\[ w = f(z) \]

corresponding to this particular case, is given, the curves along which this torsional force is constant are given by

\[ \phi = \arctan \frac{\frac{\partial y}{\partial x}}{\frac{\partial u}{\partial x}} = c. \]

Sec. 3. Examples. 1. Obtain \( R \) and \( \phi \) for the function

\[ w = \log \left[ \frac{(z - z_1)(z - z_2)}{(z - z_1)(z - z_2)} \right]. \]

We have

\[ w = \log (z - z_1) + \log (z - z_2) \]

\[ \frac{dw}{dz} = \frac{1}{z - z_1} + \frac{1}{z - z_2} = \frac{2z - (z_1 + z_2)}{(z - z_1)(z - z_2)} = \frac{z + \frac{z_1 + z_2}{2}}{(z - z_1)(z - z_2)} \]

Since \( R \) is the modulus of \( \frac{dw}{dz} \), we have

\[ \frac{\partial}{\partial z} \]

\[ R = \frac{2\rho}{r_1 r_2} \]

where \( \rho, r_1, r_2 \) are the radii vectores drawn from the points \( z \), and \( z_1, z_2 \), \( \rho \) being measured from the centroid of \( z_1 \) and \( z_2 \).

From (1) \[ \log \frac{R}{z} = \log \rho - (\log r_1 + \log r_2) \]

and it is easy to write down the conjugate functions for each of these,
Therefore the lines of equal stream strength are represented by a family of Cassinian ovals of fractional order; viz,
\[
\frac{C}{\nu_0 \nu_\alpha} = c,
\]
while the lines along which the direction of streaming is the same are represented by the hyperbolas of fractional order,
\[
\psi - (\Theta_1 + \Theta_2) = c_x.
\]

2. Consider the function
\[
\omega = \log (z - z_1) (z - z_2) (z - z_3).
\]
Here the conjugate functions are third order Cassinian ovals and hyperbolas respectively. Let the origin be the centroid of the system \( z_1, z_2, z_3 \), and then
\[
z_1 + z_2 + z_3 = 0.
\]

\[
\frac{d\omega}{dz} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \frac{1}{z - z_3}
\]
\[
= \frac{(z - z_{2,3})(z - z_{1,3})(z - z_{2,3}) + (z - z_{1,3})(z - z_{2,3})}{(z - z_1)(z - z_2)(z - z_3)}
\]
\[
= \frac{3z^2 + z_1 z + z_2 z_3 + z_3 z}{(z - z_1)(z - z_2)(z - z_3)}.
\]
\[ \frac{\omega}{1^2} = \frac{3(z - y_1)(z - y_2)}{(z - y_2)(z - y_2)(z - y_3)} \]

where \( y_1 \) and \( y_2 \) are the roots of the quadratic expression in the numerator of (2). The conjugate functions may be easily obtained in the form

\[ \log \frac{R}{3} = \frac{\rho_1 \rho_2}{\eta_1 \eta_2 \eta_3} = c, \]

and

\[ \frac{\Phi}{3} = (\psi_1 + \psi_2) - (\theta_1 + \theta_2 + \theta_3) \]

Thus the curves of equal current strength are the Cassinian ovals of fractional (2) order with foci at the points \( z_1, z_2, z_3, y_1, y_2 \). The lines of equal current-direction are the hyperbolas of fractional order which pass through the points \( z_1, z_2, z_3, y_1, y_2 \).

3. In general the function

\[ W = \log (z - z_1)(z - z_2) \cdots (z - z_n) \]

where the origin is taken at the centroid of \( z_1, z_2, \ldots, z_n \), leads to the functions

\[ \log \frac{R}{\eta} = \frac{\rho_1 \rho_2 \rho_3 \cdots \rho_{n-1}}{\eta_1 \eta_2 \eta_3 \cdots \eta_{n-1}} = c, \]

and

\[ \frac{\Phi}{\eta} = (\psi_1 + \psi_2 + \psi_3 + \cdots + \psi_{n-1}) - (\theta_1 + \theta_2 + \cdots + \theta_{n-1}) = c \]
4. The function

\[ w = \log \frac{z - z_1}{z - z_2} \]

has for its first derived function

\[ \frac{dw}{dz} = \frac{1}{z - z_1} - \frac{1}{z - z_2} = \frac{z - z_2 - z + z_1}{(z - z_1)(z - z_2)} = \frac{z_1 - z_2}{(z - z_1)(z - z_2)} \]

The function \( R \) which is the modulus of \( \frac{dw}{dz} \) is expressed by

\[ R = \frac{d}{r_1 r_2} \]

where \( d = |z_1 - z_2| \) is a constant and \( r_1, r_2 \) are the bipolar radii vectores of the point \( z \) with respect to the poles \( z_1 \) and \( z_2 \). To find the function \( \phi \), we use \( \log R \); i.e.,

\[ \log R = \log d - \log r_1 - \log r_2 \]

and set down the conjugate functions which are

\[ \phi = \theta_c - (\theta_1 + \theta_2), \]

\( \theta_c \) being used to designate the constant difference in amplitudes of the fixed points \( z_1 \) and \( z_2 \). This case is of interest physically since it represents the case of streaming in a plane where a point source and a point sink have equal strength. As already stated in the previous chapter, the stream lines are the bundle of circular arcs passing from \( z_1 \) to \( z_2 \) while the equipotential curves are the orthogonal system of circles drawn about \( z_1 \) and \( z_2 \). From equation (5) and the definition of \( R \) it follows that the curves of equal current strength are represented by the equation.
The foci of the ovals coincide with the source and sink points. The dotted lines in the figure show the ovals for three values of the parameter $k$. Suppose that a positive electrode at potential $+e$ is situated at the right hand focus and that an electrode of equal negative strength $-e$ is situated at the left hand focus and suppose that the plane of the paper is a good conductor of electricity. When the circuit is closed and the streaming becomes steady the potential will be falling at a certain rate at every point of the outer oval and at another rate at every point of the next oval and so on. These ovals should not be confused with the lines of equal potential.

From (6) we see that the lines of equal current direction are hyperbolas $\Theta_1 + \Theta_2 = k' \Theta = \mathcal{K}$, (k, being a parameter).
Since the stream lines radiate in all directions from A and converge from all directions toward B, it is clear that all these hyperbolas pass through the points A and B. The points A and B do not both lie on the same branch of any one of these hyperbolas. This may be seen by inspection. Consider the location of all points of contact for a set of parallel tangents to the circles of flow drawn in the foregoing figure when the tangents have a given direction. To each circle there are two tangents having a given slope. These tangents lie on opposite sides of the circle. By selecting two such points of contact and tracing the hyperbolic curves which pass through them it will be found that one of these curves passes through A while the other passes through B.

An illustration of the occurrence of these hyperbolic curves comes from the field of magnetic force about two long conductors of electricity where the direction of flow is opposite in the two conductors and the currents of equal strength. Suppose the conductors to be vertical and to pass through a plate of glass which serves as a plane in which to study the field of force. Let the plane be covered with a large number of compasses having short needles. Send strong currents through the conductors so that the earth's component in the field may be negligible. The direction of a compass needle will indicate the direction of the field at its pivot point. Draw a line on
the plane to serve as a reference line. Adjust a set of the compasses on the plane so that their needles will point in a direction parallel to the reference line. It will be found that the compasses are situated along a segment of a hyperbola which passes through one or the other of the points where the conductors pass through the plane.

5. Consider \( w = \log \frac{(z - z_1)(z - z_v) \ldots (z - z_n)}{(z - \xi_1)(z - \xi_v) \ldots (z - \xi_n)} \).

Here we have \( \frac{dw}{dz} = \sum_{i} \frac{1}{z - z_i} - \sum_{i} \frac{1}{z - \xi_i} \).

\[ \frac{dw}{dz} = \sum_{i} \frac{(z - z_i)(z - z_v) \ldots (z - z_{i+1}) \ldots (z - z_n)}{(z - z_i) \ldots (z - z_n)} \]

\[ \frac{1}{z - z_i} \]

\[ \frac{1}{z - \xi_i} \]

\[ \frac{1}{z - \xi_v} \ldots (z - \xi_n) \]

\[ \frac{1}{z - z_v} \cdot \frac{1}{z - \xi_v} \ldots (z - \xi_n) \]

\[ \frac{1}{z - z_v} \cdot \frac{1}{z - \xi_v} \ldots (z - \xi_n) \]

* NOTE. In his discussion of this problem in Zeit schrift fur Math. und Physik, Vol 42, page 224, Holmuller makes a statement which my work shows to be erroneous. He says, commenting on the equation marked (8) in my work, "One can unite all three terms into a single fraction whose denominator is of second degree and whose numerator is of (2n-1) degree. The numerator can be factored so that the result is \( \frac{dw}{dz} = \frac{1}{(z - \xi_v) \ldots (z - \xi_n)} \)." The error lies in the fact that the degree of the numerator is 2n-2. The terms of degree 2n-1 cancel each other when the fractions are added together.
The degree of each of the two general parts of the numerator of (9) is 2n-1. The coeff. of \( z^{2n-1} \) coming from each part of the numerator is n. Performing the indicated subtraction, it is obvious that the \( z \) terms of degree 2n-1 cancel each other, leaving the degree of the numerator 2n-2. If we denote by \( g_1, g_2, g_3, \ldots, g_{2n-2} \) the 2n-2 roots of the function in the numerator of equation (9) we get

\[
\frac{d\omega}{dz} = \frac{\sum_{i=1}^{2n-2} (z - g_i)}{\prod_{i=1}^{n} (z - z_i) \cdot \prod_{i=1}^{n} (z - g_i)}
\]

The modulus of \( \frac{d\omega}{dz} \) may be written

\[
R = \frac{\prod_{i=1}^{n} \rho_i \cdot \rho_2 \cdot \rho_3 \ldots \rho_{2n-2}}{\prod_{i=1}^{n} \rho_i \cdot \rho_2 \cdot \rho_3 \ldots \rho_n \cdot g_1 \cdot g_2 \ldots g_n}
\]

(10)

And when this is equated to a constant parameter we have Cassinian ovals of fractional order \( \frac{2n-2}{2n} \) as the curves of equal stream strength. The amplitude of \( \frac{d\omega}{dz} \); viz, \( \phi \), is given by the equation

\[
\phi = \left[ (\psi_1 + \psi_2 + \ldots + \psi_{2n-2}) \right] - \left[ \theta_1 + \theta_2 + \ldots + \theta_n \right] (\lambda_1 + \lambda_2 + \ldots + \lambda_n)
\]

(11)

When \( \phi \) is given a set of constant values we obtain as the curves of equal current direction, hyperbolas of fractional order.
6. Consider the function

\[ \omega = -\log \left( \frac{(z-z_1)^{\mu_1} \cdots (z-z_m)^{\mu_m}}{(z-\xi_1)^{\nu_1} \cdots (z-\xi_n)^{\nu_n}} \right) \]

where the \( \mu_1 \) and \( \nu_1 \) signify the relative strength of the sources and sinks at the points \( z_1, \ldots, z_m \) and \( \xi_1, \ldots, \xi_n \), respectively.

In this case

\[ \frac{d\omega}{dz} = \sum_{i=1}^{m} \frac{\mu_i}{z-z_i} - \sum_{i=1}^{n} \frac{\nu_i}{z-\xi_i}. \] -----(12)

When these fractions are all added together in one fraction

the common denominator will be

\[ \prod_{i=1}^{m} (z-z_i) \cdot \prod_{i=1}^{n} (z-\xi_i). \]

The degree of the numerator will be \( m+n-1 \) and may be expressed as

\[ (z-\eta_1) (z-\eta_2) \cdots (z-\eta_{m+n}). \]

where the \( \eta \)'s are the roots of the function found in the numerator after reduction, i. e.

\[ \frac{d\omega}{dz} = \frac{(z-\eta_1) (z-\eta_2) \cdots (z-\eta_{m+n})}{\prod_{i=1}^{m} (z-z_i) \cdot \prod_{i=1}^{n} (z-\xi_i)}. \] -----(13)

Here we have

\[ R = \frac{\rho_1 \rho_2 \cdots \rho_{m+n}}{\rho_1 \rho_2 \cdots \rho_m \cdot \eta_1 \eta_2 \cdots \eta_n} = K \] -----(14)
for the curves of equal strength of current, and

$$\Phi = \sum_{\psi_1 + \psi_2 + \cdots + \psi_{n+1}} - \sum_{\theta_1, \theta_2, \cdots + \theta_{n+1}} \chi_{1+} \lambda - \sum_{\eta_{n+1}}$$

for the curves of the same direction of current.

7. Consider the streaming for the case where a source of double strength and two sinks, each of single strength, are situated in a plane, say at the vertices of an equilateral triangle. Let the source of double strength be at $z_1$. Our analytic function becomes

$$\omega = \log \frac{(z - z_1)}{(z - z_2)(z - z_3)}$$

Then

$$\frac{d\omega}{dz} = \frac{2}{z - z_1} - \frac{1}{z - z_2} - \frac{1}{z - z_3},$$

or

$$\frac{d\omega}{dz} = \frac{2(z - z_2)(z - z_3) - (z - z_1)(z - z_3) - (z - z_1)(z - z_2)}{(z - z_1)(z - z_2)(z - z_3)}$$
Equation (16)* reduces to

\[
\frac{dw}{dz} = \frac{z (z_2 - z_\lambda - z_3) + (z_2 + z_3 - z_\lambda - z_\mu)}{(z - z_1)(z - z_\lambda)(z - z_3)},
\]

or

\[
\frac{dw}{dz} = \frac{(z^2 - z_\lambda - z_3) \left( z - \frac{z_3 + z\lambda - z_\mu}{z - z_\lambda} \right)}{(z - z_1)(z - z_\lambda)(z - z_3)} \quad \text{----- (17)}
\]

Now the expression

\[
\frac{z_3 + z\lambda - z_\mu}{z - z_\lambda}
\]

is a complex number which divides the difference between \(z_3\) and \(z\) in the ratio

\[
\lambda = \frac{z_\lambda - z_3}{z - z_\lambda}
\]

If the points \(z_1\), \(z_2\), \(z_3\) are the vertices of an equilateral triangle \(\lambda = 1\) and we have for (17)

*NOTE. Here again Holz Müller makes a mistake by leaving out the coefficient \(a\) in the numerator. With that omission of course the degree of the numerator is second and his results would follow.


\[ \frac{d\omega}{dz} = (z^1 - z^2 - z^3) \left( \frac{z - z_m}{(z - z^1)(z - z^2)(z - z^3)} \right) \]

where \( z_m \) is the centroid of the points \( z_2 \) and \( z_3 \).

Thus \( K = \frac{\kappa \rho}{\rho_1 \rho_2 \rho_3} = C \),

and \( \Phi = (X + \Psi) - (\Theta_1 + \Theta_2 + \Theta_3) = C \)

are the lines of equal current strength and of equal current direction respectively. In this case the lines of flow emanating from \( z_1 \) are all collected at one or the other of the points \( z_2 \) and \( z_3 \). From the figure it is obvious that a portion of the plane bounded by a set of stream lines may be cut out and made the surface upon which a streaming takes place. If a barrier to the passage of matter across these lines of flow is constructed and \( z_1 \) is made a source of double strength while \( z_2 \) and \( z_3 \) are sinks of single strength, the streaming will go on just as if the region of its activity were infinite.
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