Hebbert
Some Circular Curves Generated by Pencils of Stelloids and their Polars
SOME CIRCULAR CURVES GENERATED BY
PENCILS OF STELLOIDS AND THEIR POLARS

BY

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Clarence Mark Hebbert

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Head of Department

Recommendation concurred in:*

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I. INTRODUCTION.

If \( f(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n \) is a polynomial in
\( z = x + iy \), the reals may be separated from the imaginaries, and
the expression put in the form \( f(z) = u + iv \), where \( u \) and \( v \) are
polynomials of the \( n \)th degree in \( x \) and \( y \) with real coefficients.
The curves defined by the equations \( u = 0 \) and \( v = 0 \) are called
stelloids by Lucas,\(^1\) rhizic curves by Walton,\(^2\) potential curves
by Kasner,\(^3\) and orthic curves by Brooks.\(^4\) The name given by
Lucas is based upon the fact that the asymptotes are concurrent
and divide the whole angle about their common point into \( n \) equal
parts. Kasner calls them potential curves because they satisfy
Laplace's differential equation and arise in connection with the
study of potential in physical problems.\(^5\) Much of Lucas' work

\(^1\) F. Lucas. Géométrie des polynômes, Journal de l'École Polytechnique, XLVI\(^{e}\) Cahier (1879), pp. 1-33. See page six of his article for the name stelloid and his reasons for using it.

\(^2\) W. Walton. Several papers in Quarterly Journal of Mathematics XI (1871).


was done from the latter standpoint. From a purely geometric view-point the name stellloid seems most appropriate for these curves, and it will be used in this paper.

The properties of stelloids have been set forth by the men mentioned above and by Fouret. Lucas showed (l.c.,p.8) that the polar of a point with respect to a stellloid is also a stellloid. Kasner (l.c.) and Emch proved this by different methods.

Emch has shown (l.c.) that a pencil of stelloids of order n + l is completely determined by n + l distinct real points (A) and their associates. The first polars of an arbitrary point (x'y') with respect to this pencil form a pencil of stelloids of order n, whose n^2 base-points (n real points and their associates) define with (x',y') a (1,n^2) correspondence in the Cartesian plane. The real part (1,n) of this correspondence is realized in a super-

posed complex plane by the transformation

\[ z' = z - \frac{(n+1) f(z)}{f'(z)} , \]

in which the roots of \( f(z) = 0 \) represent the points (A) and \( f'(z) \) is the derivative of \( f(z) \). The product of a pencil of stelloids of order \( n+1 \) and the first polar pencil of a point \((x',y')\) with respect to this pencil is a circular curve of order \( 2n+1 \) having an \( n \)-fold point at each of the circular points. This may be extended to the \( r^{th} \) polar pencil since the polynomials \( u, v, \Delta^r u, \Delta^r v \), satisfy the conditions of the general theorem on p. 4 of the Rendiconti article referred to above, viz., "if \( u, v, \) and \( r, s \) are two pairs of irreducible polynomials in \( x \) and \( y \) with real coefficients of degree \( m \) and \( n \) respectively, (and satisfy Laplace's differential equation) each of the curves \( ru + sv = 0 \) and \( rv - su = 0 \) contains the circular points as multiple points; the degree of multiplicity is identical with that of the polynomial of lowest degree among \( u,v,r,s \)." Then we can state the

**Theorem I.** The product of the pencil of stelloids of order \( n+1 \) and the pencil of \( r^{th} \) polars (of order \( n - r + 1 \)) is a curve of order \( 2n - r + 2 \), having an \((n - r + 1)\)-fold point at each of the circular points.

The following table shows the curves generated in this manner.

<table>
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<tr>
<th>( n )</th>
<th>1st polar</th>
<th>2nd polar</th>
<th>3rd polar</th>
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<th>2nd polar</th>
<th>3rd polar</th>
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K-1 K-2 K-3
The table, in which the numbers in the columns indicate the orders of the curves, shows at once that it is impossible to generate in this way a bicircular quartic, tricircular sextic, or, in general, an \( n \)-ic having an \( \frac{n}{2} \)-fold point at each of the circular points.

This is also evident from the fact that the products are of order \( m + n \), generated by two pencils of curves of orders \( m \) and \( n \), respectively, where \( n \geq m + 1 \), and the product has an \( m \)-fold point at each of the circular points.

Emch has also made a detailed study of the stelloids connected with the circular transformation \( z' = \frac{az + b}{cz + d} \), and, in particular \( z' = \frac{1}{z} \). This is an involutoric circular transformation and is shown to be identical with a Steinerian transformation \(^1\) based upon a pencil of equilateral hyperbolas. The pencil of stelloids in this case consists of the equilateral hyperbolas through the points +1 and -1 and their associates. The first polar pencil is a pencil of straight lines and the product of the two pencils is a circular cubic, an invariant cubic in the Steinerian transformation.

It is the purpose of this paper to consider the transformation \( z' = \frac{1}{z^2} \), which has the three cube roots of unity for double points and with which is connected the pencil of stelloids (cubics) through the three cube roots of unity and their associates. Some

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properties of the quintic generated by the pencil of cubics and the first polar pencil (equilateral hyperbolas) will be derived.

The transformation \( z' = \frac{1}{z} \) will also be studied and the general form of the product of the pencil of stelloids through the \( n + 1 \)th roots of unity and their associates, and the first and second polar pencil of any point \((x', y')\) will be determined. Some properties of the asymptotes and foci of these curves will be derived. This transformation is simply the contracted form of the general transformation \( z' = z - \frac{(n+1)f(z)}{f'(z)} \), for

\[ f(z) = z^{n+1} - 1. \]

For some of the work, use will be made of the following

**Theorem II.** The product of a pencil of curves and the second polar pencil of a point \((x', y')\) is identical with the polar of the product of the pencil and the first polar pencil of \((x', y')\). \(^1\)

For, let the pencil of curves be

(1) \( P + \lambda Q = 0 \), and the first and second polar pencils,

(2) \( \Delta P + \lambda \Delta Q = 0 \), and

(3) \( \Delta^2 P + \lambda \Delta^2 Q = 0 \), respectively,

The product of pencils (1) and (2) is

(4) \( P \cdot \Delta Q - Q \cdot \Delta P = 0 \), whose polar is

\( \Delta P \cdot \Delta Q + P \cdot \Delta^2 Q - \Delta Q \cdot \Delta P - Q \cdot \Delta^2 P = 0 \) or

(5) \( P \cdot \Delta^2 Q - Q \cdot \Delta^2 P = 0 \), which is identical with the

---

\(^1\) It is understood that the pencils are projective.
product\(^1\) of (1) and (3).

\[\text{II. Transformation } z' = z - \frac{3(z^2 - 1)}{3z^2 - 1} = \frac{1}{z^2}\]

Geometrically, this transformation represents an inversion, a reflexion, doubling of the angle and squaring of the absolute value. For it may be replaced by the two transformations, \(z'' = \frac{1}{z}\) and \(z' = z''^2\), whose properties are well known. Straight lines are reflected on the x-axis and their inclinations are doubled. The unit circle corresponds to itself but only the three points (1,0), \((-\frac{1}{2}, \frac{1}{2}\sqrt{3})\), and \((-\frac{1}{2}, -\frac{1}{2}\sqrt{3})\) are invariant. The three lines joining these three points and the origin are also invariant lines but not point-wise. An equilateral hyperbola, \(xy = c\), goes into the circle, \(2c(x^2 + y^2) + y = 0\), counted twice. If \((x', y')\) describes a straight line, the point \((x, y)\) describes a locus of the fourth order, since the points corresponding to \((x', y')^*\) are the base-points of the first polar pencil of \((x', y')\) with respect to the pencil of cubics (stelloids) through the three cube roots of unity and their associates.\(^*\) Since the two transformations \(z'' = \frac{1}{z}\) and \(z' = z''^2\) are con-

---

1 On products of projective pencils see:
Cremona, Theorie der Ebenen Kurven (German by Curtze, 1865)
Paragraph 50, ff.
Sturm, Die Lehre von den Geometrischen Verwandtschaften,

\* L. Cremona. Theorie der Ebenen Kurven (German by Curtze, 1865)
p. 120, Lehrsatz XI.

\** A. Emch. (I.e. p. 2.) pp. 8 and 12.
formal around all points except 0 and \( \infty \), the result of using both of them is conformal, i.e., finite singularities of curves are preserved in the transformation \( z' = \frac{1}{z} \). Infinite points, however, are transformed into singularities at the origin.

The pencil of cubics is \( u + \lambda v = 0 \) where \( u \) and \( v \) are the real and imaginary parts, respectively, of \( z^3 - 1 \), i.e.,

\[
(1) \quad u + \lambda v = x^3 - 3xy^2 - 1 + \lambda(3x^2y - y^3) = 0.
\]

The projective pencil of first polars is

\[
(2) \quad (x^2 - y^2)x' - 2xyy' - 1 + \lambda[2xyx' + (x^2 - y^2)y'] = 0,
\]

pencil of second polars is

\[
(3) \quad (x'^2 - y'^2)x - 2x'y'y - 1 + \lambda[2x'y'x + (x'^2 - y'^2)y] = 0.
\]

The product of (1) and (2) is, as we should expect from the general theory, a bicircular quintic

\[
(4) \quad (x'y - y'x)[(x^2 + y^2)x + 2x] + (y' - y)(3x^2 - y^2) = 0.
\]

The product of (1) and (3) is

\[
(5) \quad 2(x^2 + y^2)(xx' + yy')(x'y - xy') + (x'^2 - y'^2)y + 2xx'y' - 3x'y - 3x^2y + y^3 = 0,
\]

a circular quartic; the first polar of \( (x'y') \) with respect to (4), in agreement with Theorem II.

The product of (2) and (3) is the circular cubic

\[
(6) \quad (x'^2 - y'^2)[(x'y + xy')(x^2 + y^2) + y] + 2x'y'(x^2yy' + y^1y^3 - xx'y^2 - x'x^3 + x) - 2xx'y'y'(x^2 - y^2) = 0.
\]

This cubic belongs to the class discussed by Emch in the paper referred to on p. 4, and will not be studied here.

**The Quintic (4)**

Since there are no terms of the fourth degree in equation (4),
and \((x'y - y'x)\) is a factor of the fifth degree terms, the line 
\(x'y - y'x = 0\) is an asymptote. There is a double point at the 
origin and at each of the circular points, \(I\) and \(J\), at infinity.
The curve passes through the base-points of (1) and (2), viz., the 
points 
\[
(1, 0); \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right); \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right); \text{and } \left(\frac{\sqrt{1+x'^2+y'^2}}{2}, \frac{\sqrt{1-x'^2+y'^2}}{2}\right),
\]
At the first three points above, \(\frac{dy}{dx}\) has the values 
\[
\frac{-y'}{x'} = 1, \quad \frac{2y'}{2x' + 1}, \quad \text{and } \frac{2y' + \sqrt{3}}{2x' + 1},
\]
respectively. These show that the tangents at these three points, which are the points representing the three cube roots of unity, pass through the pole \((x', y')\). This follows directly from the fact that (5), the first polar of \((x', y')\) with respect to (4), passes through these three points. At the origin
\[
\frac{dy}{dx} = \frac{x' + \sqrt{x'^2 + y'^2}}{y' \sqrt{x'^2 + y'^2}} \quad \text{i.e., the tangents to the curve at the origin are } y = \frac{x' + \sqrt{x'^2 + y'^2}}{y'} - \frac{y'}{x'} X \quad \text{which are orthogonal. If } \theta \text{ is the inclination of either of these tangents, } \tan 2 \theta = -\frac{y'}{x'} \text{. Hence, to construct the tangents to (4) at the origin, join the origin to the point } (x', -y') \text{ and bisect the angles made by this line with the x-axis. The bisectors are the required tangents. These tangents form the only real degenerate conic of the pencil (2), and are obtained also by putting } \lambda = \infty \text{ in equation (2).}

This is sufficient to enable us to make a fairly accurate drawing of the curve. (See figure, p. 9).

Some of the properties of (4) appear more readily if it is 
put into the polar form
\[
(7) \rho^2 [\rho^2(x' \sin \theta - y' \cos \theta) - \rho \sin 3\theta + x' \sin 2\theta + y' \cos 2\theta] = 0.
\]

The factor \(\rho^2\) indicates again, that the origin is a double
point. If $x' \sin 2 \theta + y' \cos 2 \theta = 0$, or $\tan 2 \theta = -\frac{y'}{x'}$, one value of $p$ is zero. The others are obtained from

$$p^2(x' \sin \theta - y' \cos \theta) = \sin^2 \theta,$$

whence

$$p = \frac{1}{\sqrt{x'^2 + y'^2}},$$

provided we consider $\sin 2 \theta$ negative and $\cos 2 \theta$ positive. An interchange of signs would make $p$ imaginary.

(The fourth root arises from the fact that the functions of $\theta$ involve the square root.) The curve cuts one of the tangents at the origin in two points equidistant from the origin. These two points are the real base-points of (2) as may be verified by making use of the coordinates of the base-points as given on p. 8. Since the coefficient of $p^2$ within the bracket is zero, the sum of the three non-vanishing segments on any ray through the origin vanishes. The origin is therefore a center\(^1\) of the curve.

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\(^1\) See Kasner (l.c. p. 1), p. 397, for this definition of center.
More than this, the polar equation (7), gives us a hint as to the form of the equation of the product curve for \( n > 2 \). This will be discussed later.

**Quadruple foci of (4).**

Foci are sect-points of tangents from the circular points to a curve.\(^1\) The tangents at I and J are of the form \( y = ix + b \) and \( y = ix + c \), respectively. Putting \( b = \beta + i\alpha \), then \( (\alpha, \beta) \) is the only real point on the tangent, i.e., it is the focus. Substituting \( y = -ix + b \) in equation (4) we have

\[
(4i + 4ib^2x' + 4b^2y')x^3 + (4ib^3y' - 6b - 2i x' - 3b^3x' + 2y')x^2 \\
+ (2bx' + 2iby' - 5ix^4x' - b^4y' - 3ib^2)x + b^3 - b^2y' + b^3x' = 0
\]

The degree reduces to 3 because the circular points are double points. In order for \( y = -ix + b \) to be tangent to I, the coefficient of \( x^3 \) must also vanish, i.e.,

\[
4i + 4ib^2x' + 4b^2y' = 0, \text{ or } b^2 = -\frac{1}{x'^2 - iy'^2}.
\]

Hence the tangents at I are \( y = -i(x + \frac{1}{\sqrt{x'^2 - iy'^2}}) \). Similarly, the tangents at J are \( y = i(x + \frac{1}{\sqrt{x'^2 + iy'^2}}) \). These intersect in the four points, (two of them real)

\[
\left( \frac{1}{2} \frac{\sqrt{x'^2 - iy'^2} + \sqrt{x'^2 + iy'^2}}{\sqrt{x'^2 + y'^2}}, \frac{1}{2} \frac{\sqrt{x'^2 + iy'^2} - \sqrt{x'^2 - iy'^2}}{\sqrt{x'^2 + y'^2}} \right)
\]

which are the base-points of the pencil (2). We have seen that

the orthogonal tangents at the origin are the two lines of the real degenerate equilateral hyperbola of the pencil (2). Hence we may state the

Theorem III. The three degenerate equilateral hyperbolas of the pencil (2) are the tangents to the curve (4) at its double points, which are their vertices. The base-points of the pencil (2) are foci of (4).

Single foci of (4).

The quintic (4) has three double points and no other singularities. Its class is therefore $5(5 - 1) - 3 \cdot 2 = 14$. Since the circular points are double points we can draw from each of them only 10 tangents touching the curve elsewhere. The 100 intersections of these ten tangents are foci of the curve but only 10 of these are real. They belong to the 196 base-points of a pencil of curves of order 14. Each of the 10 tangents from I cuts each of the two tangents at J in two coincident points (double foci), thus yielding 40 double foci; similarly, the tangents from J determine 40 double foci. The tangents at I and J determine four quadruple foci (2 real) considered above, counting for 16 points. Thus we have accounted for $100 + 40 + 40 + 16 = 196$ base-points. To determine the real single foci, impose on equation (8) the condition that it shall have equal roots, i.e., that the discriminant shall vanish. To obtain the discriminant, take the derivative with respect to $x$ and solve the quadratic so obtained for $x$. Since the double roots of the cubic (8) must also be roots of its derived
equation, we reverse the process and substitute the roots of the derived equation in (8). The two expressions thus obtained are the two factors of the discriminant. These, set equal to zero, are \( b - ix' - y' = 0 \), and

\[
(9) \quad 54b^9(x' + iy')^3 + 27b^7(x' - iy')^2 - b^6(15ix'^3 + 117x'y^2 - 45iy'y^2 - 33y'^3) + 9b^5(x' - iy') - 54ib^4(x' - iy')^2 - 12b^3(x' + iy')^3 + 27b^2 - 27ib^2(x' - iy') - 4i(x' + iy')^3 = 0.
\]

Hence, the line \( y = -ix + ix' + y' \) is a tangent to the curve (4). The real point on it is \((x', y')\), the pole, which is therefore a focus. (As is well known, the corresponding value of \( c \) is \( y' - ix' \))

By equation (9), the other nine real single foci are so situated that the origin is their centroid and the product of their distances from the origin has an absolute value equal to unity. The former follows from the fact that the eighth degree term is missing; the latter is seen by dividing through by the coefficient of \( b^9 \), when the constant term reduces to 1.

Since the inverse of a focus is a focus of the inverse curve,\(^1\) the problem of finding the foci of (4) reduces to that of finding the foci of its inverse with respect to the origin, viz., a circular quartic

\[
(10) \quad (x^2 + y^2)[2x'xy' + y'(x^2 - y^2)] + y^3 = 3x^2y + x'y - y'x = 0.
\]

This does not simplify matters, however.

\(^1\) Bassett, (l.c. p. 10) p. 50.
Isotropic Coordinates.

The problem of finding foci is much simpler when the equation of the curve is expressed in isotropic coordinates. Put $z = x + iy$, $\bar{z} = x - iy$, or $x = \frac{z + \bar{z}}{2}$, $y = \frac{z - \bar{z}}{2i}$. Equation (4) becomes

\[ f = [(x' - iy')(\bar{z}^2 - 1)]z^3 - [(x' + iy')(\bar{z}^3 - 1)]z^2 - \bar{z}^2(x' - iy') + \bar{z}^3 = 0. \tag{11} \]

\[ \frac{\partial f}{\partial z} = 3[(x' - iy')(\bar{z}^2 - 1) - 2[(x' + iy')(\bar{z}^3 - 1)]z = 0. \tag{12} \]

The roots of (12) are $z = 0$, $z = \frac{2(x' + iy')(\bar{z}^2 - 1)}{3(x' - iy')\bar{z}^2 - 1}$. Substituting $z = 0$ in (11), we get $\bar{z} = x' - iy'$, i.e., $x', y'$ is a focus. Substituting the second root of (12) in (11), we have

\[ \frac{4}{z}(x' + iy')^3(\bar{z}^3 - 1)^3 + 27\bar{z}^2(-\bar{z} + x' - iy')[(x' - iy')(\bar{z}^2 - 1)]^2 = 0. \tag{13} \]

If in equation (13), $z$ is replaced by its equivalent, $-ib$, the result is identical with equation (9), as it should be.

III. Transformation $z' = z - \frac{(n + 1)(\bar{z}^{\frac{n+1}{n+1}} - 1)}{(n + 1)\bar{z}^{\frac{n}{n+1}} - 1} = \frac{1}{z'}$

The pencil of stelloids connected with this transformation is the pencil of curves through the $n + 1^{th}$ roots of unity and their associates. The transformation represents a $(1,n)$ corres-

---

5. C. E. Brooks, (l.c. p. 1) calls these conjugate coordinates. Cayley (Collected Works VI, p. 498) uses the name circular coordinates.
Correspondence between the pole \((x',y')\) and the \(n\) real base-points of the first polar pencil of \((x',y')\). To establish the equation of the pencil and first polar in polar coordinates we have

\[ u + iv = z^{n+1} - 1 = \rho^{n+1}\cos(n+1)\theta + i\rho^{n+1}\sin(n+1)\theta - 1 = 0. \]

The pencil of stelloids is

\[ (14) \quad u + \lambda v = \rho^{n+1}\cos(n+1)\theta - 1 + \lambda\rho^{n+1}\sin(n+1)\theta = 0. \]

The first polar pencil of \((x',y')\) is

\[ (15) \quad u_{1} + \lambda v_{1} = \rho^{n}x'\cos n\theta - y'\rho^{n}\sin n\theta - 1 + \rho^{n}\lambda(x'sin\theta + y'cos\theta) = 0. \]

The product of \((14\) and \((15)\) is

\[ (16) \quad \rho^{n}(c^{n+1}(x'sin\theta - y'cos\theta) - cos(n+1)\theta + x'sin\theta + y'cos\theta) = 0, \]

which may be written in the form

\[ (17) \quad \rho^{2n}(x'y - y'x) - \rho^{n+1}\sin(n+1)\theta + x'\rho^n\sin n\theta + y'\rho^n \cos n\theta = 0. \]

This shows that in cartesian coordinates, \((x^2 + y^2)^n\) is a factor of the terms containing \(x\) and \(y\) to degree \(2n + 1\) while the next highest power of \(x\) and \(y\) is \(n + 1\). Hence the

Theorem IV: The product of the pencil of stelloids determined by the \(n+1\)th roots of unity and their associates as base-points and the first polar pencil of a point \((x',y')\), is a circular curve \((16)\) having an \(n\)-fold point at each of the circular points and at the origin.

Also since \(x'y - y'x\) is a factor of the highest degree terms, and the terms of order \(n + 2\) to \(2n\) are missing, we have

Theorem V: The line \(x'y - y'x = 0\), joining the origin and the pole is an asymptote of the product \((16)\), if \(n > 1\). The sum of segments on rays thru the origin is zero, i.e., the origin is a center.
If in the second factor of (16) we put \( \rho = 0 \), we get
\[
x'\sin n\theta + y'\cos n\theta = 0,
\]
that is
\[
Theorem VI: The tangents to (16) at the origin are the lines
\[
y = x \tan \theta, \text{ where } \tan n\theta = - \frac{y'}{x'}.
\]
These \( n \) tangents divide the whole angle about the origin into \( n \) equal parts, beginning at
\[
y = x \tan \varphi, \text{ where } \varphi = \arctan\left(\frac{-y'}{x'}\right) = n\theta + 2kn.
\]
Making use of the values
\[
\theta = \frac{\varphi}{n} - \frac{2kn}{n}, \text{ (} k = 0,1,2,\ldots,(n-1) \text{)}
\]
we find that the curve cuts the tangents at the origin in the points
\[
\rho^n(x'\sin \theta - y'\cos \theta) = \sin(n+1)\theta, \text{ or } \rho^n = \pm \frac{1}{\sqrt{x'^2 + y'^2}},
\]
according as \( \cos n\theta \) or \( \sin n\theta \) is considered positive, i.e., the curve cuts in other real points all of these tangents if \( n \) is odd and cuts only half of them elsewhere if \( n \) is even. The points of intersection are the base-points of (15), as may be easily verified by substituting their coordinates in (15).

The tangents at the origin constitute the degenerate curve obtained by making \( \lambda = \infty \) in (15). A general theorem\(^1\) states that if two corresponding curves \( C^m \) and \( C^n \) in two projective pencils of curves have a common multiple point of multiplicities \( r \) and \( s \) (\( r < s \)) respectively, their product \( K \) has there a multiple point of order \( r \) and the \( r \) tangents of \( K \) are tangents to \( C^m \). We have here an example in which both \( C^m \) and \( C^n \) are the real degenerate members of the two pencils. In fact, each of them consists of straight lines through the origin, the \( C^n \) being the \( n + 1 \) straight lines through the origin obtained by making \( \lambda = \infty \) in (14).

Quadruple Foci of (16).

In rectangular coordinates, equation (16) is

\[(x^2+y^2)^n(x'y'-y'x')-(n+1)y[x^n+i\frac{\binom{n}{n-1}}{2}x^{n-2}y^2+\ldots+i\frac{\binom{n}{n-1}}{2}(n-2r+3)\ldots]} = 0.

The terms in the bracket are the even terms of the binomial expansion \((x+iy)^{n+1}\).

Any line \(y = ix + b\) thru the circular point \(I\) cuts the curve (18) \(n\) times at \(I\) and in \(n + 1\) other points. To make the line tangent at \(I\), impose the condition that it shall have \((n + 1)\) intersections at \(I\), i.e., the coefficient of the highest power of \(x\) must vanish when \(y = ix + b\) is substituted in (18). Making the substitution and picking out the coefficient of \(x^{n+1}\), we have the condition

\[(2n+1)x - 2nb^n_iy = i\frac{\binom{n}{n+1}}{2(n-1)}(n+1)i\frac{\binom{n}{n-1}}{2n} + \ldots \]

Since in the bracket \(i\) appears only to multiples of the fourth power, we have simply the sum of the coefficients of the even terms in the binomial expansion \((x+y)^{n+1}\). This sum is \(2^n\). Hence, equation (19) becomes

\[2^n+i^{n+1}x' - 2^nib^ny' = 2^n = 0, \text{ or } b^n = \frac{1}{i\frac{\binom{n}{n+1}}{2}}(x'+iy').\]

Setting \(x'+iy' = re^i\theta\), \(b^n = re^{i(-\theta-n+2k\pi)}\) the \(n\) values of \(b\) are \(b_k = r\sqrt{n} \cdot e^{i\frac{\theta+2k\pi}{n}}, k = (0,1,2, \ldots ,n-1)\)

The tangents at \(I\) are therefore

\[y = ix + b_k = i(x - \frac{1}{\sqrt{x'+iy'}})\]

\(^1\) Hagen, Synopsis der Höheren Mathematik, p. 64.
The form of $b_k$ shows that in the complex plane the values of $b$ represent $n$ equidistant points on a circle of radius $\frac{1}{\sqrt{n}}$. This is analogous to the location of tangents at the origin which divide the whole angle about the origin into $n$ equal parts beginning at the asymptote $x'y - y'x = 0$.

The tangents at $J$ are obtained from (21) by changing the sign of $i$, since a tangent through $J$ may be written in the form $y = -ix + b$. These are

$$y = -ix + \frac{b_k}{\sqrt{x'^2 + y'^2}} = -i(x - \frac{1}{\sqrt{x'^2 + y'^2}})$$

Conjugate tangents (21) and (22) intersect in the real points (for the same value of $k$)

$$(x, y) \equiv \left[ \frac{\sqrt{x'^2 + iy'^2} + \sqrt{x'^2 - iy'^2}}{2\sqrt{x'^2 + y'^2}}, i \frac{\sqrt{x'^2 + iy'^2} - \sqrt{x'^2 - iy'^2}}{2\sqrt{x'^2 + y'^2}} \right]$$

By substitution in (15), these may easily be shown to be the base-points of the first polar pencil (15). Hence the

**Theorem VII:** The tangents to (18) at the circular points (and at the origin) pass through the base-points of the first polar pencil (15).

Putting $\lambda = i$ in equation (15) we get the degenerate form

$$c^n(\cos n\theta + i \sin n\theta) = \frac{1}{\sqrt{x'^2 + iy'^2}}$$

which reduces to $\rho(\cos \theta + i \sin \theta) = \frac{1}{\sqrt{x'^2 + iy'^2}}$, or $y = ix + \frac{1}{\sqrt{x'^2 + iy'^2}}$ which is identical with the equation (21). Similarly, putting $\lambda = -i$, we get (22). This, in connection with the fact that the tangents at the origin constitute a degenerate curve of the pencil (15), gives us

**Theorem VIII:** Three degenerate curves of the pencil (15) which break up into $n$ straight lines have their singular points at the origin and the circular points $I$ and $J$, respectively, each of
which is an $n$-fold point of the curve (16), and these $3n$ lines are
tangents to the curve at the $n$-fold points.

**Single foci of (16).**

Introducing isotropic coordinates $z = x + iy = \rho (\cos \theta + i \sin \theta)$;

\[ \bar{z} = x - iy = \rho (\cos \theta - i \sin \theta) \]

there is $z\bar{z} = \rho^2$;

\[ z^{n+1} = \rho^{n+1} [\cos(n+1)\theta + i \sin(n+1)\theta] \]
\[ \bar{z}^{n+1} = \rho^{n+1} [\cos(n+1)\theta - i \sin(n+1)\theta] \]
\[ \rho \cos \theta = \frac{z + \bar{z}}{2}; \quad \rho \sin \theta = \frac{z - \bar{z}}{2i}; \quad \rho^{n+1} = \frac{z^{n+1} - \bar{z}^{n+1}}{2iz} \]
\[ \rho^n \sin n\theta = \frac{z^n - \bar{z}^n}{2i}; \quad \rho^n \cos n\theta = \frac{z^n + \bar{z}^n}{2} \]

Substituting these values in equation (16), it reduces to

(23) $f \equiv [(x' - iy')(\bar{z}' - 1)]z^{n+1} - [(x' + iy')(\bar{z}' + 1)]z^n + \bar{z}^{n+1} - (x' - iy')\bar{z}^n = 0.$

To find the foci, impose the condition on (23) that it shall
have equal roots in $z$.\(^1\) To do this, we get

(24) $\frac{\partial f}{\partial z} = [(x' - iy')(\bar{z}' - 1)](n+1)z^n - [(x' + iy')(\bar{z}' + 1)]z^{n-1} = 0.$

If a root of (24) is also a root of (23), it is a double root
of (23). Equation (24) has $(n - 1)$ roots $z = 0$. In order for
$z = 0$ to be a root of (23), we must have $\bar{z}n[\bar{z} - (x' - iy')] = 0$,
i.e., $\bar{z} = x' - iy'$, whence the pole $(x', y')$ is a focus. $\bar{z} = 0$ sig-
nifies merely that the origin is a multiple point. The remaining
root of (24) is $z = \frac{n[(x' + iy')(\bar{z}' + 1)]}{(n+1)[(x' - iy')(\bar{z}' - 1)\bar{z}' - 1]}$. To find the condition
that this shall be a root of (23) it is substituted in (23) giving
the condition

(25) $\rho^n[(x' + iy')(\bar{z}' + 1)]^{n+1} - \bar{z}^{n+1}(n+1)^{n+1}[ar{z} - x' + iy'][(x' - iy')(\bar{z}' - 1)] = 0.$

\(^1\) See Lery or Brooks, l.c. p. 14.
The highest power of $z$ in this equation is $(n + 1)^2$ and the next highest power is $n^2 + n + 1 = (n + 1)^2 - n$. Hence, for $n > 1$, the coefficient of the next highest power of $z$ vanishes and the origin is the centroid of the roots of (25), i.e., of the single foci of (23), or (16). Also the constant term of (25) arises in the first bracket and has the same coefficient, except for sign, as the highest power of $z$, i.e., the product of the roots of (25) is $\pm 1$, according as $n$ is even or odd.

We note in passing that if in (23) we set the coefficient of $z^{n+1}$ equal to zero, we get at once the double foci obtained in the preceding section.

**First polar of (16).**

By theorem I, the product of (14) and the second polar pencil of $(x',y')$ is the first polar of (16), viz.,

\[
(y^n+1)\left[(x'^2 - y'^2)\sin 2\Theta - 2x'y'\cos 2\Theta - n^2\sin(n+1)\Theta + (x'^2 - y'^2)\sin(n-1)\Theta + 2x'y'\cos(n-1)\Theta\right] = 0.
\]

Since the difference in degree of the two highest powers of $\rho$ is $n - 1$, for $n > 2$, the asymptotes are determined by

\[
(x'^2 - y'^2)\sin 2\Theta - 2x'y'\cos 2\Theta = 0.
\]

From this, $\tan 2\Theta = \frac{-2m}{1 - m^2}$, where $m = \frac{y'}{x'}$. Moreover, since $\tan 2\Theta = \tan 2(\Theta + \frac{n}{2})$, it follows that these are the lines joining $(x',y')$ to the origin and the line normal to it at the origin.

The tangents at the origin are determined by

\[
(x'^2 - y'^2)\sin[(n-1)\Theta + 2\kappa\Theta] + 2x'y'\cos[(n-1)\Theta + 2\kappa\Theta] = 0.
\]

since in (26) this is the condition for a root $\rho = 0$. From (28)
we get \( \tan(n-1)\theta = -\frac{2x'y'}{x'^2+y'^2} = \tan2[\arctan(-\frac{y'}{x'})] \),

or \((n - 1) \theta = 2^p + m^n = -2A + m^n \), where \(A\) is the inclination of
the line joining the origin and the pole \((x',y')\). For \(\sin(n + 1)\theta = 0\), \(\rho^{n+1} = \cos(n+1)\theta\), or \(\phi = 1\). Hence the curve (26) passes
thru the \((n + 1)\text{th}\) roots of unity. But the curve (16) with respect
to which (26) is the first polar of \((x',y')\), also passes through
these points. We have therefore the

Theorem IX: The lines joining the pole \((x',y')\) to the \((n+1)\text{th}\)
roots of unity are tangents to the curve (16).

IV. The general transformation \(z' = z - \frac{(n+1)f(z)}{f'(z)}\).

In the general case\(^1\) \(f(z) = u + iv = \frac{z^{n+1}}{z} = 0\).
This may be thought of as representing \(n + 1\) lines through the cir-
cular point \(I\). The pencil of stelloids is \(u + \lambda v = 0\) and the \(n + 1\)
lines are determined by the value \(\lambda = i\).

Similarly, for \(\lambda = i\), the first polar pencil \(u_1 + \lambda v_1 = 0\)
represents \(n\) lines through \(I\) and the base-points of the first polar
pencil. Also \(u - iv = 0\) and \(u_1 - iv_1 = 0\) represent sets of lines
through \(J\).

By the general theorem regarding multiplicities of products,

\(^1\) A. Emch (l.c. p. 2.), p. 2.

\(^2\) Compare C. Segre. Le rappresentazioni reali delle forme com-
p. 15, we then have (assuming that the general theorem applies to imaginary elements)

Theorem X: The product \(uv. - u.v = 0\) of the projective pencils \(u + \lambda v\) and \(u. + \lambda v\) has an \(n\)-fold point at each of the circular points and the \(n\) lines \(u. + iv\) = \(0\) are tangents to the product at \(I\) and the \(n\) lines \(u. - iv\) are tangents to \(J\).

Since the sect-points of \(u. + iv\) = \(0\) and \(u. - iv\) = \(0\) are the base-points of the pencil \(u. + \lambda v\) = \(0\), and these lines are tangents at \(I\) and \(J\), we have

Theorem XI: The \(n^2\) base-points of the first polar pencil \(u. + \lambda v\) = \(0\) are quadruple foci of the product \(uv. - u.v = 0\).

Among these are the \(n\) real base-points forming \(n\) real foci.

Since in the special cases treated, the pole is a focus, we might expect that the pole is also, in general, a focus. This, however, is not the case.

Equation 27, p. 10 of the article by Emch referred to above, is the equation of the product in general, viz.,

\[
(29) \ (x - x')(rv - su) - (y - y')(ru + sv) = 0, \text{ where } r \text{ and } s \text{ are } \frac{u^2}{n+1} \text{ and } \frac{v^2}{n+1} \text{ respectively.} \quad \text{The form of (29) gives us the}
\]

Theorem XII: The product curve is also the product of the pencil of lines \((x - x') - \lambda (y - y') = 0\), through the pole, and the pencil of circular curves \((ru + sv) - \lambda (rv - su) = 0\).

The line \(x + iy = x' + iy'\), joining the pole and the circular point \(I\), meets this curve in \(\lambda (ru + sv) + i(rv - su) = 0\), which is identical with the expression just above equation (4), p. 4
of that article, where it is shown to be equal to

\[(a_0 + ib_0)^{\frac{n}{2} + \frac{3}{2}} \prod (x + iy - z_j) \prod (x - iy - z_k) = 0.\]

Substituting the value of \(x = x' + iy' - iy\), the first two sets of factors become constants and the third one gives \(n\) values of \(y\) which are

\[(32) \quad y = \frac{x' + iy'}{Z_1} - z_k, \quad (k=1, 2, \ldots, n).\]

Hence two values of \(y\) cannot be equal unless two points of \(\tilde{z}_k\) coincide. In the special cases treated, \(\tilde{z}_k = 0\) so that the pole is a focus, but in the general case, we have

**Theorem XIII:** In general the pole \((x', y')\) is not a focus of the product curve.

This also follows directly, since \((30)\) is independent of \((x', y')\).

From the equation of the tangent line and equation \((32)\) the point of tangency is \((the\ contact\ is\ of\ order\ n-i)\)

\[(33) \quad \left(\frac{x_1}{2} + i\frac{y_1}{2}, \frac{x_1}{2} + i\frac{y_1}{2}\right)\]

for the case \(\tilde{z}_k = 0\). This point lies on the line \(y = -ix\). In the same way it may be shown that the point of contact of the tangent joining the pole to the circular point \(J\) lies on the line \(y = ix\). Hence the

**Theorem XIV:** In the special cases of sections II and III, the circular points, the pole, the origin, and the two points of contact of the tangents joining the pole to the circular points, are the vertices of a complete quadrilateral, i.e., the points of contact of these tangents are the associate points of the pole and origin.
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