Wohlenberg

Theory Of Energetic Activities For Fluid Carriers Through Channels Of Continuous Flow Transformers.
THEORY OF ENERGETIC ACTIVITIES
FOR
FLUID CARRIERS
THROUGH
CHANNELS OF CONTINUOUS FLOW
TRANSFORMERS

BY

WALTER JACOB WOHLLENBERG

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Recommendation concurred in:

Committee
on
Final Examination
THEORY OF ENERGETIC ACTIVITIES FOR FLUID CARRIERS THROUGH CHANNELS OF CONTINUOUS FLOW TRANSFORMERS.

- by -

W. J. WOHLENBERG.
PREFACE.

While still very young in the experience of applying theory to practice the author conceived certain modes of rotary energy transformers, involving fluid carrying media, which gave promise of having great practical advantages. The required transformations and motions did not at the time seem to be at variance with the general laws involved. In fact mental picture seemed so real to the author that others were thereby convinced of the feasibility of the undertaking and accordingly plans were drawn for the construction of apparatus with which to carry on experimental work in this respect.

During the period of construction of the apparatus, (covering about eighteen months of the years 1913-14), the author was continually occupied with the attempt to more exactly coordinate and apply the theories involved and after much intense thought the real nature, of the activities involved became more evident. As the solution of the problem was forced to light it became apparent that the original conception was fallacious. The original dream was shattered but this was compensated for by a knowledge of the truth and from this began the development of the theory which is herewith presented.

It is the exact fundamental theory involved in all such continuous flow transformers as are described in the introduction, and accordingly the work is submitted for study to advanced engineering students, and as a guide to engineers for the design of the transformers mentioned.

For the fundamentals of the dynamics involved the work
of A. G. Webster, "The Dynamics of Particles and Rigid, Elastic and Fluid Bodies", has been consulted, and for the thermodynamics the books, on this subject, of G. H. Bryan and G. A. Goodenough have been consulted. In a few instances equations have been taken bodily from these works and for these the reference is given in the foot notes.

University of Oklahoma, May 1, 1916.

W. J. Wohlenberg.
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**NOTATION.**

In dealing with energetic activities, of systems involving fluid energy carriers, dimensional, motional, energy and force coordinates are employed.

In studying the motions involved we consider the components thereof. These are represented by affixing a subscript, denoting the component, to the symbol representing the motion. For example the symbol \( q_\rho \) denotes the radial component of the absolute velocity \( q \) where \( \rho \) now symbolizes the radius.

In general velocity magnitudes have been represented by small letters and the corresponding accelerations by the same capitals.

Energy magnitudes, excepting certain force coordinates, have been represented by other capital letters, and the differential transformations between forms of energy have been denoted by combinations as \( dN\Phi \) where \( N \) and \( \Phi \) represent the forms of energy active in the transformation. With the positive sign \( \Phi \) is being transformed to \( N \) and vice versa for the minus sign.

**DIMENSION COORDINATES.**

<table>
<thead>
<tr>
<th>NAME</th>
<th>SYMBOL</th>
<th>FORMULAE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>( t )</td>
<td>( \text{---} )</td>
</tr>
<tr>
<td>Mass</td>
<td>( m )</td>
<td>( \frac{w}{j} )</td>
</tr>
<tr>
<td>Volume</td>
<td>( V )</td>
<td>( \frac{1}{V} )</td>
</tr>
<tr>
<td>Density</td>
<td>( \mu )</td>
<td>( \text{---} )</td>
</tr>
<tr>
<td>Area</td>
<td>( A )</td>
<td>( \text{---} )</td>
</tr>
</tbody>
</table>
### Differential Displacements

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absolute linear displacement</td>
<td>( ds )</td>
</tr>
<tr>
<td>Absolute and relative radial displacements</td>
<td>( d\rho, d\rho )</td>
</tr>
<tr>
<td>Absolute and relative axial displacements</td>
<td>( da, ds_a )</td>
</tr>
<tr>
<td>Absolute peripheral displacement</td>
<td>( ds_e )</td>
</tr>
<tr>
<td>Absolute tangential displacement</td>
<td>( ds, dx )</td>
</tr>
<tr>
<td>Relative tangential displacement</td>
<td>( ds_r )</td>
</tr>
<tr>
<td>Absolute normal displacement</td>
<td>( ds_n )</td>
</tr>
<tr>
<td>Relative normal displacement</td>
<td>( dy )</td>
</tr>
<tr>
<td>Displacement normal to plane of fluid flow</td>
<td>( dz )</td>
</tr>
</tbody>
</table>

### Direction Angles:

- of any radius with reference radius \( \xi \)
- of channels axis \( \Phi, \eta, \zeta, \theta, \sigma \) (See Art. 19 and 1st part of Chapt. IV.)
- of channels surface element with its axis \( \tau_1 \)
- of stream line with channels axis \( \tau \)
- of vertical plane, about horizontal axis, with rotational plane \( \beta \)
- of peripheral vector with projection in rotational plane of vertical vector \( \lambda \)

### MOTIONAL COORDINATES

<table>
<thead>
<tr>
<th>Velocities</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absolute fluid</td>
<td>( q )</td>
</tr>
<tr>
<td>Relative fluid</td>
<td>( r )</td>
</tr>
<tr>
<td>Absolute peripheral velocity of channel</td>
<td>( e )</td>
</tr>
</tbody>
</table>
Velocity components
\[ q_r, q_o, q_e, q_r, q_n, r_r, r_o, r_e, r, o, o, e, e_r, e_n, \]
\[
\begin{align*}
\text{See} & \quad \text{Symbol} \\
\text{Art. 26.} & \quad \text{FORMULAE}
\end{align*}
\]

Accelerations:

Absolute fluid
\[ Q \]
\[ \frac{dq}{dt} \]

Relative fluid
\[ R \]
\[ \frac{dy}{dt} \]

Absolute peripheral of channel

Acceleration components:

Absolute radial component of fluid
\[ Q' \]
\[ \pm Q' - \frac{q_e^2}{p} \]

Relative radial component of fluid
\[ \frac{dq_e}{dt} \]

Other components
\[ Q_a, Q_e, Q_r, Q_n, \]
\[ E = E_p = E_a = \ldots \cdot o. \]
\[ R_p, R_o, R_e, R, o, e \]

Acceleration due to gravity
\[ g \]

FORCE COORDINATES

Static head
\[ h \]

Pressure
\[ p \]

Temperature
\[ T \]

Mechanical force
\[ F \]

Machanical force components
\[ F_p, F_o, F_e, F_r, F_n. \]

Centrifugal constraint function
\[ I = \frac{1}{2g} \left[ q_e^2 - q_r^2 \right] \]

ENERGY COORDINATES

Potential energy
\[ L \]
\[ G + \Phi \]

Kinetic energy
\[ N \]
\[ \frac{1}{2} m q^2 \]
Gravitational energy  
Intrinsic energy  
Flow displacement potential  
Configurational energy  
Available energy  
Unavailable energy  
Dynamic energy  
Differential of energy as heat flow from external systems  
Differential of total action or work done by central system  
Differential of total compensation in external systems  
Specific heat at constant volume  
Specific heat at constant pressure  
Transformation of efficiency  
Transformation Symbols:  
Concealed differential of kinetic energy in action  
Other transformations:  
(MISCELLANEOUS SYMBOLS:  
Expansion factor  

<table>
<thead>
<tr>
<th>NAME</th>
<th>SYMBOL</th>
<th>FORMULAE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight of fluid flow per second</td>
<td>$C_2$</td>
<td>$-\frac{\mu}{g} Y_x^2$</td>
</tr>
<tr>
<td>Relative velocity component in channel's axis</td>
<td>$C_3$</td>
<td>$-\frac{\mu}{g} Y_x \cos \sigma \frac{de}{dx}$</td>
</tr>
<tr>
<td>Relative velocity component normal to axis</td>
<td>$C_0$</td>
<td>$-A \gamma \mu$</td>
</tr>
<tr>
<td>Gas constant</td>
<td>$\gamma_x$</td>
<td></td>
</tr>
<tr>
<td>Exponent for adiabatic of $PV^n$</td>
<td>$k = n$</td>
<td>$k = \frac{\gamma_x^2}{\gamma_Y}$</td>
</tr>
</tbody>
</table>

$\mu$ - viscosity, $g$ - acceleration due to gravity, $Y_x$ - relative velocity component in channel's axis, $\gamma_x$ - relative velocity component normal to axis.
INTRODUCTION

The simplest controllable mechanical motion of any rigid body, especially for high speeds, is its rotation about its center of mass. Engineers have begun to realize the practical importance of this and accordingly we have seen in the past few decades the remarkable growth of the steam turbines, centrifugal pumps, rotary air compressors and other such types of energy transformers.

This remarkably rapid practical development of the modern continuous flow rotary energy transformer, has not however been preceded, or accompanied, by the development of a general theory expressed in terms of coordinates directly applicable to the engineering solution of the activities involved. In this respect the theory being applied to those transformers, involving centrifugal forces as a means, is especially weak and poorly coordinated.

All continuous flow transformers, have the energy carrying medium flowing continuously through moving and stationary channels, in both of which energetic transformations may occur. Stationary channels now are merely the particular cases of those moving channels having constant velocity, in which the said constant velocity has the special magnitude zero. Therefore, a general theory of energetic transformations in moving channels is likewise applicable to the stationary channels of the transformers as, for these the coordinates representing the channels motion and its rate of variation of motion merely vanish from the differential equations of the activity.

This paper represents the completion of a theoretical
research started by the author during his fellowship in the engineering experiment station of the University of Illinois and the work is occupied with the attempt to present an exact and coordinated theory of the energetic activities through any such channels (having continuous flow of fluid energy carrying media through them) as are and may be used in energy transformers; and, furthermore, to express all quantities involved in terms of coordinates readily adaptable to the engineering solution of these activities.

The development of any general theory in terms of such usable coordinates involves however the broadest principles and laws of material nature and accordingly the opening sections of this paper are devoted to the exposition of these theories in as brief and concise a manner as seemed to the author to be consistent with the general scope of the work undertaken. Common conventionalities concerning the general exposition of these laws have been sometimes disregarded for what seemed to be an easier road to an understanding and appreciation of them.

It has been found convenient, for the purpose of exposition, to consider matter in a statistical sense and then define energy as that universal property possessed by all particles whereby every particle has itself a potentiality for causing a change in the material relativity of all matter within its energetic vicinity. Force in a direction has been defined as a condition of energy in that direction determining the intensity with which an activity caused by that force or condition would take place. Energy is thus postulated as an entity whose intensity or concentration depends on material relativity and which entity furthermore has within itself
the causal origin of its own activities. The cause of the activity is called force. An activity has then been described as **MONO-ACTIVE** **BI-ACTIVE** or **POLYACTIVE** according as to whether in the activity forces are considered as active between one or more different subdivisions of matter and it has been shown that for unit efficiency of transformation an activity must be monoactive.

With this conception of the nature of the activity and the general notion of potential the whole theory of transformations has been directly deduced from Newton's Laws of Motion and Hamilton's Principle. It has also, by this means, been found possible to draw a sharp and exact distinction between conservative and unconservative activities and furthermore it is shown that conservative activities only have potentials in that, by definition the potential implies an ultimately active subdivision of matter and this is exactly the condition of conversative activity. This mode of describing the activity has thus been found to admirably lend itself in coordinating the general theory of activity and the author therefore feels satisfied in having, as he believes, the ends justify the means. Indeed it is a question if the means do not in themselves become justifiable in that they afford a more convenient and systematic description of the activity.

An attempt has been made to devise such a system of transformation symbols that all of the component transformations of the activity can be systematically represented and investigated. This has been accomplished, in a measure at least, by means of symbols of the type $dN_s$ where the subscript denotes the form of energy undergoing a differential transformation to that form represented by the differential letter proper $N$. In such a system a
general equation of conservation for the adiabatic activity is written in the form
\[ dD = dG + dN + dW \]
where \( D \), \( G \), and \( N \) represent respectively, dynamic, gravitational and kinetic forms of energy and \( W \) represents work done or energy of external systems. \( \Phi \) represents the thermodynamic potential \( U + PV \) where \( U \), \( P \), and \( V \) represent respectively intrinsic energy, pressure and volume. This potential has been called the \textit{CONFIGURATIONAL ENERGY} instead of heat content by which name it is ordinarily known. This substitution of terms is justifiable from the general definition of energy. Indeed it is necessary for the sake of consistency, in that heat is in general understood to be a mode of transmitting energy whence the term heat content is erroneous. The energy compensated for in the action \( W \) is written with the triple index as \( N'' \) to denote the components \( dN_w, d\Phi_w, \) and \( dG_w \). The symbol for kinetic energy is used to imply that for the type of transformers considered the whole action is fundamentally \textit{ZERO ORDER} kinetic where the modifying term zero order indicates our order of magnitude of active parts. Furthermore it has been shown that transformations of the type \( dW_z \), where \( Z \) is not zero order kinetic energy, are more accurately represented by the combination \( dN'_z \) with \( dW_{N'} \), where \( N' \) represents an intermediate form of energy which is kinetic and never exists in quantity separately from \( N \).

To distinguish between the constraints determining the quantity of flow the former have been named \textit{DESTRAINTS} and the latter \textit{QUANSTRAINTS}. To define the quanstraints we need merely the coordinate area, which we have assumed to vary so that the equation of the mass continuity
area = \frac{\text{Weight per second}}{\text{relative velocity}} \times \text{density}

is continuously satisfied. It is however shown in the subject matter under continuity that for continuity of flow the fluids physical configuration must be capable of undergoing the variations imposed by the above relation. The above permissible variations are thus, for strict continuity and equilibrium, limited to the condition of \text{CONFIGURATIONAL CONTINUITY} and it is discovered that a critical magnitude of the channels surface angularity for such continuity is expressed by the relation

\[
\tan \theta_c = \gamma \left[ -c_z + \frac{dP}{dx} + \frac{\sqrt{(c_z - \frac{dP}{dx})^2 - \frac{c_z}{\gamma}}} {2c_z} \right]
\]

where for the section under consideration \( \theta_c \), represents said critical surface angularity. It is such that for \( \theta_c > \theta_c \), the flow can not conform to the channel or, configurational continuity is impossible. Of the other quantities \( c_2 = \frac{\mu}{\gamma} Y_x^2 \)

where \( Y_x \) represents the relative velocity (at the section under consideration) and \( \mu \) is the fluids density, \( c_z = \frac{\mu}{\gamma} Y_x \cos \sigma \frac{de}{dx} \)

where \( \sigma \) is the angularity of channels axis with its direction of motion and \( \frac{de}{dx} \) represents the rate of variation of the channels velocity with respect the axial position \( x \), and \( \frac{dP}{dx} \) and \( \frac{dP}{dy} \) represent respectively the rate of fluid pressure variation along the axis and normal to it. This formula is generally applicable and by means of it critical relations are arrived at for flow in diverging and converging channels for both increasing and decreasing space rates of pressure variation.

With the above relations as to quantatnts satisfied
it becomes the problem of the engineer either to determine the necessary destraint variations to accomplish a certain action with a given energetic transformation; to find what action will result from a given transformation over given destraints or last; to find the transformation required for a given action over destraints. It is therefore very important that to define the destraints a comprehensive and usable system of coordinates be adopted and accordingly the whole of chapter III is given to the development of such a system of defining coordinates. The absolute and relative motions of the fluid and the absolute motion of the vane or channel are considered. General equations in terms of the destraint angularity or direction, relative fluid velocity absolute or peripheral vane velocity are arrived at for the radial, axial, peripheral, normal and tangential components. Either the first three or the last two are sufficient to specify the magnitude of a vector but whereas tangential and normal components will be found particularly applicable in investigations of the action for different force distributions they will not be so for an investigation of centrifugal effects. Here the radial component is of particular significance and accordingly in these investigations vectors are specified by radial axial and peripheral components. In addition to these a system of gravity coordinates is developed for use in investigations in those particular cases wherein gravity has an appreciable effect on the activity. The reader interested only in the use of these coordinates, and to whom the matter of development is one of indifference, is referred in Chapter III directly to articles 25, 26 and 27. These articles with the brief exposition of the coordinates at the beginning of Chapter IV should determine exactly, in the mind of the reader, the quanti-
ties which they represent.

The normal components of the fluids motion are obviously never free of constraints unless the normal channel motion should happen to be coincident with the constraint free motion of the fluid. It is discovered in Chapter IV that the normal component of any action is always wholly compensated for in the zero order kinetic energy of the fluid. The tangential component of the action and motions may be free of constraint but when not so it has been shown that three distinct conditions of the tangential force or constraint distribution are conceivable. The force considered obviously has one point of application on the fluid particle under consideration. The other point of application is called the REACTING POINT and for the mentioned conceivable force distribution this point is considered first as wholly situated in adjacent fluid particles, secondly as situated wholly in the channel or vane surface and thirdly as being partially situated in both.

It is discovered for the first case that the whole action of the fluid is compensated for in the zero order kinetic energy of the fluid and that for zero tangential constraint (and hence least kinetic variation) the tangential action vanishes. For this class of action, when centrifugal forces have zero effect, the variation of the relative velocity $\mathbf{Y}$ for least action is represented by the expression.

$$\mathbf{Y}_x = \mathbf{Y}_r + \int \cos \sigma \, de$$

where $\sigma$ represents the constraint angularities with the direction of motion and $e$ their absolute velocity in the direction of motion. The actions and hence the terminal potential variations which will occur over a path are obviously a function of the path. In a spe-
cial case the variation \( \delta e \) vanishes and we have as the relative velocity equation for least action

\[ Y_x = Y_t = \text{constant} \]

which is likewise applicable to stationary channels where now \( Y \) represents the velocity of flow \( q \).

For an action of the second class the tangential action and hence the total action is in general only partially compensated for in the kinetic energy of the fluid. The remaining compensation obviously is in the potential energy of the medium and this part of the compensation is represented in general for a particle \( dm \) by the expression

\[ dL^2 = dm \int [\gamma \cos \sigma \, \delta e - r \, d\gamma] \]

where in the special case when \( \delta e \) vanishes we have

\[ dL^2 = - \frac{dm}{2} \gamma^2 \]

which is as above applicable to stationary channels when \( Y \) is replaced by \( q \). The right hand member is then exactly the variation of the kinetic energy \( N \).

An analysis shows that an action of the third class, if it actually did exist as described involves in adjacent fluid particles a difference (due to action) of their rate of variation of potential with the result that there would exist an irreversible flow of energy between them. Further analysis shows that if the law of least action is to be obeyed universally by all the parts of the active medium that, when the terminal states of the active mediums energy are exactly as they would exist if the activity occurred as above described, that the actual mode of the activity can not
be as so described, but is exactly as that of class II, where now the ideal medium is considered replaced by one possessing a RE-
STRAINED OR IMPERFECT AVAILABLE ENERGY. That is in the actual
activity the irreversibilities occur simultaneously with the action or, adjacent differential parts exist at the same potential whence there is HOMOGENEITY OF IRREVERSIBILITY. For such an activity between given limits of potential the actual available energy is
\[ \xi_n A_n^2 \] whereas for the ideal medium between the same limits it would be \[ A_n^2 \] where the factor \( \xi_n \) is a referred efficiency of the process and will be a minimum. If the activity is bi-
active or poly-active then for each successive order of parts there will in this sense be an arranging, for lower order activity, transform
ation whose efficiency may be represented by a term \( \xi_n \). The final available energy of an activity originating in the \( n^{th} \) or-
der will be \( \xi_n, \xi_n, \cdots \xi_n A_n \) where \( A_n \) is the available energy in the \( n^{th} \) order of parts for available arrangements in the \( n-1 \) order and where \( \xi_n, \xi_n, \cdots \xi_n \) are the arranging efficiencies which will each separately be minimums.

To study the centrifugal effects a CONSTRAINT FUNCTION
\[ I \] (whose magnitude is exactly equal to the increase of the rad-
ial component of kinetic energy \( N \) necessary to cause the projec-
tion into the rotational plane, of the motion under consideration, to be rectilinar) is introduced. It is found that \( I \) is otherwise exactly proportional to the magnitude of the radial force, or head, being exerted by a unit mass due to its curvilinear motion depend-
ing upon whether the unit considered is of volume or weight. The use of this function leads to particularly interesting and useful
results in the derivation of expressions representing the pressure and force heads existing in a column of fluid in such curvilinear motion as described. The rate of variation of head is represented by the expression

\[
\frac{dh}{d\rho} = \frac{2I}{\rho} + \left(\frac{\partial I}{\partial \rho}\right)_{q_e}
\]

where \(\rho\) and \(q_e\) represent respectively the radius and the absolute peripheral velocity of the fluid and where \(I\) is represented by the expression

\[
\frac{1}{2q} \left[q_e^2 - q_\rho^2\right]
\]

in which \(q_\rho\) represents the relative radial velocity of the mass under consideration. Again the head differential is reduced to a singularly comprehensive and useful expression as follows

\[
dh = V dP = 2I \frac{d\rho}{\rho} + \left(\frac{\partial I}{\partial \rho}\right)_{q_e} d\rho
\]

where \(V\) represent the specific volume of the fluid and the other terms are as above.

For rectilinear motion the centrifugal constraints vanish whence the above derivative likewise vanishes. It has the peculiar form in which the condition for which the first derivative vanishes likewise is the condition for the vanishing of all orders of derivatives. This forces the conclusion that in a centrifugal head there are no maximum or minimum points between the terminals. This is also otherwise obvious.

From the analysis disclosing the exact nature of a centrifugal head it is found that such a head does not in itself represent energy and that therefore if energy is necessary for the
compression it must be supplied either from the fluids own previously attained kinetic energy \( N \) or from an external source of energy as heat. The terms **SIMPLE** and **COMPOUND** centrifugal compression have been introduced to distinguish the constant volume compression of incompressible media, unaccompanied by a compensating reduction in the fluids radial component of kinetic energy, from the compression in which now the force head is compensated for by a reduction in the fluids radial component of kinetic energy \( N \) and thus represents actual energy stored within the fluid. It is disclosed also that compound compression may exist for incompressible media as well as for compressible media.

The terms **CENTRI-VANESCENT** and **CENTRI-FORCE** are introduced to distinguish the field of condition in which centrifugal force resultants vanish from that in which these forces exist and it is discovered that a critical ratio

\[
q_p = q_e
\]

exists such that if \( q_p > q_e \) the state of motion is centri-vanescent and if \( q_p < q_e \) the state of motion is in a centri-force condition. This critical ration leads at a section \( X \) to a critical magnitude \( Y_x^c \) of the relative velocity represented by the expression

\[
Y_x^c = \frac{e_x}{\sin \phi_x + \cos \sigma_x}
\]

where as above if \( Y_x \geq Y_x^c \) the motion is in a centri-vanescent state and if \( Y_x < Y_x^c \) it is in a centri-force state. These conditions are also respectively interpreted as a state of motion.
in which the tangential component of the centrifugal constraint vanishes and one in which such a constraint component exists.

Of particular importance to the general knowledge of designers is the analysis on "Constraint Variations Possible for Equilibrium During the Activity". It is discovered for instance that compound centrifugal compression is impractical and hence should be avoided if possible. Considerable light is shed on the manner of restraint variations for which equilibrium is possible with various types of activities.

These general rules combined with the developed exact theory, expressed in differential equations, directly applicable to any part of the activity, will directly lead to the design formulae for the solution of the complete activity. When this solution is arrived at it will furthermore be an engineering solution because all the qualities involved have been expressed in terms of dimensional coordinates for the restraints, constraints, rates of motion and the energetic and hence physical state of the fluids used as energy carrying media.

It should be noted as before stated that the general differential equations are applicable to stationary channels as well as to moving channels. For these applications the motion coordinate \( e \) and its variation \( de \) will vanish from the equations and the relative velocity through the channel will become identical to the absolute velocity of flow \( q \). We have thus for stationary channels the foregoing equations subjected to the conditions

\[
e = de = 0
\]

and

\[
\gamma = q
\]
In concluding the presented general theory the author feels that there is likewise a place for a work presenting numerous particular applications of this theory in which design formulae are arrived at from the differential equations here presented. The results arrived at from such design formulae will in general be multiplied by certain design factors and the deviations of these factor values from unity will depend both upon the exactness of the data, in use, of the properties of the fluid media and on the degrees of perfectness of the actual channels, and the flow through, with respect the ideal or reference channels and activities to which the calculations will directly apply. Accordingly the author looks forward with pleasure to the time when circumstances will be favorable for the further development of the more particular applications of this theory.
CHAPTER I.

ACTIVE ENERGY

1. GENERAL CONCEPTION OF RELATIONS OF ENERGY AND MATTER.

Energy is the name given to that entity, whatever it is, by virtue of which variations occur in the relativity of states of material existence. It is thus purely a relative quantity.

If now we choose to consider matter as indefinitely divisible into small particles, then we may choose as the ultimate subdivisions, of any system, a particle so small that at any instant its material relativity is wholly described by its relative position, the rate at which it is varying its relative position and the influence or force state which exists between it and neighboring particles.

A system whose relativity can thus be described is called a dynamical system and the forces active therein are called dynamical forces. In this sense therefore any system whatsoever may, in the ultimate analysis, be reduced to a dynamical system and the ultimate changes therein will be according to Newton's Laws of Motion. The rate of relative positional variation is exactly the relative velocity, and the potentiality for causing variations in the relativity of material arrangements by virtue of this velocity is called kinetic energy. The potentiality for causing variations in the relativity of material arrangements because of the influence or forces existing by virtue of the mean positional arrangements is called the potential energy of the system. The energy of any system has in general therefore the two components kinetic and potential, and in any state the sum of both or the total energy will be
measured by the magnitude of the variation of relative material arrangement caused from this state to attain some standard or reference state of material relativity.

Observing again the ultimate particles considered active in the transformation we may define energy as being the universal property possessed by all of these particles whereby every particle has itself a potentiality for causing variations in the material relativity among all particles within its energetic vicinity. If now these energetic particles become so placed relatively that the influence existing between them actually cause changes in their energetic proximities, that is in their mean relative arrangements, then an energetic activity is said to have taken place.

We have given different names to these activities and the direct influence causing them depending upon what order of subdivisions of matter we consider as being effected by the activity or active influence. For instance, to the activity causing a relative rearrangement of the atoms we have given the name chemical reaction, the activity causing a change in the rate of motion of the molecules we have said is due to transference of energy as heat and the influence causing the transference is temperature difference. When the parts of matter influenced in their relative positions are of the order of magnitudes of quantities with which we can directly deal we say there has been an activity of dynamic energy, or work has been done. The direct influence causing this activity is said in this case to be a mechanical or dynamic force or just a force. In all the cases mentioned however there have been forces involved and in the ultimate analysis these forces, as before noted, can in the
above sense be considered as reduced to dynamical forces. In the first case these forces should in this sense be considered effective between atoms, in the second case between molecules and in the third case between parts of matter of our own order of magnitude. In general, force in any direction is merely the condition of energy in that direction determining the intensity with which the energetic activity resulting from that force takes place.

Every energetic group or aggregate, as a molecule considered as made up of atoms or the atom considered as made up of electrons, possesses now, as before noted, such a property that by virtue of the energetic proximity of other such groups a relative influence exists between them. Furthermore these influences are of exactly the same magnitude when the relativity of the same or same kinds of sub-divisions of matter is the same. This property is a physical realization of the fundamental equation of energetic activity.

In a materially isolated system now, the quantity of matter must be constant, for, if matter is indefinitely divisible it is certainly indestructible. If furthermore this system is energetically isolated then since the quantity of energy within a given system is a function of the existent material relativity only and since in the considered system the quantity of material is constant it follows that only such variations of material relativity can occur as will leave the energy constant.

In transferring or transforming energy it is in general not possible to have the change occur entirely into the special energetic channel into which we desire, for the purpose, to direct this change but instead there will be energetic leaks into other forms of energy. For instance if it is desired to transform some
intrinsic energy possessed by a fluid to dynamic energy available to do work, the whole quantity of intrinsic energy transformed is greater than that which appears as dynamic energy. In the process part of the intrinsic energy has been transformed to energy of the same and possibly other forms at a lower potential. The ratio of the quantity of energy transformed into the desired energetic channel to the whole quantity transformed to that and other channels, during the process, is called the transformation efficiency of the process. To have an energetic transformation therefore with unit efficiency it is necessary that the total energy of the transformation appears in that form in which we desire to have it transformed. If however we consider energy as conservatively active then all processes have an absolute efficiency of unity as the absolute considered quantity of energy within the active systems remains the same.

Work is the name for the useful dynamic activity in parts of matter of our order of magnitude. We shall call these of the ZERO-ORDER of magnitude and in general we desire to change or rearrange these for some purpose. In order that these changes may be accomplished we must first avail ourselves of an amount of zero order dynamic energy equal to or greater than that required to produce these changes and secondly have a means for directing its activity so that these changes will be brought about by said activity. The act of rearranging material parts is called working and the accomplishment of the rearrangement work done. Work done thus represents exactly a variation of the zero order dynamic energy and hence work itself is merely another name for zero order dynamic energy.

2. CONVENIENT CLASSIFICATION OF ENERGETIC SYSTEMS.
Ordinarily we speak of a dynamic system as one in which the only activity is among a single order of magnitude of material sub-divisions and that sub-division of our own order of magnitude. We will as before noted call this the zero order of magnitude and for the sake of convenience will call the considered infinitesimal parts or particles as being of higher orders. Thus the first considered order of infinitesimal with respect to us will be called of the first order, the next of the second order, etc.

What we speak of as a dynamic system is therefore really a system whose active energy causes a variation in the material relativity in parts of matter of the zero order of magnitude only and certainly a better descriptive term for such is certainly **MONOACTIVE SYSTEMS OF THE ZERO ORDER**, meaning that force is active between parts of matter of one order only and that of the zero order.

A monoactive system of any order obviously can only exist ideally, as, there will inevitably always be internal activities caused which will result in motional disturbances of parts of matter of higher orders than those involved in the activity in question. These internal disturbances occur at the expense of external disturbances because of the general influence existing between all parts of matter within energetic vicinity of each other. We can therefore only approach the condition of monoactivity as an inattainable limit. Actually all activities are at least biactive. All monoactive systems would have efficiencies of transformation equal to unity and conversely in general in order to have an efficiency of unity the activity must be so constrained as to be monoactive.
An order of material sub-divisions may now be considered as being separately composed of a finite number of parts as molecules are of atoms. Activities may take the place among an order of parts which leaves wholly undisturbed the relative arrangement of the finite sub-divisions of this order of active parts. The arrangement may then be said to be such that the finite sub-divisional arrangements are not within the energetic vicinity of each other and hence no activity occurs due to them. When such an activity does not occur simultaneously with that of the aggregate or renewed aggregates which the activity forms we will still call it a biactivity even though the orders of active parts are considered as finitely related instead of infinitely. It will then be spoken of as a biactivity of whatever order of parts it is composed.

3. MOTIONS AND EQUILIBRIUM IN MONOACTIVE SYSTEMS.
Since in the ultimate analysis all activities may be considered as being reducible to forces active between one or more orders of magnitude of mass parts we have, as before stated, as the laws which all ultimate activities obey Newton's Laws of Motion. These are stated as follows:

I. A body persists in a state of rest or uniform straight line motion unless compelled by a force to change this state.

II. The change of motion produced is proportional to the force applied and in the straight line direction of this force.

III. To every action there is an equal and opposite reaction.

By applying these laws the fundamental equations of dynamic motion may be stated as follows:
\( (X_r - m_r \frac{d^2x_r}{dt^2}) = 0 \)
\( (Y_r - m_r \frac{d^2y_r}{dt^2}) = 0 \)
\( (Z_r - m_r \frac{d^2z_r}{dt^2}) = 0 \)

where \( X_r, Y_r, \) and \( Z_r \) represent the force components caused respectively by the acceleration components \( \frac{d^2x_r}{dt^2}, \frac{d^2y_r}{dt^2}, \) and \( \frac{d^2z_r}{dt^2} \) of the mass \( m_r \) or vice versa.

If we consider the particle \( m_r \) now to undergo an arbitrary displacement from its actual path such that the displacement components are \( \delta x_r, \delta y_r, \) and \( \delta z_r \) respectively, we have as the virtual work of the particles displacement said infinitesimal amount from the actual path;

\( (X_r-m_r \frac{d^2x_r}{dt^2}) \delta x_r + (Y_r-m_r \frac{d^2y_r}{dt^2}) \delta y_r + (Z_r-m_r \frac{d^2z_r}{dt^2}) \delta z_r = 0 \)

which, since it vanishes for the arbitrary particle \( m_r \) will likewise vanish for all particles. This equation is the mathematical statement of D'Alembert's principle and is exactly the fundamental dynamical equation for any system in dynamical equilibrium.

From this equation Hamilton's principle and the Principle of Least Action may be directly deduced but which, since it is the object here rather to study their application than their origin, we will merely state and interpret. *

Hamilton's principle is thus most briefly stated as

\( \delta \int_{t_0}^{t_e} (N-W) \, dt = 0 \)

where \( N \) and \( W \) represent respectively kinetic and potential

*See A.G. Webster, The Dynamics of Particles and of Rigid, Elastic and Fluid Bodies. Chapter IV.
energy and \( t \) the time. In *words* the expression is equivalent to stating that during the change of configuration between the time limits \( t_0 \) and \( t \), the path of the actual change will be such that with respect to any neighboring path, separated from the actual path by only an infinitesimal amount, the above variation vanishes, which can obviously only be true if this difference is either a maximum or a minimum. The principle is otherwise very simply stated in *words* as: the *time mean of the difference of the kinetic and potential energies is a minimum or; in natural dynamical systems all motions take place so as to tend to equalize the kinetic and potential energies.*

When the above equation is applied to dynamically conservative systems it reduces to the expression

\[
\delta \int_{t_0}^{t} 2N \, dt = 0
\]

or the integral of kinetic energy is a minimum on the actual path as compared to that on any other path between the same time limits. The variation of kinetic energy is compensated for in dynamical systems, as will later be shown, wholly in work done and hence the above expression is exactly equivalent to stating that the work done or the action will be a minimum. It is accordingly known as the *Principle of Least Action* or *Least Work.* Furthermore if over a given path in such a system the variation of kinetic energy is a minimum, then because \( F = m \frac{dq}{dt} \) where \( q \) represents velocity, the existing forces \( F \) will be a minimum and hence the natural path is likewise the *path of Least Resistance.*

These principles are now applicable to dynamical sys-
tems and since any monoactive system can in the ultimate analysis be reduced to a dynamical system we conclude that they likewise describe the motions in any monoactive system free of constraints. The motions need not however necessarily have to have three dimensional space to move in, as if for instance, we have planar motion merely one of the coordinates $x$, $y$, or $z$ vanishes from the equation of motion and hence the corresponding equation itself vanishes. The remaining equations will be exactly of the form they were and the same results will be arrived at as regards the equilibrium of unconstrained planar motion. To apply the equations to motions partially constrained we consider the components of motion at right angles. Obviously each component will have zero resultant in the direction of the others, and hence does not effect them. If the motion is free in any sense we may find at least one such component which is in the direction in which the constraints vanish. This component will then obey the foregoing laws for equilibrium.

4. NATURE OF POLY-ACTIVITY AND GENERAL MOTION OF POTENTIAL. In polyactivities we shall observe directly the parts of zero order of magnitude. In such activities now not only are the zero order parts undergoing variations of relativity but the first and possibly higher order of parts are likewise undergoing such variations. These are such as to cause a variation of the relative influences existing in the zero order parts and consequently they effect the motions of said parts.

The motions of any single order of parts may thus be considered to be constrained by the varying influence on them due to the varying relativity of other orders of parts, and for such
the motions are not completely described by the foregoing principles. We must accordingly introduce other coordinates in terms of which we can specify at least the quantitative effect of these varying constraints on the energy coordinates used to describe the energetic state of the system, and for this purpose we introduce the general notion of potential. It is fundamentally purely a relative term just as are all our other coordinates and the zero of its numerical magnitude is, as such, based on whatever order of parts are considered as being the ultimately active parts of our active system. Thus the potential temperature is so defined that it vanishes when the molecular potentiality for activity has been reduced to zero.

We are justified by experience in assuming that such potentials exist for all energetic systems but are limited to a knowledge of how to specify them, by a comparison of the activity which their variation would cause, with the zero order measurable physical effect which they produce in some material medium. We establish a unit of activity accomplished or action and a unit of measurable effect and when we discover such a coordinate (which is measured in terms of said measurable effect) that the available action of a kind between given limits of variation of said coordinate is a function only of these limits then we call it a potential. In the proper system of units this potential is then an exact measure of the quantity of energy available for said action above the state of energy considered zero when said potential is considered as vanishing.

Such a system of units is universally chosen whence; the chosen potentials of poly-active systems have relativity it, exactly the same properties as has the potential energy of the monoactive system with respect to it for; if we considered the
absolute value of the potential in both cases and both systems are conservative we will have the relations

\[ dN = -dW \]

and

\[ dH = -dL \]

where \( dN \) and \( dH \) are respectively defined as the total action of the mono and poly active systems caused respectively by the variations \( -dW \) and \( -dL \) in their potentials. The differential action \( dH \) now represent exactly the whole work done at the expense of potential energy \( L \). It may appear wholly as zero order kinetic energy \( N \) or may simultaneously with its generation be partially or wholly converted to zero order potential or other forms of energy.

If we consider as above that the potentials \( W \) and \( L \) are absolute with respect to the activity in question then our active systems must inevitably be considered as being conservative for; by the absolute potential of the energy involved in an activity, we imply that whole quantity which would be available for a kind of action by reducing to zero the relative energies of all orders of parts considered to be active in the activity. Any lower or higher orders of parts are thus considered as having in the existent condition no energy with respect said activity or at least they have zero influence on the activity and are thus regarded as being without the energetic vicinity of the activity even though physically they may actually be within the space of the activity. It follows that any activity wherein a definite number of orders of parts only are effective is conservative and thus has a potential and we conclude that the principle of conservation itself may be but relative.
Here again we are justified however by experience in assuming that energy is conservatively active and hence, if in the system of adopted coordinates the equation of conservation does not hold for the activity as we describe it we merely conclude that additional coordinates are necessary to completely describe the activity, that is, to take account of the ultimately active particles and parts.

Since now these potentials and the forms of energy they are a measure of having been determined from the physical state only of the medium, serving as the energy carrier, they are obviously functions of the physical state of the medium only. Furthermore since the magnitude of the potential at any and every point is an exact measure of the intensity of the energy at these points its variation will be, in the proper units, exactly the variation of the energetic intensity and this is exactly the negative of the force condition existing in this direction. For obviously energy flows from a higher to a lower intensity and the rate of difference of the intensity will be an exact measure of the magnitude of the influence causing the flow. This rate of difference is exactly the rate of variation of the potential and since by a decrease of this quantity the influence causing the flow is made existent we conclude that the force condition of energy causing activity is exactly the negative space rate of variation of potential. This conclusion can otherwise be arrived at directly from the equations of motion as follows.

The product force times displacement has been called work done or action and a dynamic force has been defined as equal
to the product of mass and acceleration where acceleration is the time rate of variation of velocity caused to said mass by said force. We have thus the differential action equal to

$$F ds = m \frac{dq}{dt} ds$$

where \(m, q, t,\) and \(s\), represent respectively mass, velocity, time and distance. The differential distance \(ds\) through which the mass \(m\) moves in time \(\Delta t\) will now obviously be equal to \(q dt\) whence by substitution

$$F ds = m q dq$$

where \(m q dq\) is called the differential of the kinetic energy of the mass \(m\) whence; the kinetic energy \(N\) is equal to

$$\frac{1}{2} m q^2$$

We will now reserve the process. Differentiating \(N\) with respect to \(s\) we have;

$$\frac{dN}{ds} = m q \frac{dq}{ds}$$

Where \(ds = q dt\) whence

$$\frac{dN}{ds} = m \frac{dq}{dt} = F$$

If a mono active system possesses a potential \(W\) it must, as before explained, be conservatively active whence

$$dN = -dW$$

and it follows

$$\frac{dN}{ds} = -\frac{dW}{ds} = F$$

where \(\frac{dW}{ds}\) represents the space rate of variation of potential. Hence for conservative monoactive systems at least the above relation is true and since the variations \(dL\) of the potential \(L\)
defined for polyactive systems are relatively the poly activity \( dH \) of exactly the same nature as the variation \( dW \) is to the action \( dN \) of the monoactivity we conclude that likewise for the polyactivity the relation

\[
F = -\frac{dL}{ds}
\]

exists where \( F \) is exactly the magnitude of the influence causing the action \( dH \), or exactly the force.

5. RATES OF POTENTIAL VARIATION FOR EQUILIBRIUM IN POLYACTIVE SYSTEMS. We will consider in this respect the time and space rates of variation of the potential. We have as Hamilton's principle the relation

\[
\delta \int_{t_0}^{t} (N-W) \, dt = 0
\]

or the time mean of the difference \((N-W)\) is a minimum. Similarly since relative the poly active system \( H \) and \( L \) are related as \( N \) and \( W \) in the mono-active systems we write the general equation of activity for poly active systems as

\[
\delta \int_{t_0}^{t} (H-L) \, dt = 0
\]

or the time mean of the difference \((H-L)\) is a minimum.

It should now be understood that this equation will be applied exactly to a system in which the energy is conservatively active whence; within this system the total energy \( H+L \) will remain constant. The equation does not imply that it will be impossible, by any means whatsoever, to reduce \( H \) below a certain minimum for; obviously if we put our present system in communication with a second in which the intensities \( H'' \) and \( L'' \) are separate-
ly less than $H'$ and $L'$ in the first system then obviously energy in both forms $H$ and $L$ will flow from the first to the second so that both $H'$ and $L'$ decrease and in addition there will in general be a flow between $H$ and $L$ at such a rate that the time mean of the difference $(H - L)$ for the activity will be a minimum.

Considering now the system conservative and such that $L_t$, of the system is greater than $L_{t_2}$ of the system after an interval of activity and hence $H_{t_2} > H_t$, then, if the above time mean is to be a minimum we must have $L_t > H_t$. If this is true then when the time mean $(H - L)$ is a minimum we will have the condition

$$-\frac{dL}{dt} = \frac{dH}{dt} = \text{maximum}$$

or the time rate of decrease of potential will be a maximum. The differential $dH_L$ denotes the transformation from potential energy $L$ to action. The above condition is true when

$$-\frac{d^2L}{dt^2} = \frac{d^2H}{dt^2} = 0$$

and

$$-\frac{d^3L}{dt^3} = \frac{d^3H}{dt^3} = \text{negative}$$

and hence these relations are the criteria for the condition of equilibrium of polyactive systems in terms of the time rate of variation of the potential $L$.

To determine the space rate of variation of potential for a change we consider the equation of least action as follows: We have

$$\delta \int_{t_0}^{t_1} N dt = 0$$

or the action is a minimum over the actual path. Since now $N = \frac{1}{2} m q^2$ the above condition is obviously fulfilled when $m \frac{dq}{dt}$
is a minimum at every instant. But \( m \frac{dq}{dt} \) is exactly equal to the dynamical force \( F \) existing on the path and this is as before shown the negative of the space rate of variation of potential. We have thus as the space rate of variation of potential for equilibrium the relation

\[
F = - \frac{dW}{ds} = \text{minimum}
\]

and as before similarly

\[
- \frac{dL}{ds} = \frac{dH}{ds} = \text{minimum}
\]

where as before \( dH_L \) denotes the transformation to \( H \) from \( L \).

The above condition is true when

\[
- \frac{d^2H}{ds^2} = \frac{d^2H_L}{ds^2} = 0
\]

and

\[
- \frac{d^3L}{ds^3} = \frac{d^3H}{ds^3} = \text{positive}
\]

Collecting we have as the criteria for the equilibriums of a poly activity the relations

\[
\begin{align*}
- \frac{d^2L}{dt^2} &= - \frac{d^2L}{ds^2} = 0 \\
- \frac{d^3L}{dt^3} &= \text{negative} \quad - \frac{d^3L}{ds^3} = \text{positive}
\end{align*}
\]

In a similar manner the relations

\[
+ \frac{dL}{dt} = \text{minimum}, \quad \text{and} \quad + \frac{dL}{ds} = \text{minimum}
\]

are arrived at for flow in the direction of an increasing potential. These conditions are true when

\[
\begin{align*}
+ \frac{d^2L}{dt^2} &= + \frac{d^2L}{ds^2} = 0 \\
\frac{d^3L}{dt^3} &= + \frac{d^3L}{ds^3} = \text{positive}
\end{align*}
\]

6. RATES OF AVAILABLE ENERGY VARIATIONS FOR EQUILIBRIUM IN POLYACTIVE SYSTEMS. The conditions of equilibrium may also be stated in terms of the available energy of the system where now the available energy is that part of the total energy
of transformation which is available for direct conversion into that kind of monoaicy activity desired and of which work or dynamical activity is the special form to which the term is usually applied.

Consider any two systems at different absolute potentials and having a transformation of energy between them. A given quantity of energy transmitted is definite in value no matter between what places the transmission occurs. The amount of action which may be obtained by said transmission is however dependent entirely upon the difference in those potentials, of the two systems, whose variation causes said action. It furthermore follows directly from the definitions of potential and activity that, with respect to said receiving system, the available portion of the energy transformed is represented by the product - total quantity of energy transferred times ratio of difference of said potentials, of the two systems to that same absolute potential of the transmitting system, whose variation causes said activity.

We may consider a system as having zero availability which exists in the lowest possible practical state of potential and the availability of any given system with respect to this will be the ratio of their potential difference to that same potential of the system under consideration.

We see from the foregoing that the considered absolute potential and the available energy of a given system vary together in the same sense and are a minimum at the same time and furthermore there is a functional relation between them. The conditions of equilibrium of activity in terms of the available energy and availability are therefore exactly as those in terms of the potential. Letting A represent the available energy and B the
unavailable energy and letting \( dB_A \) represent the differential transformation from available to unavailable energy we have as the condition of equilibrium for continuous activity

\[
\frac{dB_A}{dt} = \text{maximum} \quad \frac{dB_A}{ds} = \text{minimum}
\]

which conditions are true when

\[
\frac{d^2B_A}{dt^2} = \frac{d^2B_A}{ds^2} = 0
\]

and \( \frac{d^3B_A}{dt^3} = \text{negative}, \quad \frac{d^3B_A}{ds^3} = \text{positive} \)

Similarly for the transformation \( dA_B \) (forced flow to higher potential from lower) we have as before

\[
\frac{dA_B}{dt} = \text{minimum} \quad \frac{dA_B}{ds} = \text{minimum}
\]

which condition is satisfied when

\[
\frac{d^2A_B}{dt^2} = \frac{d^2A_B}{ds^2} = 0
\]

and \( \frac{d^3A_B}{dt^3} = \frac{d^3A_B}{ds^3} = \text{positive} \)

7. EQUILIBRIUM FOR ZERO ACTIVITY. A system may be in equilibrium in a state of inactivity when for infinitesimal variations of relativity the potential and hence available energy does not vary. For stable equilibrium the further condition is imposed that this energy must be a minimum for; an activity will obviously, if in progress, not cease until a finite continuance of activity would cause an increase, or at least not a decrease of said energy.

We have thus the conditions of stable equilibrium for inactivity represented mathematically as

\[
\delta L = \delta A = 0 \quad \delta^2 L = \text{Positive} \quad \delta^2 A = -\text{Positive}
\]

8. EQUILIBRIUM FOR FLUID FLOW. We will consider
here briefly the flow of fluids through channels. In general the velocity of flow will be varying along the channels and we will first consider the velocity increasing in the direction of flow.

If the source of fluid flow is kept at a constant potential then obviously the conditions of equilibrium of first order activity will be satisfied when the maximum amount of available energy is taken from it. The total energy and hence the available energy is proportional to the weight of the fluid. Therefore the pressure and velocities along the channel will assume such values that the maximum weight of fluid is passed through the channel from the source.

If the velocity decreases in the direction of flow the potential energy and available energy of the receiver are obviously being increased at the expense of kinetic energy of the fluid. The flow will be such as to have the rate of increase of available energy in the receiver a minimum, or, the pressures and velocities will assume such values as to cause the minimum weight of fluid to be passed through the channels to a receiver at a higher potential.

Suppose that the channels are moving. We are concerned with the relative velocities and their variations. Obviously the conditions of equilibrium will be satisfied exactly as stated in the foregoing paragraphs, for those cases in which the potential varies along the path, and the exchange of a dynamic energy between the fluid and the channel walls upon which the fluids motion is effective will be according to the principle of least action.
CHAPTER II.
ENERGY COORDINATES, GENERAL EQUATIONS, AND GENERAL METHOD OF
SOLUTION OF ACTIVITY.

9. VARIATIONS DURING ACTIVITY. We will consider a mass
particle $dm$ of a fluid medium during a differential activity. Dur-
ing such an activity there will in general be a variation $dN$
of its zero order kinetic energy $N$ because of the changed rate
of motion of its zero order mass center. This mass center will
likewise in general have a variation of vertical position or grav-
ity level so that its zero order positional energy will be varied
an amount $dG$. Its temperature $T$ and its volume $V$ will un-
dergo differential variations $dT$ and $dV$ whence the intrinsic
energy $U$, being the sum of the first order kinetic and potential
energies, will undergo a differential transformation $dU$. In addi-
tion to these variations of the particles energy there is, during
flow, a displacement variation which results in work done compensat-
ed for in the energy of external systems, but which is not apparent
in any of the above variations. To discover the magnitude and na-
ture of this energy variation we will investigate the following sys-
tem.

Referring to figure 1. $E$ represents a partition of
infinite extension and separating the space $D$ from the space
$C$. A weightless solid piston $B$ extends through an opening of
the partition in such a manner that the joint around said piston
is non-leakable and frictionless. The space $D$ contains a fluid medi-
um at pressure $P$, and if now by means of a force $F$ the piston
is forced a distance $S$ into the space $D$ against said fluid
pressure, then an amount of work $PAS$ will have been done in
displacing the fluid in said space, where $A$ represents the
area of the piston. This product is otherwise represented as \( PV \) where \( V \) is the volume displaced by the piston during its motion. Since now the piston is weightless and the space \( D \) is of infinite extension this work \( PV \) is compensated for neither in the pistons energy nor in the intrinsic energy of the medium. Considering the system at rest in the two limiting states, and since the piston may be assumed horizontal this work is likewise compensated for in neither the kinetic energy \( N \) of the medium nor in its energy \( G \). Nevertheless work \( PV \) is done within the space \( D \) during the displacement. This work is inevitably compensated for in the energy of space \( C \) and furthermore since this process is obviously reversible the compensation of each activity for the other in the spaces \( C \) and \( D \) represents wholly available energy.

Suppose now that instead of the piston \( B \) we forced a fluid medium through the opening. If \( V \) is the volume of the entering fluid then likewise the work \( PV \) will have been done at the expense of external energy in forcing said fluid into the space \( D \) and this work is not apparent in the intensities of the energy components \( G, N \) or \( U \) of the medium in said space. If now this fluid volume \( V \), forced into \( D \), is again forced out into \( C \), then obviously an amount of work \( PV \) will have been done by the medium \( D \) in forcing said volume \( V \) back to the space \( C \) and this again without any apparent effect on the energy components \( G, N, \) and \( U \). We conclude that with respect the space \( C \) the medium in \( D \) possesses for a volume \( V \) exactly a potential \( PV \) not apparent in the energy components \( G, N \) and \( U \) of said volume for; obviously in forcing \( V \) into the space \( D \) at constant \( G, N \) and \( U \)
this amount of work must be done and the compensating available energy from the space $C$ can be exactly restored again if the volume $V$ is forced back to $C$ at constant $G, N$ and $U$ and furthermore the quantity of work done is a function of the magnitude of the coordinates $P$ and $V$ only.

We can if we choose call the space $D$ the central system and the space $C$ the external system when obviously the flow of volume $V$ through the opening is analogous to the flow of fluid into any chamber containing fluid under pressure. In all such cases therefore an amount of work (compensated for in external energy and not apparent in the energy components $G, N, and U$ of the central system) will be done and; during the differential displacement in any chamber, to or through which fluid is flowing, the work $d(PV)$ will likewise be compensated for in external energy but will not be apparent in the variations $dG, dN$ or $dU$.

This quantity of work is, as before stated, obviously a function of the physical state of the substance only and hence the product $PV$ is of the exact nature of a potential. We shall denote it by $J$ and call it the FLOW DISPLACEMENT potential. The differential $dJ$ of this potential is thus represented by the expression.

$$dJ = d(PV) = PdV + VdP$$

Since now both $dJ$ and $dU$ are functions of the state variation only, their sum will be a function of the state variation only and hence exactly a potential variation. We will denote this sum by $d\Phi$ and since now $\Phi$ is exactly a measure of the total energy of medium $D$, with respect to space $C$ (fig. 1.), defined by the thermodynamic configurational coordinates $(P, V, T)$ of
the medium, we will name it the **CONFIGURATIONAL ENERGY OR POTENTIAL**. It is otherwise known as the heat content but certainly this substitution of terms is justifiable for, by our definition, energy is the potentiality for causing variations in material relativity and certainly the part \((PV)\) of this potential is just as much such a potentiality as the intrinsic energy \(U\) itself. It is exactly a potentiality by virtue of zero order relativity just as \(U\) is a potentiality by virtue of first order relativity. We have thus

\[
\frac{d \Phi}{d \Phi} = dU + dJ = dU + d(PV)
\]

where \(d \Phi\) represents exactly (when \(d G\) and \(d N\) vanish within a chamber or channel too or though which the fluid is flowing) the total differential compensation for the differential external activity caused by the fluid mediums differential energy variation.

If now we have a physically inclosed space so that fluid can neither enter nor leave it then the differential \(dJ\) vanishes for; as before stated the potential \(J\) is with respect external systems, and if these be physically isolated from the central system there can obviously be no physical communication between the fluid of said central system and that of external systems and hence no variation of \(J\) whence \(dJ\) vanishes. For such a system all of the effects are apparent and accounted for in the variations \(dG, dN,\) and \(dU\) of the mediums energy.

10. **APPLICATION OF PRINCIPLE OF CONSERVATION.**

In terms of the above coordinates the sum total of the differential activity within the central system which is compensated for in external energy is thus represented by the sum
\[ dN + dG + d\Phi + \ldots = 0 \]

and this sum furthermore represents exactly the differential of the total potential defining the energetic state of the material relativity, for the activities having only first and zero order parts active.

As the activity is external systems compensating for the above differential we have the differential heat flow \( dM \) between the systems and, since work is exactly the dynamic activity between systems, we have as the remaining portion of the compensating activity, \( dW \) the external work done.

We will observe the energy of external systems for the heat flow and work done and the energy of the central system for all other transformations. Work done and heat flow will thus be considered positive when increasing the external energy and all other forms when increasing the energy of the central system. If now \( dK \) represents the total external compensation it will obviously have the components \( dM \) and \( dW \). We have by applying the principle of conservation

\[ -dK = -(dM + dW) = dN + dG + d\Phi \]

Any frictional work will directly be transformed back as low grade heat to the energy of the central system and hence it will be necessary to increase the amount of work for a given transformation by the exact amount of the frictional loss. This will reduce the amount of heat flow \( -dM \), necessary for a given \( -dK \), by exactly this amount whence the frictional transformations cancel out of the conservation equation. Equation 7 is thus
true for all processes under consideration whether with or without frictional losses.

Zero order dynamic energy $D$ is wholly available for conversion to work as is likewise the energy of volumetric displacement $J$ but the intrinsic energy $U$ is not so. We will, as before, let $A$ represent and available and $B$ unavailable energy. Then

$$dA = dG + dN + dJ + dA_u$$

and

$$dB = dB_u$$

where $dA_u$ and $dB_u$ represent respectively the portion of the differential $dU$ converted to available and unavailable energy.

11. COMPONENT TRANSFORMATIONS AND THEIR SYMBOLS. In addition to the transformations represented by the various differentials above, activities may occur between the various forms of energy which they represent. In order to represent these we will use combinations of letter as $dW_L$, $dL_w$, $d\Phi_w$ — where the two letters following the differential symbol $d$ denote what forms of energy the transformation is between. If the differential is negative as $-dW_L$ then the differential letter as $W$ denotes the form of energy being transformed into that form denoted by the subscript as $L$. If the differential is positive as $+d\Phi_N$ then the differential letter as $\Phi$ denotes the form of energy being increased at the expense of that form denoted by the subscript as $N$.

With these conventions such indentities as the following obviously exist:
\[ -dW_L \equiv +dL_w \]
\[ -dW_N \equiv +dN_w \]
\[ -dW_\Phi \equiv +d\Phi_w \]
\[ -d\Phi_N \equiv +dN_\Phi \]
\[ -dA_B \equiv +dB_A \]  
\[ +dW_L \equiv -dL_w \]
\[ +dW_N \equiv -dN_w \]
\[ +dW_\Phi \equiv -d\Phi_w \]
\[ +d\Phi_N \equiv -dN_\Phi \]  

These identities show how any transformation may be written in two different ways depending upon whether the observer is primarily noting the decreasing or the increasing forms of energy. Thus in group 9b the transformations of group 9a have been represented as occurring in the opposite sense. The transformation \[ +dA_B \equiv -dB_A \] has not been included in group because it represents a process in which the available energy is increased, and of course in an isolated system such a transformation is impossible.

12. GENERAL METHOD OF SOLUTION OF ACTIVITY WITHIN TRANSFORMER. A system involving an energy transformer, as described, is capable of three principle variations. The sets of conditions which may be so varied are the mechanical constraints, the terminal differences of total potentials and the distribution of the energy in its present existing forms.

The mechanical constraints have really two independent components of variations; those determining the direction of flow and called hereafter the DESTRAINS and those determining the quantity of flow as cross sectional areas and called hereafter QUANSTRAINTS.
We will now present the general method of solution for the case in which there is zero heat flow between the central and external systems. For this case we have \( dM = 0 \) whence \(-dK = -dW = dN + dG + d\Phi\). During an interval of time \( dt \) in which there is activity the total transformation is represented as follows:

\[
\begin{align*}
-dK &= dN + dG + d\Phi \\
dN &= dN_w + 0 + dN_\gamma + dN_\Phi \\
dG &= dG_w + dG_N + 0 + dG_\Phi \\
d\Phi &= d\Phi_w + d\Phi_N + d\Phi_\gamma + 0
\end{align*}
\]

Adding the three component equations we have because of identities \( g_a \) and \( g_b \)

\[
\begin{align*}
-dK &= dN + dG + d\Phi = dN_w + dG_w + d\Phi_w \\
-dW &= -dH
\end{align*}
\]

The differential activity is now expressed in terms of the variables \( dK, dN, dG, d\Phi, dN_w, dG_w, d\Phi_w, d\Phi_N, d\Phi_\gamma \) nine in number and we have five equations of condition. For a general solution we must therefore discover three additional determining conditions.

One such condition is offered by the path or restraint equations. This with given restraint and relative velocities will determine the work done or action over said path and which will be according to the law of least action.

A second condition is offered by the constant magnitude of the force gravity which with the restraint equation determines the variation \( \int dG \) for; with \( g \) constant, in the vertical direction, \( \int^2 dG \) depends only on the variation of vertical position.

A third condition is offered by the characteristic equa-
tion of the media being used. Obviously this with the condition of equilibrium will determine for given constraint variations the transformations $dN_\phi$ and $dG_\phi$.

These three conditions with the foregoing equations are now sufficient for a general solution of the problem. For a particular case we may assign a value to some one of the quantities not already determined in the above conditions and this will afford the fourth necessary condition for a complete solution of the special case under consideration. Such a condition is assigned if we determine the terminal potential conditions or the total work to be done.

13. INVESTIGATION INTO EXACT NATURE OF ACTION COMPONENTS.

In such a system as we have described for our transformer all forces which are dynamically active between it and external systems must have either original or final application on the fluid mass elements; and if such resultant forces actually do exist it can only be by virtue of actual accelerations of said fluid masses or by virtue of gravity; that is by virtue of the weight of the particles on the channel wall over which they are moving.

Through these channels the fluid carrying media have relative velocities. It follows that in such, the work of gravity $dW_\theta$ has possible two different paths for; either the weight of the mass can be considered as directly active on the channel as above mentioned or the transformation may be considered as having the two parts $dN'_\theta$ and $dW_N'$ where $N'$ is kinetic energy which is disappearing to $W$ as fast as it is appearing from $G$ and so never exists in finite quantity separately from or increasing $N$. 
Any elementary fluid mass $dm$ receiving $N'$ is transforming it to $W$ at the same rate, or rather; a decrease in the energy $N$ of $dm$ for work is simultaneously compensated for from $G$ and since both $W$ and $N'$ represent available energy the increase of $W$ from $N'$ is exactly equal to the quantity of $N'$ transformed and we have the equation

$$dN'_G = dW_N = dW_G$$

That is, the quantity of $G$ transformed to $W$ is (for the same quantity of action $dW$) exactly the same whether considered as being directly transformed to action or, as undergoing first an intermediate transformation through kinetic energy.

If we take into consideration now the condition of equilibrium and the principle of least action we immediately conclude that the latter path is followed by the activity for; in all activities from a potential source $Z$ the transformations $dN_Z$, as before shown, will be a maximum and the transformations $dW_Z$ a minimum. The activity will obviously take the path, if it can, tending to fulfill these conditions which tendencies are only fulfilled if the path $Z$ to $N'$ to $W$ is taken.

We conclude that for these transformers the whole action is fundamentally kinetic. It has in general the components $dW_N$, $dW_\phi$ and $dW_G$ of which the first is obviously a direct kinetic action and where both the transformations $dW_\phi$ and $dW_G$ involve, in the light of the foregoing, kinetic actions as each is the net result of two transformations which are respectively represented by the combinations $dN'_\phi$ occurring with $dW_N'$ and $dN'_G$ occurring with $dW_N'$. $N'$ represents the intermediate form of energy which is zero order kinetic and which, as before
stated, does not exist in quantity separately from $N$ but is exactly the contribution to $N$, for work done, from $\Phi$, $G$ and possibly other potential sources of the medium's energy. It may well be called the concealed zero order kinetic energy.

14. USEFUL FORMS OF GENERAL EQUATIONS. We will represent the total action $dW_N + dW_\Phi + dW_G$ by the symbol $dW_N''$ where now the subscript $N''$ indicates that fundamentally the action is kinetic but is compensated for in components of energy indicated by the triple index. If now the sum

$$dD + dW_N''$$

is equal to zero then the work done is exactly compensated for in the dynamic energy of the system. If however

$$dD + dW_N'' \neq 0$$

then other forms of energy are involved in the activity. We will now develop general expression for the latter case. We have

$$dW_N'' = dW_N + dW_\Phi + dW_G$$

and for $dD = dN + dG$ we have from equations 10

$$dN_w + dG_n + dG_\Phi + dN_\Phi$$

By substitution of 13 and 13a into 12 we have

$$dD + dW_N'' = dG_\Phi + dN_\Phi + dW_\Phi$$

as the general equation for any adiabatic activity. The right hand side of this equation is exactly equal to the total differential $-d\Phi$ whence we have

$$dD + dW_N'' = -d\Phi$$
of which the right hand member is a function of the state of 
the substance only and hence represents an exact differential. 
The left hand member must therefore likewise be an exact differ-
ential.

The form 15 of the general equation states merely that the dif-
ference from zero, of the sum of the dynamic different-
tial and work done is compensated for in the potential energy 
Φ of the medium but it does not represent the specific 
compensating transformations which take place. We therefore 
conclude, even though equation 15 is always true, to adopt the 
form 14 as more specifically representing the general differ-
ential activity.

We may now write equation 14 in the form 

\[ dD = dG_\Phi + dN_\Phi + dW_\Phi + dN'_w \]

where we note that the transformations \( dW_\Phi + dN'_w \) are oppo-
site sense for; \( dW_\Phi \) represents works done at the expense of 
configurational energy within the system and \( dN'_w \) represents 
an increase of energy within the system at the expense of 
external energy as work done. At any point on the acting 
surface now we can not have simultaneous transformations 
\( dW_\Phi \) and \( dN'_w \). The net result will be either one or 
the other and we conclude therefore that the general equation 
16 takes either of the forms 16a or 16b as follows,

\[ dD = dG_\Phi + dN_\Phi + dW_\Phi \]  \[16a\]
\[ dD = dG_\Phi + dN_\Phi + dN'_w \]  \[16b\]

where now 16a is merely the special case of 16b in which the 
whole work done is at the expense of configurational energy \( \Phi \).
For a pump action now the whole work done has a kinetic origin whence \( + dN''' \equiv dN_w \) and equation 16b takes the form

\[
dD = dG_\Phi + dN_\Phi + dN_w - - - - - - 16b'
\]

This equation may be written in the form

\[
dD = -d \Phi_p + dN_w
\]

where \(-d \Phi_p\) (the total differential of the configurational energy) represents configurational to dynamic conversion within the system. That is \( d \Phi \), if it does not vanish, is intrinsically negative during the action for this case. We conclude therefore that if in a moving pump channel we wish to have a positive increase of the configurational energy that there must be at least a component of the kinetic energy \( N \) which compensates for this gain and furthermore; that with respect this compensating component that the channel has distinct working and compensating regions in the first of which this component is increased by work done and in the latter the \( \Phi_N \) conversion occurs.

For motor action the transformation \( \pm dN''' \) must obviously have the form \( +dW_N'' \). Equation 16b will now be written as

\[
dD = dG_\Phi + dN_\Phi - dW_N'' \\
or \quad -(dD + d \Phi_G + d \Phi_N) = dW_N'' - - - - - - 17
\]

where the minus sign indicates that the variation of energy is in the sense opposite to the work done.

The activity will obviously proceed so as to adjust the energy distribution for equilibrium and the terms representing these transformations are \( dG_\Phi \) and \( dN_\Phi \) where; as before
explained, the transformation $dG_\Phi$ is exactly the net result of a transformation more accurately described by the combination transformation of $dN'_\Phi$ occurring with $dG_N'$ in such a manner that $N'$ never exists in finite kinetic quantity separately from or increasing $N$.

15. EXISTENCE CONDITIONS FOR TRANSFORMATIONS $dG_\Phi$ AND $dN_\Phi$. For the transformation $dN_\Phi$ to exist there must be a drop in the potential $\Phi$ in the direction of flow or; since $\Phi = f(P,T)$ where $P$ and $T$ vary in the same sense as $\Phi$ there must be a decrease of the pressure and temperature of the medium in the direction of flow......................... 18.

For the transformation $d\Phi_N$ to exist there must be an increase of the potential $\Phi$ in the direction of flow or; the pressure and temperature of the medium must increase in the direction of flow........................................ 19.

For the existence of the transformation $dG_\Phi$ the potential $G$ must be increasing and the potential $\Phi$ must be decreasing in the direction of flow. The potential $G$ exists by virtue of position so that its variation depends on the direction of flow and it is increasing when the flow is upwards and decreasing when it is downwards. Hence the transformation $dG_\Phi$ may exist when the pressure and temperature of the medium are decreasing in the direction of upward flow. If the flow is not upwardly in the direction $\Phi$ decreasing $P$ and $T$ or if in the upward direction $P$ and $T$ do not decrease then $\pm dG_\Phi$ vanishes.........................20.
For the existence of the transformation \( d\Phi_G \) the potential \( \Phi \) must be increasing and the potential \( G \) decreasing in the direction of flow and hence this transformation may exist when the pressure and temperature of the medium are increasing in the direction of downward flow. If the flow is not downwardly in the direction of increasing pressure and temperature or if the pressure and temperature are not increasing in the downward direction then as above the transformation \( \pm d\Phi_G \) vanishes.

Conditions 20 and 21 are fundamentally the same and may be stated as follows; for a transformation between potentials \( G \) and \( \Phi \) to exist the variation of the pressure and temperature of the medium must be of the opposite sense to that of its change of zero order position with respect to the earth's center.

16. **Time Rates of Activity for Transformations** \( \pm dG_\Phi \) AND \( \pm dN_\Phi \). From article 5 we have that for equilibrium the time rate of decrease of potential will be a maximum and the time rate of increase of potential will be a minimum. Therefore when in existence the transformation 

\[ +dN_\Phi \] will be a maximum..............22.

and the transformation 

\[ +d\Phi_N \] will be a minimum..............23.

\( G \) and \( \Phi \) now both represent potential energy but that represented by \( G \) is wholly available and that by \( \Phi \) is only partially available. From article 6 we have that for equilibrium the time rate of decrease of the available energy will be a maximum. Now the transformation \( d\Phi_G \) represents a decrease of available energy and \( dG_\Phi \) an increase thereof. Therefore
when in existence the transformation

\[ +d\Phi \text{ will be a maximum} \]

and the transformation \[ +dG \text{ will be a minimum} \]

So that we may design the dostrains to suit certain activities we must now adopt a system of relative and defining coordinates for them. Accordingly in the next chapter we shall arrive at such a system of coordinates.
CHAPTER III.

DEFORMATION AND MOTION COORDINATES

Since all practical transformers utilizing moving channels as a means, have them in rotation about a fixed stationary axis it is logical and convenient that as our origin we choose a point on the axis. Also it will be most convenient for the engineer to choose such coordinates that the physical dimensions involved are determined with respect to points and directions on the rotating element.

If now we can build up such a system of reference points lines and planes, that at any point on to the rotating element vectors in any direction may be completely specified with respect to directions on this element we will have the necessary mathematical graphics with which to represent any quantities we may wish to and hence; may arrive at the differential relations between them.

For the sake of brevity and clearness of explanation we will in the development of this coordinate system observe certain conventions and notations. Planes are denoted by letters with a bar over them. Thus \( \overline{\xi} \) represents a plane as does \( \overline{\eta}, \overline{\zeta} \) etc. The trace formed by the intersection of two planes is denoted by the letters denoting the planes, with a bar over them. Thus \( \overline{\xi}, \overline{\eta} \) represents the trace formed by the intersection of \( \overline{\xi}, \overline{\eta} \). Certain planes of reference contain traces of reference which are also contained in planes free to rotate about these traces as axis. The first named reference planes, which are irrotational with respect these traces, are denoted by
letters with the bar over them and with the subscript 0 affixed. The latter mentioned reference planes which may arbitrarily rotate about said traces as axis are denoted with the same barred letter but without the subscript affixed and the angles between said rotational and irrotational planes are denoted simply but the same letter. Thus $\xi_0$ is irrotational with respect to $\xi \xi$, about which traces as an axis $\xi$ is free to rotate and may assume any arbitrary angularity $\xi$ with respect to $\xi_0$.

17. ARBITRARY POINT ON ROTOR. Referring now to figure 2. $\xi_0$ is any planes of reference through the rotor containing the axis of rotation. $\xi$ is another plane containing the axis of rotation and having an angularity $\xi$ with $\xi_0$. $\xi_0 \xi$ is the axis of rotation and it is the trace about which as an axis $\xi$ is free to rotate and may assume any angularity $\xi$ with respect to $\xi_0$. On this axis we take a point 0 as the origin.

Let $p$ be any point on the rotor. It will have angular velocity $\omega$. Consider $\xi_0$ to be fixed in the rotor so that it also has angular velocity $\omega$ and fix $\xi$ in the rotor when it contains the point $p$. Erect $\eta_0$ so that $\xi_0 \xi$ is normal to it and place it so axially that it contains the point $p$. $\xi \eta_0$ is now a radial in the rotor, contains $p$, and the radial length to said point on said line is $ap$. The axial distance of the point from the origin is $oa$ and the angularity of the radial line $\xi \eta_0$ with respect to the reference
radial line \( \xi, \eta \) is equal to \( \xi \). The coordinates of the point \( p \) therefore are \( \alpha a, \alpha p, \) and \( \xi \) being respectively, the axial and radial distances and the angularity of the radial line through the point to a reference radial line or plane. This point was arbitrarily chosen and therefore these coordinates completely specify, with respect the origin \( o \) and a reference radius, any point on the rotor. At \( p \) there may be vectors in any direction, certain ones of which will now be specified.

18. TANGENTIAL VECTORS. Tangential vectors will be those tangent to the channels course. The channel may have any arbitrary direction in general and it is therefore necessary that we specify any arbitrary direction at the point \( p \).

Through \( p \) erect \( \xi \), so that \( \xi, \eta \) is normal to it. \( \xi, \eta \) and \( \xi \) are now mutually at right angles and form an orthogonal system of reference planes and their trances \( \xi, \eta \) and \( \xi, \eta \) an orthogonal system of axis with \( p \) as origin. These axis represent respectively radial, axial and peripheral directions and \( \xi, \eta \) and \( \xi \) are respectively radial, rotational and peripheral planes.

We next erect \( \eta \) and \( \xi \) so that \( \eta \) contains \( \xi, \eta \) and has angularity \( \eta \) with \( \eta \) and \( \xi \) contains \( \xi, \eta \) and has angularity \( \xi \) with \( \xi \). All of these and preceding planes to be (when having the desired angularity) fixed in the rotor and thus rotating with it.

The intersection of planes \( \eta \) and \( \xi \) will form \( \eta \xi \) which contains point \( p \) and since \( \eta \) and \( \xi \) were perfectly
arbitrary chosen angles, this trace represents any arbitrary directed line through \( p \). Its direction with respect to the peripheral direction on the rotor is completely specified by angles \( \eta \) and \( \xi \). This line may therefore be in the direction of the channel at point \( p \), and any vector \( bp \) contained in it will be a tangential vector whose direction, as before stated, is specific by angles \( \eta \) and \( \xi \).

19. RELATIONS CONNECTING DESTRAINT ANGLES \( \Phi, \Theta, \eta, \) and \( \xi \). The solid angles of which the traces \( \eta \xi, \eta \xi', \xi \eta' \) and \( \eta \xi \) are the vertices, are, by construction, all right angles so that the pyramid of which they are the corner lines has a rectangular cross section as shown at \( bcde \). (\( \eta \xi \) being normal to the plane containing section \( bcde \)). By construction angle \( epd \) is equal to \( \xi \) and angle \( cpd \) is equal to \( \eta \). Let angles \( bpc \) and \( bpe \) be represented by \( \phi \) and \( \Theta \) respectively.

In the further development we must know the relations existing between these angles and accordingly these will be discovered as follows:

We have

\[
\begin{align*}
\frac{bc}{bp} &= \sin \phi, & \frac{de}{ep} &= \sin \xi = \frac{bc}{ep}, & \frac{ep}{bp} &= \cos \Theta.
\end{align*}
\]

Whence

\[
\begin{align*}
\frac{bc}{bp} &= \cos \Theta \sin \xi \quad \text{and} \quad \sin \phi = \cos \Theta \sin \xi.
\end{align*}
\]

\[
\begin{align*}
\frac{be}{bp} &= \sin \Theta, & \frac{cd}{cp} &= \sin \eta = \frac{be}{cp}, & \frac{cp}{bp} &= \cos \phi.
\end{align*}
\]

Whence

\[
\begin{align*}
\frac{be}{bp} &= \cos \phi \sin \eta \quad \text{and} \quad \sin \Theta = \cos \phi \sin \eta
\end{align*}
\]
\[
\begin{align*}
\frac{dp}{bp} &= \cos \phi, \quad \frac{dp}{ep} = \cos \zeta, \quad \frac{dp}{cp} = \cos \eta, \quad \frac{ep}{bp} = \cos \theta \\
\text{Whence} \quad \frac{cp}{bp} \cos \eta &= \cos \zeta \quad \text{and} \quad \cos \phi \cos \eta = \cos \theta \cos \zeta \\
\text{Now} \quad \tan \phi &= \frac{\sin \phi}{\cos \phi} \quad \text{whence by substitution we have} \\
\tan \phi &= \cos \eta \tan \zeta \\
\text{and similarly} \quad \tan \phi &= \cos \zeta \tan \eta \\

\text{Collecting and extending these relations we have} \\
\begin{align*}
\sin \phi &= \cos \theta \sin \zeta \\
\cos \phi &= \sec \eta \cos \zeta \cos \xi \\
\tan \phi &= \cos \eta \tan \zeta \\
\cot \phi &= \sec \eta \cot \zeta \\
\sec \phi &= \cos \eta \sec \phi \sec \zeta \\
\csc \phi &= \sec \theta \csc \zeta
\end{align*}
\] 26.

\begin{align*}
\sin \phi &= \cos \eta \sin \phi \\
\cos \phi &= \sec \zeta \cos \phi \cos \eta \\
\tan \phi &= \cos \zeta \tan \eta \\
\cot \phi &= \sec \phi \cot \eta \\
\sec \phi &= \cos \zeta \sec \phi \sec \eta \\
\csc \phi &= \sec \phi \csc \eta
\end{align*}
27

In general we may write vector relations in two or more forms using either the right or left hand side of the above equations or a combination of both.

20. **TANGENTIAL VECTOR AND ITS COMPONENTS.** In general we will have to consider normal tangential axial, radial and peripheral components of a vector. For the tangential vector
In the context considered we have obviously the tangential component identical with the vector and the normal component being at right angles to it vanishes. The radial, axial and peripheral components are respectively \( bc \), \( cd \), and \( dp \).

**Radial component**

\[
bc = bp \sin \phi - \cdots - A \] \[= bp \cos \theta \sin \zeta - \cdots - B \] \[28\]

**Axial component**

\[
\begin{align*}
  cd &= cp \sin \eta = bp \cos \phi \sin \eta - A \] \[= bp \cos \theta \cos \zeta \tan \eta - - - B \]
\end{align*} \[29\]

**Peripheral component**

\[
\begin{align*}
  dp &= cp \cos \eta = bp \cos \phi \cos \eta - A \] \[= bp \cos \theta \cos \zeta - - - B \]
\end{align*} \[30\]

21. **Proof of Correctness of Relations 26 and 27.**

If these relations are true then obviously we must get a true equation if we substitute in the equation

\[
bp = \sqrt{(bc)^2 + (cd)^2 + (dp)^2}
\]

the above vector components as deduced from said relations.

Substituting the **A** relations we have

\[
bp = bp \sqrt{\sin^2 \phi + \cos^2 \phi \sin^2 \eta + \cos^2 \phi \cos^2 \eta}
\]

where the radial must be equal to unity if the relations are true. By factoring we have this radial written as

\[
\sqrt{\sin^2 \phi + \cos^2 \phi (\sin^2 \eta + \cos^2 \eta)}
\]

which is obviously equal to unity and hence relations are proven true.

Substituting for the **B** relations we have

\[
bp = bp \sqrt{\cos^2 \theta \sin^2 \zeta + \cos^2 \theta \cos^2 \zeta \tan^2 \zeta + \cos^2 \theta \cos^2 \zeta}
\]

where as above the radial must be equal to unity if the relations
are true. Squaring and factoring the radical we have \( \cos^2 \theta (1 + \cos^2 \xi \tan^2 \eta) \) and from relations 27 \( \cos \xi \tan \eta = \tan \theta \) whence this expression reduces to \( \cos^2 \theta (1 + \tan^2 \theta) \) which is obviously equal to unity and hence relations B are proven true. These could obviously only be true if the relations 26 and 27 were likewise true.

22. NORMAL AND TANGENTIAL COMPONENTS OF A PERIPHERAL VECTOR. Referring to figure 3 consider \( X \) so erected that \( \eta \xi \) is normal to it and passes through it at the point \( q \). This perpendicular plane has section \( q, k, f, j, q \) cut out by the pyramid of traces. Its intersection with \( \eta \) will form \( \eta X \) (not lettered) containing \( q, j \), and its intersection with \( \xi \) will form \( \xi X \) (not lettered) containing \( q, k \). Since \( \eta \xi \) is normal to \( X \) every line contained in \( X \) and intersecting \( \eta \xi \) must necessarily be at right angles to \( \eta \xi \). Therefore the angles \( h, q, j \) and \( l, q, k \) are respectively equal to \( \phi \) and \( \theta \).

Because \( \eta \xi \) (to which \( X \) is perpendicular) is contained in \( \xi \) the solid angle between \( X \) and \( \xi \) is a right angle. If two planes are at right angles the angle formed by the traces of the intersection with them of a third plane which is at right angles to either or both of them is a right angle. Now \( \eta_o \) is at right angles to \( \xi \) by construction and therefore the angle formed by the intersection of \( \eta_o X \) and \( \eta \xi \) is a right angle. Hence the angle \( f, k, p \) is a right angle.

Let \( f, q, \) be a vector contained in \( X \). It will certainly be at right angles to the normal of \( X \), which is \( \eta \xi \), and therefore at right angles to vector \( q, p \). \( f, p \) now is any vec-
tor whose components normal to and in the direction $\mathbf{f}$ are respectively $f, q,$ and $q, p$ where $q, p$ will become identical with $bp$ when $b$ and $q,$ coincide.

In the figure $f, p$ is taken along $\mathbf{h}$ which is peripheral in direction. If $\mathbf{h}$ is in the direction of the channel then $f, q,$ and $p, q,$ are respectively the normal and tangential components of a peripheral velocity of the channel at the point $p$ then $f, q,$ and $q, p$ will be respectively its normal and tangential velocities.

In order to arrive at the magnitude of the normal component $f, q,$ we consider its own components. Its tangential component will obviously vanish and the radial, axial and peripheral components will be respectively $h, q, ; q, h,$ and $f, q,$. If $bp$ be increased in length until $b$ coincides with $q,$ then the rectangle $bcdeb$ will coincide with $q, h, g, l, q$. Therefore the radial and axial components of $f, q,$ are of the same magnitude as those same components of $bp$ in this extended condition. They are however of opposite direction and therefore of opposite sign. The peripheral component $f, q,$ is in the same direction as the peripheral component of $bp$ and therefore it has the same sign as that component. Since $q,$ and $d$ will exactly coincide when $b$ and $q,$ coincide $f, q, + dp$ will be exactly a peripheral vector $f, p$ whose components normal to and in the direction $\mathbf{f}$ have radial and axial components of magnitude $h, q,$ and $q, h,$ respectively.

**COMPONENTS OF NORMAL VECTOR $f, q,$**

Radial component

$$h, q, = -bc = -q, p \sin \phi - \cdots - A \cdots 31$$

$$= -q, p \cos \theta \sin \xi - \cdots - B \cdots$$
Axial component
\[ g, h = -c d = -q, p \cos \phi \sin \eta \quad \text{-- -- A} \]
\[ = -q, p \cos \theta \cos \xi \tan \eta \quad \text{-- -- B} \]

Peripheral component
where
\[ f, g = f, p - g, p \]
\[ q, p = q, p \cos \phi \cos \eta \]
hence
\[ f, g = f, p - q, p \cos \phi \cos \eta \quad \text{-- -- A} \]
\[ = f, p - q, p \cos \theta \cos \xi \quad \text{-- -- B} \]
where
\[ f, p = q, p \sec \phi \sec \eta \]
hence
\[ f, g = q, p \left( \sec \phi \sec \eta - \cos \phi \cos \eta \right) \quad \text{-- -- A} \]
\[ = q, p \left( \sec \theta \sec \xi - \cos \theta \cos \xi \right) \quad \text{-- -- B} \]

NORMAL VECTOR COMPONENT \( f, q \)

We have
\[ f, q = \sqrt{\left( f, q \right)^2 + \left( g, h \right)^2 + \left( h, q \right)^2} \]
\[ = \sqrt{(q, p)^2 + (f, p)^2 - 2(q, p)(f, p) \cos \phi \cos \eta} \quad \text{-- -- A} \]
\[ = \sqrt{(q, p)^2 + (f, p)^2 - 2(q, p)(f, p) \cos \theta \cos \xi} \quad \text{-- -- B} \]
also
\[ = q, p \sqrt{\sec^2 \phi \sec^2 \eta - 1} \quad \text{-- -- A} \]
\[ = q, p \sqrt{\sec^2 \theta \sec^2 \xi - 1} \quad \text{-- -- B} \]
\[ = f, p \sqrt{1 - \cos^2 \phi \cos^2 \eta} \quad \text{-- -- A} \]
\[ = f, p \sqrt{1 - \cos^2 \theta \cos^2 \xi} \quad \text{-- -- B} \]

TANGENTIAL VECTOR COMPONENT \( q, p \)

Referring to figure 3 we have
\[ \frac{q, p}{j, p} = \cos \phi \quad , \quad \frac{j, p}{f, p} = \cos \eta \]
Whence,

\[ q_p = f_p \cos \phi \cos \eta \quad \text{(A)} \]
\[ = f_p \cos \theta \cos \xi \quad \text{(B)} \]

Expressions 36 and 37 represent respectively the normal and tangential components of the peripheral vector.

23. SYSTEM OF VECTOR REPRESENTING THE ABSOLUTE AND RELATIVE MOTIONS AT A POINT IN MOVING CHANNELS. Referring now to figure 4, at any point \( p \) the absolute peripheral motion of the channel is in the direction \( \eta_1 \xi_0 \) and of magnitude \( f_2 p \). At this point an element of fluid will have an absolute motion \( rp = q_2 f_2 \) and the relative motion of the fluid element through the channel will be \( q_2 p = f_2 r \). We have now a vector triangle \( p f_2 q_2 p \) where \( p f_2 \) and \( q_2 f_2 \) represent respectively in magnitude and direction the absolute motions of a point of the channel at \( p \) and a fluid element at \( p \), and where \( q_2 p \) represents the magnitude and direction of the motion of the fluid element relative the channel. \( q_2 p \) therefore has the direction of the channel at the point \( p \). We erect \( \eta \) and \( \xi \) so that \( \eta \xi \) contains \( q_2 p \) and therefore is in the direction of the channel at \( p \).

\( q_2 p \) is a tangential vector, as is \( b p \). The radial and axial components of \( b p \) are respectively \( bc \) and \( cd \), those of \( q_2 f_2 \) are respectively \( q_2 h_2 \) and \( h_2 g_2 \). These latter have the same direction as the corresponding components of the tangential vector \( b p \) and when the point \( b \) coincides with \( q_2 \) the tangential vector \( b p \) is exactly equal to the relative velocity vector \( q_2 p \) and the radial and axial components \( bc \) and \( cd \) will now become identically \( q_2 h_2 \) and \( h_2 q_2 \) whence; the radial and axial components of the fluids absolute and relative
motions are identical or, the radial and axial components of
the relative motion are identically the absolute radial and
axial components of the fluid elements motion.

As in figure 3 we have erected $\bar{x}$ so that $\bar{n} \xi$ is nor-
mal to it. $q'_z f_z$ therefore is the normal component of the
fluids absolute motion. $q_z q'_z$ is the tangential component
of the fluids absolute motion. The resultant effect may obviously now be obtained by considering either of two sets of compo-
nents, namely; $q_z h_z$, $h_z q_z$ and $q'_z f_z$ or $q_z q'_z$ and $q'_z f_z$
respectively the radial, axial and peripheral or tangential and
normal components of the fluids motion.

As stated $q'_z f_z$ is the normal component of the fluids
absolute motion. It is also identically the normal component
of vector $p f_z$ that is, it is the normal component of the
channels motion whence; the normal component of the channels
motion is identically the normal component of the fluids motion.
Obviously this must be true for the fluid certainly cannot pass
into the surface of the channel which it would do if there were
a difference in their normal components of motion.

COMPONENTS OF FLUIDS ABSOLUTE MOTION.

Radial component
\[ q_z h_z = q_z p \sin \phi - A \]
\[ = q_z p \cos \theta \sin \xi - B \]  
\[ q'_z f_z = q_z p \cos \theta \sin \eta - A \]
\[ = q_z p \cos \theta \cos \xi \tan \eta - B \]  
Axial component
Peripheral component
\[ h_z q_z = q_z p \cos \phi \sin \eta - A \]
\[ = q_z p \cos \theta \cos \xi \tan \eta - B \]  
\[ g_z f_z = p f_z - p g_z \]
where
\[ p g_z = - q_z p \cos \phi \cos \eta \]
whence
\[ q_z f_z = pf_z + q_z p \cos \phi \cos \eta - - - - - A \]  
\[ = pf_z + q_z p \cos \theta \cos \zeta - - - - - B \]  

where \( q_z p \) is intrinsically negative. It will be less confusing to express these relations as
\[ q_z f_z = pf_z - q_z p \cos \phi \cos \eta - - - - - A \]  
\[ = pf_z - q_z p \cos \theta \cos \zeta - - - - - B \]  

and use \( q_z p \) as a positive quantity.

Normal component
\[ q_z f_z = pf_z \sqrt{1 - \cos^2 \phi \cos^2 \eta} - - - - - A \]  
\[ = pf_z \sqrt{1 - \cos^2 \theta \cos^2 \zeta} - - - - - B \]

Tangential component
\[ q_z q_z' = pf_z \cos \eta \cos \phi + q_z p - - - - - A \]  
\[ = pf_z \cos \zeta \cos \theta + q_z p - - - - - B \]

where as before \( q_z p \) is intrinsically negative and it will therefore be less confusing to write these equations as
\[ q_z q_z' = pf_z \cos \eta \cos \phi - q_z p - - - - - A \]  
\[ = pf_z \cos \zeta \cos \theta - q_z p - - - - - B \]

24. RELATIONS CONNECTING VECTORS \( q_z f_z, pf_z \) AND \( q_z p \)

The projections of \( pf_z \) and \( q_z f_z \) on \( \xi \) are respectively \( pf_z \cos \zeta \) and \( \xi \) proj. of \( q_z f_z \). We have
\[ [\xi \text{ proj. of } q_z f_z]^2 = (pf_z \cos \xi)^2 + (q_z p)^2 - 2(q_z p)(pf_z)\cos \zeta \cos \theta \]
and
\[ (q_z f_z)^2 = (\xi \text{ proj. of } q_z f_z)^2 + (pf_z \sin \zeta)^2 \]

whence by substitution
\[ (q_z f_z)^2 = (pf_z)^2 + (q_z p)^2 - 2(q_z p)(pf_z)\cos \zeta \cos \theta - A \]  
\[ = (pf_z)^2 + (q_z p)^2 - 2(q_z p)(pf_z)\cos \eta \cos \phi - B \]
25. DESTALL NOTATION. We will let \( Q, E \) and \( R \) represent respectively the absolute accelerations of the fluid and channel, and the acceleration of the fluid relative the channel. Similarly \( q, e, \) and \( r \) will represent respectively the fluids absolute velocity, the channels absolute velocity, the channels absolute peripheral velocity and the velocity of the fluid relative the channel.

Any point \( p \) will be specified by the coordinates \( a, \rho \) and \( \xi \) where \( a \) and \( \rho \) are respectively the axial and radial distances to the point from the origin \( 0 \) and where \( \xi \) is the angularity of a radial line \( \xi \ell_0 \) through the point to a reference radial line \( \xi_0 \ell_0 \).

As previously stated a vector will in general have components in the radial, axial, peripheral, tangential and normal directions of which either the first three or the last two will completely specify the vector. These components will be denoted by the vector letters with subscripts affixed, these being \( \rho, a, e, r, \) and \( \eta \) and denoting respectively radial, axial, peripheral, tangential and normal components. A vector \( q \) will thus have the components \( q_\rho, q_a, q_e, q_r, \) and \( q_\eta \) and similarly for the other vectors.

Referring now to figure 4, \( Q \) and \( q, E \) and \( e, \) and \( R \) and \( r \) are respectively represented by \( q_2 \ell_2, p \ell_2 \) and \( q_2 \rho \). The components are thus represented as follows:

\[
\begin{align*}
Q_\rho \text{ and } q_\rho & \text{ by } q_2 h_2 \\
Q_a \text{ and } q_a & \text{ by } h_2 q_2 \\
Q_e \text{ and } q_e & \text{ by } q_2 f_2 \\
Q_r \text{ and } q_r & \text{ by } q_2 q'_2 \\
O_\eta \text{ and } q_\eta & \text{ by } q'_2 f_2
\end{align*}
\]
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\[ E_\rho \text{ and } e_\rho = 0 \]

\[ E_a = e_a = 0 \]

\[ E_e = e_e \text{ by } pt_z \text{ and } f_p \text{ of figure } 3 \]

\[ E_r = e_r = pq'_e \quad q,p \]

\[ E_n = e_n = q'_e t_z \quad t,q \]

\[ R_\rho = r_\rho = \text{ respectively } Q_\rho \text{ and } q_\rho \]

\[ R_a = r_a = Q_a = q_a \]

\[ R_e = r_e \text{ by } g_z p \text{ and } g_p \text{ of figure } 3 \]

\[ R_r = r_r = q_e p \quad q,p \]

\[ R_n = r_n = 0 \]

26. DEOTHERANT VECTOR COMPONENTS IN TERMS OF DESIGN QUANTITIES. In the design of the transformer the engineer has direct use for the quantities \( Q, E, R, q, e, r, \phi, \eta, \xi, \) and \( \Theta \) and accordingly we will express all vector components in terms of these quantities. To do this it is merely necessary to substitute the above notation in its proper place in the foregoing formulae 26 to 45. The following table is thus arrived at and in which, the encircled numbers with the component equations, denote the particular formula substituted in for that particular component.

In the table the component equations are given in terms of the velocities \( q, e, r, \) and the angles. They will likewise be true if we substitute respectively for \( q, e, \) and \( r, \) the accelerations \( Q, E, \) and \( R. \)
In addition to these relations we have the relations

\[ q^2 = e^2 + r^2 - 2er\cos\phi\cos\eta \]

between \( q, e, \) and \( r \) where as above the velocities \( q, e, \) and \( r \) may be replaced respectively by the accelerations \( Q, E, \) and \( R. \)

27. DERAINT GRAVITY COORDINATES. To determine the effect of gravity we must know in addition to the foregoing coordinates their relation to the direction in which gravity acts.

Referring to figure 5; \( \bar{\psi} \) is a vertical plane and \( \beta \) the angularity about a horizontal axis of the rotational plane \( \bar{n}_o \) with this plane. \( \omega d \) and \( \omega'd' \) are vertical lines in the vertical plane and \( \nu d \) and \( \nu'd' \) the projections of these lines into the rotational plane \( \bar{n}_o \) so that the angle between these lines and their projections is \( \beta \). \( \lambda \) represents the angle between the lines \( \nu d, \nu'd' \) and the peripheral line \( \rho f \).

The vertical component of any vector will be its...
projection into or parallel to the lines \( wd \) and \( w'd' \).

**VERTICAL COMPONENT OF RADIAL VECTOR.** - The vertical projection of the radial vector \( bc \) is the projection parallel to \( wd \) into \( \psi \) of the projection on to the line \( vd \) extended downwards as shown. Hence the radial vector \( bc \) has the vertical component

\[
bc \cos (\theta - \lambda) \cos \beta = bc \sin \lambda \cos \beta \tag{46}
\]

**VERTICAL COMPONENT OF AXIAL VECTOR.** The vertical component of the axial vector \( cd \) can easily be shown to be

\[
C d \sin \beta \tag{47}
\]

for; since \( cd \) is normal to \( \overline{n_o} \) and \( \overline{\psi} \) makes the angle \( \beta \) with \( \overline{n_o} \), then \( cd \) makes the angle \( \beta \) with a line \( uc \) (figure 5A) normal to \( \overline{\psi} \). This line \( uc \) is horizontal because \( \overline{\psi} \) is vertical and hence the vertical projection of \( cd \) will be

\[
u d = cd \sin \beta.
\]

**VERTICAL COMPONENT OF PERIPHERAL VECTOR.** The vertical component of the peripheral vector \( dp \) is obviously equal to

\[
d p \cos \lambda \cos \beta \tag{48}
\]

**VERTICAL COMPONENT OF ANY VECTOR.** The vectors \( bc, cd, \) and \( dp \) are the components of any vector \( bp \) and hence the vertical projection of a vector \( bp \) will be the sum of the above projections or components. This is equal to

\[
bc \sin \lambda \cos \beta + cd \sin \beta + dp \cos \lambda \cos \beta = bp [\sin \phi \sin \lambda \cos \theta + \cos \phi \sin \sigma \sin \beta + \cos \phi \cos \sigma \cos \lambda \cos \theta] \tag{49}
\]

**COMPONENT, IN DIRECTION OF MOTION, OF SURFACE REACTION DUE TO GRAVITY.** In order to find the actual work done due to gravity we must have the component of the surface reaction, due
to gravity, in the direction of action. If $u'd'$ represents the force of gravity in the plane $\mathbf{F}$ then its surface normal projection in the plane of rotation will be $f_1 \mathbf{k}$ equal to

$$u'd' \cos \beta \cos \left[90^\circ - (\lambda - \xi)\right]$$

$$= u'd' \cos \beta \sin (\lambda - \xi)$$

The projection of this component in the direction of motion of the surface will obviously be the effective part of the force gravity in doing work. This is equal to

$$u'd' \cos \beta \sin (\lambda - \xi) \cos (90^\circ - \xi)$$

$$= u'd' \cos \beta \sin (\lambda - \xi) \sin \xi$$

The angle $\beta$ will usually be constant for any given rotor as the plane of rotation is generally fixed as to angularity. If this plane is vertical $\beta = 0$, and if horizontal $\beta = 90^\circ$. All other planes of rotation will be between these limiting positions.

The angle $\lambda$ will not in general be a constant for all parts of the channel or rotor. Not only will it vary from point to point of a given passage but it will vary for a given point as it rotates about the center.

MANNER OF INVESTIGATION OF GRAVITY EFFECTS. To investigate for the effects of gravity it will simplify matters to consider the channel in instantaneous positions. To find the maximum variation of these effects at a given channel surface point, during the revolution, we perform the variation, for said point, corresponding to a half revolution and starting with the point in that rotational position in which $\lambda = 0$.

We will let $\lambda = \lambda_0 + \int_{\lambda_0}^{\xi} \lambda \, d\lambda$
where \( \lambda_o \) is the instantaneous value of \( \lambda \) at some reference surface point within the rotor or channel and where \( \int_{s_0}^{s} d\lambda \) is the variation of \( \lambda \) from the reference point to the point under consideration.

Since the peripheral tangent and the radius have always a constant angle between them, then the variation of angularity of the radius to a given line is equal to the variation of angularity of the peripheral tangent to that line.

We have represented the differential variation of the radius angularity by \( d\xi \) and hence \( d\lambda = d\xi \) whence

\[
\lambda = \lambda_o + \int_{s_0}^{s} d\xi \quad 51
\]

In this equation \( \int_{s_0}^{s} d\xi \) is obviously independent of the angular position of the channel and hence for the same limits \( s_0 \) and \( s \) in a given channel it is constant. The angle \( \lambda_o \) however is dependent on the angular position and is the angularity of the peripheral tangent with the projection into the rotational plane of vertical lines.
Chapter IV.

FORCE DISTRIBUTION
AND EQUILIBRIUM WITHIN CHANNELS.

VANE ACTION. By Vane Action is meant the direct action or work done between the channel surfaces and the fluid media passing there over, and the meaning of the word action, will in this sense be extended to include, as in polyactivities, all of the work done regardless of what forms of energy are compensated for. We have learned in the foregoing that such vane action is, exactly a dynamic exchange between the energy of our medium and that of external systems, that such an exchange is fundamentally wholly kinetic and that with respect the zero order dynamic activity this action will be a minimum.

In order to have the coordinates well in mind we will again refer to figure 4 and to articles 25 and 26. \( p \) represents a surface point in the channel of the transformer, \( q \) the absolute fluid velocity at this point, \( e \) the absolute peripheral velocity of said surface point, \( r \) the velocity of the fluid relative the channel and \( Q, E, R \) the corresponding accelerations. The components of these quantities are represented in the table of article 26, where the angles represent the following differences of directions.

The angle \( \Theta \) is a measure of the angularity between the direction of the channel at \( p \) and the projection of this direction on the rotational plane.

The angle \( \Phi \) is a measure of the angularity between the direction of the channel at \( p \) and the projection of this direction on to a plane normal to the radius through \( p \).
The angle $\eta$ is as shown the projection of the angle $\Theta$ on the plane normal to the radius through $p$ and the angle $\zeta$ is the projection of angle $\phi$ on to the rotational plane of the transformer.

The angles $\phi$, $\Theta$, $\eta$, and $\zeta$ are thus the directional conditions of the restraints relative the rotor, and the radius $\rho$ the positional dimension relative the center, of the point $p$ at which said directional conditions exist.

In addition to the above angles it will at times be found to be convenient to introduce the angle $\sigma = dp\,b$ (fig. 4) representing the angularity between the channels course direction $bp$ and its direction of motion $dp$. The angle $\sigma$ thus has the components $\phi$ and $\eta$ or $\Theta$ and $\zeta$ and is such that

$$\cos \sigma = \cos \phi \cos \eta = \cos \Theta \cos \zeta$$

28. RELATION BETWEEN KINETIC VARIATION AND WORK IN TERMS OF DESTRAINT COORDINATES. Work we have defined as the product-force x displacement. Letting $F$ represent a force it will have the components $F_p, F_a, F_e, F_r$, and $F_n$ representing respectively the radial, axial, peripheral tangential and normal components. If $ds$ represents an infinitesimal displacement its components will be represented by $ds_p, ds_a, ds_e, ds_r$, and $ds_n$. The component differential actions will thus be represented by the products

$$F_p \, ds_p, F_a \, ds_a, F_e \, ds_e, F_r \, ds_r, \text{ and } F_n \, ds_n$$

of which the sum of either the first three or the last two represents the whole work done. Thus
\[
F ds = F_p ds_p + F_a ds_a + F_e ds_e = F_r ds_r + F_n ds_n
\]

From Newton's Laws of Motion we have that the product-mass acceleration, is called the dynamic force where acceleration is the time rate of variation of velocity. Hence:

\[
F = dm \frac{dq}{dt}
\]

and in terms of \(q\) and \(t\)

\[
ds = q dt
\]

whence:

\[
F ds = dm q dt \cdot \frac{dq}{dt} = dm q dq
\]

because \(dt\) is the same differential of time in both terms.

Likewise we have

\[
F_p ds_p = dm q_p dq_p, \ F_a ds_a = dm q_a dq_a
\]

where the terms of the form \(dm q dq\) are, as before noted, known as the differentials of kinetic energy. We have therefore by substitution in the differential equations of work done the relations

\[
F ds = dm \left[ q_p dq_p + q_a dq_a + q_e dq_e \right] = dm \left[ q_r dq_r + q_n dq_n \right]
\]

of which the right hand members are called the total differentials of kinetic energy, or the total differential of kinetic action.

In terms of the energy coordinates of Chapter 11 we have therefore

\[
dN = F ds
\]

if the work is compensated for in the kinetic energy of the fluid.

The right hand members of equations 53 are exact differentials and are directly integrated for the particle as follows:

\[
dN^1 = \frac{dm}{2} \left[ q_p^2 + q_a^2 + q_e^2 \right],
\]

\[
dN^2 = \frac{dm}{2} \left[ q_r^2 + q_n^2 \right].
\]
where now we have replaced the term \( \int F ds \) by the term \( dN l^2 \) to denote a variation of kinetic energy which; obviously the right hand members always represent regardless of what form of energy is compensated for by this variation. In terms of the desired coordinates, of articles 25 and 26 we have by substitution in either of the equations 54 the expression

\[
dN_l^2 = \frac{dm}{2} [e^2 + r^2 - e r \cos \phi \cos \theta]^2 - - - - - - - - - 55
\]

representing the general equation for the kinetic variation or action of the particle \( dm \) between the limits 1 and 2 of the channel. Between any given limits for a given channel this variation is obviously completely determined when we know the variation of the relative velocity \( r \).

29. NORMAL AND TANGENTIAL VANE ACTIONS. Referring now to figure 6 we have represented a vane contour \( ab \) with a particle \( dm \) situated at point \( p \) thereon. \( q, \phi, \) and \( r \) represent, as before, respectively the particles absolute velocity, the vanes absolute velocity and the relative velocity at the point \( p \). After an interval of time \( dt \) the particles will have been displaced to the point \( p' \) distant from \( p \) by a differential amount \( ds_r \). We will now consider the form of the particles energy compensating for the work done or vane action, and for the sake of convenience will consider the normal and tangential components of the action.

In the figure \( F_n \) and \( F_r \) represent respectively the normal and tangential forces by virtue of which the action on \( dm \) occurs. As shown in article 25, the normal velocity
of the particle must be equal to the normal velocity of the vane and we conclude that the normal motion components can not be free of constraints if there is a variation of the normal velocity of the channels surface over which the fluid is flowing.

In general the normal component of the channels velocity will vary by an amount $de_n = de \cos \sigma$ and the restraint direction will vary by an amount $d\sigma$ having the components $d\phi$ and $d\eta$ . This variation of the restraint direction will obviously cause a normal fluid velocity variation equal to $e d(cos \sigma)$ where $e$ represents the instantaneous peripheral velocity, at the channel section having said variation of restraint direction. The normal component of the fluid particles velocity will thus vary by an amount $dq_n$ equal to $de_n + e d(cos \sigma)$ between the points $p$ and $p'$ and the force $F_n$ exerted by the particle on the channel wall be equal to

$$F_n = \frac{dq_n}{dt} dm$$

where $dt$ is the time required for the displacement from $p$ to $p'$ of said particle $dm$ . In this time interval the point in the channel on which $F_n$ acts will have been displaced an amount $ds_n = e_n dt = q_n dt$ because, as before explained, at any and every section the absolute normal velocity of the particle $dm$ must be equal to that of the channel. The work or action of the normal force will therefore be equal to

$$F_n ds_n = dm q_n dq_n$$

the right hand side of which equation is exactly equal to the differential of the normal component of the kinetic energy.
We conclude that the normal action is inevitably wholly compensated for in the zero order kinetic energy of the fluid.

For the condition of equilibrium of the tangential action and the nature of the compensating activity therefore, we must consider the tangential components of the motions. In general the channel or vane velocity \( e \) will differ from \( p \) to \( p' \) by an amount \( de \). This change will have a tangential component equal to

\[ \partial(e_r)_\sigma = \cos \sigma \, de = \cos \phi \cos \eta \, de \]

We write the above differential as a partial to distinguish it from the total differential.

\[ de_r = d(\cos \sigma \cdot e) = \cos \sigma \, de + e \cdot d(\cos \sigma) = \partial(e_r)_\sigma + \partial(e_r)_e \]

The former is obviously the partial differential of the latter total, represented by the first of term of the right hand member, in which the restraint direction is considered for the instant constant. It is important that we distinguish between the variation of the channels tangential velocity, \( \partial(e_r)_\sigma \) caused purely by the variation \( de \) of the channels absolute velocity \( e \) and the variation \( \partial(e_r)_e \) caused purely by variations \( d\sigma \) in the direction of the restraints. This latter variation of direction is, as before shown, a surface normal variation and has, for continuous channels, zero effect on the fluids relative velocity, but just as in stationary channels guides the relative flow over the surface. It effects only the normal components of the fluids motion, which change, is, as before explained, wholly compensated for in the kinetic energy of the fluid.
30. LEAST KINETIC VARIATION OF FLUID. The variations \( dq_n \) and \( dq_r \) completely specify the differential variation of the fluids velocity and \( dq_n \) is wholly determined by the normal velocity variation over the surface across which \( dm \) passes; less than which the fluids velocity can not vary and to which its variation is equal if its tangential velocity variation \( dq_r \) vanishes.

Since now normal reactions are at right angles to tangential reaction, their resultants in the tangential direction will vanish and we therefore wholly determine the tangential variation by the tangential constraints alone. If now as above the tangential variation is to vanish then obviously the tangential motions must be unconstrained and we have for consideration exactly a fluid motion partially constrained as mentioned in article 3. Considering now that a variation \( \partial (e_r)_\sigma \) actually occurs and that the tangential motion is unconstrained (That is the tangential force resultant vanishes on \( dm \) in its passage from \( p \) to \( p' \) ) then obviously its own absolute motion in the tangential direction will be unchanged whence; the relative motion between the vane or channel and \( dm \) will change by the exact amount of the change in said tangential motion of the channel. For such a condition we have therefore

\[
dr - \partial (e_r)_\sigma = 0
\]

which is an exact representation of the motion for equilibrium.
in least kinetic variation. This equation for least action will now be otherwise arrived at in the following.

We have as a general expression for the kinetic variation, equation 55, and on this expression we will impose the condition (1a) of article 5 for the space rate of variation of potential for equilibrium. For a given channel, as before states, the only variable in this equation not fixed by the destrains is the relative velocity $r$.

If the space rate of variation of potential is a minimum then obviously between two points $p$ and $p'$ in the channel, the variation of kinetic energy, $dN$, will be a minimum or; the relative velocity $r$ will vary at such a rate as will cause the variation of the kinetic energy $N$ to be a minimum. Differentiating 55 with respect to $r$ we have

$$\frac{dN}{dr} = r - e \cos \phi \cos \eta$$

where for a minimum

$$\frac{dN}{dr} = 0 \quad \text{and} \quad \frac{d^2N}{dr^2} = - - - - - \text{positive}$$

Now $\frac{dN}{dr^2} = +1$ whence; if the expression for $\frac{dN}{dr}$ is placed equal to zero we have the condition of least kinetic variation. We have

$$r - e \cos \phi \cos \eta = 0$$

or

$$r - e \cos \sigma = 0$$

Differentiating the terms of the latter expression effecting $r$ with respect to $r$ we have

$$1 - \frac{de}{dr} \cos \sigma - e \frac{d(cos \sigma)}{dr} = 0$$

but $\frac{d(cos \sigma)}{dr} = 0$ since $d\sigma$ can not cause a variation of $r$ whence;

$$dr - \cos \sigma \, de = 0$$

which is identically expression 56.
31. TANGENTIAL FORCE DISTRIBUTION. The foregoing difference $56$ is obviously, when it does not vanish, exactly equal to the differential $dq_r$ of the tangential component of the fluids absolute velocity and hence the magnitude of the force $F_r$ arising because of this variation is exactly represented, for a particle $dm$ by the expression

$$F_r = dm \frac{dq_r}{dt}$$

where $\frac{dq_r}{dt}$ is the absolute tangential acceleration of the particles motion.

The force $F_r$ exists between the particle $dm$ and some adjacent reacting or surface point. This surface point may be an adjacent fluid particle or a point on the surface of the channel. We shall name it the REACTION POINT of the force $F_r$.

During an interval of time $dt$ the reaction point moves through a distance $ds$ such that the component of $ds$ in the direction of action of $F_r$ is $ds_r$. Then obviously the least differential action of the force $F_r$ on the reaction point will be represented by the product.

$$F_r \cdot ds_r$$

which is exactly the differential of tangential work done between the reaction point and the particle $dm$ in the time $dt$. In other words it is exactly the dynamic exchange between the particle $dm$ and said point. Obviously this activity is exactly compensated for in the energy of the particle.

There are three conceivable distributions of the force $F_r$ and described as follows;
1. The reaction point may be considered as located wholly in an adjacent fluid particle so that the force $F_r$ is active wholly between $dm$ and adjacent fluid. For this case the differential displacement $ds_r$ of the reaction point will obviously be represented by the product $q_r dt$.

11. The reaction point may be considered as located wholly in the channel wall adjacent the particle $dm$ so that the force $F_r$ is active wholly between $dm$ and an adjacent point in the channel wall. For this case the differential displacement of the action point is represented by the product $e_r dt$ where $e_r$ is the component in the direction of $F_r$ of the channel's velocity at the action point.

111. The reaction point may be considered to be divided so that for a part of $dm$, $F_r$ acts between the particle and adjacent fluid and for the remaining part between it and an adjacent point in the channel wall.

32. ACTIONS OF CLASS I. For the first case the differential of tangential work is exactly

$$dW_r = dm\frac{dq_r}{dt} q_r dt = dm q_r dq_r$$

of which the right hand side is exactly equal to the differential of the fluids absolute tangential component of kinetic energy. This plus the differential of the normal work equivalent is exactly equal to the work done hence; for Case I we have

$$dW = dm (q_r dq_r + q_n dq_n)$$

or

$$dW = dW_n = -dN_w$$

For this case therefore the whole action or work
done is compensated for in the kinetic energy of the fluid.

The right hand side of this expression is exactly one of the differential expression 54 whence; in terms of the restraint co-
ordinates the action of a particle \( dm \) is expressed, for Case I, by the equation.

\[
dN_w|_i = \frac{dm}{2} [e^2 + r^2 - 2er \cos \sigma]^2
\]

58

For least action or unconstrained tangential motion we have condition (56) satisfied whence the relative velocity

\[
r_x = r_x + \int \delta(\varepsilon_r) \sigma
\]

59

Before we can integrate expression 59 we must ob-
viously know the path or restraint relation \( \sigma = f(\varepsilon) \) whence the magnitude of the least action is dependent on the path.

33. CONDITIONS FOR WHICH RELATION 59 IS TRUE.

In deducing criteria 59 for the magnitude of the action we have applied the conditions of equilibrium for an unconstrained tangential motion. If therefore the tangential constraints or resultant forces do not vanish then the above equation does not describe the motion of the particle. Obviously such resultants exist if there is a variation of the fluid pressure in the direction of flow for; the existing tangential force is then equal to

\[
F_{r_p} = - \frac{dP}{ds_r}
\]

60

where \( \frac{dP}{ds_r} \) is the rate of variation along the channel of the
null
pressure at the point where the force \( F_{r_p} \) exists. This is shown as follows:— If we consider a cube of fluid of dimension \( ds_r \) and area \( dA \) and consider that the pressure varies an amount \( dP \) along its side \( ds_r \) then obviously the resultant force on the cube due to the pressure variation is represented by the product

\[
dA \cdot dP = dA \cdot \frac{dP}{ds_r} \cdot ds_r
\]

If now we consider a cube of unit volume then obviously its area \( A \) and its side \( S_r \) will both be unity and we have as the resultant force on the unit cube exactly \( -\frac{dP}{ds_r} \) where the negative sign is used because the force acts oppositely to the rate of increase of pressure. This force would obviously cause an acceleration such that the relation

\[
F_{r_p} = -\frac{dP}{ds_r} = \frac{\mu}{g} \frac{dq_{r_p}}{dt} \quad \text{(61)}
\]

is true where \( \mu \) represents the density and \( dq_{r_p} \) the differential tangential variation of the fluids absolute velocity. If now \( F_{r_p} \) is not zero then obviously \( dq_{r_p} \) does not vanish whence relation (56) is not true and hence the expression (59), derived from it, does not hold.

The only other sources of tangential resultants arise from accelerations imposed on the particles motion by the de- straint motion; that is in pump actions only, as obviously the particles motion will not of its own accord or self actingly impose such accelerations on itself. The motions which the destraints in rotation tend to impose on the particle are ob-
viously rotary motions about the axis of rotation of the transformer. There is consequently an acceleration inward toward the center of rotation imposed on the particle under at least those conditions of motion where it does not move relatively radially away from the center of rotation, and consequently there will be force resultants radially outward exerted by particles in such states of motion. Such radial forces are called centrifugal forces and obviously these will have tangential components wherever radial projections into the tangential directions do not vanish. This is true for any channel along whose path the radius is a variable.

We conclude that for pump activities 59 is a criterion of the magnitude of the action only when \( \frac{dP}{ds_r} = 0 \) and when, for the motion as described by this equation, the centrifugal forces vanish; as obviously if the centrifugal or radial forces vanish then likewise will their tangential components vanish. For channels whose paths have a constant radius the tangential projection of any radial vector vanishes whence; for these the only condition of unconstrained tangential motion is that \( \frac{dP}{ds_r} \) vanishes.

We shall discover in Chapter V that there will be such a magnitude \( r^c \) of the relative velocity \( r \) at and above which the centrifugal forces vanish but below which they appear. That is for the magnitude \( r \geq r^c \) the inward radial accelerations of the particles motion vanishes but for all magnitudes \( r < r^c \) there is such an inward acceleration whence there will be centrifugal forces.
We will call the conditions in which centrifugal forces vanish the CENTRI-VANESCENT STATE and that in which they exist the CENTRI-FORCIENT STATE of the motion, and the mathematical relation expressing the demarcation ratio between them the CRITICAL RATIO. In this sense therefore the magnitude \( r^c \) of the relative velocity \( r \) will be its critical magnitude and it will signify that at and above this magnitude the centri-vanescent state exists. Hence for \( \frac{dP}{ds_r} = 0 \) the magnitude of the action will be determined from the criterion 59 if \( r \geq r^c \). If however \( r < r^c \) the magnitude of the action must be determined from the conditions of tangential force equilibrium and these will be further investigated in Chapter V.

34. ACTIONS OF CLASS II. We will now investigate the class of actions in which we have existent tangential constraints of the form \( \frac{dP}{ds_r} \). The differential of the tangential work is represented by the expression

\[
dW_r = dm \frac{dq_r}{dt} e_r dt = dm \ e_r \ dq_r
\]

This differs in magnitude from the tangential kinetic differential by the amount

\[
dm(e_r - q_r) dq_r
\]

and since for all activities all of the action except that represented by this difference is compensated for in the kinetic energy of the fluid we conclude; that this differential difference is likewise the equivalent of the work compensating.
differential transformation of the potential energy of the fluid. It follows that

\[-dW_L = dm(e_r - q_r) dq_r\]

which is otherwise represented in terms of the relative velocity as

\[-dW_L = -dm \cdot r [d r - \partial(e_r)_{\sigma}]\]

or for a particle \( dm \) the integrated value is

\[+dL_A = -dW_L = dm \int r \partial(e_r)_{\sigma} - dm \cdot r^2 |^2 - - - : 62\]

where \( \partial(e_r)_{\sigma} = \cos \sigma \cdot de \) so that obviously the destraint equation and the relation connecting \( r \) and \( e \) must be found before the integration can be completed.

The differential \( dW_L \) has in general the parts \( dW_G \) and \( dW_\phi \) where for the existence of the transformation \( dW_G \) there must be a vertical displacement of the fluid, and where for the existence of the transformation \( dW_\phi \) there must be a variation of the fluids pressure in the direction of flow. Transformations \( dW_G \) and \( dG_w \) do not effect the availability of the system. They will therefore proceed according to the principle of least action whence for any given activity involving such transformations we conclude that \( dW_G = -dG_w \) minimum.

For any given compensating transformation

\[+dW_L = dm \cdot r [d r - \partial(e_r)_{\sigma}]\]

the part \( dW_G \) will therefore be a minimum whence, we can immediately solve for the part \( dW_\phi \) from the equation
\[ + dW_\Phi = dm \left[ d\mathbf{r} - \partial(C_r) \right] - dW_G \]

For all ordinary cases the part \( dW_G \) will either completely vanish or be negligibly small when

\[ dW_\Phi = dm \left[ d\mathbf{r} - \partial(C_r) \right] \]

For the total differential of work done in actions of Class II we have

\[-dW = dm \, e_r \, dq_r + dm \, q_n \, dq_n \]

which may be otherwise written as

\[-dW = dm \left( e_r \, dq_r + (q_r \, dq_r + q_n \, dq_n) \right) = dm \left[ \partial(C_r) - d\mathbf{r} \right] + dm \left( q_r \, dq_r + q_n \, dq_n \right) \]

of which each term excepting \( dm \, r \, \partial(C_r) \) is separately exact.

For Case II therefore the relation between the restraint coordinates and the relative velocity is necessary for the complete integration of the action equation. Integrating the exact terms and replacing \( q_r, q_n, \) and \( e_r \) in terms of the restraint coordinates we have as the action for a particle \( dm \) the following expression:

\[-dW = dm \int r \cos \sigma \cdot de - \frac{dm \, r^2}{2} + \frac{dm}{2} \left[ e^2 + r^2 - 2 \, e \, r \, \cos \sigma \right]^2 \]

In this expression the first two terms of the right hand member exactly represent the magnitude of the action compensating for the potential energy variation and the third term that compensating for the kinetic energy variation. The equation is exactly true for all activities in which the action is dynamically conservative and in which the potential energy transformations are adiabatic so that in the activity there is
zero dissipation of available energy.

Since now as before shown all pump actions (for the types of transformers here considered) have a strictly kinetic origin and since in such, as also before shown, transformations of the type \( +dL_N \) can not exist simultaneously at the same place with the action \( +dN_w \); we conclude that the action can only have all the components of equation 65 if the component transformation between potential and kinetic energy is of the sense \( +dN_L \). In practice this would generally only occur when the action or work done \( dW \) is of the sense \( dW_N \) which is exactly a motor action.

35. EQUILIBRIUM FOR ACTIONS OF CLASS II. The variations for equilibrium in this class of activity will obviously be wholly dependent on the terminal potentials.

We have as the general relation connecting these the equation

\[
dm L_i^2 = -dm W_L_i^2 + dm N_L_i^2
\]

or

\[
L_i^2 = W_L_i^2 + N_L_i^2
\]

where for pump actions the term \( N_L_i^2 \) vanishes and for motor actions will, when it is in existence, be a maximum.

Its kinetic variation is exactly equal to

\[
\frac{q_e^2 - q_i^2}{2g}
\]

but subject to the condition that \( q_2 > q_i \); obviously if \( q_2 < q_i \) in a motor action it can not have been because
of a compensating transformation $L_N^2$. Substituting 62 and the foregoing in the above expression we have as the equilibrium equation

$$L_N^2 - \frac{q^2}{2\gamma} = \frac{1}{g} \int r \cos \sigma \, de - \frac{r^2}{2\gamma}, \quad - - - - - - - 66$$

In this equation the term $\frac{q^2}{2\gamma}$ vanishes if $q_2 \geq q_1$, as the negative difference does not, for this class of activity, represent compensation for a transformation $L_N^2$, but exactly the compensation for the transformation $dW_N^2$, which is otherwise taken care of by the last term of equation 65. This latter term vanishes for the condition $q_2 > q_1$ if the transformation $dW_N^2$ is existent during the action for; in a motor action a net gain of the kinetic energy of the medium is compensated for, not in word done, but from a potential transformation $N_L$. With these restrictions in mind equation 66 with 65 is sufficient to solve the actions of Class II. In special cases the potential transformation might vanish when equation 65 would become identical to the general action equation of Case I. The latter case is therefore merely a special application of Case II and applicable for dynamical conservative activities to either motor or pump actions.

In Case I now the force $F_r$, due to the tangential acceleration of the particle $dm$ has been considered as acting entirely between $dm$ and adjacent fluid so that only the fluids velocity is considered in arriving at the displacement $ds$ and in Case II said force has been considered as acting entirely between $dm$ and the channel wall whence; we consider the tangential velocity of said wall in determining $ds$. 
The only other conceivable distribution of said force application is its partial application between the channel wall and the fluid, as mentioned for Case III.

36. ACTIVITIES OF CLASS III. For this class of action the particle \( dm \) is to be considered as having the part \( dm' \) whose acceleration causes a tangential force to exist between it and adjacent fluid and a part \( dm'' \) whose acceleration causes a tangential force to exist between it and the channel wall. Since now \( dm' \) is compensating for work done, from its kinetic energy only, and \( dm'' \) from both its kinetic and potential energy they can not continuously exist at the same potential. If such an activity actually did occur there would inevitably be an irreversible flow of heat from the parts \( dm' \) to the parts \( dm'' \). We may choose the particle \( dm \) as small as we please and in the limit it would shrink up into a mass point. Obviously within such a point two different potentials can not exist at the same instant and we conclude that the above is not an exact interpretation of the actual variation of conditions. We arrive at the proper interpretation by taking into consideration the principle of least action as follows;

If the part \( dm'' \) of the element \( dm \) compensates for action from both its kinetic and potential energy whereas the part \( dm' \) compensates for action from its kinetic energy only then since, the kinetic variation of both parts is the same, the action of the part \( dm'' \) will be greater than that of
the part $dm'$. The force conditions influencing both parts are however the same and consequently there can only be one variation resulting in least action. We conclude that the action and energy variation of both parts must conform to this condition whence; both, the potential energy transformations compensating therefore and the action are to be considered as being taken part in equally by all fluid elements in one cross-section of uniform force conditions.

We will let $\xi_a$ represent such a factor that, when multiplying the quantitative representation $dA_L^2$ of the actual compensating potential transformations for a particle $dm$ between pressure limits 1 and 2, the produce will be such that the equation

$$\xi_a \cdot dA_L^2 = dm' \int r \, dq_r$$

is true where; the right hand side represents exactly the dynamic activity, of the particle $dm$ compensated for in the potential energy of the particle and the left hand side represents exactly the realized part, for such action, of the available energy of the particle between these limits. If we consider a unit weight of the fluid the equation becomes

$$\xi_a A_L^2 = m'' \int r \, dq_r$$

where $m' = \frac{w''}{g}$

whence

$$\xi_a A_L^2 = \frac{w''}{g} \int r \, dq_r$$

where now $\xi_a A_L^2$ is equal to the realized part, for action, of the potential transformation per unit weight of the
medium, where \( \frac{w^u}{g} \int r \, dq_r \) represents exactly the action per unit weight to be compensated for from the potential energy per unit weight of the medium and, where in view of the foregoing conclusions as regards the nature of the activity \( W^u \) must be unity if the equation is to be satisfied when accurately representing the activity. We have therefore as the equation representing this part of the action

\[
\xi_a A |_1^2 = \frac{1}{g} \int r \, dq_r
\]

where \( dq_r \) may be replaced by \( [dr - \partial (e_r)_\sigma] \) and \( r \) by \( (q_r - e_r) \) and where \( e_r = e \cos \sigma \). The factor \( \xi_a \) now represents exactly the efficiency of the compensating potential transformation referred to the adiabatic process as unity and we conclude; since the whole fluid mass is considered homogeneously active therewith, that the above system of coordinates and method of attack lead to accurate quantitative representations of the energetic variations during the activity.

We have thus shown that for the only final energetic states of the fluid which could result from a partial force distribution as described for Case III, that in the actual activity the dynamic action can not be such as to have this force distribution, but that if such a final energetic state exists that the path will have been such as to have had the least action for each particle. This path is different from the above path for at least part of the fluid mass and can be exactly determined for a given potential energy variation between limits 1 and 2 by the equation 67.
The activity as described is exactly biactive in the sense of the classification of activities of article 2. Forces are active in two senses namely; in the zero order or ordinary dynamic sense and in the first order or molecular sense. By the principle of least action now the dynamic forces will be least active in doing work and the molecular force will be least active in arranging parts of the first order for zero order activity or work. The factor $\xi_0$ is now exactly the efficiency of the arranging process and as another criterion of the direction of the activity we have the condition that $\xi_0$ will be a minimum.

37. HOMOGENEITY OF IRREVERSIBILITY AND RESTRAINED AVAILABLE ENERGY. It follows also from the foregoing conclusions that if the mechanical constraints are so arranged as to have possible for the given activity any internal irreversibilities, such as free expansions or internal friction, that for homogeneous fluids these irreversibilities will occur simultaneously in adjacent fluid particle and that no one differential part of the fluid can be regarded as having its potential energy vary at a greater rate than that of the immediate neighboring particles. In other words there is HOMOGENEITY OF IRREVERSIBILITY and the medium active in transforming its energy is to be regarded as having for every particle a RESTRAINED transformation making only available a portion $dA_L \cdot \xi_0$ of energy, for action, where for ideal conditions the portion $dA_L$ would be available between the
same potential limits. The force distribution for such cases is therefore to be regarded as that under ideal conditions of an equivalent medium whose available energy between potential limits 1 and 2 is now $\xi_a A_L$, as shown in equation 67, instead of $A_L^2$ as shown by equation 62. The activity with this modifications will be treated exactly as for Case II, and to make the equations of Case II applicable it is merely necessary to modify the energy terms as above.

The same argument may be extended to include dynamic or other activities which in some sense involve irreversibilities. We need therefore in all such cases merely to consider for the active medium an equivalent medium whose available energy is equal to the restrained available energy of the actual medium. Algebraically this function is of course performed by merely multiplying the available energy of the actual medium between the potential limits in question, by a factor so that the product equals the available energy of the equivalent medium between the same potential limits. Obviously the product represents the realized, for action, available energy between said limits and the multiplying factor a referred efficiency of the process.

38. AVAILABILITY IN POLYACTIVITIES. If the activity were polyactive, then a general activity each order of parts would have within itself an arranging activity which would cause an activity of the next lower order of parts. This part of the activity is exactly of the nature of the fore-
going biactivity and the relations between the parts of the activity will be exactly as for the foregoing. If, therefore $\xi_a'$ represents the efficiency of the arranging activity it will be a minimum. For any polyactivity therefore the subdivisions of active particles may be taken in pairs as above and for each there will be an arranging efficiency $\xi_a'$. If the activity of second order of particles with respect the first has an arranging efficiency for activity of first order of particles, equal to $\xi_a''$ and the first with respect to the zero order a referred efficiency of $\xi_a'$ then the efficiency of the second order parts in arranging for zero order activity will be represented by the product

$$\xi_a'' \cdot \xi_a'$$

where as before the activity will be such that each efficiency is separately a minimum. If now $dA'$ and $dA'$ represent respectively the available energy, for a particle $dm$ between given potential limits, of the second order with respect to the first and of the first with respect to the zero order when the efficiencies $\xi_a''$ and $\xi_a'$ are unity then; during a transformation between the above potential limits, the zero order available energy will be

$$dA'' \xi_a'' \xi_a' + dA' \xi_a'$$

and the average efficiency of the zero order activity referred to the adiabatic process as unity will be
\[ \xi' = \frac{dA' \xi_a + dA' \xi_a}{dA'' + dA'} \]

Such parts as \( dA' \) are exactly the energy in an order of parts which may cause available activity in the next lower order parts without the aid of transformations as \( dA'' \xi_a \). They represent thus exactly kinetic energy, or more specifically the motional energy of an order of parts which is available for next lower order activity and in this respect they are exactly a measure of the degree of orderliness of the motions of said parts.

An example of such an activity is that, within a gas engine cylinder during the explosion stroke, before previous to explosion the gases have been compressed whence there is available some first order kinetic energy \( A' \) by virtue of the work of compression. Then during the explosion stroke combustion occurs and both the energy components \( A'' \) and \( A' \) are active. \( A'' \) representing the chemical energy and \( A' \) the molecular available energy. Obviously the greater part of the energy \( A'' \) is used in merely giving ultimately unavailable motion to the first order parts. The activity here cited is exactly triactive in the first order but as a whole triactive since atomic, molecular and zero order dynamic forces are active therein.

39. COMPONENT TRANSFORMATION DUE TO GRAVITY.

It is of course evident that in general it is impractical to build rotary transformers so that their passages will in all positions conform to all changes including those due to gravity.
for; in order to do this the passage on the rotating element would have to have their mechanical constraint vary during rotation to respond to variations of the fluids physical configuration caused by variations of gravity effects during said rotation. Nevertheless it may in some cases be important to know the magnitude of these effects and their variations so that such constraints may be planned as will give for a complete rotation the best average results.

**TOTAL DIFFERENTIAL \( dG \) OF GRAVITATIONAL ENERGY.**

This total differential will be equal to the product mass \( x \) \( g \) \( x \) vertical displacement of the mass. The fluid within the transformer has at any instant the absolute velocity \( \mathbf{q} \) which has the components \( r \sin \phi, r \cos \phi \sin \eta \) and \( e - r \cos \phi \cos \eta \)

In time \( dt \) a fluid element with absolute velocity \( \mathbf{q} \) will be displaced by an amount equal to \( \mathbf{q} \ dt \) and the components of the displacement will be \( r \ dt \sin \phi, \ r \ dt \cos \phi \sin \eta \) and \( e \ dt - r \ dt \cos \phi \cos \eta \) respectively, the radial, axial and peripheral displacements. Now \( r \ dt \) is equal to the relative displacement in the channel and we will denote it by \( ds_r \). \( e \ dt \) is the peripheral displacement in the same time of the fixed point in the channel at which \( dm \) is situated and whose peripheral velocity is \( e \). We will denote this displacement by \( ds_e \).

The components of displacement are thus represented as follows

\[
\begin{align*}
\text{Radial} & \quad ds_r \sin \phi \\
\text{axial} & \quad ds_r \cos \phi \sin \eta \\
\text{and peripheral} & \quad ds_e - ds_r \cos \phi \cos \eta
\end{align*}
\]
Referring now to figures 5 and 5a we have, as before shown, the angles $\beta$ and $\lambda$ measuring the angularity of any peripheral line in the plane of rotation with the vertical direction and hence; expressions 46 to 49 inclusive represent the vertical projections of radial, axial, peripheral and general vectors. Expression 50 represents the projection in the direction of motion of the surface normal projection of a vertical vector. From formulae 46 to 48 inclusive we have thus, as representing the vertical projections of displacements 68, the expressions

\[
\begin{aligned}
\text{radial} & \quad ds_r \sin \phi \sin \lambda \cos \beta \\
\text{axial} & \quad ds_r \cos \phi \sin \eta \sin \beta \\
\text{peripheral} & \quad ds_e \cos \lambda \cos \beta - ds_r \cos \phi \cos \eta \cos \lambda \cos \beta \\
\end{aligned}
\]

and the total vertical displacement will be

\[
\left[ds_r \left( \sin \phi \sin \lambda \cos \beta + \cos \phi \sin \eta \sin \beta - \cos \phi \cos \eta \cos \lambda \cos \beta \right) + ds_e \cos \lambda \cos \beta \right] \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad 70.
\]

For a mass element $dm$ the change $dG$ will thus be

\[
\left[ds_r \left( \sin \phi \sin \lambda \cos \beta + \cos \phi \sin \eta \sin \beta - \cos \phi \cos \eta \cos \lambda \cos \beta \right) + ds_e \cos \lambda \cos \beta \right] \quad g \quad dm \quad - \quad - \quad - \quad - \quad - \quad 71.
\]

in which $\beta$ is the general constant, $\lambda$ is constant along stationary radial lines but varies from one to another and $\phi$ and $\eta$ are in general functions of the radius $\rho$. $ds_e$ is obviously directly dependent on the radius and equal to $\rho d\alpha$ where $d\alpha$ is the angular displacement, about the axis of rotation, of the rotor during time $dt$.

Important practical cases are when the rotor is
horizontal or vertical. In the first case $\beta = 90^\circ$ and $71$ reduces to

$$dG = ds_r \cos \phi \sin \eta \ gdm$$

In the second case $\beta = 0$ and $71$ reduces to

$$dG = [ds_r (\sin \phi \sin \lambda - \cos \phi \cos \eta \cos \lambda) + ds_e \cos \lambda] g \ dm$$

**Component Parts of Differential Change $dG$**. We are next confronted with the problem of discovering the division of $dG$ into its component parts. The total differential is represented by the expression

$$dG_w + dG_N + dG_\phi$$

To determine the magnitude of the transformation $dG_w$, we have to guide us the principle of least action and the constancy of the force gravity. Thus; the least action due to gravity, which can be transmitted between the rotor and fluid element is represented by the product: resolution of surface reaction due to gravity in direction of motion $X$ displacement in direction of motion.

Referring to formula 50 and replacing the vertical vector $u'd'$ therein by $g \ dm$ the force of gravity, we have as the resolution of the surface reaction, due to gravity, in the direction of motion

$$g \ dm \cos \beta \sin(\lambda - \xi) \sin \xi$$

and as the work done due to gravity

$$dG_w = g \ dm \cos \beta \sin(\lambda - \xi) \sin \xi \cdot ds_e$$
If \( \xi \) is zero we note that \( dG_w \) vanishes and hence for this condition all of the differential \( dG \) is compensated for in energy \( N \) and \( \Phi \) within the system.

We have from the foregoing

\[
dG_N + dG_\Phi = dG - dG_w
\]

whence by substitution of 71 and 72 the following expression is arrived at

\[
dG_N + dG_\Phi = \left\{ \sin \phi \sin \lambda \cos \beta + \cos \phi \sin \eta \sin \beta - \cos \phi \cos \eta \cos \lambda \cos \beta \right\} ds_e + \left[ \cos \lambda \cos \beta - \cos \phi \sin \xi \sin(\lambda - \beta) \right] ds_e \]

In this expression the transformation \( dG_\Phi \) will, according to condition 25 of article 16, be a minimum. For all cases this equation with the existence conditions of article 15 and the rates for equilibrium as disclosed in article 16 will afford a general solution of the effects of gravity.

For particular cases we use these formulae together with the expression

\[
\lambda = \lambda_0 + \int_{s_0}^{s} d\xi \quad - - (51)
\]

with a relation

\[
\xi = f (\rho, s, \phi, \eta)
\]

exists and where, as before explained, \( \xi \) represents the angularity of the radius, to the point under consideration, with respect to a reference radius and where \( s \) is the distance along the channel, to said point, from a reference point.

At a point \( \rho \) we have

\[
\rho d\xi = ds \cos \phi \cos \eta = ds \cos \sigma = ds_e
\]

and hence
\[ d\xi = \frac{ds \cos \sigma}{\rho} = \frac{d\xi_e}{\rho} \]

whence (1) may be written in the forms

\[ \lambda = \lambda_0 + \int_{\xi_0}^{\xi} \frac{\cos \sigma}{\rho} \, ds = \lambda_0 + \int_{\xi_0}^{\xi} \frac{d\xi_e}{\rho} \]

We must be careful not to confuse the coordinate \( s \) here used with the coordinate \( s_r \). \( ds \) is exactly the differential tangential distance along the channel of which the peripheral projection is \( \rho d\xi \) or \( d\xi_e \) whereas \( ds_r \) is the relative tangential displacement in the channel of the particle considered, during a peripheral displacement \( d\xi_e \) of a point in the channel at which the particle was considered as located at the beginning of the differential activity.
Chapter V.

CENTRIFUGAL EFFECTS.

40. NATURE OF CENTRIFUGAL COMPRESSION. The tendency to move in a straight line, will cause mass elements in curvilinear motion, to exert, radially outwards, forces proportional to the instantaneous radially inward accelerations to their curvilinear motions. Such motions have thus imposed on them radial constraints causing, or at least always existing with, said curvilinear states of motion. Those constraints or radial forces are as before mentioned, called the centrifugal forces of the motion.

Referring to figure 7 there is represented a channel A, (through which fluid is flowing) in rotation about the stationary center 0. This system is such as to have the increase of fluid pressure, radially outwards, exist by virtue of the centrifugal forces as \( \Delta F' \) and these, as above noted, are exerted by mass elements \( \Delta m \) because of their curvilinear state of motion.

If the fluid pressure increases with the radial distance from the center 0 then in general the intensity of the fluids potential energy will likewise increase with said radial displacement. We are thus forced to investigate the exact nature of the transformations resulting in said increase of potential energy, and will, for the present, further specify the system by imposing the condition, that energy from external systems can only be imparted to the fluid medium as work done on it, through means of the transformer, and obviously at the expense of external energy. We must now first discover in what forms of the fluids energy this work done, may be compensated for.

KINETIC COMPENSATION FOR WORK DONE IN PUMP ACTION. It was
discovered in Article 13 that, for all continuous flow rotary transformers, the whole action, or work done is fundamentally kinetic, and in Article 14, that for pump transformers of this type, not only is the action fundamentally kinetic but the whole energy of the fluid compensated for in the action, is of the zero order kinetic form. In the foregoing analysis all the forces involved were not directly considered, but we shall now directly consider these forces, and shall show that even with centrifugal forces and force heads present, that the whole energy compensated for, in work done, is of the zero order kinetic form. The radial, axial and peripheral components of the resultant forces on the fluid mass elements will be considered and it will be remembered that in a pump action the reaction points are acting on the fluid mass elements.

Between an element $\Delta m$ and a reaction point a peripheral component of force can obviously only exist by virtue of an actual peripheral acceleration of said element. If this is true, then the least work which can be done to the element in a peripheral displacement is exactly equivalent to the change of the peripheral component of the kinetic energy $N$ of the element. The action will of course be a minimum and we conclude that the peripheral component of force increases, in a pump action, only the zero order kinetic energy of the fluid.

The centrifugal forces $\Delta F'$ exist in the radial direction, and if these should have one point of application on the reaction points and the latter are displaced in the line of action of said forces, that is radially, then certainly work could be done on said mass elements even though they remained radially stationary.
or moved with a constant velocity only. That is work could be done from the reaction point on the mass elements which would increase the potential energy of the latter. Such displacements of the reaction points do not occur however for; these are in rotation about a stationary center and hence their motion components in both the radial and axial directions vanish.

As shown in the forgoing, the peripheral component of action can not directly result in an increase of the potential energy of the fluid, and we have shown that radial and axial components of action directly increasing the potential energy are non-existent. We conclude that work done can not (in a pump action) result in an increase of the potential energy of the medium or; as in Article 14, the whole action is directly compensated for in the zero order kinetic energy only of the fluid medium.

Centrifugal forces and likewise axial forces can therefore only result in work done on the mass elements if relative radial and axial acceleration components are allowed to exist. Under these conditions, the kinetic energy of the fluid is caused to be increased by work done because of resultant surface reactions brought into play in the direction of the absolute variation of the fluids motion, and obviously the least action will be exactly equal to the variation of kinetic energy caused thereby.

**KINETIC COMPENSATION FOR CENTRIFUGAL COMPRESSION. SIMPLE AND COMPOUND COMPRESSION.** Since now the work done can not possibly (in a pump activity) result in a direct increase of the potential energy of the fluid medium, then for the increase of this energy for centrifugal compression, we must discover another
source. If there is zero thermal transmission between the central and external systems then, in view of the preceding, obviously the only source left for said potential increase is the radial component of said fluid mediums zero order kinetic energy. The increase of potential energy for centrifugal compression will therefore (with zero thermal communication) be wholly compensated for in this kinetic energy, and if there is not a decrease of the latter, there can not possibly be any such work of compression. It is to be noted that we call it work of compression even though it is not directly compensated for in energy of external systems. The kinetic energy compensating for the compression was, during its period of generation, however directly compensated for in external systems and thus the energy of centrifugal compression represents work done during the preceding period of time in which the compensating energy \( N \) was generated.

Now obviously if there is an increase of pressure in the line of progression of the element \( \Delta m \) then the only possible way in which the work of compression can vanish, that is be the equal to zero, is for the volume to remain constant. This does not however necessarily mean, that in all cases in which the volume remains constant there can not be a conversion of kinetic energy to potential energy by a reduction, in said radial component of the kinetic energy.

We will call such a constant volume centrifugal pressure increase, when it is not accompanied by a decrease of the radial component of the kinetic energy, a pure or SIMPLE CENTRIFUGAL COMPRESSION to distinguish it from the COMPOUND CENTRIFUGAL COMPRESSION wherein, the compression is compensated for by a
reduction in the radial component of the kinetic energy simultaneously with the variation of pressure because of centrifugal forces.

For incompressible media therefore a simple compression exists when the increase of centrifugal pressure is unaccompanied by a reduction of kinetic energy, and this is the ordinary centrifugal head for liquids. For compressible media however we can attain simple compression only by the addition of energy as heat, during compression, from an external source, at such a rate that with the increasing pressure the specific volume is maintained exactly constant. All ordinary centrifugal compressions of these media must therefore inevitably be compound compressions or, for such, if there is not a conversion of kinetic energy to work of compression, an increase of pressure due to centrifugal forces cannot exist.

41. CENTRIFUGAL CONSTRAINT FUNCTION I. We shall now build up a defining function for the radial constraints on the fluid mediums motion, known as centrifugal forces.

Suppose that any mass point $dm$ (figure 8) possesses such a motion that its components in the orthogonal system of axes $e, a,$ and $\rho$, are $q_e, q_a,$ and $q_\rho$ respectively. The magnitude of the function $I$, for the particle in any particular state of its motion, we will define as equal to the kinetic energy equivalent of the change in velocity $q_\rho$ necessary to cause the projection in the $e \rho$ plane, of the particles motion, to be rectilinear.

The differential $dl$ of $I$ is thus represented for a
particle $dm$ by the expression

$$dI = dm \, q_{\rho}' \, dq_{\rho}'$$

for a unit weight of the fluid by the expression

$$dI = \frac{q_{\rho}'}{g} \, dq_{\rho}'$$

and lastly for a unit volume mass of the fluid by the expression

$$dI = \frac{\mu}{g} \, q_{\rho} \, dq_{\rho}$$

where $\mu$ represents the density.

Suppose now that the particles motion is considered as it passes through the transformer, and that the $e$, $\alpha$, and $\rho$, axes represent respectively the peripheral, axial and radial directions on the rotor. The plane $e\rho$ is now the rotational plane, and since axial variations are normal to it and hence have only zero resultant components therein, then certainly when the components of motion in the rotational plane are rectilinear the centrifugal forces vanish. But when this is true then likewise the function $I$ vanishes and we conclude; that the function $I$ is for a given transformer a measure of the centrifugal forces. For the condition of motion in which $I$ vanishes there is certainly existent the state of motion called in Article 33 the centri-vanescent state, and when the motion in the rotational plane is curvilinear then, with certain restrictions as to the causes of the curvilinearity of the motion, the centri force state of motion is in existence.

42. FUNCTION $I$ EXPRESSED IN TERMS OF COORDINATES $q_{\rho}$ AND $q_e$. The center of rotation of the transformer is
regarded as being stationary and when this is true then the translatory components \( q_\rho, q_u \) and \( q_c \) of the fluids motion are respectively exactly equal to its absolute translatory components \( q'_\rho, q'_u \) and \( q'_c \).

This being true, then if from any cause whatsoever the relative radial velocity \( q_\rho \) is changed by an amount \( dq_\rho \) then the differential \( dm q_\rho dq_\rho \) for the particle \( dm \) will be exactly to its differential \( dm q'_\rho dq'_\rho \) for the same radial change. We conclude that the differential change of the function caused by a variation \( dq_\rho \) is represented by the expression

\[
dI_\rho = - dm q_\rho dq_\rho
\]

Expression 78 where \( dI_\rho \) denotes the partial variation of I with respect to \( q_\rho \) and where \( dq_\rho \) denotes an increase in the direction radially outwards of the fluids velocity. The partial differential symbol is used to express the variation I because, as will be shown below, the magnitude of I depends upon both the components \( q_\rho \) and \( q_c \) and hence a variation of either taken separately is only a part of the possible total variation which may be caused to it.

If now in addition to the change \( dq_\rho \) there is a variation \( + dq_c \) from a centri-force cause (Art. 43), in the peripheral component \( q_c \) then, as will be shown below, the partial change in I caused thereby is represented by the expression

\[
\delta I_c = dm q_c dq_c
\]

Expression 79 is stated in words as; if in a curvilinear motion, the peripheral component of velocity is caused by the rotors motion to be changed, then the necessary magnitude of the
radial component of velocity for rectilinear motion is varied by exactly the same amount.

PROOF OF RELATION 79. If the absolute motion of the particle is considered it will have peripheral, radial and axial components. Of these the axial component is, as above stated, normal to the plane of rotation and hence its variations can have only zero effect on centrifugal magnitudes. It is necessary to consider therefore, concerning the latter, only the radial and peripheral components of the motion, whence; the considered component of the motion is planar, and it is our object to find the conditions of motion which will cause said planar component of motion to become rectilinear.

Any motion which is rectilinear will have zero angular velocity and likewise any components of such a rectilinear motion, in its own plane, will certainly possess zero angular velocity. Furthermore if any component, in its plane of a planar motion has zero angular velocity then likewise will all its components, in that plane, and hence the motion itself possess zero angular velocity.

If therefore the condition can be discovered, which will cause the angular velocity of either the radial or peripheral components of the motion to vanish, this will likewise be a condition for which the component of motion in the rotational plane will be rectilinear.

We now choose to find the condition which will cause the angular velocity of the peripheral component to vanish. Referring to figure 9a, there is represented a particle $dm$ in pure
Fig. 9a.

Fig. 9b.
circular motion about a center \( O \). Its path \( pb \) is now described as such that at every point its distance from the fixed center \( O \) is constant. It is readily proven that the radial acceleration \( Q'_\rho \) existing by virtue of the departure of the motion from the straight line path \( pc \) (that is due to the angular motion of the particle), is represented by the expression

\[
-\frac{q_e^2}{\rho^2}
\]

where \( q_e \) represents the peripheral velocity of the particle at the instant of consideration, and where the minus sign indicates the direction toward the center \( O \). If the motion is purely circular as here described then, since \( \rho \) is constant the relative radial acceleration \( Q'_\rho = \frac{dq_e}{dt} \) vanishes.

Suppose now that instead of the pure circular motion that it is as represented by the path \( p'b' \) of figure 9b, so that at the point \( p' \) the absolute velocity \( q \) has, referred to the rotor circle \( p'b \) a peripheral component \( q_e \). It has in addition a component of radial velocity \( q_\rho \) so that the particle varies its relative radial position in the rotor as well as its circumferential position. It may thus in general likewise possess a relative radial acceleration \( Q_\rho = \frac{dq_\rho}{dt} \) and, when this is true, then its absolute radial acceleration \( Q'_\rho \) is no longer represented by \( 80 \).

To arrive at the exact magnitude of the resultant radial acceleration for the latter conditions of motion we consider the source of the motion component \( q_e \). We are considering here a pump action whence this component is regarded as being wholly generated by virtue of the rotary motion imparted from the channel surface points to the fluid particle under consideration. If now the motion imparted to the fluid particle is identical with
the motion of the surface point then \( q = q_e = e \) and the motion will be purely circular. For this condition, as mentioned above, the acceleration component \( Q_\rho \) vanishes and the resultant radial acceleration \( Q'_\rho \) is represented by the expression \(-\frac{q^2}{\rho}\). Suppose now that on the particle in purely circular motion, a relative radial acceleration \( Q'_\rho \) is imposed. It will be directed along that radial line in which \(-\frac{q^2}{\rho}\) is directed at the instant of the appearance of \( Q_\rho \) and it follows; in view of this and since \( Q'_\rho = -\frac{q^2}{\rho} \) when \( Q_\rho \) vanishes, that when \( Q_\rho \) exists, that the resultant radial acceleration \( Q'_\rho \) is represented by the algebraic sum of the former two components as follows:

\[
Q'_\rho = \pm Q_\rho - \frac{q^2}{\rho}
\]

This expression exactly represents the centrifugal acceleration causing the existent centrifugal forces. It is likewise a measure of the curvilinearity of the motion due to the rotational motion of the source from which the motion is derived. It follows that when the acceleration \( Q'_\rho \) in the rotational plane vanishes that then the motion therein is rectilinear. Now 81 can obviously only vanish if we have \( Q_\rho \) positive whence; as a representation of the condition of rectilinearity of the motions we have the relation

\[
+ Q_\rho - \frac{q^2}{\rho} = 0 \quad \text{or} \quad + \frac{dq_\rho}{dt} - \frac{q^2}{\rho} = 0
\]

The latter expression may be stated in terms of \( \rho, q_\rho, n \) where \( n \) represents the number of revolutions per unit time for a peripheral velocity \( q_e \) at the radius \( \rho \).

We have, since \( q_\rho dt = d\rho \) by substitution in 82
\[ q_\rho dq_\rho = 4 \pi^2 n^2 \rho d\rho \]  

whence by integration

\[ \frac{1}{2} q_\rho^2 = \frac{1}{2} q_e^2 \]

and

\[ q_\rho = q_e \]

Relation 84 is now exactly an expression of the condition for the rectilinearity of the projection of the particles motion into the rotational plane. It may be otherwise stated as; the peripheral velocity component \( q_e \) must not be greater than the relative radial velocity component \( q_\rho \).

Suppose now that for a condition of motion described by the relation \( q_\rho \geq q_e \) we increase \( q_e \), from a centri-force cause, by an amount \( dq_e \). Obviously, in view of 84, we must now increase the magnitude of \( q_\rho \) necessary for rectilinear motion in the \( \rho \) plane, by exactly the same amount whence; expression 79 is proven true.

43. CRITICAL RATIO AND FIELD OF CONDITIONS FOR CENTRI-VANESCENT AND CENTRI-FORCE STATES. The above relation 84 between the relative radial and peripheral components of velocity represents, as below shown, exactly the critical ration between them so that for motions described by the relation

\[ q_\rho \geq q_e \]

the state of centri-vanescence exists. For those described by the relation

\[ q_\rho < q_e \]

obviously, in view of the foregoing, the state of centri-force exists, if as before, \( q_e \) has been generated from a centri-
PROOF OF TRUTH OF CONTION 85a FOR CENTRI-VANESCENCE.

Any component of a rectilinear motion may be increased by a rectilinear amount and the new resultant motion will be rectilinear. If $q_\rho = q_\epsilon$ then as above shown the motion is rectilinear and the centri-vanescent state exists. If now $q_\rho$ is increased in magnitude, over its magnitude $q_\rho = q_\epsilon$ the only cause of this increase can be from gravity or from the force existing by virtue of a negative variation of the pressure potential in the line of flow. These forces will obviously have a constant straight line direction whence; they can, of themselves, cause only rectilinear variations in the motion components and we conclude; that the resultant motions, if mechanically free, will be thereby varied only be rectilinear magnitudes. Of course these latter variations, which will be rectilinear if they can, might, by the varying direction of mechanical walls, (mechanically constrained, or not mechanically free) be constrained to generate into curvilinear variations. In such cases however the constraining forces (being surface normal reactions) are all normal to the direction of fluid flow, and hence for such, there would still be zero tangential resultants due to centrifugal forces as; the latter themselves, if existent, will be identically the above mentioned surface normal reactions, and hence can have only zero resultants in the line of flow. We conclude therefore that the state of centri-vanescent motion (or non-existence of tangential resultants due to centrifugal forces) exists for all motions which are such as to fulfill the relation 85a.
The term centri-vanescent is thus extended to include all motions free of tangential resultants due to centrifugal forces. It is a broader defining term than the term rectilinear and includes all rectilinear motions and in addition those curvilinear motions whose curvilinearity is due wholly to surface flow reactions normal to the direction of motion. In short a motion is in the centri-vanescent state of generation if, at a point, the force causing the motion has a constant straight line direction, and such a force condition is properly called a centri-vanescent cause. A force of varying directions would thus be termed a centri-force cause.

Obviously after the variation of motion from a centri-vanescent cause or simultaneously with its generation, surface normal reactions may come into play so as to deflect the motion from the straight line path, but the motion will still be in a centri-vanescent state because, as noted, said normal reactions have zero resultants in the direction of flow.

LIMITATIONS OF FIELD OF APPLICATION FOR CRITERIA 84, 85a and 85b. When a motion is such as to fulfill relation 84 then obviously its components \( q_e \) and \( q_\rho \) become respectively identical to its components \( q'_e \) and \( q'_\rho \) or; the angular velocity in the plane of rotation of the particle under consideration vanishes, and all components of its motion in the rotational plane are rectilinear. Now, regardless of the previous motional history of the particle, when the state of motion described by the relation \( q_\rho = q_e \) is attained then its angular velocity in the rotational plane will vanish and its action in
this plane will be rectilinear. We conclude; that relation
84 is a criterion for the rectilinear state, in the rotational
plane of any fluid action whatsoever through the transformer.

In 85a the above criterion is included in the more
general relation expressed by \( q_\rho \geq q_e \). Since now, as above
noted, the relation \( q_\rho = q_e \) is always a criterion for the
rectilinearity of the motion, and since, as before shown, any
variation \( dq_\rho \) above the magnitude of \( q_\rho = q_e \), has as the only
source a centri-vanescent force condition, then likewise the
relation \( q_\rho > q_e \) is a general criterion of the centri-vanescent state
of the motion. We conclude; that relation 85a just as 84 is a
general criterion for the centri-vanescent state of the fluid
motion component in the rotational plane of the transformer.

The criterion 85b may as before mentioned, not however
in general indicate the existence of a centrifugal force. For
instance it is conceivable that a motion may have a component
of motion \( q'_e < q'_\rho \) where now both components are rectilinear. In
that case the motion is obviously in the centri-vanescent state,
but if the relation
\[
q'_e < q'_\rho
\]
is true then likewise, for any instant, the relation
\[
q_e < q_\rho
\]
for the motion will be satisfied. This is exactly the criterion
85b for the centri-force state. We conclude that whether or not
85b is a criterion of the centri-force state depends on the
previous history of the motion and in order to discover that
history for which the criterion is applicable we need merely to
examine the conditions for which the theory has been deduced.

In building up the theory for the criteria of these states of motion we have considered the component \( q_e \) as wholly imparted to the fluid by the rotary motion of the channels working surfaces. For motions generated in this way (that is for pump actions) therefore 85b is exactly a criterion of the existence of the centri-force state. In such cases the motion is generated from a centri-force cause and we may say that criterion 85b is applicable only when the component \( q_e \) has a centri-force origin. We conclude; that criterion 85b, and the theory for centri-force states of motion, are only applicable when the motion component \( q_e \) has, as above mentioned, a centri-force origin or more specifically, when it is generated from rotating surface reaction points.

44. EQUILIBRIUM AND MAGNITUDE OF ACTION FOR CENTRI-VANESCENT AND FROM CENTRI-FORCE STATES OF MOTION IN PUMP ACTIVITIES. For such activities, as before shown, there will, during action, not be a positive variation of the pressure potentials and certainly a negative variation would not be desirable whence, the pressure will in general not vary.

CENTRI-VANESCENT STATE OF MOTION. If the motions satisfy relation 85a the condition of equilibrium is, as before shown, represented by the equation 59 of least action. We have

\[
\frac{r_x}{r_i} + \int \cos \sigma \, dr = (59)
\]

which with the general equation, for kinetic variation,

\[
\frac{dN_w}{N} = \frac{d}{dm} \left[ e^2 + r^2 - 2er \cos \sigma \right]
\]
will determine the magnitude of the action when the fluid pressure remains constant.

CRITICAL RATIO 84 IN TERMS OF RELATIVE VELOCITY \( r^* \).

We have as the critical relation 84 between the relative radial and the peripheral velocities the expression

\[
q_p = q_e - - - - (84)
\]

whence; by, substitution of their equivalents in terms of the relative velocities and the channel angularity the following relation is deduced.

\[
r^* \sin \phi = e - r^* \cos \sigma
\]

Solving for the critical magnitude \( r^* \) of the relative velocity \( r \) we have

\[
r^* = \frac{e}{\sin \phi + \cos \sigma} = \frac{e}{\sin \phi + \cos \phi \cos \sigma} - - - - - - 86a
\]

As shown in Article 23 the relative radial velocity \( q_p \) is exactly the radial component of the relative velocity \( r \) and therefore the magnitudes of these two vary in the same sense when either is effected only by the others variation. We therefore conclude that if the relative velocity \( r \) as solved for from the equation of least action 59 is greater than the value \( r^* \) from equation 86a that the activity will occur in the centri-vanescent states of the motions, and that therefore, the purely kinetic action is exactly specified by said equation of least action 59. If however the value of \( r \) as above solved for is less than \( r^* \) the motions will be in the centri-force state and hence in solving for the action these must be taken into consideration.
EQUILIBRIUM FROM THE CENTRI-FORCE STATE. With zero variation of the fluids pressure, in the direction of flow, the condition of equilibrium from a centri-force state will obviously be reached exactly when the motion has attained the centri-vanescent state and the least variation transforming to this state will be that occurring. The equation of equilibrium is represented by \( 86a \).

\[
\frac{e}{\sin \phi + \cos \sigma} = (86a)
\]

and for any continuous channel at whose exit

\[
r^c_e = (r_c + \int \cos \sigma \, d\epsilon)
\]

the final relative velocity \( r^c_e \) will be exactly equal to

\[
\frac{e_2}{\sin \phi_2 + \cos \sigma_2} = (86b)
\]

The action will for such cases thus have exactly the critical magnitude.

If \( r^c \) reaches a maximum value \( r^c_x \) at a section \( x \) (such that \( r^c_x > r^c_z \) ) before arriving at the terminal of the channel, then the equation of equilibrium is represented by

\[
r^c = r^c_x + \int_x^z \cos \sigma \, d\epsilon = \frac{e_x}{\sin \phi_x + \cos \sigma_x} + \int_x^z \cos \sigma \, d\epsilon = (86c)
\]

for; beyond the section \( x \), the relative velocity will again obey the law of variation of least pure kinetic action, subject to the condition, that in said latter region the state of centri-vanescence exists.

If now the condition of centri-force exists so that the terminal fluid pressure is greater than the initial, then the flow through the channel must be such as to have equilibrium between the existing forces. To discover the conditions
of equilibrium we must first know the magnitudes of these forces and these are investigated in the following.

45. TOTAL DIFFERENTIAL AND INTEGRAL OF CONSTRAINT FUNCTION I. The total differential variation $dI$ caused to the function $I$ by a variation

$$dN = dm \left( q_e dq_e + q_\rho dq_\rho + q_\alpha dq_\alpha \right)$$

in the particles kinetic energy is, as shown in Article 42 represented by the expression

$$dI = \partial I_e + \partial I_\rho$$

where the particle $\partial I_\alpha$ does not appear because of the absolute ineffectiveness of axial motional variations on the component of the motion in the rotational plane. We have by substituting 78 and 79 in the above relation

$$dI = dm \left( q_e dq_e - q_\rho dq_\rho \right)$$

whereas before, $dq_e$ denotes the increase of peripheral velocity due to the increase $de$ of the rotational velocity of the active surface reacting points, and $dq_\rho$ the increase of relative radial velocity outwardly from the center of rotation, of an elementary mass $dm$.

For a given mass $dm$, 87 is obviously exact and hence the variation of $I$ is independent of the path and depends only on the initial and final conditions of velocity of the given mass. This of course with the restriction, that $+ dq_e$ must have a centri-force cause.

We have as the absolute value of $dI$ for a given element
or between the limits 1 and 2 the variation is
\[ dI = \int dq \left( \frac{q^2}{2} - q_{r}^2 \right) + I_0 \]

These expressions are, as above, subject to the restriction of having a centri-force cause for the variation of \( q_e \) and to the limiting condition that \( q_r \leq q_e \) for obviously if \( q_r > q_e \) then \( I = dI = 0 \).

46. PRESSURE DIFFERENTIAL AND RADIAL RATE OF VARIATION OF CENTRIFUGAL COMPRESSION HEAD. Referring again to figure 7, we will consider the forces \( \Delta F' \) and \( \Delta P \). Of these the former is exerted radially outwards by the element \( \Delta m \) because of the radially inward acceleration of its curvilinear motion, and the latter is exerted on the element because of the inward variation of the pressure potential. The resultant force in the direction radially outwards will be
\[ \Delta F' - \Delta P = \Delta F \]
and this force will tend to cause a relative radial acceleration \( \frac{dq_r}{dt} \) of the element, such that the relation
\[ \Delta F = \Delta m \frac{dq_r}{dt} \]
is true.

This acceleration is directed radially outwards and hence, as before shown, decreases the magnitude \( q'_r \), (existing before its appearance) of the absolute radial acceleration by exactly the amount of said appearing acceleration component. Thus an increase of this component is not accompanied by an
increase of radial force but exactly by a decrease of this force. When, within the motional limits \( q_e \equiv q_f \), this component acceleration has attained the magnitude

\[
\frac{dq_e}{dt} = \frac{\Delta F}{\Delta m}
\]

where \( \Delta F \) represents the difference \( \Delta F'_1 - \Delta P \) at the original instant when \( \frac{dq_e}{dt} \) was zero, then the force \( \Delta F \) has been reduced to zero, and the equation

\[
\Delta F'_1 - \Delta F = \Delta F'_2
\]

is true, where \( \Delta F'_2 \) represents exactly the total force exerted radially outwards in the final state of curvilinear motion and \( \Delta F'_1 \), as before, exactly that at the considered initial instant when \( \frac{dq_e}{dt} \) was zero.

We may write the foregoing equations in terms of differentials as

\[
\begin{align*}
dF'_1 - dP &= dF \\
dF &= dm \frac{dq_e}{dt} \\
dF'_1 - dF &= dF'_2 \\
and \quad dP &= dF'_1 - dm \frac{dq_e}{dt}
\end{align*}
\]

Consider now an element of volume of unit cross-sectional area and depth \( d\rho \). The differential \( dP \) of 90 is for such exactly the differential of pressure, and if \( F'_1 \) is the equivalent centrifugal force exerted by a unit volume under average conditions equal to the instantaneous conditions of the element in consideration at the initial instant when \( \frac{dq_e}{dt} = \) zero, then, in terms of this force that exerted by the differential volume element at the same instant will obviously be

\[
dF'_1 = F'_1 d\rho
\]
The differential $\frac{dt}{d\xi}$ may be represented by $\frac{d\rho}{q\rho}$ and $d\mu$ by $\frac{\mu d\rho}{q\rho}$ whence, by substitution in the last equation of 90 we have

$$\frac{d\rho}{d\xi} = F_1' - \frac{\mu}{g} \frac{q\rho}{d\rho}$$

where $\mu$ represents the fluid density.

The magnitude of the force coordinate $F_1'$ in the above expression is, as before noted, exactly equal to the radial force magnitude for the given motional state if the relative radial acceleration $\frac{dq\rho}{dt}$ vanishes or is reduced to zero. To discover this magnitude we investigate first, the magnitude of the function $I$ for a pure circular motion, and to avoid confusion we will in the following measure the intensity of $I$ by its magnitude per unit weight. We have thus

$$I = \frac{1}{2g} \left[ q^2 - q^2 \right]$$

whence, for a unit volume mass the magnitude of $I$, in terms of the above units, will be

$$\mu I = \frac{\mu}{2g} \left[ q^2 - q^2 \right]$$

For the pure circular motion the relative radial acceleration $\frac{dq\rho}{dt}$ vanishes whence, for such, the magnitude of the centrifugal force is certainly a special case of the magnitude of the force $F_1'$. For such a motion the absolute radial acceleration is, as before, equal to

$$-\frac{q^2}{\rho}$$

and the centrifugal force $F_1'$ of a unit volume mass is for this case equal to

$$\frac{\mu}{g} \frac{q^2}{\rho}$$
But for a pure circular motion \( q_{r} = 0 \) whence;
\[
I = \frac{1}{2y} q_{r}^{2}
\]
and the above centrifugal force is for this case expressed in terms of \( I \) and \( \rho \) as
\[
F'_{r} = -\frac{\epsilon \mu I}{\rho}
\]

The question immediately arises: is this a general form of the expression representing \( F'_{r} \)? To discover the answer we investigate the origin of the force \( F'_{r} \) so that it, and the result of its activity, may be classified. Obviously this force has resulted directly from an ordinary (Newtonian) acceleration, directed now along radial lines toward the center of rotation, and the magnitude of the existing force will be proportional to said existing radial acceleration. Let us now briefly examine the relation of the force and motional conditions in a channel through which a variation of the fluid pressure is compensated for in the kinetic energy of the fluid.

Obviously for such an activity the acceleration of the fluid is exactly proportional to the force condition existing, but this is exactly a description of the force condition existing because of said radial acceleration above. We conclude that a motion is certainly varied in the same degree by a variation of the intensity of the force condition in either activity. In the latter activity the following relations exist:
\[
\frac{dh}{dq} = \frac{1}{2} q dq
\]
\[
\mu h_{r}^{e} = \frac{1}{2g} q^{2} l_{r}^{2}
\]
or \( F = \alpha q^{e} \)

where \( \alpha \) represents the proportionality factor connecting the
force $F$, due to a column of fluid of height $h$, and the velocity $q$ caused by converting the static head $h$ to a kinetic head.

$q^2$ will thus represent for this column the change of velocity caused through the channel with the difference of head $h$. In the channel section where the velocity $q$ has attained a maximum value, the head $h$ will have been reduced to a minimum whence, the last equation may be stated as: the decrease of force head, compensated for in kinetic head, is proportional to the square of the velocity increase caused thereby.

The force head $F'$ obviously tends to increase the velocity component $q^2$ and in view of the above we now conclude; that the magnitude of the decrease of $F'$ less than its magnitude with $q^2=0$ must be proportional to the square of the existent magnitude of $q^2$. With $q^2=0$ we have

$$F' \propto \frac{q^2}{\rho}$$

whence in general with $q^2>0$ we have, in view of the above,

$$F' \propto \left[\frac{q^2}{\rho} - C'q^2\right]$$

where it is necessary to disclose the form and value of the factor $C'$. In Article 42 it was discovered that with $q^2 = q^2$ that the force $F'$ must vanish, which can only in general be true for the above expression if $C'$ has the value

$$\frac{1}{\rho}$$

whence

$$F' = \left[\frac{q^2}{\rho} - \frac{q^2}{\rho}\right]$$
or
\[ F' = C' \left[ \frac{q^e - q^f}{\rho} \right] \]
in which the constant \( C' \) is readily evaluated by assigning a special value to \( q_\rho \). Thus with \( q_\rho = \text{zero} \) \( C' = \frac{\mu}{g} \) and by substitution we have
\[
F' = \frac{\mu}{g} \left[ \frac{q^e - q^f}{\rho} \right]
= \mu \frac{2\mathbf{I}}{\rho}
\]
which is perfectly general and identically expression 91.

We note by inspection that the second term of the right hand member of expression 90a is exactly equal to the partial derivative of \( I \) with respect to \( \rho \) holding \( q_e \) constant and multiplied by \( \mu \) whence by substitution
\[
\frac{dP}{d\rho} = \mu \left[ \frac{2\mathbf{I}}{\rho} + \left( \frac{\partial}{\partial \rho} q_e \right) \right]
\]

If the rate of variation of head \( h \) with respect to \( \rho \) is considered this expression reduces to
\[
\frac{dh}{d\rho} = \left[ \frac{2\mathbf{I}}{\rho} + \left( \frac{\partial}{\partial \rho} q_e \right) \right]
\]

Expressions 92a and 92b represent respectively the variation of pressure and head of the fluid within a rotating channel wherein the component \( q_e \) has been generated from a centri-force cause, and with this restriction they are perfectly general and applicable to all uses whether the compression is simple or compound.

47. CONDITION FOR WHICH \( I \) VANISHES AND MANNER OF VARIATION OF PRESSURE IN CENTRIFUGAL HEAD. We will now investigate the condition \( \frac{dP}{d\rho} = \frac{dh}{d\rho} = 0 \) for these and higher order derivatives. For this condition
\[
\frac{2I}{\rho} = -\left(\frac{\partial I}{\partial \rho}\right) q_e
\]

Differentiating 93 with respect to \( \rho \) there results

\[
\frac{d\left(\frac{2I}{\rho}\right)}{d\rho} = -\frac{d\left(\frac{\partial I}{\partial \rho}\right)}{d\rho} q_e
\]

where now the right hand side is exactly equal to

\[
\left(\frac{\partial^2 I}{\partial \rho^2}\right) q_e
\]

because the partial \( \left(\frac{\partial I}{\partial \rho}\right) q_e \) contains no terms involving \( q_e \)

Thus

\[
\frac{d\left(\frac{2I}{\rho}\right)}{d\rho} = -\left(\frac{\partial^2 I}{\partial \rho^2}\right) q_e
\]

Similarly

\[
\frac{d^2\left(\frac{2I}{\rho}\right)}{d\rho^2} = -\left(\frac{\partial^3 I}{\partial \rho^3}\right) q_e
\]

and

\[
\frac{d^{n-1}\left(\frac{2I}{\rho}\right)}{d\rho^{n-1}} = -\left(\frac{\partial^n I}{\partial \rho^n}\right) q_e
\]

Writing these equations in the implicit form we have

\[
\frac{d\left(\frac{2I}{\rho}\right)}{d\rho} + \left(\frac{\partial^2 I}{\partial \rho^2}\right) q_e = 0
\]

\[
\frac{d^{n-1}\left(\frac{2I}{\rho}\right)}{d\rho^{n-1}} + \left(\frac{\partial^n I}{\partial \rho^n}\right) q_e = 0
\]

of which the left hand members are exactly equal to the derivatives

\[
\frac{dh}{d\rho}, \quad \frac{d^2 h}{d\rho^2} - \ldots - \frac{d^{n-1} h}{d\rho^{n-1}}
\]

This forces the conclusion that for the condition in which the first derivative \( \frac{dh}{d\rho} \) vanishes that likewise all higher derivatives vanish. This condition can only be satisfied if I vanishes whence; the state of motion is exactly centri-
vanescent or the force vanishes as obviously it should if the potential variation $\frac{dh}{d\rho}$ vanishes. Since now if any derivative of the forms $\frac{dh}{d\rho}, \frac{dP}{d\rho}$ vanishes all others of higher orders likewise vanish we are forced to the further conclusion; that the centrifugal compression curve has no maximum or minimum points. The centrifugal pressure within a continuous rotating channel cannot therefore under any circumstances have a maximum or minimum value between the ends of the channel and furthermore; if in a rotating channel the fluid pressure has a maximum or minimum value between the ends of the channel, there is a disconintuity in this sense, whence; any one such differential equation as above will not fit the variation for the whole channel. For complete solution such channels must be divided into separate regions in some of which centrifugal head variations may exist and certainly in some others pressure variations will be caused from other than centrifugal effects. These conclusions are consistent with those arrived at in Article 14 as regards the nature of the transformations $d\Phi_N$ in a pump activity for; if we have such a transformation terminally, it certainly will be possible to have a minimum pressure between the inlet and exit of said channel.

48. PRESSURE INTEGRAL FOR CENTRIFUGAL COMPRESSION.

We have from formula 92b the differential

$$dh = \frac{2I}{\rho} d\rho + \left( \frac{\partial I}{\partial \rho} q_e \right) d\rho$$

as representing the differential increase of head due to centrifugal compression. If there is an actual volume compres-
sion it must be from kinetic energy and hence the differential of head \(dh\) must have the form of a differential of kinetic energy. For \(dh\) we may therefore substitute the product \(\nabla dP\) whence we have

\[\nabla dP = \frac{2I}{\rho} d\rho + \left(\frac{\partial I}{\partial \rho}\right)_{q_e} d\rho - - - - - - 94\]

as the general differential expression where \(I\) is the functional magnitude per unit weight.

From the formula 92a we have the same differential involving the density \(\mu\) as follows:

\[dP = \mu \left[\frac{2I}{\rho} d\rho + \left(\frac{\partial I}{\partial \rho}\right)_{q_e} d\rho\right]\]

and writing the density term \(\mu\) with the differential \(dP\) the expression becomes

\[\nabla dP = \left[\frac{2I}{\rho} d\rho + \left(\frac{\partial I}{\partial \rho}\right)_{q_e} d\rho\right] - - - - 95\]

where the left-hand term represents the kinetic compression which is thus consistent with our deductions as regards this compression.

For complete integration of the right-hand side of the equation we must know the relation existing between \(q_e, q_P\) and \(\rho\). This integral is thus dependent on the path of the integration. To integrate the left-hand side of the expression we must obviously know the relation existing between the volume and pressure variations of the fluid for the kind of compression involved.

49. CONSERVATION EQUATIONS OF ACTIVITY, AND VARIA-
TIONS OF PHYSICAL STATE OF MEDIA DURING COMPRESSION. For
the simple compression of compressible media we must have an
addition of heat \(-dM\) from external systems. Accordingly
we must before building up and applying our conservation
equations to include the above activities, have a means of
representing and defining this differential addition of energy
as heat.

During simple compression the volume of the medium
remains constant whence, no external work is done by it and
all the energy added as heat appears as increased intrinsic
and hence likewise configurational energy of the medium. The
quantity of heat flow is over the given path, now purely a
function of the variation of the physical state of the medium
caus ed thereby. It will be exactly an algebraic addition
to the configurational energy of the medium.

The differential conservation equation of activity
with zero heat flow can now be made applicable to the case
wherein heat flow takes place, as above specified, by merely
adding with the proper sign the above differential of heat
flow. For a positive flow as \(+dU_m\) to the central system,
the intrinsic energy will obviously be increased by exactly
this amount over its intensity without this addition. Without
this addition the most general equation 16 may be written
in the form

\[ dD + d\Phi_g + d\Phi_n + d\Phi_w = dN_w'' \]

or

\[ dD + d\Phi = dN_w''' \]
The text on this page is not legible due to the image quality. It appears to be a page from a document, but the content is not discernible.
adding \( dU_m = d\Phi_m \) to both sides we have

\[
dD + d\Phi_g + d\Phi_n + d\Phi_w + d\Phi_m = dN_w'' + dU_m
\]

or

\[
dD + d\Phi = dN_w'' + dU_m
\]

In these forms the equation may be used. For any pump activity the term \( d\Phi_w \) will vanish and for simple compression with zero decrease of pressure in the direction of flow the transformation \( dN_w \) will likewise vanish.

The kind of compression during the activity will immediately, from the physical state variations of the media during compression, determine the quantity \( dU_m \) and the required work of compression will determine the transformation \( \int d\Phi_n \). The transformation \( \int dG_q \) will as explained in Article 15 be determined by the variation of vertical position of the media, and the total variation of the dynamic energy desired will as a rule be known, so that the solution of the equation reduces to an advantageous choice of the general relations to exist between the restraint coordinates involved in the terms \( dD \), and \( dN_w'' \).

**Differential State Equations During Compression Cycle.**

We will next investigate the variations of physical state during centrifugal compression. Since the energy of the medium is a function of its physical state only we can obviously determine the variation of the physical state if we know the variation of its energy.

As the total differential \( d\Phi \) of the configurational energy we have
\[ d\Phi = d\Phi_w + d\Phi_N + d\Phi_m \]
\[ = d\Phi_w + d\Phi_N + dU_m \]
where \( d\Phi_N \) represents the transformations \( d\Phi_G + d\Phi_N \).

From the thermodynamics* we have
\[ dU = Y_v dT + \left( T \frac{dP}{dT} - P \right) dV \]
and for \( d\Phi = dU + d(PV) \)
\[ d\Phi = Y_v dT + T \frac{dP}{dT} dV + vdP. \]
as the general differentials of the intrinsic and configurational energy respectively, and where \( Y_v \) represents the specific heat at constant volume.

We have therefore the expression
\[ d\Phi_w + d\Phi_N + dU_m = Y_v dT + T \frac{dP}{dT} dV + vdP \]
as the general differential equation of the configurational energy for the effects specified, where, since for any but motor activities \( d\Phi_w \) vanishes, the term \( d\Phi_N \) represents exactly the transformation of the compression cycle which must be compensated for in the kinetic energy, and which is thus otherwise represented by the terms \(-\frac{gP}{g}da_d\text{ and } -vdP\).

We arrive at the general relation
\[ d\Phi + dU_m = -vdP + dU_m = Y_v dT + T \frac{dP}{dT} dV + vdP \]
the left hand side of which represents exactly the total compensating energy for the compression cycle variation represented by the right hand side. This equation is applicable to all pump activities having energy transformations between the central and external systems occur as work, and energy flow as heat wholly compensated for in the intrinsic energy of the central system.

*G. H. Bryan Thermodynamics, Art. 133.
For an adiabatic compression \( dU_m = 0 \) whence 97b reduces to
\[
d\Phi = -\nu dP = \gamma_v dT + T \frac{\partial P}{\partial T} d\nu + \nu dP - - - - - - 97c.
\]

For an isodynamic compression the intrinsic energy \( U \) remains constant whence the heat flow must be such that
\[
+ dU_N = - dU_m = dM_0 \quad \text{and the sum} \quad (dU_N + dM_0) \quad \text{of their variation vanishes.} \quad \text{In other words} \quad dU \quad \text{vanishes and the left hand side of 97b reduces to the differential} \quad d\Phi = dJ \quad \text{. To discover the form of the left hand side we will first investigate the general differential} \quad dU \quad . \quad \text{We have from the foregoing, since} \quad dU \quad \text{vanishes}
\]
\[
\gamma_v dT + (T \frac{\partial P}{\partial T} - P) d\nu = 0 \quad - - - - - - - - 97d.
\]
as a general relation to be satisfied for the isodynamic variations. This can only be generally satisfied if each term is separately equal to zero. The first term can obviously only be equal to zero if \( dT = 0 \) whence; during such a compression the temperature must remain constant. The general equation of the configurational energy variation reduces to the expression
\[
d\Phi = -\nu dP = T \frac{\partial P}{\partial T} d\nu + \nu dP - - - - - - 97d'.
\]
From the other term of 97d we have the relation
\[
T \frac{\partial P}{\partial T} = P \quad - - - - - - - - - - - - 97d''
\]
as another condition of variation which will be satisfied during the compression. From this equation we may obtain the pressure volume relation obeyed on the compression curve for the medium in question.

For a simple centrifugal compression now \( d\nu = 0 \) whence the general expression for \( dU \) reduces to \( \gamma_v dT \) and the
quantity of heat which must be added is represented by the expression

\[ dU_m = \gamma_v dT \]

where now \( dT \) represents the change of temperature necessary to maintain a constant specific volume with the pressure increase \( dP \).

50. RELATION \( f(P, V) \) DURING COMPRESSION AND INTEGRAL \( \int VdP \) FOR PARTICULAR MEDIA.

INCOMPRESSIBLE MEDIA. For incompressible media such as liquids the volume \( V \) will obviously be constant under all conditions and hence the integral \( \int VdP \) will always be represented by

\[ V\int^e dP = VP|_1^2 \]

For such media the adiabatic, isodynamic and simple compressions will obviously be the same and each will require zero work of compression. It does not follow however that therefore a compound compression, as we have defined it, cannot exist.

COMPOUND COMPRESSION OF LIQUIDS. Since liquids are incompressible their intrinsic energy will be uneffected by a reversible change of their kinetic energy. Therefore if we have an actual continuous decrease of velocity of a liquid in one part of the channel, without doing work thereby and with no irreversible processes then; the decrease of kinetic energy represented thereby must be compensated for in some other part of the systems energy. This compensating transformation may occur in three different ways.

We may suppose the channel to be horizontal and to have
its area decreased farther on so that the increase of kinetic energy thereby is exactly equal to the preceding decrease. Secondly the channel may be vertical with the flow upwards when the kinetic energy is converted to potential energy of the zero order, and; thirdly the kinetic energy might be transformed to intrinsic energy within a gaseous media by a volume displacement method of compressing said media.

Such systems have in general three regions, namely; that of the primary transformation, that of the compensating transformation and that connecting the primary and compensating regions called the transmitting region.

In the transmitting region the liquid is in a state of force connecting the primary and compensating regions. The energy for the displacement of the forces within the transmitting region is supplied from the primary region and is given to the compensating region at the same rate. The liquid within the transmitting region therefore may theoretically be reduced to a state of force of zero energy and still be the only means of conveying energy from the primary to the compensating region for; obviously if we enlarged a channels section until its area is infinite, the liquid within this section will have zero kinetic energy whereas that on either side of this section, where the area is finite, will possess a finite amount of kinetic energy. Since we have a reversible conversion of kinetic energy the intrinsic energy is constant throughout and may thus be regarded as non-existent.

The liquid within the transmitting region is therefore
to be regarded merely as a means of conveying energy from the primary to the compensating regions, and whose energetic state is independent of that of the liquid in the primary region. The energetic state of the liquid in the compensating region is likewise independent of that in the transmitting region, but wholly dependent on that in the primary region.

For a compound centrifugal compression of a liquid the energy of compression now can obviously not appear as intrinsic energy within the liquid. We conclude therefore that we have exactly such a system as above described, and in which the primary region is that in which the kinetic energy is decreasing, where the liquid in the rotating channel beyond this section of decrease, and possibly also in it, is that of the transmitting region and where in general the compensating region will be located in the stationary parts beyond the rotating channel, unless; said channel has repeated regions in which the radial velocity is first increased and then decreased.

In order to realize compound compression for a liquid we must of course design the destraints and quanstraints so that it will be a forced condition of activity. The foregoing general design equations with the analysis in Chapter II offer sufficient direction with which to proceed.

**ADIABATIC COMPRESSION OF PERFECT CASES.** As the general law of the compression cycle we have
\[ d\Phi = -\nu dP = \gamma \nu dT + T \frac{\partial P}{\partial T} d\nu + \nu dP \]

The characteristic equation of the gas is
\[ P V = B T \]
whence \( \frac{\partial P}{\partial T} = \frac{B}{V} \)

We have, also, from the thermodynamics, the relation

\[ B = Y_p - Y_v \]

With these substitutions in the above expression we obtain after integration

\[ \int d\Phi = \int V dP = \frac{h}{k-1} (PV) \]

where \( k = Y_r \) and where \( Y_p \) represents the specific heat at constant pressure. Before we can substitute the limiting values of the product \( (PV) \) in the above expression, we must have the law of pressure-volume variation during adiabatic compression. We arrive at this law as follows.

On the compression curve the kinetic variation \(- V dP\) will be exactly compensated for in the intrinsic energy \( dU \). For this period of compression therefore the relation

\[ -V dP = dU \]

will be satisfied, or,

\[ -V dP = Y_v dT + (T \frac{\partial P}{\partial T} - P) dV \]

After substituting the foregoing conditions and integrating we arrive at the expression

\[ PV^k = \text{constant} \]

This is the law of compression. From it we can determine for any pressure variation \( \int dP \) likewise the corresponding volume variation.

In a similar manner we might develop expressions for the kinetic compensation and the pressure-volume relations during the adiabatic compression cycle for other substances.
The above relations are however sufficiently accurate, in the present state of our knowledge of the properties of them, for most media which would be so compressed practically.

In this expression \( k \) takes the following values for the fluids specified

<table>
<thead>
<tr>
<th>Fluid</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perfect gases</td>
<td>1.41</td>
</tr>
<tr>
<td>Vapor of carbon dioxide (superheated)</td>
<td>1.29</td>
</tr>
<tr>
<td>Vapor of sulphur dioxide (superheated)</td>
<td>1.28</td>
</tr>
<tr>
<td>Vapor of ammonia (superheated)</td>
<td>1.33</td>
</tr>
<tr>
<td>Vapor of water (superheated)</td>
<td>1.31</td>
</tr>
</tbody>
</table>

**ISODYNAMIC COMPRESSION OF PERFECT CASES.** As the differential state equation for the compression cycle we have from the foregoing

\[
d\phi = -vdP = T \frac{\partial P}{\partial T} dv + vdP - - - (97 d')
\]

After substituting for \( \frac{\partial P}{\partial T} \) from the perfect gas equation

\[ PV = BT \]

and integrating we obtain the expression

\[
\int^2 d\phi = -\int^2 vdP = (PV)_{1}^2 - - - - - 100 a
\]

which represents the energy equation for the compression cycle.

For the general relation satisfied on the compression curve we have

\[
T \frac{\partial P}{\partial T} = P - - - (97 d'')
\]

where for a perfect gas \( \frac{\partial P}{\partial T} = \frac{R}{V} \) whence by substitution the equation

\[ T B = PV \]

is arrived at, where now \( T \) is a constant. The pressure volume law during compression is therefore represented by the equation
\[ PV = \text{constant} = BT \]

where \( B \) represents the gas constant and \( T \) the temperature at which the compression takes place. The value of \( B \) for any gas or highly superheated vapor may be calculated in English units (ft. lbs.) from the relation

\[ B = \frac{R}{m} = \frac{1544}{m} \]

where \( m \) is the molecular weight of the medium.

**NON-EXISTENCE OF ISODYNAMIC COMPRESSION FOR SATURATED VAPORS.** A saturated vapor is incapable of undergoing such a compression. This conclusion may be arrived by inspection of expression 97d" together with the characteristic equation of a saturated vapor. This for a vapor is of the form

\[ f(P, T) = 0 \]

whence, the partial \( \frac{\partial P}{\partial T} \) will be in terms of constants and the temperature only, and the latter is also constant for an isodynamic compression. The left hand side of 97d" will for this case therefore be constant whence the right hand side must be constant. We conclude that no pressure volume relation for isodynamic compression exists for saturated vapors.

**SIMPLE COMPRESSION OF COMPRESSIBLE MEDIA.** Since the volume \( V \) is constant we have

\[ V \int \frac{dP}{P} = V \left. P \right|_i \]

and as the heat which must be added during compression to keep the volume constant

\[ \int \frac{dU}{U_m} = \int \gamma \, dT \]

which expression has for integration limits the temperatures accompanying the pressure variations when the volume is maintained constant.
Most actual compressions will fall between the isodynamic and the adiabatic compression. That is some heat will leave the fluid so that the compression will not be adiabatic. Neither will it be isodynamic as, to maintain a constant temperature during compression would impose impractical condition of construction.

If we knew for the practical case the exact quantity and its rate of flow of the heat \( dM_u \) we could by a complete integration of expression 97b arrive at the exact pressure volume law for the case in question. For most practical cases it will however be convenient and sufficiently accurate to assume that it will be of the form

\[ PV^n \text{ constant} \]

where it will be necessary to discover the value of \( n \) for which the equation most nearly represents the change. This value will obviously in general lie between unity and \( k \) and may be said to depend on the efficiency of compression referred either to the isothermal or adiabatic compressions as a standard.

Having determined the kind of compression so that as above we may integrate the left hand side of the centrifugal pressure equation 95, it is merely necessary to substitute on the right hand side for \( q_e \) and \( q_\rho \) their equivalents \( e^{-r\cos\sigma} \) and \( r \sin\phi \) respectively, assign the general relations of the path and integrate. The resulting equation will be a design formula for the particular field it covers. We must however in choosing the general path relations confine our choice to the region within which equilibrium is possible.
CHAPTER VI.

CONSTRAINT VARIATIONS FOR WHICH EQUILIBRIUM IS POSSIBLE.

51. CONSTRAINT RESTRICTIONS FOR EQUILIBRIUM OF ACTIVITIES WHEREIN FLUID MOTIONS ARE IN CENTRI-VANESCENT STATE.

If during flow, the variation of the fluids physical configuration is such as to exactly conform to the channel through which it is flowing and zero dissipation of energy available for zero order activity occurs, then the activity, or flow, may be said to exist in the condition of transformation equilibrium, or, in a state of continuous physical continuity.

If the zero order fluid motions are in the centri-vanescent state, then certainly physical continuity is possible for any continuous destraint contour of such length, that the fluid is physically capable of undergoing (while in said state of transformation equilibrium) the variations of physical configuration necessary to have the flow exactly conform to the variation of quanstraint of the channel. We shall call such continuity the CONFIGURATIONAL CONTINUITY to distinguish it from the ordinary mass continuity, and shall arrive at the relations between the coordinates determining these limiting rates of quanstraint variation, for configurational continuity, in the following articles on continuity.

52. MASS CONTINUITY. Obviously through any continuous channel the weight of fluid per unit time must be the same for all cross sections, and the fundamental equation of continuity has the form

\[ A \frac{\partial \mu}{\partial t} = C_0 \]
where $A$ represents the area of cross section, $r$ the relative velocity and $\mu$ the density. This equation may be written as $A = \frac{V}{r}$, whence by differentiation with respect to the distance along the channel we have

$$\frac{dA}{dx} = \frac{r \frac{dV}{dx} - V \frac{dr}{dx}}{r^2} \mu.$$

This with the relation

$$-VdP = \frac{1}{g} q dq,$$

and the equation

$$f(P,V,T,\ldots)$$

of the transformation, will be sufficient to completely determine the constraint variation if we assume that the fluid is going to be physically capable of undergoing, at the rates assigned, the variations of physical configuration necessary to have the flow, in the equilibrium state, exactly conform to the channel.

If it is conceivable that if the fluid were weightless and hence its mass inertia vanished, that then its flow could, even during the equilibrium state, conform to the channel for any rates of constraint variation consistent with the condition of mass continuity 103. Since however all actual fluids have mass inertia, then, certainly the second condition, which we have called the configurational continuity, must likewise be satisfied for such flow.

53. CONFIGURATIONAL CONTINUITY. Referring to figure 10a, there is represented a lamina of fluid containing an elementary mass $m$ flowing through the channel.
of which $a' - a'$ represents a tangent to the surface contour line $a - a$. In the position $p$ the element $m$ has a motional direction coinciding with the stream line tangent $c - c$ whose angularity with the instantaneous direction of the axis $x - x$ is represented by $l$. For configurational continuity of flow, that is to have the flow conform to the quanstraint variation, the element $m$ must obviously have a component of relative velocity $r_y$ at right angles to the axis $x - x$ as well as the component $r_x$ in the direction of said axis.

Only two motional components of the fluid have been considered, namely; that along any diameter $y - y$ and that at right angles to this and along the axis $x - x$. In general, motion in space has three orthogonal components whence, we must for a complete description of the motion also investigate the motion component normal to the $xy$ plane.

Referring to figure 10b we have there represented a cross-section of the channel normal to the axis $x - x$ so that the axis $z - z$ is now normal to any plane $xy$. The axis $y - y$ may be coincident with any diameter of the channel, and the axis $z - z$ will thus be normal to the plane containing this diameter and the channels axis.

For continuity of flow through the channel the element $m$ will have, at any and every instant, equal pressures on its opposite faces normal to the axis $z - z$ for; certainly with a homogeneous fluid flowing through a channel whose cross-section is normal to the axis $x - x$ there will be concentric linear regions, as $l_1$ and $l_2$ of equal pressure about the said axis. At any point on such lines the axis $z - z$ is tangent to them whence,
in the direction \( z \) the variation of pressure vanishes.

If this is true there is zero force condition causing the relative motion in the axis \( z - z \) and we conclude, that the relative whirling motion of the particle \( m \) about the axis \( x - x \) (if not zero) is vanishingly small. It follows that the centers of gravity of all particles \( m \) will move in diametrical planes \( xy \) stationery with respect the channel. Since each and every particle may in the limit be taken so small as to shrink into a mass point, then it follows that for such flow, every particle of the fluid moves only radially and axially relative the channel and, the fluids motion is completely described in terms of its motion along the \( x \) and \( y \) axes.

The angularity \( \mathcal{L} \) of the stream line \( c - c \) will, for configurational continuity, obviously be zero for the point \( p \) in the axis \( x - x \) and equal to \( \mathcal{L} \) when it is at the surface element \( a-a \). Certainly now for continuity the stream line direction will vary so that a relation

\[
\tan \mathcal{L} = c, f\left(\frac{\mathcal{L}}{y^2}\right)
\]

exists; where \( c = \tan \mathcal{L}, y \) represents the radial distance to the channels surface from its axis at the cross-section under consideration, and \( y \) represents the radial distance from said axis to the elementary mass of said section under consideration.

The function \( f\left(\frac{\mathcal{L}}{y^2}\right) \) must obviously have such a form, if continuity is to exist, that its magnitude increases in a finite ratio continuously with \( y \) and that for \( y = 0 \) it likewise vanishes and for \( y = y \), it assumes the value of unity. For any form of the function fulfilling these conditions, (even though the assumed form may not at all points exactly lead to the
actual stream line tangent) certainly at the channel surface, or at a differential distance therefrom, the tangent value arrived at by the assumed form will, to the first order of small quantities, be exactly the actual tangent and hence here errors vanish to said first order of small quantities.

Accordingly we assume for the surface conditions, that this function has the form

\[ f(y, \frac{y}{r}) = \frac{y}{r} \quad - - - - - - - - - - - - 107 \ a \]

whence the relation

\[ \tan \theta = c_1 \frac{y}{r} \quad - - - - - - - - - - - - - - 107 \ b \]

is true.

Letting \( r_x \) and \( r_y \) represent the components of the elements relative velocity, we have for configurational continuity the relation

\[ r_y = r_x c_1 f(y, \frac{y}{r}) \quad - - - - - - - - - - - - - - 108 \]

Differentiating with respect to time

\[
\frac{dr_y}{dt} = c_1 f(y, \frac{y}{r}) \frac{dr_x}{dt} + r_x c_1 d f(y, \frac{y}{r}) \frac{dy}{dt}
\]

\[
= c_1 f(y, \frac{y}{r}) \frac{dr_x}{dt} + c_1 r_x d f(y, \frac{y}{r}) r_y
\]

Multiplying by \( \frac{\mu}{\varrho} \) and substituting from 108 for \( r_y \) expression 109 is arrived at

\[
\frac{\mu}{\varrho} \frac{dr_y}{dt} = c_1 f(y, \frac{y}{r}) \frac{\mu}{\varrho} \frac{dr_x}{dt} + \frac{\mu}{\varrho} c_1^2 r_x^2 f(y, \frac{y}{r}) d f(y, \frac{y}{r}) \quad -109
\]

We have discovered in Article 29 that the normal velocity \( q_n \) of any particle in the axis must be exactly equal to the normal velocity of the channels axis \( x - x \) whence;

\[ q_n = \sin \sigma \cdot e \]

where \( \sigma \) is the angularity of said axis with the direction of motion \( e \). If now during an infinitesimal displacement there
is a variation \((d\sigma, de)\) in the direction and velocity of the channel then obviously a particle in the axis will have, as before shown, its absolute normal velocity varied by an amount

\[
dq_n = \sin \sigma \, de + e \, \cos \sigma \, de
\]

This variation is however, as before explained, wholly caused by surface normal reactions and the energy compensated for is wholly zero order kinetic. The configurational state \((U+PV)\) of the fluid is thus not effected by such a variation. If now there is a variation \(dr_y\) of the relative velocity normal to the axis it will (since the normal velocity of the axis \(x-x\) is exactly the absolute normal velocity, the particle has when \(y_y\) is equal to zero) obviously be exactly a variation \(dq_y\) of the absolute velocity normal to the axis, and it can only be caused by a force existing by virtue of the normal pressure variation of the fluid medium of which the considered element is a part. Hence the expression

\[
\frac{\mu}{g} \frac{dr_y}{dt} = \frac{\mu}{g} \frac{dq_y}{dt} = F_y - - - - - 110
\]

represents the normal force on the particle causing the acceleration \(dr_y\).

We have furthermore from Article 50, when \(dq_r\) vanishes, the relation,

\[
dr - \partial (e_r) = 0 - - - (56)
\]

and in general,

\[
dr - \partial (e_r) = dq_r
\]

or

\[
dr - \cos \sigma \, de = dq_r
\]

where \(\sigma\) is, as before mentioned, the angularity of the channels axis whence, \(\gamma\) and \(\eta\) would likewise be considered in exact-
ly this direction. It follows that the relation
\[ dr_x - \cos \sigma \, ds = dq_x \]
is true.

Solving for \( dr_x \) and dividing by \( dt \) we have
\[ \frac{dr_x}{dt} = \cos \sigma \frac{ds}{dt} + \frac{dq_x}{dt} \]

Now \( dt = \frac{dx}{dr_x} \) and \( \frac{\mu}{g} \frac{dq_x}{dt} = F_x \) whence by substitution
\[ \frac{\mu}{g} \frac{dr_x}{dt} = \frac{\mu}{g} r_x \cos \sigma \frac{ds}{dx} + F_x \]

Substituting 110 and 111 in 109, the expression
\[ F_y = c_s f(\frac{y}{y}) \left[ \frac{\mu}{g} r_x \cos \sigma \frac{ds}{dx} + F_x \right] + \frac{\mu}{g} c_s r_x^2 f(\frac{y}{y}) \frac{df(\frac{y}{y})}{dy} \]
is arrived at, where \( F_y = -\frac{dP}{dy} \) and \( F_x = -\frac{dP}{dx} \)

Substituting these equivalents in this expression it is now represented as follows:
\[ \frac{\mu}{g} r_x^2 f(\frac{y}{y}) \frac{df(\frac{y}{y})}{dy} c_s + \left[ \frac{\mu}{g} r_x \cos \sigma \frac{ds}{dx} - \frac{dP}{dx} \right] f(\frac{y}{y}) c_s + \frac{dP}{dy} = 0 \]

At the channels surface the function \( f(\frac{y}{y}) \) may, as before stated, be assumed to have the form \( \frac{y}{y_s} \) whence for the value \( y = y_s \) the above expression reduces to
\[ \frac{\mu}{g} r_x^2 c_s + \left( \frac{\mu}{g} r_x \cos \sigma \frac{ds}{dx} - \frac{dP}{dx} \right) c_s + \frac{dP}{dy} = 0 \]

We will now denote the expression \( \frac{\mu}{g} r_x^2 \) by \( \frac{c_s}{y_s} \) and the expression \( \frac{\mu}{g} r_x \cos \sigma \frac{ds}{dx} \) by \( c_3 \) whence;
\[ \frac{c_s}{y_s} c_s + (c_3 - \frac{dP}{dx}) c_s + \frac{dP}{dy} = 0 \]

Solving for \( c_s \) and dividing by \( y_s \), we have
\[ \frac{c_s}{y_s} = \frac{-c_3 + \frac{dP}{dx} \pm \sqrt{(c_3 - \frac{dP}{dx})^2 - \frac{c_s}{y_s} \frac{dP}{dy}}}{2 \frac{c_s}{y_s}} \]

---

112
As \( y \) approaches zero, the point \( p \) approaches the axis \( x \) and for this condition the angularity \( \tan \frac{y}{x} \) approaches zero. The ratio \( \frac{c}{y} = \tan \frac{\tan \frac{y}{x}}{y} \) approaches the value zero since, for an infinitesimally small angle \( d\tan \frac{y}{x} \) the adjacent side of the angle is parallel to the hypotenuse and hence of infinite length. Thus in the limit when \( y = dy \), then tangent \( d\tan \frac{y}{x} = \frac{dy}{\infty} \) whence;

\[
\text{limit of } \frac{c}{y} = \frac{\tan dx}{dy} = \frac{dy}{\infty dy} = \text{zero.}
\]

For an expression, \( \frac{dP}{dx} \) is negative whence the right hand side of 112 can only vanish if the plus sign is used with the radical, and 112 is written as

\[
\frac{c}{y} = -c + \frac{dP}{dx} + \sqrt{(c - \frac{dP}{dx})^2 - 4c_2 \frac{dP}{dy}} - \frac{1}{2c_2} 112 a
\]

This is the form of expression to be applied to channels through which the pressure is dropping.

For a compression \( \frac{dP}{dx} \) is positive whence in general the right member can only vanish if the negative sign is used with the radical and we have

\[
\frac{c}{y} = -c + \frac{dP}{dx} - \sqrt{(c - \frac{dP}{dx})^2 - 4c_2 \frac{dP}{dy}} - \frac{1}{2c_2} 112 b
\]

as the form of the expression to be applied to channels through which the pressure is increasing.

Before expressions 112a and 112b can be solved the relation existing between the rates \( \frac{dP}{dx} \) and \( \frac{dP}{dy} \) must be known. To determine these the following four cases are investigated.
EXPANSION IN DIVERGING CHANNEL. For this case we use formula (112a) where now obviously the rates \( \frac{dP}{dx} \) and \( \frac{dP}{dy} \) will be negative. Referring to figure 11, \( m \) represents an element of fluid having velocity \( \gamma \) in some direction as shown. At a given instant of time the pressure variation is such that in the direction \( x \) the rate of variation is \( \frac{dP}{dx} \) and in the direction \( y \) it is \( \frac{dP}{dy} \). Obviously these rates of variation are dependent on the difference in pressure on the sides of the cube and its dimensions only, and are wholly independent of its direction of motion and its velocity. It is true of course that the motion \( q \) of the element \( m \) is dependent on the forces, and the time of their action, to which it has been subjected up to the time of consideration from some reference time and state, but the forces \( F_x \) and \( F_y \) to which it is subjected at any instant are dependent only on the pressure conditions surrounding it at that instant.

Thus, \( F_y = -\frac{dP}{dy} \), \( F_x = -\frac{dP}{dx} \) and for the flow under consideration

\[
F_z = -\frac{dP}{dz} = 0
\]

If we consider, of a perfect fluid, an element in space, and suddenly cause the surrounding pressure to vanish, then certainly it will expand at equal space rates along all lines radiating from its center. In the flow through the channel an expansion occurs, in which the space rate of pressure variation is, as shown, limited to the \( xy \) planes. With respect the element \( m \) now the expansion can certainly not occur at a greater space rate in the direction \( y \) than it can in the direction \( x \) but in the limiting ideal conditions it can, in view of the above
obviously occur in this direction at just as great a rate. For the ideal condition of expansion through the channel, the critical ratio of the space rates of pressure variation are thus expressed as

\[- \frac{dP}{dy} = - \frac{dP}{dx}\]

or

\[\frac{dP}{dy} = \frac{dP}{dx} = 1\]

For the actual conditions now, in which the imperfections of the real fluids and channel surfaces enter, it is reasonable to suppose since the \(X - X\) axis is the principle axis of pressure drop, that it will be impossible to attain a rate \(- \frac{dP}{dy}\) as great as \(- \frac{dP}{dx}\) and accordingly the critical ratio is represented by the expression

\[\frac{dP}{dy} = \psi\frac{dP}{dx}\]

where the factor \(\psi\) will be less than unity. In this sense it will be a measure of the degree of perfection of the actual conditions referred to the condition \(\frac{dP}{dy} = \frac{dP}{dx}\) as a standard of perfection for the expansion and it may properly be called the EXPANSION FACTOR. Obviously the magnitude of \(\psi\) can be determined from data obtained under the actual condition of practice.

Substituting the critical ratio in 112a the expression

\[\frac{C_1}{Y_1} = -c_3 - \frac{dP}{dx} + \sqrt{(c_3 - \frac{dP}{dx})^2 + \frac{4c_2}{\psi} \cdot \frac{dP}{dx}}\]

is arrived at.

The magnitude \(\tau_{1c}\) of \(\tau_1\) as solved for from this expression will be called its critical magnitude to denote that for all angularities \(\tau_{1c} < \tau_{1c}\), configurational continuity of the
fluid is possible whereas, for angularities \( I, > I^c \), configurational continuity is not possible.

In a special case in which the channel velocity \( e \) is constant the variation \( de \) vanishes whence \( C_g = 0 \) and expression 113a reduces to

\[
\frac{C}{y} = \frac{-dP}{dx} + \sqrt{(\frac{dP}{dx})^2 + \frac{4C}{y} \cdot \frac{dP}{dx} \cdot y} \geq C^c
\]

in which form it is exactly applicable to all channels wherein \( e \) is constant. In a further specification \( e \) will be constant at zero, when the channel is stationary. For this case the relative velocity \( Y_x \) in the constant

\[
C^c = \frac{M}{Q} Y_x^c
\]

will be exactly the velocity of flow \( q_x \) at the section under consideration.

If for such a system we consider a fluid of unit density flowing with unit velocity when the rate of pressure variation is unity we arrive for ideal conditions \( (\gamma = 1) \) at the critical magnitude of 62° at a radius \( Y_1 \) of unity. Obviously as the velocity \( Y_x \) and the rate \( \frac{dP}{dx} \) increases this critical magnitude decreases.

For mass continuity the rate \( \frac{dP}{dx} \) is itself of course a function of the angularity. In the solution we may assume however a given length of channel with the assumed pressure drop there through. This will determine, for the example, the rate \( \frac{dP}{dx} \). With this rate then, and the velocity \( Y_x \) as calculated for various sections we may by substitution in the critical relation determine the critical magnitude \( \frac{T_1}{c} \) for the various sections. If these are greater than the actually required
angularity \( I \), of the surface as calculated from the condition of mass continuity, then in the actual channel the configurational continuity of the fluid medium is at least, for the condition considered, possible when the flow conforms to the channel. If however the actual angularities as calculated are greater than then configurational continuity of the fluid is impossible when the flow conforms to the channel and we conclude that the flow can not in this case conform to the channel.

55. II. CONDITION OF CONFIGURATIONAL CONTINUITY FOR EXPANSION IN CONVERGING CHANNEL. For this case the angularity \( I \), is obviously negative. Equation 112a can only satisfy this condition with a negative rate \( \frac{dP}{dx} \) if the magnitude of the radical term is less than \( \left[ c_3 + \frac{dP}{dx} \right] \) whence, \( \frac{dP}{dy} \) must be positive, or the pressure must increase radially outwards from the axis of the converging channel.

The rate of pressure variation \( + \frac{dP}{dy} \) is caused by the decrease of velocity component \( -v_y \) toward the axis \( x \) of the channel. This condition of normal flow is of exactly the nature of fluid flow from a region of low pressure to one of higher pressure and in the limit the rate \( \frac{dr_y}{dt} \) and hence the rate \( \frac{dP}{dy} \) may for such flow have any magnitude whatsoever. If however for the channel in consideration the rate \( \frac{dP}{dy} \) is greater than its value determined from the relation

\[
\frac{dP}{dy} = \frac{v_y}{4c_2} \left[ c_3 + \frac{dP}{dx} \right]^2
\]

then the radical of expression 112a becomes imaginary and we conclude that for such a condition configurational continuity and hence equilibrium is impossible. The critical value of
\( \tau_c \) is thus for this case determined from the relation causing said radical to vanish, whence we have

\[
\frac{c_i}{y_i} = \frac{-c_3 + \frac{dP}{dx}}{2c_2}
\]  

as the critical relation between the coordinates. Suppose however that \( -\tau_i \) is actually greater than its critical magnitude as calculated from the above and that there is actually an opening at the end of the converging nozzle. In the limit \( \tau_i = 90^\circ \) and the opening would be exactly an orifice in a flat plate. Relation 113b therefore determines the limiting angularity of approach to an orifice, for configurational continuity of flow.

Here as before we have \( C_3 \) vanishing for constant \( e \) whence the above expressions reduce to

\[
\frac{dP}{dy} = \frac{y_i}{4c_2} \left( \frac{dP}{dx} \right)^2
\]

and

\[
\frac{c_i}{y_i} = \frac{-\frac{dP}{dx}}{2c_2}
\]  

in which forms they apply also to stationary channels.

56. III. CONDITION OF CONFIGURATIONAL CONTINUITY FOR COMPRESSION IN A DIVERGING CHANNEL. For this case we have the general relation 112b in which both the rates \( \frac{dP}{dx} \) and \( \frac{dP}{dy} \) would be considered positive. We have, as before, the maximum value of \( \frac{dP}{dy} \) for configurational continuity represented by

\[
\frac{dP}{dy} = \frac{y_i}{4c_2} \left[ c_3 + \frac{dP}{dx} \right]^2
\]

and as the critical ratio

\[
\frac{c_i}{y_i} = \frac{+c_3 + \frac{dP}{dx}}{2c_2}
\]  

which is exactly expression 113b with the positive instead of
the minus sign. We conclude that expansion through a converging channel is exactly the reverse of compression in a diverging channel. The critical magnitudes of the angles are equal for both cases but of opposite sense with respect to the direction of flow.

57. IV. IMPOSSIBILITY OF CONTINUITY FOR COMPRESSION IN A CONVERGING CHANNEL. As above both $\frac{dP}{dx}$ and $\frac{dP}{dy}$ are positive whence $\left(\frac{c_3}{y}\right)$ can only be negative if the radical of $112b$ is greater than $-\left[c_3 + \frac{dP}{dx}\right]$. Such a real result is not obtainable from this expression, and we conclude that configurational continuity, and hence equilibrium, does not exist for such flow.

58. CONSTRAINT RESTRICTIONS FOR EQUILIBRIUM OF ACTIVITIES WHEREIN FLUID MOTIONS ARE IN CENTRI-FORCE STATE.

INCOMPRESSIBLE FLUIDS. For incompressible fluids the intrinsic energy is unaffected by the compression, and we have for all cases a purely zero order dynamic activity. All of the active forces and relative motions with which we are concerned here, are therefore of the zero order only. Under such conditions equilibrium is always possible when the relative motions of the zero order are continuous, and this is obviously always a possible condition for any restraint contour which is continuous and as before, of sufficient length for the configurational continuity.

COMPRESSIBLE FLUIDS FOR SIMPLE COMPRESSION. For simple compression of a compressible fluid the volume is
maintained constant at exactly the pressure variation caused by centrifugal compression. The relation of zero order physical coordinates is for these cases therefore maintained exactly as though the fluid were incompressible. It follows that the relation of physical variations of this order for possible equilibrium will be exactly as that for the incompressible.

COMPRESSIBLE FLUIDS FOR COMPOUND COMPRESSION. For the compound compression of compressible fluids there is an actual decrease of volume with increase of pressure of the fluid. We have shown that the only source, for the necessary work of compression in such cases, is from the radial component of the fluids kinetic energy. The kinetic compression is represented by formula 95 as follows:

\[ \mathcal{V} dP = \frac{[q_e^2 - q_p^2]}{g} \cdot \frac{dP}{\beta} - \frac{1}{g} q_p dq_p = - (95) \]

where \( \frac{1}{g} q_p dq_p \) represents exactly the differential of the radial component of the kinetic energy, and which for the case of kinetic compression will be positive, as \( dq_p \) will be negative whence; 95 would be written as:

\[ - \mathcal{V} dP = - \frac{[q_e^2 - q_p^2]}{g} \frac{dP}{\beta} + \frac{1}{g} q_p dq_p \]

Now the work of kinetic compression \( \mathcal{V} dP \) must, for equilibrium, be exactly equal to the above differential of the radial component of the kinetic energy or; for equilibrium without irreversibilities the equation

\[ - \mathcal{V} dP = \frac{1}{g} q_p dq_p \]

must be true.

This relation can obviously only be true if the
relation \[ \frac{q_e^2 - q_p^2}{g} \frac{d \rho}{\rho} = 0 \]
is satisfied during the change. The change occurs during a radial displacement \( d \rho \) of the fluid and hence \( d \rho \) can not be zero whence; as the general restraint condition for possible equilibrium for the compression we have

\[ q_p \geq q_e \]

This is identically the ratio 85a between the peripheral and relative radial velocities of the fluid for centri-vanescence and the vanishing of \( I \) whence; for compound compression of compressible fluids equilibrium does not exist.

If now it is desirable to have a kinetic compression in the rotating element then for equilibrium it will be necessary to first increase the component \( q_p \) to such a magnitude that the relation

\[ q_p \geq q_e \]
is satisfied. In this state of motion centri-vanescence exists whence; the motion component \( q_e \) has generated into a rectilinear component. If now without increasing \( q_e \) from a centri-force origin, we increase the pressure of the fluid at the expense of \( q_p \) then during the compression equilibrium is certainly possible. Furthermore if after having reached the centri-vanescent state the component \( q_e \) is maintained constant it will certainly be uneffected from a centri-force cause and we conclude; that in general, during the kinetic compression of fluid media in rotating channels the motion component \( q_e \) should remain constant.
If at the beginning of the compression the component $q_P$ be greater by a sufficient amount then its magnitude $q_P = q_e$ and the compression ceases when the relation $q_P \geq q_e$ is still satisfied, then obviously the state of motion will be centri-vanescent regardless of how $q_e$ is varied during the compression, and for such cases $q_e$ need not be maintained constant. In general however it will not be possible to, form a centri-force cause, increase the magnitude of $q_P$ over its magnitude $q_P = q_e$ whence, the foregoing conditions must be satisfied during the compression.

59. EQUILIBRIUM DURING VARIATION OF TERMINAL CONDITIONS. In designing the transformer for equilibrium of a given activity we involve the coordinates $e, \sigma, A, r, q, I, P,$ and $V$ of which the first three pertain to the mechanical constraints, the next three to the fluids motions and the last two to the physical configuration of the fluid. There exists such a relation between these coordinates that none of them may be varied independently of the others and still maintain the activity in equilibrium. For instance if the terminal pressure $P$ of a compressible fluid is varied, then likewise in general will the specific volume $V$ be varied and the velocities $r$ and $q$.

The only way therefore, in which the activity can be in equilibrium in the varied state, is by having the mechanical constraints vary with the other coordinates so that continuity is forced condition for; obviously the probability of finding a medium whose physical variation, for the activity men-
tioned, would be such as to have continuity, for the same mechanical constraints, for all rates of activity is very small. We conclude that for equilibrium to exist during variation of the rate of activity in a given transformer, that the mechanical constraints must be varied in such a manner, that continuity is a forced condition of the activity.

60. CONTINUITY AT DISCHARGE BETWEEN MOVING AND STATIONARY CHANNELS. At discharge between moving and stationary channels discontinuities occur in the channel walls and possibly in the force state of the motion. For instance, if in the moving channels the motions are in the centri-force state then certainly such a discontinuity will occur for; in the stationary channels all motions are in the centri-vanescent state.

For continuity of flow the fluid stream lines must certainly be continuous and we conclude that between the stationary and moving element the angularity \( \eta \) of the walls must vary so as to maintain the continuity of the stream lines.

Since now, as above otherwise stated, the function or constraint \( I \) vanishes in stationary channels, then, in zero time interval beyond either terminal of the moving channel, centrifugal forces vanish and hence, the differential of time \( dt \) of their action or effect likewise vanishes within these regions. It follows that the product; rate of effect into time of effect vanishes at any and all instants in which the fluid particle is outside the moving channel. We are therefore concerned, in the region between stationary and moving channels, with
exactly the conditions of equilibrium and continuity as for stationary channels.