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RAMSEY THEORY AND ITS APPLICATION

BY

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DISSERTATION

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Abstract

In this dissertation, we study three problems about Ramsey theory. First, we prove a self-dual Ramsey theorem for parameter systems which is a generalization of the self-dual Ramsey theorem developed by Solecki. Second, we prove a Ramsey theorem for finite sets equipped with a partial order and a fixed number of linear orders extending the partial order. Third, we study the relations between Ramsey theorems which have points in common with the classical Ramsey theorem and the dual Ramsey theorem by the concept of interpretation.

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Chapter 1

Introduction

1.1 Overview

Ramsey theory is a branch of combinatorics. Two of the classical theorems in this area are the classical Ramsey theorem and the dual Ramsey theorem. In this thesis, we study Ramsey theory and its applications, in particular, we study the following three topics: a self-dual Ramsey theorem for parameter systems; a Ramsey theorem for partial orders with linear extensions; some other Ramsey theorems, the notions of retraction and interpretation. In the rest of this introduction, first, we describe our main tools: the abstract approach to finite Ramsey theory developed by Solecki in [1] and the language of injections and surjections from [1]; then we give an introduction for each topic mentioned above.

1.2 The abstract approach to finite Ramsey theory

1.2.1 Algebraic structure

Actoid. By an *actoid*, we understand two sets A, X, a partial function from $A \times A$ to A

$$(a, b) \to a \cdot b,$$

and a partial function from $A \times X$ to X

$$(a, x) \to a \,.\, x,$$

such that for $a, b \in A$ and $x \in X$, if $a \,.\, (b \,.\, x)$ and $(a \cdot b) \,.\, x$ are both defined, then

$$a \,.\, (b \,.\, x) = (a \cdot b) \,.\, x. \tag{1.2.1}$$

The binary operation $\cdot$ on an actoid as above will be called multiplication and the binary operation $\,.\,$ will be called action.

Set actoid. Let (A, X) be an actoid and $\mathcal{F}$ be a family of subsets of A, and $\mathcal{S}$ be a family of subsets of X. Let

$$(F, G) \to F \bullet G$$

be a partial function from $\mathcal{F} \times \mathcal{F}$ to $\mathcal{F}$ and let

$$(F, S) \to F \bullet S$$

be a partial function from $\mathcal{F} \times \mathcal{S}$ to $\mathcal{S}$. We say that $(\mathcal{F}, \mathcal{S})$ with these two operations is a *set actoid* over (A, X) provided that whenever $F \bullet G$ is defined, then $f \cdot g$ is defined for all $f \in F$ and $g \in G$ and

$$F \bullet G = \{f \cdot g : f \in F, g \in G\},$$

and whenever $F \bullet S$ is defined, then $f\,.\,s$ is defined for all $f \in F$, $s \in S$ and

$$F \bullet S = \{f\,.\,s : f \in F, s \in S\}.$$

Normed composition space. A normed composition space is an actoid (A, X) together with a function $\partial : X \to X$, a function $|\cdot| : X \to L$ where L is a partial order such that

1. for $a \in A, x \in X$, if $a\,.\,x$ and $a\,.\,\partial x$ are defined, then

$$a\,.\,\partial x = \partial(a\,.\,x);$$

2. for all $x \in X$, $|\partial x| \leq |x|$;

3. for $x, y \in X$, $|x| \leq |y|$ implies that for all $a \in A$

$$a\,.\,y \text{ defined} \Rightarrow a\,.\,x \text{ defined and } |a\,.\,x| \leq |a\,.\,y|.$$

The function ∂ will be called a truncation and $|\cdot|$ will be called a norm.

For $a, b \in A$, we say that b *extends* a if for each $x \in X$ for which $a \,.\, x$ is defined, we have that $b \,.\, x$ is defined and $a \,.\, x = b \,.\, x$. For $F \subseteq A$, $S \subseteq X$, let

1. $F_a = \{b \in F \mid b \text{ extends } a\}$;
2. $(S)_x = \{s \in S \mid \partial s = x\}$;
3. $\partial S = \{\partial s \mid s \in S\}$;
4. if for all $f \in F$, $s \in S$, $f \,.\, s$ is defined, then we let

$$F \,.\, S = \{f \,.\, s : f \in F, s \in S\}.$$

and we use ∂^t to denote the t^{th} iteration of ∂ for $t \in \mathbb{N}$.

Ramsey domain. A set actoid $(\mathcal{F}, \mathcal{S})$ over a normed composition space is called a Ramsey domain, if each set in $\mathcal{S}$ is non-empty and the following conditions hold for all $F, G \in \mathcal{F}$, $S \in \mathcal{S}$,

(A) if $F \bullet (G \bullet S)$ is defined, then so is $(F \bullet G) \bullet S$;

(B) $\partial S \in \mathcal{S}$;

(C) if $F \bullet \partial S$ is defined, then there exists $H \in \mathcal{F}$, such that $H \bullet S$ is defined and for each $f \in F$ there is $h \in H$ extending f.

A Ramsey domain $(\mathcal{F}, \mathcal{S})$ is called vanishing if for each $S \in \mathcal{S}$, there is $t \in \mathbb{N}$ such that $\partial^t S$ consists of one element; is called linear if the image of S under the norm is linear for each $S \in \mathcal{S}$.

1.2.2 An abstract Ramsey theorem.

For a natural number $d > 0$, a *d-coloring* is a coloring with d colors. We call the following condition the Ramsey condition for a set actoid $(\mathcal{F}, \mathcal{S})$:

(R) Given $d > 0$. For each $S \in \mathcal{S}$, there exists $F \in \mathcal{F}$ with $F \bullet S$ defined such that for each d-coloring of it there exists $f \in F$ such that $f \,.\, S$ is monochromatic.

We call the following condition the pigeonhole principle for a Ramsey domain $(\mathcal{F}, \mathcal{S})$:

(P) Given $d > 0$. For each $S \in \mathcal{S}$ and $x \in \partial S$, there is $F \in \mathcal{F}$ and $a \in A$ such that $F \bullet S$ and $a \,.\, x$ are defined, and for every d-coloring of $F_a \,.\, (S)_x$ there is $f \in F_a$ such that $f \,.\, (S)_x$ is monochromatic.

Theorem 1.2.1 (Solecki)**.** *Let $(\mathcal{F}, \mathcal{S})$ be a Ramsey domain over a normed composition space. If it is linear, vanishing, and each $S \in \mathcal{S}$ is finite, then* **(P)** *implies* **(R)**.

1.3 The language of injections and surjections

For $n \in \mathbb{N}$, let $[n] = \{1, \ldots, n\}$, in particular $[0] = \emptyset$.

1.3.1 Classes of injections and surjections

An increasing injection is an injective function $s : [k] \to [n]$ such that if $y_1 < y_2 \in [k]$, then $s(y_1) < s(y_2)$. Let

$$\mathrm{II} = \{s : s \text{ an increasing injection}\}.$$

A rigid surjection is a surjective function $s : [n] \to [k]$ such that for each $y \in [k]$, there exists $x \in [n]$ with $s([x]) = [y]$. Let

$$\mathrm{RS} = \{s : s \text{ a rigid surjection}\}.$$

An increasing surjection is a surjective function $s : [n] \to [k]$ such that if $y_1 < y_2 \in [n]$, then $s(y_1) \leq s(y_2)$, so strictly speaking, s is a non-decreasing surjection. Let

$$\mathrm{IS} = \{s : s \text{ an increasing surjection}\}.$$

An augmented surjection is an ordered pair whose elements are a rigid surjection and an increasing surjection with an appropriate interaction. Let $s, p : [l] \to [k]$ with $p \in \mathrm{IS}, s \in \mathrm{RS}$, $s \leq p$ such that for each $x \in [k]$,

$$s(\max p^{-1}(x)) = x.$$

Then we say (s, p) is an augmented surjection. Let

$$\mathrm{AS} = \{(s, p) : (s, p) \text{ an augmented surjection}\}.$$

1.3.2 Canonical compositions

For II. Let $v : [k] \to [l] \in \mathrm{II}$ and $s : [m] \to [n] \in \mathrm{II}$. The canonical composition of s and v which we denote by $s \circ v$ is defined if and only if $l \leq m$ and let it be the usual composition of $s \restriction [l]$ and v.

For RS. Let $v : [l] \to [k] \in \mathrm{RS}$ and $s : [n] \to [m] \in \mathrm{RS}$. The canonical composition of v and s which we denote by $v \circ s$ is defined if and only if $l \leq m$ and let it be

$$v \circ (s \restriction [n_0])$$

where

$$n_0 = \max\{i : s(i) \leq l\}.$$

Note that $v \circ s \in \mathrm{RS}$.

For AS. Let $(s, p), (t, q) \in \mathrm{AS}$, $s, p : [l] \to [k]$ and $t, q : [n] \to [m]$. The canonical composition of (s, p) and (t, q) which we denote by $(s, p) \circ (t, q)$ is defined if and only if $l \leq m$ and let it be

$$((s \circ t) \restriction \mathrm{dom}(p \circ q), p \circ q).$$

Note that $(s, p) \circ (t, q) \in \mathrm{AS}$.

1.3.3 Canonical truncations

For II. The canonical truncation for II is defined as follows. Let $s : [k] \to [l] \in \mathrm{II}$, define

$$\partial(s) = s \restriction [k - 1].$$

For RS. The canonical truncation for RS is the forgetful truncation in [1] which is defined as follows. Let $s : [l] \to [k] \in \mathrm{RS}$, and let $l_0 = \min s^{-1}(k)$. Define

$$\partial(s) = s \restriction [l_0 - 1].$$

Note that $\partial S \in \mathrm{RS}$.

For AS. The canonical truncation for AS is defined as follows. Let $(s,p) \in \mathrm{AS}$, define

$$\partial(s,p) = (s \restriction \mathrm{dom}(\partial p), \partial p),$$

where ∂ in ∂p is the canonical truncation for RS. Note that $\partial(s,p) \in \mathrm{AS}$.

1.3.4 Canonical norms

For II. The canonical norm for II is defined as follows. Let $|\cdot| : \mathrm{II} \to \mathbb{N}$ be defined by

$$|s| = s(k)$$

where $s \in \mathrm{II}$ is from $[k]$ to $[l]$.

For RS. The canonical norm for RS is defined as follows. Let $|\cdot| : \mathrm{RS} \to \mathbb{N}$ be defined by

$$|s| = l$$

where $s \in \mathrm{RS}$ is from $[l]$ to $[k]$.

For AS. The canonical norm for AS is defined as follows. Let $|\cdot| : \mathrm{AS} \to \mathbb{N}$ be defined by

$$|(s,p)| = l$$

where $(s,p) \in \mathrm{AS}$ and s, p are from $[l]$ to $[k]$.

1.4 A self-dual Ramsey theorem for parameter systems

In Chapter 2, we prove a self-dual Ramsey theorem for parameter systems. Graham, Leeb, Rothschild proved a Ramsey theorem for vector/affine spaces in [6]. Spencer simplified the proof and extended their Ramsey theorem to more general objects which are called parameter systems in [5].

Solecki develops a self-dual Ramsey theorem generalizing both the classical Ramsey theorem and the dual Ramsey theorem in [1]. The classical Ramsey theorem is about coloring sets. The dual Ramsey theorem is about coloring partitions. The self-dual Ramsey theorem is about coloring pairs, which consist of a partition and a set with appropriate interactions.

We generalize the Ramsey theorem for parameter systems in the self-dual sense. We find a Ramsey theorem whose objects are pairs consisting of a space and a set with appropriate interactions. The self-dual Ramsey theorem in [1] is a particular case of our result and the Ramsey theorem for parameter systems in [5] is a simple corollary of it. Moreover, the proof of our theorem also works for the Ramsey theorem for parameter systems. This proof makes the Ramsey theorem for parameter systems a part of the abstract approach outlined in Section 1.2.

In the context of the abstract approach, if we apply some Ramsey theorem as a pigeonhole principle to prove another Ramsey theorem, then we say the resulting Ramsey theorem has higher rank than the one playing the role of the pigeonhole principle. By [1], we can build a rank picture for all exisiting pure finite Ramsey theorems. In the sense above, the Ramsey theorem we find in Chapter 2 sits on the top of the picture.

In the rest of this section, we will introduce the related background and state the self-dual Ramsey theorem for parameter systems.

1.4.1 The Ramsey theorem for parameter systems

In this section, we give the definition of parameter system and the Ramsey theorem for parameter systems from [5]. Let D be a finite set and $F = \bigcup_{i=0}^{\infty} F_i$ where F_i is a family of functions $f\colon D^i \to D$. Note that a function f has domain D^0 means it has arity 0. A set $V \subseteq D^n$ is called an m-space if there exist $I = \{i_1, \ldots, i_m\} \subseteq [n]$ with $|I| = m$, and for $i \notin I$, functions $f_i \in F_m$, so that

$$V = \{(x_1, \ldots, x_n) : x_i = f_i(x_{i_1}, \ldots, x_{i_m}), i \notin I\}.$$

We call I a basis of V. Generally I is not determined by V, but $|V| = |D|^{|I|}$, $m = |I|$ is determined. We call m the dimension of V and write dim(V) = m. Note that if $|V| = 1$, then $\dim(V) = 0$ and $\emptyset$ is the unique basis of V. For each $J = \{j_1, \ldots, j_k\} \subseteq [n]$ with $j_1 < \cdots < j_k$, we define $p_J\colon D^n \to D^k$ by

$$p_J(x_1, \ldots, x_n) = (x_{j_1}, \ldots, x_{j_k}).$$

We call $p\colon D^n \to D^k$ a projection, if $p = p_J$ for some J. Note that if $J = \emptyset$, then $p_J(D^n) = \emptyset$.

We call F a parameter system on D, if it satisfies $(A1)-(A6)$ as follows.

(A1) Constants: for all $a \in D$, m, the constant function $f(x_1, \dots, x_m) = a$ is in F_m;

(A2) Identity: F_1 contains the identity function $f(x) = x$;

(A3) Extension: if $f \in F_m$ and $p\colon D^n \to D^m$ is a projection, then

$$f^{'} = f \circ p \in F_n;$$

(A4) Composition: if $f_1, \dots, f_k \in F_m$, $f \in F_k$, then $f^{'} \in F_m$ where

$$f^{'}(x_1, \dots, x_m) = f(f_1(x_1, \dots, x_m), \dots, f_k(x_1, \dots, x_m));$$

(A5) Basis: if $V \subseteq D^n$ is a space, $\emptyset \neq I \subseteq [n]$ and $p_I \upharpoonright V$ is bijective, then I is a basis of V;

(A6) Projection: if $V \subseteq D^n$ is a space, and $\emptyset \neq J \subseteq [n]$, then $p_J(V)$ is a space of $D^{|J|}$.

Example 1.4.1. *Let D be a finite field and F_i be the family of affine linear functions $f(x_1, \dots, x_i) = c + \Sigma d_j x_j$. One can check that, in this case, F is a parameter system on D, and the spaces in this parameter system are affine spaces.*

Theorem 1.4.2 (Spencer)**.** *Given $d > 0, m \geq k$, let F be a parameter system on D. Then, there exists n such that, for each d-coloring of all k-spaces of D^n, there exists an m-space V of D^n, such that all k-spaces of D^n contained in V get the same color.*

1.4.2 The self-dual Ramsey theorem

In this section, we present the self-dual Ramsey theorem from [1]. Let $\mathcal{R}$ be a partition of $[n]$ and C be a subset of $[n]$. Let $m \in \mathbb{N}$. We say that $(\mathcal{R}, C)$ is an *m-connection* of $[n]$ if $\mathcal{R}$ and C have m elements each and, upon listing $\mathcal{R}$ as $R_1, \dots, R_m$ with $\min R_i < \min R_{i+1}$ and C as $c_1, \dots, c_m$ with $c_i < c_{i+1}$, we have $c_i \in R_i$ for $i \leq m$ and $c_i < \min R_{i+1}$ for $i < m$. We say that a k-connection $(\mathcal{Q}, B)$ is a *k-subconnection* of an m-connection $(\mathcal{R}, C)$ if $\mathcal{Q}$ is a coarser partition than $\mathcal{R}$ and $B \subseteq C$.

Theorem 1.4.3 (Solecki)**.** *Given $d > 0, m \geq k$, there exists n such that, for each d-coloring of all k-connections of $[n]$, there exists an m-connection all of whose k-subconnections get the same color.*

1.4.3 The self-dual Ramsey theorem for parameter systems

Let F be a parameter system on D. Let V be an m-space of D^n. For $v \in V$, and $i, j \in [n]$, $J \subseteq [n]$,

1. let $v(i)$ denote the i-th coordinate of v;
2. let $v \upharpoonright J$ denote $p_J(v)$.

We say coordinate i *depends on* the coordinates J through the function $f \in F_{|J|}$ in V, if for each $v \in V$, $v(i) = f(v \upharpoonright J)$. For $K \subseteq \mathbb{N}$, and $s \leq |K|$, let $K(s)$ denote the s-th smallest number in K.

Definition 1.4.4 (Minimal basis)**.** *A basis I of V is called a minimal basis, if for each $i \in [n] \setminus I$, the i-th coordinate of each element of V depends on coordinates $I \cap [i]$ through some function in $F_{|I \cap [i]|}$.*

Lemma 1.4.5. *V has a unique minimal basis.*

Proof. For the case $m = 0$, by the definition of basis, the unique basis of V is the empty set. Then by $(A1)$ in Section 1.4.1 and the definition of minimal basis, we see the empty set is also a minimal basis of V. So we assume $m > 0$. By $(A6)$ in Section 1.4.1, for each $i \in [n]$, $p_{[i]}(V)$ is a space. We also know $\dim(V) = m$. So, for each $s \in [m]$, there exists a smallest number $i_s \in [n]$ such that $\dim(p_{[i_s]}(V)) = s$. By the choice of i_s, we observe that for each $i \in [n]$,

$$p_{I \cap [i]} \upharpoonright p_{[i]}(V) \text{ is } 1 - 1 \tag{1.4.1}$$

where $I = \{i_1, \ldots, i_m\}$.

To show that I is a basis. Condition $(A5)$ in Section 1.4.1 and (1.4.1) for $i = n$ imply that I is a basis of V.

To show that I is a minimal basis. Let $i \in [n] \setminus I$. If $I \cap [i] \neq \emptyset$, then according to (1.4.1) and $(A5)$ in Section 1.4.1, $I \cap [i]$ is a basis of $p_{[i]}(V)$. It follows that there exists $f \in F_{|I \cap [i]|}$ such that coordinate i depends on coordinates $I \cap [i]$ through f in $p_{[i]}(V)$ which implies that coordinate i depends on coordinates $I \cap [i]$ through f in V. Now we assume $I \cap [i] = \emptyset$. By the choice of I, $\dim(p_{[i]}(V)) = 0$ which implies that $\dim(p_{\{i\}}(V)) = 0$. It follows that $p_{\{i\}}(V)$ is a constant c. By $(A1)$ in Section 1.4.1, the constant function c is in F_0. So, coordinate i depends on coordinates $I \cap [i]$ which is the empty set through the constant function c in V.

To show the uniqueness. Let $J \subseteq [n]$ be another minimal basis of V. We prove by induction that $I(s) = J(s)$ for $s \in [m]$.

Base: $s = 1$. Since J is a minimal basis, we have that if $i < J(1)$, then coordinate i depends on coordinates $J \cap [i]$ which is the empty set through some function in $F_{|J\cap[i]|}$. That means $p_{\{i\}}(V)$ is a constant. It follows that $J(1)$ is the smallest number such that $\dim(p_{[J(1)]})(V) = 1$ which implies that $I(1) = J(1)$.

Move from s to $s+1$. By induction, we have $I(s) = J(s)$ which implies that $\dim(p_{[J(s)]}(V)) = s$. Since J is a minimal basis, we have that for each $J(s) < i < J(s+1)$, coordinate i depends on coordinates $\{J(1), \ldots, J(s)\}$ through some function $f \in F_s$ in V. It follows that $\dim(p_{[i]}(V)) = s$ for each $J(s) < i < J(s+1)$ which implies that $\dim(p_{[J(s+1)]}(V)) \leq s+1$. It is obvious that $\dim(p_{[J(s+1)]}(V)) \geq s+1$. So we have that $J(s+1)$ is the smallest number such that $\dim(p_{[J(s+1)]}(V)) = s+1$ which implies that $I(s+1) = J(s+1)$. □

Remark 1.4.1. By the above proof, we see that there is another equivalent definition for minimal basis, that is, $I \subseteq [n]$ with $|I| = m$ is the minimal basis of V, if for each $s \in [m]$, $I(s)$ is the smallest number such that $\dim(p_{[I(s)]}(V)) = s$.

Now we assume I is the minimal basis of V.

Definition 1.4.6 (Associate coordinate)**.** *For a minimal basis coordinate $i \in I$, and an arbitrary coordinate $j \in [n]$, we say that coordinate j is an associate coordinate of i in V if the following conditions hold.*

R.1 $I \cap [j] = I \cap [i]$*;*

R.2 $\forall v \in V$*,* $v(j) = v(i)$*.*

Definition 1.4.7 (Connection)**.** *Let $P \subseteq [n]$ with $|P| = m$ be such that $P(s)$ is an associate coordinate of $I(s)$ for each $s \in [m]$, then we say (V, P) is an m-connection of D^n.*

If V is a 0-space of D^n, then by the above definition, $(V, \emptyset)$ is a 0-connection of D^n.

Let (W, Q), (V, P) be two connections, if $W \subseteq V$ and $Q \subseteq P$, we write $(W, Q) \leq (V, P)$, and say (W, Q) is a *subconnection* of (V, P).

The self-dual Ramsey theorem for parameter systems is stated as below.

Theorem 1.4.8. Given $d > 0, m \geq k$, there exists n such that for each d-coloring of all k-connections of D^n, there exists an m-connection of D^n all of whose k-subconnections get the same color.

Applying this theorem to the parameter system in Example 1.4.1 gives a Ramsey theorem that is neither a particular case of the self-dual Ramsey theorem, nor a particular case of the Ramsey theorem for parameter systems.

1.5 A Ramsey theorem for partial orders with linear extensions

In Chapter 3, we prove a Ramsey theorem for finite sets equipped with a partial order and a fixed number of linear orders extending the partial order. This is a common generalization of two recent Ramsey theorems due to Sokić. In this section, all orders are strict orders, and we fix a natural number $p > 0$.

By a *structure* we understand a set X equipped with a partial order P and p linear orders $L_0, \ldots, L_{p-1}$ each of which extends P. We write

$$\vec{L}$$

for $(L_0, \ldots, L_{p-1})$ and

$$(X, P, \vec{L})$$

for the whole structure. A structure is called finite if X is a finite set. Given two structures $\mathcal{X} = (X, P^X, \vec{L}^X)$ and $\mathcal{Y} = (Y, P^Y, \vec{L}^Y)$, a function $f\colon X \to Y$ is an *embedding* if for all $x_1, x_2 \in X$

$$x_1 P^X x_2 \Longleftrightarrow f(x_1) P^Y f(x_2)$$

and, for each $i < p$,

$$x_1 L_i^X x_2 \Longleftrightarrow f(x_1) L_i^Y f(x_2).$$

By a *copy* we understand the image of an embedding.

Theorem 1.5.1 (Solecki, Z.)**.** *Let $d > 0$, and let $\mathcal{X} = (X, P^X, \vec{L}^X)$ and $\mathcal{Y} = (Y, P^Y, \vec{L}^Y)$ be finite structures. There exists a finite structure $\mathcal{Z} = (X, P^Z, \vec{L}^Z)$ with the following property: for each d-coloring of all copies of $\mathcal{X}$ in $\mathcal{Z}$, there exists a copy $\mathcal{Y}'$ of $\mathcal{Y}$ in $\mathcal{Z}$ such that all copies of $\mathcal{X}$ in $\mathcal{Y}'$ have the same color.*

1.6 Some Ramsey theorems, the notions of retraction and interpretation

In Chapter 4, we study the following three problems. For natural numbers n, m, let $[m, n] = \{i : m \le i \le n\}$; let $(m, n) = \{i : m < i < n\}$; let $(m, n] = \{i : m < i \le n\}$; let $[m, n) = \{i : m \le i < n\}$.

1.6.1 Problem 1

Let A be a set of functions with domain and range contained in $\mathbb{N}$. For k, m, let

$$H^A_{m,k} = \{s \in A \mid s : [m] \to [k]\}.$$

If $A \subseteq \mathrm{RS}$, then we say A is a *Ramsey set* if for $d > 0, k \leq m$, there exists n, such that for each d-coloring of $H^A_{n,k}$, there exists $t \in H^A_{n,m}$ such that $\{s \circ t : s \in H^A_{m,k}\}$ is monochromatic.

In [1], Solecki proposes a problem: can we characterize the following set?

$$\{A : \mathrm{IS} \subsetneq A \subsetneq \mathrm{RS}, A \text{ a Ramsey set}\}.$$

In Section 4.1, we make progress on the above question. We find three Ramsey sets between IS and RS as follows.

(1) Let

$$A_1 = \{s \colon [m] \to [k] \mid s \in \mathrm{RS},\ s \restriction s^{-1}([k] \setminus [1]) \text{ non decreasing}\}.$$

(2) Let $u_0 : [3] \to [2]$ be such that $u_0(1) = u_0(3) = 1, u_0(2) = 2$. Let

$$A_2 = \{s \in \mathrm{RS} \mid \text{ if } \mathrm{dom}(s) \geq 3, \text{ then } s \restriction [3] \neq u_0\}.$$

(3) Let A_3 consist of all rigid surjections $s : [m] \to [k]$ satisfying the following properties:

(i) for each $j \leq k-1$, if $I = (\min s^{-1}(j), \min s^{-1}(j+1)) \neq \emptyset$, then there exists $i \in I$, such that $s(i) = j$;

(ii) if $I = (\min s^{-1}(k), m] \neq \emptyset$, then there exists $i \in I$, such that $s(i) = k$.

We prove that A_1, A_2, A_3 are Ramsey sets. The Ramsey theorems corresponding to A_1, A_2, A_3 have points in common with the classical Ramsey theorem and the dual Ramsey theorem. We are interested in knowing how they relate to each other. This is the motivation of Problem 3.

1.6.2 Problem 2

Solecki finds that the function $\pi : \mathrm{RS} \to \mathrm{IS}$ defined by sending $s : [m] \to [k] \in \mathrm{RS}$ to $s' : [m] \to [k] \in \mathrm{IS}$ where for $j \in [k]$,

$$\min s'^{-1}(j) = \min s^{-1}(j)$$

satisfies the following properties:

1. $\pi \restriction \mathrm{IS} = id \restriction \mathrm{IS}$;

2. for each $s_1, s_2 \in \mathrm{RS}$ with $s_1 \circ s_2$ defined, we have $\pi(s_1) \circ \pi(s_2)$ is defined, and $\pi(s_1 \circ s_2) = \pi(s_1) \circ \pi(s_2)$.

In general, for A with $\mathrm{IS} \subseteq A \subseteq \mathrm{RS}$, we say A is a *retraction* of RS, if there exists a $\phi : \mathrm{RS} \to A$ such that

1. $\phi \restriction A = id \restriction A$;

2. for each $s_1, s_2 \in \mathrm{RS}$ with $s_1 \circ s_2$ defined, we have $\phi(s_1) \circ \phi(s_2)$ is defined, and $\phi(s_1 \circ s_2) = \phi(s_1) \circ \phi(s_2)$.

It is easy to observe that each such A is a Ramsey set as the retraction preserves Ramseyness. Solecki ask if we can characterize the following set

$$\{A : \mathrm{IS} \subseteq A \subseteq \mathrm{RS}, A \text{ a retraction of RS}\}.$$

If we can characterize the above set, then we find a subclass of Ramsey sets. In Section 4.2, we show that IS and RS are the only elements of the above set.

1.6.3 Problem 3

Beside the Ramsey theorems corresponding to A_1, A_2, A_3, we have some other Ramsey theorems in Section 4.1 which are related to the classical Rasmey theorem and the dual Ramsey theorem. We want to explore the relations between them.

In [1], Solecki introduced a concept of interpretation. This concept provides a way to characterize the relation between different Ramsey theorems. In particular, when two Ramsey theorems can be interpreted by each other, then they are, in essence, the same Ramsey theorem.

There are two different types of interpretation: the ∂-interpretation and the relaxed interpretation. The relaxed interpretation is a weak version of the ∂-interpretation.

Definition 1.6.1 (∂-interpretation [1]). *Let $(\mathcal{F}, \mathcal{R})$ and $(\mathcal{G}, \mathcal{S})$ be Ramsey domains over composition spaces (A, X) and (B, Y), respectively . We say that S is ∂-interpretable in $(\mathcal{F}, \mathcal{R})$ if there exists $R \in \mathcal{R}$ and a function $\alpha : S \to R$ such that*

(i) for $y_1, y_2 \in S$,

$$\partial y_1 = \partial y_2 \Rightarrow \partial \alpha(y_1) = \partial \alpha(y_2); \tag{1.6.1}$$

(ii) if $F \bullet R$ is defined for some $F \in \mathcal{F}$, then there exists $G \in \mathcal{G}$, with $G \bullet S$ defined, and a function $\phi : F \to G$ such that for $f_1, f_2 \in F$ and $y_1, y_2 \in S$,

$$f_1 \,.\, \alpha(y_1) = f_2 \,.\, \alpha(y_2) \Rightarrow \phi(f_1) \,.\, y_1 = \phi(f_2) \,.\, y_2. \tag{1.6.2}$$

Definition 1.6.2 (Relaxed interpretation). *Let $(\mathcal{F}, \mathcal{R})$ and $(\mathcal{G}, \mathcal{S})$ be set actoids over actoids (A, X) and (B, Y), respectively. Let $S \in \mathcal{S}$. We say that S is relaxed interpretable in $(\mathcal{F}, \mathcal{R})$ if there exists $R \in \mathcal{R}$ and a function $\alpha : S \to R$ such that if $F \bullet R$ is defined for some $F \in \mathcal{F}$, then there exists $G \in \mathcal{G}$, with $G \bullet S$ defined, and a function $\phi : F \to G$ such that for $f_1, f_2 \in F$ and $y_1, y_2 \in S$,*

$$f_1 \,.\, \alpha(y_1) = f_2 \,.\, \alpha(y_2) \Rightarrow \phi(f_1) \,.\, y_1 = \phi(f_2) \,.\, y_2. \tag{1.6.3}$$

For Ramsey domains $(\mathcal{G}, \mathcal{S})$ and $(\mathcal{F}, \mathcal{R})$, we say $(\mathcal{G}, \mathcal{S})$ can be *∂-interpreted* by $(\mathcal{F}, \mathcal{R})$ if for each $S \in \mathcal{S}$, S is ∂-interpretable in $(\mathcal{F}, \mathcal{R})$.

For set actoids $(\mathcal{G}, \mathcal{S})$ and $(\mathcal{F}, \mathcal{R})$, we say $(\mathcal{G}, \mathcal{S})$ can be *relaxed interpreted* by $(\mathcal{F}, \mathcal{R})$ if for each $S \in \mathcal{S}$, S is relaxed interpretable in $(\mathcal{F}, \mathcal{R})$.

We apply the concept of interpretation to characterize the relations between Ramsey theorems stated in Section 4.1 and get the following results.

(1) [Theorem 4.3.7] The Ramsey domain for the classical Ramsey theorem for increasing injections and the Ramsey domain for the classical Ramsey theorem for increasing surjections can be ∂-interpreted by each other.

(2) [Theorem 4.3.8] The Ramsey domain for the Ramsey theorem corresponding to A_2 and the Ramsey domain for the dual Ramsey theorem can be ∂-interpreted by each other.

(3) [Theorem 4.3.9] The Ramsey domain for the Ramsey theorem corresponding to A_3 and the Ramsey domain for the dual Ramsey theorem can be ∂-interpreted by each other.

(4) [Theorem 4.3.12] The set actoid for the classical Ramsey theorem for increasing injections can not be relaxed interpreted by the set actoid for the standard pigeonhole principle.

(5) [Theorem 4.3.14] The Ramsey domain for the dual Ramsey theorem can not be ∂-interpreted by the Ramsey domain for the classical Ramsey theorem for increasing injections.

(6) [Theorem 4.3.15] The Ramsey domain for the Ramsey theorem corresponding to A_1 can not be ∂-interpreted by the Ramsey domain for the classical Ramsey theorem for increasing injections.

(7) [Theorem 4.3.17] The Ramsey domain for the spiral Ramsey theorem can be relaxed interpreted but not ∂-interpreted by the Ramsey domain for the classical Ramsey theorem for increasing injections.

By Theorem 4.3.8, Theorem 4.3.9, we see the Ramsey theorems corresponding to A_2, A_3 and the dual Ramsey theorem are, in essence, the same Ramsey theorem.

Chapter 2

A self-dual Ramsey theorem for parameter systems

In this chapter, we prove the self-dual Ramsey theorem for parameter systems. We organize this chapter as follows: in Section 2.1, we state a variation of the self-dual Ramsey theorem and present the Hales-Jewett theorem in terms of rigid surjections; in Section 2.2, we give a proof of Theorem 1.4.8 by the abstract approach where the above variation of the self-dual Ramsey theorem and the Hales-Jewett theorem play the role of pigeonhole principle; in Section 2.3, we show the Ramsey theorem for parameter systems is a simple corollary of Theorem 1.4.8; in Section 2.4, we show the self-dual Ramsey theorem is a particular case of Theorem 1.4.8; in Section 2.5, we show a natural extension of Theorem 1.4.8 does not hold.

2.1 A variation of the self-dual Ramsey theorem and the Hales-Jewett Theorem

In Section 8.3 of [1], Solecki gives the self-dual Ramsey theorem in terms of augmented surjections. Also Solecki presents the Hales-Jewett theorem in terms of rigid surjections in [1]. In this section, we give a variation of the self-dual Ramsey theorem. Also for completeness, we include the Hales-Jewett theorem in terms of rigid surjections in this section. These two theorems play the role of pigeonhole principle in the proof of Theorem 1.4.8 by the abstract approach.

Theorem 2.1.1 (Solecki)**.** *Given $d > 0$, $k \leq m$, there exists n with the following property. For each d-coloring of*

$$\{(s,h) \in \mathrm{AS} \mid (s,h) : [n+1] \to [k+1] \text{ and } s^{-1}(k+1) = \{n+1\}\},$$

there is an augmented surjection $(t,g) : [n+1] \to [m+1]$ such that $t^{-1}(m+1) = \{n+1\}$ and

$$\{(t,g) \,.\, (s,h) \mid (s,h) : [m+1] \to [k+1], (s,h) \in \mathrm{AS} \text{ and } s^{-1}(k+1) = \{m+1\}\}$$

is monochromatic.

Following almost the same proof of the above theorem, we get a variation as follows.

Given $d > 0$*,* $m_0 < k \leq m$*, and* $v_0 = (id \restriction [m_0+1], id \restriction [m_0+1])$*, there exists* n *with the following property. For each* d*-coloring of*

$$\{(s,h) \in \mathrm{AS} \mid (s,h) : [n+1] \to [k+1], (s,h) \restriction [m_0+1] = v_0, s^{-1}(k+1) = \{n+1\}\},$$

there is an augmented surjection $(t,g) : [n+1] \to [m+1]$ *such that* $(t,g) \restriction [m_0+1] = v_0$*,* $t^{-1}(m+1) = \{n+1\}$ *and*

$$\{(t,g) \,.\, (s,h) \mid (s,h) : [m+1] \to [k+1], (s,h) \restriction [m_0+1] = v_0,$$
$$(s,h) \in \mathrm{AS}, s^{-1}(k+1) = \{m+1\}\}$$

is monochromatic.

Beside introducing connection in terms of partitions, Solecki also introduces another kind of connection in terms of rigid surjections and increasing injections in Section 2.1 of [1]. A pair (s,i) is called a *connection* between m and k if $s : [m] \to [k] \in \mathrm{RS}$, $i : [k] \to [m] \in \mathrm{II}$ and for each $x \in [k]$

$$s(i(x)) = x \text{ and } \forall y < i(x) \ s(y) \leq x.$$

We write $(s,i) : [m] \leftrightarrow [k]$.

Note that there is a natural $1-1$ correspondence between the specific augmented surjections in the above statement and connections defined above. Let $(s,h) : [m+1] \to [k+1]$ be an augmented surjection with $s^{-1}(k+1) = \{m+1\}$. Let $s' = s \restriction [m]$. Let $i : [k] \to [m]$ be defined by letting, for each $x \in [k]$, $i(x) = \max h^{-1}(x)$. Then it is not difficult to check that (s,i) is a connection between m and k. By this relation, we can translate the above statement to the following one.

Given $d > 0$*,* $m_0 < k \leq m$*, there exists* n *with the following property. For each* d*-coloring of*

$$\{(s,h) \mid (s,h) : [n] \leftrightarrow [k], s \restriction [m_0+1] = id \restriction [m_0+1], h \restriction [m_0] = id \restriction [m_0]\},$$

there is a connection $(t,g) : [n] \leftrightarrow [m]$ *such that* $t \restriction [m_0+1] = id \restriction [m_0+1]$, $g \restriction [m_0] = id \restriction [m_0]$ *and*

$$\{(t,g) \,.\, (s,h) \mid (s,h) : [m] \leftrightarrow [k], s \restriction [m_0+1] = id \restriction [m_0+1], h \restriction [m_0] = id \restriction [m_0]\}$$

is monochromatic.

Given a set A, let $\text{lin}_A(\mathbb{N})$ consist of all linear orders L on $A \cup \{\mathbb{N}\}$ with the property that, for each $a \in A$, $n \in \mathbb{N}$, aLn and $L \restriction \mathbb{N}$ is the usual order.

For m, n two natural numbers, we say $t : A \cup [n] \to A \cup [m]$ is a rigid surjection with respect to $L \in \text{lin}_A(\mathbb{N})$, if it is a rigid surjection when $A \cup [n]$ and $A \cup [m]$ are equipped with linear orders inherited from L; we say $(t,g) : A \cup [n] \leftrightarrow A \cup [m]$ is a connection with respect to $L \in \text{lin}_A(\mathbb{N})$, if it is a connection when $A \cup [n]$ and $A \cup [m]$ are equipped with linear orders inherited from L.

Now the above statement can be translated into the following form.

Given $d > 0$, $1 \leq k' \leq m'$, *a finite set* A *with* $|A| = m_0$, *a linear order* $L \in \text{lin}_A(\mathbb{N})$, *there exists* n' *with the following property. For each* d*-coloring of*

$$\{(s,h) \mid (s,h) : A \cup [n'] \leftrightarrow A \cup [k'], s \restriction A' = id \restriction A', h \restriction A = id \restriction A\},$$

there is a connection $(t,g) : A \cup [n'] \leftrightarrow A \cup [m']$ *such that* $t \restriction A' = id \restriction A'$, $g \restriction A = id \restriction A$ *and*

$$\{(t,g) \,.\, (s,h) \mid (s,h) : A \cup [m'] \leftrightarrow A \cup [k'], s \restriction A' = id \restriction A', h \restriction A = id \restriction A\}$$

is monochromatic, where $A' = A \cup \{1\}$ *and all connections are with respect to* L.

To be convenient for the following proof, we restate the above statement. Let $U^{AS}_{A,n',m'}$ denote

$$\{(t,g) \mid (t,g) : A \cup [n'] \leftrightarrow A \cup [m'], t \restriction A' = id \restriction A', g \restriction A = id \restriction A\}.$$

Let $V^{AS}_{A,m',k'}$ denote

$$\{(s,h) \mid (s,h) : A \cup [m'] \leftrightarrow A \cup [k'], s \restriction A' = id \restriction A', h \restriction A = id \restriction A\}.$$

Theorem 2.1.2. *Given* $d > 0$, $1 \leq k' \leq m'$, *a finite set* A *and* $L \in \text{lin}_A(\mathbb{N})$, *there exists* n' *with the following* *property. For each d-coloring of* $U^{AS}_{A,n',m'} \,.\, V^{AS}_{A,m',k'}$ *there is a* $(t,g) \in U^{AS}_{A,n',m'}$ *such that* $(t,g) \,.\, V^{AS}_{A,m',k'}$ *is monochromatic.*

Theorem 2.1.3 (Hales-Jewett)**.** *Given* $d > 0, m$*, a finite set* A *and* $L \in \mathrm{lin}_A(\mathbb{N})$*, there exists* n *with the following property. For each d-coloring of the set*

$$\{f : A \cup [n] \to A \mid f \restriction A = id \restriction A\}$$

there exists a rigid surjection $g : A \cup [n] \to A \cup [m]$ *with respect to* L*, such that* $g \restriction A = id \restriction A$ *and*

$$\{f \circ g \mid f : A \cup [m] \to A, f \restriction A = id \restriction A\}$$

is monochromatic.

To be convinient for the following proof, we restate the above statement. Let $U^{RS}_{A,n,m}$ denote

$$\{g : A \cup [n] \to A \cup [m] \mid g \text{ a rigid surjection with respect to } L, g \restriction A = id \restriction A\}.$$

Let $V^{RS}_{A,m}$ denote

$$\{f : A \cup [m] \to A\} \mid f \restriction A = id \restriction A.\}$$

Then the Hales-Jewett theorem can be restated as follows.

Theorem 2.1.4 (Hales-Jewett)**.** *Given* $d > 0, m$*, a finite set* A *and* $L \in \mathrm{lin}_A(\mathbb{N})$*, there exists* n *with the following property. For each d-coloring of* $U^{RS}_{A,n,m} \,.\, V^{RS}_{A,m}$*, there exists* $g \in U^{RS}_{A,n,m}$ *such that* $g \,.\, V^{RS}_{A,m}$ *is monochromatic.*

2.2 Proof of the self-dual Ramsey theorem for parameter systems

In this section, we prove Theorem 1.4.8. We fix F a parameter system on a finite set D. For P, Q, two subsets of $\mathbb{N}$ with $\max Q \leq |P|$, let $P \circ Q$ denote $\{P(Q(s)) : s \leq |Q|\}$; let ∂P denote the set obtained by removing the largest element from P. For V, an m-space of D^n, let $|V|$ denote the number of coordinates, that is n; let $b(V)$ denote the minimal basis of V; let ∂V denote the set $p_{\{i:i<i_0\}}(V)$ where i_0 is the smallest number such that

$$\dim(p_{[i_0]}(V)) = \dim(V).$$

Formally speaking, $\emptyset$ is not a space, but for technical reason, we let $\partial\emptyset = \emptyset$ and $|\emptyset| = 0$.

Minimal representation. Let V be an m-space of D^n with the minimal basis I. By Definition 1.4.4, there exists $f_i \in F_{|I\cap[i]|}$ for each $i \in [n] \setminus I$ such that V can be expressed as follows.

$$\{v \in D^n \mid v(i) = f_i(v \upharpoonright I \cap [i]), i \in [n] \setminus I\}. \tag{2.2.1}$$

We call it the minimal respresentation of V.

Composition of two spaces. Again for technical reason, we let $V' \circ \emptyset$ be defined and equal to $\emptyset$ for every space V', also let $\emptyset \circ \emptyset$ be defined and equal to $\emptyset$. In addition to V, let W be a k-space of D^l with the minimal basis J. Then $V \circ W$ is defined precisely when $l \leq m$, and is defined as follows. Let

$$\dim = \max\{s : \dim(p_{[s]}(V)) = l\}. \tag{2.2.2}$$

Define $V \circ W \subseteq D^{\dim}$as follows:

$$\begin{aligned}\{u \in D^{\dim} \mid &\exists w \in W,\ u \upharpoonright I([l]) = w, \\ &u(i) = f_i(u \upharpoonright I \cap [i]) \text{ for } i \in [\dim] \setminus I\}.\end{aligned} \tag{2.2.3}$$

where f_i are as in (2.2.1).

Lemma 2.2.1. *Assume $V \circ W$ is defined, then it is a k-space of $D^{\dim}$ with the minimal basis $I(J)$.*

Proof. If $k = 0$, then it is easy to check that the statement holds. So we assume $k > 0$. By (2.2.3), we observe that for $s \in [l]$, if coordinate s depends on the coordinates $J \cap [s]$ through $h \in F_{|J\cap[s]|}$ in W, then coordinate $I(s)$ depends on the coordinates $I(J \cap [s])$ through h in $V \circ W$. We want to point out that if $s \in J$, then in W, coordinate s depends on the coordinates $J \cap [s]$ through the projection on the $|J \cap [s]|$-th coordinate from $D^{|J\cap[s]|}$ to D.

Now for $s \in [l]$, let $h_s \in F_{|J\cap[s]|}$ be the function through which coordinate s depends on the coordinates $J \cap [s]$ in W.

Claim 2.2.2. *For each $i \in [\dim] \setminus I(J)$, coordinate i depends on the coordinates $I(J) \cap [i]$ through some function $f \in F_{|I(J)\cap[i]|}$ in $V \circ W$.*

Proof. Let $i \in [\dim] \setminus I(J)$. There are two cases to deal with.

Case 1: $i \in I \setminus I(J)$. Assume $i = I(s)$ for some $s \in [l]$, by the above observation, we have that for each $u \in V \circ W$

$$u(i) = h_s(u \upharpoonright I(J \cap [s])).$$

Note that

$$I(J \cap [s]) = I(J) \cap I([s]) = I(J) \cap [I(s)] = I(J) \cap [i].$$

It follows that coordinate i depends on the coordinates $I(J) \cap [i]$ through $h_s \in F_{|I(J)\cap[i]|}$ in $V \circ W$.

Case 2: $i \in [\dim] \setminus I$. Let f_i be the function in $F_{|I\cap[i]|}$ through which coordinate i depends on the coordinates $I \cap [i]$ in V. Let $l' = |I \cap [i]|$, note that $l' \leq l$. By (2.2.3), we see for each $u \in V \circ W$,

$$u(i) = f_i(h_1(u \upharpoonright I(J \cap [1])), \ldots, h_{l'}(u \upharpoonright I(J \cap [l']))).$$

By $A(2) - A(4)$ in Section 1.4.1, we have that $f_i(h_1, \ldots, h_{l'}) \in F$. Note that

$$I(J \cap [l']) = I(J) \cap I([l']) = I(J) \cap [I(l')] = I(J) \cap [i].$$

It follows that coordinate i depends on the coordinates $I(J) \cap [i]$ through $f_i(h_1, \ldots, h_{l'}) \in F_{|I(J)\cap[i]|}$ in $V \circ W$. □

By (2.2.3), the above claim and Definition 1.4.4, we see $V \circ W$ is a k-space of $D^{\dim}$ with the minimal basis $I(J)$. □

Normed composition space (A, X)**.** Let A, X both consist of all connections and $(\emptyset, \emptyset)$. Let $P, Q \subseteq [n]$ be such that $(V, P), (W, Q) \in A$. Then $(V, P) \cdot (W, Q)$ and $(V, P) \,.\, (W, Q)$ are defined precisely when $V \circ W$ is defined, and let

$$(V, P) \cdot (W, Q) = (V, P) \,.\, (W, Q) = (V \circ W, P \circ Q).$$

Lemma 2.2.3. *Assume $(V, P) \circ (W, Q)$ is defined, then it is a k-connection of $D^{\dim}$.*

Proof. By Lemma 2.2.1, $V \circ W$ is a k-space of $D^{\dim}$ with the minimal basis $I(J)$. It remains to show that, for each $s \in [k]$, $P(Q(s))$ is an associate coordinate of $I(J(s))$ in $V \circ W$. The case for $k = 0$ is trivial, so we assume $k > 0$. Let $s \in [k]$.

First, we show R.2 holds for $P(Q(s)), I(J(s))$ in $V \circ W$. Since $Q(s)$ is an associate coordinate of $J(s)$ in W, by R.2, we have that for each $w \in W$, $w(Q(s)) = w(J(s))$. Then by (2.2.3), we have that for each $u \in V \circ W$, $u(I(Q(s))) = u(I(J(s)))$. Since $P(Q(s))$ is an associate coordinate of $I(Q(s))$ in V, by R.2 and the fact $V \circ W \subseteq V$, we have that for each $u \in V \circ W$, $u(P(Q(s))) = u(I(Q(s)))$. It follows that for each $u \in V \circ W$, $u(P(Q(s))) = u(I(J(s)))$.

Second, we show R.1 holds for $P(Q(s)), I(J(s))$ in $V \circ W$. It is obvious that

$$I(J(s)) \leq P(Q(s)). \tag{2.2.4}$$

If $s = k$, then by (2.2.4), we have that

$$I(J) \cap [P(Q(k))] = I(J) = I(J) \cap [I(J(k))].$$

Now we assume $s < k$. Since $Q(s)$ is an associate coordinate of $J(s)$ in W, by R.1, we have that

$$Q(s) < J(s+1). \tag{2.2.5}$$

Since $P(Q(s))$ is an associate coordinate of $I(Q(s))$ in V, by R.1 and (2.2.5), we have that

$$P(Q(s)) < I(Q(s)+1) \leq I(J(s+1)), \tag{2.2.6}$$

Then by (2.2.4) and (2.2.6), we have that

$$I(J(s)) \leq P(Q(s)) < I(J(s+1)),$$

which implies that

$$I(J) \cap [P(Q(s))] = I(J) \cap [I(J(s))].$$

□

Define the norm and truncation of (W, Q) as follows.

1. $|(W, Q)| = |W|$;

2. $\partial(W, Q) = (\partial W, \partial Q)$.

It is straightforward to check that (A, X) with the operations defined above is a normed composition space.

Ramsey domain $(\mathcal{F}, \mathcal{S})$ over (A, X). For $0 \leq k \leq m$, let $\binom{m}{k}$ denote all k-connections of D^m if $m > 0$; denote $\{(\emptyset, \emptyset)\}$ if $m = 0$.

Lemma 2.2.4. *Let V, V' be spaces of D^n with the minimal basises I, I', respectively and $V' \subseteq V$, then $I' \subseteq I$.*

Proof. Let $i \in [n] \setminus I$, note that it suffices to show $i \notin I'$. If $i = 1$, then $p_{\{1\}}(V)$ is a constant which implies by the fact $V' \subseteq V$, that $p_{\{1\}}(V')$ is also a constant. So $1 \notin I'$. Now we assume $i > 1$. Since I is the minimal basis of V, coordinate i depends on the coordinates $I \cap [i]$ through some function $f \in F_{|I \cap [i]|}$ in V. By the fact $V' \subseteq V$ again, the same property holds for V', which implies that $\dim(p_{[i]}(V')) = \dim(p_{[i-1]}(V'))$. By the equivalent definition of minimal basis(see Remark 1.4.1 after Lemma 1.4.5), we have that $i \notin I'$. $\square$

Lemma 2.2.5. *For $0 \leq k \leq m \leq n$ with $m > 0$, let $(V, P) \in \binom{n}{m}$, then $(V, P) \,.\, \binom{m}{k} = \{(V', P') \in \binom{n}{k} \mid (V', P') \leq (V, P)\}$.*

Proof. By Lemma 2.2.3 the left hand side is a subset of the right hand side. It remains to show the other direction. Assume $(V', P') \in \binom{n}{k}$ and $(V', P') \leq (V, P)$. We need to construct $(W, Q) \in \binom{m}{k}$ such that $(V', P') = (V, P) \,.\, (W, Q)$.

Let I, I' be the minimal basises of V and V', respectively. By Lemma 2.2.4 and the fact $V' \subseteq V$, we have $I' \subseteq I$ which implies that $\dim(p_I(V')) = \dim(V')$. So $p_I(V')$ is a k-space of D^m. Let $W = p_I(V')$. Then by (2.2.3) and the fact $V' \subseteq V$, we see $V' = V \circ W$. Let $h : I \to [m]$ be an increasing injection, then we see that the minimal basis J of W is $h(I')$. Next we construct associate coordinate for each minimal basis coordinate in $h(I')$.

Let $i' \in I'$, a minimal basis coordinate of V', and assume its associate coordinate in V' is $j \in P'$. Note that $P' \subseteq P$, so there exists $i \in I$ whose associate coordinate in V is j. We claim that $h(i)$ is an assoicate coordinate of $h(i')$ in W.

To show R.2 holds for $h(i'), h(i)$. By R.2 for i, j and i', j and the fact that $V' \subseteq V$, we have that for each $v \in V'$, $v(i) = v(j)$ and $v(i') = v(j)$. It follows that, for each $v \in V'$, $v(i') = v(i)$, which implies that for each $w \in W$, $w(h(i')) = w(h(i))$.

To show R.1 holds for $h(i'), h(i)$. By R.1 for i, j and i', j, we have

$$I \cap [i] = I \cap [j] \text{ and} \tag{2.2.7}$$

$$I' \cap [i'] = I' \cap [j]. \tag{2.2.8}$$

By (2.2.7) and the fact $i \in I$, we have $i \leq j$. By (2.2.8) and the fact $i' \in I'$, we have $i' \leq j$. Note that if $i < i'$, then by the facts $i' \in I, i' \leq j$, we have $i' \notin I \cap [i]$ and $i' \in I \cap [j]$, contradicting with (2.2.7). So we have $i' \leq i \leq j$. It follows by (2.2.8), that

$$I' \cap [i'] = I' \cap [i]$$

which implies that

$$h(I') \cap h([i']) = h(I') \cap h([i]).$$

It follows that

$$h(I') \cap [h(i')] = h(I') \cap [h(i)].$$

So $h(i)$ is an associate coordinate of $h(i')$ in W. Let Q consist of all such kind of associate coordinates for the minimal basis coordinates of W. By the constructions, we see $P' = P \circ Q$. So, we have $(V', P') = (V, P) . (W, Q)$.

□

Lemma 2.2.6. *For $0 \leq k \leq m \leq n$ with $m > 0$, $\binom{n}{m} . \binom{m}{k} = \binom{n}{k}$.*

Proof. By Lemma 2.2.3 the left hand side is a subset of the right hand side. It remains to show the other direction. Let (V', P') be a k-connection of D^n. Observe that there exists an m-connection (V, P) of D^n such that $(V', P') \leq (V, P)$. By Lemma 2.2.5, there exists a k-connection (W, Q) of D^m such that

$$(V', P') = (V, P) \circ (W, Q),$$

which implies that the right hand side is a subset of the left hand side. □

For $1 \leq m \leq n$, an m-connection (V, P) of D^n is called a *rigid m-connection* of D^n if $\dim(p_{[2,n]}(V)) = m - 1$ in the case $n > 1$ and $1 \in b(V)$. Let (W, Q) be a rigid k-connection of D^n, we say (W, Q) is a *rigid k-subconnection* of (V, P) if $(W, Q) \leq (V, P)$.

Lemma 2.2.7. *Let (V, P) be a rigid m-connection of D^n, then $1 \in P$ and $b(V) = \{1\} \cup \{1 + i : i \in b(p_{[2,n]}(V))\}$ in the case $n > 1$.*

Proof. The case for $n = 1$ is trivial. So we assume $n > 1$. It is not difficult to see that, the only associate coordinate of coordiate 1 in V is 1, because, otherwise, we have $\dim(V) = \dim(p_{[2,n]}(V))$, a contradiction. So $1 \in P$.

By Lemma 1.4.5 and the fact $\dim(p_{[2,n]}(V)) = m - 1$, there exists $I' \subseteq [2, n]$ with $|I'| = m - 1$, such that for each $i \in [2, n] \setminus I'$, there exists $f_i \in F_{|I' \cap [2,i]|}$ such that coordinate i depends on the coordinates $I' \cap [2, i]$ through f_i in V. So we have

$$\dim(p_{\{1\} \cup I'}(V)) = \dim(V) = m.$$

It follows that $b(V) = \{1\} \cup I'$. Note that $I' = \{1 + i : i \in b(p_{[2,n]}(V))\}$, so the statement holds. □

For a rigid connection (V, P) as above, let $\mathrm{map}(V, P)$ denote

$$(p_{[2,n]}(V), P([2, m])).$$

The following three lemmas are easy consequences of Lemma 2.2.7.

Lemma 2.2.8. *The function map is a bijection from the set of rigid connections to A and if $(V, P) \in \binom{n}{m}^r$ for some $1 \leq m \leq n$, then $map(V, P) \in \binom{n-1}{m-1}$.*

Lemma 2.2.9. *Let (V, P), (W, Q) be two rigid connections with $(V, P) \circ (W, Q)$ defined and* $\mathrm{map}(V, P) \neq (\emptyset, \emptyset)$*,* $\mathrm{map}(W, Q) \neq (\emptyset, \emptyset)$*, then*

$$\mathrm{map}((V, P) \circ (W, Q)) = \mathrm{map}(V, P) \circ \mathrm{map}(W, Q).$$

Lemma 2.2.10. *Let (V', P'), (W', Q') be two non-$(\emptyset, \emptyset)$ connections with $(V', P') \circ (W', Q')$ defined, then*

$$\mathrm{map}^{-1}((V', P') \circ (W', Q')) = \mathrm{map}^{-1}(V', P') \circ \mathrm{map}^{-1}(W', Q').$$

For $1 \leq m \leq n$, let $\binom{n}{m}^r$ consist of all rigid m-connections of D^n, let $\begin{bmatrix} n \\ m \end{bmatrix}^r$ consist of all rigid m-connections of $D^{n'}$ for each $m \leq n' \leq n$. Note that $\begin{bmatrix} n \\ m \end{bmatrix}^r = \cup_{m'=m}^{n} \binom{m'}{m}^r$. Let $\binom{0}{0}^r$ and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}^r$ denote $\{(\emptyset, \emptyset)\}$.

Put

$$\mathcal{F} = \mathcal{S} = \{\binom{n}{m}^r, \begin{bmatrix} n \\ m \end{bmatrix}^r : 1 \leq m \leq n \text{ or } m = n = 0\}.$$

For non-$\{(\emptyset, \emptyset)\}$ elements in $\mathcal{F}$, we define $\bullet$ and $\bullet$ in the following situations.

$$\binom{n}{m}^r \bullet \binom{m}{k}^r = \binom{n}{m}^r \bullet \binom{m}{k}^r = \binom{n}{k}^r;$$
$$\begin{bmatrix} n \\ m \end{bmatrix}^r \bullet \begin{bmatrix} m \\ k \end{bmatrix}^r = \begin{bmatrix} n \\ m \end{bmatrix}^r \bullet \begin{bmatrix} m \\ k \end{bmatrix}^r = \begin{bmatrix} n \\ k \end{bmatrix}^r.$$

For $\{(\emptyset, \emptyset)\}$, we let

$$\{(\emptyset, \emptyset)\} \bullet \{(\emptyset, \emptyset)\} = \{(\emptyset, \emptyset)\} \bullet \{(\emptyset, \emptyset)\} = \{(\emptyset, \emptyset)\}$$

Lemma 2.2.11. *For $1 \leq k \leq m \leq n$, $\binom{n}{m}^r \cdot \binom{m}{k}^r = \binom{n}{k}^r$; $\begin{bmatrix} n \\ m \end{bmatrix}^r \cdot \begin{bmatrix} m \\ k \end{bmatrix}^r = \begin{bmatrix} n \\ k \end{bmatrix}^r$.*

Proof. To prove the first part. It is trivial for the case $n = m$ or $m = k$. So we assume $n > m$ and $m > k$. This case is an immediate consequence of Lemma 2.2.6, Lemma 2.2.8, Lemma 2.2.9 and Lemma 2.2.10.

For the second part, it is straightforward to check it based on the first part. □

By Lemma 2.2.11, the operations $\bullet$ and $\bullet$ are defined pointwise. And it is straightforward to check that the properties (A), (B) and (C) hold. So $(\mathcal{F}, \mathcal{S})$ is a Ramsey domain over (A, X).

Pigeonhole condition. Let S, G, A be three sets with a function from $G \times S$ to A: $(g, s) \to g \,.\, s$. Let F, R, B be three sets with a function from $F \times R$ to B: $(f, r) \to f \,.\, r$. Given $d > 0$, we say (F, R) satisfies *d-Ramsey condition*, if for each d-coloring of $\{f \,.\, r : f \in F, r \in R\}$, there exists $f \in F$ such that $\{f \,.\, r : r \in R\}$ is monochromatic. We say (G, S) is *interpretable* in (F, R) if there are functions $\alpha : S \to R$ and $\phi : F \to G$ such that for $y_1, y_2 \in S$ and $f_1, f_2 \in F$, we have

$$f_1 \,.\, \alpha(y_1) = f_2 \,.\, \alpha(y_2) \Rightarrow \phi(f_1) \,.\, y_1 = \phi(f_2) \,.\, y_2. \tag{2.2.9}$$

The following lemma can be checked without difficulty.

Lemma 2.2.12. *Given $d > 0$. If (G, S) is interpretable in (F, R) and (F, R) satisfies the d-Ramsey condition, then (G, S) satisfies the d-Ramsey condition.*

For $i, j \in [n]$, V a space of D^n, we say coordinate i *agrees with* coordinate j in V, if for each $v \in V$, $v(i) = v(j)$.

Lemma 2.2.13. *Given $d > 0$, $S = \binom{m}{0}$ with $m > 0$, there exists n such that (G, S) satisfies the d-Ramsey condition where $G = \binom{n}{m}$.*

Proof. Let $L \in \mathrm{lin}_D(\mathbb{N})$. Let n in terms of d, m, D, L be big enough so that Theorem 2.1.4 works for n. Let $G = \binom{n}{m}$.

Define $\alpha : S \to V^{RS}_{D,m}$ as follows. For $(W, Q) \in S$, note that W consists of only one element w and $Q = \emptyset$. Let $f : D \cup [m] \to D$ be defined as below.

$$f(i) = \begin{cases} i & \text{if } i \in D; \\ w(i), & \text{if } i \in [m]. \end{cases}$$

Note that $f \in V^{RS}_{D,m}$, we let $\alpha((W, Q)) = f$.

Define $\phi : U^{RS}_{D,n,m} \to G$ as follows. For $g \in U^{RS}_{D,n,m}$, there exists an m-connection (V, P) of D^n such that

1. $b(V) = \{\min g^{-1}(j) : j \in [m]\}$ and $P = b(V)$;
2. for $i \in [n] \setminus b(V)$, if $g(i) \in D$, then for each $v \in V$, $v(i) = g(i)$; if $g(i) \in [m]$, then coordinate i agrees with coordinate $\min g^{-1}(g(i))$ in V.

We let $\phi(g) = (V, P)$. It is straightforward to check that for the unique element $u \in V \circ W$, we have $u(i) = f \circ g(i)$ for each $i \in [n]$. Also note that for each $(W, Q) \in S$, $Q = \emptyset$. It follows that (2.2.9) holds. Note that $(U^{RS}_{D,n,m}, V^{RS}_{D,m})$ satisfies the d-Ramsey condition. It follows by Lemma 2.2.12, that (G, S) satisfies the d-Ramsey condition. □

Lemma 2.2.14. *Given $d > 0$, $S = \binom{m}{k}$ with $k > 0$, $x = (X, R) \in \partial S$ with $|X| = m_0$, there exists n such that $(G_a, (S)_x)$ satisfies the d-Ramsey condition where $G = \binom{n}{m}$ and $a = (D^{m_0}, [m_0])$.*

Proof. Let $B = F_k$(a family of functions from D^k to D) and $L \in \mathrm{lin}_B(\mathbb{N})$. Let $k_0 = |b(X)|$, $m' = m - m_0$ and $k' = k - k_0$ which is equal to 1. Let n' be big enough in terms of d, k', m', B and L such that Theorem 2.1.2 works for n'. Let $G = \binom{n'+m_0}{m'+m_0}$. Note that $(U^{AS}_{B,n',m'}, V^{AS}_{B,m',k'})$ satisfies the d-Ramsey condition, then by Lemma 2.2.12, it suffices to show $(G_a, (S)_x)$ is interpretable in $(U^{AS}_{B,n',m'}, V^{AS}_{B,m',k'})$.

To be convinient in the following proof, we rename the coordinates as follows. For a space with number of coordinates greater than or equal to m_0, when we say coordinate c_i for $i \leq m_0$, we mean the i-th coordinate; when we say coordinate $j \in \mathbb{N}$, we mean the $(m_0 + j)$-th coordinate. Let $C = \{c_1, \ldots, c_{m_0}\}$ and $J = b(X) \cup \{1\}$.

First, we show a natural relation between $(S)_x$ and $V^{AS}_{B,m',k'}$(see Section 2.1). For $(W, Q) \in (S)_x$, we have $\partial(W, Q) = (X, R)$ which implies that $\partial W = X$ and $\partial Q = R$. It follows that $b(W) = J$ and $Q = R \cup Q'$ for some one element subset Q' of $[m']$. Thus we can translate (W, Q) to $(s, h) \in V^{AS}_{B,m',k'}$ which satisfies the following conditions.

S.1 For $i \in [m']$, if coordinate i agrees with coordinate 1 in W, then $s(i) = 1$; if not, then $s(i) = g$ where coordinate i depends on the coordinates J through g in W;

S.2 $h(1) = Q'(1)$.

Second, we show a natural relation between $U^{AS}_{B,n',m'}$ and G_a. For $(t, g) \in U^{AS}_{B,n',m'}$, we construct (V, P) satisfying the following conditions.

G.1 Minimal basis coordinates: $b(V) = C \cup I'$, $I' = \{\min t^{-1}(j) : j \in [m']\}$;

G.2 Coordinates $[n']$: For $i \in [n']$

(a) if $t(i) \in B$, then coordinate i depends on the coordinates J through $t(i)$ in V;

(b) if $t(i) \in [m']$, then coordinate i agrees with coordinate $\min t^{-1}(t(i))$ in V.

G.3 Associate coordinates: $P = C \cup P'$ where $P' \subseteq [n']$ such that for $j \in [m']$, $P'(j) = g(j)$.

Since $t \restriction B \cup \{1\} = id \restriction B \cup \{1\}$, we have $1 \in I'$. Since (t, g) is a connection with respect to rigid surjections and increasing injections, we observe that (V, P) is an m-connection of D^n. Furthermore, since $C \cup \{1\} \subseteq b(V)$, we have $(V, P) \in G_a$.

Define $\alpha : (S)_x \to V^{AS}_{B,m',k'}$, $\phi : U^{AS}_{B,n',m'} \to G_a$ by sending $(W, Q) \in (S)_x$, $(t, g) \in U^{AS}_{B,n',m'}$ to (s, h), (V, P) obtained in the above ways, respectively.

Claim 2.2.15. *(2.2.9) holds.*

Proof. Let $(t, g) \in U^{AS}_{B,n',m'}$ and $(W, Q) \in (S)_x$. Assume $\phi((t, g)) = (V, P)$ and $\alpha((W, Q)) = (s, h)$. We show (2.2.9) holds by analyzing $(V, P) \, . \, (W, Q)$ from the perspectives of minimal basis coordinates, coordinates $[n']$ and assoicate coordiates.

Minimal basis coordinates: Note that $C \cup \{1\} \subseteq b(V)$ and $b(W) = J \subseteq C \cup \{1\}$, then by (2.2.3), we have that $b(V \circ W) = J$ which does not depend on the choices of (t, g) and (W, Q).

Coordinates $[n']$**:** Note that $\partial W = X$, so we have $p_C(W) = X$. Also we have $C \subseteq b(V)$. Then by (2.2.3), we have that $p_C(V \circ W) = X$ which is given. That means in $V \circ W$, coordinates in C does not depend on the choices of (t, g) and (W, Q). So we only need to consider coordinates $[n']$ in $V \circ W$.

We claim that for $i \in [n']$, if $s \circ t(i) \in B$, then coordinate i depends on the coordinates J in $V \circ W$ through $s \circ t(i)$; if $s \circ t(i) \notin B$, then coordinate i agrees with coordinate $s \circ t(i)$ which equals 1 in $V \circ W$.

Case 1: $t(i) \in B$. By (a) of G.2, coordinate i depends on the coordinates J through $t(i)$ in V. Note that $b(V \circ W) = J$, so by (2.2.3) the same thing happens to coordinate i in $V \circ W$, that is, coordinate i depends on the coordinates J through $t(i)$ in $V \circ W$. Since $s \upharpoonright B \cup \{1\} = id \upharpoonright B \cup \{1\}$, we have $t(i) = s \circ t(i)$ which implies that coordinate i depends on the coordinates J through $s \circ t(i)$ in $V \circ W$. So the claim holds in this case.

Case 2: $t(i) \in [m']$. By (b) of G.2 , we have that

$$\text{coordinate } i \text{ agrees with coordinate } \min t^{-1}(t(i)) \text{ in } V. \tag{2.2.10}$$

By G.1, we have that

$$\text{coordinate } \min t^{-1}(t(i)) \text{ is the } (m_0 + t(i)) - \text{th element in } b(V). \tag{2.2.11}$$

Subcase 1: $s \circ t(i) \in B$. By S.1, coordinate $t(i)$ depends on the coordinates J through $s \circ t(i)$ in W. Then by (2.2.11) and (2.2.3), coordinate $\min t^{-1}(t(i))$ depends on the coordinates J through $s \circ t(i)$ in $V \circ W$. By (2.2.10) and the fact $V \circ W \subseteq V$, coordinate i depends on the coordinates J through $s \circ t(i)$ in $V \circ W$. So the claim holds in this subcase.

Subcase 2: $s \circ t(i) \notin B$, that is, $s \circ t(i) = 1$. By S.1, coordinate $t(i)$ agrees with coordinate 1 in W. Then by (2.2.11) and (2.2.3), coordinate $\min t^{-1}(t(i))$ agrees with coordinate 1 in $V \circ W$. By (2.2.10) and the fact $V \circ W \subseteq V$, coordinate i agrees with coordinate 1 which equals $s \circ t(i)$ in $V \circ W$. So the claim holds in this subcase.

Associate coordinates: Assume $Q = R \cup Q'$, then by S.2, $Q'(1) = h(1)$. Assume $P = C \cup P'$, then by G.3, $P'(j) = g(j)$ for $j \in [m']$. It follows that $P'(Q'(1)) = g(h(1))$. Note that $P \circ Q = R \cup P'(Q'(1))$ which implies that $P \circ Q = R \cup g(h(1))$.

Now combining all of the above properties, we see (2.2.9) holds. □

□

Lemma 2.2.16. $(\mathcal{F}, \mathcal{S})$ *satisfies the pigeonhole condition.*

Proof. Given $d > 0$. Let $S \in \mathcal{S}$. The case for $S = \binom{0}{0}^r = \begin{bmatrix}0\\0\end{bmatrix}^r = \{(\emptyset, \emptyset)\}$ is trivial. So we assume $S \neq \{(\emptyset, \emptyset)\}$.

First let $S = \binom{m+1}{k+1}^r$ for some $0 \leq k \leq m$. There are three cases to deal with depending on the vaules of m and k.

Case 1: $m = 0$. This case is aslo trivial.

Case 2: $m > 0$ and $k = 0$. Let $x = (X, R) \in \partial S$. Note that $\partial S = \{(\emptyset, \emptyset)\}$, so $x = (\emptyset, \emptyset)$. Let $a = (\emptyset, \emptyset)$, then $a\,.\,x$ is defined.

Let $S' = \mathrm{map}(S)$ which is $\binom{m}{0}$ with $m > 0$. By Lemma 2.2.13, there exists n such that (G', S') satisfies the d-Ramsey condition where $G' = \binom{n}{m}$. Let $G = \binom{n+1}{m+1}^r$, then $G \bullet S$ is defined. By Lemma 2.2.8, $\mathrm{map}(G) = G'$, $\mathrm{map}(S) = S'$. It follows, by Lemma 2.2.9, Lemma 2.2.10, that the d-Ramsey condition holds for (G, S).

Note that $G_a = G$ and $(S)_x = S$, so the d-Ramsey condition holds for $(G_a, (S)_x)$.

Case 3: $k > 0$. Let $x = (X, R) \in \partial S$ with $|X| = m_0 + 1$. Let $S' = \mathrm{map}(S)$ which is $\binom{m}{k}$, $x' = (X', R') = \mathrm{map}(X, R)$. Note that $x' \in \partial S'$ with $|X'| = m_0$. By Lemma 2.2.14, there exists n such that $(G'_{a'}, (S')_{x'})$ satisfies the d-Ramsey condition where $G' = \binom{n}{m}$ and $a' = (D^{m_0}, [m_0])$. Let $G = \binom{n+1}{m+1}^r$, $a = (D^{m_0+1}, [m_0+1])$, then $G \bullet S$ and $a\,.\,x$ are defined. By Lemma 2.2.8, $\mathrm{map}(G_a) = G'_{a'}$ and $\mathrm{map}((S)_x) = (S')_{x'}$. Then by Lemma 2.2.9, Lemma 2.2.10, the d-Ramsey condition also holds for $(G_a, (S)_x)$.

Second, let $S = \begin{bmatrix}m\\k\end{bmatrix}^r$ for some $1 \leq k \leq m$. Let $x = (X, R) \in \partial S$ with $|(X, R)| = m_0$, and $a = (D^{m_0}, [m_0])$, then $a\,.\,x$ is defined. It suffices to show the following statement.

(A) There exists $n \geq m$, such that for each d-coloring of $\begin{bmatrix}n\\k\end{bmatrix}^r$, there exists $(V, P) \in \begin{bmatrix}n\\m\end{bmatrix}^r$ with $[m_0 + 1] \subseteq b(V)$ and $(V, P)\,.\,(S)_x$ is monochromatic. Note (V, P) extends a because $[m_0 + 1] \subseteq b(V)$.

We produce a sequence of statemens where the last one imples (A). Fix m_1 with $m_1 \geq k$ and $m_1 > m_0$. We will specify how large m_1 should be when we prove (A) from (C).

(B) There exists n, such that for each d-coloring of $\binom{n}{k}^r$, there exists $(V, P) \in \binom{n}{m_1}^r$, such that $[m_0+1] \subseteq b(V)$, and $(V, P)\,.\,\binom{m_1}{k}^r_x$ is monochromatic. (B) follows from the first part.

(C) There exists n, such that for each d-coloring of $\begin{bmatrix}n\\k\end{bmatrix}^r$, there exists $(V, P) \in \binom{n}{m_1}^r$ such that $[m_0 + 1] \subseteq b(V)$ and $(V, P)\,.\,\binom{m'}{k}^r_x$ is monochromatic for $m_0 < m' \leq m_1$.

We prove (C) from (B) by induction. It is obvious that (C) holds for $m_1 = m_0 + 1$. Move from m_1 to $m_1 + 1$. Apply induction hypothesis to get n_1 that works for m_1. Apply (B) to $n_1 + 1$ (playing the role of m_1) to get n.

Claim 2.2.17. *n works for $m_1 + 1$ in (C).*

Proof. Let c be a d-coloring of $\left[{n \atop k}\right]^r$. By the choice of n, there exists $(V_1, P_1) \in \binom{n}{n_1+1}^r$, such that $[m_0+1] \subseteq b(V_1)$, and $(V_1, P_1) \,.\, \binom{n_1+1}{k}^r_x$ is monochromatic.

Induce a coloring c' of $\left[{n_1 \atop k}\right]^r$ by letting

$$c^{'}((W,Q)) = c((V_1, P_1) \,.\, (W,Q)) \text{ for } (W,Q) \in \begin{bmatrix} n_1 \\ k \end{bmatrix}^r.$$

By the choice of n_1, there exists $(V_2, P_2) \in \binom{n_1}{m_1}^r$, such that $[m_0 + 1] \subseteq b(V_2)$, and $(V_2, P_2) \,.\, \binom{m'}{k}^r_x$ is monochromatic with respect to c' for $m_0 < m' \leq m_1$.

Let $(V_3, P_3) \in \binom{n_1+1}{m_1+1}^r$ be such that $\partial(V_3, P_3) = (V_2, P_2)$. We show that $(V_1, P_1) \cdot (V_3, P_3)$ works.

1. It is obvious that $(V_1, P_1) \cdot (V_3, P_3) \in \binom{n}{m_1+1}^r$ and $[m_0 + 1] \subseteq b(V_1 \circ V_3)$.

2. For $m' = m_1 + 1$. If $(W, Q) \in \binom{m_1+1}{k}^r_x$, then $(V_3, P_3) \,.\, (W, Q) \in \binom{n_1+1}{k}^r$, and $\partial((V_3, P_3) \,.\, (W, Q)) = x$. By the choice of (V_1, P_1), we have that

$$((V_1, P_1) \cdot (V_3, P_3)) \,.\, \binom{m_1 + 1}{k}^r_x$$

   is monochromatic with respect to c.

3. For $m_0 < m' \leq m_1$. If $(W, Q) \in \binom{m'}{k}^r$, then by the choice of (V_3, P_3)

$$((V_1, P_1) \cdot (V_3, P_3)) \,.\, (W, Q) = (V_1, P_1) \,.\, ((V_2, P_2) \,.\, (W, Q)).$$

   It follows, by the choices of c' and (V_2, P_2), that for $m_0 < m' \leq m_1$

$$((V_1, P_1) \cdot (V_3, P_3)) \,.\, \binom{m'}{k}^r_x$$

   is monchromatic with respect to c.

□

Now we derive (A) from (C). Note that $m > m_0$. Pick m_1 so that for each d-coloring of (m_0, m_1), there exists a subset with $m - m_0$ elements getting the same color. Apply (C) to m_1 to obtain n.

Claim 2.2.18. *n works for (A).*

Proof. Let c be a d-coloring of $\begin{bmatrix} n \\ k \end{bmatrix}^r$. By (C), there exists $(V_1, P_1) \in \binom{n}{m_1}^r$ with $[m_0+1] \subseteq b(V_1)$, such that for $m_0 < m' \leq m_1$, $(V_1, P_1) \,.\, \binom{m'}{k}^r_x$ is monochromatic.

By the choice of m_1, there exists $m_0 < i_1 < \cdots < i_{m-m_0} < m_1$ such that

$$(V_1, P_1) \,.\, \bigcup_{j=1}^{m-m_0} \binom{i_j}{k}^r_x \tag{2.2.12}$$

is monochromatic.

Let $(V_2, P_2) \in \binom{m_1}{m+1}^r$ be such that

$$b(V_2) = [m_0+1] \cup \{i_j + 1 \mid 1 \leq j \leq m - m_0\}.$$

Note that $(V_1, P_1) \cdot (V_2, P_2)$ satisfies the following conditions.

1. $(V_1, P_1) \cdot (V_2, P_2) \in \binom{n}{m+1}^r$;

2. $[m_0+1] \subseteq b(V_1 \circ V_2)$;

3. $((V_1, P_1) \cdot (V_2, P_2)) \,.\, \begin{bmatrix} m \\ k \end{bmatrix}^r_x$ is monochromatic. Indeed, if $(W, Q) \in \begin{bmatrix} m \\ k \end{bmatrix}^r_x$, then we have that

$$(V_2, P_2) \,.\, (W, Q) \in \bigcup_{j=1}^{m-m_0} \binom{i_j}{k}^r_x.$$

   It follows that this condition holds by (2.2.12).

Finally, let $(V_3, P_3) = \partial((V_1, P_1) \cdot (V_2, P_2)) \in \begin{bmatrix} n \\ m \end{bmatrix}^r$. We observe that, for each $(W, Q) \in \begin{bmatrix} m \\ k \end{bmatrix}^r_x$,

$$(V_3, P_3) \,.\, (W, Q) = ((V_1, P_1) \cdot (V_2, P_2)) \,.\, (W, Q),$$

which implies (A) holds by conditions (2) and (3) of $(V_1, P_1) \cdot (V_2, P_2)$. □

□

Proposition 2.2.19. *$(\mathcal{F}, \mathcal{S})$ satisfies the Ramsey condition.*

Proof. Note that $(\mathcal{F}, \mathcal{S})$ is a Ramsey domain and satisfies the pigeonhole condition due to Lemma 2.2.16. And it is obvious that $(\mathcal{F}, \mathcal{S})$ is linear and vanishing. Then by Theorem 1.2.1, $(\mathcal{F}, \mathcal{S})$ satisfies the Ramsey condition. □

Finally, we come to the proof of Theorem 1.4.8.

Proof. Given $d > 0$. By Proposition 2.2.19, $(\mathcal{F}, \mathcal{S})$ satisfies the Ramsey condition which implies that for $k \leq m$, there exists n such that the d-Ramsey condition holds for $(\binom{n+1}{m+1}^r, \binom{m+1}{k+1}^r)$.

By Lemma 2.2.8, $\text{map}(\binom{n+1}{m+1}^r) = \binom{n}{m}$ and $\text{map}(\binom{m+1}{k+1}^r) = \binom{m}{k}$. Then by Lemma 2.2.9, Lemma 2.2.10, the d-Ramsey condition holds for $(\binom{n}{m}, \binom{m}{k})$.

By Lemma 2.2.6, $\binom{n}{m} \cdot \binom{m}{k} = \binom{n}{k}$ denotes all k-connections of D^n. By Lemma 2.2.5, for each $(V, P) \in \binom{m}{k}$, $(V, P) \cdot \binom{m}{k}$ denotes all k-subconnections of (V, P). So Theorem 1.4.8 follows. □

2.3 A new proof of the Ramsey theorem for parameter systems

First, let us change the definition of associate coordinate as follows.

Definition 2.3.1. *Let V be an m-space of D^n with $b(V) = \{i_1, \ldots, i_m\}$. For each $s \leq m$, we call i_s an associate coordinate of i_s.*

We want to point out that, in Definition 1.4.6, a minimal basis coordinate might have multiple associate coordinates, however, in the current definition, a minimal basis coordinate has only one associate coordinate, that is, itself.

Note that the self-dual Ramsey theorem for parameter systems with the above new definition is a restatement of the Ramsey theorem for parameter systems. One can check that following the same proof with slight modification, we get the self-dual Ramsey theorem for paramter systems with the new definition. That means, we have a new proof for the Ramsey theorem for parameter systems by the abstract approach.

We can also derive the Ramsey theorem for parameter systems from Theroem 1.4.8 by a simple argument as follows.

Proof. Given $d > 0, m \geq k$, let n works for d, m, k in Theorem 1.4.8. Let c be a d-coloring of all k-spaces of D^n. Induce a coloring c' to all k-connections of D^n by letting, for (W, Q), a k connection of D^n,

$$c'((W, Q)) = c(W).$$

By the choice of n, there exists an m-connection (V, P) of D^n all of whose k-subconnections get the same color with respect to c'. By the choices of c' and (V, P), it suffices to show that for each k-space W of D^n with $W \subseteq V$, there exists $Q \subseteq [n]$ such that (W, Q) is a k-subconnection of (V, P).

Let W be a k-space of D^n with $W \subseteq V$. Let I, J be the minimal basises of V and W, respectively. Let $h : I \to P$ be the function such that $h(i)$ is an associate coordinate of i in V for $i \in I$. By Lemma 2.2.4, $J \subseteq I$, so J is a subset of the domain of h.

Claim 2.3.2. *For each $j \in J$, $h(j)$ is an assoicate coordinate of j in W.*

Proof. To show R.2 holds for $j, h(j)$ in W. Since R.2 holds for $j, h(j)$ in V and $W \subseteq V$, we see R.2 holds for $j, h(j)$ in W.

To show R.1 holds for $j, h(j)$ in W. Since R.1 holds for $j, h(j)$ in V, we have that $I \cap [h(j)] = I \cap [j]$ which implies that $J \cap I \cap [h(j)] = J \cap I \cap [j]$. Then by the fact $J \subseteq I$, we have that $J \cap [h(j)] = J \cap [j]$. So R.1 holds for $j, h(j)$ in W. □

Now let $Q = h(J)$, then (W, Q) is a k-subconnection of (V, P). □

2.4 The self-dual Ramsey theorem is a particular case of Theorem 1.4.8

Let D be a finite set and $F = \bigcup_{i=0}^{\infty} F_i$ where F_i is a family of functions $f \colon D^i \to D$.

Parameter systems. In Section 1.4.1, $(A1)$ says that F contains all constant functions. Let $(A1')$ be the condition as follows

$(A1')$ Constants: $\exists A \subseteq D$, such that for each m, we have $A = \{\mathrm{img}(f) : f \in F_m,$ a constant function$\}$ and for each m, $f \in F_m$, we have $f(A^m) \subseteq A$.

We still call F a parameter system on D if it satisfieds $(A1')$ and $(A2)-(A6)$. One can check that following almost the same proof, Theorem 1.4.8 still holds.

Note that if condition $(A1)$ holds, then condition $(A1')$ automatically holds. So, parameter systems defined here is a generalization of the definition of it in Introduction chapter.

Parameter system for partitions. Let D be a finite set with $|D| > 1$, and $F = \{f\colon D^n \to D \mid n \in \mathbb{N}, f \text{ a projection}\}$. It is straightforward to check that F is a parameter system on D.

We show that Theorem 1.4.3 is, in essence, the same as the self-dual Ramsey theorem for the above parameter system.

Proof. First, we explain a canonical $1-1$ correspondence between all spaces with respect to the above parameter system and all partitions as follows. Given $n \geq m$. Let V be an m-space of D^n with the minimal basis I and for each $i \in [n] \setminus I$, coordinate i depends on the coordinates $I \cap [i]$ through f_i in V, then

$$V = \{v \in D^n \mid v_i = f_i(v \restriction I \cap [i]), i \in [n] \setminus I\}.$$

Note that for each $i \in [n] \setminus I$, f_i is a one-dimensional projection, that means coordinate i agrees with a minimal basis coordinate $I(s)$ in V for some $s \in [m]$. So for each $s \in [m]$, we can define a set R_s as follows.

$$\{I(s)\} \cup \{i \in [n] \setminus I : \text{ coordinate } i \text{ agrees with coordinate } I(s) \text{ in } V\}.$$

By the minimality of I, we see $\mathcal{R} = \{R_1, \ldots, R_m\}$ is an m-partition of $[n]$ with $\min R_s < \min R_{s+1}$ for $s < m$.

Note that the above procedure is invertable, so the above correspondence is $1-1$. Then we see that Theorem 1.4.3 is, in essence, the same as Theorem 1.4.8 for the parameter system for partitions.

□

2.5 A natural extension of Theorem 1.4.8 is false

There is a natural way to extend the concept of associate coordinate as follows. However the corresponding Ramsey theorem does not hold.

Let F be a parameter system on D, and V be an m-space of D^n with the minimal basis I.

Definition 2.5.1 (Relaxed associate coordinate)**.** *For a minimal basis coordinate i, and an arbitrary co-*

ordinate $j \in [n]$. *We say that coordinate* j *is a relaxed associate coordinate of* i *if the following conditions hold.*

1. $I \cap [j] = I \cap [i]$;
2. $\exists f \in F_1$, *a non constant function, such that for each* $v \in V$, $v(j) = f(v(i))$.

Definition 2.5.2 (Relaxed connection). *Let* $P \subseteq [n]$ *be such that* $P(s)$ *is a relaxed associate coordinate of* $I(s)$ *for each* $s \in [m]$, *then we call* (V, P) *a relaxed* m*-connection of* D^n.

If (W, Q) is a relaxed k-connection of D^n and $W \subseteq V$, $Q \subseteq P$, then we say (W, Q) is a *relaxed* k*-subconnectoin* of (V, P). Based on the above definitions, we can generalize the self-dual Ramsey theorem for parameter systems to the below statement.
Given $d > 0, m \geq k$, *there exists* n *such that for each* d*-coloring of all relaxed* k*-connections of* D^n, *there exists a relaxed* m*-connection of* D^n *all of whose relaxed* k*-subconnections get the same color.*

Now we assume D and F are as in Example 1.4.1. Then we have the following theorem.

Theorem 2.5.3. *The above statement does not hold with respect to the parameter system for affine spaces.*

Proof. Let $d = |D|$, $m = 2, k = 1$ and D be the set of colors. Suppose the statement holds. Let n works for d, m, k in the above statement.

Let us construct a specific coloring c to all relaxed 1-connections of D^n. Let (V', P') be a relaxed 1-connection of D^n with $P' = \{p_1\}$, and $b(V') = \{i_1\}$. Then since F consists of all affine linear functions and p_1 is a relaxed assoicate coordinate of i_1, there exist $a, b \in D, b \neq 0$ such that for each $v \in V'$,

$$v(p_1) = a + b \cdot v(i_1).$$

We let $c((V', P')) = b$.

Claim 2.5.4. *There is no relaxed* 2*-connection of* D^n *all of whose relaxed* 1*-subconnections get the same color.*

Proof. Suppose not, let (V, P) be a desired one with $P = \{p_1, p_2\}$ and $b(V) = \{i_1, i_2\}$ such that coordinate p_2 depends on the coordinate $\{i_2\}$ through some function $a + b \cdot x$ where $a, b \in D$ and $b \neq 0$.

Let $c_1, c_2 \in D$ be such that $c_1 \neq c_2, c_1 \neq 0, c_2 \neq 0$. For $i = 1, 2$, let (W_i, Q_i) be a relaxed 1-connection of D^2 such that $b(W_i) = \{1\}$, $Q_i = \{2\}$ and coordinate 2 depends on the coordinate $\{1\}$ through function $c_i \cdot x$ in

W_i. Then for $i = 1, 2$, $(V, P) \circ (W_i, Q_i)$ is a relaxed 1-subconnection of (V, P) and $c((V, P) \circ (W_i, Q_i)) = b \cdot c_i$, contradicting with the choice of (V, P). □

□

Chapter 3

A Ramsey theorem for partial orders with linear extensions

In this Chapter, all orders are strict orders and we fix a natural number $p > 0$. Theorem 1.5.1 gives a common generalization of the following two of its known special cases.

The first one is the case $p = 1$, that is, the case when structures are equipped with a partial order and a single linear order extending it. This case was proved by Sokić [10, Theorem 7(6)] using results of Paoli, Trotter and Walker [9] and Fouché [8]. Because of certain peculiar features of Sokić's argument (for example, the usage of the ordering property to prove the Ramsey property), there has been some interest in finding a more direct proof. Our argument for Theorem 1.5.1 specialized to the case $p = 1$ gives just such a short and direct proof.

The second case is the case of finite sets endowed only with p linear orders. This situation corresponds to $P^X = P^Y = \emptyset$ (when one can obviously make $P^Z = \emptyset$) in Theorem 1.5.1. It was proved by Sokić in [11, Theorem 10]. Our proof here also specializes to an argument different from the one in [11].

In our proofs, we use some ideas from [8] and [9]. We connect them with a special case of the main theorem from [4].

In Section 3.1, we prove a product Ramsey theorem that is the Ramsey theoretic core of Theorem 1.5.1. In Sections 3.2 and 3.3, we make explicit certain canonical structures and morphisms important to the proof. Once these structures are properly defined and their natural properties are established, the theorem quickly follows (Section 3.4). In Section 3.5, we make precise the relationship between the product Ramsey theorem and Theorem 1.5.1 using some notions from [1].

3.1 A product Ramsey theorem

In this section, we prove a consequence of two known Ramsey results, this is Proposition 3.1.1.

We adopt the notational convention that each natural number is equal to the set of its predecessors, that is,

$$m = \{i \colon i < m\}.$$

In particular, $0 = \emptyset$. The set m is considered to be linearly ordered with its natural order inherited from $\mathbb{N}$. For a set X and a natural number k,

$$\binom{X}{k}$$

is the family of all k element subsets of X. The set X can itself be a natural number m, and then $\binom{m}{k}$ is the family of all k element subsets of m.

We formulate all our results in terms of rigid surjections, rather than partitions, as this form fits the applications better; see Lemma 3.3.1 and the proof of Lemma 3.3.4(ii). Let A, B be two finite linearly ordered sets. A function $r \colon B \to A$ is a *rigid surjection* if it is a surjection and the images of initial segments of B are initial segments of A, in other words, if for all $a_1, a_2 \in A$, with a_1 preceding a_2 in A, we have that a_1 is first attained by r before a_2 is first attained by r. See Section 1.3 or [1] for information on the language of rigid surjections.

Recall that we have fixed a natural number $p > 0$. A sequence $\vec{a} = (a_0, \ldots, a_{p-1})$ of length p of elements of A is called *anchored* if a_0 is the smallest element of A.

We will be considering linearly ordered sets A and B with anchored sequences $\vec{a} = (a_0, \ldots, a_{p-1})$ in A and $\vec{b} = (b_0, \ldots, b_{p-1})$ in B. Let

$$\binom{B, \vec{b}}{A, \vec{a}}_{\mathrm{rs}}$$

be the set of all rigid surjections $r \colon B \to A$ such that $r(b_i) = a_i$ for all $i < p$. Note that having anchored sequences $\vec{a} = (a_0, a_1, \ldots, a_{p-1})$ and $\vec{b} = (b_0, b_1 \ldots, b_{p-1})$ is equivalent, in this context, to having arbitrary sequences $(a_1, \ldots, a_{p-1})$ and $(b_1 \ldots, b_{p-1})$ since r automatically maps the smallest element of A to the smallest element of B. However, in view of our applications in Section 3.3, it will be notationally convenient to keep the elements a_0 and b_0 in the sequences.

Let m be a natural number. Let $\vec{i} = (i_0, \ldots, i_{p-1})$ be an anchored sequence of elements of m. For finite subsets $S_0, \ldots, S_{m-1}, T_0, \ldots, T_{m-1}$ of $\mathbb{N}$ and $s \in \binom{m, \vec{i}}{A, \vec{a}}_{\mathrm{rs}}$ and $t \in \binom{m, \vec{i}}{B, \vec{b}}_{\mathrm{rs}}$, we write

$$(S_0, \ldots, S_{m-1}, s) \ll (T_0, \ldots, T_{m-1}, t)$$

if for each $i < m$, $S_i \subseteq T_i$ and there is $r \in \binom{B, \vec{b}}{A, \vec{a}}_{\mathrm{rs}}$ with $s = r \circ t$.

Proposition 3.1.1. *Assume we are given $d > 0$, finite linearly ordered sets A, B, anchored sequences $\vec{a}$ and $\vec{b}$ of length p of elements of A and B, respectively, and two natural numbers k, l. Then there exist natural numbers m, n and an anchored sequence $\vec{i}$ of length p of elements of m such that for each d-coloring of $\binom{n}{k}^m \times \binom{m,\vec{i}}{A,\vec{a}}_{\rm rs}$ there exists $(T_0, \ldots, T_{m-1}, t) \in \binom{n}{l}^m \times \binom{m,\vec{i}}{B,\vec{b}}_{\rm rs}$ such that*

$$\{(S_0, \ldots, S_{m-1}, s) \in \binom{n}{k}^m \times \binom{m,\vec{i}}{A,\vec{a}}_{\rm rs} : (S_0, \ldots, S_{m-1}, s) \ll (T_0, \ldots, T_{m-1}, t)\}$$

is monochromatic.

Proposition 3.1.1 is a quick consequence of two known Ramsey statements, which we now recall. The first statement is the product of the classical Ramsey theorem, see [7]. For $S_0, \ldots, S_{m-1}, T_0, \ldots, T_{m-1}$ finite subsets of $\mathbb{N}$, we write

$$(S_0, \ldots, S_{m-1}) \leq (T_0, \ldots, T_{m-1}),$$

if for each $i < m$, $S_i \subseteq T_i$.

Product Ramsey Theorem. *Given $d > 0$ and natural numbers k, l, m, there exists a natural number n such that for each d-coloring of $\binom{n}{k}^m$, there exists $(T_0, \ldots, T_{m-1}) \in \binom{n}{l}^m$ such that*

$$\{(S_0, \ldots, S_{m-1}) \in \binom{n}{k}^m : (S_0, \ldots, S_{m-1}) \leq (T_0, \ldots, T_{m-1})\}$$

is monochromatic.

The following result is a particular case of [4, Theorem 1]. (One considers [4, Theorem 1] for the language consisting of $p-1$ constants, that is, $p-1$ function symbols of arity 0.) The case $p = 1$ of this result is just the dual Ramsey theorem.

Dual Ramsey Theorem with Constants. *Assume we are given $d > 0$ and finite linearly ordered sets A, B with anchored sequences $\vec{a}$ and $\vec{b}$ of length p in A and B, respectively. Then there exist a natural number m and an anchored sequence $\vec{i}$ of length p of elements of m such that for each d-coloring of $\binom{m,\vec{i}}{A,\vec{a}}_{\rm rs}$ there exists $t \in \binom{m,\vec{i}}{B,\vec{b}}_{\rm rs}$ with*

$$\{s \circ t \colon s \in \binom{B,\vec{b}}{A,\vec{a}}_{\rm rs}\}$$

monochromatic.

Proof of Proposition 3.1.1. Choose $(m, \vec{i})$ in terms of d, p, $(A, \vec{a})$, $(B, \vec{b})$ so the Dual Ramsey Theorem with Constants holds for $(m, \vec{i})$. Let n be large in terms of A, d, k, l, m so the Product Ramsey Theorem with $d^{|\binom{m,\vec{i}}{A,\vec{a}}_{\mathrm{rs}}|}$ colors holds for n.

Let ϕ be a coloring with d colors of $\binom{n}{k}^m \times \binom{m,\vec{i}}{A,\vec{a}}_{\mathrm{rs}}$. Let ψ be a coloring with $d^{|\binom{m,\vec{i}}{A,\vec{a}}_{\mathrm{rs}}|}$ colors of $\binom{n}{k}^m$ such that for each $(S_0, \ldots, S_{m-1})$, $(S'_0, \ldots, S'_{m-1}) \in \binom{n}{k}^m$

$$\begin{aligned}&\psi(S_0, \ldots, S_{m-1}) = \psi(S'_0, \ldots, S'_{m-1}) \iff \\ &\forall s \in \binom{m, \vec{i}}{A, \vec{a}}_{\mathrm{rs}} \phi(S_0, \ldots, S_{m-1}, s) = \phi(S'_0, \ldots, S'_{m-1}, s).\end{aligned}$$

Then by the choice of n, there exists $(T_0, \ldots, T_{m-1}) \in \binom{n}{l}^m$, such that ψ is constant on

$$\{(S_0, \ldots, S_{m-1}) \in \binom{n}{k}^m : (S_0, \ldots, S_{m-1}) \leq (T_0, \ldots, T_{m-1})\},$$

which implies for $(S_0, \ldots, S_{m-1}) \in \binom{n}{k}^m$ with $(S_0, \ldots, S_{m-1}) \leq (T_0, \ldots, T_{m-1})$, the color $\phi(S_0, \ldots, S_{m-1}, s)$ only depends on s. Then by the choice of $(m, \vec{i})$, there exists $t \in \binom{m,\vec{i}}{B,\vec{b}}_{\mathrm{rs}}$ such that ϕ is constant on the set from the conclusion of the proposition. □

3.2 Linear orders and a twisted product Ramsey theorem

We will need a general definition. Let K be a linear order on a set X, as usual assumed to be a strict order, and let $x \in X$. Put

$$(K)_x = (\{y \in X : yKx\}, K \restriction \{y \in X : yKx\}). \tag{3.2.1}$$

Let L be a linear order on a finite set Y. By

$$\mathrm{lin}_L \tag{3.2.2}$$

we denote the set of all linear orders on Y, which we order as follows. Let $L_1, L_2 \in \mathrm{lin}_L$. We put L_1 *below* L_2 if there exist $x, y \in Y$ such that $(L_1)_x = (L_2)_y$ and xLy. (By $(L_1)_x = (L_2)_y$ here we mean the literal equality, not just an isomorphism.) In other words, let $|Y| = n$ and let $(x_i)_{i<n}$ and $(y_i)_{i<n}$ be enumerations of Y in the L_1- and L_2-increasing order, respectively. We put L_1 below L_2 if $(x_i)_{i<n}$ is smaller than $(y_i)_{i<n}$

in the lexicographic order with respect to L.

The proof of the following lemma is straightforward.

Lemma 3.2.1. lin_L *is linearly ordered by the above defined relation and* L *is its smallest element.*

Assume we are given a natural number m and $B \subseteq \mathrm{lin}_L$. Let $\vec{i}$, $\vec{b}$ be anchored sequences of length p of elements of m and B, respectively. For

$$\tau = (T_0, \ldots, T_{m-1}, t) \in \binom{\mathbb{N}}{|Y|}^m \times \binom{m, \vec{i}}{B, \vec{b}}_{\mathrm{rs}} \tag{3.2.3}$$

and $i < m$, let

$$\pi_i^\tau : (Y, t(i)) \to (T_i, <\restriction T_i) \tag{3.2.4}$$

be the unique isomorphism. Assume we are additionally given a linear order K on a finite set X, $A \subseteq \mathrm{lin}_K$, and an anchored sequence $\vec{a}$ of length p of elements of A. Let τ be as in (3.2.3) and let

$$\sigma = (S_0, \ldots, S_{m-1}, s) \in \binom{Y}{|X|}^m \times \binom{B, \vec{b}}{A, \vec{a}}_{\mathrm{rs}}.$$

Define

$$\tau \cdot \sigma = (\pi_0^\tau(S_0), \ldots, \pi_{m-1}^\tau(S_{m-1}),\ s \circ t) \in \binom{\mathbb{N}}{|X|}^m \times \binom{m, \vec{i}}{A, \vec{a}}_{\mathrm{rs}}. \tag{3.2.5}$$

If n is a natural number taken with the linear order $<\restriction \mathbb{N}$ inherited from $\mathbb{N}$, we let

$$\mathrm{lin}_n = \mathrm{lin}_{<\restriction n}.$$

Consider the situation when (X, K) is the natural number k with the natural order and (Y, L) is the natural number l with the natural order. Note that directly from (3.2.5), $\tau \cdot \sigma \ll \tau$, so the following result is an immediate consequence of Proposition 3.1.1.

Assume we are given $d > 0$*, and natural numbers* k, l*. Let* $A \subseteq \mathrm{lin}_k$ *and* $B \subseteq \mathrm{lin}_l$*, and let* $\vec{a}$*,* $\vec{b}$ *be anchored sequences of length* p *of elements of* A *and* B*, respectively. Then there exist natural numbers* m, n *and an anchored sequence* $\vec{i}$ *of length* p *of elements of* m *such that for each* d*-coloring of* $\binom{n}{k}^m \times \binom{m,\vec{i}}{A,\vec{a}}_{\mathrm{rs}}$ *there exists* $\tau_0 \in \binom{n}{l}^m \times \binom{m,\vec{i}}{B,\vec{b}}_{\mathrm{rs}}$ *such that*

$$\{\tau_0 \cdot \sigma \colon \sigma \in \binom{l}{k}^m \times \binom{B, \vec{b}}{A, \vec{a}}_{\mathrm{rs}} \}$$

is monochromatic.

Since arbitrary finite linear orders (X, K) and (Y, L) can be identified with k and l, respectively, the result above can be restatement as Proposition 3.2.2 below.

Proposition 3.2.2. *Assume we are given $d > 0$, and linear orders K, L on finite sets X and Y, respectively. Let $A \subseteq \mathrm{lin}_K$ and $B \subseteq \mathrm{lin}_L$, and let $\vec{a}$, $\vec{b}$ be anchored sequences of length p of elements of A and B, respectively. Then there exist natural numbers m, n and an anchored sequence $\vec{i}$ of length p of elements of m such that for each d-coloring of $\binom{n}{|X|}^m \times \binom{m,\vec{i}}{A,\vec{a}}_{\mathrm{rs}}$ there exists $\tau_0 \in \binom{n}{|Y|}^m \times \binom{m,\vec{i}}{B,\vec{b}}_{\mathrm{rs}}$ such that*

$$\{\tau_0 \cdot \sigma \colon \sigma \in \binom{Y}{|X|}^m \times \binom{B,\vec{b}}{A,\vec{a}}_{\mathrm{rs}}\}$$

is monochromatic.

3.3 Certain canonical structures

In this section, P is a partial order on a finite set Y, and L is a linear order on Y extending P. Let

$$\mathrm{lin}_L(P) \subseteq \mathrm{lin}_L$$

be the set of all linear orders of Y extending P. The set $\mathrm{lin}_L(P)$ is equipped with the linear order inherited from lin_L. Let $X \subseteq Y$. Note that the linear order $L \restriction X$ extends the partial order $P \restriction X$. Define

$$\mathrm{res}_X \colon \mathrm{lin}_L(P) \to \mathrm{lin}_{L\restriction X}(P \restriction X),\ \mathrm{res}_X(L') = L' \restriction X.$$

Now, in addition to Y, P, and L, we fix linear orders $L_1, \ldots, L_{p-1}$ on Y that extend P, and let

$$\vec{L} = (L, L_1, \ldots, L_{p-1}).$$

By Lemma 3.2.1, $\vec{L}$ is an anchored sequence in $\mathrm{lin}_L(P)$. We set

$$\vec{L} \restriction X = (L \restriction X, L_1 \restriction X, \ldots, L_{p-1} \restriction X).$$

The following lemma is essentially [9, Lemma 14]. We include a proof of it for completeness.

Lemma 3.3.1. res_X *is an element of* $\binom{\mathrm{lin}_L(P),\vec{L}}{\mathrm{lin}_{L\restriction X}(P\restriction X),\vec{L}\restriction X}_{\mathrm{rs}}$.

Proof. By the definition of res_X, it suffices to show that res_X is a rigid surjection from $\mathrm{lin}_L(P)$ to $\mathrm{lin}_{L\restriction X}(P \restriction X)$. Recall (3.2.1).

Fix $L_1, L_2 \in \mathrm{lin}_L(P)$, $M_1 \in \mathrm{lin}_{L\restriction X}(P \restriction X)$, and $x, y \in Y$ with $(L_1)_x = (L_2)_y$. Assume that L_1 is the smallest element of $\mathrm{lin}_L(P)$ such that $L_1 \restriction X = M_1$.

Claim 3.3.2. *If $y \notin X$, then xLy or $x = y$.*

Proof. Towards a contradiction, assume that yLx. Define a linear order L_1' on Y by

(a) $L_1' \restriction (Y \setminus \{y\}) = L_1 \restriction (Y \setminus \{y\})$;

(b) y is the L_1'-immediate predecessor of x.

Note that L_1' extends P. Indeed, since L_1 extends P, condition (a) is compatible with P. Also we have

$$(L_1')_y = (L_1)_x = (L_2)_y.$$

So for $z \neq y$, if $z \in (L_1')_y$, then $z \in (L_2)_y$, and if $z \notin (L_1')_y$, then $z \notin (L_2)_y$, therefore, since L_2 extends P, condition (b) is compatible with P. Thus, $L_1' \in \mathrm{lin}_L(P)$. We have that L_1' is below L_1 in $\mathrm{lin}_L(P)$ since $(L_1')_y = (L_1)_x$ and yLx. Since $y \notin X$, $X \subseteq Y \setminus \{y\}$, so by (a)

$$L_1' \restriction X = L_1 \restriction X = M_1,$$

contradicting the choice of L_1 and proving the claim. □

Claim 3.3.3. *If $x \notin X$, xL_1y, and there is no $z \in X$ with xL_1zL_1y, then xLy.*

Proof. Note that by assumption $x \neq y$, so if the conclusion fails, then yLx. There are z_1, z_2 such that

(i) (xL_1z_1 or $x = z_1$) and z_1L_1y;

(ii) z_1 is an L_1-immediate predecessor of z_2;

(iii) z_2Lz_1.

To get such z_1 and z_2, let $x = v_0, v_1, \ldots, v_k = y$ be such that v_i is the L_1-immediate predecessor of v_{i+1} for $i < k$. If for each $i < k$, $v_i L v_{i+1}$, then we would have xLy contradicting yLx. So for some $i < k$, $v_{i+1} L v_i$, and we take $z_2 = v_{i+1}$ and $z_1 = v_i$.

Note that, by (i) and by our assumptions, $z_1 \notin X$.

Define a linear order L_1' on Y by

(a) $L_1' \restriction (Y \setminus \{z_1\}) = L_1 \restriction (Y \setminus \{z_1\})$;

(b) z_2 is the L_1'-immediate predecessor of z_1.

The linear order L_1' extends P. Indeed, since L_1 extends P, condition (a) is compatible with P; by (ii) and (iii), condition (b) is compatible with P as L and L_1 extend P. So $L_1' \in \mathrm{lin}_L(P)$. Since $(L_1')_{z_2} = (L_1)_{z_1}$ and $z_2 L z_1$, L_1' is below L_1. Since $z_1 \notin X$, we get $L_1' \restriction X = L_1 \restriction X = M_1$ contradicting our choice of L_1 and proving the claim. □

Now assume that $x \neq y$. Let $M_2 = L_2 \restriction X$ and assume that M_1 is below M_2 in $\mathrm{lin}_{L\restriction X}(P)$. We need to show that L_1 is below L_2 in $\mathrm{lin}_L(P)$.

If $y \notin X$, by Claim 3.3.2, we have xLy, so L_1 is below L_2, as required.

So assume $y \in X$. If $x \in X$, then $(M_1)_x = (M_2)_y$ and $x \neq y$. So xLy by our assumption that M_1 is below M_2. Thus, L_1 is below L_2 as required.

So assume that $y \in X$ and $x \notin X$. Let $y' \in X$ be such that xL_1y' and $z \notin X$ for all xL_1zL_1y'. Such a y' exists since xL_1y (as $(L_1)_x = (L_2)_y$ and $x \neq y$) and $y \in X$. By Claim 3.3.3, xLy'. If yLx, then yLy'. Note that $(M_1)_{y'} = (M_2)_y$ since $(L_1)_x = (L_2)_y$. So we have that M_2 is below M_1, contradiction. Thus, xLy and L_1 is below L_2, as required. □

The set $\mathbb{N}$ is equipped with its natural linear order, which we denote by $<$. Let m be a natural number. We define a partial order $<_{\mathrm{pr}}$ on $\mathbb{N}^m$ by letting

$$(k_0, \ldots, k_{m-1}) <_{\mathrm{pr}} (l_0, \ldots, l_{m-1})$$

if and only if $k_i < l_i$ for each $0 \leq i < m$. For $i < m$, let $<_{\mathrm{lx},i}$ be the linear order in $\mathbb{N}^m$ defined by letting

$$(k_0, \ldots, k_{m-1}) <_{\mathrm{lx},i} (l_0, \ldots, l_{m-1})$$

if and only if there exists $j \geq 0$ such that $k_{i+_m j} < l_{i+_m j}$ and $k_{i+_m j'} = l_{i+_m j'}$ for all $0 \leq j' < j$, where $+_m$ stands for addition modulo m. In particular, $<_{\mathrm{lx},0}$ is the usual lexicographic order. Note that each $<_{\mathrm{lx},i}$ extends $<_{\mathrm{pr}}$.

Fix an anchored sequence

$$\vec{i} = (i_0, \dots, i_{p-1}).$$

of elements of m. Let

$$\vec{<}_{\mathrm{lx},\vec{i}} = (<_{\mathrm{lx},i_0}, \dots, <_{\mathrm{lx},i_{p-1}}).$$

Then

$$(\mathbb{N}^m, <_{\mathrm{pr}}, \vec{<}_{\mathrm{lx},\vec{i}})$$

is a structure.

Let $\tau \in \binom{\mathbb{N}}{|Y|}^m \times \binom{m,\vec{i}}{\mathrm{lin}_L(P),\vec{L}}_{\mathrm{rs}}$. Recall (3.2.4) and define

$$\pi^\tau : Y \to \mathbb{N}^m,\ \pi^\tau(y) = (\pi^\tau_0(y), \dots, \pi^\tau_{m-1}(y)). \tag{3.3.1}$$

Lemma 3.3.4. *(i) π^τ is an embedding from $(Y, P, \vec{L})$ to $(\mathbb{N}^m, <_{\mathrm{pr}}, \vec{<}_{\mathrm{lx},\vec{i}})$.*

(ii) Let $X' \subseteq \pi^\tau(Y)$. Then, for $X = (\pi^\tau)^{-1}(X')$, we have

$$X' = \pi^{\tau\cdot\sigma}(X),$$

for some $\sigma \in \binom{Y}{|X|}^m \times \binom{\mathrm{lin}_L(P),\vec{L}}{\mathrm{lin}_{L\restriction X}(P\restriction X),\vec{L}\restriction X}_{\mathrm{rs}}$.

Proof. (i) Since each partial order is the intersection of all the linear orders containing it, we have that, for $y_1, y_2 \in Y$,

$$y_1 P y_2 \iff y_1 t(i) y_2 \text{ for all } i < m.$$

It follows that π^τ preserves P. Since

$$t(0) = L,\, t(i_1) = L_1, \dots,\, t(i_{p-1}) = L_{p-1}$$

we see that π^τ preserves each linear order in $\vec{L}$.

(ii) Let

$$\sigma = ((\pi_0^\tau)^{-1}(p_0(X')), \ldots, (\pi_{m-1}^\tau)^{-1}(p_{m-1}(X')), \mathrm{res}_X)$$

where p_i, $i < m$, is the i-th projection from $\mathbb{N}^m$ to $\mathbb{N}$. By Lemma 3.3.1, we have $\sigma \in \binom{Y}{|X|}^m \times \binom{\mathrm{lin}_L(P), \vec{L}}{\mathrm{lin}_{L\restriction X}(P\restriction X), \vec{L}\restriction X}_{\mathrm{rs}}$. The remainder of the conclusion, follows from the observation, made by a direct computation, that for $i < m$

$$\pi_i^{\tau\cdot\sigma} = \pi_i^\tau \restriction X. \qquad \square$$

3.4 Proof of Theorem 1.5.1

Let $\mathcal{X} = (X, P^X, \vec{L}^X)$ and $\mathcal{Y} = (Y, P^Y, \vec{L}^Y)$ be given. We assume, as we can, that $\mathcal{X}$ is a substructure of $\mathcal{Y}$. Fix the number of colors d. Set $K = L_0^X$, $L = L_0^Y$, $A = \mathrm{lin}_{L_0^X}(P^X)$, $B = \mathrm{lin}_{L_0^Y}(P^Y)$, $\vec{a} = \vec{L}^X$ and $\vec{b} = \vec{L}^Y$. Apply Proposition 3.2.2 to this data obtaining m, n and $\vec{i}$. We claim that the structure

$$(n^m, <_{\mathrm{pr}}\restriction n^m, \vec{<}_{\mathrm{lx},\vec{i}} \restriction n^m)$$

does the job. Color with d colors all substructures of this structure isomorphic to $(X, P^X, \vec{L}^X)$. By Lemma 3.3.4(i), this induces a coloring of all $\sigma \in \binom{n}{|X|}^m \times \binom{m,\vec{i}}{A,\vec{a}}_{\mathrm{rs}}$ by coloring σ with the color of the structure $\pi^\sigma(X)$. By our choice of m, n, and $\vec{i}$, there exists $\tau_0 \in \binom{n}{|Y|}^m \times \binom{m,\vec{i}}{B,\vec{b}}_{\mathrm{rs}}$ such that all $\tau_0 \cdot \sigma$, with $\sigma \in \binom{Y}{|X|}^m \times \binom{B,\vec{b}}{A,\vec{a}}_{\mathrm{rs}}$, get the same color. Consider the structure

$$\pi^{\tau_0}(Y) \subseteq n^m.$$

By Lemma 3.3.4(i), it is isomorphic to $(Y, P^Y, \vec{L}^Y)$. By Lemma 3.3.4(ii), each substructure of $\pi^{\tau_0}(Y)$ that is isomorphic to $(X, P^X, \vec{L}^X)$ is of the form $\pi^{\tau_0\cdot\sigma}(X)$ for $\sigma \in \binom{Y}{|X|}^m \times \binom{B,\vec{b}}{A,\vec{a}}_{\mathrm{rs}}$. So all of them have the same color.

3.5 On the relationship between Propositions 3.1.1 and 3.2.2 and Theorem 1.5.1

The arguments in Sections 3.3 and 3.4 show that Theorem 1.5.1 is, in a sense, a translation of Proposition 3.2.2, which, in turn, is just a particular case of Proposition 3.1.1. In the present section, we make the notion of translation precise using a variation of the concept of interpretation from [1]. As argued in [1], many particular Ramsey statements are instances of a general Ramsey statement formulated for certain algebraic structures. Interpretation is a precise notion of "homomorphism" that allows one to transfer the Ramsey statement from one such algebraic structure to another. We explain details of this setup below. Further, we define such algebraic structures for the statements in Proposition 3.2.2 and Theorem 1.5.1 and show that the first one interprets the second one. So Propositions 3.1.1 and 3.2.2 are the Ramsey theoretic essence of the main result Theorem 1.5.1.

Consider a set A with a partial function from $A \times A$ to A: $(a, b) \to a \cdot b$. Let $\mathcal{F}$ and $\mathcal{R}$ be families of subsets of A. Let $(F, R) \to F \bullet R$ be a function whose domain is a subset of $\mathcal{F} \times \mathcal{R}$, whose values are subsets of A, and which is such that whenever $F \bullet R$ is defined, then $f \cdot r$ is defined for all $f \in F$ and $r \in R$ and $F \bullet R = \{f \cdot r\colon f \in F, r \in R\}$. We say that $(\mathcal{F}, \mathcal{R}, \bullet)$ is a *pair of families over* $(A, \cdot)$.

Let $(\mathcal{F}, \mathcal{R}, \bullet)$ and $(\mathcal{G}, \mathcal{S}, \bullet)$ be pairs of families over $(A, \cdot)$ and $(B, \cdot)$, respectively. We say that $S \in \mathcal{S}$ is *interpretable in* $(\mathcal{F}, \mathcal{R})$ if there exists $R \in \mathcal{R}$ and a function $\alpha : S \to R$ such that if $F \bullet R$ is defined for $F \in \mathcal{F}$, then there exists $G \in \mathcal{G}$ with $G \bullet S$ defined, and a function $\phi : F \to G$ such that for $f_1, f_2 \in F$ and $s_1, s_2 \in S$,

$$f_1 \cdot \alpha(s_1) = f_2 \cdot \alpha(s_2) \Longrightarrow \phi(f_1) \cdot s_1 = \phi(f_2) \cdot s_2. \tag{3.5.1}$$

Now, we formulate the Ramsey condition for a pair of families. Let $(\mathcal{F}, \mathcal{R}, \bullet)$ be a pair of families and let $d > 0$. We say the *d-Ramsey condition* holds for $(\mathcal{F}, \mathcal{R}, \bullet)$ if for each $R \in \mathcal{R}$, there exists $F \in \mathcal{F}$ such that for each d-coloring of $F \bullet R$, there exists $f \in F$ with $\{f \cdot r : r \in R\}$ is monochromatic.

The following proposition can be checked without difficulty.

Proposition 3.5.1. *Let $(\mathcal{F}, \mathcal{R}, \bullet)$ and $(\mathcal{G}, \mathcal{S}, \bullet)$ be pairs of families, and let $d > 0$. If the d-Ramsey condition holds for $(\mathcal{F}, \mathcal{R}, \bullet)$ and each $S \in \mathcal{S}$ is interpretable in $(\mathcal{F}, \mathcal{R}, \bullet)$, then the d-Ramsey condition holds for $(\mathcal{G}, \mathcal{S}, \bullet)$.*

From now on, we fix d, the number of colors.

A pair of families for Proposition 3.2.2. Let A_1 consist of all τ belonging to $\binom{\mathbb{N}}{l}^m \times \binom{C,\vec{c}}{B,\vec{b}}_{\rm rs}$ for some natural numbers l and m, $B \subseteq \mathrm{lin}_l$, a linearly ordered set C, and anchored sequences $\vec{b}$ and $\vec{c}$ of length p of elements of B and C, respectively. If $\sigma, \tau \in A_1$, then $\tau \cdot \sigma$ is defined precisely when $\sigma \in \binom{l}{k}^m \times \binom{B,\vec{b}}{A,\vec{a}}_{\rm rs}$ and $\tau \in \binom{\mathbb{N}}{l}^m \times \binom{m,\vec{i}}{B,\vec{b}}_{\rm rs}$, and $\tau \cdot \sigma$ is defined by formula (3.2.5).

Let $\mathcal{F}_1$ consist of all sets of the form $F = \binom{n}{l}^q \times \binom{q,\vec{i}}{C,\vec{c}}_{\rm rs}$ for some natural numbers $l \le n$ and q, $C \subseteq \mathrm{lin}_l$, and anchored sequences $\vec{i}$ and $\vec{c}$ of length p of elements of q and C, respectively. Let $\mathcal{S}_1^d$ consist of all sets of the form $S = \binom{r}{k}^m \times \binom{B,\vec{b}}{A,\vec{a}}_{\rm rs}$ for some natural numbers $k \le r$, m, $A \subseteq \mathrm{lin}_k$ and $B \subseteq \mathrm{lin}_r$ and anchored sequences $\vec{a}$, $\vec{b}$ of length p of elements of A and B, respectively, where m is large enough so that the Dual Ramsey Theorem with Constants, as stated in Section 3.1, holds with d colors for m, $(A,\vec{a})$ and $(B,\vec{b})$. For $F \in \mathcal{F}_1$ and $S \in \mathcal{S}_1^d$ as above, $F \bullet S$ is defined when $r = l$, $q = m$, and $(B,\vec{b}) = (C,\vec{c})$, and is then equal to $\{\tau \cdot \sigma \colon \tau \in F, \sigma \in S\}$.

Following the proof of Proposition 3.2.2 one gets Proposition 3.5.2 below.

Proposition 3.5.2. *The d-Ramsey condition holds for $(\mathcal{F}_1, \mathcal{S}_1^d, \bullet)$.*

A pair of families for Theorem 1.5.1. Let A_2 consist of all embeddings between structures of the form $(X, P^X, \vec{L}^X)$ as in Section **??**. For $f, g \in A_2$, $f \cdot g$ is defined precisely when the range structure of f is equal to the domain structure of g and then we let $f \cdot g = f \circ g$.

Let $\mathcal{F}_2$ consist of all sets $F = \binom{n^m, <_{\rm pr}, <_{\rm lx,\vec{i}}}{Y, P^Y, \vec{L}^Y}$, and let $\mathcal{S}_2$ consist of all $S = \binom{Z, P^Z, \vec{L}^Z}{X, P^X, \vec{L}^X}$. For $F \in \mathcal{F}_2$ and $S \in \mathcal{S}_2$ as above, $F \bullet S$ is defined precisely when $(Y, P^Y, \vec{L}^Y) = (Z, P^Z, \vec{L}^Z)$ and is then equal to $\{f \cdot g \colon f \in F, g \in S\}$.

Proposition 3.5.3. *For each $d > 0$, each $S \in \mathcal{S}_2$ is interpretable in $(\mathcal{F}_1, \mathcal{S}_1^d, \bullet)$.*

Proof. Let $S \in \mathcal{S}_2$ be $\binom{Y, P^Y, \vec{L}^Y}{X, P^X, \vec{L}^X}$. Set $k = |X|$ and $l = |Y|$. Observe that we can assume $(X, P^X, \vec{L}^X) = (k, P^k, \vec{L}^k)$ and $(Y, P^Y, \vec{L}^Y) = (l, P^l, \vec{L}^l)$ where P^k, P^l are partial orders on k, l and $\vec{L}^k, \vec{L}^l$ are sequences of linear orders of length p extending P^k, P^l with $L_0^k = <\restriction k$, $L_0^l = <\restriction l$, respectively. Fix m such that $R = \binom{l}{k}^m \times \binom{\mathrm{lin}_{L_0^l}(P^l), \vec{L}^l}{\mathrm{lin}_{L_0^k}(P^k), \vec{L}^k}_{\rm rs}$ is in $\mathcal{S}_1^d$. Define $\alpha : S \to R$ by letting, for $s \in S$,

$$\alpha(s) = (s[k], \ldots, s[k], r \circ \mathrm{res}_{s[k]}) \tag{3.5.2}$$

where $r : \mathrm{lin}_{L_0^l \restriction s[k]}(P^l \restriction s[k]) \to \mathrm{lin}_{L_0^k}(P^k)$ is the unique isomorphism. By Lemma 3.3.1, $\alpha(s) \in R$.

Assume $F \bullet R$ is defined. Then $F = \binom{n}{l}^m \times \binom{m,\vec{i}}{\mathrm{lin}_{L_0^l}(P^l), \vec{L}^l}_{\rm rs}$ for some n and an anchored sequence $\vec{i}$ of length p of elements of m. Let $G = \binom{n^m, <_{\mathrm{pr}}, <_{\mathrm{lx}, \vec{\imath}}}{l, P^l, \vec{L}^l}$. So $G \bullet S$ is defined. Define $\phi \colon F \to G$ by $\phi(\tau) = \pi^\tau$, where π^τ is as in (3.3.1). Note that by Lemma 3.3.4(i), $\pi^\tau \in G$.

If $\tau = (T_0, \ldots, T_{m-1}, t) \in F$ and $s \in S$, then by (3.2.5) and (3.5.2) we have

$$\tau \cdot \alpha(s) = ((\pi_0^\tau \circ s)[k], \ldots, (\pi_{m-1}^\tau \circ s)[k], r \circ \mathrm{res}_{s[k]} \circ t).$$

Now, one checks, using (3.3.1), that $\pi^{\tau \cdot \alpha(s)} = \phi(\tau) \cdot s$, which implies (3.5.1), as required. $\square$

By Propositions 3.5.1, 3.5.2 and 3.5.3, the d-Ramsey condition holds for $(\mathcal{F}_2, \mathcal{S}_2, \bullet)$, so Theorem 1.5.1 follows.

Chapter 4

Some Ramsey theorems, the notions of retraction and interpretation

4.1 Some Ramsey theorems

In this section, first, we present three Ramsey sets between IS and RS; second, we include some other Ramsey theorems having points in common with the classical Ramsey theorem and the dual Ramsey theorem. For $k \leq m$ and a set A of function with domain and range in $\mathbb{N}$, let $E^A_{m,k} = \{s \in A \mid s : [m'] \to [k], k \leq m' \leq m\}$.

4.1.1 The Ramsey theorem for A_1

Normed composition space (A_1, X_1)**.** Recall A_1 and let $X_1 = A_1$. We equip (A_1, X_1) with the canonical composition for RS as multiplication and action; the canonical truncation for RS as truncation; the canonical norm for RS as norm. It is straightforward to check that (A_1, X_1) with the operations defined above is a normed composition space.

Ramsey domain $(\mathcal{F}_1, \mathcal{S}_1)$ **over** (A_1, X_1)**.** Both $\mathcal{F}_1$ and $\mathcal{S}_1$ consist of all sets of the form $H^{A_1}_{m,k}, E^{A_1}_{m,k}$.

The operations $\bullet$ and ${}_\bullet$ are defined only in the following situations: precisely when $m = l$,

(1) $H^{A_1}_{n,m} \bullet H^{A_1}_{l,k} = H^{A_1}_{n,k}$, $E^{A_1}_{n,m} \bullet E^{A_1}_{l,k} = E^{A_1}_{n,k}$;

(2) $H^{A_1}_{n,m} {}_\bullet H^{A_1}_{l,k} = H^{A_1}_{n,k}$, $E^{A_1}_{n,m} {}_\bullet E^{A_1}_{l,k} = E^{A_1}_{n,k}$;

It is straightforward to check that $(\mathcal{F}_1, \mathcal{S}_1)$ with the operations defined above is a Ramsey domain over (A_1, X_1).

Theorem 4.1.1 ([7], Finite union theorem)**.** *Given $d > 0, k$, there exists n such that, for each d-coloring of the family of all subsets of $[n]$, there exist k pairwise disjoint subsets $D_1, \ldots, D_k$ of $[n]$ with $\max D_i < \min D_{i+1}$ for $i < k$, such that the family of all non empty finite unions from $\{D_1, \ldots, D_k\}$ is monochromatic.*

We translate the above theorem into the following form.

Corollary 4.1.2. *Given $d > 0, k$, there exists n such that, for each d-coloring of the space 2^n, there exists k pairwise disjoint subsets $D_1, \dots, D_k$ of $[n]$ with $\max D_i < \min D_{i+1}$ for $i < k$, such that the subspace consisting of elements $v \in 2^n$ with the following properties is monochromatic,*

(1) there exists $i \leq n$ such that $v(i) \neq 0$;

(2) for each $i \leq n$, if $i \notin \cup_{j=1}^{k} D_j$, then $v(i) = 0$;

(3) for each $j \leq k$, v is constant on D_j.

Let $M(d, k)$ denote the smallest such n.

Note that the below theorem implies that A_1 is a Ramsey set.

Theorem 4.1.3. *$(\mathcal{F}_1, \mathcal{S}_1)$ satisfies the Ramsey condition.*

Proof. Note that $(\mathcal{F}_1, \mathcal{S}_1)$ is a Ramsey domain over (A_1, X_1) and it is obvious that $(\mathcal{F}_1, \mathcal{S}_1)$ is linear and vanishing. Then by Theorem 1.2.1, it remains to check the pigeonhole principle. Given $d > 0$. There are two cases to deal with.

Case 1: Assume $P = H^{A_1}_{m,k}$ and $x \in \partial P$ with $|x| = m_0$. Note $m_0 < m$, let $m' = m - m_0$. Let $a = id \upharpoonright [m_0]$, and $F = H^{A_1}_{n,m}$, where n is defined as follows:

$$n_1 = M(d, m'); \qquad n = m_0 + 1 + n_1.$$

Then $F \bullet P$ and $a \,.\, x$ are defined.

Now, we show that there is a bijection between $F_a \,.\, (P)_x$ and 2^{n_1}. Observe that the following two facts.

$$F_a = \{t \in F : t \upharpoonright [m_0 + 1] = id \upharpoonright [m_0 + 1]\} \tag{4.1.1}$$

$$(P)_x = \{s \in P : s \upharpoonright [m_0] = x, s(m_0 + 1) = k\}. \tag{4.1.2}$$

Let $x' : [m_0 + 1] \to [k]$ be such that $x' \upharpoonright [m_0] = x$ and $x'(m_0 + 1) = k$.

It follows by (4.1.1), (4.1.2), that

$$F_a \,.\, (P)_x = \{h : [n] \to [k] \mid h \upharpoonright [m_0 + 1] = x', h(m_0 + 1 + i) \in \{1, k\} \text{ for } i \leq n_1\}.$$

Define a function α from 2^{n_1} to $F_a \,.\, (P)_x$ by sending $v \in 2^{n_1}$ to $h \in F_a \,.\, (P)_x$ where $h \restriction [m_0+1] = x'$ and for $i \leq n_1$

$$h(m_0+1+i) = \begin{cases} k, & \text{if } v(i) = 1; \\ 1, & \text{if } v(i) = 0. \end{cases} \tag{4.1.3}$$

It is obvious that α is a bijection.

Now, let c be a d-coloring of $F_a \,.\, (P)_x$. Induce a d-coloring c' of 2^{n_1} by letting, for $v \in 2^{n_1}$

$$c'(v) = c(\alpha(v)). \tag{4.1.4}$$

By Corollary 4.1.2 and the choice of n_1, there exist m' pairwise disjoint subsets $D_1, \dots, D_{m'}$ of $[n_1]$ such that

$$\max D_j < \min D_{j+1} \text{ for } j < m' \tag{4.1.5}$$

and the subspace generated by $(D_j)_{j \leq m'}$ as in Corollary 4.1.2 is monochromatic with respect to c'.

Construct $t : [n] \to [m]$ based on $(D_j)_{j \leq m'}$ such that $t \restriction [m_0+1] = id \restriction [m_0+1]$ and for $i \leq n_1$

$$t(m_0+1+i) = \begin{cases} m_0 + j, & \text{if } i \in D_j \text{ for some } j \leq m'; \\ 1, & \text{otherwise.} \end{cases} \tag{4.1.6}$$

By (4.1.5), we see $t \in F_a$. We claim that $t \,.\, (P)_x$ is monochromatic with respect to c. Indeed let $s \in (P)_x$, then by (4.1.6) and (4.1.2) we have

(1) for each $i \in D_1$, $(t \,.\, s)(m_0+1+i) = k$;

(2) for each $i \leq n_1$, if $i \notin \cup_{j=1}^{m'} D_j$ then $(t \,.\, s)(m_0+1+i) = 1$;

(3) for each $j \leq m'$, $t \,.\, s$ is constant on $\{m_0+1+i : i \in D_j\}$.

It follows by (4.1.6) and (4.1.3), that for $v = \alpha^{-1}(t \,.\, s)$, we have

(1) for each $i \in D_1$, $v(i) = 1$;

(2) for each $i \notin \cup_{j=1}^{m'} D_j$, $v(i) = 0$;

(3) for each $j \leq m'$, v is constant on D_j.

So v is in the subspace generated by $(D_j)_{j\leq m'}$ as in Corollary 4.1.2. Then by the choices of c' and $(D_j)_{j\leq m'}$, we have $t\,.\,(P)_x$ is monochromatic with respect to c.

Case 2: Assume $P = E^{A_1}_{m,k}$ and $x \in \partial P$ with $|x| = m_0$. Let $a = id \restriction [m_0]$, then $a\,.\,x$ is defined. It suffices to show the following statement.

(A) There exists n, such that for each d-coloring of $E^{A_1}_{n,k}$, there exists $t \in (E^{A_1}_{n,m})_a$, that is $t \restriction [m_0+1] = id \restriction [m_0+1]$, such that $t\,.\,(P)_x$ is monochromatic.

We prove the following statements (B) and (C) first, then derive (A) from (C). Fix $m_1 > m_0$.

(B) There exists n, such that for each d-coloring of $H^{A_1}_{n,k}$, there exists $t \in H^{A_1}_{n,m_1}$ with $t \restriction [m_0+1] = id \restriction [m_0+1]$ such that $t\,.\,(H^{A_1}_{m_1,k})_x$ is monochromatic. This follows from Case 1.

(C) There exists n, such that for each d-coloring of $E^{A_1}_{n,k}$, there exists $t \in H^{A_1}_{n,m_1}$ with $t \restriction [m_0+1] = id \restriction [m_0+1]$ such that $t\,.\,(H^{A_1}_{m',k})_x$ is monochromatic for $m_0 < m' \leq m_1$. We prove this by induction.

Base: $m_1 = m_0 + 1$. In this case, $(H^{A_1}_{m_1,k})_x$ contains only one element, so the statement is trivially true.

Induction: move from m_1 to $m_1 + 1$. By induction, obtain n_1 that works for m_1 in (C). Apply (B) to $n_1 + 1$ (play the role of m_1) to get n.

Claim 4.1.4. *n works for $m_1 + 1$ in (C).*

Proof. Let c be a d-coloring of $E^{A_1}_{n,k}$. By the choice of n, there exists $t_1 \in H^{A_1}_{n,n_1+1}$ with $t_1 \restriction [m_0+1] = id \restriction [m_0+1]$ such that $t_1\,.\,(H^{A_1}_{n_1+1,k})_x$ is monochromatic with respect to c.

Induce a coloring c' of $E^{A_1}_{n_1,k}$ by letting, for $s \in E^{A_1}_{n_1,k}$,

$$c'(s) = c(t_1\,.\,s).$$

By the choice of n_1, there exists $t_2 \in H^{A_1}_{n_1,m_1}$ with $t_2 \restriction [m_0+1] = id \restriction [m_0+1]$ such that $t_2\,.\,(H^{A_1}_{m',k})_x$ is monochromatic for $m_0 < m' \leq m_1$ with respect to c'.

Let $t_3\colon [n_1+1] \to [m_1+1]$ be $t_3 \restriction [n_1] = t_2$ and $t_3(n_1+1) = m_1+1$. Now we show $t_1 \cdot t_3$ works for $m_1 + 1$ in (C). Note that $(t_1 \cdot t_3) \restriction [m_0+1] = id \restriction [m_0+1]$ and $t_1 \cdot t_3 \in H^{A_1}_{n,m_1+1}$.

If $m' = m_1 + 1$, then

$$\begin{aligned}(t_1 \cdot t_3) \,.\, (H^{A_1}_{m_1+1,k})_x &= (t_1 \cdot t_3) \,.\, \{s \in H^{A_1}_{m_1+1,k} | \partial s = x\} \\ &= t_1 \,.\, \{t_3 \,.\, s | s \in H^{A_1}_{m_1+1,k}, \partial s = x\} \subseteq t_1 \,.\, \{s \in H^{A_1}_{n_1+1,k} \mid \partial s = x\} \\ &= t_1 \,.\, (H^{A_1}_{n_1+1,k})_x.\end{aligned} \tag{4.1.7}$$

The $\subseteq$ holds because $\partial(t_3 \,.\, s) = x$ which is due to $t_3 \restriction [m_0 + 1] = id \restriction [m_0 + 1]$ and $\partial s = x$. Then by (4.1.7) and the choice of t_1, we have that $(t_1 \cdot t_3) \,.\, (H^{A_1}_{m_1+1,k})_x$ is monochromatic with respect to c.

If $m_0 < m' \leq m_1$, then we have the following equality.

$$(t_1 \cdot t_3) \,.\, (H^{A_1}_{m',k})_x = (t_1 \cdot t_2) \,.\, (H^{A_1}_{m',k})_x. \tag{4.1.8}$$

By the choices of c' and t_2, we have that $t_1 \,.\, (t_2 \,.\, (H^{A_1}_{m',k})_x)$ is monochromatic with respect to c which implies, by (4.1.8), that $(t_1 \cdot t_3) \,.\, (H^{A_1}_{m',k})_x$ is monochromatic with respect to c. □

Now let us prove (A). Note that $m > m_0$. Pick m_1 so that for each d-coloring of (m_0, m_1), there is a $(m - m_0)$-subset getting the same color. Apply (C) to m_1 to get n.

Claim 4.1.5. *n works for m in (A).*

Proof. Let c be a d-coloring of $E^{A_1}_{n,k}$. By the choice of n, there exists $t_1 \in H^{A_1}_{n,m_1}$ such that $t_1 \,.\, (H^{A_1}_{m',k})_x$ is monochromatic for $m_0 < m' \leq m_1$.

By the choice of m_1, there exist

$$m_0 < i_1 < \cdots < i_{m-m_0} < m_1$$

such that

$$t_1 \,.\, \cup_{j=1}^{m-m_0} (H^{A_1}_{i_j,k})_x \text{ is monochromatic.} \tag{4.1.9}$$

Let $t_2 : [m_1] \to [m + 1] \in \mathrm{IS}$ be such that $t_2 \restriction [m_0 + 1] = id \restriction [m_0 + 1]$ and

$$i_j + 1 = \min t_2^{-1}(m_0 + 1 + j) \text{ for } j \leq m - m_0.$$

By the choice of t_2, we have

$$\partial t_2 \,.\, (H^{A_1}_{m_0+j,k})_x \subseteq (H^{A_1}_{i_j,k})_x \text{ for } j \leq m - m_0. \tag{4.1.10}$$

We also have

$$(E^{A_1}_{m,k})_x = \cup_{j=1}^{m-m_0} (H^{A_1}_{m_0+j,k})_x$$

which implies by (4.1.10) and (4.1.9) that $(t_1 \cdot \partial t_2) \,.\, (E^{A_1}_{m,k})_x$ is monochromatic. Note that $t_1 \cdot \partial t_2 \in (E^{A_1}_{n,m})_a$, so (A) holds. □

□

4.1.2 The Ramsey theorem for A_2

Normed composition space (A_2, X_2)**.** Recall A_2 and let $X_2 = A_2$. We equip (A_2, X_2) with the canonical composition for RS as multiplication and action; the canonical truncation for RS as truncation; the canonical norm for RS as norm. It is straightforward to check that (A_2, X_2) with the operations defined above is a normed composition space.

Ramsey domain $(\mathcal{F}_2, \mathcal{S}_2)$ **over** (A_2, X_2)**.** Both $\mathcal{F}_2$ and $\mathcal{S}_2$ consist of all sets of the form $H^{A_2}_{m,k}, E^{A_2}_{m,k}$. The operations $\bullet$ and ${}_\bullet$ are defined only in the following situations: precisely when $m = l$,

(1) $H^{A_2}_{n,m} \bullet H^{A_2}_{l,k} = H^{A_2}_{n,k},\ E^{A_2}_{n,m} \bullet E^{A_2}_{l,k} = E^{A_2}_{n,k}$;

(2) $H^{A_2}_{n,m} {}_\bullet H^{A_2}_{l,k} = H^{A_2}_{n,k},\ E^{A_2}_{n,m} {}_\bullet E^{A_2}_{l,k} = E^{A_2}_{n,k}$;

It is straightforward to check that $(\mathcal{F}_2, \mathcal{S}_2)$ with the operations defined above is a Ramsey domain over (A_2, X_2).

Note that the below theorem implies that A_2 is a Ramsey set. And we will prove it in Section 4.3(see Remark 4.3.2 after Theorem 4.3.8).

Theorem 4.1.6. $(\mathcal{F}_2, \mathcal{S}_2)$ *satisfies the Ramsey condition.*

4.1.3 The Ramsey theorem for A_3

Normed composition space (A_3, X_3)**.** Recall A_3 and let $X_3 = A_3$. We equip (A_3, X_3) with the canonical composition for RS as multiplication and action; the canonical truncation for RS as truncation; the canonical norm for RS as norm. It is straightforward to check that (A_3, X_3) with the operations defined above is a normed composition space.

Ramsey domain $(\mathcal{F}_3, \mathcal{S}_3)$ **over** (A_3, X_3)**.** Put

$$\mathcal{F}_3 = \mathcal{S}_3 = \{P \mid P \subseteq H^{A_3}_{m,k}, \text{ or } P \subseteq E^{A_3}_{m,k}, k \leq m\}.$$

For $P \in \mathcal{S}$, let $r(P)$ denote the range of functions in P; and let

$$d(P) = \max\{\mathrm{dom(s)} \mid \mathrm{s} \in \mathrm{P}\}.$$

We say P has *common domain* if all functions in P have the same domain.

For $F, P \in \mathcal{S}_3$, declare $F {}_\bullet P$, $F \bullet P$ to be defined precisely when the following conditions hold.

1. $r(F) = d(P)$;

2. F has common domain $\Leftrightarrow$ P has common domain.

And we let $F {}_\bullet P = F \,.\, P$, and $F \bullet P = F \cdot P$.

Lemma 4.1.7. $(\mathcal{F}_3, \mathcal{S}_3)$ *with the operations defined above is a Ramsey domain over* (A_3, X_3)*.*

Proof. It is obvious that $\bullet$ and ${}_\bullet$ are defined pointwise and property (B) holds. To prove property (C). Assume $F {}_\bullet \partial P$ is defined. Let

$$n = d(F), \;\; m = d(P), \;\; m' = d(\partial P).$$

Note that $m' \leq m$. Let $G = H^{A_3}_{n+m-m',m}$ for the case P has common domain; otherwise, let $G = E^{A_3}_{n+m-m',m}$. So $G {}_\bullet P$ is defined. By the fact that $F {}_\bullet \partial P$ is defined, we have $r(F) = m'$ which implies that $F \subseteq E^{A_3}_{n,m'}$. Then we see that for each $f \in F$, there exists $g \in G$ extending it.

To prove property (A). Observe that for $F \in \mathcal{F}_3$, $P \in \mathcal{S}_3$, if $F {}_\bullet P$ or $F \bullet P$ is defined, then both $F {}_\bullet P$ and $F \bullet P$ are defined and

1. $d(F) = d(F {}_\bullet P) = d(F \bullet P)$;

2. Both F and P have common domains $\Leftrightarrow$ $F {}_\bullet P$ has common domain $\Leftrightarrow$ $F \bullet P$ has common domain.

By the above observation, one can check that property (A) holds. $\square$

Note that the below theorem implies that A_3 is a Ramsey set. And we will prove it in Section 4.3(see Remark 4.3.3 after Theorem 4.3.9).

Theorem 4.1.8. $(\mathcal{F}_3, \mathcal{S}_3)$ *satisfies the Ramsey condition.*

4.1.4 The spiral Ramsey theorem

In this section, we fix $l \in \mathbb{N}$

Normed composition space (A_4, X_4). Let $(s_1, \ldots, s_l) : [k] \to [m]$ denote a sequence of increasing injections of length l from $[k]$ to $[m]$, we say it satisfies the *spiral property* if the following conditions hold:

(1) for each $i \leq k$, $j < l$, $s_j(i) \leq s_{j+1}(i)$;

(2) for each $i < k$, $s_l(i) < s_1(i+1)$.

Let A_4, X_4 both consist of sequences of increasing injections $(s_1, \ldots, s_l) : [k] \to [m]$ satisfying the spiral property. We say (A_4, X_4) has *dimension l*.

Define the multiplication on A_4 by

$$(t_1, \ldots, t_l) \cdot (s_1, \ldots, s_l) = (t_1 \circ s_1, \ldots, t_l \circ s_l),$$

for $(t_1, \ldots, t_l) : [m] \to [n] \in A_4$, $(s_1, \ldots, s_l) : [k] \to [m'] \in A_4$ with $s_l(k) \leq m$, and the action of A_4 on X_4 is the same as the multiplication.

Define $\partial : X_4 \to X_4$ and $|\cdot| : X_4 \to \mathbb{N}$ as follows. For $(s_1, \ldots, s_l) : [k] \to [m] \in X_4$, let

$$\partial(s_1, \ldots, s_l) = (s_1 \upharpoonright [k-1], \ldots, s_l \upharpoonright [k-1]),$$

and

$$|(s_1, \ldots, s_l)| = s_l(k).$$

It is straightforwad to check that (A_4, X_4) with the operations defined above is a normed composition space.

Ramsey domain $(\mathcal{F}_4, \mathcal{S}_4)$ **over** (A_4, X_4)**.** As it will not cause confuse, we use $H^{A_4}_{k,m}$ to denote all sequences of increasing injections of length l from $[k]$ to $[m]$ satisfying the spiral property. Both $\mathcal{F}_4$ and $\mathcal{S}_4$

consist of all sets of the form $H^{A_4}_{k,m}$. The operations $\bullet$ and $\bullet$ are defined only in the following situations:

$$H^{A_4}_{m,n} \bullet H^{A_4}_{k,m} = H^{A_4}_{k,n}, H^{A_4}_{m,n} \bullet H^{A_4}_{k,m} = H^{A_4}_{k,n}.$$

It is straightforward to check that $(\mathcal{F}_4, \mathcal{S}_4)$ with the operations defined above is a Ramsey domain over (A_4, X_4).

We will prove the below theorem in Section 4.3(see Remark 4.3.4 after Theorem 4.3.17).

Theorem 4.1.9. $(\mathcal{F}_4, \mathcal{S}_4)$ *satisfies the Ramsey condition.*

4.1.5 The classical Ramsey theorem

In [1], the classical Ramsey theorem is represented in the form of increasing injections, and increasing surjections, respectively. We include the corresponding two Ramsey domains here.

First, let us introduce the one for increasing injections.

Normed composition space (A_5, X_5)**.** Let $A_5 = X_5 = \text{II}$. We equip (A_5, X_5) with the canonical composition for II as multiplication and action; the canonical truncation for II as truncation; the canonical norm for II as norm. It is straightforward to check that (A_5, X_5) is a normed composition space.

Ramsey domain $(\mathcal{F}_5, \mathcal{S}_5)$ **over** (A_5, X_5)**.** Both $\mathcal{F}_5$ and $\mathcal{S}_5$ consist of all sets of the form $H^{A_5}_{k,m}$. The operations $\bullet$ and $\bullet$ are defined only in the following situations: precisely when $m = l$,

$$H^{A_5}_{m,n} \bullet H^{A_5}_{k,l} = H^{A_5}_{k,n}, H^{A_5}_{m,n} \bullet H^{A_5}_{k,l} = H^{A_5}_{k,n}.$$

It is straightforward to check that $(\mathcal{F}_5, \mathcal{S}_5)$ with the operations defined above is a Ramsey domain over (A_5, X_5).

Theorem 4.1.10 ([1])**.** $(\mathcal{F}_5, \mathcal{S}_5)$ *satisfies the Ramsey condition.*

Second, let us introduce the one for increasing surjections.

Normed composition space (A_6, X_6)**.** Let $A_6 = X_6 = \text{IS}$. We equip (A_6, X_6) with the canonical composition for RS as multiplication and action; the canonical truncation for RS as truncation; the canonical norm for RS as norm. It is straightforward to check that (A_6, X_6) is a normed composition space.

Ramsey domain $(\mathcal{F}_6, \mathcal{S}_6)$ **over** (A_6, X_6)**.** Both $\mathcal{F}_6$ and $\mathcal{S}_6$ consist of all sets of the form $H^{A_6}_{m,k}, E^{A_6}_{m,k}$.

The operations $\bullet$ and $_\bullet$ are defined only in the following situations: precisely when $m = l$,

$$H^{A_6}_{n,m} \bullet H^{A_6}_{l,k} = H^{A_6}_{n,k}, \quad E^{A_6}_{n,m} \bullet E^{A_6}_{l,k} = E^{A_6}_{n,k};$$
$$H^{A_6}_{n,m} {}_\bullet H^{A_6}_{l,k} = H^{A_6}_{n,k}, \quad E^{A_6}_{n,m} {}_\bullet E^{A_6}_{l,k} = E^{A_6}_{n,k}.$$

It is straightforward to check that $(\mathcal{F}_6, \mathcal{S}_6)$ with the operations defined above is a Ramsey domain over (A_6, X_6).

Theorem 4.1.11 ([1])**.** *$(\mathcal{F}_6, \mathcal{S}_6)$ satisfies the Ramsey condition.*

4.1.6 The dual Ramsey theorem

In [1], the dual Ramsey theorem is represented in the form of surjection and proved by the abstract approach. We include the corresponding Ramsey domain here.

Normed composition space (A_7, X_7)**.** Let $A_7 = X_7 = \mathrm{RS}$. We equip (A_7, X_7) with the canonical composition for RS as multiplication and action; the canonical truncation for RS as truncation; the canonical norm for RS as norm. It is straightforward to check that (A_7, X_7) is a normed composition space.

Ramsey domain $(\mathcal{F}_7, \mathcal{S}_7)$ **over** (A_7, X_7)**.** Both $\mathcal{F}_7$ and $\mathcal{S}_7$ consist of all sets of the form $H^{A_7}_{m,k}, E^{A_7}_{m,k}$. The operations $\bullet$ and $_\bullet$ are defined only in the following situations: precisely when $m = l$,

$$H^{A_7}_{n,m} \bullet H^{A_7}_{l,k} = H^{A_7}_{n,k}, \quad E^{A_7}_{n,m} \bullet E^{A_7}_{l,k} = E^{A_7}_{n,k};$$
$$H^{A_7}_{n,m} {}_\bullet H^{A_7}_{l,k} = H^{A_7}_{n,k}, \quad E^{A_7}_{n,m} {}_\bullet E^{A_7}_{l,k} = E^{A_7}_{n,k}.$$

It is straightforward to check that $(\mathcal{F}_7, \mathcal{S}_7)$ with the operations defined above is a Ramsey domain over (A_7, X_7).

Theorem 4.1.12 ([1])**.** *$(\mathcal{F}_7, \mathcal{S}_7)$ satisfies the Ramsey condition.*

4.1.7 The standard pigeonhole principle

In this section, we state the standard pigeonhole principle in terms of set actoid.

Actoid (A_8, X_8)**.** Let $A_8 = X_8 = \mathrm{II}$. We equip (A_8, X_8) with the canonical composition for II as multiplication and action. It is easy to see that (A_8, X_8) is an actoid.

Set actoid $(\mathcal{F}_8, \mathcal{S}_8)$ **over** (A_8, X_8) Put

$$\mathcal{F}_8 = \{H^{A_8}_{k,m} \mid k \leq m\};\ \mathcal{S}_8 = \{H^{A_8}_{1,k} \mid k \geq 1\}.$$

The operations $\bullet$ and $\bullet$ are defined only in the following situations: precisely when $m = l$,

$$H^{A_8}_{m,n} \bullet H^{A_8}_{k,l} = H^{A_8}_{k,n}, H^{A_8}_{m,n} \bullet H^{A_8}_{1,l} = H^{A_8}_{1,n}.$$

It is straightforward to check that $(\mathcal{F}_8, \mathcal{S}_8)$ is a set actoid over (A_8, X_8).

Theorem 4.1.13 (Standard pigeonhole principle)**.** $(\mathcal{F}_8, \mathcal{S}_8)$ *satisfies the Ramsey condition.*

4.2 Retraction

In the previous section, we present three Ramsey sets. In this section, we show there is no proper retraction between IS and RS. Note that dom and codom stand for domain and codomain, respectively.

Lemma 4.2.1. *Let $m \leq n$, IS $\subseteq A \subseteq$ RS, and A is a retraction of RS through ϕ, then for each $f \in H^{RS}_{n,m}$, $\phi(f) \in H^{A}_{n,m}$.*

Proof. Let $f \in H^{RS}_{n,m}$, $g \in H^{RS}_{m,1}$, $h \in H^{RS}_{n,n}$.

First, we show $\mathrm{codom}(\phi(\mathrm{f})) \geq \mathrm{codom}(\mathrm{f}), \mathrm{dom}(\phi(\mathrm{f})) \geq \mathrm{dom}(\mathrm{f})$. Note

$$g, f \,.\, g \in \mathrm{IS} \subseteq A.$$

It follows that $f \,.\, g = \phi(f) \,.\, g$. Then $\phi(f) \,.\, g$ is defined which implies

$$\mathrm{codom}(\phi(\mathrm{f})) \geq \mathrm{dom}(\mathrm{g}) = \mathrm{m} = \mathrm{codom}(\mathrm{f}),$$

and

$$\mathrm{dom}(\phi(\mathrm{f}) \,.\, \mathrm{g}) = \mathrm{dom}(\mathrm{f} \,.\, \mathrm{g}) = \mathrm{n} = \mathrm{dom}(\mathrm{f}),$$

which implies $\mathrm{dom}(\phi(\mathrm{f})) \geq \mathrm{dom}(\mathrm{f})$.

Second, we show $\mathrm{dom}(\phi(\mathrm{f})) \leq \mathrm{dom}(\mathrm{f})$, $\mathrm{codom}(\phi(\mathrm{f})) \leq \mathrm{codom}(\mathrm{f})$. Note $h \,.\, f$ is defined, so is $\phi(h) \,.\, \phi(f) =$

$h \,.\, \phi(f)$ which implies

$$\mathrm{dom}(\phi(\mathrm{f})) \leq \mathrm{codom}(\mathrm{h}) = \mathrm{n} = \mathrm{dom}(\mathrm{f}).$$

Now, we have $\mathrm{dom}(\phi(\mathrm{f})) = \mathrm{dom}(\mathrm{f})$. It remains to show $\mathrm{codom}(\phi(\mathrm{f})) \leq \mathrm{codom}(\mathrm{f})$. Suppose $\mathrm{codom}(\phi(\mathrm{f})) >$ $\mathrm{codom}(\mathrm{f})$, note

$$\mathrm{codom}(\mathrm{f}) = \mathrm{m} = \mathrm{dom}(\mathrm{g}),$$

so $\mathrm{codom}(\phi(\mathrm{f})) > \mathrm{dom}(\mathrm{g})$. It follows that

$$\mathrm{dom}(\phi(\mathrm{f}) \,.\, \mathrm{g}) < \mathrm{dom}(\phi(\mathrm{f})) = \mathrm{dom}(\mathrm{f}) = \mathrm{n} = \mathrm{dom}(\mathrm{f} \,.\, \mathrm{g}) = \mathrm{dom}(\phi(\mathrm{f}) \,.\, \mathrm{g}).$$

The last equality is due to the fact $\phi(f) \,.\, g = f \,.\, g$, thus a contradiction, so $\mathrm{codom}(\phi(\mathrm{f})) \leq \mathrm{codom}(\mathrm{f})$. □

Theorem 4.2.2. *Let IS $\subseteq A \subseteq$ RS, if A is a retraction of RS through ϕ, then either $A = IS$ or $A = RS$. Moreover, if $A = IS$, then ϕ should be π defined in Section 1.6.2.*

Proof. Let $u : [3] \to [2]$ be $u(1) = u(3) = 1, u(2) = 2$. We will show that if $u \in A$, then A must be equal to RS, otherwise A must be equal to IS and $\phi = \pi$.

Since each $f \in$ RS can be expressed as a composition of a sequence of functions in RS with the difference of domain and codomain as 1, and $\phi(g \,.\, h) = \phi(g) \,.\, \phi(h)$, $\pi(g \,.\, h) = \pi(g) \,.\, \pi(h)$, for each $g, h \in$ RS, we only need to treat the following two cases.

First, if $u \in A$, then we show for each n, each $f \in H^{RS}_{n+1,n}$, $\phi(f) = f$; second, if $u \notin A$, then we show for each n, each $f \in H^{RS}_{n+1,n}$, $\phi(f) = \pi(f)$. Note that, for each $f \in H^{IS}_{n+1,n}$, $\phi(f) = \pi(f) = f$, we only need to deal with $f \in H^{RS}_{n+1,n} \setminus H^{IS}_{n+1,n}$.

Let $n \geq 1$. For each $f \in H^{RS}_{n+2,n+1} \setminus H^{IS}_{n+2,n+1}$, there exists a unique pair $x < y \in [n+2]$ such that $f(x) = f(y)$. Then based on the difference of x and y and the property of rigid surjection, f should be in one of the following forms. In the first two forms the difference between x and y is 2; in the third form, the difference between x and y is greater than 2.

$$f_1(i) = \begin{cases} i, & \text{if } 1 \leq i \leq l+1; \\ l, & \text{if } i = l+2; \\ i-1, & \text{if } l+2 < i \leq n+2. \end{cases}$$

$$f_2(i) = \begin{cases} i, & \text{if } 1 \le i \le 2; \\ 1, & \text{if } i = 3; \\ i-1, & \text{if } 3 < i \le n+2. \end{cases}$$

$$f_3(i) = \begin{cases} i, & \text{if } 1 \le i \le m+1; \\ k, & \text{if } i = m+2; \\ i-1, & \text{if } m+2 < i \le n+2. \end{cases}$$

where $1 < l \le n, 1 \le k < m \le n$.

We also need a sequence of auxiliary functions from $[n+1]$ to $[n]$ and a sequence of auxiliary functions from $[n+2]$ to $[n+1]$ as follows which will be used in the following proof.

$$g_1(i) = \begin{cases} i, & \text{if } 1 \le i \le l-1; \\ i-1, & \text{if } l \le i \le n+1. \end{cases}$$

$$g_2(i) = \begin{cases} i, & \text{if } 1 \le i \le l; \\ l-1, & \text{if } i = l+1; \\ i-1, & \text{if } l+1 < i \le n+1. \end{cases}$$

$$g_3(i) = \begin{cases} i, & \text{if } 1 \le i \le l; \\ i-1, & \text{if } l < i \le n+1. \end{cases}$$

$$g_4(i) = \begin{cases} i, & \text{if } 1 \le i \le 2; \\ i-1, & \text{if } 2 < i \le n+1. \end{cases}$$

$$g_5(i) = \begin{cases} i, & \text{if } 1 \leq i \leq 2; \\ 1, & \text{if } i = 3; \\ i-1, & \text{if } 3 < i \leq n+1. \end{cases}$$

$$g_6(i) = \begin{cases} i, & \text{if } 1 \leq i \leq m; \\ i-1, & \text{if } m < i \leq n+1. \end{cases}$$

$$g_7(i) = \begin{cases} 1, & \text{if } 1 \leq i \leq 2; \\ i-1, & \text{if } 2 < i \leq n+1. \end{cases}$$

$$g_8(i) = \begin{cases} i, & \text{if } 1 \leq i \leq m; \\ k, & \text{if } i = m+1; \\ i-1, & \text{if } m+1 < i \leq n+1. \end{cases}$$

$$g_9(i) = \begin{cases} i, & \text{if } 1 \leq i \leq m-1; \\ k, & \text{if } i = m; \\ i-1, & \text{if } m < i \leq n+1. \end{cases}$$

$$h_1(i) = \begin{cases} i, & \text{if } 1 \leq i \leq l-1; \\ i-1, & \text{if } l \leq i \leq n+2. \end{cases}$$

$$h_2(i) = \begin{cases} i, & \text{if } 1 \leq i \leq l+1; \\ l-1, & \text{if } i = l+2; \\ i-1, & \text{if } l+2 < i \leq n+2. \end{cases}$$

$$h_3(i) = \begin{cases} i, & \text{if } 1 \leq i \leq 3; \\ 2, & \text{if } i = 4; \\ i-1, & \text{if } 4 < i \leq n+2. \end{cases}$$

$$h_4(i) = \begin{cases} i, & \text{if } 1 \leq i \leq m; \\ i-1, & \text{if } m < i \leq n+2. \end{cases}$$

$$h_5(i) = \begin{cases} i, & \text{if } 1 \leq i \leq 3; \\ i-1, & \text{if } 3 < i \leq n+2. \end{cases}$$

$$h_6(i) = \begin{cases} i, & \text{if } 1 \leq i \leq 2; \\ i-1, & \text{if } 3 \leq i \leq n+2. \end{cases}$$

$$h_7(i) = \begin{cases} i, & \text{if } 1 \leq i \leq 3; \\ 2, & \text{if } i = 4; \\ i-1, & \text{if } 4 < i \leq n+2. \end{cases}$$

$$h_8(i) = \begin{cases} i, & \text{if } 1 \leq i \leq m+1; \\ m, & \text{if } i = m+2; \\ i-1, & \text{if } m+2 < i \leq n+2. \end{cases}$$

We prove it by induction on n. Note that, by Lemma 4.2.1, we have

$$\text{for each } f \in H^{\mathrm{RS}}_{n+2,n+1}, \phi(f) \in H^{\mathrm{RS}}_{n+2,n+1}. \tag{4.2.1}$$

First, let us assume $u \in A$. If $n = 1$, note that u is the only element in $H^{RS}_{3,2} \setminus H^{IS}_{3,2}$, then we only need to show $\phi(u) = u$ which is true because $u \in A$.

Let $n \geq 2$, move from $n-1$ to n. Let $f \in H^{RS}_{n+2,n+1} \setminus H^{IS}_{n+2,n+1}$, there are three cases.

Case 1: $f = f_1$. Observe that $f_1 \,.\, g_1 = h_1 \,.\, g_2$, and $h_1 \in \text{IS}$, then by induction and retraction property, $\phi(f_1) \,.\, g_1 = h_1 \,.\, g_2$ which implies, by (4.2.1), that $\phi(f_1) \in \{f_1, h_2\}$. It suffices to show $\phi(f_1) \neq h_2$. Suppose not, then $\phi(f_1) = h_2$, note that $g_3, f_1 \,.\, g_3 \in \text{IS}$, then by retraction propery, $f_1 \,.\, g_3 = h_2 \,.\, g_3$ which does not hold, thus a contradiction.

Case 2: $f = f_2$. Observe that $f_2 \,.\, g_4 = h_3 \,.\, g_5$. Note that h_3 is in the first form. Then by Case 1, $\phi(h_3) = h_3$, then by induction and retraction property, $\phi(f_2) \,.\, g_4 = h_3 \,.\, g_5$. Then by (4.2.1), we see $\phi(f_2)$ has to be f_2.

Case 3: $f = f_3$. Observe that $f_3 \,.\, g_6 = h_4 \,.\, g_8, h_4 \in \text{IS}$, then by induction and retraction property, $\phi(f_3) \,.\, g_6 = h_4 \,.\, g_8$. Then by (4.2.1), we see $\phi(f_3)$ has to be f_3.

Second, assume $u \notin A$. If $n = 1$, note that u is the only element in $H^{RS}_{3,2} \setminus H^{IS}_{3,2}$, we only need to show $\phi(u) = \pi(u)$. By Lemma 4.2.1, $\phi(u) \in H^{A}_{3,2}$, so if $\phi(u) \neq \pi(u)$, then it has to be g where $g : [3] \to [2]$ with $g(1) = g(2) = 1, g(3) = 2$.

Let $g' = \pi(u)$, and $h : [4] \to [3]$ be

$$h(i) = \begin{cases} i, & \text{if } 1 \leq i \leq 2; \\ i - 1, & \text{if } 2 < i \leq 4, \end{cases}$$

$h' : [4] \to [3]$ be

$$h'(i) = \begin{cases} i, & \text{if } 1 \leq i \leq 3; \\ 1, & \text{if } i = 4. \end{cases}$$

Observe that $h \,.\, u = h' \,.\, g'$ which implies $\phi(h) \,.\, \phi(u) = \phi(h') \,.\, \phi(g')$. Since $\phi(h) = h, \phi(u) = g$, we have

$$(\phi(h) \,.\, \phi(u))^{-1}(2) = \{4\}.$$

Since $\phi(g') = g'$, we have $\phi(g')^{-1}(2) = \{2, 3\}$ which implies that

$$(\phi(h') \,.\, \phi(g'))^{-1}(\{2\})$$

includes at least two elements, thus a contradiction.

Let $n \geq 2$, move from $n - 1$ to n. Let $f \in H^{RS}_{n+2,n+1} \setminus H^{IS}_{n+2,n+1}$, there are three cases.

Case 1: $f = f_1$. Observe that $f_1 . g_1 = h_1 . g_2$, and $h_1, g_1 \in$ IS, then by induction and retraction property, $\phi(f_1) . g_1 = h_1 . \pi(g_2)$. Then by (4.2.1), we see $\phi(f_1)$ has to be $\pi(f_1)$.

Case 2: $f = f_2$. Observe that $f_2 . g_5 = h_5 . g_5$, and $h_5 \in$ IS, then by induction and retraction property, $\phi(f_2) . \pi(g_5) = h_5 . \pi(g_5)$ which implies, by (4.2.1), that $\phi(f_2) \in \{h_5, h_6, h_7\}$. Note that h_7 is in the first form, then by Case 1, $\phi(h_7) = \pi(h_7)$ which is not equal to h_7. Then by retraction property, we have $h_7 \notin A$. So $\phi(f_2)$ could not be h_7. Note that $\pi(f_2) = h_6$, so it suffices to show $\phi(f_2) \neq h_5$. Suppose not, then $\phi(f_2) = h_5$. Observe that $f_2 . g_7 = h_6 . g_7$, and $g_7, h_6 \in$ IS. Then by retraction property, $h_5 . g_7 = h_6 . g_7$ which does not hold, thus a contradiction.

Case 3: $f = f_3$. Observe that $f_3 . g_9 = h_8 . g_9$. Note that h_8 is in the first form, then by Case 1, $\phi(h_8) = \pi(h_8)$. By induction and retraction property, we have $\phi(f_3) . \pi(g_9) = \pi(h_8) . \pi(g_9)$. Then by (4.2.1), we see $\phi(f_3)$ has to be $\pi(f_3)$. □

4.3 Interpretation

Recall the definitions of ∂-interpretation and relaxed interpretation. In this section, we study the relations between Ramsey theorems stated in Section 4.1 from the perspective of interpretation. The following theorem can be checked without difficulty.

Theorem 4.3.1. *Let $(\mathcal{F}, \mathcal{R})$ and $(\mathcal{G}, \mathcal{S})$ be set actoids over actoids (A, X) and (B, Y), respectively. If $(\mathcal{G}, \mathcal{S})$ can be relaxed interpreted by $(\mathcal{F}, \mathcal{R})$ and $(\mathcal{F}, \mathcal{R})$ satisfies the Ramsey condition, then $(\mathcal{G}, \mathcal{S})$ also satisfies the Ramsey condition.*

By the above theorem, we can understand the interpretation from two different views.

(i) It provides a way to prove a Ramsey theorem from another Ramsey theorem. For example, we prove Theorem 4.1.6 and Theorem 4.1.8 from Theorem 4.1.12 by applying Theorem 4.3.1.

(ii) It provides a precise way to describe the relations between Ramsey theorems. For two Ramsey theorems T_1, T_2 with set actoids $(\mathcal{F}, \mathcal{R})$ and $(\mathcal{G}, \mathcal{S})$ respectively, if $(\mathcal{G}, \mathcal{S})$ can be relaxed interpreted by $(\mathcal{F}, \mathcal{R})$, then by Theorem 4.3.1, we see T_2 is a simple corollary of T_1, so T_2 has a close relation with T_1; in particular, if $(\mathcal{F}, \mathcal{R})$ and $(\mathcal{G}, \mathcal{S})$ can be relaxed interpreted by each other, then in essence, T_1 and T_2 are the same Ramey theorem; if $(\mathcal{G}, \mathcal{S})$ can not be relaxed interpreted by $(\mathcal{F}, \mathcal{R})$, then we see, in some sense, T_2 is a new Ramsey theorem with respect to T_1.

In order to check that a set actoid can be interpreted by another one, one follows the definitions.

To check that a set actoid can not be interpreted by another one, we develop Theorem 4.3.5 and Theorem 4.3.6.

Definition 4.3.2 (Left cancellation)**.** *Let (A, X) be an actoid. We say (A, X) satisfies left cancellation property, if for $f \in A, x_1, x_2 \in X$,*

$$f \,.\, x_1 = f \,.\, x_2 \Rightarrow x_1 = x_2.$$

Definition 4.3.3 (Left defined)**.** *Let $(\mathcal{F}, \mathcal{R})$ be a set actoid. We say $(\mathcal{F}, \mathcal{R})$ satisfies left defined property, if for each $R \in \mathcal{R}$, there exists $F \in \mathcal{F}$ such that $F \bullet R$ is defined.*

Lemma 4.3.4. *Let $(\mathcal{F}, \mathcal{R})$ and $(\mathcal{G}, \mathcal{S})$ be set actoids over actoids (A, X) and (B, Y), respectively, such that $(\mathcal{F}, \mathcal{R})$ satisfies the left defined property and (B, Y) satisfies the left cancellation property. For $S \in \mathcal{S}$ which is relaxed interpretable in $(\mathcal{F}, \mathcal{R})$, let R and $\alpha : S \to R$ be as in Definition 1.6.2. Then α is an injection.*

Proof. Suppose not, then there exist $y_1 \neq y_2 \in S$ with $\alpha(y_1) = \alpha(y_2)$. By the left defined property of $(\mathcal{F}, \mathcal{R})$, there exists $F \in \mathcal{F}$ with $F \bullet R$ defined. Then find $G \in \mathcal{G}$ with $G \bullet S$ defined and $\phi : F \to G$ for which (1.6.3) holds. Let $f \in F$, note that $f \,.\, \alpha(y_1) = f \,.\, \alpha(y_2)$. Then by (1.6.3), we have $\phi(f) \,.\, y_1 = \phi(f) \,.\, y_2$ which implies, by the left cancellation property of (B, Y) that $y_1 = y_2$. Thus, a contradiction. □

Let $r \in \mathbb{N}$. We say a directed graph H with vertices $\{v_1, \ldots, v_m\}$, edges $\{(p_1, q_1), \ldots, (p_n, q_n)\}$ is *based* on $[r]$, if

$$\{(p_1, q_1), \ldots, (p_n, q_n)\} \subseteq [r] \times [r].$$

We want to point out that (p_i, q_i) is the name of some edge and p_i, q_i are not vertices of H.

Let (A, X) be an actoid, $X' \subseteq X$ with $|X'| = r$ and $x_1 \ldots x_r$ be a permutation of X'. We call an object obtained from H by replacing each vertex v_i by an element $f_i \in A$, and each edge (p_j, q_j) by an ordered pair

$$(x_{p_j}, x_{q_j}) \in X' \times X',$$

an *instance* of H with respect to (A, X). Let $H(f_1 \ldots f_m, x_1 \ldots x_r)$ denote it. Furthermore, if for each edge (x_{p_j}, x_{q_j}) with starting vertex f, ending vertex f', we have $f \,.\, x_{p_j}$, $f' \,.\, x_{q_j}$ are defined and equal, then we say the instance *commutes*.

Theorem 4.3.5. *Let H be a directed graph with m vertices, n edges based on $[r]$. Let $(\mathcal{F}, \mathcal{R})$ and $(\mathcal{G}, \mathcal{S})$ be set atoids over actoids (A, X) and (B, Y), respectively, such that $(\mathcal{F}, \mathcal{R})$ satisfies the left defined property* *and (B, Y) satisfies the left cancellation property. Let $S \in \mathcal{S}$, then S is not relaxed interpretable in $(\mathcal{F}, \mathcal{R})$, if the following conditions hold:*

(i) for each $G \in \mathcal{G}$ with $G \bullet S$ defined, there exists $S' \subseteq S$ with $|S'| = r$, such that for each permutation $y_1 \dots y_r$ of S', there do not exist $g_1, \dots, g_m \in G$ such that

$$H(g_1 \dots g_m, y_1 \dots y_r) \text{ commutes;}$$

(ii) for each $R \in \mathcal{R}$, there exists $F \in \mathcal{F}$ with $F \bullet R$ defined, such that for each $R' \subseteq R$ with $|R'| = r$, there exist a permutation $x_1 \dots x_r$ of R' and $f_1, \dots, f_m \in F$ such that

$$H(f_1 \dots f_m, x_1 \dots x_r) \text{ commutes.}$$

Proof. Suppose S is relaxed interpretable in $(\mathcal{F}, \mathcal{R})$, find R and $\alpha : S \to R$ as in Definition 1.6.2. Since $(\mathcal{F}, \mathcal{R})$, (B, Y) satisfy the left defined property, the left cancellation property, respectively, by Lemma 4.3.4, α is an injection.

For R given above, find F as in condition (ii). Then Find $G \in \mathcal{G}$ with $G \bullet S$ defined and $\phi : F \to G$ for which (1.6.3) holds.

For G given above, find $S' \subseteq S$ as in condition (i) with $|S'| = r$. Let $R' = \alpha(S')$. Note that $|R'| = r$ because α is an injection.

For R' and F, find a permutation $x_1 \dots x_r$ of R' and $f_1, \dots, f_m \in F$ as in condition (ii). So $H(f_1 \dots f_m, x_1 \dots x_r)$ commutes which implies, by (1.6.3) that

$$H(\phi(f_1) \dots \phi(f_m), \alpha^{-1}(x_1) \dots \alpha^{-1}(x_r))$$

commutes. Contradicting with the choices of G and S', so S is not relaxed interpretable in $(\mathcal{F}, \mathcal{R})$. □

Theorem 4.3.6. *Let H be a directed graph with m vertices, n edges based on $[r]$. Let $(\mathcal{F}, \mathcal{R})$ and $(\mathcal{G}, \mathcal{S})$ be Ramsey domains over normed composition spaces (A, X) and (B, Y), respectively, such that $(\mathcal{F}, \mathcal{R})$ satisfies the left defined property and (B, Y) satisfies the left cancellation property. Let $S \in \mathcal{S}$, then S is not ∂-interpretable in $(\mathcal{F}, \mathcal{R})$, if the following conditions hold:*

(i) for each $G \in \mathcal{G}$ with $G \bullet S$ defined, there exists $S' \subseteq S$ with $|S'| = r$ and $|\partial S'| = 1$, such that for each permutation $y_1 \dots y_r$ of S', there do not exist $g_1, \dots, g_m \in G$ such that

$$H(g_1 \dots g_m, y_1 \dots y_r) \text{ commutes;}$$

(ii) for each $R \in \mathcal{R}$, there exists $F \in \mathcal{F}$ with $F \bullet R$ defined, such that for each $R' \subseteq R$ with $|R'| = r$ and $|\partial R'| = 1$, there exist a permutation $x_1 \dots x_r$ of R' and $f_1, \dots, f_m \in F$ such that

$$H(f_1 \dots f_m, x_1 \dots x_r) \text{ commutes.}$$

Remark 4.3.1. Theorems 4.3.5 and Theorem 4.3.6 are two parallel theorems. Theorem 4.3.5 is for relaxed interpretation and Theorem 4.3.6 is for ∂-interpretation. It is easy to see that Theorem 4.3.6 follows from amost the same proof of Theorem 4.3.5.

In the rest of this chapter, we apply the concept of interpretation and Theorem 4.3.5, Theorem 4.3.6 to explore relations between Ramsey theorems stated in Section 4.1.

Theorem 4.3.7. *$(\mathcal{F}_5, \mathcal{S}_5)$ and $(\mathcal{F}_6, \mathcal{S}_6)$ can be ∂-interpreted by each other.*

Proof. First, we show $(\mathcal{F}_5, \mathcal{S}_5)$ can be ∂-interpreted by $(\mathcal{F}_6, \mathcal{S}_6)$. For $k \leq m$, let $S = H^{A_5}_{k,m}$, take $R = E^{A_6}_{m,k}$. Define $\alpha : S \to R$ be as follows. For $y \in S$, let $\alpha(y) : [y(k)] \to [k]$ be an increasing surjection such that

$$\forall j \leq k, y(j) = \max \alpha(y)^{-1}(j). \tag{4.3.1}$$

We claim that R with α works for S. It is obvious that (1.6.1) holds. So it remains to check (1.6.2). Given $F = E^{A_6}_{n,m}$ with $F \bullet R$ defined, let $G = H^{A_5}_{m,n}$, then $G \bullet S$ is defined. Define $\phi : F \to G$ as follows. For $f \in F$, let $\phi(f) : [m] \to [n]$ be as follows.

$$\forall i \leq m, \phi(f)(i) = \max f^{-1}(i). \tag{4.3.2}$$

Let $f \in F, y \in S$, by (4.3.1) and (4.3.2), we have for $i \leq k$,

$$(\phi(f)\,.\,y)(i) = \phi(f)(y(i)) = \max f^{-1}(y(i))$$
$$= \max f^{-1}(\max \alpha(y)^{-1}(i)) = \max(f\,.\,\alpha(y))^{-1}(i).$$

The second equality is due to (4.3.2); the third equality is due to (4.3.1); the last equality holds because $f, \alpha(y)$ are non-decreasing. It follows that (1.6.2) holds.

Second, we show $(\mathcal{F}_6, \mathcal{S}_6)$ can be ∂-interpreted by $(\mathcal{F}_5, \mathcal{S}_5)$. There are two cases to deal with.

Case 1: $S = E^{A_6}_{m,k}$. Take $R = H^{A_5}_{k,m}$ and define $\alpha : S \to R$ as follows. For $y \in S$, let $\alpha(y) : [k] \to [m]$ be as follows.

$$\forall i \leq k, \alpha(y)(i) = \max y^{-1}(i). \tag{4.3.3}$$

We claim that R with α works for S. It is obvious that (1.6.1) holds. So it remains to check (1.6.2). Given $F = H^{A_5}_{m,n}$ with $F \bullet R$ defined, let $G = E^{A_6}_{n,m}$, then $G \bullet S$ is defined. Define $\phi : F \to G$ as follows. For $f \in F$, let $\phi(f) : [f(m)] \to [m]$ be an increasing surjection such that.

$$\forall j \leq m, f(j) = \max \phi(f)^{-1}(j). \tag{4.3.4}$$

Let $f \in F, y \in S$, by (4.3.3) and (4.3.4) we have, for $i \leq k$,

$$\max(\phi(f)\,.\,y)^{-1}(i) = \max \phi(f)^{-1}(\max y^{-1}(i))$$
$$= \max \phi(f)^{-1}(\alpha(y)(i)) = (f\,.\,\alpha(y))(i).$$

The first equality holds because $\phi(f)$ and y are non-deceasing; the second equality is due to (4.3.3); the third equality is due to (4.3.4). It follows that (1.6.2) holds.

Case 2: $S = H^{A_6}_{m,k}$. The subcase for $k = 1$ is trivial. So we assume $k \geq 2$, take $R = H^{A_5}_{k-1,m-1}$ and define $\alpha : S \to R$ as follows. For $y \in S$, let $\alpha(y) : [k-1] \to [m-1]$ be as follows.

$$\forall i \leq k-1, \alpha(y)(i) = \min y^{-1}(i+1) - 1. \tag{4.3.5}$$

We claim that R with α works for S. It is obvious that (1.6.1) holds. So it remains to check (1.6.2). Given $F = H^{A_5}_{m-1,n-1}$ with $F \bullet R$ defined , let $G = H^{A_6}_{n,m}$, then $G \bullet S$ is defined. Define $\phi : F \to G$ as follows.

For $f \in F$, let $\phi(f) : [n] \to [m]$ be an increasing surjection such that

$$\forall j \leq m-1, f(j)+1 = \min \phi(f)^{-1}(j+1). \tag{4.3.6}$$

Let $f \in F, y \in S$, by (4.3.5) and (4.3.6), we have for $2 \leq i \leq k$,

$$\begin{aligned}
&\min(\phi(f)\,.\,y)^{-1}(i) = \min \phi(f)^{-1}(\min y^{-1}(i)) \\
&= \min \phi(f)^{-1}(\alpha(y)(i-1)+1) = (f\,.\,\alpha(y))(i-1)+1.
\end{aligned}$$

The first equality holds because $\phi(f)$ and y are non-deceasing; the second equality is due to (4.3.5); the third equality is due to (4.3.6). It follows that (1.6.2) holds. □

Theorem 4.3.8. *$(\mathcal{F}_2, \mathcal{S}_2)$ and $(\mathcal{F}_7, \mathcal{S}_7)$ can be ∂-interpreted by each other.*

Proof. First, we show that $(\mathcal{F}_2, \mathcal{S}_2)$ can be ∂-interpreted by $(\mathcal{F}_7, \mathcal{S}_7)$. Let $S = H^{A_2}_{m,k}(E^{A_2}_{m,k}$ respectively), take $R = H^{A_7}_{m,k}(E^{A_7}_{m,k}$ respectively). Let $\alpha\colon S \to R$ be $id \restriction S$.

We claim that R with α works for S. It is obvious that (1.6.1) holds. So it remains to check (1.6.2). Given $F = H^{A_7}_{n,m}(E^{A_7}_{n,m}$ respectively) with $F \bullet R$ defined, let $G = H^{A_2}_{n+1,m}(E^{A_2}_{n+1,m}$ respectively), then $G \bullet S$ is defined.

Define $\phi\colon F \to G$ as follows. For $f \in F$, let $\phi(f) : [\mathrm{dom}(f)+1] \to [m]$ be as follows.

$$\phi(f)(i) = \begin{cases} 1, & \text{if } i = 1; \\ f(i-1), & \text{if } 2 \leq i \leq \mathrm{dom(f)}+1. \end{cases} \tag{4.3.7}$$

Note that $\phi(f) \in G$, because $\phi(f)(1) = \phi(f)(2) = 1$.

Observe that for $f \in F, y \in S$, $\phi(f\,.\,y) = \phi(f)\,.\,y$. Then since α is an identity function, we have $\phi(f\,.\,\alpha(y)) = \phi(f)\,.\,y$ which implies that (1.6.2) holds.

Second, we show that $(\mathcal{F}_7, \mathcal{S}_7)$ can be ∂-interpreted by $(\mathcal{F}_2, \mathcal{S}_2)$.

Let $S = H^{A_7}_{m,k}(E^{A_7}_{m,k}$ respectively), take $R = H^{A_2}_{m+1,k}(E^{A_2}_{m+1,k}$ respectively). Define $\alpha\colon S \to R$ as follows.

For $y \in S$, let $\alpha(y) : [\mathrm{dom}(y) + 1] \to [k]$ be as follows.

$$\alpha(y)(i) = \begin{cases} 1, & \text{if } i = 1; \\ y(i-1), & \text{if } 2 \le i \le \mathrm{dom}(y) + 1. \end{cases} \tag{4.3.8}$$

Note that $\alpha(y) \in R$, because $\alpha(y)(1) = \alpha(y)(2) = 1$.

We claim that R with α works for S. It is obvious that (1.6.1) holds. So it remains to check (1.6.2). Given $F = H^{A_2}_{n,m+1}(E^{A_2}_{n,m+1}$ respectively) with $F \bullet R$ defined, let $G = H^{A_7}_{n,m}(E^{A_7}_{n,m}$ respectively), then $G \bullet S$ is defined.

Define ϕ: $F \to G$ as follows. For $f \in F$, let $\phi(f) : [\mathrm{dom}(f)] \to [m]$ be as follows.

$$\phi(f)(i) = \begin{cases} 1, & \text{if } f(i) = 1; \\ f(i) - 1, & \text{if } f(i) > 1. \end{cases} \tag{4.3.9}$$

Note that $\phi(f) \in G$.

Let $f \in F, y \in S$, it suffices to show $\phi(f) \,.\, y = f \,.\, \alpha(y)$ because it implies (1.6.2) holds.

(i) Show $\mathrm{dom}(\phi(f) \,.\, y) = \mathrm{dom}(f \,.\, \alpha(y))$. Let $m_1 = \mathrm{dom}(y)$, then $\mathrm{dom}(\alpha(y)) = m_1 + 1$.

If $m_1 = m$, then it is obvious that

$$\mathrm{dom}(f \,.\, \alpha(y)) = \mathrm{dom}(f) = \mathrm{dom}(\phi(f) \,.\, y).$$

If $m_1 < m$, then

$$\begin{aligned} \mathrm{dom}(f \,.\, \alpha(y)) &= \min f^{-1}(m_1 + 2) - 1 \\ &= \min \phi(f)^{-1}(m_1 + 1) - 1 = \mathrm{dom}(\phi(f) \,.\, y). \end{aligned}$$

The second equality holds because $f^{-1}(m_1 + 2) = \phi(f)^{-1}(m_1 + 1)$ which is due to (4.3.9).

(ii)Show for $i \le \mathrm{dom}(\phi(f) \,.\, y)$, $(\phi(f) \,.\, y)(i) = (f \,.\, \alpha(y))(i)$.

If $f(i) = 1$, then

$$(\phi(f)\,.\,y)(i) = y(\phi(f)(i)) = y(1) = 1$$
$$= \alpha(y)(f(i)) = (f\,.\,\alpha(y))(i).$$

If $f(i) > 1$, then

$$(\phi(f)\,.\,y)(i) = y(\phi(f)(i)) = y(f(i) - 1)$$
$$= \alpha(y)(f(i)) = (f\,.\,\alpha(y))(i).$$

The second equality is due to (4.3.9); the third equality is due to (4.3.8)

□

Remark 4.3.2. By Theorem 4.3.1 and the above theorem, we see Theorem 4.1.6 holds.

Theorem 4.3.9. *$(\mathcal{F}_3, \mathcal{S}_3)$ and $(\mathcal{F}_7, \mathcal{S}_7)$ can be ∂-interpreted by each other.*

Proof. First, we show $(\mathcal{F}_3, \mathcal{S}_3)$ can be ∂-interpreted by $(\mathcal{F}_7, \mathcal{S}_7)$. Let $S = H^{A_3}_{m,k}(E^{A_3}_{m,k}$ respectively), take $R = H^{A_7}_{m,k}(E^{A_7}_{m,k}$ respectively). Let $\alpha\colon S \to R$ be $id \restriction S$.

We claim that R with α works for S. It is obvious that (1.6.1) holds. So it remains to check (1.6.2). Given $F = H^{A_7}_{n,m}(E^{A_7}_{n,m}$ respectively) with $F \bullet R$ defined, let $G = H^{A_3}_{2n,m}(E^{A_3}_{2n,m}$ respectively), then $G \bullet S$ is defined.

Define $\phi\colon F \to G$ as follows. For $f \in F$, let $\phi(f) : [2 \times \mathrm{dom}(f)] \to [m]$ be as follows.

$$\phi(f)(i) = \begin{cases} f((i+1)/2), & \text{if } i \text{ is odd;} \\ f(i/2), & \text{if } i \text{ is even.} \end{cases} \tag{4.3.10}$$

Note that $\phi(f) \in G$, because $\phi(f)(2i-1) = \phi(f)(2i)$ for $i \le \mathrm{dom}(f)$.

We show that for $f \in F, y \in S$, $\phi(f\,.\,y) = \phi(f)\,.\,y$. By (4.3.10), we have that for $i \le m$,

$$\min \phi(f)^{-1}(i) = 2 \times \min f^{-1}(i) - 1. \tag{4.3.11}$$

(i) Show $\mathrm{dom}(\phi(f\,.\,y)) = \mathrm{dom}(\phi(f)\,.\,y)$. Let $m_1 = \mathrm{dom}(y)$.

If $m_1 = m$, then it is obvious that the equality holds.

If $m_1 < m$ then

$$\operatorname{dom}(\phi(f\,.\,y)) = 2 \times (\min f^{-1}(m_1+1) - 1) =$$
$$\min \phi(f)^{-1}(m_1+1) - 1 = \operatorname{dom}(\phi(f)\,.\,y).$$

The second equality is due to (4.3.11).

(ii) Show for $j \leq \operatorname{dom}(\phi(f)\,.\,y)$, $\phi(f\,.\,y)(j) = (\phi(f)\,.\,y)(j)$.

Let $i \leq \operatorname{dom}(\phi(f\,.\,y))/2$, we have

$$\phi(f\,.\,y)(2i) = (f\,.\,y)(i) = y(f(i)) =$$
$$y(\phi(f)(2i)) = (\phi(f)\,.\,y)(2i).$$

The first and third equalities are due to (4.3.10). Similarly, we have $\phi(f\,.\,y)(2i-1) = (\phi(f)\,.\,y)(2i-1)$.

Since α is an identity function, we have $\phi(f\,.\,\alpha(y)) = \phi(f)\,.\,y$ which implies that (1.6.2) holds.

Second, we show $(\mathcal{F}_7, \mathcal{S}_7)$ can be ∂-interpreted by $(\mathcal{F}_3, \mathcal{S}_3)$. Let $S = H^{A_7}_{m,k}(E^{A_7}_{m,k}$ respectively), take $R = H^{A_3}_{2m,k}(E^{A_3}_{2m,k}$ respectively). Define $\alpha\colon S \to R$ as follows. For $y \in S$, let $\alpha(y) : [2 \times \operatorname{dom}(y)] \to [k]$ be as follows.

$$\alpha(y)(i) = \begin{cases} y((i+1)/2), & \text{if } i \text{ is odd;} \\ y(i/2), & \text{if } i \text{ is even.} \end{cases} \tag{4.3.12}$$

Note $\alpha(y) \in R$, because $\alpha(y)(2i-1) = \alpha(y)(2i)$ for $i \leq \operatorname{dom}(y)$.

We claim that R with α works for S. It is obvious that (1.6.1) holds. So it remains to check (1.6.2). Given $F = H^{A_3}_{n,2m}(E^{A_3}_{n,2m}$ respectively) with $F \bullet R$ defined, let $G = H^{A_7}_{n,m}(E^{A_7}_{n,m}$ respectively), then $G \bullet S$ is defined. Define $\phi\colon F \to G$ as follows. For $f \in F$, let $\phi(f) : [\operatorname{dom}(f)] \to [m]$ be as follows.

$$\phi(f)(i) = \begin{cases} (f(i)+1)/2, & \text{if } f(i) \text{ is odd;} \\ f(i)/2, & \text{if } f(i) \text{ is even.} \end{cases} \tag{4.3.13}$$

Note that $\phi(f) \in G$. So it suffices to show that, for $f \in F, y \in S$,

$$\phi(f) \,.\, y = f \,.\, \alpha(y).$$

which implies (1.6.2) holds.

(i) Show $\mathrm{dom}(\phi(f) \,.\, y) = \mathrm{dom}(f \,.\, \alpha(y))$. Let $m_1 = \mathrm{dom}(y)$, then $\mathrm{dom}(\alpha(y)) = 2m_1$.

If $m_1 = m$, then it is obvious that

$$\mathrm{dom}(f \,.\, \alpha(y)) = \mathrm{dom}(f) = \mathrm{dom}(\phi(f) \,.\, y).$$

If $m_1 < m$, then

$$\mathrm{dom}(f \,.\, \alpha(y)) = \min f^{-1}(2m_1 + 1) - 1$$
$$= \min \phi(f)^{-1}(m_1 + 1) - 1 = \mathrm{dom}(\phi(f) \,.\, y).$$

The second equality is due to the facts that

$$\phi(f)^{-1}(m_1 + 1) = f^{-1}(2m_1 + 1) \cup f^{-1}(2m_1 + 2),$$

and

$$\min f^{-1}(2m_1 + 1) < \min f^{-1}(2m_1 + 2).$$

(ii) Show for $i \leq \mathrm{dom}(\phi(f) \,.\, y)$, $(\phi(f) \,.\, y)(i) = (f \,.\, \alpha(y))(i)$.

If $f(i)$ is even, then

$$f \,.\, \alpha(y)(i) = \alpha(y)(f(i)) = \alpha(y)(2\phi(f)(i)) =$$
$$y(\phi(f)(i)) = (\phi(f) \,.\, y)(i).$$

The two equalities in the middle are due to (4.3.13) and (4.3.12), respectively.

If $f(i)$ is odd, similarly, we have $f \,.\, \alpha(y)(i) = (\phi(f) \,.\, y)(i)$. □

Remark 4.3.3. By Theorem 4.3.1 and the above theorem, we see Theorem 4.1.8 holds.

Lemma 4.3.10. *Given $k \leq m$, l, and $x_1, \dots, x_l \in H^{II}_{k,m}$ with*

$$\partial x_1 = \cdots = \partial x_l, \text{ and } x_1(k) < \cdots < x_l(k).$$

there exist n and $f_1, f_2 \in H^{II}_{m,n}$, such that $f_1 \,.\, x_i = f_2 \,.\, x_{i+1}$ for $i < l$.

Proof. Observe that if n is big enough, then there exist $f_1, f_2 \in H^{II}_{m,n}$ such that

$$f_1(x_i(k)) = f_2(x_{i+1}(k)) \text{ for } i < l$$

and

$$f_1 \restriction [x_1(k-1)] = f_2 \restriction [x_1(k-1)].$$

For such f_1, f_2, we have $f_1 \,.\, x_i = f_2 \,.\, x_{i+1}$ for $i < l$.

□

Lemma 4.3.11. (A_5, X_5) *satisfies the left cancellation property.*

Proof. Let $x_1, x_2 \in X_5$ and $f \in A_5$ such that $f \,.\, x_1$ and $f \,.\, x_2$ are defined and equal. It follows that $\mathrm{dom}(x_1) = \mathrm{dom}(x_2)$. Suppose $x_1 \neq x_2$, then there exists $i \in \mathrm{dom}(x_1)$ such that $x_1(i) \neq x_2(i)$. Because f is an injection, we have

$$f(x_1(i)) \neq f(x_2(i))$$

which implies $f \,.\, x_1 \neq f \,.\, x_2$, a contradiction. □

Theorem 4.3.12. $(\mathcal{F}_5, \mathcal{S}_5)$ *can not be relaxed interpreted by* $(\mathcal{F}_8, \mathcal{S}_8)$.

Proof. Let H be a directed graph based on $[3]$ with vertices $\{v_1, v_2\}$, edges $\{(1,2),(2,3)\}$ that both connect from v_1 to v_2. Let $m > 2$, $S = H^{A_5}_{2,m} \in S_5$. We show that S is not relaxed interpretable in $(\mathcal{F}_8, \mathcal{S}_8)$ by applying Theorem 4.3.5.

First, by Lemma 4.3.11, (A_5, X_5) satisfies the left cancellation property. And it is obvious that $(\mathcal{F}_8, \mathcal{S}_8)$ satisfies the left defined property.

Second, show condition (ii) holds. Given $R = H^{A_8}_{1,m'} \in S_8$, let $F = H^{A_8}_{m',n'}$ where n' is big enough. Note that $F \bullet R$ is defined. Let $R' \subseteq R$ with $|R'| = 3$, $x_1x_2x_3$ be a permutation of R' such that $x_1(1) < x_2(1) <$

$x_3(1)$. Since n' is big enough, by Lemma 4.3.10, there exist $f_1, f_2 \in F$ such that

$$f_1 . x_1 = f_2 . x_2 \text{ and } f_1 . x_2 = f_2 . x_3,$$

which implies that $H(f_1 f_2, x_1 x_2 x_3)$ commutes.

Third, show condition (i) holds. Given $G \in \mathcal{F}_5$ with $G \bullet S$ defined, then $G = H^{A_5}_{m,n}$ for some n. Let $S' = \{x_1, x_2, x_3\} \subseteq S$ be as follows.

$$x_1(1) = 2, x_1(2) = 3;\; x_2(1) = 1, x_2(2) = 3;\; x_3(1) = 1, x_3(2) = 2.$$

Need to show, for each permutation $y_1 y_2 y_3$ of S', there do not exist $g_1, g_2 \in G$ such that $H(g_1 g_2, y_1 y_2 y_3)$ commutes. Suppose not, then there exist a permutation $y_1 y_2 y_3$ of S' and $g_1, g_2 \in G$ such that $H(g_1 g_2, y_1 y_2 y_3)$ commutes which implies that

$$g_1 . y_1 = g_2 . y_2 \text{ and } g_1 . y_2 = g_2 . y_3. \tag{4.3.14}$$

Since $y_1 y_2 y_3$ is a permutation of S', there are 6 possibilities. We apply the same method to deal with them, that is, checking

$$g_1 . y_1(2),\; g_2 . y_2(2),\; g_1 . y_2(2),\; g_2 . y_3(2).$$

We only check the case $y_1 y_2 y_3 = x_1 x_2 x_3$, all other cases are similar. By (4.3.14), we have

$$g_1(y_1(2)) = g_2(y_2(2)) \Rightarrow g_1(3) = g_2(3),$$

and

$$g_1(y_2(2)) = g_2(y_3(2)) \Rightarrow g_1(3) = g_2(2).$$

which leads a contradiction because g_2 is an injection. □

Lemma 4.3.13. (A_7, X_7) *satisfies the left cancellation property.*

Proof. Let $x_1, x_2 \in X_7$ and $f \in A_7$ such that $f . x_1$ and $f . x_2$ are defined and equal. It follows that $\text{dom}(x_1) = \text{dom}(x_2)$ and $\text{codom}(x_1) = \text{codom}(x_2)$. Suppose $x_1 \neq x_2$, then there exists $j \in \text{dom}(x_1)$ such that $x_1(j) \neq x_2(j)$. Since $j \in \text{codom}(f)$, there exists $i \in \text{dom}(f)$ such that $f(i) = j$. It follows that $f . x_1(i) \neq f . x_2(i)$, a contradiction. Thus x_1 should equal x_2. □

Theorem 4.3.14. *$(\mathcal{F}_7, \mathcal{S}_7)$ can not be ∂-interpreted by $(\mathcal{F}_5, \mathcal{S}_5)$.*

Proof. Let H be a directed graph based on [3] with vertices $\{v_1, v_2\}$, edges $\{(1,2),(2,3)\}$ that both connect from v_1 to v_2. Let $S = H^{A_7}_{k+1,k} \in S_7$ with $k > 3$. We show that S is not ∂-interpretable in $(\mathcal{F}_5, \mathcal{S}_5)$ by applying Theorem 4.3.6.

First, by Lemma 4.3.13, (A_7, X_7) satisfies the left cancellation property. And it is obvious that $(\mathcal{F}_5, \mathcal{S}_5)$ satisfies the left defined property.

Second, show condition (ii) holds. Given $R = H^{A_5}_{k',m'} \in \mathcal{S}_5$, let $F = H^{A_5}_{m',n'} \in \mathcal{F}_5$ where n' is big enough. Note that $F \bullet R$ is defined. Let $R' \subseteq R$ with $|R'| = 3$ and $|\partial R'| = 1$. It follows that R' has a permutation $x_1x_2x_3$ with $x_1(k') < x_2(k') < x_3(k')$. Then by Lemma 4.3.10 and the fact that n' is big enough, there exist $f_1, f_2 \in F$ such that

$$f_1 \,.\, x_1 = f_2 \,.\, x_2 \text{ and } f_1 \,.\, x_2 = f_2 \,.\, x_3,$$

which implies that $H(f_1f_2, x_1x_2x_3)$ commutes.

Third, show condition (i) holds. Given $G \in \mathcal{F}_7$ with $G \bullet S$ defined, then $G = H^{A_7}_{n,k+1}$ for some n. Let $S' = \{x_1, x_2, x_3\} \subseteq S$ be as follows.

$$x_1 \restriction [k] = id \restriction [k], x_1(k+1) = 1;$$

$$x_2 \restriction [k] = id \restriction [k], x_2(k+1) = 2;$$

$$x_3 \restriction [k] = id \restriction [k], x_3(k+1) = 3.$$

Note that $|\partial S'| = 1$.

Need to show, for each permutation $y_1y_2y_3$ of S', there do not exist $g_1, g_2 \in G$ such that $H(g_1g_2, y_1y_2y_3)$ commutes. Suppose not, then there exist a permutation $y_1y_2y_3$ of S' and $g_1, g_2 \in G$ such that $H(g_1g_2, y_1y_2y_3)$ commutes which implies that

$$g_1 \,.\, y_1 = g_2 \,.\, y_2 \text{ and } g_1 \,.\, y_2 = g_2 \,.\, y_3. \tag{4.3.15}$$

Assume $y_1(k+1) = j_1$; $\quad y_2(k+1) = j_2$; $\quad y_3(k+1) = j_3$. Note j_1, j_2, j_3 are pairwise distinct.

From the first equality of (4.3.15), we have

$$(g_1 \,.\, y_1)^{-1}(j_1) = (g_2 \,.\, y_2)^{-1}(j_1)$$

which implies that

$$g_1^{-1}(\{k+1, j_1\}) = g_2^{-1}(j_1).$$

From the second equality of (4.3.15), we have

$$(g_1 \,.\, y_2)^{-1}(j_1) = (g_2 \,.\, y_3)^{-1}(j_1)$$

which implies that

$$g_1^{-1}(j_1) = g_2^{-1}(j_1).$$

It follows that $g_1^{-1}(\{k+1, j_1\}) = g_1^{-1}(j_1)$, contradicting with the facts $k > 3$ and $j_1 \leq 3$.

□

Theorem 4.3.15. $(\mathcal{F}_1, \mathcal{S}_1)$ *can not be ∂-interpreted by* $(\mathcal{F}_5, \mathcal{S}_5)$*.*

Proof. Let H be a directed graph based on [3] with vertices $\{v_1, v_2\}$, edges $\{(1,2),(2,3)\}$ that both connect from v_1 to v_2. Let $S = H_{4,2}^{A_1} \in S_1$. We show that S is not ∂-interpretable in $(\mathcal{F}_5, \mathcal{S}_5)$ by applying Theorem 4.3.6.

First, by Lemma 4.3.13 and the fact $A_1 \subseteq A_7, X_1 \subseteq X_7$, we have (A_1, X_1) satisfies the left cancellation property. And it is obvious that $(\mathcal{F}_5, \mathcal{S}_5)$ satisfies the left defined property.

Second, by the same proof of proving conditon (ii) holds as in the proof of Theorem 4.3.14, we know condition (ii) also holds here.

Third, show condition (i) holds. Given $G \in \mathcal{F}_1$ with $G \bullet S$ defined, then $G = H_{n,4}^{A_1}$ for some n. Let $S' = \{x_1, x_2, x_3\} \subseteq S$ be as follows.

$$x_1 \restriction [2] = id \restriction [2], \;\; x_1(3) = x_1(4) = 2;$$

$$x_2 \restriction [2] = id \restriction [2], \;\; x_2(3) = 1, \;\; x_2(4) = 2;$$

$$x_3 \restriction [2] = id \restriction [2], \;\; x_3(3) = 2, \;\; x_3(4) = 1.$$

Note that $|\partial S'| = 1$.

Need to show, for each permutation $y_1y_2y_3$ of S', there do not exist $g_1, g_2 \in G$ such that $H(g_1g_2, y_1y_2y_3)$ commutes. Suppose not, then there exist a permuation $y_1y_2y_3$ of S' and $g_1, g_2 \in G$ such that $H(g_1g_2, y_1y_2y_3)$

commutes which implies that

$$g_1 \,.\, y_1 = g_2 \,.\, y_2 \text{ and } g_1 \,.\, y_2 = g_2 \,.\, y_3. \tag{4.3.16}$$

Let

$$B_i = g_1^{-1}(i+1), \quad C_i = g_2^{-1}(i+1) \text{ for } i \leq 3.$$

Note B_1, B_2, B_3 are pairwise disjoint, so is C_1, C_2, C_3.

Since $y_1y_2y_3$ is a permutation of S', there are 6 possibilities. We apply the same method to deal with them, that is, checking

$$(g_1 \,.\, y_1)^{-1}(2), \;\; (g_1 \,.\, y_2)^{-1}(2), \;\; (g_2 \,.\, y_2)^{-1}(2), \;\; (g_2 \,.\, y_3)^{-1}(2).$$

We only check the case $y_1y_2y_3 = x_1x_2x_3$, all other cases are similar. By the first equality of (4.3.16), we have

$$B_1 \cup B_2 \cup B_3 = C_1 \cup C_3.$$

By the second equality of (4.3.16), we have

$$B_2 \cup B_3 = C_1 \cup C_2.$$

It follows that

$$B_1 \cup C_1 \cup C_2 = C_1 \cup C_3,$$

contradicting with the fact that C_1, C_2, C_3 are pairwise disjoint.

□

Lemma 4.3.16. (A_4, X_4) *satisfies the left cancellation propery.*

Proof. Assume (A_4, X_4) has dimension l. Let $x = (x_1, \ldots, x_l), y = (y_1, \ldots, y_l) \in X_4$ and $f = (f_1, \ldots, f_l) \in A_4$ such that $f \,.\, x$ and $f \,.\, y$ are defined and equal. It follows that, for each $i \leq l$, $f_i \,.\, x_i$ and $f_i \,.\, y_i$ are defined and equal. Then by Lemma 4.3.11, $x_i = y_i$ for $i \leq l$. Thus $x = y$. □

Theorem 4.3.17. $(\mathcal{F}_4, \mathcal{S}_4)$ *can be relaxed interpreted but not ∂-interpreted by* $(\mathcal{F}_5, \mathcal{S}_5)$.

Proof. Assume (A_4, X_4) has dimension l. To prove the first part. Let $S = H^{A_4}_{k,m}$, take $R = H^{A_5}_{kl,ml}$. Define

$\alpha\colon S \to R$ as follows. For $y = (y_1, \ldots, y_l) \in S$, let $\alpha(y) : [kl] \to [ml]$ be such that for $i \leq l$, $j \leq k$,

$$\alpha(y)(i + (j-1) \times l) = i + (y_i(j) - 1) \times l. \tag{4.3.17}$$

Note that $\alpha(y) \in R$, because $y \in H^{A_4}_{k,m}$.

We claim that R with α works for S. Given $F = H^{A_5}_{ml,n}$ with $F \bullet R$ defined. Let $G = H^{A_4}_{m,n}$, then $G \bullet S$ is defined.

Define $\phi\colon F \to G$ as follows. For $f \in F$, let $\phi(f) = (f_1, \ldots, f_l)$ where, for $i \leq l, j \leq m$,

$$f_i(j) = f(i + (j-1) \times l). \tag{4.3.18}$$

Since f is an increasing injection, it is easy to check that $\phi(f) \in H^{A_4}_{m,n}$.

Let $f \in F, y = (y_1, \ldots, y_l) \in S$, assume $\phi(f) = (f_1, \ldots, f_l)$. Then $\phi(f) \,.\, y = (f_1 \,.\, y_1, \ldots, f_l \,.\, y_l)$. Observe that for $i \leq l, j \leq k$,

$$(f_i \,.\, y_i)(j) = f_i(y_i(j)) = f(i + (y_i(j) - 1) \times l) =$$
$$f(\alpha(y)(i + (j-1) \times l)).$$

The second equality is due to (4.3.18); the third equality is due to (4.3.17). It follows that (1.6.3) holds.

To prove the second part, without loss of generality, assume $l = 2$.
Let H be a directed graph based on $[r]$ with vertices $\{v_1, v_2\}$, edges

$$\{(1,2), (2,3), \ldots, (r-1, r)\}$$

that connect from v_1 to v_2. r will be defined later.

Let $S = H^{A_4}_{k,m}$. We show that S is not ∂-interpretable in $(\mathcal{F}_5, \mathcal{S}_5)$ by applying Theorem 4.3.6.

First, by Lemma 4.3.16, (A_4, X_4) satisfies the left cancellation property. And it is obvious that $(\mathcal{F}_5, \mathcal{S}_5)$ satisfies the left defined property.

Second, show condition (i) holds. Given $G \in \mathcal{F}_4$ with $G \bullet S$ defined, then $G = H^{A_4}_{n,m}$ for some n. Let S' be as follows.

$$S' = \{y \in S : \partial y = (id \restriction [k-1], id \restriction [k-1])\}.$$

Let $r = |S'|$.

Need to show, for each permutation $y_1 \dots y_r$ of S', there do not exist $g_1, g_2 \in G$ such that $H(g_1 g_2, y_1 \dots y_r)$ commutes. Suppose not, then there exist a permutation $y_1 \dots y_r$ of S' and $g_1, g_2 \in G$ such that $H(g_1 g_2, y_1 \dots y_r)$ commutes which implies that

$$g_1 \,.\, y_i = g_2 \,.\, y_{i+1} \text{ for } i < r. \tag{4.3.19}$$

It follows that

$$g_1 \,.\, y_i(k) = g_2 \,.\, y_{i+1}(k) \text{ for } i < r. \tag{4.3.20}$$

Assume

$$g_1 = (s_1, t_1); \quad g_2 = (s_2, t_2); \quad y_i(k) = (p_i, q_i) \text{ for } i \leq r.$$

We claim that $p_1 \dots p_r$ is a constant or monotone sequence. By (4.3.20), we have

$$s_1(p_i) = s_2(p_{i+1}) \text{ for } i < r. \tag{4.3.21}$$

Now based on the values of p_1 and p_2, we have three cases to deal with.

Case 1 : $p_1 = p_2$. Then by induction, we have $p_1 = \dots = p_r$. Indeed, suppose $p_i = p_{i+1}$, then by (4.3.21), $s_2(p_{i+1}) = s_2(p_{i+2})$ which implies that $p_{i+1} = p_{i+2}$ because s_2 is an injection.

Case 2 : $p_1 < p_2$. Then by induction, we have $p_1 < \dots < p_r$. Indeed, suppose $p_i < p_{i+1}$, then since s_1 is an increasing injection, we have $s_1(p_i) < s_1(p_{i+1})$. It follows, by (4.3.21), that $s_2(p_{i+1}) < s_2(p_{i+2})$ which implies that $p_{i+1} < p_{i+2}$ because s_2 is an increasing injection.

Case 3 : $p_1 > p_2$. Then by a similar argument as in Case 2, we have $p_1 > \dots > p_r$.

Let $y_1, y_2, y_3 \in S$ be such that $\partial y_i = (id \restriction [k-1], id \restriction [k-1])$ for $i \leq 3$ and

$$y_1(k) = (k, k), y_2(k) = (k, m), y_3(k) = (m, m).$$

It is obvious that $\{y_1, y_2, y_3\} \subseteq S'$. So there exist i_1, i_2, i_3 such that $p_{i_1} = k$, $p_{i_2} = p_{i_3} = m$ which implies that $p_1 \dots p_r$ could not be a constant or monotone sequence, contradicting with the above claim.

Third, show condition (ii) holds. Given $R = H^{A_5}_{k',m'}$, let $F = H^{A_5}_{m',n'}$ with $F \bullet R$ defiend, where n' is big enough. Let $R' \subseteq R$ with $|R'| = r$ and $|\partial R'| = 1$. Then there exists a permutation $x_1 \dots x_r$ of R' such that

$x_1(k') < \cdots < x_r(k')$. Since n' is big enough, by Lemma 4.3.10, there exist $f_1, f_2 \in F$ such that

$$f_1 \,.\, x_i = f_2 \,.\, x_{i+1} \text{ for } i \leq r,$$

which implies that $H(f_1 f_2, x_1 \ldots x_r)$ commutes. □

Remark 4.3.4. By Theorem 4.3.1 and the first part of the above theorem, we see Theorem 4.1.9 holds.

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