HYPERGEOMETRIC FUNCTIONS, CONTINUED FRACTIONS FOR PRODUCTS OF GAMMA FUNCTIONS, AND q-ANALOGUES

BY

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DISSERTATION

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Abstract

Some of the most interesting of Ramanujan's continued fraction identities are those involving ratios of Gamma functions in Chapter 12 of his second notebook. This thesis develops a method for deriving such identities, using hypergeometric functions as the main tool. We begin by deriving a continued fraction identity, use it to prove Ramanujan's Entry 34, and then use the method to obtain new identities and relate them to two of Ramanujan's identities. We next prove Ramanujan's Entries 36 and 39. Finally, we rework the method for use with basic hypergeometric functions and use it to find q-analogues of the earlier new results.

To my family

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Table of Contents

Chapter 1

Introduction

Several entries in Chapter 12 of S. Ramanujan's second notebook [4] express a quotient of Gamma functions as a continued fraction. These continued fractions range from reasonably simple to quite complicated. The entries have been proven by a variety of means, some elementary, some hypergeometric, some complex-analytic. This thesis explores a method of deriving these identities and others using hypergeometric methods. We derive several of Ramanujan's entries and some new identities and q-analogues.

The goal when developing this method was to find a technique that did not require full complex analytic methods to derive new continued fraction identities. Instead, many of the identities found so far using this method have already been found by Ramanujan. This fact, coupled with the ordering of entries in Chapter 12 of his second notebook, makes it seem reasonable that he used a method very similar to this one to obtain identities when he was working on Chapter 12. This insight into a possible method of Ramanujan may further our understanding of how he obtained his results and what mathematical tools were used "behind the scenes" in the writing of his notebooks.

1.1 Background

Throughout this thesis, entries from Ramanujan's notebooks without a chapter number given are from Chapter 12. All other entries are given as Entry A.B, where A is the chapter number and B is the entry number. The simplest of Ramanujan's continued fractions that can be written as a quotient of Gamma functions is Entry 25 [4].

Theorem 1.1.1 (Entry 25)**.** *Suppose that either n is an odd integer and x is a complex number or that $\Re(x) > 0$ and n is an arbitrary complex number. Then*

$$\frac{\Gamma(\frac{1}{4}(x+n+1))\Gamma(\frac{1}{4}(x-n+1))}{\Gamma(\frac{1}{4}(x+n+3))\Gamma(\frac{1}{4}(x-n+3))} = \frac{4}{x+}\,\frac{1^2-n^2}{2x}\,{+}\,\frac{3^2-n^2}{2x}\,{+}\,\frac{5^2-n^2}{2x}\,{+}\cdots. \tag{1.1}$$

In order to fully understand these identities, some notation and definitions are needed. The Gamma functions on the left side of Entry 25 are defined in the usual way by

$$\Gamma(z) := \int_0^\infty e^{-t}t^{z-1}\,dt, \qquad \Re(z) > 0. \tag{1.2}$$

The fundamental property of the Gamma function is that $\Gamma(z+1) = z\Gamma(z)$. This property will be used repeatedly throughout this thesis, usually without comment.

A continued fraction is an expression of the form

$$\mathop{\mathbf{K}}_{n=1}^{\infty} \frac{a_n}{b_n} = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ddots}}}}.$$

We will use the space-saving notation

$$\mathop{\mathbf{K}}_{n=1}^{\infty} \frac{a_n}{b_n} = \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\,\frac{a_4}{b_4+}\cdots,$$

where the "+" sign in the denominator indicates a continued fraction rather than a sum of ordinary fractions.

The hidden link between Gamma functions and continued fractions in these identities is the hypergeometric function. We define

$$ {}_2F_1\left(\begin{matrix} a,\, b \\ c \end{matrix};\, z\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \qquad |z| < 1 \text{ or } |z| = 1 \text{ and } \Re(c-a-b) > 0, \tag{1.3} $$

$$ {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix};\, z\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n n!} z^n, \qquad |z| < 1 \text{ or } |z| = 1 \text{ and } \Re(d+e-a-b-c) > 0, \tag{1.4} $$

and in general

$$ {}_pF_q\left(\begin{matrix} a_1,\, a_2,\, \ldots,\, a_p \\ b_1,\, b_2,\, \ldots,\, b_q \end{matrix};\, z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n n!} z^n, \tag{1.5} $$

where $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$ is the so-called Pochhammer symbol. These series converge absolutely if $|z| < 1$ or if $|z| = 1$ and $\Re(\sum b_i - \sum a_i) > 0$.

There are now hypergeometric proofs of many of Ramanujan's entries and other similar continued fractions that can be written as quotients of Gamma functions. The proofs currently known, along with the type of hypergeometric function used in each proof, are listed below in Table 1.1. The proofs in this thesis are given by theorem number, while those proved elsewhere refer to the author of the proof and the relevant journal article. There are multiple hypergeometric proofs known for Entries 25 and 39. The two proofs of Entry 25, by K.G. Ramanathan in [15] and R.L. Lamphere in [11], are comparable in complexity, as they both use ${}_2F_1$ hypergeometric series. It should be noted, however, that the proof of Entry 39 given in this thesis is simpler than the earlier proof of Entry 39 by D.R. Masson in [14] and this new proof also shows that Entry 39 is a companion to Entry 36.

Identity	Proof	Hypergeometric Type
Entry 25	Ramanathan, 1988 [15]	${}_2F_1$
Entry 25	Lamphere, 2000 [11]	${}_2F_1$
companion to Entry 25	Theorem 3.2.1	${}_2F_1$
Entry 26	none	
Entry 33	none	
Entry 34	Theorem 2.2.1	${}_3F_2$
variation on Entry 34	Theorem 3.1.1	${}_3F_2$
Entry 35	Masson, 1991 [14]	Wilson polynomials (${}_4F_3$)
Entry 36	Theorem 4.1.1	${}_3F_2$
Entry 39	Theorem 4.2.1	${}_3F_2$
Entry 39	Masson, 1991 [14]	Wilson polynomials (${}_4F_3$)
Entry 40	Masson, 1991 [13]	${}_9F_8$
companion to Entry 40	Masson, 1991 [13]	${}_9F_8$
Entry 16.10	none	

Table 1.1: Known hypergeometric proofs of Ramanujan's entries and similar identities

In his paper [15], Ramanathan used a three-term relation for ${}_2F_1$ hypergeometric functions to prove Entry 25. He used iterated division of the recurrence to generate the continued fraction side of the equality, and then used Kummer's Identity (stated by Ramanujan [4] as Corollary 13 to Entry 10.7) to evaluate the ratio of ${}_2F_1$ hypergeometric functions and obtain a ratio of Gamma functions. In general, the contiguous relations between ${}_3F_2$ hypergeometric functions have four terms, but they reduce to three terms in the case $z = 1$, and we use some of these to extend Ramanathan's method to ${}_3F_2$ hypergeometric functions and to obtain several other continued fraction identities.

Iterated division of a three-term recurrence is a common method used to generate continued fractions. As an example of its use, we now find what is known as Euler's continued fraction. The recurrence used is

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \frac{c+(a-b+1)x}{c}\, {}_2F_1\left(\begin{matrix} a+1, b \\ c+1 \end{matrix}; x\right) - \frac{(a+1)(c-b+1)x}{c(c+1)}\, {}_2F_1\left(\begin{matrix} a+2, b \\ c+2 \end{matrix}; x\right). \tag{1.6} $$

This relation is also due to Euler and is given as Equation (2.5.3) on page 94 of [2]. Dividing both sides of the recurrence by $(1/c)_2F_1\left(\begin{smallmatrix} a+1,b \\ c+1 \end{smallmatrix}; x\right)$ yields

$$ \frac{c\,{}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix}; x\right)}{{}_2F_1\left(\begin{smallmatrix} a+1,b \\ c+1 \end{smallmatrix}; x\right)} = c+(a-b+1)x - \frac{(a+1)(c-b+1)x}{c+1} \frac{{}_2F_1\left(\begin{smallmatrix} a+2,b \\ c+2 \end{smallmatrix}; x\right)}{{}_2F_1\left(\begin{smallmatrix} a+1,b \\ c+1 \end{smallmatrix}; x\right)}, \tag{1.7} $$

which can be written as

$$ \frac{c\,{}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix}; x\right)}{{}_2F_1\left(\begin{smallmatrix} a+1,b \\ c+1 \end{smallmatrix}; x\right)} = c+(a-b+1)x - \frac{(a+1)(c-b+1)x}{(c+1)\frac{{}_2F_1\left(\begin{smallmatrix} a+1,b \\ c+1 \end{smallmatrix}; x\right)}{{}_2F_1\left(\begin{smallmatrix} a+2,b \\ c+2 \end{smallmatrix}; x\right)}}. \tag{1.8} $$

Replacing a by $a+1$ and c by $c+1$ in (1.8) yields

$$ \frac{(c+1)\,{}_2F_1\left(\begin{smallmatrix} a+1,b \\ c+1 \end{smallmatrix}; x\right)}{{}_2F_1\left(\begin{smallmatrix} a+2,b \\ c+2 \end{smallmatrix}; x\right)} = c+1+(a-b+2)x - \frac{(a+2)(c-b+2)x}{(c+2)\frac{{}_2F_1\left(\begin{smallmatrix} a+2,b \\ c+2 \end{smallmatrix}; x\right)}{{}_2F_1\left(\begin{smallmatrix} a+3,b \\ c+3 \end{smallmatrix}; x\right)}}, \tag{1.9} $$

which is then substituted into (1.8) to obtain

$$ \begin{aligned} \frac{c\,{}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix}; x\right)}{{}_2F_1\left(\begin{smallmatrix} a+1,b \\ c+1 \end{smallmatrix}; x\right)} &= c+(a-b+1)x - \cfrac{(a+1)(c-b+1)x}{c+1+(a-b+2)x - \cfrac{(a+2)(c-b+2)x}{(c+2)\frac{{}_2F_1\left(\begin{smallmatrix} a+2,b \\ c+2 \end{smallmatrix}; x\right)}{{}_2F_1\left(\begin{smallmatrix} a+3,b \\ c+3 \end{smallmatrix}; x\right)}}} \\ &= c+(a-b+1)x - \frac{(a+1)(c-b+1)x}{c+1+(a-b+2)x-} \, \frac{(a+2)(c-b+2)x}{(c+2)\frac{{}_2F_1\left(\begin{smallmatrix} a+2,b \\ c+2 \end{smallmatrix}; x\right)}{{}_2F_1\left(\begin{smallmatrix} a+3,b \\ c+3 \end{smallmatrix}; x\right)}}. \end{aligned} \tag{1.10} $$

This process is iterated, at the k^{th} step using the recurrence

$$\frac{(c+k)\,{}_2F_1\left({a+k,b \atop c+k};\, x\right)}{{}_2F_1\left({a+k+1,b \atop c+k+1};\, x\right)} = c+k+(a-b+k+1)x - \frac{(a+k+1)(c-b+k+1)x}{(c+k+1)\frac{{}_2F_1\left({a+k+1,b \atop c+k+1};\, x\right)}{{}_2F_1\left({a+k+2,b \atop c+k+2};\, x\right)}}, \tag{1.11}$$

which is (1.8) with a replaced by $a+k$ and c replaced by $c+k$. The end product is Euler's continued fraction,

$$\frac{c\,{}_2F_1\left({a,b \atop c};\, x\right)}{{}_2F_1\left({a+1,b \atop c+1};\, x\right)} = c+(a-b+1)x - \frac{(a+1)(c-b+1)x}{c+1+(a-b+2)x-}\,\frac{(a+2)(c-b+2)x}{c+2+(a-b+3)x}$$
$$-\frac{(a+3)(c-b+3)x}{c+3+(a-b+4)x-}\,\frac{(a+4)(c-b+4)x}{c+4+(a-b+5)x-\cdots}. \tag{1.12}$$

Ramanujan lists Euler's continued fraction in a slightly different form as Entry 22 in his second Notebook [4], and we use it in the proof of Theorem 3.2.1.

1.2 Convergence of continued fractions

It is important when dealing with continued fractions to prove convergence carefully, as the convergence of continued fractions has rather different properties than that of infinite series or products. L. Lorentzen and H. Waadeland's book [12] provides much insight on continued fractions in general, and L. Jacobsen's paper [10] was invaluable in understanding the requirements for convergence and domains of validity of Ramanujan's continued fraction identities.

In proving the convergence of the new continued fractions obtained in Chapter 3, we use Theorem 2.3 in [10].

Theorem 1.2.1 (Jacobsen)**.** *Let*

$$a_{2k-1}(x) = \sum_{j=0}^{p} \alpha_j(x) k^j, \qquad a_{2k}(x) = \sum_{j=0}^{p} \gamma_j(x) k^j,$$
$$b_{2k-1}(x) = \sum_{j=0}^{q} \beta_j(x) k^j, \qquad b_{2k}(x) = \sum_{j=0}^{r} \delta_j(x) k^j, \tag{1.13}$$

for $k = 1, 2, 3, \ldots,$ be polynomials in k, where all α_j, γ_j, β_j, and δ_j are entire functions of x, and $\alpha_p(x)\gamma_p(x)\beta_q(x)\delta_r(x) \not\equiv 0$. Further let $\widetilde{D} = \{x \in \mathbb{C} : \beta_q(x)\delta_r(x) \neq 0$ and all $a_k(x) \neq 0\}$, and let D be defined in the following way:

(i) if $q + r > p$, then $D = \widetilde{D}$;

(ii) if $q + r = p$, then

$$D = \left\{ x \in \widetilde{D} : \frac{\alpha_p(x)\gamma_p(x)}{(\alpha_p(x) + \gamma_p(x) + \beta_q(x)\delta_r(x))^2} \notin \left[\frac{1}{4}, \infty\right) \right\};$$

(iii) if $q + r = p - 1$ and $\alpha_p(x) \equiv \gamma_p(x)$, then

$$D = \left\{ x \in \widetilde{D} : \frac{\beta_q(x)\delta_r(x)}{\alpha_p(x)} \notin (-\infty, 0] \right\};$$

(iv) if $q + r = p - 2$ and $\alpha_p(x) \equiv \gamma_p(x)$, then

$$D = \left\{ x \in \widetilde{D} : \frac{4\beta_q(x)\delta_r(x)}{\alpha_p(x)} + \left(r + 1 + \frac{\alpha_{p-1}(x) - \gamma_{p-1}(x)}{\alpha_p(x)} \right)^2 \notin (-\infty, 0] \right\};$$

(v) if $q + r = p - 2$, $\alpha_p(x) \equiv \gamma_p(x)$, and $\alpha_{p-1}(x) \equiv \gamma_{p-1}(x)$, then

$$D = \left\{ x \in \widetilde{D} : \frac{\beta_q(x)\delta_r(x)}{\alpha_p(x)} \notin \left(-\infty, -\frac{(r+1)^2}{4} \right] \right\}.$$

Then (under such additional conditions) $\mathbf{K}(a_k(x)/b_k(x))$ converges in D to a function f, *meromorphic or identically ∞ in each component of D. The convergence is uniform on compact subsets $C \subseteq D$ for which $\infty \notin f(C)$. Here $\alpha_{-1}(x) \equiv \gamma_{-1}(x) \equiv 0$.*

The q-continued fractions obtained in Chapter 5 require a somewhat different approach. We use the Parabola Theorem in their proofs, as well as two remarks from Jacobsen's paper [10].

Theorem 1.2.2 (Parabola Theorem)**.** *Let θ be a fixed number, $-\pi/2 < \theta < \pi/2$, and let $\mathbf{K}(a_n/1)$ be a continued fraction with all elements a_n contained in the parabolic region $P_\theta = \{z \in \mathbb{C} : |z| - \Re(ze^{-2i\theta}) \leq \frac{1}{2}\cos^2\theta\}$. Then the following hold.*

A. The approximants f_n are all finite and contained in the half plane $V_\theta = \{z \in \mathbb{C} : \Re(ze^{-i\theta}) \geq -\frac{1}{2}\cos\theta\}$.

B. $\{f_{2n+1}\}$ and $\{f_{2n}\}$ both converge to finite values.

C. If all $a_n \neq 0$, then $\mathbf{K}(a_n/1)$ converges (to a finite value) if and only if

$$\sum_{n=1}^{\infty} |d_n| = \infty, \text{ where } d_n = \prod_{k=1}^{n} a_k^{(-1)^{n+k-1}}.$$

Remark 1.2.3 (Jacobsen). If $a_N = 0$ and $a_n \neq 0$ for $n < N$, then both f_{2n+1} and f_{2n} converge to f_{N-1}, the value of the continued fraction. This follows since the approximants $f_n^{(N)}$ of the Nth tail of $\mathbf{K}(a_n/1)$,

$$\frac{a_{N+1}}{1} + \frac{a_{N+2}}{1} + \frac{a_{N+3}}{1} + \cdots,$$

are bounded away from -1 by part A of the Parabola Theorem.

Remark 1.2.4 (Jacobsen). If $\mathbf{K}(a_n/1)$ has holomorphic elements $a_n : D \to \mathbb{C}$ such that $\lim a_n(z) = a(z)$ locally uniformly in D, then $\mathbf{K}(a_n/1)$ converges to a meromorphic function f in $D_0 = \{x \in D : \text{ all } a_n(z) \neq 0 \text{ and } |\arg(a(z) + 1/4)| < \pi\}$, or to $f(z) \equiv \infty$ in D_0. The convergence is uniform on compact subsets $C \subseteq D_0$ such that $\infty \notin f(C)$.

Chapter 2

Derivation of the main continued fraction and proof of Entry 34

2.1 Preliminaries

Before proving Entry 34, we begin with some preliminary identities and recurrences which will be used throughout this thesis. The first of these is best known as Dixon's Identity and is also found as Entry 10.7 in Ramanujan's notebooks [4].

Theorem 2.1.1 (Dixon's Identity)**.** *If* $\Re(\frac{1}{2}a - b - c + 1) > 0$*, then*

$$ {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ a-b+1,\, a-c+1 \end{matrix}; 1\right) = \frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}a-b-c+1)}{\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)\Gamma(a-b-c+1)}. \tag{2.1}$$

Two recurrences of hypergeometric functions will also be needed and are proved here. The second of these recurrences is used immediately in the proof of Lemma 2.1.4, which is the main continued fraction used to obtain the continued fraction identities with three variables.

Proposition 2.1.2. *If* $\Re(d + e - a - b - c) > 0$*, then*

$$\begin{aligned} \frac{ab(e-c)}{e}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right) &= (d-a)(d-b)\,{}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d+1,\, e \end{matrix}; 1\right) \\ &\quad + d(a+b-d)\,{}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right). \end{aligned} \tag{2.2}$$

Proof. We begin with Equation (2.5.10) on page 97 of Andrews–Askey–Roy [2], which states that

$$ab(1-x)_2F_1\left(\begin{matrix}a+1,\,b+1\\d+1\end{matrix};x\right)=(d-a)(d-b)_2F_1\left(\begin{matrix}a,\,b\\d+1\end{matrix};x\right)$$
$$+\,d(a+b-d)_2F_1\left(\begin{matrix}a,\,b\\d\end{matrix};x\right). \tag{2.3}$$

Assume that $\Re(c)>0$ and $\Re(e-c)>0$, multiply both sides by $x^{c-1}(1-x)^{e-c-1}$, and integrate with respect to x from 0 to 1. We obtain

$$\begin{aligned}&\int_0^1 abx^{c-1}(1-x)^{e-c}{}_2F_1\left(\begin{matrix}a+1,\,b+1\\d+1\end{matrix};x\right)dx\\&\quad=\int_0^1(d-a)(d-b)x^{c-1}(1-x)^{e-c-1}{}_2F_1\left(\begin{matrix}a,\,b\\d+1\end{matrix};x\right)dx\\&\qquad+\int_0^1 d(a+b-d)x^{c-1}(1-x)^{e-c-1}{}_2F_1\left(\begin{matrix}a,\,b\\d\end{matrix};x\right)dx.\end{aligned} \tag{2.4}$$

We integrate each of these term-by-term. Thus

$$\begin{aligned}&\int_0^1 abx^{c-1}(1-x)^{e-c}{}_2F_1\left(\begin{matrix}a+1,\,b+1\\d+1\end{matrix};x\right)dx\\&=\int_0^1 abx^{c-1}(1-x)^{e-c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(d+1)_n n!}x^n\,dx\\&=ab\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(d+1)_n n!}\int_0^1 x^{n+c-1}(1-x)^{e-c}\,dx\\&=ab\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(d+1)_n n!}\mathrm{B}(n+c,e-c+1)\\&=ab\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(d+1)_n n!}\frac{\Gamma(n+c)\Gamma(e-c+1)}{\Gamma(n+c+e-c+1)}\\&=\frac{ab\Gamma(e-c+1)\Gamma(c)}{\Gamma(e+1)}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(d+1)_n n!}\frac{\Gamma(n+c)}{\Gamma(c)}\frac{\Gamma(e+1)}{\Gamma(n+e+1)}\\&=\frac{ab\Gamma(e-c+1)\Gamma(c)}{\Gamma(e+1)}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n(c)_n}{(d+1)_n n!(e+1)_n}\end{aligned}$$

$$
\begin{aligned}
&= \frac{ab\Gamma(e-c+1)\Gamma(c)}{\Gamma(e+1)} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right) \\
&= \frac{ab(e-c)\Gamma(e-c)\Gamma(c)}{e\Gamma(e)} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right), \qquad (2.5)
\end{aligned}
$$

where we make use of the fact that the Beta function is given by

$$
\mathrm{B}(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \Re(\alpha) > 0,\, \Re(\beta) > 0. \qquad (2.6)
$$

Similarly,

$$
\begin{aligned}
&\int_0^1 (d-a)(d-b)x^{c-1}(1-x)^{e-c-1} {}_2F_1\left(\begin{matrix} a,\, b \\ d+1 \end{matrix}; x\right) dx \\
&\quad = (d-a)(d-b)\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(d+1)_n n!} \int_0^1 x^{n+c-1}(1-x)^{e-c-1}\,dx \\
&\quad = (d-a)(d-b)\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(d+1)_n n!} \frac{\Gamma(n+c)\Gamma(e-c)}{\Gamma(n+e)} \\
&\quad = \frac{(d-a)(d-b)\Gamma(e-c)\Gamma(c)}{\Gamma(e)} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d+1)_n n!(e)_n} \\
&\quad = \frac{(d-a)(d-b)\Gamma(e-c)\Gamma(c)}{\Gamma(e)} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d+1,\, e \end{matrix}; 1\right) \qquad (2.7)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 d(a+b-d)x^{c-1}(1-x)^{e-c-1} {}_2F_1\left(\begin{matrix} a,\, b \\ d \end{matrix}; x\right) dx \\
&\quad = d(a+b-d)\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(d)_n n!} \int_0^1 x^{n+c-1}(1-x)^{e-c-1}\,dx \\
&\quad = d(a+b-d)\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(d)_n n!} \frac{\Gamma(n+c)\Gamma(e-c)}{\Gamma(n+e)} \\
&\quad = \frac{d(a+b-d)\Gamma(e-c)\Gamma(c)}{\Gamma(e)} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n n!(e)_n} \\
&\quad = \frac{d(a+b-d)\Gamma(e-c)\Gamma(c)}{\Gamma(e)} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right), \qquad (2.8)
\end{aligned}
$$

so (2.4) becomes

$$\begin{aligned}
&\frac{ab(e-c)\Gamma(e-c)\Gamma(c)}{e\Gamma(e)}{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c\\ d+1,\,e+1\end{matrix};1\right)\\
&\quad=\frac{(d-a)(d-b)\Gamma(e-c)\Gamma(c)}{\Gamma(e)}{}_3F_2\left(\begin{matrix}a,\,b,\,c\\ d+1,\,e\end{matrix};1\right)\\
&\qquad+\frac{d(a+b-d)\Gamma(e-c)\Gamma(c)}{\Gamma(e)}{}_3F_2\left(\begin{matrix}a,\,b,\,c\\ d,\,e\end{matrix};1\right).
\end{aligned}\tag{2.9}$$

Dividing both sides of (2.9) by $\Gamma(e-c)\Gamma(c)/\Gamma(e)$ yields (2.2). We use analytic continuation to remove the restrictions $\Re(c)>0$ and $\Re(e-c)>0$ and obtain Proposition 2.1.2. □

Proposition 2.1.3. *If* $\Re(d+e-a-b-c)>0$*, then*

$$ {}_3F_2\left(\begin{matrix}a,\,b,\,c\\ d,\,e\end{matrix};1\right)=\frac{d-b}{d}{}_3F_2\left(\begin{matrix}a+1,\,b,\,c\\ d+1,\,e\end{matrix};1\right)+\frac{b(e-c)}{de}{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c\\ d+1,\,e+1\end{matrix};1\right). \tag{2.10}$$

Proof. Begin with Equation (9) in Ramanathan's paper [15],

$$ {}_2F_1\left(\begin{matrix}a,\,b\\ d\end{matrix};x\right)=\frac{d-b}{d}{}_2F_1\left(\begin{matrix}a+1,\,b\\ d+1\end{matrix};x\right)+\frac{b(1-x)}{d}{}_2F_1\left(\begin{matrix}a+1,\,b+1\\ d+1\end{matrix};x\right). \tag{2.11}$$

As in the proof of the previous proposition, we assume $\Re(c)>0$ and $\Re(e-c)>0$, multiply both sides by $x^{c-1}(1-x)^{e-c-1}$, and then integrate with respect to x from 0 to 1. This yields

$$\begin{aligned}
&\int_0^1 x^{c-1}(1-x)^{e-c-1}{}_2F_1\left(\begin{matrix}a,\,b\\ d\end{matrix};x\right)dx\\
&\quad=\int_0^1\frac{d-b}{d}x^{c-1}(1-x)^{e-c-1}{}_2F_1\left(\begin{matrix}a+1,\,b\\ d+1\end{matrix};x\right)dx\\
&\qquad+\int_0^1\frac{b}{d}x^{c-1}(1-x)^{e-c}{}_2F_1\left(\begin{matrix}a+1,\,b+1\\ d+1\end{matrix};x\right)dx.
\end{aligned}\tag{2.12}$$

We evaluate each of these integrals term-by-term, again making use of the Beta function in the calculations. We find that

$$\begin{aligned}
&\int_0^1 x^{c-1}(1-x)^{e-c-1}{}_2F_1\left(\begin{matrix} a,\, b \\ d \end{matrix}; x\right) dx \\
&\quad = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(d)_n n!} \int_0^1 x^{n+c-1}(1-x)^{e-c-1}\, dx \\
&\quad = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(d)_n n!} \frac{\Gamma(n+c)\Gamma(e-c)}{\Gamma(n+e)} \\
&\quad = \frac{\Gamma(e-c)\Gamma(c)}{\Gamma(e)} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n n!(e)_n} \\
&\quad = \frac{\Gamma(e-c)\Gamma(c)}{\Gamma(e)} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right),
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
&\int_0^1 \frac{d-b}{d} x^{c-1}(1-x)^{e-c-1}{}_2F_1\left(\begin{matrix} a+1,\, b \\ d+1 \end{matrix}; x\right) dx \\
&\quad = \frac{d-b}{d} \sum_{n=0}^{\infty} \frac{(a+1)_n(b)_n}{(d+1)_n n!} \int_0^1 x^{n+c-1}(1-x)^{e-c-1}\, dx \\
&\quad = \frac{d-b}{d} \sum_{n=0}^{\infty} \frac{(a+1)_n(b)_n}{(d+1)_n n!} \frac{\Gamma(n+c)\Gamma(e-c)}{\Gamma(n+e)} \\
&\quad = \frac{(d-b)\Gamma(e-c)\Gamma(c)}{d\Gamma(e)} \sum_{n=0}^{\infty} \frac{(a+1)_n(b)_n(c)_n}{(d+1)_n n!(e)_n} \\
&\quad = \frac{(d-b)\Gamma(e-c)\Gamma(c)}{d\Gamma(e)} {}_3F_2\left(\begin{matrix} a+1,\, b,\, c \\ d+1,\, e \end{matrix}; 1\right),
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
&\int_0^1 \frac{b}{d} x^{c-1}(1-x)^{e-c}{}_2F_1\left(\begin{matrix} a+1,\, b+1 \\ d+1 \end{matrix}; x\right) dx \\
&\quad = \frac{b}{d} \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(d+1)_n n!} \int_0^1 x^{n+c-1}(1-x)^{e-c}\, dx \\
&\quad = \frac{b}{d} \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(d+1)_n n!} \frac{\Gamma(n+c)\Gamma(e-c+1)}{\Gamma(n+e+1)}
\end{aligned}$$

$$= \frac{b\Gamma(e-c+1)\Gamma(c)}{d\Gamma(e+1)} \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n(c)_n}{(d+1)_n n!(e+1)_n}$$

$$= \frac{b(e-c)\Gamma(e-c)\Gamma(c)}{de\Gamma(e)} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right). \tag{2.15}$$

Hence

$$\frac{\Gamma(e-c)\Gamma(c)}{\Gamma(e)} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right) = \frac{(d-b)\Gamma(e-c)\Gamma(c)}{d\Gamma(e)} {}_3F_2\left(\begin{matrix} a+1,\, b,\, c \\ d+1,\, e \end{matrix}; 1\right) + \frac{b(e-c)\Gamma(e-c)\Gamma(c)}{de\Gamma(e)} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right), \tag{2.16}$$

and Proposition 2.1.3 is obtained by dividing both sides of this equation by $\Gamma(e-c)\Gamma(c)/\Gamma(e)$ and using analytic continuation to remove the conditions $\Re(c) > 0$ and $\Re(e-c) > 0$. □

Lemma 2.1.4.

$$d\frac{{}_3F_2\left(\begin{smallmatrix} a,b,c \\ d,e \end{smallmatrix}; 1\right)}{{}_3F_2\left(\begin{smallmatrix} a,b+1,c \\ d+1,e \end{smallmatrix}; 1\right)} = (d-c) + \frac{c(e-a)}{e-b-1+} \frac{(b+1)(d+1-a)}{d-c} + \frac{(c+1)(e+1-a)}{e-b-1} + \frac{(b+2)(d+2-a)}{d-c} + \frac{(c+2)(e+2-a)}{e-b-1} + \cdots$$

$$= (d-c) + \frac{c(e-a)}{e-b-1+} \mathop{\mathbf{K}}_{k=1}^{\infty} \frac{p_k}{q_k}, \tag{2.17}$$

where

$$p_{2k-1} = (b+k)(d+k-a),$$

$$q_{2k-1} = d-c,$$

$$p_{2k} = (c+k)(e+k-a),$$

$$q_{2k} = e-b-1.$$

The proof given here is a formal proof. We will discuss convergence of the identities resulting from this lemma later.

Proof. Simultaneously replace a by b, b by c, and c by a in Proposition 2.1.3 to obtain

$$
{}_3F_2\left(\begin{matrix} b,\, c,\, a \\ d,\, e \end{matrix};1\right) = \frac{d-c}{d}\,{}_3F_2\left(\begin{matrix} b+1,\, c,\, a \\ d+1,\, e \end{matrix};1\right) + \frac{c(e-a)}{de}\,{}_3F_2\left(\begin{matrix} b+1,\, c+1,\, a \\ d+1,\, e+1 \end{matrix};1\right). \tag{2.18}
$$

We rearrange (2.18) into the form

$$
d\frac{{}_3F_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix};1\right)}{{}_3F_2\left(\begin{matrix} a,b+1,c \\ d+1,e \end{matrix};1\right)} = (d-c) + \frac{c(e-a)}{e}\,\frac{{}_3F_2\left(\begin{matrix} a,b+1,c+1 \\ d+1,e+1 \end{matrix};1\right)}{{}_3F_2\left(\begin{matrix} a,b+1,c \\ d+1,e \end{matrix};1\right)}, \tag{2.19}
$$

and iterate this to generate

$$
\begin{aligned}
d\frac{{}_3F_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix};1\right)}{{}_3F_2\left(\begin{matrix} a,b+1,c \\ d+1,e \end{matrix};1\right)} &= (d-c) + \frac{c(e-a)}{1}\,\frac{1}{e\frac{{}_3F_2\left(\begin{matrix} a,b+1,c \\ d+1,e \end{matrix};1\right)}{{}_3F_2\left(\begin{matrix} a,b+1,c+1 \\ d+1,e+1 \end{matrix};1\right)}} \\
&= (d-c) + \frac{c(e-a)}{1}\,\frac{1}{e\frac{{}_3F_2\left(\begin{matrix} a,c,b+1 \\ e,d+1 \end{matrix};1\right)}{{}_3F_2\left(\begin{matrix} a,c+1,b+1 \\ e+1,d+1 \end{matrix};1\right)}} \\
&= (d-c) + \frac{c(e-a)}{1}\,\frac{1}{(e-(b+1)) + \frac{(b+1)((d+1)-a)}{d+1}\,\frac{{}_3F_2\left(\begin{matrix} a,c+1,b+2 \\ e+1,d+2 \end{matrix};1\right)}{{}_3F_2\left(\begin{matrix} a,c+1,b+1 \\ e+1,d+1 \end{matrix};1\right)}} \\
&= (d-c) + \cfrac{c(e-a)}{e-b-1+\cfrac{(b+1)(d+1-a)}{(d+1)\frac{{}_3F_2\left(\begin{matrix} a,b+1,c+1 \\ d+1,e+1 \end{matrix};1\right)}{{}_3F_2\left(\begin{matrix} a,b+2,c+1 \\ d+2,e+1 \end{matrix};1\right)}}} \\
&= (d-c) + \frac{c(e-a)}{e-b-1\,+}\,\frac{(b+1)(d+1-a)}{d-c}\,+\,\frac{(c+1)(e+1-a)}{(e+1)\frac{{}_3F_2\left(\begin{matrix} a,c+1,b+2 \\ e+1,d+2 \end{matrix};1\right)}{{}_3F_2\left(\begin{matrix} a,c+2,b+2 \\ e+2,d+2 \end{matrix};1\right)}} \\
&= (d-c) + \frac{c(e-a)}{e-b-1\,+}\,\frac{(b+1)(d+1-a)}{d-c}\,+\,\frac{(c+1)(e+1-a)}{e-b-1}\,+\cdots.
\end{aligned}
$$

We finally obtain the continued fraction

$$
d\frac{{}_3F_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix};1\right)}{{}_3F_2\left(\begin{matrix} a,b+1,c \\ d+1,e \end{matrix};1\right)} = (d-c) + \frac{c(e-a)}{e-b-1\,+}\,\mathop{\mathbf{K}}_{k=1}^{\infty}\frac{p_k}{q_k}, \tag{2.20}
$$

where

$$
\begin{aligned}
p_{2k-1} &= (b+k)(d+k-a), \\
q_{2k-1} &= d-c, \\
p_{2k} &= (c+k)(e+k-a), \\
q_{2k} &= e-b-1.
\end{aligned}
$$

□

We are now ready to state and prove Entry 34 of Ramanujan's second notebook.

2.2 Proof of Entry 34

Theorem 2.2.1 (Entry 34)**.** *Suppose that n is an odd integer or m is an even integer, or that $\Re(x) > 0$ with m and n arbitrary complex numbers. Define*

$$
P = \frac{\Gamma(\frac{1}{4}(x+m+n+1))\Gamma(\frac{1}{4}(x+m-n+1))\Gamma(\frac{1}{4}(x-m+n+3))\Gamma(\frac{1}{4}(x-m-n+3))}{\Gamma(\frac{1}{4}(x-m+n+1))\Gamma(\frac{1}{4}(x-m-n+1))\Gamma(\frac{1}{4}(x+m+n+3))\Gamma(\frac{1}{4}(x+m-n+3))}.
$$

Then

$$
\frac{1-P}{1+P} = \frac{m}{x+}\,\frac{1^2-n^2}{x}\,\frac{}{+}\,\frac{2^2-m^2}{x}\,\frac{}{+}\,\frac{3^2-n^2}{x}\,\frac{}{+}\,\frac{4^2-m^2}{x}\,\frac{}{+\cdots}.
$$

Proof. First, we solve (2.2) for ${}_3F_2\left({a,b,c \atop d,e}; 1\right)$. This yields

$$
\begin{aligned}
{}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right) &= \frac{ab(e-c)}{de(a+b-d)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right) \\
&\quad - \frac{(d-a)(d-b)}{d(a+b-d)}\,{}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d+1,\, e \end{matrix}; 1\right),
\end{aligned} \tag{2.21}
$$

which can be combined with (2.10) to find that

$$
\begin{aligned}
&{}_3F_2\left({a,\, b,\, c \atop d,\, e};1\right)\\
&\quad= \frac{d-b}{d}\,{}_3F_2\left({a+1,\, b,\, c \atop d+1,\, e};1\right) + \frac{b(e-c)}{de}\,{}_3F_2\left({a+1,\, b+1,\, c \atop d+1,\, e+1};1\right)\\
&\quad= \frac{ab(e-c)}{de(a+b-d)}\,{}_3F_2\left({a+1,\, b+1,\, c \atop d+1,\, e+1};1\right) - \frac{(d-a)(d-b)}{d(a+b-d)}\,{}_3F_2\left({a,\, b,\, c \atop d+1,\, e};1\right).
\end{aligned}
\tag{2.22}
$$

Hence

$$
\begin{aligned}
\frac{d-b}{d}\,{}_3F_2\left({a+1,\, b,\, c \atop d+1,\, e};1\right) &+ \frac{(d-a)(d-b)}{d(a+b-d)}\,{}_3F_2\left({a,\, b,\, c \atop d+1,\, e};1\right)\\
&= \left[\frac{ab(e-c)}{de(a+b-d)} - \frac{b(e-c)}{de}\right]{}_3F_2\left({a+1,\, b+1,\, c \atop d+1,\, e+1};1\right)\\
&= \frac{b(e-c)(d-b)}{de(a+b-d)}\,{}_3F_2\left({a+1,\, b+1,\, c \atop d+1,\, e+1};1\right).
\end{aligned}
\tag{2.23}
$$

Divide (2.23) by $(d-b)/d$, exchange a and b, and isolate ${}_3F_2\left({a,b+1,c \atop d+1,e};1\right)$ to see that

$$
\begin{aligned}
{}_3F_2\left({a,\, b+1,\, c \atop d+1,\, e};1\right) = &\frac{a(e-c)}{e(a+b-d)}\,{}_3F_2\left({a+1,\, b+1,\, c \atop d+1,\, e+1};1\right)\\
&- \frac{d-b}{a+b-d}\,{}_3F_2\left({a,\, b,\, c \atop d+1,\, e};1\right).
\end{aligned}
\tag{2.24}
$$

We now let $d = a - b$ and $e = a - c + 1$ in (2.24) to obtain

$$
\begin{aligned}
{}_3F_2\left({a,\, b+1,\, c \atop a-b+1,\, a-c+1};1\right) = &\frac{a(a-2c+1)}{(a-c+1)(2b)}\,{}_3F_2\left({a+1,\, b+1,\, c \atop a-b+1,\, a-c+2};1\right)\\
&- \frac{a-2b}{2b}\,{}_3F_2\left({a,\, b,\, c \atop a-b+1,\, a-c+1};1\right).
\end{aligned}
\tag{2.25}
$$

When we replace a by $a+1$ and b by $b+1$ in Dixon's Identity (2.1), we find that

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix};1\right) \\
&\quad = \frac{\Gamma(a-b+1)\Gamma(a-c+2)\Gamma(\frac{1}{2}(a+1)+1)\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c+1)\Gamma(a+2)\Gamma(a-b-c+1)},
\end{aligned}
\tag{2.26}
$$

so (2.25) becomes

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b+1,\, a-c+1 \end{matrix};1\right) \\
&= \frac{a(a-2c+1)\Gamma(a-b+1)\Gamma(a-c+2)\Gamma(\frac{1}{2}(a+1)+1)\Gamma(\frac{1}{2}(a+1)-b-c)}{(a-c+1)(2b)\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c+1)\Gamma(a+2)\Gamma(a-b-c+1)} \\
&\quad - \frac{(a-2b)\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}a-b-c+1)}{2b\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)\Gamma(a-b-c+1)} \\
&= \frac{a(a-2c+1)\Gamma(a-b+1)(a-c+1)\Gamma(a-c+1)\frac{1}{2}(a+1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{(a-c+1)(2b)\Gamma(\frac{1}{2}(a+1)-b)(\frac{1}{2}(a+1)-c)\Gamma(\frac{1}{2}(a+1)-c)(a+1)a\Gamma(a)\Gamma(a-b-c+1)} \\
&\quad - \frac{(a-2b)\Gamma(a-b+1)\Gamma(a-c+1)\frac{1}{2}a\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{2b(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)a\Gamma(a)\Gamma(a-b-c+1)} \\
&= \frac{(a-2c+1)\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{2b\Gamma(\frac{1}{2}(a+1)-b)(a+1-2c)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a)\Gamma(a-b-c+1)} \\
&\quad - \frac{(a-2b)\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{2b(a-2b)\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)\Gamma(a)\Gamma(a-b-c+1)} \\
&= \frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{2b\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a)\Gamma(a-b-c+1)} \\
&\quad - \frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{2b\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)\Gamma(a)\Gamma(a-b-c+1)} \\
&= \frac{\Gamma(a-b+1)\Gamma(a-c+1)}{2b\Gamma(a)\Gamma(a-b-c+1)}\left[\frac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)} - \frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)}\right] \\
&= \frac{\Gamma(a-b+1)\Gamma(a-c+1)}{2b\Gamma(a)\Gamma(a-b-c+1)}\left[S-T\right],
\end{aligned}
\tag{2.27}
$$

where

$$S = \frac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)},$$
$$T = \frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)}.$$

Similarly, under $d = a - b$ and $e = a - c + 1$, (2.21) becomes

$$\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right) \\
&= \frac{ab(a-2c+1)}{(a-b)(a-c+1)(2b)}\, {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix}; 1\right) \\
&\quad - \frac{(-b)(a-2b)}{(a-b)(2b)}\, {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ a-b+1,\, a-c+1 \end{matrix}; 1\right) \\
&= \frac{a(a-2c+1)\Gamma(a-b+1)\Gamma(a-c+2)\Gamma(\frac{1}{2}(a+1)+1)\Gamma(\frac{1}{2}(a+1)-b-c)}{2(a-b)(a-c+1)\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c+1)\Gamma(a+2)\Gamma(a-b-c+1)} \\
&\quad + \frac{(a-2b)\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}a-b-c+1)}{2(a-b)\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)\Gamma(a-b-c+1)} \\
&= \frac{2a(\frac{1}{2}(a+1)-c)\Gamma(a-b+1)(a-c+1)\Gamma(a-c+1)}{2(a-b)(a-c+1)\Gamma(\frac{1}{2}(a+1)-b)(\frac{1}{2}(a+1)-c)\Gamma(\frac{1}{2}(a+1)-c)} \times \\
&\qquad \frac{\frac{1}{2}(a+1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{(a+1)a\Gamma(a)\Gamma(a-b-c+1)} \\
&\quad + \frac{2(\frac{1}{2}a-b)\Gamma(a-b+1)\Gamma(a-c+1)\frac{1}{2}a\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{2(a-b)(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)a\Gamma(a)\Gamma(a-b-c+1)} \\
&= \frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{2(a-b)\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a)\Gamma(a-b-c+1)} \\
&\quad + \frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{2(a-b)\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)\Gamma(a)\Gamma(a-b-c+1)} \\
&= \frac{\Gamma(a-b+1)\Gamma(a-c+1)}{2(a-b)\Gamma(a)\Gamma(a-b-c+1)} \times \\
&\qquad \left[\frac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)} + \frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)}\right] \\
&= \frac{\Gamma(a-b+1)\Gamma(a-c+1)}{2(a-b)\Gamma(a)\Gamma(a-b-c+1)}\left[S+T\right]. \qquad (2.28)
\end{aligned}$$

Finally, using (2.27) and (2.28), we find that

$$(a-b)\frac{{}_3F_2\left({a,b,c \atop a-b,a-c+1};1\right)}{{}_3F_2\left({a,b+1,c \atop a-b+1,a-c+1};1\right)} = \frac{(a-b)\frac{\Gamma(a-b+1)\Gamma(a-c+1)}{2(a-b)\Gamma(a)\Gamma(a-b-c+1)}[S+T]}{\frac{\Gamma(a-b+1)\Gamma(a-c+1)}{2b\Gamma(a)\Gamma(a-b-c+1)}[S-T]}$$
$$= \frac{b(S+T)}{S-T}$$
$$= b\frac{1+R}{1-R}, \tag{2.29}$$

where

$$R = \frac{T}{S} = \frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}.$$

Note that if $a = \frac{1}{2}x+b+c$, $b = \frac{1}{2}m$, and $c = \frac{1}{2}(n+1)$,

$$R = \frac{\Gamma(\frac{1}{4}(x+m+n+1))\Gamma(\frac{1}{4}(x+m-n+1))\Gamma(\frac{1}{4}(x-m+n+3))\Gamma(\frac{1}{4}(x-m-n+3))}{\Gamma(\frac{1}{4}(x-m+n+1))\Gamma(\frac{1}{4}(x-m-n+1))\Gamma(\frac{1}{4}(x+m+n+3))\Gamma(\frac{1}{4}(x+m-n+3))}$$
$$= P.$$

Substituting $d = a-b$ and $e = a-c+1$ into (2.17), we find that

$$(a-b)\frac{{}_3F_2\left({a,b,c \atop a-b,a-c+1};1\right)}{{}_3F_2\left({a,b+1,c \atop a-b+1,a-c+1};1\right)} = (a-b-c) + \frac{c(1-c)}{a-b-c+} \mathop{\mathbf{K}}_{k=1}^{\infty} \frac{r_k}{s_k}, \tag{2.30}$$

where

$$r_{2k-1} = (k+b)(k-b),$$
$$s_{2k-1} = a-b-c,$$
$$r_{2k} = (k+c)(k+1-c),$$
$$s_{2k} = a-b-c.$$

Again letting $a = \frac{1}{2}x + b + c$, $b = \frac{1}{2}m$, $c = \frac{1}{2}(n+1)$, we see that

$$\begin{aligned}
s_{2k-1} = s_{2k} &= \frac{1}{2}x, \\
r_{2k-1} &= \left(k + \frac{1}{2}m\right)\left(k - \frac{1}{2}m\right) \\
&= \frac{1}{4}\left((2k)^2 - m^2\right), \\
r_{2k} &= \left(k + \frac{1}{2}(n+1)\right)\left(k + 1 - \frac{1}{2}(n+1)\right) \\
&= \frac{1}{4}\left((2k+1)^2 - n^2\right),
\end{aligned}$$

so the continued fraction in (2.30) can be written as

$$\begin{aligned}
&(a-b-c) + \frac{c(1-c)}{a-b-c+} \mathop{\mathbf{K}}_{k=1}^{\infty} \frac{r_k}{s_k} \\
&= \frac{x}{2} + \frac{\frac{1}{4}(1^2-n^2)}{x/2\ +} \frac{\frac{1}{4}(2^2-m^2)}{x/2\ +} \frac{\frac{1}{4}(3^2-n^2)}{x/2\ +} \frac{\frac{1}{4}(4^2-m^2)}{x/2\ +} \cdots \\
&= \frac{x}{2} + \frac{\frac{1}{2}(1^2-n^2)}{x\ +} \frac{2^2-m^2}{x\ +} \frac{3^2-n^2}{x\ +} \frac{4^2-m^2}{x\ +} \cdots.
\end{aligned} \tag{2.31}$$

Combining (2.29), (2.30), and (2.31), we find that

$$\frac{m}{2}\frac{1+P}{1-P} = \frac{x}{2} + \frac{\frac{1}{2}(1^2-n^2)}{x\ +} \frac{2^2-m^2}{x\ +} \frac{3^2-n^2}{x\ +} \frac{4^2-m^2}{x\ +} \cdots. \tag{2.32}$$

Multiplying both sides of (2.32) by 2 yields

$$m\frac{1+P}{1-P} = x + \frac{1^2-n^2}{x\ +} \frac{2^2-m^2}{x\ +} \frac{3^2-n^2}{x\ +} \frac{4^2-m^2}{x\ +} \cdots. \tag{2.33}$$

Taking reciprocals of both sides and then multiplying by m, we obtain

$$\frac{1-P}{1+P} = \frac{m}{x\ +} \frac{1^2-n^2}{x\ +} \frac{2^2-m^2}{x\ +} \frac{3^2-n^2}{x\ +} \frac{4^2-m^2}{x\ +} \cdots, \tag{2.34}$$

which is Entry 34. The convergence of the continued fraction to $(1-P)/(1+P)$ is discussed by Jacobsen in [10]. $\square$

Chapter 3

Identities related to Entries 34 and 25

In obtaining Entry 34, we used an initial substitution of $d = a - b$, $e = a - c + 1$ in the main continued fraction (2.17). Starting with the substitution $d = a - b - 1$, $e = a - c$ yields a similar identity, given here as Theorem 3.1.1. Theorem 3.1.1 has a corollary that is interesting in that it appears to be a companion to Entry 25. This corollary, Theorem 3.2.1, is discussed and proved in the second section of this chapter.

3.1 An identity related to Entry 34

Theorem 3.1.1. *Suppose that $m \equiv 0 (mod\ 4)$ or $n \equiv 2 (mod\ 4)$, or that $\Re(x) > 0$ with m and n arbitrary complex numbers. Define*

$$P = \frac{\Gamma(\frac{1}{8}(x+m+n+2))\Gamma(\frac{1}{8}(x+m-n+2))\Gamma(\frac{1}{8}(x-m+n+6))\Gamma(\frac{1}{8}(x-m-n+6))}{\Gamma(\frac{1}{8}(x-m+n+2))\Gamma(\frac{1}{8}(x-m-n+2))\Gamma(\frac{1}{8}(x+m+n+6))\Gamma(\frac{1}{8}(x+m-n+6))}.$$

Then

$$\frac{1-P}{1+P} = \frac{m}{x+}\,\frac{2^2-n^2}{x}\,\underset{+}{}\,\frac{4^2-m^2}{x}\,\underset{+}{}\,\frac{6^2-n^2}{x}\,\underset{+}{}\,\frac{8^2-m^2}{x}\,\underset{+}{}\cdots.$$

We first prove this identity without regard for its region of convergence, and then prove convergence later.

Proof. As in the proof of Entry 34, we use the relations (2.2) and (2.10) and the continued fraction (2.17). In (2.2), we exchange b and c and then d and e, and then replace b by $b + 1$

and d by $d+1$ and rearrange to obtain

$$
{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right) = \frac{ac(d-b)}{(d+1)e(a+c-e)}\, {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c+1 \\ d+2,\, e+1 \end{matrix}; 1\right) - \frac{(e-a)(e-c)}{e(a+c-e)}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right). \tag{3.1}
$$

In (2.10), we exchange a and b to obtain

$$
{}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right) = \frac{d-a}{d}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right) + \frac{a(e-c)}{de}\, {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right). \tag{3.2}
$$

Also in (2.10), we exchange b and c and then d and e, and then replace b by $b+1$ and d by $d+1$ and rearrange to see that

$$
{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right) = \frac{e}{e-c}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right) - \frac{c(d-b)}{(d+1)(e-c)}\, {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c+1 \\ d+2,\, e+1 \end{matrix}; 1\right). \tag{3.3}
$$

We substitute (3.3) into (3.2) to obtain

$$
\begin{aligned}
{}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right) &= \frac{d-a}{d}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right) \\
&\quad + \frac{a(e-c)}{de}\left[\frac{e}{e-c}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right) \right. \\
&\quad \left. - \frac{c(d-b)}{(d+1)(e-c)}\, {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c+1 \\ d+2,\, e+1 \end{matrix}; 1\right)\right] \\
&= \frac{d-a}{d}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right) + \frac{a}{d}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right) \\
&\quad - \frac{ac(d-b)}{de(d+1)}\, {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c+1 \\ d+2,\, e+1 \end{matrix}; 1\right) \\
&= {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right) - \frac{ac(d-b)}{de(d+1)}\, {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c+1 \\ d+2,\, e+1 \end{matrix}; 1\right),
\end{aligned} \tag{3.4}
$$

and then substitute (3.1) into (3.4) to eliminate ${}_3F_2\left(\begin{smallmatrix}a,b+1,c\\d+1,e\end{smallmatrix};1\right)$. This yields

$$\begin{aligned}
{}_3F_2\left(\begin{matrix}a,\,b,\,c\\d,\,e\end{matrix};1\right) &= \frac{ac(d-b)}{(d+1)e(a+c-e)}{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\d+2,\,e+1\end{matrix};1\right)\\
&\quad-\frac{(e-a)(e-c)}{e(a+c-e)}{}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\d+1,\,e+1\end{matrix};1\right)\\
&\quad-\frac{ac(d-b)}{de(d+1)}{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\d+2,\,e+1\end{matrix};1\right)\\
&=\left[\frac{ac(d-b)}{(d+1)e(a+c-e)}-\frac{ac(d-b)}{de(d+1)}\right]{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\d+2,\,e+1\end{matrix};1\right)\\
&\quad-\frac{(e-a)(e-c)}{e(a+c-e)}{}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\d+1,\,e+1\end{matrix};1\right)\\
&=\frac{ac(d-b)}{e(d+1)}\left[\frac{1}{(a+c-e)}-\frac{1}{d}\right]{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\d+2,\,e+1\end{matrix};1\right)\\
&\quad-\frac{(e-a)(e-c)}{e(a+c-e)}{}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\d+1,\,e+1\end{matrix};1\right)\\
&=\frac{ac(d-b)}{e(d+1)}\left[\frac{d+e-a-c}{d(a+c-e)}\right]{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\d+2,\,e+1\end{matrix};1\right)\\
&\quad-\frac{(e-a)(e-c)}{e(a+c-e)}{}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\d+1,\,e+1\end{matrix};1\right)\\
&=\frac{ac(d-b)(d+e-a-c)}{de(d+1)(a+c-e)}{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\d+2,\,e+1\end{matrix};1\right)\\
&\quad-\frac{(e-a)(e-c)}{e(a+c-e)}{}_3F_2\left(\begin{matrix}a,\,b+1,\,c\\d+1,\,e+1\end{matrix};1\right). \qquad (3.5)
\end{aligned}$$

We let $d=a-b-1$ and $e=a-c$, so (2.17) becomes

$$(a-b-1)\frac{{}_3F_2\left(\begin{smallmatrix}a,b,c\\a-b-1,a-c\end{smallmatrix};1\right)}{{}_3F_2\left(\begin{smallmatrix}a,b+1,c\\a-b,a-c\end{smallmatrix};1\right)}=(a-b-c-1)-\frac{c^2}{a-b-c-1+\underset{k=1}{\overset{\infty}{\mathbf{K}}}\frac{r_k}{s_k}}, \qquad (3.6)$$

where

$$r_{2k-1}=(k+b)(k-1-b),$$

$$s_{2k-1}=a-b-c-1,$$

$$r_{2k} = (k+c)(k-c),$$

$$s_{2k} = a-b-c-1.$$

With this substitution, (3.5) becomes

$$\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b,\, c \\ a-b-1,\, a-c \end{matrix}; 1\right) \\
&\quad = \frac{a(a-2b-1)(a-b-2c-1)}{2(a-b-1)(a-c)(a-b)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c+1 \\ a-b+1,\, a-c+1 \end{matrix}; 1\right) \\
&\qquad + \frac{(a-2c)}{2(a-c)}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right),
\end{aligned} \tag{3.7}$$

and (3.1) becomes

$$\begin{aligned}
{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c \end{matrix}; 1\right) = &\frac{a(a-2b-1)}{2(a-b)(a-c)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c+1 \\ a-b+1,\, a-c+1 \end{matrix}; 1\right) \\
&+ \frac{(a-2c)}{2(a-c)}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right).
\end{aligned} \tag{3.8}$$

Note that both ${}_3F_2\left(\begin{smallmatrix} a+1,\, b+1,\, c+1 \\ a-b+1,\, a-c+1 \end{smallmatrix}; 1\right)$ and ${}_3F_2\left(\begin{smallmatrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{smallmatrix}; 1\right)$ can be evaluated using Dixon's Identity (2.1). Thus

$$\begin{aligned}
&\frac{(a-2c)}{2(a-c)}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right) \\
&\quad = \frac{(a-2c)\Gamma(a-b)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}a-b-c)}{2(a-c)\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)\Gamma(a-b-c)} \\
&\quad = \frac{\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{2\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)\Gamma(a)\Gamma(a-b-c)},
\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
&\frac{a(a-2b-1)(a-b-2c-1)}{2(a-b-1)(a-c)(a-b)}{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\a-b+1,\,a-c+1\end{matrix};1\right)\\
&\quad=\frac{a(a-2b-1)(a-b-2c-1)}{2(a-b-1)(a-c)(a-b)}\times\\
&\qquad\quad\frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}(a+1)+1)\Gamma(\frac{1}{2}(a+1)-b-c-1)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a+2)\Gamma(a-b-c)}\\
&\quad=\frac{(a-b-2c-1)\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}{2(a-b-1)\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a)\Gamma(a-b-c)},
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
&\frac{a(a-2b-1)}{2(a-b)(a-c)}{}_3F_2\left(\begin{matrix}a+1,\,b+1,\,c+1\\a-b+1,\,a-c+1\end{matrix};1\right)\\
&\quad=\frac{a(a-2b-1)\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}(a+1)+1)\Gamma(\frac{1}{2}(a+1)-b-c-1)}{2(a-b)(a-c)\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a+2)\Gamma(a-b-c)}\\
&\quad=\frac{\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}{2\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a)\Gamma(a-b-c)}.
\end{aligned}
\tag{3.11}
$$

Hence we have, after combining the last five equations,

$$
\begin{aligned}
&(a-b-1)\frac{{}_3F_2\left(\begin{matrix}a,b,c\\a-b-1,a-c\end{matrix};1\right)}{{}_3F_2\left(\begin{matrix}a,b+1,c\\a-b,a-c\end{matrix};1\right)}\\
&=(a-b-1)\frac{\frac{a(a-2b-1)(a-b-2c-1)}{2(a-b-1)(a-c)(a-b)}{}_3F_2\left(\begin{matrix}a+1,b+1,c+1\\a-b+1,a-c+1\end{matrix};1\right)+\frac{(a-2c)}{2(a-c)}{}_3F_2\left(\begin{matrix}a,b+1,c\\a-b,a-c+1\end{matrix};1\right)}{\frac{a(a-2b-1)}{2(a-b)(a-c)}{}_3F_2\left(\begin{matrix}a+1,b+1,c+1\\a-b+1,a-c+1\end{matrix};1\right)+\frac{(a-2c)}{2(a-c)}{}_3F_2\left(\begin{matrix}a,b+1,c\\a-b,a-c+1\end{matrix};1\right)}\\
&=(a-b-1)\frac{\frac{(a-b-2c-1)\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}{2(a-b-1)\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a)\Gamma(a-b-c)}+\frac{\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{2\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)\Gamma(a)\Gamma(a-b-c)}}{\frac{\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}{2\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a)\Gamma(a-b-c)}+\frac{\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{2\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)\Gamma(a)\Gamma(a-b-c)}}\\
&=(a-b-1)\frac{\frac{(a-b-2c-1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}{(a-b-1)\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)}+\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)}}{\frac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}{\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)}+\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)}}\\
&=(a-b-1)\frac{\frac{a-b-2c-1}{a-b-1}+\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}}{1+\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}}
\end{aligned}
$$

$$= (a-b-1)\frac{\frac{a-b-2c-1}{a-b-1}+R}{1+R}$$
$$= (a-b-1)\left[-\frac{\frac{2c}{a-b-1}}{1+R}+1\right]$$
$$= -\frac{2c}{1+R}+(a-b-1)$$
$$= (a-b-c-1)-c\frac{1-R}{1+R}, \tag{3.12}$$

where

$$R = \frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)\Gamma(\frac{1}{2}(a+1)-b-1)\Gamma(\frac{1}{2}(a+1)-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c-1)}.$$

Thus (3.6) becomes

$$(a-b-c-1)-c\frac{1-R}{1+R} = (a-b-c-1)-\frac{c^2}{a-b-c-1+\underset{k=1}{\overset{\infty}{\mathbf{K}}}\frac{r_k}{s_k}},$$

or

$$\frac{1-R}{1+R} = \frac{c}{a-b-c-1+\underset{k=1}{\overset{\infty}{\mathbf{K}}}\frac{r_k}{s_k}}, \tag{3.13}$$

where

$$r_{2k-1} = (k+b)(k-1-b),$$
$$s_{2k-1} = a-b-c-1,$$
$$r_{2k} = (k+c)(k-c),$$
$$s_{2k} = a-b-c-1.$$

Now let $a = \frac{1}{4}x+b+c+1$, $b = \frac{1}{4}(n-2)$, and $c = \frac{1}{4}m$. We find that

$$R = \frac{\Gamma(\frac{1}{8}(x+m+n+2))\Gamma(\frac{1}{8}(x-m-n+6))\Gamma(\frac{1}{8}(x+m-n+2))\Gamma(\frac{1}{8}(x-m+n+6))}{\Gamma(\frac{1}{8}(x+m-n+6))\Gamma(\frac{1}{8}(x-m+n+2))\Gamma(\frac{1}{8}(x+m+n+6))\Gamma(\frac{1}{8}(x-m-n+2))}$$
$$= P.$$

Also,

$$
\begin{aligned}
s_{2k-1} = s_{2k} &= \left(\frac{1}{4}x + b + c + 1\right) - b - c - 1 \\
&= \frac{1}{4}x, \\
r_{2k-1} &= \left(k + \frac{1}{4}(n-2)\right)\left(k - 1 - \frac{1}{4}(n-2)\right) \\
&= \frac{1}{16}\left(4k + n - 2\right)\left(4k - 4 - (n-2)\right) \\
&= \frac{1}{16}\left((4k-2)^2 - n^2\right), \\
r_{2k} &= \left(k + \frac{1}{4}m\right)\left(k - \frac{1}{4}m\right) \\
&= \frac{1}{16}\left((4k)^2 - m^2\right),
\end{aligned}
$$

so

$$
\begin{aligned}
\frac{1-P}{1+P} &= \frac{\frac{1}{4}m}{x/4+}\,\frac{\frac{1}{16}(2^2-n^2)}{x/4}\,+\,\frac{\frac{1}{16}(4^2-m^2)}{x/4}\,+\,\frac{\frac{1}{16}(6^2-n^2)}{x/4}\,+\,\frac{\frac{1}{16}(8^2-m^2)}{x/4}\,+\cdots \\
&= \frac{m}{x+}\,\frac{2^2-n^2}{x}\,+\,\frac{4^2-m^2}{x}\,+\,\frac{6^2-n^2}{x}\,+\,\frac{8^2-m^2}{x}\,+\cdots,
\end{aligned}
$$

as desired. □

The domain of validity claimed for Theorem 3.1.1 is that $m \equiv 0 (\mathrm{mod}\ 4)$ or $n \equiv 2 (\mathrm{mod}\ 4)$, or that $\Re(x) > 0$ with m and n arbitrary complex numbers.

Proof of convergence. If either $m \equiv 0 (\mathrm{mod}\ 4)$ or $n \equiv 2 (\mathrm{mod}\ 4)$, the continued fraction terminates, so it converges. In this case, there are only finitely many iterations in the repeated division process of Lemma 2.1.4, so the above proof shows that the continued fraction converges to the function given on the left side of the identity.

Now suppose that m and n are arbitrary complex numbers. We apply Theorem 1.2.1 to the continued fraction

$$F(x,m,n) = \mathop{\mathbf{K}}_{j=1}^{\infty} \frac{a_j}{b_j} = \frac{2^2-n^2}{x} \genfrac{}{}{0pt}{}{}{+} \frac{4^2-m^2}{x} \genfrac{}{}{0pt}{}{}{+} \frac{6^2-n^2}{x} \genfrac{}{}{0pt}{}{}{+} \frac{8^2-m^2}{x} \genfrac{}{}{0pt}{}{}{+\cdots}.$$

We have $a_{2k-1} = 16k^2-16k+4-n^2$, $a_{2k} = 16k^2-m^2$, and $b_{2k-1} = b_{2k} = x$, so $p = 2$, $q = r = 0$, $\alpha_2 = \gamma_2$, $\alpha_1 \neq \gamma_1$, and $\widetilde{D} = \{x \in \mathbb{C} : x^2 \neq 0 \text{ and } m \not\equiv 0(\text{mod } 4) \text{ and } n \not\equiv 2(\text{mod } 4)\}$. Thus $q + r = p - 2$, so Case (iv) of Theorem 1.2.1 applies. This means that

$$\begin{aligned} D &= \left\{x \in \widetilde{D} : \frac{4(x)(x)}{16} + \left(0+1+\frac{-16-0}{16}\right)^2 \notin (-\infty,0]\right\} \\ &= \left\{x \in \widetilde{D} : \frac{x^2}{4} + 0 \notin (-\infty,0]\right\} \\ &= \{x \in \mathbb{C} : \Re(x) \neq 0 \text{ and } m \not\equiv 0(\text{mod } 4) \text{ and } n \not\equiv 2(\text{mod } 4)\}. \end{aligned}$$

By Theorem 1.2.1, $F(x,m,n)$ converges in D to a function that is meromorphic or identically ∞ in each component of D. Hence the continued fraction of Theorem 3.1.1 also converges to a function f, meromorphic or identically ∞ in each component of D. This function f coincides with $(1-P)/(1+P)$ for $x > 0$, so by analytic continuation the continued fraction of this theorem converges to $(1-P)/(1+P)$ for $\Re(x) > 0$. Note that the continued fraction is an odd function of x, since

$$\begin{aligned} &\frac{m}{-x} \genfrac{}{}{0pt}{}{}{+} \frac{2^2-n^2}{-x} \genfrac{}{}{0pt}{}{}{+} \frac{4^2-m^2}{-x} \genfrac{}{}{0pt}{}{}{+} \frac{6^2-n^2}{-x} \genfrac{}{}{0pt}{}{}{+} \frac{8^2-m^2}{-x} \genfrac{}{}{0pt}{}{}{+\cdots} \\ &\qquad = \frac{-m}{x} \genfrac{}{}{0pt}{}{}{+} \frac{-(2^2-n^2)}{-x} \genfrac{}{}{0pt}{}{}{+} \frac{4^2-m^2}{-x} \genfrac{}{}{0pt}{}{}{+} \frac{6^2-n^2}{-x} \genfrac{}{}{0pt}{}{}{+} \frac{8^2-m^2}{-x} \genfrac{}{}{0pt}{}{}{+\cdots} \\ &\qquad = \cdots \\ &\qquad = -\frac{m}{x} \genfrac{}{}{0pt}{}{}{+} \frac{2^2-n^2}{x} \genfrac{}{}{0pt}{}{}{+} \frac{4^2-m^2}{x} \genfrac{}{}{0pt}{}{}{+} \frac{6^2-n^2}{x} \genfrac{}{}{0pt}{}{}{+} \frac{8^2-m^2}{x} \genfrac{}{}{0pt}{}{}{+\cdots}. \end{aligned}$$

Since $(1-P)/(1+P)$ is not an odd function of x, the continued fraction does not converge to $(1-P)/(1+P)$ when $x<0$, so the identity is not valid for $\Re(x)<0$. $\square$

Theorem 3.1.1 can actually be shown to be equivalent to Entry 34 by replacing n by $2n$, m by $2m$, and x by $2x$ in Theorem 3.1.1 and then simplifying. However, the format given in Theorem 3.1.1 makes the corollary, Theorem 3.2.1, more obvious. The substitutions $d=a-b+1$, $e=a-c+2$ and $d=a-b-2$, $e=a-c-1$ in the main continued fraction (2.17), followed by an appropriate second substitution, also give identities that are equivalent to Entry 34. This suggests the following conjecture.

Conjecture 3.1.2. *An initial substitution of the form $d=a-b+l$, $e=a-c+l+1$, where l is an integer, followed by the substitution $a=\frac{1}{2}(x+m+n+1)$, $b=\frac{1}{2}(m+l)$, $c=\frac{1}{2}(n+l+1)$, always gives an identity equivalent to Entry 34.*

3.2 A Corollary of Theorem 3.1.1

Theorem 3.2.1. *Suppose that N is an even integer, or that $\Re(x)>0$ with N an arbitrary complex number. Define*

$$R=\frac{\Gamma(\frac{1}{4}(x+N+1))\Gamma(\frac{1}{4}(x-N+3))}{\Gamma(\frac{1}{4}(x+N+3))\Gamma(\frac{1}{4}(x-N+1))}.$$

Then

$$\frac{1-R}{1+R}=\frac{N}{2x+}\,\frac{2^2-N^2}{2x}\;{+}\;\frac{4^2-N^2}{2x}\;{+}\;\frac{6^2-N^2}{2x}\;{+}\;\frac{8^2-N^2}{2x}\;{+\cdots}.$$

This theorem is a nice companion to Ramanujan's Entry 25. It contains the same Gamma functions in a different configuration, and the numerators of the continued fraction are of a similar form: $(2k)^2-N^2$ instead of $(2k-1)-N^2$. Two proofs are provided; one shows this theorem to be an easy corollary of Theorem 3.1.1, while the other proves it directly using the method Ramanathan used in [15] to prove Entry 25.

Proof 1. In Theorem 3.1.1, replace x by $2x$ and let $m = n = N$. Theorem 3.2.1 follows immediately as a corollary. $\square$

Proof 2. The continued fraction used in this proof is Entry 22 from Ramanujan's second Notebook [4],

$$\frac{\beta x}{\gamma}\frac{{}_2F_1\left(\begin{smallmatrix}-\alpha,\beta+1\\ \gamma+1\end{smallmatrix};-x\right)}{{}_2F_1\left(\begin{smallmatrix}-\alpha,\beta\\ \gamma\end{smallmatrix};-x\right)} = \frac{\beta x}{\gamma-(\alpha+\beta+1)x+}\,\frac{(\beta+1)(\alpha+\gamma+1)x}{\gamma+1-(\alpha+\beta+2)x+}\,\frac{(\beta+2)(\alpha+\gamma+2)x}{\gamma+2-(\alpha+\beta+3)x+\cdots}. \tag{3.14}$$

We let $\alpha = -a$, $\beta = b$, $\gamma = a - b + 2$, and $x = 1$ in (3.14). This yields

$$\frac{{}_2F_1\left(\begin{smallmatrix}a,b+1\\ a-b+3\end{smallmatrix};-1\right)}{{}_2F_1\left(\begin{smallmatrix}a,b\\ a-b+2\end{smallmatrix};-1\right)} = \frac{a-b+2}{2a-2b+1+}\,\frac{(b+1)(3-b)}{2a-2b+1}\,{+}\,\frac{(b+2)(4-b)}{2a-2b+1}\,{+}\,\frac{(b+3)(5-b)}{2a-2b+1}\,{+}\cdots. \tag{3.15}$$

Note that

$$(b+k)(k+2-b) = \frac{1}{4}\left((2k+2)^2-(2b-2)^2\right),$$

so (3.15) can be written as

$$\frac{{}_2F_1\left(\begin{smallmatrix}a,b+1\\ a-b+3\end{smallmatrix};-1\right)}{{}_2F_1\left(\begin{smallmatrix}a,b\\ a-b+2\end{smallmatrix};-1\right)} = \frac{a-b+2}{2a-2b+1+}\,\frac{\frac{1}{4}(4^2-(2b-2)^2)}{2a-2b+1}\,{+}\,\frac{\frac{1}{4}(6^2-(2b-2)^2)}{2a-2b+1}\,{+}\,\frac{\frac{1}{4}(8^2-(2b-2)^2)}{2a-2b+1}\,{+}\cdots. \tag{3.16}$$

We will evaluate ${}_2F_1\left(\begin{matrix}a,b+1\\ a-b+3\end{matrix};-1\right)$ and ${}_2F_1\left(\begin{matrix}a,b\\ a-b+2\end{matrix};-1\right)$ by rewriting them in a form that allows the use of Kummer's Identity, which is stated as Corollary 3.1.2 in [2]:

$${}_2F_1\left(\begin{matrix}a,b\\ a-b+1\end{matrix};-1\right) = \frac{\Gamma(a-b+1)\Gamma(\frac{1}{2}a+1)}{\Gamma(\frac{1}{2}a-b+1)\Gamma(a+1)}. \tag{3.17}$$

To do this, we use the relations

$$ {}_2F_1\left(\begin{matrix}\alpha+1,\,\beta+1\\ \gamma+1\end{matrix};x\right)=\frac{\gamma}{\beta(1-x)}{}_2F_1\left(\begin{matrix}\alpha,\,\beta\\ \gamma\end{matrix};x\right)-\frac{\gamma-\beta}{\beta(1-x)}{}_2F_1\left(\begin{matrix}\alpha+1,\,\beta\\ \gamma+1\end{matrix};x\right), \tag{3.18} $$

$$ {}_2F_1\left(\begin{matrix}\alpha,\,\beta\\ \gamma\end{matrix};x\right)={}_2F_1\left(\begin{matrix}\alpha,\,\beta+1\\ \gamma\end{matrix};x\right)-\frac{\alpha x}{\gamma}{}_2F_1\left(\begin{matrix}\alpha+1,\,\beta+1\\ \gamma+1\end{matrix};x\right), \tag{3.19} $$

and

$$ \begin{aligned} {}_2F_1\left(\begin{matrix}\alpha+1,\,\beta+1\\ \gamma+2\end{matrix};x\right)&=\frac{\gamma(\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)x}{}_2F_1\left(\begin{matrix}\alpha,\,\beta\\ \gamma\end{matrix};x\right)\\ &\quad-\frac{\gamma(\gamma+1)(1-x)}{(\gamma-\alpha)(\gamma-\beta)x}{}_2F_1\left(\begin{matrix}\alpha+1,\,\beta+1\\ \gamma+1\end{matrix};x\right), \end{aligned} \tag{3.20} $$

where (3.19), due to C.F. Gauss, is equation (17) on page 133 in [7] and (3.18) and (3.20) were obtained by Ramanathan in [15] from other results of Gauss in [7].

In (3.18), let $\alpha=a-1$, $\beta=b-1$, $\gamma=a-b+1$, and $x=-1$ to obtain

$$ \begin{aligned} {}_2F_1\left(\begin{matrix}a,\,b\\ a-b+2\end{matrix};-1\right)&=\frac{a-b+1}{2(b-1)}{}_2F_1\left(\begin{matrix}a-1,\,b-1\\ a-b+1\end{matrix};-1\right)\\ &\quad-\frac{a-2b+2}{2(b-1)}{}_2F_1\left(\begin{matrix}a,\,b-1\\ a-b+2\end{matrix};-1\right). \end{aligned} \tag{3.21} $$

Both of the hypergeometric series on the right side of (3.21) can be evaluated by (3.17), so

$$ \begin{aligned} &{}_2F_1\left(\begin{matrix}a,\,b\\ a-b+2\end{matrix};-1\right)\\ &\quad=\frac{(a-b+1)\Gamma(a-b+1)\Gamma(\frac{1}{2}(a-1)+1)}{2(b-1)\Gamma(\frac{1}{2}(a-1)-b+2)\Gamma(a)}-\frac{(a-2b+2)\Gamma(a-b+2)\Gamma(\frac{1}{2}a+1)}{2(b-1)\Gamma(\frac{1}{2}a-b+2)\Gamma(a+1)}\\ &\quad=\frac{\Gamma(a-b+2)\Gamma(\frac{1}{2}(a+1))}{2(b-1)\Gamma(\frac{1}{2}(a+1)-b+1)\Gamma(a)}-\frac{(a-2b+2)\Gamma(a-b+2)\frac{1}{2}a\Gamma(\frac{1}{2}a)}{2(b-1)(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-b+1)a\Gamma(a)}\\ &\quad=\frac{\Gamma(a-b+2)\Gamma(\frac{1}{2}(a+1))}{2(b-1)\Gamma(\frac{1}{2}(a+1)-b+1)\Gamma(a)}-\frac{(a-2b+2)\Gamma(a-b+2)\Gamma(\frac{1}{2}a)}{2(b-1)(a-2b+2)\Gamma(\frac{1}{2}a-b+1)\Gamma(a)}\\ &\quad=\frac{\Gamma(a-b+2)}{2(b-1)\Gamma(a)}\left[\frac{\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a+1)-b+1)}-\frac{\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}a-b+1)}\right]. \end{aligned} \tag{3.22} $$

To evaluate ${}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+3 \end{matrix}; -1\right)$, we begin with (3.20). Let $\alpha = a-1$, $\beta = b$, $\gamma = a-b+1$, and $x = -1$ to obtain

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+3 \end{matrix}; -1\right) &= \frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)} {}_2F_1\left(\begin{matrix} a-1,\, b \\ a-b+1 \end{matrix}; -1\right) \\ &\quad - \frac{2(a-b+1)(a-b+2)}{(b-2)(a-2b+1)} {}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+2 \end{matrix}; -1\right). \end{aligned} \tag{3.23}$$

We now need to evaluate ${}_2F_1\left(\begin{matrix} a-1,\, b \\ a-b+1 \end{matrix}; -1\right)$ and ${}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+2 \end{matrix}; -1\right)$. For the first of these, we let $\alpha = b$, $\beta = a-1$, $\gamma = a-b+1$, and $x = -1$ in (3.19). This yields

$${}_2F_1\left(\begin{matrix} b,\, a-1 \\ a-b+1 \end{matrix}; -1\right) = {}_2F_1\left(\begin{matrix} b,\, a \\ a-b+1 \end{matrix}; -1\right) + \frac{b}{a-b+1} {}_2F_1\left(\begin{matrix} b+1,\, a \\ a-b+2 \end{matrix}; -1\right). \tag{3.24}$$

To evaluate ${}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+2 \end{matrix}; -1\right)$, we apply (3.20) with $\alpha = a-1$, $\beta = b$, $\gamma = a-b$, and $x = -1$ to obtain

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+2 \end{matrix}; -1\right) \\ &\quad = \frac{(a-b)(a-b+1)}{(b-1)(a-2b)} {}_2F_1\left(\begin{matrix} a-1,\, b \\ a-b \end{matrix}; -1\right) \\ &\qquad - \frac{2(a-b)(a-b+1)}{(b-1)(a-2b)} {}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+1 \end{matrix}; -1\right) \\ &\quad = \frac{(a-b)(a-b+1)}{(b-1)(a-2b)} \left[{}_2F_1\left(\begin{matrix} a-1,\, b \\ a-b \end{matrix}; -1\right) - 2\,{}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+1 \end{matrix}; -1\right) \right]. \end{aligned} \tag{3.25}$$

Next, (3.19) with $\alpha = a$, $\beta = b$, $\gamma = a-b$, and $x = -1$ yields

$${}_2F_1\left(\begin{matrix} a,\, b \\ a-b \end{matrix}; -1\right) = {}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b \end{matrix}; -1\right) + \frac{a}{a-b} {}_2F_1\left(\begin{matrix} a+1,\, b+1 \\ a-b+1 \end{matrix}; -1\right) \tag{3.26}$$

and (3.18) with $\alpha = b$, $\beta = a$, $\gamma = a - b$, and $x = -1$ yields

$$ {}_2F_1\left({b+1,\, a+1 \atop a-b+1}; -1\right) = \frac{a-b}{2a} {}_2F_1\left({b,\, a \atop a-b}; -1\right) + \frac{b}{2a} {}_2F_1\left({b+1,\, a \atop a-b+1}; -1\right) \tag{3.27} $$

or, isolating ${}_2F_1\left({a, b+1 \atop a-b+1}; -1\right)$, we have

$$ \begin{aligned} {}_2F_1\left({a,\, b+1 \atop a-b+1}; -1\right) &= \frac{2a}{b} {}_2F_1\left({a+1,\, b+1 \atop a-b+1}; -1\right) - \frac{a-b}{b} {}_2F_1\left({a,\, b \atop a-b}; -1\right) \\ &= \frac{2a}{b} {}_2F_1\left({a+1,\, b+1 \atop a-b+1}; -1\right) - \frac{a-b}{b}\Bigg[{}_2F_1\left({a,\, b+1 \atop a-b}; -1\right) \\ &\quad + \frac{a}{a-b} {}_2F_1\left({a+1,\, b+1 \atop a-b+1}; -1\right)\Bigg] \\ &= \frac{2a}{b} {}_2F_1\left({a+1,\, b+1 \atop a-b+1}; -1\right) - \frac{a-b}{b} {}_2F_1\left({a,\, b+1 \atop a-b}; -1\right) \\ &\quad - \frac{a}{b} {}_2F_1\left({a+1,\, b+1 \atop a-b+1}; -1\right) \\ &= \frac{a}{b} {}_2F_1\left({a+1,\, b+1 \atop a-b+1}; -1\right) - \frac{a-b}{b} {}_2F_1\left({a,\, b+1 \atop a-b}; -1\right), \end{aligned} \tag{3.28} $$

where in the second equality we use (3.26). Substituting (3.28) into (3.25), we find that

$$ \begin{aligned} &{}_2F_1\left({a,\, b+1 \atop a-b+2}; -1\right) \\ &= \frac{(a-b)(a-b+1)}{(b-1)(a-2b)}\left[{}_2F_1\left({a-1,\, b \atop a-b}; -1\right) - 2\, {}_2F_1\left({a,\, b+1 \atop a-b+1}; -1\right)\right] \\ &= \frac{(a-b)(a-b+1)}{(b-1)(a-2b)}\Bigg[{}_2F_1\left({a-1,\, b \atop a-b}; -1\right) - 2\Bigg[\frac{a}{b} {}_2F_1\left({a+1,\, b+1 \atop a-b+1}; -1\right) \\ &\quad - \frac{a-b}{b} {}_2F_1\left({a,\, b+1 \atop a-b}; -1\right)\Bigg]\Bigg] \\ &= \frac{(a-b)(a-b+1)}{(b-1)(a-2b)}\Bigg[{}_2F_1\left({a-1,\, b \atop a-b}; -1\right) - \frac{2a}{b}\frac{(a-b)}{2a} {}_2F_1\left({a-1,\, b \atop a-b}; -1\right) \\ &\quad + \frac{2(a-b)}{b} {}_2F_1\left({a,\, b+1 \atop a-b}; -1\right)\Bigg] \end{aligned} $$

$$= \frac{(a-b)(a-b+1)}{(b-1)(a-2b)}\left[\frac{2b-a}{b}\,{}_2F_1\left(\begin{matrix}a-1,\, b\\ a-b\end{matrix}; -1\right) + \frac{2(a-b)}{b}\,{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b\end{matrix}; -1\right)\right], \tag{3.29}$$

where in the third equality we use

$$\begin{aligned}
{}_2F_1\left(\begin{matrix}a+1,\, b+1\\ a-b+1\end{matrix}; -1\right) &= \frac{\Gamma(a-b+1)\Gamma(\frac{1}{2}(a+1)+1)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(a+2)} \\
&= \frac{(a-b)\Gamma(a-b)\frac{1}{2}(a+1)\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a+1)-b)(a+1)a\Gamma(a)} \\
&= \frac{a-b}{2a}\,{}_2F_1\left(\begin{matrix}a-1,\, b\\ a-b\end{matrix}; -1\right),
\end{aligned} \tag{3.30}$$

which is obtained by applying Kummer's Identity (3.17) twice. Substituting (3.24) into (3.23), we find that

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b+3\end{matrix}; -1\right) \\
&= \frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\left[{}_2F_1\left(\begin{matrix}a,\, b\\ a-b+1\end{matrix}; -1\right) + \frac{b}{a-b+1}\,{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b+2\end{matrix}; -1\right)\right] \\
&\quad - \frac{2(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\,{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b+2\end{matrix}; -1\right) \\
&= \frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\,{}_2F_1\left(\begin{matrix}a,\, b\\ a-b+1\end{matrix}; -1\right) \\
&\quad + \frac{b(a-b+2)}{(b-2)(a-2b+1)}\,{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b+2\end{matrix}; -1\right) \\
&\quad - \frac{2(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\,{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b+2\end{matrix}; -1\right) \\
&= \frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\,{}_2F_1\left(\begin{matrix}a,\, b\\ a-b+1\end{matrix}; -1\right) \\
&\quad + \left[\frac{b(a-b+2)}{(b-2)(a-2b+1)} - \frac{2(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\right]{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b+2\end{matrix}; -1\right) \\
&= \frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\,{}_2F_1\left(\begin{matrix}a,\, b\\ a-b+1\end{matrix}; -1\right) \\
&\quad + \frac{(a-b+2)(3b-2a-2)}{(b-2)(a-2b+1)}\,{}_2F_1\left(\begin{matrix}a,\, b+1\\ a-b+2\end{matrix}; -1\right).
\end{aligned} \tag{3.31}$$

Next, substituting (3.29) into the last result, we obtain

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+3 \end{matrix}; -1\right) \\
&\quad = \frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\,{}_2F_1\left(\begin{matrix} a,\, b \\ a-b+1 \end{matrix}; -1\right) \\
&\qquad + \frac{(a-b+2)(3b-2a-2)}{(b-2)(a-2b+1)}\left[\frac{(a-b)(a-b+1)}{(b-1)(a-2b)}\left[\frac{2b-a}{b}\,{}_2F_1\left(\begin{matrix} a-1,\, b \\ a-b \end{matrix}; -1\right)\right.\right. \\
&\qquad \left.\left. + \frac{2(a-b)}{b}\,{}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b \end{matrix}; -1\right)\right]\right] \\
&\quad = \frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\left[{}_2F_1\left(\begin{matrix} a,\, b \\ a-b+1 \end{matrix}; -1\right)\right. \\
&\qquad \left. + \frac{(a-b)(3b-2a-2)}{(b-1)(a-2b)}\left[\frac{2b-a}{b}\,{}_2F_1\left(\begin{matrix} a-1,\, b \\ a-b \end{matrix}; -1\right) + \frac{2(a-b)}{b}\,{}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b \end{matrix}; -1\right)\right]\right] \\
&\quad = \frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\left[{}_2F_1\left(\begin{matrix} a,\, b \\ a-b+1 \end{matrix}; -1\right)\right. \\
&\qquad - \frac{(a-b)(3b-2a-2)}{b(b-1)}\,{}_2F_1\left(\begin{matrix} a-1,\, b \\ a-b \end{matrix}; -1\right) \\
&\qquad \left. + \frac{2(a-b)^2(3b-2a-2)}{b(b-1)(a-2b)}\,{}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b \end{matrix}; -1\right)\right].
\end{aligned} \tag{3.32}$$

Note that, by Kummer's Identity (3.17),

$$\begin{aligned}
{}_2F_1\left(\begin{matrix} a,\, b \\ a-b+1 \end{matrix}; -1\right) &= \frac{\Gamma(a-b+1)\Gamma(\frac{1}{2}a+1)}{\Gamma(\frac{1}{2}a-b+1)\Gamma(a+1)} \\
&= \frac{(a-b)\Gamma(a-b)\Gamma(\frac{1}{2}a+1)}{(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-b)\Gamma(a+1)} \\
&= \frac{2(a-b)}{a-2b}\,{}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b \end{matrix}; -1\right),
\end{aligned} \tag{3.33}$$

so (3.32) becomes

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+3 \end{matrix}; -1\right) \\
&\quad = \frac{(a-b+1)(a-b+2)}{(b-2)(a-2b+1)}\left[\frac{2(a-b)}{a-2b}\,{}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b \end{matrix}; -1\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{(a-b)(3b-2a-2)}{b(b-1)} {}_2F_1\left(\begin{matrix} a-1,\, b \\ a-b \end{matrix}; -1\right) \\
&\quad + \frac{2(a-b)^2(3b-2a-2)}{b(b-1)(a-2b)} {}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b \end{matrix}; -1\right) \Bigg] \\
&= \frac{2(a-b)(a-b+1)(a-b+2)}{(b-2)(a-2b)(a-2b+1)} \left[1 + \frac{(a-b)(3b-2a-2)}{b(b-1)}\right] {}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b \end{matrix}; -1\right) \\
&\quad - \frac{(a-b)(a-b+1)(a-b+2)(3b-2a-2)}{b(b-1)(b-2)(a-2b+1)} {}_2F_1\left(\begin{matrix} a-1,\, b \\ a-b \end{matrix}; -1\right) \\
&= \frac{2(a-b)(a-b+1)(a-b+2)}{(b-2)(a-2b)(a-2b+1)} \frac{(b-2a)(a-2b+1)}{b(b-1)} {}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b \end{matrix}; -1\right) \\
&\quad - \frac{(a-b)(a-b+1)(a-b+2)(3b-2a-2)}{b(b-1)(b-2)(a-2b+1)} {}_2F_1\left(\begin{matrix} a-1,\, b \\ a-b \end{matrix}; -1\right) \\
&= \frac{2(a-b)(a-b+1)(a-b+2)(b-2a)}{b(b-1)(b-2)(a-2b)} {}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b \end{matrix}; -1\right) \\
&\quad - \frac{(a-b)(a-b+1)(a-b+2)(3b-2a-2)}{b(b-1)(b-2)(a-2b+1)} {}_2F_1\left(\begin{matrix} a-1,\, b \\ a-b \end{matrix}; -1\right). \qquad (3.34)
\end{aligned}
$$

Applying (3.17) to (3.34) yields

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix} a,\, b+1 \\ a-b+3 \end{matrix}; -1\right) \\
&\quad = \frac{2(a-b)(a-b+1)(a-b+2)(b-2a)}{b(b-1)(b-2)(a-2b)} \frac{\Gamma(a-b)\Gamma(\frac{1}{2}a+1)}{\Gamma(\frac{1}{2}a-b)\Gamma(a+1)} \\
&\quad\quad - \frac{(a-b)(a-b+1)(a-b+2)(3b-2a-2)}{b(b-1)(b-2)(a-2b+1)} \frac{\Gamma(a-b)\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(a)} \\
&\quad = \frac{2(b-2a)\Gamma(a-b+3)\frac{1}{2}a\Gamma(\frac{1}{2}a)}{b(b-1)(b-2)2(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-b)a\Gamma(a)} \\
&\quad\quad - \frac{(3b-2a-2)\Gamma(a-b+3)\Gamma(\frac{1}{2}(a+1))}{b(b-1)(b-2)2(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-b)\Gamma(a)} \\
&\quad = \frac{(b-2a)\Gamma(a-b+3)\Gamma(\frac{1}{2}a)}{2b(b-1)(b-2)\Gamma(\frac{1}{2}a-b+1)\Gamma(a)} \\
&\quad\quad - \frac{(3b-2a-2)\Gamma(a-b+3)\Gamma(\frac{1}{2}(a+1))}{2b(b-1)(b-2)\Gamma(\frac{1}{2}(a+1)-b+1)\Gamma(a)} \\
&\quad = \frac{\Gamma(a-b+3)}{2b(b-1)(b-2)\Gamma(a)} \left[\frac{(b-2a)\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}a-b+1)} + \frac{(2a-3b+2)\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a+1)-b+1)}\right]. \qquad (3.35)
\end{aligned}
$$

Then, if we set

$$P = \frac{\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a+1)-b+1)},$$
$$Q = \frac{\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}a-b+1)},$$

we finally find that

$$\begin{aligned}\frac{{}_2F_1\left({a,b+1 \atop a-b+3};-1\right)}{{}_2F_1\left({a,b \atop a-b+2};-1\right)} &= \frac{\frac{\Gamma(a-b+3)}{2b(b-1)(b-2)\Gamma(a)}\left[\frac{(b-2a)\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}a-b+1)}+\frac{(2a-3b+2)\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a+1)-b+1)}\right]}{\frac{\Gamma(a-b+2)}{2(b-1)\Gamma(a)}\left[\frac{\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a+1)-b+1)}-\frac{\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}a-b+1)}\right]}\\ &= \frac{\frac{(a-b+2)\Gamma(a-b+2)}{b(b-2)}\left[(b-2a)Q+(2a-3b+2)P\right]}{\Gamma(a-b+2)\left[P-Q\right]}\\ &= \frac{(a-b+2)}{b(b-2)}\frac{\left[(2a-b)\left[P-Q\right]+(2-2b)P\right]}{P-Q}\\ &= \frac{(a-b+2)}{b(b-2)}\left[2a-b+\frac{(2-2b)P}{P-Q}\right], \end{aligned} \tag{3.36}$$

so (3.15) becomes

$$\begin{aligned}&\frac{(a-b+2)}{b(b-2)}\left[2a-b+\frac{(2-2b)P}{P-Q}\right]\\ &\quad = \frac{a-b+2}{2a-2b+1+}\,\frac{(b+1)(3-b)}{2a-2b+1}\,+\,\frac{(b+2)(4-b)}{2a-2b+1}\,+\,\frac{(b+3)(5-b)}{2a-2b+1}\,+\cdots. \end{aligned} \tag{3.37}$$

Multiplying both sides of (3.37) by $-b(b-2)/(a-b+2)$ and adding $2a-b$ to both sides of the result yields

$$\begin{aligned}&\frac{2(b-1)P}{P-Q}\\ &\quad = 2a-b+\frac{b(2-b)}{2a-2b+1+}\,\frac{(b+1)(3-b)}{2a-2b+1}\,+\,\frac{(b+2)(4-b)}{2a-2b+1}\,+\,\frac{(b+3)(5-b)}{2a-2b+1}\,+\cdots. \end{aligned} \tag{3.38}$$

We now let $a = \frac{1}{2}(x+N+1)$ and $b = \frac{1}{2}N+1$ to obtain

$$\frac{N\Gamma(\frac{1}{4}(x+N+3))\Gamma(\frac{1}{4}(x-N+1))}{\Gamma(\frac{1}{4}(x+N+3))\Gamma(\frac{1}{4}(x-N+1)) - \Gamma(\frac{1}{4}(x+N+1))\Gamma(\frac{1}{4}(x-N+3))}$$
$$= x + \frac{N}{2} + \frac{\frac{1}{4}(2^2-N^2)}{x} \genfrac{}{}{0pt}{}{}{+} \frac{\frac{1}{4}(4^2-N^2)}{x} \genfrac{}{}{0pt}{}{}{+} \frac{\frac{1}{4}(6^2-N^2)}{x} \genfrac{}{}{0pt}{}{}{+\cdots}. \tag{3.39}$$

If we define

$$S = \Gamma\left(\frac{1}{4}(x+N+3)\right)\Gamma\left(\frac{1}{4}(x-N+1)\right),$$
$$T = \Gamma\left(\frac{1}{4}(x+N+1)\right)\Gamma\left(\frac{1}{4}(x-N+3)\right),$$

we see that

$$\begin{aligned} x &+ \frac{\frac{1}{4}(2^2-N^2)}{x} \genfrac{}{}{0pt}{}{}{+} \frac{\frac{1}{4}(4^2-N^2)}{x} \genfrac{}{}{0pt}{}{}{+} \frac{\frac{1}{4}(6^2-N^2)}{x} \genfrac{}{}{0pt}{}{}{+\cdots} \\ &= \frac{NS}{S-T} - \frac{N}{2} \\ &= \frac{N(S+T)}{2(S-T)} \\ &= \frac{N(1+\frac{T}{S})}{2(1-\frac{T}{S})}. \end{aligned} \tag{3.40}$$

Taking reciprocals, we obtain

$$\begin{aligned} \frac{2(1-\frac{T}{S})}{N(1+\frac{T}{S})} &= \frac{1}{x+} \frac{\frac{1}{4}(2^2-N^2)}{x} \genfrac{}{}{0pt}{}{}{+} \frac{\frac{1}{4}(4^2-N^2)}{x} \genfrac{}{}{0pt}{}{}{+} \frac{\frac{1}{4}(6^2-N^2)}{x} \genfrac{}{}{0pt}{}{}{+\cdots} \\ &= \frac{2}{2x+} \frac{2^2-N^2}{2x} \genfrac{}{}{0pt}{}{}{+} \frac{4^2-N^2}{2x} \genfrac{}{}{0pt}{}{}{+} \frac{6^2-N^2}{2x} \genfrac{}{}{0pt}{}{}{+\cdots}. \end{aligned} \tag{3.41}$$

Multiplying both sides of (3.41) by $\frac{1}{2}N$ and setting $R = T/S$ yields Theorem 3.2.1. □

Proof of convergence. The domain of validity for Theorem 3.2.1 follows directly from that of Theorem 3.1.1. We let $m = n = N$ and replace x by $2x$ in Theorem 3.1.1. If $N = m \equiv$

$0 \pmod 4$, N is even, and if $N = n \equiv 2 \pmod 4$, N is even. In both cases the identity holds. If N is an arbitrary complex number, we must have $\Re(2x) > 0$, so $\Re(x) > 0$ is the necessary condition. $\square$

Chapter 4

Other entries of Ramanujan

Entry 34 and Theorem 3.1.1 both result from an initial substitution in the main continued fraction (2.17) satisfying $d-e=c-b-1$, so they could be considered to be relatives of each other even before they were discovered to be equivalent. Entries 36 and 39 in Ramanujan's second Notebook [4] are related in a similar way, but are not equivalent to each other. For Entries 36 and 39, the initial substitutions satisfy $d-e=c-b$. This chapter contains their proofs. It should be noted that Entries 36 and 39 appear to be the only identities that come about from initial substitutions of the form $d=a-b+l$, $e=a-c+l$, where l is an integer.

4.1 Entry 36

Theorem 4.1.1 (Entry 36)**.** *Suppose that n or m is an even integer, or that $\Re(x)>0$ with m and n arbitrary complex numbers. Define*

$$R=\frac{\Gamma(\frac{1}{4}(x+m+n+3))\Gamma(\frac{1}{4}(x-m-n+3))\Gamma(\frac{1}{4}(x-m+n+1))\Gamma(\frac{1}{4}(x+m-n+1))}{\Gamma(\frac{1}{4}(x+m+n+1))\Gamma(\frac{1}{4}(x-m-n+1))\Gamma(\frac{1}{4}(x-m+n+3))\Gamma(\frac{1}{4}(x+m-n+3))}.$$

Then

$$\frac{1-R}{1+R}=\frac{mn}{x^2-1-n^2+}\,\frac{2^2-m^2}{1}\,\frac{2^2-n^2}{+\;x^2-1\;+}\,\frac{4^2-m^2}{1}\,\frac{4^2-n^2}{+\;x^2-1\;+}\,\frac{6^2-m^2}{1}\,{+\cdots}.$$

Proof. In the continued fraction (2.17), we let $d=a-b$ and $e=a-c$. This produces the continued fraction

$$(a-b)\frac{{}_3F_2\left(\substack{a,b,c\\a-b,a-c};1\right)}{{}_3F_2\left(\substack{a,b+1,c\\a-b+1,a-c};1\right)} = (a-b-c)+\frac{c(-c)}{a-b-c-1+}\underset{k=1}{\overset{\infty}{\mathbf{K}}}\frac{p_k}{q_k}$$
$$= (a-b-c)+\frac{\frac{1}{4}(0^2-(2c)^2)}{a-b-c-1+}\underset{k=1}{\overset{\infty}{\mathbf{K}}}\frac{p_k}{q_k}, \tag{4.1}$$

where

$$\begin{aligned} p_{2k-1} &= (k+b)(k-b) \\ &= \frac{1}{4}\left((2k)^2-(2b)^2\right), \\ q_{2k-1} &= a-b-c, \\ p_{2k} &= (k+c)(k-c) \\ &= \frac{1}{4}\left((2k)^2-(2c)^2\right), \\ q_{2k} &= a-b-c-1. \end{aligned}$$

We evaluate ${}_3F_2\left(\substack{a,b,c\\a-b,a-c};1\right)$ and ${}_3F_2\left(\substack{a,b+1,c\\a-b+1,a-c};1\right)$ by first rewriting them in a form that allows the use of Dixon's Identity (2.1). In this process, we again use the recurrences (2.10) and (2.24), as well as the relations

$${}_3F_2\left(\begin{matrix}a+1,\, b,\, c\\ d,\, e\end{matrix};1\right) = \frac{a-d+1}{a}{}_3F_2\left(\begin{matrix}a,\, b,\, c\\ d,\, e\end{matrix};1\right)+\frac{d-1}{a}{}_3F_2\left(\begin{matrix}a,\, b,\, c\\ d-1,\, e\end{matrix};1\right) \tag{4.2}$$

and

$${}_3F_2\left(\begin{matrix}a,\, b,\, c\\ d,\, e-1\end{matrix};1\right) = \frac{d-1}{e-1}{}_3F_2\left(\begin{matrix}a,\, b,\, c\\ d-1,\, e\end{matrix};1\right)+\frac{e-d}{e-1}{}_3F_2\left(\begin{matrix}a,\, b,\, c\\ d,\, e\end{matrix};1\right). \tag{4.3}$$

Equations (4.2) and (4.3) are both due to J.A. Wilson, and are given as Equations (17) and (24) respectively in [16].

In (2.10), exchange a with b and d with e to obtain

$$ {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix};1\right) = \frac{e-a}{e}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d,\, e+1 \end{matrix};1\right) + \frac{a(d-c)}{de}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix};1\right). \tag{4.4} $$

Now let $d = a - b$ and $e = a - c$. This yields

$$ \begin{aligned} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ a-b,\, a-c \end{matrix};1\right) &= \frac{(a-c)-a}{a-c}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right) \\ &\quad + \frac{a(a-b-c)}{(a-b)(a-c)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+1 \end{matrix};1\right) \\ &= -\frac{c}{a-c}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right) \\ &\quad + \frac{a(a-b-c)}{(a-b)(a-c)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+1 \end{matrix};1\right). \end{aligned} \tag{4.5} $$

In (4.2), replace b by $b+1$, d by $d+1$, and e by $e+1$ to obtain

$$ \begin{aligned} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+1 \end{matrix};1\right) &= \frac{a-(d+1)+1}{a}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e+1 \end{matrix};1\right) \\ &\quad + \frac{(d+1)-1}{a}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d,\, e+1 \end{matrix};1\right) \\ &= \frac{a-d}{a}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e+1 \end{matrix};1\right) + \frac{d}{a}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d,\, e+1 \end{matrix};1\right), \end{aligned} \tag{4.6} $$

then let $d = a - b$ and $e = a - c$. This yields

$$ \begin{aligned} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+1 \end{matrix};1\right) &= \frac{a-(a-b)}{a}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b+1,\, a-c+1 \end{matrix};1\right) \\ &\quad + \frac{a-b}{a}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right) \\ &= \frac{b}{a}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b+1,\, a-c+1 \end{matrix};1\right) \\ &\quad + \frac{a-b}{a}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right). \end{aligned} \tag{4.7} $$

In (2.24), replace e by $e+1$ to obtain

$$
{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e+1 \end{matrix};1\right) = \frac{a(e+1-c)}{(e+1)(a+b-d)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d+1,\, e+2 \end{matrix};1\right) - \frac{d-b}{a+b-d}\,{}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d+1,\, e+1 \end{matrix};1\right). \tag{4.8}
$$

Again let $d = a-b$ and $e = a-c$. We find that

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b+1,\, a-c+1 \end{matrix};1\right) \\
&\quad = \frac{a(a-2c+1)}{2b(a-c+1)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix};1\right) \\
&\qquad - \frac{a-2b}{2b}\,{}_3F_2\left(\begin{matrix} a,\, b,\, c \\ a-b+1,\, a-c+1 \end{matrix};1\right) \\
&\quad = \frac{a(a-2c+1)}{2b(a-c+1)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix};1\right) \\
&\qquad - \frac{(a-2b)(a-b)(a-2b-2c)}{2b(a-2b)(a-b-c)}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right) \\
&\quad = \frac{a(a-2c+1)}{2b(a-c+1)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix};1\right) \\
&\qquad - \frac{(a-b)(a-2b-2c)}{2b(a-b-c)}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right),
\end{aligned} \tag{4.9}
$$

where in the second equality we use Dixon's Identity (2.1) twice to obtain

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b,\, c \\ a-b+1,\, a-c+1 \end{matrix};1\right) \\
&\quad = \frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}a-b-c+1)}{\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)\Gamma(a-b-c+1)} \\
&\quad = \frac{(a-b)\Gamma(a-b)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)(\frac{1}{2}a-b-c)\Gamma(\frac{1}{2}a-b-c)}{(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)(a-b-c)\Gamma(a-b-c)} \\
&\quad = \frac{(a-b)(a-2b-2c)}{(a-2b)(a-b-c)}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right).
\end{aligned} \tag{4.10}
$$

Substituting (4.9) into (4.7) yields

$$
\begin{aligned}
&{}_3F_2\left({a+1,\, b+1,\, c \atop a-b+1,\, a-c+1}; 1\right) \\
&\quad = \frac{b}{a}\left[\frac{a(a-2c+1)}{2b(a-c+1)}{}_3F_2\left({a+1,\, b+1,\, c \atop a-b+1,\, a-c+2}; 1\right)\right. \\
&\qquad \left. - \frac{(a-b)(a-2b-2c)}{2b(a-b-c)}{}_3F_2\left({a,\, b+1,\, c \atop a-b,\, a-c+1}; 1\right)\right] + \frac{a-b}{a}{}_3F_2\left({a,\, b+1,\, c \atop a-b,\, a-c+1}; 1\right) \\
&\quad = \frac{(a-2c+1)}{2(a-c+1)}{}_3F_2\left({a+1,\, b+1,\, c \atop a-b+1,\, a-c+2}; 1\right) \\
&\qquad - \frac{(a-b)(a-2b-2c)}{2a(a-b-c)}{}_3F_2\left({a,\, b+1,\, c \atop a-b,\, a-c+1}; 1\right) + \frac{a-b}{a}{}_3F_2\left({a,\, b+1,\, c \atop a-b,\, a-c+1}; 1\right) \\
&\quad = \frac{(a-2c+1)}{2(a-c+1)}{}_3F_2\left({a+1,\, b+1,\, c \atop a-b+1,\, a-c+2}; 1\right) \\
&\qquad - \frac{a-b}{a}\left[\frac{a-2b-2c}{2(a-b-c)} - 1\right]{}_3F_2\left({a,\, b+1,\, c \atop a-b,\, a-c+1}; 1\right) \\
&\quad = \frac{(a-2c+1)}{2(a-c+1)}{}_3F_2\left({a+1,\, b+1,\, c \atop a-b+1,\, a-c+2}; 1\right) \\
&\qquad - \frac{a-b}{a}\left[\frac{-a}{2(a-b-c)}\right]{}_3F_2\left({a,\, b+1,\, c \atop a-b,\, a-c+1}; 1\right) \\
&\quad = \frac{(a-2c+1)}{2(a-c+1)}{}_3F_2\left({a+1,\, b+1,\, c \atop a-b+1,\, a-c+2}; 1\right) + \frac{a-b}{2(a-b-c)}{}_3F_2\left({a,\, b+1,\, c \atop a-b,\, a-c+1}; 1\right).
\end{aligned}
\tag{4.11}
$$

We next substitute (4.11) into (4.5) to see that

$$
\begin{aligned}
&{}_3F_2\left({a,\, b,\, c \atop a-b,\, a-c}; 1\right) \\
&\quad = -\frac{c}{a-c}{}_3F_2\left({a,\, b+1,\, c \atop a-b,\, a-c+1}; 1\right) \\
&\qquad + \frac{a(a-b-c)}{(a-b)(a-c)}\left[\frac{(a-2c+1)}{2(a-c+1)}{}_3F_2\left({a+1,\, b+1,\, c \atop a-b+1,\, a-c+2}; 1\right)\right. \\
&\qquad \left. + \frac{a-b}{2(a-b-c)}{}_3F_2\left({a,\, b+1,\, c \atop a-b,\, a-c+1}; 1\right)\right] \\
&\quad = -\frac{c}{a-c}{}_3F_2\left({a,\, b+1,\, c \atop a-b,\, a-c+1}; 1\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{a(a-b-c)(a-2c+1)}{2(a-b)(a-c)(a-c+1)} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix}; 1\right) \\
&+ \frac{a}{2(a-c)} {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right) \\
= &\left[-\frac{c}{a-c} + \frac{a}{2(a-c)}\right] {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right) \\
&+ \frac{a(a-b-c)(a-2c+1)}{2(a-b)(a-c)(a-c+1)} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix}; 1\right) \\
= &\frac{a-2c}{2(a-c)} {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right) \\
&+ \frac{a(a-b-c)(a-2c+1)}{2(a-b)(a-c)(a-c+1)} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix}; 1\right). \qquad (4.12)
\end{aligned}
$$

Now, by Dixon's Identity (2.1),

$$
\begin{aligned}
&\frac{a-2c}{2(a-c)} {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right) \\
&= \frac{(a-2c)}{2(a-c)} \frac{\Gamma(a-b)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}a-b-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)\Gamma(a-b-c)} \\
&= \frac{(a-2c)\Gamma(a-b)(a-c)\Gamma(a-c)\frac{1}{2}a\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{2(a-c)\Gamma(\frac{1}{2}a-b)(\frac{1}{2}a-c)\Gamma(\frac{1}{2}a-c)a\Gamma(a)\Gamma(a-b-c)} \\
&= \frac{\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{2\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)\Gamma(a)\Gamma(a-b-c)} \qquad (4.13)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{a(a-b-c)(a-2c+1)}{2(a-b)(a-c)(a-c+1)} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix}; 1\right) \\
&= \frac{a(a-b-c)(a-2c+1)}{2(a-b)(a-c)(a-c+1)} \frac{\Gamma(a-b+1)\Gamma(a-c+2)\Gamma(\frac{1}{2}(a+1)+1)\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c+1)\Gamma(a+2)\Gamma(a-b-c+1)} \\
&= \frac{2a(a-b-c)(\frac{1}{2}(a+1)-c)(a-b)\Gamma(a-b)(a-c+1)(a-c)\Gamma(a-c)}{2(a-b)(a-c)(a-c+1)\Gamma(\frac{1}{2}(a+1)-b)(\frac{1}{2}(a+1)-c)\Gamma(\frac{1}{2}(a+1)-c)} \times \\
&\qquad \frac{\frac{1}{2}(a+1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{(a+1)a\Gamma(a)(a-b-c)\Gamma(a-b-c)} \\
&= \frac{\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{2\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a)\Gamma(a-b-c)}, \qquad (4.14)
\end{aligned}
$$

so (4.12) can be written as

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b,\, c \\ a-b,\, a-c \end{matrix}; 1\right) \\
&\quad = \frac{\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{2\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)\Gamma(a)\Gamma(a-b-c)} \\
&\qquad + \frac{\Gamma(a-b)\Gamma(a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{2\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a)\Gamma(a-b-c)} \\
&\quad = \frac{\Gamma(a-b)\Gamma(a-c)}{2\Gamma(a)\Gamma(a-b-c)}\left[\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)} + \frac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)}\right] \\
&\quad = \frac{\Gamma(a-b)\Gamma(a-c)}{2\Gamma(a)\Gamma(a-b-c)}\left[P+Q\right], \qquad (4.15)
\end{aligned}
$$

where

$$
\begin{aligned}
P &= \frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c)}, \\
Q &= \frac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)}.
\end{aligned}
$$

Next, replace b by $b+1$, d by $d+1$, and e by $e+1$ in (4.3) to obtain

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix}; 1\right) \\
&\quad = \frac{d+1-1}{e+1-1}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d,\, e+1 \end{matrix}; 1\right) + \frac{e+1-(d+1)}{e+1-1}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right) \\
&\quad = \frac{d}{e}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d,\, e+1 \end{matrix}; 1\right) + \frac{e-d}{e}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e+1 \end{matrix}; 1\right). \qquad (4.16)
\end{aligned}
$$

Let $d = a-b$ and $e = a-c$. This yields

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b+1,\, a-c \end{matrix}; 1\right) \\
&\quad = \frac{a-b}{a-c}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right) \\
&\qquad + \frac{(a-c)-(a-b)}{a-c}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b+1,\, a-c+1 \end{matrix}; 1\right) \\
&\quad = \frac{a-b}{a-c}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right) + \frac{b-c}{a-c}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b+1,\, a-c+1 \end{matrix}; 1\right),
\end{aligned}
\tag{4.17}
$$

and now substitute (4.9) into this to see that

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b+1,\, a-c \end{matrix}; 1\right) \\
&\quad = \frac{a-b}{a-c}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right) \\
&\qquad + \frac{b-c}{a-c}\left[\frac{a(a-2c+1)}{2b(a-c+1)}\, {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix}; 1\right)\right. \\
&\qquad \left. - \frac{(a-b)(a-2b-2c)}{2b(a-b-c)}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right)\right] \\
&\quad = \frac{a-b}{a-c}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right) \\
&\qquad + \frac{a(b-c)(a-2c+1)}{2b(a-c)(a-c+1)}\, {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix}; 1\right) \\
&\qquad - \frac{(a-b)(b-c)(a-2b-2c)}{2b(a-c)(a-b-c)}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right) \\
&\quad = \frac{a-b}{a-c}\left[1 - \frac{(b-c)(a-2b-2c)}{2b(a-b-c)}\right] {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right) \\
&\qquad + \frac{a(b-c)(a-2c+1)}{2b(a-c)(a-c+1)}\, {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix}; 1\right) \\
&\quad = \frac{(a-b)(a-2c)(b+c)}{2b(a-c)(a-b-c)}\, {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix}; 1\right) \\
&\qquad + \frac{a(b-c)(a-2c+1)}{2b(a-c)(a-c+1)}\, {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix}; 1\right).
\end{aligned}
\tag{4.18}
$$

By Dixon's Identity (2.1),

$$\begin{aligned}
&\frac{(a-b)(a-2c)(b+c)}{2b(a-c)(a-b-c)}{}_3F_2\left(\begin{matrix}a,\, b+1,\, c\\ a-b,\, a-c+1\end{matrix};1\right)\\
&\quad=\frac{(a-b)(a-2c)(b+c)}{2b(a-c)(a-b-c)}\frac{\Gamma(a-b)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}a-b-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)\Gamma(a-b-c)}\\
&\quad=\frac{(a-2c)(b+c)\Gamma(a-b+1)(a-c)\Gamma(a-c)\frac{1}{2}a\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{2b(a-c)(a-b-c)\Gamma(\frac{1}{2}a-b)(\frac{1}{2}a-c)\Gamma(\frac{1}{2}a-c)a\Gamma(a)\Gamma(a-b-c)}\\
&\quad=\frac{(a-2c)(b+c)\Gamma(a-b+1)\Gamma(a-c)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c)}{2b\Gamma(\frac{1}{2}a-b)(a-2c)\Gamma(\frac{1}{2}a-c)\Gamma(a)\Gamma(a-b-c+1)}\\
&\quad=\frac{(b+c)\Gamma(a-b+1)\Gamma(a-c)}{2b\Gamma(a)\Gamma(a-b-c+1)}P
\end{aligned}\tag{4.19}$$

and

$$\begin{aligned}
&\frac{a(b-c)(a-2c+1)}{2b(a-c)(a-c+1)}{}_3F_2\left(\begin{matrix}a+1,\, b+1,\, c\\ a-b+1,\, a-c+2\end{matrix};1\right)\\
&\quad=\frac{a(b-c)(a-2c+1)}{2b(a-c)(a-c+1)}\frac{\Gamma(a-b+1)\Gamma(a-c+2)\Gamma(\frac{1}{2}(a+1)+1)\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c+1)\Gamma(a+2)\Gamma(a-b-c+1)}\\
&\quad=\frac{2a(b-c)(\frac{1}{2}(a+1)-c)\Gamma(a-b+1)(a-c+1)(a-c)\Gamma(a-c)}{2b(a-c)(a-c+1)\Gamma(\frac{1}{2}(a+1)-b)(\frac{1}{2}(a+1)-c)\Gamma(\frac{1}{2}(a+1)-c)}\times\\
&\qquad\quad\frac{\frac{1}{2}(a+1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{(a+1)a\Gamma(a)\Gamma(a-b-c+1)}\\
&\quad=\frac{(b-c)\Gamma(a-b+1)\Gamma(a-c)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{2b\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a)\Gamma(a-b-c+1)}\\
&\quad=\frac{(b-c)\Gamma(a-b+1)\Gamma(a-c)}{2b\Gamma(a)\Gamma(a-b-c+1)}Q,
\end{aligned}\tag{4.20}$$

so (4.18) becomes

$$\begin{aligned}
&{}_3F_2\left(\begin{matrix}a,\, b+1,\, c\\ a-b+1,\, a-c\end{matrix};1\right)\\
&\quad=\frac{(b+c)\Gamma(a-b+1)\Gamma(a-c)}{2b\Gamma(a)\Gamma(a-b-c+1)}P+\frac{(b-c)\Gamma(a-b+1)\Gamma(a-c)}{2b\Gamma(a)\Gamma(a-b-c+1)}Q\\
&\quad=\frac{\Gamma(a-b+1)\Gamma(a-c)}{2b\Gamma(a)\Gamma(a-b-c+1)}\left[(b+c)P+(b-c)Q\right]
\end{aligned}$$

$$= \frac{\Gamma(a-b+1)\Gamma(a-c)}{2b\Gamma(a)\Gamma(a-b-c+1)} \left[b(P+Q) + c(P-Q)\right]. \tag{4.21}$$

Multiplying (4.15) by $(a-b)$ and dividing by (4.21), we obtain

$$\begin{aligned}
\frac{(a-b)\,{}_3F_2\left({a, b, c \atop a-b, a-c}; 1\right)}{{}_3F_2\left({a, b+1, c \atop a-b+1, a-c}; 1\right)} &= \frac{(a-b)\frac{\Gamma(a-b)\Gamma(a-c)}{2\Gamma(a)\Gamma(a-b-c)}\left[P+Q\right]}{\frac{\Gamma(a-b+1)\Gamma(a-c)}{2b\Gamma(a)\Gamma(a-b-c+1)}\left[b(P+Q)+c(P-Q)\right]} \\
&= \frac{\frac{\Gamma(a-b+1)}{\Gamma(a-b-c)}\left[P+Q\right]}{\frac{\Gamma(a-b+1)}{b(a-b-c)\Gamma(a-b-c)}\left[b(P+Q)+c(P-Q)\right]} \\
&= \frac{b(a-b-c)[P+Q]}{b[P+Q]+c[P-Q]}.
\end{aligned} \tag{4.22}$$

Thus (4.1) becomes

$$\begin{aligned}
&\frac{b(a-b-c)[P+Q]}{b[P+Q]+c[P-Q]} \\
&\quad = (a-b-c) - \frac{c^2}{a-b-c-1+} \frac{\frac{1}{4}(2^2-(2b)^2)}{a-b-c} \frac{\frac{1}{4}(2^2-(2c)^2)}{+a-b-c-1+} \frac{\frac{1}{4}(4^2-(2b)^2)}{a-b-c} \\
&\qquad \frac{\frac{1}{4}(4^2-(2c)^2)}{+a-b-c-1+} \frac{\frac{1}{4}(6^2-(2b)^2)}{a-b-c} \frac{\frac{1}{4}(6^2-(2c)^2)}{+a-b-c-1+\cdots} \\
&\quad = (a-b-c) - \frac{2c^2}{2(a-b-c-1)+} \frac{2^2-(2b)^2}{2(a-b-c)+} \frac{2^2-(2c)^2}{2(a-b-c-1)+} \frac{4^2-(2b)^2}{2(a-b-c)} \\
&\qquad \frac{4^2-(2c)^2}{+2(a-b-c-1)+} \frac{6^2-(2b)^2}{2(a-b-c)+} \frac{6^2-(2c)^2}{2(a-b-c-1)+\cdots}.
\end{aligned} \tag{4.23}$$

Now,

$$\begin{aligned}
\frac{b(a-b-c)[P+Q]}{b[P+Q]+c[P-Q]} - (a-b-c) &= (a-b-c)\left[\frac{b[P+Q]}{b[P+Q]+c[P-Q]} - 1\right] \\
&= (a-b-c)\left[-\frac{c[P-Q]}{b[P+Q]+c[P-Q]}\right],
\end{aligned} \tag{4.24}$$

so subtracting $(a-b-c)$ from both sides of (4.23) and applying (4.24) to the result yields

$$\begin{aligned}
&-(a-b-c)\left[\frac{c[P-Q]}{b[P+Q]+c[P-Q]}\right] \\
&\quad = -\frac{2c^2}{2(a-b-c-1)+}\,\frac{2^2-(2b)^2}{2(a-b-c)+}\,\frac{2^2-(2c)^2}{2(a-b-c-1)+}\,\frac{4^2-(2b)^2}{2(a-b-c)} \\
&\qquad +\frac{4^2-(2c)^2}{2(a-b-c-1)+}\,\frac{6^2-(2b)^2}{2(a-b-c)+}\,\frac{6^2-(2c)^2}{2(a-b-c-1)+\cdots}, \qquad (4.25)
\end{aligned}$$

and upon dividing by $-2c^2$ we obtain

$$\begin{aligned}
&\frac{(a-b-c)}{2c}\left[\frac{P-Q}{b[P+Q]+c[P-Q]}\right] \\
&\quad = \frac{1}{2(a-b-c-1)+}\,\frac{2^2-(2b)^2}{2(a-b-c)+}\,\frac{2^2-(2c)^2}{2(a-b-c-1)+}\,\frac{4^2-(2b)^2}{2(a-b-c)} \\
&\qquad +\frac{4^2-(2c)^2}{2(a-b-c-1)+}\,\frac{6^2-(2b)^2}{2(a-b-c)+}\,\frac{6^2-(2c)^2}{2(a-b-c-1)+\cdots}. \qquad (4.26)
\end{aligned}$$

Taking reciprocals of both sides, we see that

$$\begin{aligned}
&\frac{2c}{(a-b-c)}\left[\frac{b[P+Q]+c[P-Q]}{P-Q}\right] \\
&\quad = 2(a-b-c-1)+\frac{2^2-(2b)^2}{2(a-b-c)+}\,\frac{2^2-(2c)^2}{2(a-b-c-1)+}\,\frac{4^2-(2b)^2}{2(a-b-c)} \\
&\qquad +\frac{4^2-(2c)^2}{2(a-b-c-1)+}\,\frac{6^2-(2b)^2}{2(a-b-c)+}\,\frac{6^2-(2c)^2}{2(a-b-c-1)+\cdots}. \qquad (4.27)
\end{aligned}$$

Since

$$\begin{aligned}
&2c\left[\frac{b[P+Q]+c[P-Q]}{P-Q}\right] \\
&\quad = 2c\left[\frac{b[P+Q]}{P-Q}+c\right] \\
&\quad = \frac{2bc[P+Q]}{P-Q}+2c^2, \qquad (4.28)
\end{aligned}$$

we can multiply (4.27) by $(a-b-c)$ and apply (4.28) to the resulting equation to find that

$$\begin{aligned}\frac{2bc[P+Q]}{P-Q}&+2c^2\\ &=2(a-b-c-1)(a-b-c)+\frac{(a-b-c)(2^2-(2b)^2)}{2(a-b-c)}\genfrac{}{}{0pt}{}{}{+}\frac{2^2-(2c)^2}{2(a-b-c-1)}\\ &\quad\genfrac{}{}{0pt}{}{}{+}\frac{4^2-(2b)^2}{2(a-b-c)}\genfrac{}{}{0pt}{}{}{+}\frac{4^2-(2c)^2}{2(a-b-c-1)}\genfrac{}{}{0pt}{}{}{+}\frac{6^2-(2b)^2}{2(a-b-c)}\genfrac{}{}{0pt}{}{}{+}\frac{6^2-(2c)^2}{2(a-b-c-1)}\genfrac{}{}{0pt}{}{}{+}\cdots\end{aligned}\tag{4.29}$$

or

$$\begin{aligned}\frac{2bc[P+Q]}{P-Q}&\\ &=2((a-b-c-1)(a-b-c)-c^2)+\frac{(a-b-c)(2^2-(2b)^2)}{2(a-b-c)}\genfrac{}{}{0pt}{}{}{+}\frac{2^2-(2c)^2}{2(a-b-c-1)}\\ &\quad\genfrac{}{}{0pt}{}{}{+}\frac{4^2-(2b)^2}{2(a-b-c)}\genfrac{}{}{0pt}{}{}{+}\frac{4^2-(2c)^2}{2(a-b-c-1)}\genfrac{}{}{0pt}{}{}{+}\frac{6^2-(2b)^2}{2(a-b-c)}\genfrac{}{}{0pt}{}{}{+}\frac{6^2-(2c)^2}{2(a-b-c-1)}\genfrac{}{}{0pt}{}{}{+}\cdots.\end{aligned}\tag{4.30}$$

Taking reciprocals again and multiplying both sides by $2bc$, we obtain

$$\begin{aligned}\frac{P-Q}{P+Q}&\\ &=\frac{2bc}{2((a-b-c-1)(a-b-c)-c^2)}\genfrac{}{}{0pt}{}{}{+}\frac{(a-b-c)(2^2-(2b)^2)}{2(a-b-c)}\genfrac{}{}{0pt}{}{}{+}\frac{2^2-(2c)^2}{2(a-b-c-1)}\\ &\quad\genfrac{}{}{0pt}{}{}{+}\frac{4^2-(2b)^2}{2(a-b-c)}\genfrac{}{}{0pt}{}{}{+}\frac{4^2-(2c)^2}{2(a-b-c-1)}\genfrac{}{}{0pt}{}{}{+}\frac{6^2-(2b)^2}{2(a-b-c)}\genfrac{}{}{0pt}{}{}{+}\frac{6^2-(2c)^2}{2(a-b-c-1)}\genfrac{}{}{0pt}{}{}{+}\cdots.\end{aligned}\tag{4.31}$$

Finally, let $a=\frac{1}{2}(x+m+n+1)$, $b=\frac{1}{2}m$, and $c=\frac{1}{2}n$. We now have

$$\begin{aligned}P&=\frac{\Gamma(\frac{1}{4}(x+m+n+1))\Gamma(\frac{1}{4}(x-m-n+1))}{\Gamma(\frac{1}{4}(x-m+n+1))\Gamma(\frac{1}{4}(x+m-n+1))},\\ Q&=\frac{\Gamma(\frac{1}{4}(x+m+n+3))\Gamma(\frac{1}{4}(x-m-n+3))}{\Gamma(\frac{1}{4}(x-m+n+3))\Gamma(\frac{1}{4}(x+m-n+3))}.\end{aligned}$$

Thus $Q/P = R$, and our continued fraction becomes

$$\begin{aligned}
\frac{1-R}{1+R} &= \frac{2(\frac{1}{2}m)(\frac{1}{2}n)}{2\left((\frac{1}{2}(x-1))(\frac{1}{2}(x+1))-(\frac{1}{2}n)^2\right)+}\ \frac{\frac{1}{2}(x+1)(2^2-m^2)}{x+1}\ {}_{+}\ \frac{2^2-n^2}{x-1}\ {}_{+}\ \frac{4^2-m^2}{x+1} \\
&\quad {}_{+}\ \frac{4^2-n^2}{x-1}\ {}_{+}\ \frac{6^2-m^2}{x+1}\ {}_{+}\ \frac{6^2-n^2}{x-1}\ {}_{+}\cdots \\
&= \frac{mn}{(x-1)(x+1)-n^2+}\ \frac{(x+1)(2^2-m^2)}{x+1}\ {}_{+}\ \frac{2^2-n^2}{x-1}\ {}_{+}\ \frac{4^2-m^2}{x+1}\ {}_{+}\ \frac{4^2-n^2}{x-1} \\
&\quad {}_{+}\ \frac{6^2-m^2}{x+1}\ {}_{+}\ \frac{6^2-n^2}{x-1}\ {}_{+}\cdots \\
&= \frac{mn}{x^2-1-n^2+}\ \frac{2^2-m^2}{1}\ {}_{+}\ \frac{2^2-n^2}{x^2-1+}\ \frac{4^2-m^2}{1}\ {}_{+}\ \frac{4^2-n^2}{x^2-1+}\ \frac{6^2-m^2}{1}\ {}_{+}\ \frac{6^2-n^2}{x^2-1+}\cdots,
\end{aligned} \tag{4.32}$$

which is Entry 36. The convergence of the continued fraction to $(1-R)/(1+R)$ is discussed by Jacobsen in [10]. □

If instead of beginning with the substitution $d = a-b$, $e = a-c$ in (2.17), we begin with $d = a-b+2l$, $e = a-c+2l$, where l is an integer, we can obtain Entry 36, as shown in the next theorem.

Theorem 4.1.2. *The substitution $d = a-b+2l$, $e = a-c+2l$ in Lemma 2.1.4, where l is an integer, followed by the substitution $a = \frac{1}{2}(x+m+n+1)$, $b = \frac{1}{2}m+l$, $c = \frac{1}{2}n+l$, results in a continued fraction that is equivalent to Entry 36 after manipulation.*

Proof. We proceed by induction on l.

Case $l = 0$: The proof of Entry 36 given above is the case where $l = 0$. Note that if all the substitutions into Lemma 2.1.4 ($d = a-b$, $e = a-c$ followed by $a = \frac{1}{2}(x+m+n+1)$, $b = \frac{1}{2}m$, $c = \frac{1}{2}n$) are performed immediately, the first identity obtained is

$$\frac{\frac{1}{2}(x+n+1)\,{}_3F_2\left(\begin{matrix}\frac{1}{2}(x+m+n+1),\frac{1}{2}m,\frac{1}{2}n\\ \frac{1}{2}(x+n+1),\frac{1}{2}(x+m+1)\end{matrix};1\right)}{{}_3F_2\left(\begin{matrix}\frac{1}{2}(x+m+n+1),\frac{1}{2}m+1,\frac{1}{2}n\\ \frac{1}{2}(x+n+1)+1,\frac{1}{2}(x+m+1)\end{matrix};1\right)}$$
$$= \frac{x+1}{2} + \frac{0^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \frac{1^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} + \frac{1^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \frac{2^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} + \frac{2^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \cdots, \tag{4.33}$$

before various manipulations and Dixon's Identity are used to obtain the statement of Entry 36. We will show that other values of l produce equations identical to (4.33) after manipulation.

Case $l > 0$: Suppose that the theorem is true for $l-1$; that is, that the equality

$$\frac{\frac{1}{2}(x+n+2l-1)\,{}_3F_2\left(\begin{matrix}\frac{1}{2}(x+m+n+1),\frac{1}{2}m+l-1,\frac{1}{2}n+l-1\\ \frac{1}{2}(x+n+1)+l-1,\frac{1}{2}(x+m+1)+l-1\end{matrix};1\right)}{{}_3F_2\left(\begin{matrix}\frac{1}{2}(x+m+n+1),\frac{1}{2}m+l,\frac{1}{2}n+l-1\\ \frac{1}{2}(x+n+1)+l,\frac{1}{2}(x+m+1)+l-1\end{matrix};1\right)}$$
$$= \frac{x+1}{2} + \frac{(l-1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \frac{l^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} + \frac{l^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \frac{(l+1)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} + \cdots, \tag{4.34}$$

which is produced by the substitutions $d = a-b+2(l-1)$, $e = a-c+2(l-1)$ followed by $a = \frac{1}{2}(x+m+n+1)$, $b = \frac{1}{2}m+l-1$, $c = \frac{1}{2}n+l-1$ in Lemma 2.1.4, is equivalent to (4.33). We will show that (4.34) can be converted to the equality

$$\frac{\frac{1}{2}(x+n+2l+1)\,{}_3F_2\left(\begin{matrix}\frac{1}{2}(x+m+n+1),\frac{1}{2}m+l,\frac{1}{2}n+l\\ \frac{1}{2}(x+n+1)+l,\frac{1}{2}(x+m+1)+l\end{matrix};1\right)}{{}_3F_2\left(\begin{matrix}\frac{1}{2}(x+m+n+1),\frac{1}{2}m+l+1,\frac{1}{2}n+l\\ \frac{1}{2}(x+n+1)+l+1,\frac{1}{2}(x+m+1)+l\end{matrix};1\right)}$$
$$= \frac{x+1}{2} + \frac{l^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \frac{(l+1)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} + \frac{(l+1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \frac{(l+2)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} + \cdots \tag{4.35}$$

by algebraic manipulations, and thus that the identity produced by the substitutions $d = a-b+2l$, $e = a-c+2l$ followed by $a = \frac{1}{2}(x+m+n+1)$, $b = \frac{1}{2}m+l$, $c = \frac{1}{2}n+l$ in Lemma

2.1.4 is also equivalent to (4.33).

For ease of notation, let $A = \frac{1}{2}(x+m+n+1)$, $B = \frac{1}{2}m+l$, $C = \frac{1}{2}n+l$, $D = \frac{1}{2}(x+n+1)+l$, and $E = \frac{1}{2}(x+m+1)+l$. Then (4.34) becomes

$$\frac{(D-1)_3F_2\left({A,B-1,C-1 \atop D-1,E-1};1\right)}{_3F_2\left({A,B,C-1 \atop D,E-1};1\right)} = \frac{x+1}{2} + \frac{(l-1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} \, {}_{+} \, \frac{l^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} \, {}_{+} \, \frac{l^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} \, {}_{+} \, \frac{(l+1)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} \, {}_{+\cdots}. \tag{4.36}$$

Subtract $\frac{1}{2}(x+1)$ from both sides of (4.36) to obtain

$$\frac{(D-1)_3F_2\left({A,B-1,C-1 \atop D-1,E-1};1\right) - \frac{x+1}{2}{}_3F_2\left({A,B,C-1 \atop D,E-1};1\right)}{_3F_2\left({A,B,C-1 \atop D,E-1};1\right)} = \frac{(l-1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} \, {}_{+} \, \frac{l^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} \, {}_{+} \, \frac{l^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} \, {}_{+} \, \frac{(l+1)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} \, {}_{+} \, \frac{(l+1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} \, {}_{+\cdots}. \tag{4.37}$$

Now consider Proposition 2.1.3, with $a = B-1$, $b = C-1$, $c = A$, $d = D-1$, and $e = E-1$. We obtain the relation

$$\begin{aligned}(D-1)_3F_2&\left({A,\, B-1,\, C-1 \atop D-1,\, E-1};1\right) \\ &= (D-C)_3F_2\left({A,\, B,\, C-1 \atop D,\, E-1};1\right) + \frac{(C-1)(E-1-A)}{E-1}{}_3F_2\left({A,\, B,\, C \atop D,\, E};1\right).\end{aligned} \tag{4.38}$$

Since

$$\begin{aligned} D-C &= \frac{1}{2}(x+n+1)+l-\left(\frac{1}{2}n+l\right) \\ &= \frac{1}{2}(x+1) \end{aligned}$$

and

$$\begin{aligned}(C-1)(E-1-A) &= \left(\frac{1}{2}n+l-1\right)\left(\frac{1}{2}(x+m+1)+l-1-\frac{1}{2}(x+m+n+1)\right)\\ &= \left(\frac{1}{2}n+l-1\right)\left(l-1-\frac{1}{2}n\right)\\ &= (l-1)^2-\left(\frac{1}{2}n\right)^2,\end{aligned}$$

we can rewrite (4.38) as

$$\begin{aligned}(D-1)\,{}_3F_2\left(\begin{matrix}A,\, B-1,\, C-1\\ D-1,\, E-1\end{matrix};1\right) &- \frac{x+1}{2}\,{}_3F_2\left(\begin{matrix}A,\, B,\, C-1\\ D,\, E-1\end{matrix};1\right)\\ &= \frac{(l-1)^2-\left(\frac{1}{2}n\right)^2}{E-1}\,{}_3F_2\left(\begin{matrix}A,\, B,\, C\\ D,\, E\end{matrix};1\right),\end{aligned} \tag{4.39}$$

and (4.37) becomes

$$\begin{aligned}&\frac{\left((l-1)^2-\left(\frac{1}{2}n\right)^2\right)\,{}_3F_2\left(\begin{smallmatrix}A,B,C\\ D,E\end{smallmatrix};1\right)}{(E-1)\,{}_3F_2\left(\begin{smallmatrix}A,B,C-1\\ D,E-1\end{smallmatrix};1\right)}\\ &= \frac{(l-1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\frac{l^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}+\frac{l^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\frac{(l+1)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}+\frac{(l+1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\cdots.\end{aligned} \tag{4.40}$$

Divide both sides of (4.40) by $(l-1)^2-(\frac{1}{2}n)^2$ and take reciprocals of both sides of the resulting equality to obtain

$$\begin{aligned}&\frac{(E-1)\,{}_3F_2\left(\begin{smallmatrix}A,B,C-1\\ D,E-1\end{smallmatrix};1\right)}{{}_3F_2\left(\begin{smallmatrix}A,B,C\\ D,E\end{smallmatrix};1\right)}\\ &= \frac{x-1}{2}+\frac{l^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}+\frac{l^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\frac{(l+1)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}+\frac{(l+1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\cdots.\end{aligned} \tag{4.41}$$

We repeat the above process for one more term of the continued fraction. Subtract $\frac{1}{2}(x-1)$

from both sides of (4.41) to obtain

$$\frac{(E-1){}_3F_2\left({A,B,C-1 \atop D,E-1};1\right)-\frac{x-1}{2}{}_3F_2\left({A,B,C \atop D,E};1\right)}{{}_3F_2\left({A,B,C \atop D,E};1\right)}$$
$$=\frac{l^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}+\frac{l^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\frac{(l+1)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}+\frac{(l+1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\frac{(l+2)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}+\cdots. \tag{4.42}$$

Letting $a=C-1$, $b=B$, $c=A$, $d=E-1$, and $e=D$ in Proposition 2.1.3 yields

$$\begin{aligned}(E-1){}_3F_2&\left({A,\,B,\,C-1 \atop D,\,E-1};1\right)\\ &=(E-1-B){}_3F_2\left({A,\,B,\,C \atop D,\,E};1\right)+\frac{B(D-A)}{D}{}_3F_2\left({A,\,B+1,\,C \atop D+1,\,E};1\right).\end{aligned} \tag{4.43}$$

Since

$$\begin{aligned}E-1-B&=\frac{1}{2}(x+m+1)+l-1-\left(\frac{1}{2}m+l\right)\\ &=\frac{1}{2}(x-1)\end{aligned}$$

and

$$\begin{aligned}B(D-A)&=\left(\frac{1}{2}m+l\right)\left(\frac{1}{2}(x+n+1)+l-\frac{1}{2}(x+m+n+1)\right)\\ &=\left(\frac{1}{2}m+l\right)\left(l-\frac{1}{2}m\right)\\ &=l^2-\left(\frac{1}{2}m\right)^2,\end{aligned}$$

we can rewrite (4.43) as

$$(E-1){}_3F_2\left(\begin{matrix}A,\,B,\,C-1\\D,\,E-1\end{matrix};1\right)-\frac{x-1}{2}{}_3F_2\left(\begin{matrix}A,\,B,\,C\\D,\,E\end{matrix};1\right)$$
$$=\frac{l^2-\left(\frac{1}{2}m\right)^2}{D}{}_3F_2\left(\begin{matrix}A,\,B+1,\,C\\D+1,\,E\end{matrix};1\right), \tag{4.44}$$

and (4.42) becomes

$$\frac{\left(l^2-\left(\frac{1}{2}m\right)^2\right){}_3F_2\left(\begin{smallmatrix}A,B+1,C\\D+1,E\end{smallmatrix};1\right)}{D\,{}_3F_2\left(\begin{smallmatrix}A,B,C\\D,E\end{smallmatrix};1\right)}$$
$$=\frac{l^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}+\frac{l^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\frac{(l+1)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}+\frac{(l+1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\cdots. \tag{4.45}$$

Divide both sides of (4.45) by $l^2-(\frac{1}{2}m)^2$ and take reciprocals of both sides of the resulting equality to obtain

$$\frac{D\,{}_3F_2\left(\begin{smallmatrix}A,B,C\\D,E\end{smallmatrix};1\right)}{{}_3F_2\left(\begin{smallmatrix}A,B+1,C\\D+1,E\end{smallmatrix};1\right)}$$
$$=\frac{x+1}{2}+\frac{l^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\frac{(l+1)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}+\frac{(l+1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\frac{(l+2)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}+\cdots. \tag{4.46}$$

This is just (4.35) in our simplified notation.

Case $l<0$: Suppose that the theorem is true for $l+1$; that is, that the equality

$$\frac{\frac{1}{2}(x+n+2l+3){}_3F_2\left(\begin{smallmatrix}\frac{1}{2}(x+m+n+1),\,\frac{1}{2}m+l+1,\,\frac{1}{2}n+l+1\\\frac{1}{2}(x+n+1)+l+1,\,\frac{1}{2}(x+m+1)+l+1\end{smallmatrix};1\right)}{{}_3F_2\left(\begin{smallmatrix}\frac{1}{2}(x+m+n+1),\,\frac{1}{2}m+l+2,\,\frac{1}{2}n+l+1\\\frac{1}{2}(x+n+1)+l+2,\,\frac{1}{2}(x+m+1)+l+1\end{smallmatrix};1\right)}$$
$$=\frac{x+1}{2}+\frac{(l+1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\frac{(l+2)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}+\frac{(l+2)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\cdots \tag{4.47}$$

produced by the substitutions $d=a-b+2(l+1)$, $e=a-c+2(l+1)$ followed by

$a = \frac{1}{2}(x+m+n+1)$, $b = \frac{1}{2}m+l+1$, $c = \frac{1}{2}n+l+1$ is equivalent to (4.33). We again let $A = \frac{1}{2}(x+m+n+1)$, $B = \frac{1}{2}m+l$, $C = \frac{1}{2}n+l$, $D = \frac{1}{2}(x+n+1)+l$, and $E = \frac{1}{2}(x+m+1)+l$, so our starting point is

$$\begin{aligned}\frac{(D+1)\,{}_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)}{{}_3F_2\left({A, B+2, C+1 \atop D+2, E+1}; 1\right)} \\ = \frac{x+1}{2} + \frac{(l+1)^2 - (\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} \genfrac{}{}{0pt}{}{}{+} \frac{(l+2)^2 - (\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} \genfrac{}{}{0pt}{}{}{+} \frac{(l+2)^2 - (\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} \genfrac{}{}{0pt}{}{}{+\cdots}.\end{aligned} \tag{4.48}$$

We will manipulate (4.48) to obtain (4.46).

Take reciprocals of both sides of (4.48). This yields

$$\begin{aligned}\frac{{}_3F_2\left({A, B+2, C+1 \atop D+2, E+1}; 1\right)}{(D+1)\,{}_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)} \\ = \frac{1}{\frac{1}{2}(x+1) +} \frac{(l+1)^2 - (\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} \genfrac{}{}{0pt}{}{}{+} \frac{(l+2)^2 - (\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} \genfrac{}{}{0pt}{}{}{+} \frac{(l+2)^2 - (\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} \genfrac{}{}{0pt}{}{}{+\cdots}.\end{aligned} \tag{4.49}$$

Rearrange Proposition 2.1.3 into the form

$${}_3F_2\left({a+1,\, b+1,\, c \atop d+1,\, e+1}; 1\right) = \frac{de}{b(e-c)}\,{}_3F_2\left({a,\, b,\, c \atop d,\, e}; 1\right) - \frac{e(d-b)}{b(e-c)}\,{}_3F_2\left({a+1,\, b,\, c \atop d+1,\, e}; 1\right) \tag{4.50}$$

and let $a = C$, $b = B+1$, $c = A$, $d = E$, and $e = D+1$ in (4.50) to obtain

$$\begin{aligned}{}_3F_2\left({A,\, B+2,\, C+1 \atop D+2,\, E+1}; 1\right) = {} & \frac{E(D+1)}{(B+1)(D+1-A)}\,{}_3F_2\left({A,\, B+1,\, C \atop D+1,\, E}; 1\right) \\ & - \frac{(D+1)(E-B-1)}{(B+1)(D+1-A)}\,{}_3F_2\left({A,\, B+1,\, C+1 \atop D+1,\, E+1}; 1\right).\end{aligned} \tag{4.51}$$

Then

$$
\begin{aligned}
&\frac{{}_3F_2\left({A, B+2, C+1 \atop D+2, E+1}; 1\right)}{(D+1)_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)} \\
&\quad = \frac{\frac{E(D+1)}{(B+1)(D+1-A)}{}_3F_2\left({A, B+1, C \atop D+1, E}; 1\right) - \frac{(D+1)(E-B-1)}{(B+1)(D+1-A)}{}_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)}{(D+1)_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)} \\
&\quad = \frac{1}{(B+1)(D+1-A)}\left[\frac{E_3F_2\left({A, B+1, C \atop D+1, E}; 1\right)}{{}_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)} - (E-B-1)\right] \\
&\quad = \frac{1}{(l+1)^2 - \left(\frac{1}{2}m\right)^2}\left[\frac{E_3F_2\left({A, B+1, C \atop D+1, E}; 1\right)}{{}_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)} - \frac{x-1}{2}\right],
\end{aligned}
\tag{4.52}
$$

since

$$
\begin{aligned}
(B+1)(D+1-A) &= \left(\frac{1}{2}m + l + 1\right)\left(\frac{1}{2}(x+n+1) + l + 1 - \frac{1}{2}(x+m+n+1)\right) \\
&= \left(l + 1 + \frac{1}{2}m\right)\left(l + 1 - \frac{1}{2}m\right) \\
&= (l+1)^2 - \left(\frac{1}{2}m\right)^2
\end{aligned}
$$

and

$$
\begin{aligned}
E - B - 1 &= \frac{1}{2}(x+m+1) + l - \left(\frac{1}{2}m + l\right) - 1 \\
&= \frac{1}{2}(x-1).
\end{aligned}
$$

Hence (4.49) becomes

$$\frac{1}{(l+1)^2-\left(\frac{1}{2}m\right)^2}\left[\frac{E\,{}_3F_2\left({A,B+1,C \atop D+1,E};1\right)}{{}_3F_2\left({A,B+1,C+1 \atop D+1,E+1};1\right)}-\frac{x-1}{2}\right]$$
$$=\frac{1}{\frac{1}{2}(x+1)+}\,\frac{(l+1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)\quad+}\,\frac{(l+2)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)\quad+}\,\frac{(l+2)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)\quad+\cdots}. \tag{4.53}$$

Multiply both sides of (4.53) by $(l+1)^2-(\frac{1}{2}m)^2$ and then add $\frac{1}{2}(x-1)$ to both sides of the resulting equality to obtain

$$\frac{E\,{}_3F_2\left({A,B+1,C \atop D+1,E};1\right)}{{}_3F_2\left({A,B+1,C+1 \atop D+1,E+1};1\right)}$$
$$=\frac{x-1}{2}+\frac{(l+1)^2-\left(\frac{1}{2}m\right)^2}{\frac{1}{2}(x+1)\quad+}\,\frac{(l+1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)\quad+}\,\frac{(l+2)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)\quad+\cdots}. \tag{4.54}$$

Taking reciprocals of both sides of (4.54) yields

$$\frac{{}_3F_2\left({A,B+1,C+1 \atop D+1,E+1};1\right)}{E\,{}_3F_2\left({A,B+1,C \atop D+1,E};1\right)}$$
$$=\frac{1}{\frac{1}{2}(x-1)+}\,\frac{(l+1)^2-\left(\frac{1}{2}m\right)^2}{\frac{1}{2}(x+1)\quad+}\,\frac{(l+1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)\quad+}\,\frac{(l+2)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)\quad+\cdots}. \tag{4.55}$$

In (4.50), let $a=B$, $b=C$, $c=A$, $d=D$, and $e=E$ to obtain

$${}_3F_2\left({A,\,B+1,\,C+1 \atop D+1,\,E+1};1\right)=\frac{DE}{C(E-A)}\,{}_3F_2\left({A,\,B,\,C \atop D,\,E};1\right)$$
$$-\frac{E(D-C)}{C(E-A)}\,{}_3F_2\left({A,\,B+1,\,C \atop D+1,\,E};1\right). \tag{4.56}$$

Then

$$
\begin{aligned}
&\frac{{}_3F_2\left({A,B+1,C+1 \atop D+1,E+1};\,1\right)}{E\,{}_3F_2\left({A,B+1,C \atop D+1,E};\,1\right)} \\
&\quad= \frac{\frac{DE}{C(E-A)}{}_3F_2\left({A,B,C \atop D,E};\,1\right) - \frac{E(D-C)}{C(E-A)}{}_3F_2\left({A,B+1,C \atop D+1,E};\,1\right)}{E\,{}_3F_2\left({A,B+1,C \atop D+1,E};\,1\right)} \\
&\quad= \frac{1}{C(E-A)}\left[\frac{D\,{}_3F_2\left({A,B,C \atop D,E};\,1\right)}{{}_3F_2\left({A,B+1,C \atop D+1,E};\,1\right)} - (D-C)\right] \\
&\quad= \frac{1}{l^2-\left(\frac{1}{2}n\right)^2}\left[\frac{D\,{}_3F_2\left({A,B,C \atop D,E};\,1\right)}{{}_3F_2\left({A,B+1,C \atop D+1,E};\,1\right)} - \frac{x+1}{2}\right],
\end{aligned}
\tag{4.57}
$$

since

$$
\begin{aligned}
C(E-A) &= \left(\frac{1}{2}n+l\right)\left(\frac{1}{2}(x+m+1)+l-\frac{1}{2}(x+m+n+1)\right) \\
&= \left(l+\frac{1}{2}n\right)\left(l-\frac{1}{2}n\right) \\
&= l^2-\left(\frac{1}{2}n\right)^2
\end{aligned}
$$

and

$$
\begin{aligned}
D-C &= \frac{1}{2}(x+n+1)+l-\left(\frac{1}{2}n+l\right) \\
&= \frac{1}{2}(x+1).
\end{aligned}
$$

Thus (4.55) can be rewritten as

$$
\frac{1}{l^2-\left(\frac{1}{2}n\right)^2}\left[\frac{D\,{}_3F_2\left({A,B,C \atop D,E};\,1\right)}{{}_3F_2\left({A,B+1,C \atop D+1,E};\,1\right)} - \frac{x+1}{2}\right]
$$

$$= \frac{1}{\frac{1}{2}(x-1)+} \frac{(l+1)^2-\left(\frac{1}{2}m\right)^2}{\frac{1}{2}(x+1)} \quad + \frac{(l+1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} \quad + \frac{(l+2)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} \quad + \cdots. \tag{4.58}$$

Multiply both sides of (4.58) by $l^2 - \left(\frac{1}{2}n\right)^2$ and add $\frac{1}{2}(x+1)$ to both sides of the result to obtain (4.46).

Note that for all l, the tail of the continued fraction created by the algorithm of Theorem 4.1.2 matches the tail of (4.33),

$$\frac{k^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} \quad + \frac{k^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} \quad + \frac{(k+1)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} \quad + \frac{(k+1)^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} \quad + \frac{(k+2)^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} \quad + \cdots,$$

from $k = N$ onward, where N is some positive integer. Thus the convergence of the continued fraction in Entry 36 implies the convergence of all the continued fractions in this proof. □

4.2 Entry 39

Theorem 4.2.1 (Entry 39)**.** *Suppose that n or m is an odd integer, or that $\Re(x) > 0$ with m and n arbitrary complex numbers. Define*

$$R = \frac{\Gamma(\frac{1}{4}(x+m+n+1))\Gamma(\frac{1}{4}(x-m-n+1))\Gamma(\frac{1}{4}(x-m+n+1))\Gamma(\frac{1}{4}(x+m-n+1))}{\Gamma(\frac{1}{4}(x+m+n+3))\Gamma(\frac{1}{4}(x-m-n+3))\Gamma(\frac{1}{4}(x-m+n+3))\Gamma(\frac{1}{4}(x+m-n+3))}.$$

Then

$$R = \frac{8}{\frac{1}{2}(x^2+m^2-n^2-1)+} \frac{1^2-m^2}{1} \quad + \frac{1^2-n^2}{x^2-1} + \frac{3^2-m^2}{1} \quad + \frac{3^2-n^2}{x^2-1} + \frac{5^2-m^2}{1} \quad + \cdots.$$

Proof. We again use the continued fraction (2.17), now letting $d = a-b+1$ and $e = a-c+1$.

The resulting continued fraction is

$$(a-b+1)\frac{{}_3F_2\left({a,b,c \atop a-b+1,a-c+1};1\right)}{{}_3F_2\left({a,b+1,c \atop a-b+2,a-c+1};1\right)} = (a-b-c+1) + \frac{c(1-c)}{a-b-c+\underset{k=1}{\overset{\infty}{\mathbf{K}}}\frac{p_k}{q_k}}$$
$$= (a-b-c+1) + \frac{\frac{1}{4}(1^2-(2c-1)^2)}{a-b-c} + \underset{k=1}{\overset{\infty}{\mathbf{K}}}\frac{p_k}{q_k}, \quad (4.59)$$

where

$$\begin{aligned} p_{2k-1} &= (k+b)(k+1-b) \\ &= \frac{1}{4}\left((2k+1)^2-(2b-1)^2\right), \\ q_{2k-1} &= a-b-c+1, \\ p_{2k} &= (k+c)(k+1-c) \\ &= \frac{1}{4}\left((2k+1)^2-(2c-1)^2\right), \\ q_{2k} &= a-b-c. \end{aligned}$$

We next evaluate ${}_3F_2\left({a,b,c \atop a-b+1,a-c+1};1\right)$ and ${}_3F_2\left({a,b+1,c \atop a-b+2,a-c+1};1\right)$. The first of these can immediately be evaluated by Dixon's Identity (2.1). In order to evaluate ${}_3F_2\left({a,b+1,c \atop a-b+2,a-c+1};1\right)$, we employ the relation (2.24) and the relation

$$\begin{aligned} &{}_3F_2\left({a,\, b,\, c \atop d+1,\, e};1\right) \\ &\quad = \left[\frac{abc+d(d-1)(d+e-a-b-c-1)}{(d-a)(d-b)(d-c)}+1\right]{}_3F_2\left({a,\, b,\, c \atop d,\, e};1\right) \\ &\quad\quad - \frac{d(d-1)(d+e-a-b-c-1)}{(d-a)(d-b)(d-c)}{}_3F_2\left({a,\, b,\, c \atop d-1,\, e};1\right), \end{aligned} \quad (4.60)$$

which is Equation (21) in [16].

In (4.60), replace b by $b+1$ to obtain

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d+1,\, e \end{matrix};1\right) \\
&\quad = \left[\frac{a(b+1)c + d(d-1)(d+e-a-b-c-2)}{(d-a)(d-b-1)(d-c)} + 1\right] {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d,\, e \end{matrix};1\right) \\
&\qquad - \frac{d(d-1)(d+e-a-b-c-2)}{(d-a)(d-b-1)(d-c)} {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d-1,\, e \end{matrix};1\right).
\end{aligned}
\tag{4.61}
$$

Now let $d = a-b+1$ and $e = a-c+1$. This yields

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b+2,\, a-c+1 \end{matrix};1\right) \\
&\quad = \left[\frac{a(b+1)c + (a-b+1)(a-b)(a-2b-2c)}{(1-b)(a-2b)(a-b-c+1)} + 1\right] {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b+1,\, a-c+1 \end{matrix};1\right) \\
&\qquad - \frac{(a-b+1)(a-b)(a-2b-2c)}{(1-b)(a-2b)(a-b-c+1)} {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right) \\
&\quad = \frac{(a-b+1)(a-2b-2c+1)}{(1-b)(a-b-c+1)} {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b+1,\, a-c+1 \end{matrix};1\right) \\
&\qquad - \frac{(a-b+1)(a-b)(a-2b-2c)}{(1-b)(a-2b)(a-b-c+1)} {}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right).
\end{aligned}
\tag{4.62}
$$

In (2.24), replace d by $d-1$ to obtain

$$
\begin{aligned}
{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ d,\, e \end{matrix};1\right) &= \frac{a(e-c)}{e(a+b-d+1)} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ d,\, e+1 \end{matrix};1\right) \\
&\quad - \frac{d-b-1}{a+b-d+1} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix};1\right).
\end{aligned}
\tag{4.63}
$$

Again let $d = a-b+1$ and $e = a-c+1$. This yields

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b+1,\, a-c+1 \end{matrix};1\right) \\
&\quad = \frac{a(a-2c+1)}{2b(a-c+1)} {}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix};1\right) \\
&\qquad - \frac{a-2b}{2b} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ a-b+1,\, a-c+1 \end{matrix};1\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{a(a-2c+1)}{2b(a-c+1)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix};1\right) \\
&\quad - \frac{(a-2b)(a-b)(a-2b-2c)}{2b(a-2b)(a-b-c)}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right) \\
&= \frac{a(a-2c+1)}{2b(a-c+1)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix};1\right) \\
&\quad - \frac{(a-b)(a-2b-2c)}{2b(a-b-c)}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right), \qquad (4.64)
\end{aligned}
$$

where in the second equality we use Dixon's Identity (2.1) to obtain

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b,\, c \\ a-b+1,\, a-c+1 \end{matrix};1\right) \\
&\quad = \frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}a-b-c+1)}{\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)\Gamma(a-b-c+1)} \\
&\quad = \frac{(a-b)\Gamma(a-b)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)(\frac{1}{2}a-b-c)\Gamma(\frac{1}{2}a-b-c)}{(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)(a-b-c)\Gamma(a-b-c)} \\
&\quad = \frac{(a-b)(a-2b-2c)}{(a-2b)(a-b-c)}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right). \qquad (4.65)
\end{aligned}
$$

Substituting (4.64) into (4.62), we see that

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b+2,\, a-c+1 \end{matrix};1\right) \\
&= \frac{(a-b+1)(a-2b-2c+1)}{(1-b)(a-b-c+1)}\left[\frac{a(a-2c+1)}{2b(a-c+1)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix};1\right)\right. \\
&\quad \left. - \frac{(a-b)(a-2b-2c)}{2b(a-b-c)}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right)\right] \\
&\quad - \frac{(a-b+1)(a-b)(a-2b-2c)}{(1-b)(a-2b)(a-b-c+1)}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right) \\
&= \frac{(a-b+1)(a-2b-2c+1)a(a-2c+1)}{(1-b)(a-b-c+1)2b(a-c+1)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix};1\right) \\
&\quad - \frac{(a-b+1)(a-2b-2c+1)(a-b)(a-2b-2c)}{(1-b)(a-b-c+1)2b(a-b-c)}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right) \\
&\quad - \frac{(a-b+1)(a-b)(a-2b-2c)}{(1-b)(a-2b)(a-b-c+1)}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(a-b+1)(a-2b-2c+1)a(a-2c+1)}{(1-b)(a-b-c+1)2b(a-c+1)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix};1\right) \\
&\quad - \frac{(a-b+1)(a-b)(a-2b-2c)}{(1-b)(a-b-c+1)}\times \\
&\qquad \left[\frac{(a-2b-2c+1)}{2b(a-b-c)}+\frac{1}{(a-2b)}\right]{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right) \\
&= \frac{(a-b+1)(a-2b-2c+1)a(a-2c+1)}{(1-b)(a-b-c+1)2b(a-c+1)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix};1\right) \\
&\quad - \frac{(a-b+1)(a-b)(a-2b-2c)}{(1-b)(a-b-c+1)}\times \\
&\qquad \frac{(a-2b-2c+1)(a-2b)+2b(a-b-c)}{2b(a-b-c)(a-2b)}\,{}_3F_2\left(\begin{matrix} a,\, b+1,\, c \\ a-b,\, a-c+1 \end{matrix};1\right).
\end{aligned} \tag{4.66}
$$

Now, by Dixon's Identity (2.1),

$$
\begin{aligned}
&\frac{(a-b+1)(a-2b-2c+1)a(a-2c+1)}{(1-b)(a-b-c+1)2b(a-c+1)}\,{}_3F_2\left(\begin{matrix} a+1,\, b+1,\, c \\ a-b+1,\, a-c+2 \end{matrix};1\right) \\
&= \frac{(a-b+1)(a-2b-2c+1)a(a-2c+1)}{(1-b)(a-b-c+1)2b(a-c+1)}\times \\
&\qquad \frac{\Gamma(a-b+1)\Gamma(a-c+2)\Gamma(\frac{1}{2}(a+1)+1)\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c+1)\Gamma(a+2)\Gamma(a-b-c+1)} \\
&= \frac{4a(a-b+1)(\frac{1}{2}(a+1)-b-c)(\frac{1}{2}(a+1)-c)}{2b(1-b)(a-b-c+1)(a-c+1)}\times \\
&\qquad \frac{\Gamma(a-b+1)(a-c+1)\Gamma(a-c+1)\frac{1}{2}(a+1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c)}{\Gamma(\frac{1}{2}(a+1)-b)(\frac{1}{2}(a+1)-c)\Gamma(\frac{1}{2}(a+1)-c)a(a+1)\Gamma(a)\Gamma(a-b-c+1)} \\
&= \frac{\Gamma(a-b+2)\Gamma(a-c+1)\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c+1)}{b(1-b)\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)\Gamma(a)\Gamma(a-b-c+2)} \\
&= \frac{\Gamma(a-b+2)\Gamma(a-c+1)}{b(1-b)\Gamma(a)\Gamma(a-b-c+2)}Q,
\end{aligned} \tag{4.67}
$$

where

$$
Q = \frac{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(a+1)-b-c+1)}{\Gamma(\frac{1}{2}(a+1)-b)\Gamma(\frac{1}{2}(a+1)-c)},
$$

and

$$
\begin{aligned}
&\frac{(a-b+1)(a-b)(a-2b-2c)}{(1-b)(a-b-c+1)2b(a-b-c)(a-2b)}{}_3F_2\left(\begin{matrix}a,\, b+1,\, c\\ a-b,\, a-c+1\end{matrix};1\right)\\
&=\frac{(a-b+1)(a-b)2(\frac{1}{2}a-b-c)}{(1-b)(a-b-c+1)2b(a-b-c)2(\frac{1}{2}a-b)}\frac{\Gamma(a-b)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}a-b-c)}{\Gamma(\frac{1}{2}a-b)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)\Gamma(a-b-c)}\\
&=\frac{\Gamma(a-b+2)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}a-b-c+1)}{(1-b)2b\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)\Gamma(a-b-c+2)}\\
&=\frac{\Gamma(a-b+2)\Gamma(a-c+1)\frac{1}{2}a\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{(1-b)2b\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-c+1)a\Gamma(a)\Gamma(a-b-c+2)}\\
&=\frac{\Gamma(a-b+2)\Gamma(a-c+1)}{4b(1-b)\Gamma(a)\Gamma(a-b-c+2)}P, \qquad (4.68)
\end{aligned}
$$

where

$$P=\frac{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-c+1)}.$$

Inserting (4.67) and (4.68) into (4.66), we find that

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix}a,\, b+1,\, c\\ a-b+2,\, a-c+1\end{matrix};1\right)\\
&\quad=\frac{\Gamma(a-b+2)\Gamma(a-c+1)}{b(1-b)\Gamma(a)\Gamma(a-b-c+2)}Q\\
&\qquad-\left[(a-2b-2c+1)(a-2b)+2b(a-b-c)\right]\frac{\Gamma(a-b+2)\Gamma(a-c+1)}{4b(1-b)\Gamma(a)\Gamma(a-b-c+2)}P\\
&\quad=\frac{\Gamma(a-b+2)\Gamma(a-c+1)}{b(1-b)\Gamma(a)\Gamma(a-b-c+2)}\left[Q-\frac{1}{4}\left[(a-2b-2c+1)(a-2b)+2b(a-b-c)\right]P\right].
\end{aligned}
\qquad (4.69)
$$

Using Dixon's Identity (2.1) again, we obtain

$$
\begin{aligned}
&(a-b+1){}_3F_2\left(\begin{matrix}a,\, b,\, c\\ a-b+1,\, a-c+1\end{matrix};1\right)\\
&\quad=\frac{(a-b+1)\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}a-b-c+1)}{\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-c+1)\Gamma(a+1)\Gamma(a-b-c+1)}
\end{aligned}
$$

$$= \frac{\Gamma(a-b+2)\Gamma(a-c+1)\frac{1}{2}a\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)(a-b-c+1)}{\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-c+1)a\Gamma(a)\Gamma(a-b-c+1)(a-b-c+1)}$$
$$= \frac{\Gamma(a-b+2)\Gamma(a-c+1)}{\Gamma(a)\Gamma(a-b-c+2)}\frac{\frac{1}{2}(a-b-c+1)\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b-c+1)}{\Gamma(\frac{1}{2}a-b+1)\Gamma(\frac{1}{2}a-c+1)}$$
$$= \frac{\Gamma(a-b+2)\Gamma(a-c+1)}{\Gamma(a)\Gamma(a-b-c+2)}\frac{1}{2}(a-b-c+1)P, \tag{4.70}$$

and upon dividing (4.70) by (4.69), we find that

$$\frac{(a-b+1)_3F_2\left({a,b,c \atop a-b+1,a-c+1};1\right)}{{}_3F_2\left({a,b+1,c \atop a-b+2,a-c+1};1\right)}$$
$$= \frac{\frac{\Gamma(a-b+2)\Gamma(a-c+1)}{\Gamma(a)\Gamma(a-b-c+2)}\frac{1}{2}(a-b-c+1)P}{\frac{\Gamma(a-b+2)\Gamma(a-c+1)}{b(1-b)\Gamma(a)\Gamma(a-b-c+2)}\left[Q-\frac{1}{4}\left[(a-2b-2c+1)(a-2b)+2b(a-b-c)\right]P\right]}$$
$$= \frac{\frac{1}{2}b(1-b)(a-b-c+1)P}{Q-\frac{1}{4}\left[(a-2b-2c+1)(a-2b)+2b(a-b-c)\right]P}. \tag{4.71}$$

Thus (4.59) can be written as

$$\frac{\frac{1}{2}b(1-b)(a-b-c+1)P}{Q-\frac{1}{4}\left[(a-2b-2c+1)(a-2b)+2b(a-b-c)\right]P}$$
$$= (a-b-c+1)+\frac{\frac{1}{2}(1^2-(2c-1)^2)}{2(a-b-c)}+\frac{3^2-(2b-1)^2}{2(a-b-c+1)}$$
$$+\frac{3^2-(2c-1)^2}{2(a-b-c)}+\frac{5^2-(2b-1)^2}{2(a-b-c+1)}+\frac{5^2-(2c-1)^2}{2(a-b-c)}+\cdots. \tag{4.72}$$

Taking reciprocals of both sides of (4.72) and multiplying by $\frac{1}{2}b(1-b)(a-b-c+1)$, we obtain

$$\frac{Q}{P}-\frac{1}{4}\left[(a-2b-2c+1)(a-2b)+2b(a-b-c)\right]$$
$$= \frac{\frac{1}{2}b(1-b)(a-b-c+1)}{(a-b-c+1)}+\frac{\frac{1}{2}(1^2-(2c-1)^2)}{2(a-b-c)}+\frac{3^2-(2b-1)^2}{2(a-b-c+1)}$$
$$+\frac{3^2-(2c-1)^2}{2(a-b-c)}+\frac{5^2-(2b-1)^2}{2(a-b-c+1)}+\frac{5^2-(2c-1)^2}{2(a-b-c)}+\cdots$$

$$= \cfrac{b(1-b)(a-b-c+1)}{2(a-b-c+1)} + \cfrac{1^2-(2c-1)^2}{2(a-b-c)} + \cfrac{3^2-(2b-1)^2}{2(a-b-c+1)} + \cfrac{3^2-(2c-1)^2}{2(a-b-c)} + \cfrac{5^2-(2b-1)^2}{2(a-b-c+1)} + \cfrac{5^2-(2c-1)^2}{2(a-b-c)} + \cdots$$

$$= \cfrac{\frac{1}{4}(1^2-(2b-1)^2)(a-b-c+1)}{2(a-b-c+1)} + \cfrac{1^2-(2c-1)^2}{2(a-b-c)} + \cfrac{3^2-(2b-1)^2}{2(a-b-c+1)} + \cfrac{3^2-(2c-1)^2}{2(a-b-c)} + \cfrac{5^2-(2b-1)^2}{2(a-b-c+1)} + \cfrac{5^2-(2c-1)^2}{2(a-b-c)} + \cdots. \tag{4.73}$$

Moving $\frac{1}{4}\left[(a-2b-2c+1)(a-2b)+2b(a-b-c)\right]$ to the right side of the equation and taking reciprocals of both sides again yields

$$\frac{P}{Q} = \cfrac{1}{\frac{1}{4}\left[(a-2b-2c+1)(a-2b)+2b(a-b-c)\right] + \cfrac{\frac{1}{4}(a-b-c+1)(1^2-(2b-1)^2)}{2(a-b-c+1)} + \cfrac{1^2-(2c-1)^2}{2(a-b-c)} + \cfrac{3^2-(2b-1)^2}{2(a-b-c+1)} + \cfrac{3^2-(2c-1)^2}{2(a-b-c)} + \cfrac{5^2-(2b-1)^2}{2(a-b-c+1)} + \cdots}$$

$$= \cfrac{8}{2\left[(a-2b-2c+1)(a-2b)+2b(a-b-c)\right] + \cfrac{2(a-b-c+1)(1^2-(2b-1)^2)}{2(a-b-c+1)} + \cfrac{1^2-(2c-1)^2}{2(a-b-c)} + \cfrac{3^2-(2b-1)^2}{2(a-b-c+1)} + \cfrac{3^2-(2c-1)^2}{2(a-b-c)} + \cfrac{5^2-(2b-1)^2}{2(a-b-c+1)} + \cdots}. \tag{4.74}$$

Finally, let $a = \frac{1}{2}(x+m+n+1)$, $b = \frac{1}{2}(m+1)$, and $c = \frac{1}{2}(n+1)$. We now see that

$$R = \frac{\Gamma(\frac{1}{4}(x+m+n+1))\Gamma(\frac{1}{4}(x-m-n+1))\Gamma(\frac{1}{4}(x-m+n+1))\Gamma(\frac{1}{4}(x+m-n+1))}{\Gamma(\frac{1}{4}(x+m+n+3))\Gamma(\frac{1}{4}(x-m-n+3))\Gamma(\frac{1}{4}(x-m+n+3))\Gamma(\frac{1}{4}(x+m-n+3))}$$

$$= \frac{P}{Q}$$

$$= \cfrac{8}{2\left[\frac{1}{2}(x-m-n-1)\frac{1}{2}(x-m+n-1)+2\frac{1}{2}(m+1)\frac{1}{2}(x-1)\right] + \cfrac{(x+1)(1^2-m^2)}{x+1} + \cfrac{1^2-n^2}{x-1} + \cfrac{3^2-m^2}{x+1} + \cfrac{3^2-n^2}{x-1} + \cfrac{5^2-m^2}{x+1} + \cfrac{5^2-n^2}{x-1} + \cdots}$$

$$= \cfrac{8}{\frac{1}{2}(x^2+m^2-n^2-1) + \cfrac{(x+1)(1^2-m^2)}{x+1} + \cfrac{1^2-n^2}{x-1} + \cfrac{3^2-m^2}{x+1} + \cfrac{3^2-n^2}{x-1} + \cdots}$$

$$= \cfrac{8}{\frac{1}{2}(x^2+m^2-n^2-1) + \cfrac{(x+1)(1^2-m^2)}{x+1} + \cfrac{\frac{x+1}{x+1}(1^2-n^2)}{x-1} + \cfrac{3^2-m^2}{x+1} + \cfrac{3^2-n^2}{x-1} + \cdots}$$

$$= \frac{8}{\frac{1}{2}(x^2+m^2-n^2-1)+} \frac{1^2-m^2}{1\quad +} \frac{\frac{1}{x+1}(1^2-n^2)}{x-1\quad +} \frac{3^2-m^2}{x+1\quad +} \frac{3^2-n^2}{x-1\quad +\cdots}$$

$$= \frac{8}{\frac{1}{2}(x^2+m^2-n^2-1)+} \frac{1^2-m^2}{1\quad +} \frac{1^2-n^2}{x^2-1\quad +} \frac{(x+1)(3^2-m^2)}{x+1\quad +} \frac{3^2-n^2}{x-1\quad +\cdots}$$

$$= \frac{8}{\frac{1}{2}(x^2+m^2-n^2-1)+} \frac{1^2-m^2}{1\quad +} \frac{1^2-n^2}{x^2-1\quad +} \frac{3^2-m^2}{1\quad +} \frac{3^2-n^2}{x^2-1\quad +\cdots}, \tag{4.75}$$

and we have proven Entry 39. The convergence of the continued fraction to R is discussed by Jacobsen in [10]. □

Theorem 4.2.2. *The substitution* $d = a-b+2l+1$, $e = a-c+2l+1$ *in Lemma 2.1.4, where* l *is an integer, followed by the substitution* $a = \frac{1}{2}(x+m+n+1)$, $b = \frac{1}{2}(m+1)+l$, $c = \frac{1}{2}(n+1)+l$, *results in a continued fraction that is equivalent to Entry 39 after manipulation.*

Proof. We proceed by induction on l.

Case $l = 0$: The proof of Entry 39 given above is the case where $l = 0$. Note that if all the substitutions into Lemma 2.1.4 ($d = a-b+1$, $e = a-c+1$ followed by $a = \frac{1}{2}(x+m+n+1)$, $b = \frac{1}{2}(m+1)$, $c = \frac{1}{2}(n+1)$) are performed immediately, the first identity obtained is

$$\frac{\left(\frac{1}{2}(x+n)+1\right) {}_3F_2\left({\frac{1}{2}(x+m+n+1), \frac{1}{2}(m+1), \frac{1}{2}(n+1) \atop \frac{1}{2}(x+n)+1, \frac{1}{2}(x+m)+1}; 1\right)}{{}_3F_2\left({\frac{1}{2}(x+m+n+1), \frac{1}{2}(m+1)+1, \frac{1}{2}(n+1) \atop \frac{1}{2}(x+n)+2, \frac{1}{2}(x+m)+1}; 1\right)}$$

$$= \frac{x+1}{2} + \frac{(\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)\quad +} \frac{(\frac{3}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)\quad +} \frac{(\frac{3}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)\quad +} \frac{(\frac{5}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)\quad +\cdots}, \tag{4.76}$$

before various manipulations and Dixon's Identity are used to obtain the statement of Entry 39. We will show that other values of l produce equations identical to (4.76) after manipulation.

Case $l > 0$: Suppose that the theorem is true for $l-1$; that is, that the equality

$$\frac{\left(\frac{1}{2}(x+n)+l\right) {}_3F_2\left({\frac{1}{2}(x+m+n+1), \frac{1}{2}(m+1)+l-1, \frac{1}{2}(n+1)+l-1 \atop \frac{1}{2}(x+n)+l, \frac{1}{2}(x+m)+l}; 1\right)}{{}_3F_2\left({\frac{1}{2}(x+m+n+1), \frac{1}{2}(m+1)+l, \frac{1}{2}(n+1)+l-1 \atop \frac{1}{2}(x+n)+l+1, \frac{1}{2}(x+m)+l}; 1\right)}$$
$$= \frac{x+1}{2} + \frac{(l-\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \frac{(l+\frac{1}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} + \frac{(l+\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \cdots, \tag{4.77}$$

which is produced by the substitutions $d = a - b + 2(l-1) + 1$, $e = a - c + 2(l-1) + 1$ followed by $a = \frac{1}{2}(x+m+n+1)$, $b = \frac{1}{2}(m+1)+l-1$, $c = \frac{1}{2}(n+1)+l-1$ in Lemma 2.1.4, is equivalent to (4.76). We will show that (4.77) can be converted to the equality

$$\frac{\left(\frac{1}{2}(x+n)+l+1\right) {}_3F_2\left({\frac{1}{2}(x+m+n+1), \frac{1}{2}(m+1)+l, \frac{1}{2}(n+1)+l \atop \frac{1}{2}(x+n)+l+1, \frac{1}{2}(x+m)+l+1}; 1\right)}{{}_3F_2\left({\frac{1}{2}(x+m+n+1), \frac{1}{2}(m+1)+l+1, \frac{1}{2}(n+1)+l \atop \frac{1}{2}(x+n)+l+2, \frac{1}{2}(x+m)+l+1}; 1\right)}$$
$$= \frac{x+1}{2} + \frac{(l+\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \frac{(l+\frac{3}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} + \frac{(l+\frac{3}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \cdots \tag{4.78}$$

by algebraic manipulations and thus that the identity produced by the substitutions $d = a - b + 2l + 1$, $e = a - c + 2l + 1$ followed by $a = \frac{1}{2}(x+m+n+1)$, $b = \frac{1}{2}(m+1)+l$, $c = \frac{1}{2}(n+1)+l$ in Lemma 2.1.4 is also equivalent to (4.76).

For ease of notation, let $A = \frac{1}{2}(x+m+n+1)$, $B = \frac{1}{2}(m+1)+l$, $C = \frac{1}{2}(n+1)+l$, $D = \frac{1}{2}(x+n)+l+1$, and $E = \frac{1}{2}(x+m)+l+1$. Then (4.77) becomes

$$\frac{(D-1){}_3F_2\left({A, B-1, C-1 \atop D-1, E-1}; 1\right)}{{}_3F_2\left({A, B, C-1 \atop D, E-1}; 1\right)}$$
$$= \frac{x+1}{2} + \frac{(l-\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \frac{(l+\frac{1}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} + \frac{(l+\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \cdots. \tag{4.79}$$

Subtract $\frac{1}{2}(x+1)$ from both sides of (4.79) to obtain

$$\frac{(D-1)_3F_2\left({A,B-1,C-1 \atop D-1,E-1};1\right)-\frac{x+1}{2}{}_3F_2\left({A,B,C-1 \atop D,E-1};1\right)}{{}_3F_2\left({A,B,C-1 \atop D,E-1};1\right)}$$
$$=\frac{(l-\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\frac{(l+\frac{1}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}+\frac{(l+\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}+\frac{(l+\frac{3}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}+\cdots. \tag{4.80}$$

Now consider Proposition 2.1.3, with $a=B-1$, $b=C-1$, $c=A$, $d=D-1$, and $e=E-1$. We obtain the relation

$$(D-1)_3F_2\left({A,\, B-1,\, C-1 \atop D-1,\, E-1};1\right)$$
$$=(D-C)_3F_2\left({A,\, B,\, C-1 \atop D,\, E-1};1\right)+\frac{(C-1)(E-1-A)}{E-1}{}_3F_2\left({A,\, B,\, C \atop D,\, E};1\right). \tag{4.81}$$

Since

$$D-C=\frac{1}{2}(x+n)+l+1-\left(\frac{1}{2}(n+1)+l\right)$$
$$=\frac{1}{2}(x+1)$$

and

$$(C-1)(E-1-A)=\left(\frac{1}{2}(n+1)+l-1\right)\left(\frac{1}{2}(x+m)+l+1-1-\frac{1}{2}(x+m+n+1)\right)$$
$$=\left(\frac{1}{2}n+l-\frac{1}{2}\right)\left(l-\frac{1}{2}-\frac{1}{2}n\right)$$
$$=\left(l-\frac{1}{2}\right)^2-\left(\frac{1}{2}n\right)^2,$$

we can rewrite (4.81) as

$$(D-1)_3F_2\left(\begin{matrix}A,\,B-1,\,C-1\\ D-1,\,E-1\end{matrix};1\right)-\frac{x+1}{2}{}_3F_2\left(\begin{matrix}A,\,B,\,C-1\\ D,\,E-1\end{matrix};1\right)$$
$$=\frac{(l-\frac{1}{2})^2-\left(\frac{1}{2}n\right)^2}{E-1}{}_3F_2\left(\begin{matrix}A,\,B,\,C\\ D,\,E\end{matrix};1\right), \tag{4.82}$$

and (4.80) becomes

$$\frac{\left((l-\frac{1}{2})^2-\left(\frac{1}{2}n\right)^2\right){}_3F_2\left({A,B,C \atop D,E};1\right)}{(E-1)_3F_2\left({A,B,C-1 \atop D,E-1};1\right)}$$
$$=\frac{(l-\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}\,+\,\frac{(l+\frac{1}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}\,+\,\frac{(l+\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}\,+\,\frac{(l+\frac{3}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}\,+\cdots. \tag{4.83}$$

Divide both sides of (4.83) by $(l-\frac{1}{2})^2-(\frac{1}{2}n)^2$ and take reciprocals of both sides of the resulting equality to obtain

$$\frac{(E-1)_3F_2\left({A,B,C-1 \atop D,E-1};1\right)}{{}_3F_2\left({A,B,C \atop D,E};1\right)}$$
$$=\frac{x-1}{2}+\frac{(l+\frac{1}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}\,+\,\frac{(l+\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}\,+\,\frac{(l+\frac{3}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}\,+\cdots. \tag{4.84}$$

We repeat the above process for one more term of the continued fraction. Subtract $\frac{1}{2}(x-1)$ from both sides of (4.84) to obtain

$$\frac{(E-1)_3F_2\left({A,B,C-1 \atop D,E-1};1\right)-\frac{x-1}{2}{}_3F_2\left({A,B,C \atop D,E};1\right)}{{}_3F_2\left({A,B,C \atop D,E};1\right)}$$
$$=\frac{(l+\frac{1}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}\,+\,\frac{(l+\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}\,+\,\frac{(l+\frac{3}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}\,+\,\frac{(l+\frac{3}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}\,+\cdots. \tag{4.85}$$

Letting $a = C - 1$, $b = B$, $c = A$, $d = E - 1$, and $e = D$ in Proposition 2.1.3 yields

$$\begin{aligned}
&(E-1)\,{}_3F_2\left(\begin{matrix}A,\, B,\, C-1\\ D,\, E-1\end{matrix};1\right)\\
&\quad = (E-1-B)\,{}_3F_2\left(\begin{matrix}A,\, B,\, C\\ D,\, E\end{matrix};1\right) + \frac{B(D-A)}{D}\,{}_3F_2\left(\begin{matrix}A,\, B+1,\, C\\ D+1,\, E\end{matrix};1\right).
\end{aligned} \tag{4.86}$$

Since

$$\begin{aligned}
E - 1 - B &= \frac{1}{2}(x+m) + l + 1 - 1 - \left(\frac{1}{2}(m+1) + l\right)\\
&= \frac{1}{2}(x-1)
\end{aligned}$$

and

$$\begin{aligned}
B(D-A) &= \left(\frac{1}{2}(m+1) + l\right)\left(\frac{1}{2}(x+n) + l + 1 - \frac{1}{2}(x+m+n+1)\right)\\
&= \left(\frac{1}{2}m + l + \frac{1}{2}\right)\left(l + \frac{1}{2} - \frac{1}{2}m\right)\\
&= \left(l + \frac{1}{2}\right)^2 - \left(\frac{1}{2}m\right)^2,
\end{aligned}$$

we can rewrite (4.86) as

$$\begin{aligned}
&(E-1)\,{}_3F_2\left(\begin{matrix}A,\, B,\, C-1\\ D,\, E-1\end{matrix};1\right) - \frac{x-1}{2}\,{}_3F_2\left(\begin{matrix}A,\, B,\, C\\ D,\, E\end{matrix};1\right)\\
&\quad = \frac{(l+\frac{1}{2})^2 - \left(\frac{1}{2}m\right)^2}{D}\,{}_3F_2\left(\begin{matrix}A,\, B+1,\, C\\ D+1,\, E\end{matrix};1\right),
\end{aligned} \tag{4.87}$$

and (4.85) becomes

$$\frac{\left((l+\frac{1}{2})^2-\left(\frac{1}{2}m\right)^2\right){}_3F_2\left({A,B+1,C \atop D+1,E};1\right)}{D\,{}_3F_2\left({A,B,C \atop D,E};1\right)}$$
$$=\frac{(l+\frac{1}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}\,\underset{+}{}\,\frac{(l+\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}\,\underset{+}{}\,\frac{(l+\frac{3}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}\,\underset{+}{}\,\frac{(l+\frac{3}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}\,\underset{+}{}\cdots. \tag{4.88}$$

Divide both sides of (4.88) by $(l+\frac{1}{2})^2-(\frac{1}{2}m)^2$ and take reciprocals of both sides of the resulting equality to obtain

$$\frac{D\,{}_3F_2\left({A,B,C \atop D,E};1\right)}{{}_3F_2\left({A,B+1,C \atop D+1,E};1\right)}$$
$$=\frac{x+1}{2}+\frac{(l+\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}\,\underset{+}{}\,\frac{(l+\frac{3}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}\,\underset{+}{}\,\frac{(l+\frac{3}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}\,\underset{+}{}\cdots. \tag{4.89}$$

This is just (4.78) in our simplified notation.

Case $l<0$: Suppose that the theorem is true for $l+1$; that is, that the equality

$$\frac{\left(\frac{1}{2}(x+n)+l+2\right){}_3F_2\left({\frac{1}{2}(x+m+n+1),\,\frac{1}{2}(m+1)+l+1,\,\frac{1}{2}(n+1)+l+1 \atop \frac{1}{2}(x+n)+l+2,\,\frac{1}{2}(x+m)+l+2};1\right)}{{}_3F_2\left({\frac{1}{2}(x+m+n+1),\,\frac{1}{2}(m+1)+l+2,\,\frac{1}{2}(n+1)+l+1 \atop \frac{1}{2}(x+n)+l+3,\,\frac{1}{2}(x+m)+l+2};1\right)}$$
$$=\frac{x+1}{2}+\frac{(l+\frac{3}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}\,\underset{+}{}\,\frac{(l+\frac{5}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}\,\underset{+}{}\,\frac{(l+\frac{5}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}\,\underset{+}{}\cdots \tag{4.90}$$

produced by the substitutions $d=a-b+2(l+1)+1$, $e=a-c+2(l+1)+1$ followed by $a=\frac{1}{2}(x+m+n+1)$, $b=\frac{1}{2}(m+1)+l+1$, $c=\frac{1}{2}(n+1)+l+1$ is equivalent to (4.76). We again let $A=\frac{1}{2}(x+m+n+1)$, $B=\frac{1}{2}(m+1)+l$, $C=\frac{1}{2}(n+1)+l$, $D=\frac{1}{2}(x+n)+l+1$, and $E = \frac{1}{2}(x+m) + l + 1$, so our starting point is

$$\frac{(D+1){}_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)}{{}_3F_2\left({A, B+2, C+1 \atop D+2, E+1}; 1\right)} = \frac{x+1}{2} + \frac{(l+\frac{3}{2})^2 - (\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \frac{(l+\frac{5}{2})^2 - (\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} + \frac{(l+\frac{5}{2})^2 - (\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \cdots. \tag{4.91}$$

We will manipulate (4.91) to obtain (4.89).

Take reciprocals of both sides of (4.91). This yields

$$\frac{{}_3F_2\left({A, B+2, C+1 \atop D+2, E+1}; 1\right)}{(D+1){}_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)} = \frac{1}{\frac{1}{2}(x+1)} + \frac{(l+\frac{3}{2})^2 - (\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \frac{(l+\frac{5}{2})^2 - (\frac{1}{2}m)^2}{\frac{1}{2}(x+1)} + \frac{(l+\frac{5}{2})^2 - (\frac{1}{2}n)^2}{\frac{1}{2}(x-1)} + \cdots. \tag{4.92}$$

Let $a = C$, $b = B+1$, $c = A$, $d = E$, and $e = D+1$ in (4.50) (the rearranged form of Proposition 2.1.3) to obtain

$$\begin{aligned} {}_3F_2\left({A,\ B+2,\ C+1 \atop D+2,\ E+1}; 1\right) &= \frac{E(D+1)}{(B+1)(D+1-A)} {}_3F_2\left({A,\ B+1,\ C \atop D+1,\ E}; 1\right) \\ &\quad - \frac{(D+1)(E-B-1)}{(B+1)(D+1-A)} {}_3F_2\left({A,\ B+1,\ C+1 \atop D+1,\ E+1}; 1\right). \end{aligned} \tag{4.93}$$

Then

$$\begin{aligned} \frac{{}_3F_2\left({A, B+2, C+1 \atop D+2, E+1}; 1\right)}{(D+1){}_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)} &= \frac{\frac{E(D+1)}{(B+1)(D+1-A)} {}_3F_2\left({A, B+1, C \atop D+1, E}; 1\right) - \frac{(D+1)(E-B-1)}{(B+1)(D+1-A)} {}_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)}{(D+1){}_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)} \\ &= \frac{1}{(B+1)(D+1-A)} \left[\frac{E\, {}_3F_2\left({A, B+1, C \atop D+1, E}; 1\right)}{{}_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)} - (E-B-1) \right] \end{aligned}$$

$$= \frac{1}{(l+\frac{3}{2})^2 - \left(\frac{1}{2}m\right)^2} \left[\frac{E\, {}_3F_2\left({A, B+1, C \atop D+1, E}; 1\right)}{{}_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)} - \frac{x-1}{2} \right] \tag{4.94}$$

since

$$\begin{aligned}(B+1)(D+1-A) &= \left(\frac{1}{2}(m+1)+l+1\right)\left(\frac{1}{2}(x+n)+l+1+1-\frac{1}{2}(x+m+n+1)\right)\\ &= \left(l+\frac{3}{2}+\frac{1}{2}m\right)\left(l+\frac{3}{2}-\frac{1}{2}m\right)\\ &= \left(l+\frac{3}{2}\right)^2-\left(\frac{1}{2}m\right)^2\end{aligned}$$

and

$$\begin{aligned}E-B-1 &= \frac{1}{2}(x+m)+l+1-\left(\frac{1}{2}(m+1)+l\right)-1\\ &= \frac{1}{2}(x-1).\end{aligned}$$

Hence (4.92) becomes

$$\begin{aligned}&\frac{1}{(l+\frac{3}{2})^2 - \left(\frac{1}{2}m\right)^2} \left[\frac{E\, {}_3F_2\left({A, B+1, C \atop D+1, E}; 1\right)}{{}_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)} - \frac{x-1}{2} \right]\\ &= \frac{1}{\frac{1}{2}(x+1)+}\;\frac{(l+\frac{3}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)\quad+}\;\frac{(l+\frac{5}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)\quad+}\;\frac{(l+\frac{5}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)\quad+\cdots}.\end{aligned} \tag{4.95}$$

Multiply both sides of (4.95) by $(l+\frac{3}{2})^2-(\frac{1}{2}m)^2$ and then add $\frac{1}{2}(x-1)$ to both sides of the resulting equality to obtain

$$\begin{aligned}&\frac{E\, {}_3F_2\left({A, B+1, C \atop D+1, E}; 1\right)}{{}_3F_2\left({A, B+1, C+1 \atop D+1, E+1}; 1\right)}\\ &= \frac{x-1}{2}+\;\frac{(l+\frac{3}{2})^2-\left(\frac{1}{2}m\right)^2}{\frac{1}{2}(x+1)\quad+}\;\frac{(l+\frac{3}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)\quad+}\;\frac{(l+\frac{5}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)\quad+\cdots}.\end{aligned} \tag{4.96}$$

Taking reciprocals of both sides of (4.96) yields

$$\frac{{}_3F_2\left({A,B+1,C+1 \atop D+1,E+1};1\right)}{E\,{}_3F_2\left({A,B+1,C \atop D+1,E};1\right)} = \frac{1}{\frac{1}{2}(x-1)+}\,\frac{(l+\frac{3}{2})^2-\left(\frac{1}{2}m\right)^2}{\frac{1}{2}(x+1)}\,\underset{+}{}\,\frac{(l+\frac{3}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)}\,\underset{+}{}\,\frac{(l+\frac{5}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)}\,\underset{+\cdots}{}. \tag{4.97}$$

In (4.50), let $a=B$, $b=C$, $c=A$, $d=D$, and $e=E$ to obtain

$$\begin{aligned}{}_3F_2\left({A,\,B+1,\,C+1 \atop D+1,\,E+1};1\right) &= \frac{DE}{C(E-A)}{}_3F_2\left({A,\,B,\,C \atop D,\,E};1\right)\\ &\quad-\frac{E(D-C)}{C(E-A)}{}_3F_2\left({A,\,B+1,\,C \atop D+1,\,E};1\right).\end{aligned} \tag{4.98}$$

Then

$$\begin{aligned}\frac{{}_3F_2\left({A,B+1,C+1 \atop D+1,E+1};1\right)}{E\,{}_3F_2\left({A,B+1,C \atop D+1,E};1\right)} &= \frac{\frac{DE}{C(E-A)}{}_3F_2\left({A,B,C \atop D,E};1\right)-\frac{E(D-C)}{C(E-A)}{}_3F_2\left({A,B+1,C \atop D+1,E};1\right)}{E\,{}_3F_2\left({A,B+1,C \atop D+1,E};1\right)}\\ &= \frac{1}{C(E-A)}\left[\frac{D\,{}_3F_2\left({A,B,C \atop D,E};1\right)}{{}_3F_2\left({A,B+1,C \atop D+1,E};1\right)}-(D-C)\right]\\ &= \frac{1}{(l+\frac{1}{2})^2-\left(\frac{1}{2}n\right)^2}\left[\frac{D\,{}_3F_2\left({A,B,C \atop D,E};1\right)}{{}_3F_2\left({A,B+1,C \atop D+1,E};1\right)}-\frac{x+1}{2}\right],\end{aligned} \tag{4.99}$$

since

$$\begin{aligned}
C(E-A) &= \left(\frac{1}{2}(n+1)+l\right)\left(\frac{1}{2}(x+m)+l+1-\frac{1}{2}(x+m+n+1)\right) \\
&= \left(l+\frac{1}{2}+\frac{1}{2}n\right)\left(l+\frac{1}{2}-\frac{1}{2}n\right) \\
&= \left(l+\frac{1}{2}\right)^2-\left(\frac{1}{2}n\right)^2
\end{aligned}$$

and

$$\begin{aligned}
D-C &= \frac{1}{2}(x+n)+l+1-\left(\frac{1}{2}(n+1)+l\right) \\
&= \frac{1}{2}(x+1).
\end{aligned}$$

Thus (4.97) can be rewritten as

$$\begin{aligned}
&\frac{1}{(l+\frac{1}{2})^2-\left(\frac{1}{2}n\right)^2}\left[\frac{D\,{}_3F_2\left({A,B,C \atop D,E};1\right)}{{}_3F_2\left({A,B+1,C \atop D+1,E};1\right)}-\frac{x+1}{2}\right] \\
&\quad = \frac{1}{\frac{1}{2}(x-1)+}\;\frac{(l+\frac{3}{2})^2-\left(\frac{1}{2}m\right)^2}{\frac{1}{2}(x+1)\quad+}\;\frac{(l+\frac{3}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)\quad+}\;\frac{(l+\frac{5}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)\quad+\cdots}. \qquad (4.100)
\end{aligned}$$

Multiply both sides of (4.100) by $(l+\frac{1}{2})^2-\left(\frac{1}{2}n\right)^2$ and add $\frac{1}{2}(x+1)$ to both sides of the result to obtain (4.89).

Note that for all l, the tail of the continued fraction created by the algorithm of Theorem 4.2.2 matches the tail of (4.76),

$$\frac{(k+\frac{1}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)\quad+}\;\frac{(k+\frac{1}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)\quad+}\;\frac{(k+\frac{3}{2})^2-(\frac{1}{2}m)^2}{\frac{1}{2}(x+1)\quad+}\;\frac{(k+\frac{3}{2})^2-(\frac{1}{2}n)^2}{\frac{1}{2}(x-1)\quad+\cdots},$$

from $k=N$ onward, where N is some positive integer. Thus the convergence of the continued fraction in Entry 39 implies the convergence of all the continued fractions in this proof. □

4.3 Other entries

All of the related continued fractions in Chapters 12 and 16 of Ramanujan's notebooks can be derived from Entries 34, 39, and 40. Thus it is desirable to have a proof of Entry 40 that is similar in style to the proofs of Entries 34 and 30 given here. Entry 40, however, has five variables instead of the two in Entry 25 and three in Entries 34, 36, and 39.

Theorem 4.3.1 (Entry 40). *Let* $P = \prod \Gamma(\frac{1}{2}(\alpha \pm \beta \pm \gamma \pm \delta \pm \epsilon + 1))$, *where the product contains eight distinct Gamma functions and where the argument of each Gamma function contains an even number of minus signs. Let* $Q = \prod \Gamma(\frac{1}{2}(\alpha \pm \beta \pm \gamma \pm \delta \pm \epsilon + 1))$, *where the product contains eight distinct Gamma functions and where the argument of each Gamma function contains an odd number of minus signs. Suppose that at least one of the parameters* β, γ, δ, ϵ *is equal to a nonzero integer. Then*

$$\frac{P-Q}{P+Q} = \cfrac{8\alpha\beta\gamma\delta\epsilon}{1\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\epsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2-1)^2-2^2\} + \cfrac{64(\alpha^2-1^2)(\beta^2-1^2)(\gamma^2-1^2)(\delta^2-1^2)(\epsilon^2-1^2)}{3\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\epsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2-5)^2-6^2\} + \cfrac{64(\alpha^2-2^2)(\beta^2-2^2)(\gamma^2-2^2)(\delta^2-2^2)(\epsilon^2-2^2)}{5\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\epsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2-13)^2-14^2\} + \cdots}}}. \tag{4.101}$$

If Entry 40 can be proven in a similar way, the proof would require a three-term recurrence of ${}_5F_4$ hypergeometric functions. As the contiguous relations involving ${}_5F_4$ hypergeometric functions in general have six terms, it is unknown whether an appropriate three-term recurrence of this type exists. There is a hypergeometric proof of Entry 40 that exists. Masson [13] used a three-term recurrence of very well-poised balanced ${}_9F_8$ hypergeometric functions to generate a continued fraction and he then applied a limiting process on this continued fraction to obtain Entry 40 and a companion to it. It may be that this is the necessary method for obtaining Entry 40 and other identities with more than three variables.

Chapter 5

q-Analogues

5.1 Introduction

There is a theory of basic hypergeometric functions that parallels the theory of hypergeometric functions. The base commonly used is q, where $|q| < 1$, and the function corresponding to ${}_2F_1\left(\begin{smallmatrix}a,b\\c\end{smallmatrix};x\right)$ is

$$
{}_2\phi_1\left(\begin{matrix}\alpha,\,\beta\\ \gamma\end{matrix};\,q;\,x\right) := \sum_{n=0}^{\infty} \frac{(\alpha;q)_n(\beta;q)_n}{(\gamma;q)_n(q;q)_n} x^n, \qquad |x| < 1,
$$

where $(\alpha;q)_n := (1-\alpha)(1-\alpha q)(1-\alpha q^2)\cdots(1-\alpha q^{n-1})$. In this chapter, we use the shorthand ${}_2\phi_1(\alpha,\beta;\gamma;x)$ for ${}_2\phi_1\left(\begin{smallmatrix}\alpha,\beta\\ \gamma\end{smallmatrix};\,q;\,x\right)$. It should be noted that many hypergeometric identities and relations have what are called q-analogues. The original identity can be recovered by taking the limit as q approaches 1 from below, though the evaluation of this limit requires care when continued fractions are involved.

The q-analogue of the recurrence

$$
{}_2F_1\left(\begin{matrix}a,\,b\\ c\end{matrix};\,x\right) = \frac{c+(a-b+1)x}{c}\,{}_2F_1\left(\begin{matrix}a+1,\,b\\ c+1\end{matrix};\,x\right) - \frac{(a+1)(c-b+1)x}{c(c+1)}\,{}_2F_1\left(\begin{matrix}a+2,\,b\\ c+2\end{matrix};\,x\right) \tag{5.1}
$$

is

$$
\begin{aligned}
{}_2\phi_1(\alpha,\beta;\gamma;x) &= \frac{q(1-\gamma)+(\beta-\alpha q)x}{q(1-\gamma)}\,{}_2\phi_1(\alpha q,\beta;\gamma q;x) \\
&\quad - \frac{(1-\alpha q)(\beta-\gamma q)x}{q(1-\gamma)(1-\gamma q)}\,{}_2\phi_1\left(\alpha q^2,\beta;\gamma q^2;x\right). \qquad (5.2)
\end{aligned}
$$

Just as Ramanujan's Entry 22 (3.14) can be derived from the three-term recurrence (5.1) by iteration (as in the proof of Lemma 2.1.4), (5.2) can be used to produce a continued fraction. In this case, the result is Equation (42) in [15],

$$
\begin{aligned}
\frac{{}_2\phi_1(\alpha q,\beta;\gamma q;x)}{{}_2\phi_1(\alpha,\beta;\gamma;x)} &= \frac{q(1-\gamma)}{q(1-\gamma)+(\beta-\alpha q)x\,+}\,\frac{(1-\alpha q)(\gamma q-\beta)xq}{q(1-\gamma q)+(\beta-\alpha q^2)x} \\
&\quad \frac{(1-\alpha q^2)(\gamma q^2-\beta)xq}{+\,q(1-\gamma q^2)+(\beta-\alpha q^3)x\,+}\,\frac{(1-\alpha q^3)(\gamma q^3-\beta)xq}{q(1-\gamma q^3)+(\beta-\alpha q^4)x\,+\cdots}. \qquad (5.3)
\end{aligned}
$$

Ramanathan indicates in [15] that Ramanujan knew this continued fraction, which is the q-analogue of Entry 22, but it does not appear in any of his notebooks. Ramanathan uses (5.3) in [15] to prove Entry 16.12 of Ramanujan's second notebook, which can also be found in [5] and is stated here.

Theorem 5.1.1 (Entry 16.12)**.** *Suppose that α, β, and q are complex numbers with $|\alpha\beta|<1$ and $|q|<1$ or that $\alpha=\beta q^{2m+1}$ for some integer m. Then*

$$
\frac{(\alpha^2q^3;q^4)_\infty(\beta^2q^3;q^4)_\infty}{(\alpha^2q;q^4)_\infty(\beta^2q;q^4)_\infty} = \frac{1}{1-\alpha\beta\,+}\,\frac{(\alpha-\beta q)(\beta-\alpha q)}{(1-\alpha\beta)(1+q^2)\,+}\,\frac{(\alpha-\beta q^3)(\beta-\alpha q^3)}{(1-\alpha\beta)(1+q^4)\,+\cdots}. \qquad (5.4)
$$

This continued fraction identity is a q-analogue of Entry 25 (1.1). In this chapter, two q-analogues of Theorem 3.2.1 are given, and their proofs also rely on (5.3). We begin by stating and proving several recurrences which will be used in these proofs. These recurrences can all be derived from the following three recurrences of E. Heine [9].

Theorem 5.1.2 (Heine)**.** *Suppose that* $|q| < 1$ *and* $|x| < 1$*. Then*

$$ {}_2\phi_1\left(a, b; c/q; x\right) - {}_2\phi_1\left(a, b; c; x\right) = cx\frac{(1-a)(1-b)}{(q-c)(1-c)}{}_2\phi_1\left(aq, bq; cq; x\right), \tag{5.5} $$

$$ {}_2\phi_1\left(aq, b; c; x\right) - {}_2\phi_1\left(a, b; c; x\right) = ax\frac{1-b}{1-c}{}_2\phi_1\left(aq, bq; cq; x\right), \tag{5.6} $$

and

$$ {}_2\phi_1\left(aq, b; cq; x\right) - {}_2\phi_1\left(a, b; c; x\right) = ax\frac{(1-b)(1-c/a)}{(1-c)(1-cq)}{}_2\phi_1\left(aq, bq; cq^2; x\right). \tag{5.7} $$

Proposition 5.1.3. *If* $|q| < 1$ *and* $|x| < 1$*, then*

$$ {}_2\phi_1\left(\alpha, \beta q; \gamma q; x\right) = \frac{(1-\gamma)(\alpha-\beta)}{(1-\beta)(\alpha-\gamma)}{}_2\phi_1\left(\alpha, \beta; \gamma; x\right) + \frac{(1-\alpha)(\beta-\gamma)}{(1-\beta)(\alpha-\gamma)}{}_2\phi_1\left(\alpha q, \beta; \gamma q; x\right). \tag{5.8} $$

Proof. We first rearrange (5.7) as

$$ \frac{x}{(1-c)(1-cq)}{}_2\phi_1\left(aq, bq; cq^2; x\right) = \frac{1}{(1-b)(a-c)}\left[{}_2\phi_1\left(aq, b; cq; x\right) - {}_2\phi_1\left(a, b; c; x\right)\right] \tag{5.9} $$

and then substitute into this equation twice. The substitution $a = \alpha$, $b = \beta$, $c = \gamma$ yields

$$ \frac{x}{(1-\gamma)(1-\gamma q)}{}_2\phi_1\left(\alpha q, \beta q; \gamma q^2; x\right) = \frac{1}{(1-\beta)(\alpha-\gamma)}\left[{}_2\phi_1\left(\alpha q, \beta; \gamma q; x\right) - {}_2\phi_1\left(\alpha, \beta; \gamma; x\right)\right], \tag{5.10} $$

while the substitution $a = \beta$, $b = \alpha$, $c = \gamma$ yields

$$ \frac{x}{(1-\gamma)(1-\gamma q)}{}_2\phi_1\left(\alpha q, \beta q; \gamma q^2; x\right) = \frac{1}{(1-\alpha)(\beta-\gamma)}\left[{}_2\phi_1\left(\alpha, \beta q; \gamma q; x\right) - {}_2\phi_1\left(\alpha, \beta; \gamma; x\right)\right] \tag{5.11} $$

when the symmetry of the numerator arguments of ${}_2\phi_1\left(a, b; c; x\right)$ is exploited. Since the left sides of (5.10) and (5.11) are equal, we may set the right sides equal to each other to obtain

$$\frac{1}{(1-\beta)(\alpha-\gamma)}\left[{}_2\phi_1\left(\alpha q,\beta;\gamma q;x\right)-{}_2\phi_1\left(\alpha,\beta;\gamma;x\right)\right]$$
$$=\frac{1}{(1-\alpha)(\beta-\gamma)}\left[{}_2\phi_1\left(\alpha,\beta q;\gamma q;x\right)-{}_2\phi_1\left(\alpha,\beta;\gamma;x\right)\right]. \tag{5.12}$$

Upon multiplying both sides by $(1-\alpha)(\beta-\gamma)$ and adding ${}_2\phi_1\left(\alpha,\beta;\gamma;x\right)$ to both sides of the equation, we find that

$$\begin{aligned}{}_2\phi_1\left(\alpha,\beta q;\gamma q;x\right)&={}_2\phi_1\left(\alpha,\beta;\gamma;x\right)+\frac{(1-\alpha)(\beta-\gamma)}{(1-\beta)(\alpha-\gamma)}\left[{}_2\phi_1\left(\alpha q,\beta;\gamma q;x\right)-{}_2\phi_1\left(\alpha,\beta;\gamma;x\right)\right]\\&=\left[1-\frac{(1-\alpha)(\beta-\gamma)}{(1-\beta)(\alpha-\gamma)}\right]{}_2\phi_1\left(\alpha,\beta;\gamma;x\right)+\frac{(1-\alpha)(\beta-\gamma)}{(1-\beta)(\alpha-\gamma)}{}_2\phi_1\left(\alpha q,\beta;\gamma q;x\right)\\&=\frac{(1-\gamma)(\alpha-\beta)}{(1-\beta)(\alpha-\gamma)}{}_2\phi_1\left(\alpha,\beta;\gamma;x\right)+\frac{(1-\alpha)(\beta-\gamma)}{(1-\beta)(\alpha-\gamma)}{}_2\phi_1\left(\alpha q,\beta;\gamma q;x\right).\end{aligned} \tag{5.13}$$

□

Proposition 5.1.4. *If $|q|<1$ and $|x|<1$, then*

$${}_2\phi_1\left(\alpha,\beta;\gamma;x\right)={}_2\phi_1\left(\alpha,\beta;\gamma q;x\right)+\frac{\gamma x(1-\alpha)(1-\beta)}{(1-\gamma)(1-\gamma q)}{}_2\phi_1\left(\alpha q,\beta q;\gamma q^2;x\right). \tag{5.14}$$

Proof. In (5.5), let $a=\alpha$, $b=\beta$, and $c=\gamma q$. The resulting equation is

$$\begin{aligned}{}_2\phi_1\left(\alpha,\beta;\gamma;x\right)-{}_2\phi_1\left(\alpha,\beta;\gamma q;x\right)&=\gamma qx\frac{(1-\alpha)(1-\beta)}{(q-\gamma q)(1-\gamma q)}{}_2\phi_1\left(\alpha q,\beta q;\gamma q^2;x\right)\\&=\gamma x\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\gamma q)}{}_2\phi_1\left(\alpha q,\beta q;\gamma q^2;x\right),\end{aligned} \tag{5.15}$$

so adding ${}_2\phi_1\left(\alpha,\beta;\gamma q;x\right)$ to both sides yields the desired recurrence. □

Proposition 5.1.5. *If $|q| < 1$ and $|x| < 1$, then*

$$ {}_2\phi_1(\alpha, \beta; \gamma q; x) = \frac{\beta(1-\gamma)}{\beta-\gamma} {}_2\phi_1(\alpha, \beta; \gamma; x) - \frac{\gamma(1-\beta)}{\beta-\gamma} {}_2\phi_1(\alpha, \beta q; \gamma q; x). \tag{5.16} $$

Proof. We begin by making the substitutions $a = \beta$, $b = \alpha$, and $c = \gamma$ in (5.7). This yields the relation

$$ {}_2\phi_1(\alpha, \beta q; \gamma q; x) - {}_2\phi_1(\alpha, \beta; \gamma; x) = \beta x \frac{(1-\alpha)(1-\gamma/\beta)}{(1-\gamma)(1-\gamma q)} {}_2\phi_1\left(\alpha q, \beta q; \gamma q^2; x\right). \tag{5.17} $$

We rearrange (5.17) into the form

$$ \frac{x(1-\alpha)}{(1-\gamma)(1-\gamma q)} {}_2\phi_1\left(\alpha q, \beta q; \gamma q^2; x\right) = \frac{1}{\beta-\gamma}\left[{}_2\phi_1(\alpha, \beta q; \gamma q; x) - {}_2\phi_1(\alpha, \beta; \gamma; x)\right] \tag{5.18} $$

and also rearrange Proposition 5.1.4 into the form

$$ \frac{x(1-\alpha)}{(1-\gamma)(1-\gamma q)} {}_2\phi_1\left(\alpha q, \beta q; \gamma q^2; x\right) = \frac{1}{\gamma(1-\beta)}\left[{}_2\phi_1(\alpha, \beta; \gamma; x) - {}_2\phi_1(\alpha, \beta; \gamma q; x)\right]. \tag{5.19} $$

Note that the left sides of these rearrangements are the same, so we may combine them to obtain

$$ \frac{1}{\beta-\gamma}\left[{}_2\phi_1(\alpha, \beta q; \gamma q; x) - {}_2\phi_1(\alpha, \beta; \gamma; x)\right] = \frac{1}{\gamma(1-\beta)}\left[{}_2\phi_1(\alpha, \beta; \gamma; x) - {}_2\phi_1(\alpha, \beta; \gamma q; x)\right]. \tag{5.20} $$

Multiplying both sides of (5.20) by $-\gamma(1-\beta)$ and adding ${}_2\phi_1(\alpha, \beta; \gamma; x)$ to both sides yields

$$\begin{aligned} {}_2\phi_1(\alpha, \beta; \gamma q; x) &= {}_2\phi_1(\alpha, \beta; \gamma; x) - \frac{\gamma(1-\beta)}{\beta-\gamma}\left[{}_2\phi_1(\alpha, \beta q; \gamma q; x) - {}_2\phi_1(\alpha, \beta; \gamma; x)\right] \\ &= \left[1 + \frac{\gamma(1-\beta)}{\beta-\gamma}\right] {}_2\phi_1(\alpha, \beta; \gamma; x) - \frac{\gamma(1-\beta)}{\beta-\gamma} {}_2\phi_1(\alpha, \beta q; \gamma q; x) \\ &= \frac{\beta(1-\gamma)}{\beta-\gamma} {}_2\phi_1(\alpha, \beta; \gamma; x) - \frac{\gamma(1-\beta)}{\beta-\gamma} {}_2\phi_1(\alpha, \beta q; \gamma q; x). \end{aligned} \tag{5.21}$$

□

Proposition 5.1.6. *If* $|q| < 1$ *and* $|x| < 1$*, then*

$${}_2\phi_1(\alpha q, \beta q; \gamma q; x) = \frac{\beta(1-\gamma)}{(1-\beta)(\gamma-\alpha\beta x)} {}_2\phi_1(\alpha, \beta; \gamma; x) - \frac{\beta-\gamma}{(1-\beta)(\gamma-\alpha\beta x)} {}_2\phi_1(\alpha q, \beta; \gamma q; x). \tag{5.22}$$

Proof. Replace α by αq in Proposition 5.1.5 to obtain

$${}_2\phi_1(\alpha q, \beta; \gamma q; x) = \frac{\beta(1-\gamma)}{\beta-\gamma} {}_2\phi_1(\alpha q, \beta; \gamma; x) - \frac{\gamma(1-\beta)}{\beta-\gamma} {}_2\phi_1(\alpha q, \beta q; \gamma q; x). \tag{5.23}$$

In (5.6), let $a = \alpha$, $b = \beta$, and $c = \gamma$. This yields

$${}_2\phi_1(\alpha q, \beta; \gamma; x) - {}_2\phi_1(\alpha, \beta; \gamma; x) = \alpha x \frac{1-\beta}{1-\gamma} {}_2\phi_1(\alpha q, \beta q; \gamma q; x). \tag{5.24}$$

Rewrite this as

$${}_2\phi_1(\alpha q, \beta; \gamma; x) = {}_2\phi_1(\alpha, \beta; \gamma; x) + \alpha x \frac{1-\beta}{1-\gamma} {}_2\phi_1(\alpha q, \beta q; \gamma q; x), \tag{5.25}$$

and substitute it into (5.23) to find that

$$\begin{aligned}
&{}_2\phi_1(\alpha q, \beta; \gamma q; x) \\
&= \frac{\beta(1-\gamma)}{\beta-\gamma}\left[{}_2\phi_1(\alpha, \beta; \gamma; x) + \alpha x \frac{1-\beta}{1-\gamma}{}_2\phi_1(\alpha q, \beta q; \gamma q; x)\right] - \frac{\gamma(1-\beta)}{\beta-\gamma}{}_2\phi_1(\alpha q, \beta q; \gamma q; x) \\
&= \frac{\beta(1-\gamma)}{\beta-\gamma}{}_2\phi_1(\alpha, \beta; \gamma; x) + \frac{\alpha\beta x(1-\beta)}{\beta-\gamma}{}_2\phi_1(\alpha q, \beta q; \gamma q; x) - \frac{\gamma(1-\beta)}{\beta-\gamma}{}_2\phi_1(\alpha q, \beta q; \gamma q; x) \\
&= \frac{\beta(1-\gamma)}{\beta-\gamma}{}_2\phi_1(\alpha, \beta; \gamma; x) + \left[\frac{\alpha\beta x(1-\beta)}{\beta-\gamma} - \frac{\gamma(1-\beta)}{\beta-\gamma}\right]{}_2\phi_1(\alpha q, \beta q; \gamma q; x) \\
&= \frac{\beta(1-\gamma)}{\beta-\gamma}{}_2\phi_1(\alpha, \beta; \gamma; x) - \frac{(1-\beta)(\gamma-\alpha\beta x)}{\beta-\gamma}{}_2\phi_1(\alpha q, \beta q; \gamma q; x). \qquad (5.26)
\end{aligned}$$

Solving this equation for ${}_2\phi_1(\alpha q, \beta q; \gamma q; x)$ yields the desired relation. □

5.2 A q-analogue of Theorem 3.2.1

Theorem 5.2.1. *Suppose that α, β, and q are complex numbers with $|\alpha\beta/q| < 1$ and $0 < |q| < 1$ or that $q \neq 0$ and $\alpha = \beta q^{2m}$, where m is an integer. Define $P = (\alpha^2 q^3; q^4)_\infty(\beta^2 q; q^4)_\infty$ and $Q = (\alpha^2 q; q^4)_\infty(\beta^2 q^3; q^4)_\infty$. Then*

$$\begin{aligned}
\frac{\alpha P - \beta Q}{P+Q} = &\frac{q(\alpha-\beta)}{q(1+q^0) - \alpha\beta(1+q^2)} + \frac{q(\beta-\alpha q^2)(\alpha-\beta q^2)}{q(1+q^2) - \alpha\beta(1+q^4)} \\
&+ \frac{q(\beta-\alpha q^4)(\alpha-\beta q^4)}{q(1+q^4) - \alpha\beta(1+q^6)} + \frac{q(\beta-\alpha q^6)(\alpha-\beta q^6)}{q(1+q^6) - \alpha\beta(1+q^8)} + \cdots.
\end{aligned}$$

We again prove the result first and discuss convergence later.

Proof. In (5.3), let $\alpha = b$, $\beta = a$, $\gamma = aq^2/b$ and $x = -q/b$. This results in the continued fraction

$$\begin{aligned}
&\frac{{}_2\phi_1\left({a,bq \atop aq^3/b};\, q;\, -q/b\right)}{{}_2\phi_1\left({a,b \atop aq^2/b};\, q;\, -q/b\right)} \\
&\quad = \frac{q(1-aq^2/b)}{q(1-aq^2/b)+(a-bq)(-q/b)+}\,\frac{(1-bq)(aq^3/b-a)(-q/b)q}{q(1-aq^3/b)+(a-bq^2)(-q/b)} \\
&\qquad \frac{(1-bq^2)(aq^4/b-a)(-q/b)q}{+\,q(1-aq^4/b)+(a-bq^3)(-q/b)+}\,\frac{(1-bq^3)(aq^5/b-a)(-q/b)q}{q(1-aq^5/b)+(a-bq^4)(-q/b)+\cdots} \\
&\quad = \frac{(q/b)(b-aq^2)}{(q/b)(b-aq^2-(a-bq))+}\,\frac{(aq^2/b^2)(1-bq)(b-q^3)}{(q/b)(b-aq^3-(a-bq^2))} \\
&\qquad \frac{(aq^2/b^2)(1-bq^2)(b-q^4)}{+\,(q/b)(b-aq^4-(a-bq^3))+}\,\frac{(aq^2/b^2)(1-bq^3)(b-q^5)}{(q/b)(b-aq^5-(a-bq^4))+\cdots} \\
&\quad = \frac{b-aq^2}{b(1+q)-a(1+q^2)+}\,\frac{a(1-bq)(b-q^3)}{b(1+q^2)-a(1+q^3)} \\
&\qquad \frac{a(1-bq^2)(b-q^4)}{+\,b(1+q^3)-a(1+q^4)+}\,\frac{a(1-bq^3)(b-q^5)}{b(1+q^4)-a(1+q^5)+\cdots}.
\end{aligned} \tag{5.27}$$

We evaluate ${}_2\phi_1\left({a,bq \atop aq^3/b};\, q;\, -q/b\right)$ and ${}_2\phi_1\left({a,b \atop aq^2/b};\, q;\, -q/b\right)$, rewriting them in terms of ${}_2\phi_1\left({a,b \atop aq/b};\, q;\, -q/b\right)$, ${}_2\phi_1\left({aq,b \atop aq^2/b};\, q;\, -q/b\right)$, and ${}_2\phi_1\left({aq^2,b \atop aq^3/b};\, q;\, -q/b\right)$, which allows a closed-form evaluation by applying

$${}_2\phi_1\left({\alpha,\,\beta \atop \alpha q/\beta};\, q;\, -q/\beta\right) = \frac{(\alpha q;q^2)_\infty(-q;q)_\infty(\alpha q^2/\beta^2;q^2)_\infty}{(\alpha q/\beta;q)_\infty(-q/\beta;q)_\infty}, \tag{5.28}$$

the q-analogue of Kummer's Identity [1]. We will use Propositions 5.1.3 and 5.1.5 to rewrite the q-hypergeometric functions.

In Proposition 5.1.5, let $\alpha = a$, $\beta = b$, and $\gamma = aq/b$ to obtain

$$\begin{aligned}
{}_2\phi_1\left(a,b;aq^2/b;x\right) &= \frac{b(1-aq/b)}{b-aq/b}{}_2\phi_1\left(a,b;aq/b;x\right) - \frac{aq/b(1-b)}{b-aq/b}{}_2\phi_1\left(a,bq;aq^2/b;x\right) \\
&= \frac{b(b-aq)}{b^2-aq}{}_2\phi_1\left(a,b;aq/b;x\right) - \frac{aq(1-b)}{b^2-aq}{}_2\phi_1\left(a,bq;aq^2/b;x\right).
\end{aligned} \tag{5.29}$$

In Proposition 5.1.3, let $\alpha = a$, $\beta = b$, and $\gamma = aq/b$ to obtain

$$\begin{aligned}
&{}_2\phi_1\left(a, bq; aq^2/b; x\right) \\
&\quad = \frac{(1-aq/b)(a-b)}{(1-b)(a-aq/b)}{}_2\phi_1\left(a, b; aq/b; x\right) + \frac{(1-a)(b-aq/b)}{(1-b)(a-aq/b)}{}_2\phi_1\left(aq, b; aq^2/b; x\right) \\
&\quad = \frac{(b-aq)(a-b)}{a(1-b)(b-q)}{}_2\phi_1\left(a, b; aq/b; x\right) + \frac{(1-a)(b^2-aq)}{a(1-b)(b-q)}{}_2\phi_1\left(aq, b; aq^2/b; x\right), \qquad (5.30)
\end{aligned}$$

and substitute (5.30) into (5.29). This yields

$$\begin{aligned}
&{}_2\phi_1\left(a, b; aq^2/b; x\right) \\
&\quad = \frac{b(b-aq)}{b^2-aq}{}_2\phi_1\left(a, b; aq/b; x\right) - \frac{aq(1-b)}{b^2-aq}\left[\frac{(b-aq)(a-b)}{a(1-b)(b-q)}{}_2\phi_1\left(a, b; aq/b; x\right)\right. \\
&\qquad \left. + \frac{(1-a)(b^2-aq)}{a(1-b)(b-q)}{}_2\phi_1\left(aq, b; aq^2/b; x\right)\right] \\
&\quad = \frac{b(b-aq)}{b^2-aq}{}_2\phi_1\left(a, b; aq/b; x\right) - \frac{q(b-aq)(a-b)}{(b^2-aq)(b-q)}{}_2\phi_1\left(a, b; aq/b; x\right) \\
&\qquad - \frac{q(1-a)}{b-q}{}_2\phi_1\left(aq, b; aq^2/b; x\right) \\
&\quad = \frac{b-aq}{b^2-aq}\left[b - \frac{q(a-b)}{b-q}\right]{}_2\phi_1\left(a, b; aq/b; x\right) - \frac{q(1-a)}{b-q}{}_2\phi_1\left(aq, b; aq^2/b; x\right) \\
&\quad = \frac{b-aq}{b^2-aq}\left[\frac{b(b-q)-q(a-b)}{b-q}\right]{}_2\phi_1\left(a, b; aq/b; x\right) - \frac{q(1-a)}{b-q}{}_2\phi_1\left(aq, b; aq^2/b; x\right) \\
&\quad = \frac{(b-aq)(b^2-aq)}{(b^2-aq)(b-q)}{}_2\phi_1\left(a, b; aq/b; x\right) - \frac{q(1-a)}{b-q}{}_2\phi_1\left(aq, b; aq^2/b; x\right) \\
&\quad = \frac{b-aq}{b-q}{}_2\phi_1\left(a, b; aq/b; x\right) - \frac{q(1-a)}{b-q}{}_2\phi_1\left(aq, b; aq^2/b; x\right). \qquad (5.31)
\end{aligned}$$

Let $x = -q/b$. We now have written ${}_2\phi_1\left(\begin{smallmatrix} a,b \\ aq^2/b \end{smallmatrix}; q; -q/b\right)$ in terms of ${}_2\phi_1\left(\begin{smallmatrix} a,b \\ aq/b \end{smallmatrix}; q; -q/b\right)$ and ${}_2\phi_1\left(\begin{smallmatrix} aq,b \\ aq^2/b \end{smallmatrix}; q; -q/b\right)$.

Next, replace a by aq in (5.31) to obtain

$$ {}_2\phi_1\left(aq, b; aq^3/b; x\right) = \frac{b-aq^2}{b-q}{}_2\phi_1\left(aq, b; aq^2/b; x\right) - \frac{q(1-aq)}{(b-q)}{}_2\phi_1\left(aq^2, b; aq^3/b; x\right), \qquad (5.32)$$

and in (5.8), let $\alpha = a$, $\beta = b$, and $\gamma = aq^2/b$ to obtain

$$\begin{aligned}
&{}_2\phi_1\left(a, bq; aq^3/b; x\right) \\
&\quad = \frac{(1-aq^2/b)(a-b)}{(1-b)(a-aq^2/b)}{}_2\phi_1\left(a, b; aq^2/b; x\right) + \frac{(1-a)(b-aq^2/b)}{(1-b)(a-aq^2/b)}{}_2\phi_1\left(aq, b; aq^3/b; x\right) \\
&\quad = \frac{(b-aq^2)(a-b)}{a(1-b)(b-q^2)}{}_2\phi_1\left(a, b; aq^2/b; x\right) + \frac{(1-a)(b^2-aq^2)}{a(1-b)(b-q^2)}{}_2\phi_1\left(aq, b; aq^3/b; x\right). \qquad (5.33)
\end{aligned}$$

Substituting (5.31) and (5.32) into (5.33) yields

$$\begin{aligned}
&{}_2\phi_1\left(a, bq; aq^3/b; x\right) \\
&\quad = \frac{(b-aq^2)(a-b)}{a(1-b)(b-q^2)}\left[\frac{b-aq}{b-q}{}_2\phi_1\left(a, b; aq/b; x\right) - \frac{q(1-a)}{b-q}{}_2\phi_1\left(aq, b; aq^2/b; x\right)\right] \\
&\qquad + \frac{(1-a)(b^2-aq^2)}{a(1-b)(b-q^2)}\left[\frac{b-aq^2}{b-q}{}_2\phi_1\left(aq, b; aq^2/b; x\right) - \frac{q(1-aq)}{b-q}{}_2\phi_1\left(aq^2, b; aq^3/b; x\right)\right] \\
&\quad = \frac{(b-aq^2)(a-b)(b-aq)}{a(1-b)(b-q^2)(b-q)}{}_2\phi_1\left(a, b; aq/b; x\right) \\
&\qquad + \frac{(1-a)(b-aq^2)}{a(1-b)(b-q^2)(b-q)}\left[-q(a-b) + (b^2-aq^2)\right]{}_2\phi_1\left(aq, b; aq^2/b; x\right) \\
&\qquad - \frac{q(1-a)(1-aq)(b^2-aq^2)}{a(1-b)(b-q^2)(b-q)}{}_2\phi_1\left(aq^2, b; aq^3/b; x\right) \\
&\quad = \frac{1}{a(1-b)(b-q^2)(b-q)}\Bigg[(b-aq^2)(a-b)(b-aq){}_2\phi_1\left(a, b; aq/b; x\right) \\
&\qquad + (1-a)(b-aq^2)\left[b(b+q) - aq(1+q)\right]{}_2\phi_1\left(aq, b; aq^2/b; x\right) \\
&\qquad - q(1-a)(1-aq)(b^2-aq^2){}_2\phi_1\left(aq^2, b; aq^3/b; x\right)\Bigg]. \qquad (5.34)
\end{aligned}$$

Let $x = -q/b$. We now have ${}_2\phi_1\left({a, bq \atop aq^3/b}; q; -q/b\right)$ written in terms of ${}_2\phi_1\left({a, b \atop aq/b}; q; -q/b\right)$, ${}_2\phi_1\left({aq, b \atop aq^2/b}; q; -q/b\right)$, and ${}_2\phi_1\left({aq^2, b \atop aq^3/b}; q; -q/b\right)$.

Define

$$
\begin{aligned}
A &= (b-aq^2)(a-b)(b-aq)\,{}_2\phi_1\left(\begin{matrix} a,\, b \\ aq/b \end{matrix};\, q;\, -q/b\right), \\
B &= (1-a)(b-aq^2)\left[b(b+q)-aq(1+q)\right]{}_2\phi_1\left(\begin{matrix} aq,\, b \\ aq^2/b \end{matrix};\, q;\, -q/b\right), \\
C &= -q(1-a)(1-aq)(b^2-aq^2)\,{}_2\phi_1\left(\begin{matrix} aq^2,\, b \\ aq^3/b \end{matrix};\, q;\, -q/b\right), \\
D &= (b-aq)\,{}_2\phi_1\left(\begin{matrix} a,\, b \\ aq/b \end{matrix};\, q;\, -q/b\right), \\
E &= -q(1-a)\,{}_2\phi_1\left(\begin{matrix} aq,\, b \\ aq^2/b \end{matrix};\, q;\, -q/b\right),
\end{aligned}
$$

so

$$
\begin{aligned}
\frac{{}_2\phi_1\left(\begin{smallmatrix} a,bq \\ aq^3/b \end{smallmatrix};\, q;\, \frac{-q}{b}\right)}{{}_2\phi_1\left(\begin{smallmatrix} a,b \\ aq^2/b \end{smallmatrix};\, q;\, \frac{-q}{b}\right)} &= \frac{\frac{1}{a(1-b)(b-q^2)(b-q)}(A+B+C)}{\frac{1}{b-q}(D+E)} \\
&= \frac{A+B+C}{a(1-b)(b-q^2)(D+E)}. \qquad (5.35)
\end{aligned}
$$

Applying (5.28), we find that

$$
\begin{aligned}
A &= (a-b)(b-aq)(b-aq^2)\frac{(aq;q^2)_\infty(-q;q)_\infty(aq^2/b^2;q^2)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty} \\
&= \frac{(a-b)b(1-aq/b)b(1-aq^2/b)(aq;q^2)_\infty(-q;q)_\infty(aq^2/b^2;q^2)_\infty}{(1-aq/b)(1-aq^2/b)(aq^3/b;q)_\infty(-q/b;q)_\infty} \\
&= \frac{(-q;q)_\infty}{(aq^3/b;q)_\infty(-q/b;q)_\infty}b^2(a-b)(aq;q^2)_\infty(aq^2/b^2;q^2)_\infty, \qquad (5.36)
\end{aligned}
$$

$$
\begin{aligned}
B &= (1-a)(b-aq^2)\left[b(b+q)-aq(1+q)\right]\frac{(aq^2;q^2)_\infty(-q;q)_\infty(aq^3/b^2;q^2)_\infty}{(aq^2/b;q)_\infty(-q/b;q)_\infty}\\
&= \frac{(1-a)b(1-aq^2/b)\left[b(b+q)-aq(1+q)\right](aq^2;q^2)_\infty(-q;q)_\infty(aq^3/b^2;q^2)_\infty}{(1-aq^2/b)(aq^3/b;q)_\infty(-q/b;q)_\infty}\\
&= \frac{(-q;q)_\infty}{(aq^3/b;q)_\infty(-q/b;q)_\infty}b\left[b(b+q)-aq(1+q)\right](a;q^2)_\infty(aq^3/b^2;q^2)_\infty, \qquad (5.37)
\end{aligned}
$$

and

$$
\begin{aligned}
C &= -q(1-a)(1-aq)(b^2-aq^2)\frac{(aq^3;q^2)_\infty(-q;q)_\infty(aq^4/b^2;q^2)_\infty}{(aq^3/b;q)_\infty(-q/b;q)_\infty}\\
&= \frac{-q(1-a)(1-aq)b^2(1-aq^2/b^2)(aq^3;q^2)_\infty(-q;q)_\infty(aq^4/b^2;q^2)_\infty}{(aq^3/b;q)_\infty(-q/b;q)_\infty}\\
&= \frac{(-q;q)_\infty}{(aq^3/b;q)_\infty(-q/b;q)_\infty}(-b^2q)(1-a)(aq;q^2)_\infty(aq^2/b^2;q^2)_\infty. \qquad (5.38)
\end{aligned}
$$

Hence

$$
\begin{aligned}
A+B+C &= \frac{b(-q;q)_\infty}{(aq^3/b;q)_\infty(-q/b;q)_\infty}\Big[b(a-b)(aq;q^2)_\infty(aq^2/b^2;q^2)_\infty\\
&\quad + \left[b(b+q)-aq(1+q)\right](a;q^2)_\infty(aq^3/b^2;q^2)_\infty\\
&\quad - bq(1-a)(aq;q^2)_\infty(aq^2/b^2;q^2)_\infty\Big]\\
&= \frac{b(-q;q)_\infty}{(aq^3/b;q)_\infty(-q/b;q)_\infty}\Big[\left[b(a-b)-bq(1-a)\right](aq;q^2)_\infty(aq^2/b^2;q^2)_\infty\\
&\quad + \left[b(b+q)-aq(1+q)\right](a;q^2)_\infty(aq^3/b^2;q^2)_\infty\Big]\\
&= \frac{b(-q;q)_\infty}{(aq^3/b;q)_\infty(-q/b;q)_\infty}\Big[b\left[a(1+q)-(b+q)\right](aq;q^2)_\infty(aq^2/b^2;q^2)_\infty\\
&\quad + \left[b(b+q)-aq(1+q)\right](a;q^2)_\infty(aq^3/b^2;q^2)_\infty\Big]. \qquad (5.39)
\end{aligned}
$$

Next, again applying (5.28), we find that

$$\begin{aligned}
D &= (b-aq)\frac{(aq;q^2)_\infty(-q;q)_\infty(aq^2/b^2;q^2)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty} \\
&= \frac{b(1-aq/b)(aq;q^2)_\infty(-q;q)_\infty(aq^2/b^2;q^2)_\infty}{(1-aq/b)(aq^2/b;q)_\infty(-q/b;q)_\infty} \\
&= \frac{(-q;q)_\infty}{(aq^2/b;q)_\infty(-q/b;q)_\infty} b(aq;q^2)_\infty(aq^2/b^2;q^2)_\infty \qquad (5.40)
\end{aligned}$$

and

$$\begin{aligned}
E &= -q(1-a)\frac{(aq^2;q^2)_\infty(-q;q)_\infty(aq^3/b^2;q^2)_\infty}{(aq^2/b;q)_\infty(-q/b;q)_\infty} \\
&= \frac{(-q;q)_\infty}{(aq^2/b;q)_\infty(-q/b;q)_\infty}(-q)(a;q^2)_\infty(aq^3/b^2;q^2)_\infty, \qquad (5.41)
\end{aligned}$$

so

$$D+E = \frac{(-q;q)_\infty}{(aq^2/b;q)_\infty(-q/b;q)_\infty}\left[b(aq;q^2)_\infty(aq^2/b^2;q^2)_\infty - q(a;q^2)_\infty(aq^3/b^2;q^2)_\infty\right]. \qquad (5.42)$$

Thus, if we define

$$\begin{aligned}
R &:= R(a,b;q) = (aq;q^2)_\infty(aq^2/b^2;q^2)_\infty, \\
S &:= S(a,b;q) = (a;q^2)_\infty(aq^3/b^2;q^2)_\infty,
\end{aligned}$$

we see that

$$\begin{aligned}
&\frac{A+B+C}{D+E} \\
&= \frac{\frac{b(-q;q)_\infty}{(aq^3/b;q)_\infty(-q/b;q)_\infty}\left[b\left[a(1+q)-(b+q)\right]R + \left[b(b+q)-aq(1+q)\right]S\right]}{\frac{(-q;q)_\infty}{(1-aq^2/b)(aq^3/b;q)_\infty(-q/b;q)_\infty}\left[bR-qS\right]} \\
&= b(1-aq^2/b)\frac{ab(1+q)R - b(b+q)R + b(b+q)S - aq(1+q)S}{bR-qS}
\end{aligned}$$

$$
\begin{aligned}
&= (b-aq^2)\frac{a(1+q)[bR-qS]-b(b+q)[R-S]}{bR-qS} \\
&= (b-aq^2)\left[a(1+q)-b(b+q)\frac{R-S}{bR-qS}\right] \\
&= (b-aq^2)\left[a(1+q)-2b+2b-b(b+q)\frac{R-S}{bR-qS}\right] \\
&= (b-aq^2)\left[a(1+q)-b(1+q^0)+b\frac{2(bR-qS)-(b+q)(R-S)}{bR-qS}\right] \\
&= (b-aq^2)\left[a(1+q)-b(1+q^0)+b\frac{bR-qS+bS-qR}{bR-qS}\right] \\
&= (b-aq^2)\left[a(1+q)-b(1+q^0)+b\frac{(b-q)(R+S)}{bR-qS}\right], \qquad (5.43)
\end{aligned}
$$

so

$$
\frac{{}_2\phi_1\left(\begin{smallmatrix}a,bq\\ aq^3/b\end{smallmatrix};\, q;\, -q/b\right)}{{}_2\phi_1\left(\begin{smallmatrix}a,b\\ aq^2/b\end{smallmatrix};\, q;\, -q/b\right)} = \frac{b-aq^2}{a(1-b)(b-q^2)}\left[a(1+q)-b(1+q^0)+\frac{b(b-q)(R+S)}{bR-qS}\right], \quad (5.44)
$$

and (5.27) becomes

$$
\begin{aligned}
&\frac{b-aq^2}{a(1-b)(b-q^2)}\left[a(1+q)-b(1+q^0)+\frac{b(b-q)(R+S)}{bR-qS}\right] \\
&\quad = \frac{b-aq^2}{b(1+q)-a(1+q^2)}+\frac{a(1-bq)(b-q^3)}{b(1+q^2)-a(1+q^3)} \\
&\qquad +\frac{a(1-bq^2)(b-q^4)}{b(1+q^3)-a(1+q^4)}+\frac{a(1-bq^3)(b-q^5)}{b(1+q^4)-a(1+q^5)}+\cdots. \qquad (5.45)
\end{aligned}
$$

By multiplying both sides of (5.45) by $a(1-b)(b-q^2)/(b-aq^2)$ and adding $b(1+q^0)-a(1+q)$ to both sides of the resulting equality, we obtain

$$
\begin{aligned}
\frac{b(b-q)(R+S)}{bR-qS} &= b(1+q^0)-a(1+q)+\frac{a(1-b)(b-q^2)}{b(1+q)-a(1+q^2)} \\
&\quad +\frac{a(1-bq)(b-q^3)}{b(1+q^2)-a(1+q^3)}+\frac{a(1-bq^2)(b-q^4)}{b(1+q^3)-a(1+q^4)}+\cdots. \qquad (5.46)
\end{aligned}
$$

Upon taking reciprocals of both sides and then multiplying both sides by $b(b-q)$, we find that

$$\frac{bR-qS}{R+S}=\frac{b(b-q)}{b(1+q^0)-a(1+q)}\,\frac{a(1-b)(b-q^2)}{+\,b(1+q)-a(1+q^2)}\,\frac{a(1-bq)(b-q^3)}{+\,b(1+q^2)-a(1+q^3)}\,\frac{a(1-bq^2)(b-q^4)}{+\,b(1+q^3)-a(1+q^4)+\cdots}. \tag{5.47}$$

Now replace q by q^2 to obtain

$$\frac{bR(a,b;q^2)-q^2S(a,b;q^2)}{R(a,b;q^2)+S(a,b;q^2)}=\frac{b(b-q^2)}{b(1+q^0)-a(1+q^2)}\,\frac{a(1-b)(b-q^4)}{+\,b(1+q^2)-a(1+q^4)}\,\frac{a(1-bq^2)(b-q^6)}{+\,b(1+q^4)-a(1+q^6)}\,\frac{a(1-bq^4)(b-q^8)}{+\,b(1+q^6)-a(1+q^8)+\cdots}. \tag{5.48}$$

Let $a=\alpha^2q$, $b=\alpha q^2/\beta$, $P=R(\alpha^2q,\alpha q^2/\beta;q^2)$, and $Q=S(\alpha^2q,\alpha q^2/\beta;q^2)$ to see that

$$\begin{aligned}P&=(\alpha^2qq^2;q^4)_\infty(\alpha^2qq^4/(\alpha q^2/\beta)^2;q^4)_\infty\\&=(\alpha^2q^3;q^4)_\infty(\beta^2q;q^4)_\infty,\\Q&=(\alpha^2q;q^4)_\infty(\alpha^2qq^6/(\alpha q^2/\beta)^2;q^4)_\infty\\&=(\alpha^2q;q^4)_\infty(\beta^2q^3;q^4)_\infty,\end{aligned}$$

and

$$\begin{aligned}&\frac{(\alpha q^2/\beta)P-q^2Q}{P+Q}\\&=\frac{(\alpha q^2/\beta)(\alpha q^2/\beta-q^2)}{(\alpha q^2/\beta)(1+q^0)-\alpha^2q(1+q^2)}\,\frac{\alpha^2q(1-\alpha q^2/\beta)(\alpha q^2/\beta-q^4)}{+\,(\alpha q^2/\beta)(1+q^2)-\alpha^2q(1+q^4)}\\&\quad\frac{\alpha^2q(1-(\alpha q^2/\beta)q^2)(\alpha q^2/\beta-q^6)}{+\,(\alpha q^2/\beta)(1+q^4)-\alpha^2q(1+q^6)}\,\frac{\alpha^2q(1-(\alpha q^2/\beta)q^4)(\alpha q^2/\beta-q^8)}{+\,(\alpha q^2/\beta)(1+q^6)-\alpha^2q(1+q^8)}+\cdots\\&=\frac{(\alpha q^4/\beta)(\alpha-\beta)}{\alpha q(q(1+q^0)-\alpha\beta(1+q^2))}\,\frac{\alpha^2q^3(\beta-\alpha q^2)(\alpha-\beta q^2)}{+\,\alpha q(q(1+q^2)-\alpha\beta(1+q^4))}\\&\quad\frac{\alpha^2q^3(\beta-\alpha q^4)(\alpha-\beta q^4)}{+\,\alpha q(q(1+q^4)-\alpha\beta(1+q^6))}\,\frac{\alpha^2q^3(\beta-\alpha q^6)(\alpha-\beta q^6)}{+\,\alpha q(q(1+q^6)-\alpha\beta(1+q^8))+\cdots}\end{aligned}$$

$$= \cfrac{(q^3/\beta)(\alpha-\beta)}{q(1+q^0)-\alpha\beta(1+q^2)} + \cfrac{q(\beta-\alpha q^2)(\alpha-\beta q^2)}{q(1+q^2)-\alpha\beta(1+q^4)} + \cfrac{q(\beta-\alpha q^4)(\alpha-\beta q^4)}{q(1+q^4)-\alpha\beta(1+q^6)} + \cfrac{q(\beta-\alpha q^6)(\alpha-\beta q^6)}{q(1+q^6)-\alpha\beta(1+q^8)} + \cdots. \tag{5.49}$$

Upon dividing both sides by q^2/β, we arrive at

$$\frac{\alpha P-\beta Q}{P+Q} = \cfrac{q(\alpha-\beta)}{q(1+q^0)-\alpha\beta(1+q^2)} + \cfrac{q(\beta-\alpha q^2)(\alpha-\beta q^2)}{q(1+q^2)-\alpha\beta(1+q^4)} + \cfrac{q(\beta-\alpha q^4)(\alpha-\beta q^4)}{q(1+q^4)-\alpha\beta(1+q^6)} + \cfrac{q(\beta-\alpha q^6)(\alpha-\beta q^6)}{q(1+q^6)-\alpha\beta(1+q^8)} + \cdots, \tag{5.50}$$

which is Theorem 5.2.1. □

The domain of validity claimed for Theorem 5.2.1 is $|\alpha\beta/q| < 1$ and $0 < |q| < 1$, or $q \neq 0$ and $\alpha = \beta q^{2m}$, where m is an integer. The proof of its convergence in this domain runs very similarly to the proof of Entry 16.12 in [10].

Proof of convergence. Let $D = \{a \in \mathbb{C}\}$, and suppose that b and q are fixed, arbitrary complex numbers with $|q| < 1$. If $q = 0$, the right side of the identity is zero while the left side equals $\frac{1}{2}(\alpha-\beta)$, so the identity is not true when $q = 0$.

Let T be the set of values of α for which the continued fraction terminates. In this case, $T = \{\beta q^{2m} : m \in \mathbb{Z}\}$. Then the closure of T is $Cl(T) = T \cup \{0\}$. We next prove the identity for α in $Cl(T)$. Assume that $\beta \neq 0$ and $q \neq 0$.

If $\alpha = 0$, the identity reduces to

$$-\frac{\beta(\beta^2q^3;q^4)_\infty}{(\beta^2q;q^4)_\infty+(\beta^2q^3;q^4)_\infty} = \cfrac{-\beta q}{q(1+q^0)} - \cfrac{\beta^2q^3}{q(1+q^2)} - \cfrac{\beta^2q^5}{q(1+q^4)} - \cfrac{\beta^2q^7}{q(1+q^6)} - \cdots$$
$$= -\cfrac{\beta}{1+q^0} - \cfrac{\beta^2q}{1+q^2} - \cfrac{\beta^2q^3}{1+q^4} - \cfrac{\beta^2q^5}{1+q^6} - \cdots. \tag{5.51}$$

This is simply a rearrangement of Entry 16.12 in [5] with $\alpha = 0$, so the identity is true in this case.

If $\alpha = \beta q^{2m}$, the continued fraction terminates, so the identity is true by the above proof and the fact that (5.3) was derived by iterated division.

If $\beta = 0$, then $\alpha = 0$ also, and the identity holds trivially.

Let $N = N(p)$ be the smallest index n for which $a_n(p) = 0$. We next show that $b_N + f^{(N)} \neq 0$ for all $p \in T$, where $f^{(N)}$ is the value of the Nth tail. We look at the equivalent continued fraction $\mathbf{K}(c_n/1)$ and show that for $k > N$, $1 + g^{(N)} \neq 0$, where $g^{(N)}$ is the Nth tail of this equivalent continued fraction. Thus

$$f^{(N)} = \mathop{\mathbf{K}}_{k=N+1}^{\infty} \frac{a_k}{b_k} = \frac{a_{N+1}}{b_{N+1}} + \frac{a_{N+2}}{b_{N+2}} + \frac{a_{N+3}}{b_{N+3}} + \cdots \tag{5.52}$$

and

$$g^{(N)} = \mathop{\mathbf{K}}_{k=N+1}^{\infty} \frac{c_k}{1} = \frac{c_{N+1}}{1} + \frac{c_{N+2}}{1} + \frac{c_{N+3}}{1} + \cdots, \tag{5.53}$$

where

$$c_1 = \frac{a_1}{b_1} \text{ and } c_k = \frac{a_k}{b_k b_{k-1}} \text{ for } k > 1.$$

When $\alpha = \beta q^{2m}$, $N = m + 1$, and we have

$$\begin{aligned} c_k &= \frac{q(\beta - \beta q^{2m+2k-2})(\beta q^{2m} - \beta q^{2k-2})}{[q(1+q^{2k-2}) - \beta^2 q^{2m}(1+q^{2k})][q(1+q^{2k-4}) - \beta^2 q^{2m}(1+q^{2k-2})]} \\ &= \frac{\beta^2 q^{2m-1}(1 - q^{2m+2k-2})(1 - q^{2k-2m-2})}{[1+q^{2k-2} - \beta^2 q^{2m-1}(1+q^{2k})][1+q^{2k-4} - \beta^2 q^{2m-1}(1+q^{2k-2})]} \end{aligned} \tag{5.54}$$

for $k > m + 2$. Then for $|q| < 1$, there exists an $M \in \mathbb{R}$ such that for $k > m + 1 \geq M$, $c_k \in P_0 = \{z \in \mathbb{C} : |z| - \Re(z) \leq \frac{1}{2}\}$. This is the parabolic region P_θ of the Parabola Theorem (Theorem 1.2.2) with $\theta = 0$. By Remark 1.2.3, the approximants $g_n^{(N)}$ of the Nth tail are bounded away from -1, so $1 + g^{(N)} \neq 0$. Thus the approximants

$$f_k(p) = \frac{a_1(p)}{b_1(p)} + \frac{a_2(p)}{b_2(p)} + \cdots + \frac{a_k(p)}{b_k(p)}, \qquad k = 1, 2, 3, \ldots$$

are a sequence of meromorphic functions in D with

$$\lim_{k\to\infty} f_k(p) = f_{N(p)-1}(p) =: F(p)$$

for all $p \in Cl(T)$.

Let $D_0 = D$, $m \geq 0$, and $0 < |q| < 1$. Then

$$\begin{aligned}
\lim_{k\to\infty} c_k &= \lim_{k\to\infty} \frac{q(\beta - \alpha q^{2k-2})(\alpha - \beta q^{2k-2})}{[q(1+q^{2k-2}) - \alpha\beta(1+q^{2k})][q(1+q^{2k-4}) - \alpha\beta(1+q^{2k-2})]} \\
&= \frac{q\beta\alpha}{(q-\alpha\beta)^2} \\
&= \frac{\alpha\beta/q}{(1-\alpha\beta/q)^2},
\end{aligned} \tag{5.55}$$

and the convergence is locally uniform with respect to a, b, and q. The identity follows by Remark 1.2.4 if $(\alpha\beta/q)/(1-\alpha\beta/q)^2 \notin (-\infty, -1/4]$. Since $f(z) = z/(1-z)^2$ is the Koebe function that maps the unit disk onto $\mathbb{C} \setminus (-\infty, -1/4]$, this condition is equivalent to $|\alpha\beta/q| < 1$.

Note that $f(z)$ also maps $\{z \in \mathbb{C} : |z| > 1\}$ onto $\mathbb{C}\setminus(-\infty, -1/4]$. In this case, $|\alpha\beta/q| > 1$. Let $a = 1/\alpha$ and $b = 1/\beta$. Then $|abq| < 1$, and an equivalence transformation converts the continued fraction of Theorem 5.2.1 into

$$\frac{q(a-b)}{(1+q^2) - abq(1+q^0)} + \frac{q(a-bq^2)(b-aq^2)}{(1+q^4) - abq(1+q^2)} + \frac{q(a-bq^4)(b-aq^4)}{(1+q^6) - abq(1+q^4)} + \cdots. \tag{5.56}$$

By Theorem 5.3.1, this continued fraction converges to $(P' - Q')/(\alpha P' + \beta Q')$, where $P' = (q^3/\alpha^2; q^4)_\infty (q/\beta^2; q^4)_\infty$ and $Q' = (q/\alpha^2; q^4)_\infty (q^3/\beta^2; q^4)_\infty$. Since this is not in general equal to $(\alpha P - \beta Q)/(P + Q)$, Theorem 5.2.1 is not true for $|\alpha\beta/q| > 1$. $\square$

5.3 A second q-analogue

The next theorem is another q-analogue of Theorem 3.2.1. It corresponds to the corollary of Entry 34 found by letting $m = n$. Ramanathan mentions this q-analogue in [15] but does not derive it. Its similarity to Theorem 5.2.1 led to the discovery that Theorem 3.2.1 can be found as a corollary of Entry 34 and that Theorem 3.1.1 and Entry 34 are equivalent to each other.

Theorem 5.3.1. *Suppose that α, β, and q are complex numbers with $|\alpha\beta q| < 1$ and $|q| < 1$ or that $\alpha = \beta q^{2m}$, where m is an integer. Define $P = (\alpha^2 q^3; q^4)_\infty (\beta^2 q; q^4)_\infty$ and $Q = (\alpha^2 q; q^4)_\infty (\beta^2 q^3; q^4)_\infty$. Then*

$$\frac{P-Q}{\alpha P + \beta Q} = \frac{q(\alpha-\beta)}{(1+q^2) - \alpha\beta q(1+q^0)} + \frac{q(\beta-\alpha q^2)(\alpha-\beta q^2)}{(1+q^4) - \alpha\beta q(1+q^2)} + \frac{q(\beta-\alpha q^4)(\alpha-\beta q^4)}{(1+q^6) - \alpha\beta q(1+q^4)} + \frac{q(\beta-\alpha q^6)(\alpha-\beta q^6)}{(1+q^8) - \alpha\beta q(1+q^6)} + \cdots.$$

Proof. In (5.3), we let $\alpha = b$, $\beta = a$, $\gamma = a/b$ and $x = -q/b$. This results in the continued fraction

$$\begin{aligned}
&\frac{{}_2\phi_1\left({a, bq \atop aq/b}; q; -q/b\right)}{{}_2\phi_1\left({a, b \atop a/b}; q; -q/b\right)} \\
&= \frac{q(1-a/b)}{q(1-a/b) + (a-bq)(-q/b)} + \frac{(1-bq)(aq/b-a)(-q/b)q}{q(1-aq/b) + (a-bq^2)(-q/b)} \\
&\quad + \frac{(1-bq^2)(aq^2/b-a)(-q/b)q}{q(1-aq^2/b) + (a-bq^3)(-q/b)} + \frac{(1-bq^3)(aq^3/b-a)(-q/b)q}{q(1-aq^3/b) + (a-bq^4)(-q/b)} + \cdots \\
&= \frac{(q/b)(b-a)}{(q/b)(b-a-(a-bq))} + \frac{(-aq^2/b^2)(1-bq)(q-b)}{(q/b)(b-aq-(a-bq^2))} \\
&\quad + \frac{(-aq^2/b^2)(1-bq^2)(q^2-b)}{(q/b)(b-aq^2-(a-bq^3))} + \frac{(-aq^2/b^2)(1-bq^3)(q^3-b)}{(q/b)(b-aq^3-(a-bq^4))} + \cdots \\
&= \frac{b-a}{b(1+q) - a(1+q^0)} + \frac{a(1-bq)(b-q)}{b(1+q^2) - a(1+q)}
\end{aligned}$$

$$+\cfrac{a(1-bq^2)(b-q^2)}{b(1+q^3)-a(1+q^2)+\cfrac{a(1-bq^3)(b-q^3)}{b(1+q^4)-a(1+q^3)+\cdots}}. \tag{5.57}$$

We next evaluate ${}_2\phi_1\left(\begin{smallmatrix}a,bq\\aq/b\end{smallmatrix};\, q;\, -q/b\right)$ and ${}_2\phi_1\left(\begin{smallmatrix}a,b\\a/b\end{smallmatrix};\, q;\, -q/b\right)$ by rewriting them in terms of ${}_2\phi_1\left(\begin{smallmatrix}a,b\\aq/b\end{smallmatrix};\, q;\, -q/b\right)$ and ${}_2\phi_1\left(\begin{smallmatrix}aq,b\\aq^2/b\end{smallmatrix};\, q;\, -q/b\right)$, which allows a closed-form evaluation by applying (5.28). We use Propositions 5.1.4, 5.1.5, and 5.1.6 to rewrite the q-hypergeometric functions.

In Proposition 5.1.4, let $\alpha = a$, $\beta = b$, and $\gamma = a/b$ to obtain

$$\begin{aligned}
{}_2\phi_1\left(a,b;a/b;x\right) &= {}_2\phi_1\left(a,b;aq/b;x\right) + \frac{(a/b)x(1-a)(1-b)}{(1-a/b)(1-aq/b)}{}_2\phi_1\left(aq,bq;aq^2/b;x\right)\\
&= {}_2\phi_1\left(a,b;aq/b;x\right) + \frac{abx(1-a)(1-b)}{(b-a)(b-aq)}{}_2\phi_1\left(aq,bq;aq^2/b;x\right). \qquad (5.58)
\end{aligned}$$

In Proposition 5.1.6, let $\alpha = a$, $\beta = b$, and $\gamma = aq/b$ to obtain

$$\begin{aligned}
{}_2\phi_1\left(aq,bq;aq^2/b;x\right) &= \frac{b(1-aq/b)}{(1-b)(aq/b-abx)}{}_2\phi_1\left(a,b;aq/b;x\right)\\
&\quad - \frac{b-aq/b}{(1-b)(aq/b-abx)}{}_2\phi_1\left(aq,b;aq^2/b;x\right)\\
&= \frac{b(b-aq)}{a(1-b)(q-b^2x)}{}_2\phi_1\left(a,b;aq/b;x\right)\\
&\quad - \frac{b^2-aq}{a(1-b)(q-b^2x)}{}_2\phi_1\left(aq,b;aq^2/b;x\right), \qquad (5.59)
\end{aligned}$$

and substitute (5.59) into (5.58). This yields

$$\begin{aligned}
{}_2\phi_1\left(a,b;a/b;x\right) &= {}_2\phi_1\left(a,b;aq/b;x\right) + \frac{abx(1-a)(1-b)}{(b-a)(b-aq)}\Bigg[\frac{b(b-aq)}{a(1-b)(q-b^2x)}{}_2\phi_1\left(a,b;aq/b;x\right)\\
&\quad - \frac{b^2-aq}{a(1-b)(q-b^2x)}{}_2\phi_1\left(aq,b;aq^2/b;x\right)\Bigg]\\
&= {}_2\phi_1\left(a,b;aq/b;x\right) + \frac{b^2x(1-a)}{(b-a)(q-b^2x)}{}_2\phi_1\left(a,b;aq/b;x\right)\\
&\quad - \frac{bx(1-a)(b^2-aq)}{(b-a)(b-aq)(q-b^2x)}{}_2\phi_1\left(aq,b;aq^2/b;x\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left[1 + \frac{b^2 x(1-a)}{(b-a)(q-b^2x)}\right] {}_2\phi_1\left(a, b; aq/b; x\right) \\
&\quad - \frac{bx(1-a)(b^2-aq)}{(b-a)(b-aq)(q-b^2x)} {}_2\phi_1\left(aq, b; aq^2/b; x\right) \\
&= \frac{bq - aq - b^3x + ab^2x + b^2x - ab^2x}{(b-a)(q-b^2x)} {}_2\phi_1\left(a, b; aq/b; x\right) \\
&\quad - \frac{bx(1-a)(b^2-aq)}{(b-a)(b-aq)(q-b^2x)} {}_2\phi_1\left(aq, b; aq^2/b; x\right) \\
&= \frac{q(b-a) + b^2x(1-b)}{(b-a)(q-b^2x)} {}_2\phi_1\left(a, b; aq/b; x\right) \\
&\quad - \frac{bx(1-a)(b^2-aq)}{(b-a)(b-aq)(q-b^2x)} {}_2\phi_1\left(aq, b; aq^2/b; x\right). \qquad (5.60)
\end{aligned}
$$

When we let $x = -q/b$ in (5.60), we find that

$$
\begin{aligned}
{}_2\phi_1\left(\begin{matrix} a, b \\ a/b \end{matrix}; q; -q/b\right) &= \frac{q(b-a) + b^2(-q/b)(1-b)}{(b-a)(q-b^2(-q/b))} {}_2\phi_1\left(\begin{matrix} a, b \\ aq/b \end{matrix}; q; -q/b\right) \\
&\quad - \frac{b(-q/b)(1-a)(b^2-aq)}{(b-a)(b-aq)(q-b^2(-q/b))} {}_2\phi_1\left(\begin{matrix} aq, b \\ aq^2/b \end{matrix}; q; -q/b\right) \\
&= \frac{q(b-a) - bq(1-b)}{(b-a)(q+bq)} {}_2\phi_1\left(\begin{matrix} a, b \\ aq/b \end{matrix}; q; -q/b\right) \\
&\quad + \frac{q(1-a)(b^2-aq)}{(b-a)(b-aq)(q+bq)} {}_2\phi_1\left(\begin{matrix} aq, b \\ aq^2/b \end{matrix}; q; -q/b\right) \\
&= \frac{q(b^2-a)}{q(b-a)(1+b)} {}_2\phi_1\left(\begin{matrix} a, b \\ aq/b \end{matrix}; q; -q/b\right) \\
&\quad + \frac{q(1-a)(b^2-aq)}{q(b-a)(b-aq)(1+b)} {}_2\phi_1\left(\begin{matrix} aq, b \\ aq^2/b \end{matrix}; q; -q/b\right) \\
&= \frac{b^2-a}{(b-a)(1+b)} {}_2\phi_1\left(\begin{matrix} a, b \\ aq/b \end{matrix}; q; -q/b\right) \\
&\quad + \frac{(1-a)(b^2-aq)}{(b-a)(b-aq)(1+b)} {}_2\phi_1\left(\begin{matrix} aq, b \\ aq^2/b \end{matrix}; q; -q/b\right). \qquad (5.61)
\end{aligned}
$$

Next, rearrange Proposition 5.1.5 into the form

$$
{}_2\phi_1\left(\alpha, \beta q; \gamma q; x\right) = \frac{\beta(1-\gamma)}{\gamma(1-\beta)} {}_2\phi_1\left(\alpha, \beta; \gamma; x\right) - \frac{\beta-\gamma}{\gamma(1-\beta)} {}_2\phi_1\left(\alpha, \beta; \gamma q; x\right) \qquad (5.62)
$$

and let $\alpha = a$, $\beta = b$, and $\gamma = a/b$ to obtain

$$\begin{aligned}
{}_2\phi_1\left(a, bq; aq/b; x\right) &= \frac{b(1-a/b)}{(a/b)(1-b)}{}_2\phi_1\left(a, b; a/b; x\right) - \frac{b-a/b}{(a/b)(1-b)}{}_2\phi_1\left(a, b; aq/b; x\right) \\
&= \frac{b(b-a)}{a(1-b)}{}_2\phi_1\left(a, b; a/b; x\right) - \frac{b^2-a}{a(1-b)}{}_2\phi_1\left(a, b; aq/b; x\right). \qquad (5.63)
\end{aligned}$$

Substitute (5.60) into this to see that

$$\begin{aligned}
{}_2\phi_1\left(a, bq; aq/b; x\right) &= \frac{b(b-a)}{a(1-b)}\left[\frac{q(b-a)+b^2x(1-b)}{(b-a)(q-b^2x)}{}_2\phi_1\left(a, b; aq/b; x\right)\right. \\
&\quad \left. - \frac{bx(1-a)(b^2-aq)}{(b-a)(b-aq)(q-b^2x)}{}_2\phi_1\left(aq, b; aq^2/b; x\right)\right] \\
&\quad - \frac{b^2-a}{a(1-b)}{}_2\phi_1\left(a, b; aq/b; x\right) \\
&= \frac{bq(b-a)+b^3x(1-b)}{a(1-b)(q-b^2x)}{}_2\phi_1\left(a, b; aq/b; x\right) \\
&\quad - \frac{b^2x(1-a)(b^2-aq)}{a(1-b)(b-aq)(q-b^2x)}{}_2\phi_1\left(aq, b; aq^2/b; x\right) \\
&\quad - \frac{b^2-a}{a(1-b)}{}_2\phi_1\left(a, b; aq/b; x\right) \\
&= \left[\frac{bq(b-a)+b^3x(1-b)}{a(1-b)(q-b^2x)} - \frac{b^2-a}{a(1-b)}\right]{}_2\phi_1\left(a, b; aq/b; x\right) \\
&\quad - \frac{b^2x(1-a)(b^2-aq)}{a(1-b)(b-aq)(q-b^2x)}{}_2\phi_1\left(aq, b; aq^2/b; x\right) \\
&= \frac{bq(b-a)+b^3x(1-b)-(b^2-a)(q-b^2x)}{a(1-b)(q-b^2x)}{}_2\phi_1\left(a, b; aq/b; x\right) \\
&\quad - \frac{b^2x(1-a)(b^2-aq)}{a(1-b)(b-aq)(q-b^2x)}{}_2\phi_1\left(aq, b; aq^2/b; x\right) \\
&= \frac{b^2q-abq+b^3x-b^4x-b^2q+aq+b^4x-ab^2x}{a(1-b)(q-b^2x)}{}_2\phi_1\left(a, b; aq/b; x\right) \\
&\quad - \frac{b^2x(1-a)(b^2-aq)}{a(1-b)(b-aq)(q-b^2x)}{}_2\phi_1\left(aq, b; aq^2/b; x\right) \\
&= \frac{aq(1-b)+b^2x(b-a)}{a(1-b)(q-b^2x)}{}_2\phi_1\left(a, b; aq/b; x\right) \\
&\quad - \frac{b^2x(1-a)(b^2-aq)}{a(1-b)(b-aq)(q-b^2x)}{}_2\phi_1\left(aq, b; aq^2/b; x\right). \qquad (5.64)
\end{aligned}$$

Let $x = -q/b$ in (5.64) to obtain

$$\begin{aligned}
{}_2\phi_1\left(\begin{matrix}a,\, bq\\ aq/b\end{matrix}; q;\, -q/b\right) &= \frac{aq(1-b)+b^2(-q/b)(b-a)}{a(1-b)(q-b^2(-q/b))}\,{}_2\phi_1\left(\begin{matrix}a,\, b\\ aq/b\end{matrix}; q;\, -q/b\right)\\
&\quad - \frac{b^2(-q/b)(1-a)(b^2-aq)}{a(1-b)(b-aq)(q-b^2(-q/b))}\,{}_2\phi_1\left(\begin{matrix}aq,\, b\\ aq^2/b\end{matrix}; q;\, -q/b\right)\\
&= \frac{aq-abq-b^2q+abq}{aq(1-b)(1+b)}\,{}_2\phi_1\left(\begin{matrix}a,\, b\\ aq/b\end{matrix}; q;\, -q/b\right)\\
&\quad + \frac{bq(1-a)(b^2-aq)}{aq(1-b)(b-aq)(1+b)}\,{}_2\phi_1\left(\begin{matrix}aq,\, b\\ aq^2/b\end{matrix}; q;\, -q/b\right)\\
&= \frac{a-b^2}{a(1-b)(1+b)}\,{}_2\phi_1\left(\begin{matrix}a,\, b\\ aq/b\end{matrix}; q;\, -q/b\right)\\
&\quad + \frac{b(1-a)(b^2-aq)}{a(1-b)(b-aq)(1+b)}\,{}_2\phi_1\left(\begin{matrix}aq,\, b\\ aq^2/b\end{matrix}; q;\, -q/b\right). \qquad (5.65)
\end{aligned}$$

Now define

$$\begin{aligned}
A &= (a-b^2)\,{}_2\phi_1\left(\begin{matrix}a,\, b\\ aq/b\end{matrix}; q;\, -q/b\right),\\
B &= \frac{b(1-a)(b^2-aq)}{b-aq}\,{}_2\phi_1\left(\begin{matrix}aq,\, b\\ aq^2/b\end{matrix}; q;\, -q/b\right),\\
C &= (b^2-a)\,{}_2\phi_1\left(\begin{matrix}a,\, b\\ aq/b\end{matrix}; q;\, -q/b\right),\\
D &= \frac{(1-a)(b^2-aq)}{b-aq}\,{}_2\phi_1\left(\begin{matrix}aq,\, b\\ aq^2/b\end{matrix}; q;\, -q/b\right),
\end{aligned}$$

so

$$\begin{aligned}
\frac{{}_2\phi_1\left(\begin{smallmatrix}a,bq\\ aq/b\end{smallmatrix}; q;\, -q/b\right)}{{}_2\phi_1\left(\begin{smallmatrix}a,b\\ a/b\end{smallmatrix}; q;\, -q/b\right)} &= \frac{\frac{1}{a(1-b)(1+b)}(A+B)}{\frac{1}{(b-a)(1+b)}(C+D)}\\
&= \frac{(b-a)(A+B)}{a(1-b)(C+D)}. \qquad (5.66)
\end{aligned}$$

Applying (5.28), we find that

$$\begin{aligned}A &= (a-b^2)\frac{(aq;q^2)_\infty(-q;q)_\infty(aq^2/b^2;q^2)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}\\ &= \frac{-b^2(1-a/b^2)(aq;q^2)_\infty(-q;q)_\infty(aq^2/b^2;q^2)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}\\ &= -\frac{(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}b^2(aq;q^2)_\infty(a/b^2;q^2)_\infty, \end{aligned} \tag{5.67}$$

$$\begin{aligned}B &= \frac{b(1-a)(b^2-aq)}{b-aq}\frac{(aq^2;q^2)_\infty(-q;q)_\infty(aq^3/b^2;q^2)_\infty}{(aq^2/b;q)_\infty(-q/b;q)_\infty}\\ &= \frac{b(1-a)b^2(1-aq/b^2)(aq^2;q^2)_\infty(-q;q)_\infty(aq^3/b^2;q^2)_\infty}{b(1-aq/b)(aq^2/b;q)_\infty(-q/b;q)_\infty}\\ &= \frac{(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}b^2(a;q^2)_\infty(aq/b^2;q^2)_\infty, \end{aligned} \tag{5.68}$$

$$\begin{aligned}C &= (b^2-a)\frac{(aq;q^2)_\infty(-q;q)_\infty(aq^2/b^2;q^2)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}\\ &= \frac{b^2(1-a/b^2)(aq;q^2)_\infty(-q;q)_\infty(aq^2/b^2;q^2)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}\\ &= \frac{(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}b^2(aq;q^2)_\infty(a/b^2;q^2)_\infty, \end{aligned} \tag{5.69}$$

and

$$\begin{aligned}D &= \frac{(1-a)(b^2-aq)}{b-aq}\frac{(aq^2;q^2)_\infty(-q;q)_\infty(aq^3/b^2;q^2)_\infty}{(aq^2/b;q)_\infty(-q/b;q)_\infty}\\ &= \frac{(1-a)b^2(1-aq/b^2)(aq^2;q^2)_\infty(-q;q)_\infty(aq^3/b^2;q^2)_\infty}{b(1-aq/b)(aq^2/b;q)_\infty(-q/b;q)_\infty}\\ &= \frac{(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}b(a;q^2)_\infty(aq/b^2;q^2)_\infty. \end{aligned} \tag{5.70}$$

Hence

$$\begin{aligned} A+B &= -\frac{(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty} b^2 (aq;q^2)_\infty (a/b^2;q^2)_\infty \\ &\quad + \frac{(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty} b^2 (a;q^2)_\infty (aq/b^2;q^2)_\infty \\ &= \frac{b^2(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty} \left[-(aq;q^2)_\infty (a/b^2;q^2)_\infty + (a;q^2)_\infty (aq/b^2;q^2)_\infty\right] \end{aligned} \tag{5.71}$$

and

$$\begin{aligned} C+D &= \frac{(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty} b^2 (aq;q^2)_\infty (a/b^2;q^2)_\infty \\ &\quad + \frac{(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty} b (a;q^2)_\infty (aq/b^2;q^2)_\infty \\ &= \frac{b(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty} \left[b(aq;q^2)_\infty (a/b^2;q^2)_\infty + (a;q^2)_\infty (aq/b^2;q^2)_\infty\right]. \end{aligned} \tag{5.72}$$

Thus, if we define

$$R := R(a,b;q) = (aq;q^2)_\infty (a/b^2;q^2)_\infty,$$

$$S := S(a,b;q) = (a;q^2)_\infty (aq/b^2;q^2)_\infty,$$

we see that

$$\begin{aligned} \frac{A+B}{C+D} &= \frac{\frac{b^2(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}\left[-(aq;q^2)_\infty (a/b^2;q^2)_\infty + (a;q^2)_\infty (aq/b^2;q^2)_\infty\right]}{\frac{b(-q;q)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}\left[b(aq;q^2)_\infty (a/b^2;q^2)_\infty + (a;q^2)_\infty (aq/b^2;q^2)_\infty\right]} \\ &= \frac{b(S-R)}{bR+S}. \end{aligned} \tag{5.73}$$

Substituting this into (5.66), we find that

$$\frac{{}_2\phi_1\left(\begin{matrix}a,bq\\aq/b\end{matrix};\,q;\,-q/b\right)}{{}_2\phi_1\left(\begin{matrix}a,b\\a/b\end{matrix};\,q;\,-q/b\right)}=\frac{(b-a)b(S-R)}{a(1-b)(bR+S)},\tag{5.74}$$

so (5.57) can be written as

$$\begin{aligned}\frac{(b-a)b(S-R)}{a(1-b)(bR+S)}&=\frac{b-a}{b(1+q)-a(1+q^0)}+\frac{a(1-bq)(b-q)}{b(1+q^2)-a(1+q)}\\&\quad+\frac{a(1-bq^2)(b-q^2)}{b(1+q^3)-a(1+q^2)}+\frac{a(1-bq^3)(b-q^3)}{b(1+q^4)-a(1+q^3)}+\cdots.\end{aligned}\tag{5.75}$$

Multiplying both sides of (5.75) by $-a(1-b)/(b-a)$, we obtain

$$\begin{aligned}\frac{b(R-S)}{bR+S}&=\frac{-a(1-b)}{b(1+q)-a(1+q^0)}+\frac{a(1-bq)(b-q)}{b(1+q^2)-a(1+q)}\\&\quad+\frac{a(1-bq^2)(b-q^2)}{b(1+q^3)-a(1+q^2)}+\frac{a(1-bq^3)(b-q^3)}{b(1+q^4)-a(1+q^3)}+\cdots.\end{aligned}\tag{5.76}$$

We replace q by q^2 to see that

$$\begin{aligned}\frac{b(R(a,b;q^2)-S(a,b;q^2))}{bR(a,b;q^2)+S(a,b;q^2)}&=\frac{a(b-1)}{b(1+q^2)-a(1+q^0)}+\frac{a(1-bq^2)(b-q^2)}{b(1+q^4)-a(1+q^2)}\\&\quad+\frac{a(1-bq^4)(b-q^4)}{b(1+q^6)-a(1+q^4)}+\frac{a(1-bq^6)(b-q^6)}{b(1+q^8)-a(1+q^6)}+\cdots.\end{aligned}\tag{5.77}$$

Let $a=\alpha^2q$, $b=\alpha/\beta$, $P=R(\alpha^2q,\alpha/\beta;q^2)$, and $Q=S(\alpha^2q,\alpha/\beta;q^2)$. Then

$$\begin{aligned}P&=(\alpha^2qq^2;q^4)_\infty(\alpha^2q/(\alpha/\beta)^2;q^4)_\infty\\&=(\alpha^2q^3;q^4)_\infty(\beta^2q;q^4)_\infty,\\Q&=(\alpha^2q;q^4)_\infty(\alpha^2qq^2/(\alpha/\beta)^2;q^4)_\infty\\&=(\alpha^2q;q^4)_\infty(\beta^2q^3;q^4)_\infty,\end{aligned}$$

and

$$\begin{aligned}\frac{(\alpha/\beta)(P-Q)}{(\alpha/\beta)P+Q} &= \frac{\alpha^2 q(\alpha/\beta-1)}{(\alpha/\beta)(1+q^2)-\alpha^2 q(1+q^0)+}\,\frac{\alpha^2 q(1-\alpha q^2/\beta)(\alpha/\beta-q^2)}{(\alpha/\beta)(1+q^4)-\alpha^2 q(1+q^2)}\\ &\quad +\frac{\alpha^2 q(1-\alpha q^4/\beta)(\alpha/\beta-q^4)}{(\alpha/\beta)(1+q^6)-\alpha^2 q(1+q^4)+}\,\frac{\alpha^2 q(1-\alpha q^6/\beta)(\alpha/\beta-q^6)}{(\alpha/\beta)(1+q^8)-\alpha^2 q(1+q^6)+\cdots}\\ &= \frac{\alpha^2 q(\alpha-\beta)}{\alpha((1+q^2)-\alpha\beta q(1+q^0))+}\,\frac{\alpha^2 q(\beta-\alpha q^2)(\alpha-\beta q^2)}{\alpha((1+q^4)-\alpha\beta q(1+q^2))}\\ &\quad +\frac{\alpha^2 q(\beta-\alpha q^4)(\alpha-\beta q^4)}{\alpha((1+q^6)-\alpha\beta q(1+q^4))+}\,\frac{\alpha^2 q(\beta-\alpha q^6)(\alpha-\beta q^6)}{\alpha((1+q^8)-\alpha\beta q(1+q^6))+\cdots}\\ &= \frac{\alpha q(\alpha-\beta)}{(1+q^2)-\alpha\beta q(1+q^0)+}\,\frac{q(\beta-\alpha q^2)(\alpha-\beta q^2)}{(1+q^4)-\alpha\beta q(1+q^2)}\\ &\quad +\frac{q(\beta-\alpha q^4)(\alpha-\beta q^4)}{(1+q^6)-\alpha\beta q(1+q^4)+}\,\frac{q(\beta-\alpha q^6)(\alpha-\beta q^6)}{(1+q^8)-\alpha\beta q(1+q^6)+\cdots}. \end{aligned} \tag{5.78}$$

Therefore

$$\begin{aligned}\frac{\alpha(P-Q)}{\alpha P+\beta Q} &= \frac{\alpha q(\alpha-\beta)}{(1+q^2)-\alpha\beta q(1+q^0)+}\,\frac{q(\beta-\alpha q^2)(\alpha-\beta q^2)}{(1+q^4)-\alpha\beta q(1+q^2)}\\ &\quad +\frac{q(\beta-\alpha q^4)(\alpha-\beta q^4)}{(1+q^6)-\alpha\beta q(1+q^4)+}\,\frac{q(\beta-\alpha q^6)(\alpha-\beta q^6)}{(1+q^8)-\alpha\beta q(1+q^6)+\cdots}\end{aligned} \tag{5.79}$$

and

$$\begin{aligned}\frac{P-Q}{\alpha P+\beta Q} &= \frac{q(\alpha-\beta)}{(1+q^2)-\alpha\beta q(1+q^0)+}\,\frac{q(\beta-\alpha q^2)(\alpha-\beta q^2)}{(1+q^4)-\alpha\beta q(1+q^2)}\\ &\quad +\frac{q(\beta-\alpha q^4)(\alpha-\beta q^4)}{(1+q^6)-\alpha\beta q(1+q^4)+}\,\frac{q(\beta-\alpha q^6)(\alpha-\beta q^6)}{(1+q^8)-\alpha\beta q(1+q^6)+\cdots},\end{aligned} \tag{5.80}$$

which is Theorem 5.3.1 as desired. □

The domain of validity claimed for Theorem 5.3.1 is $|\alpha\beta q| < 1$ and $|q| < 1$, or $\alpha = \beta q^{2m}$, where m is an integer.

Proof of convergence. Let $D = \{a \in \mathbb{C}\}$, and suppose that b and q are fixed, arbitrary complex numbers with $|q| < 1$. If $q = 0$, the identity holds trivially.

Let T be the set of values of α for which the continued fraction terminates. In this case, $T = \{\beta q^{2m} : m \in \mathbb{Z}\}$. Then the closure of T is $Cl(T) = T \cup \{0\}$. We next prove the identity for α in $Cl(T)$. Assume that $\beta \neq 0$ and $q \neq 0$.

If $\alpha = 0$, the identity reduces to

$$\frac{(\beta^2 q; q^4)_\infty - (\beta^2 q^3; q^4)_\infty}{\beta(\beta^2 q^3; q^4)_\infty} = -\frac{\beta q}{1+q^2} - \frac{\beta^2 q^3}{1+q^4} - \frac{\beta^2 q^5}{1+q^6} - \frac{\beta^2 q^7}{1+q^8} - \cdots. \tag{5.81}$$

This is a rearrangement of Entry 16.12 in [5] with $\alpha = 0$, so the identity is true in this case for $\alpha = 0$.

If $\alpha = \beta q^{2m}$, the continued fraction terminates, so the identity is true by the above proof and the fact that (5.3) was derived by iterated division.

If $\beta = 0$, then $\alpha = 0$ also, and the identity holds trivially.

Let $N = N(p)$ be the smallest index n for which $a_n(p) = 0$. We next show that $b_N + f^{(N)} \neq 0$ for all $p \in T$, where $f^{(N)}$ is the value of the Nth tail. We look at the equivalent continued fraction $\mathbf{K}(c_n/1)$ and show that for $k > N$, $1 + g^{(N)} \neq 0$, where $g^{(N)}$ is the Nth tail of this equivalent continued fraction. Thus

$$f^{(N)} = \mathop{\mathbf{K}}_{k=N+1}^{\infty} \frac{a_k}{b_k} = \frac{a_{N+1}}{b_{N+1}} + \frac{a_{N+2}}{b_{N+2}} + \frac{a_{N+3}}{b_{N+3}} + \cdots \tag{5.82}$$

and

$$g^{(N)} = \mathop{\mathbf{K}}_{k=N+1}^{\infty} \frac{c_k}{1} = \frac{c_{N+1}}{1} + \frac{c_{N+2}}{1} + \frac{c_{N+3}}{1} + \cdots, \tag{5.83}$$

where

$$c_1 = \frac{a_1}{b_1} \text{ and } c_k = \frac{a_k}{b_k b_{k-1}} \text{ for } k > 1.$$

When $\alpha = \beta q^{2m}$, $N = m + 1$, and we have

$$\begin{aligned} c_k &= \frac{q(\beta - \beta q^{2m+2k-2})(\beta q^{2m} - \beta q^{2k-2})}{[1 + q^{2k} - \beta^2 q^{2m+1}(1 + q^{2k-2})][1 + q^{2k-2} - \beta^2 q^{2m+1}(1 + q^{2k-4})]} \\ &= \frac{\beta^2 q^{2m+1}(1 - q^{2m+2k-2})(1 - q^{2k-2m-2})}{[1 + q^{2k} - \beta^2 q^{2m+1}(1 + q^{2k-2})][1 + q^{2k-2} - \beta^2 q^{2m+1}(1 + q^{2k-4})]} \end{aligned} \tag{5.84}$$

for $k > m + 2$. Then for $|q| < 1$, there exists an $M \in \mathbb{R}$ such that for $k > m + 1 \geq M$, $c_k \in P_0 = \{z \in \mathbb{C} : |z| - \Re(z) \leq \frac{1}{2}\}$. This is the parabolic region P_θ of the Parabola Theorem (Theorem 1.2.2) with $\theta = 0$. By Remark 1.2.3, the approximants $g_n^{(N)}$ of the Nth tail are bounded away from -1, so $1 + g^{(N)} \neq 0$. Thus the approximants

$$f_k(p) = \frac{a_1(p)}{b_1(p)} + \frac{a_2(p)}{b_2(p)} + \cdots + \frac{a_k(p)}{b_k(p)}, \qquad k = 1, 2, 3, \ldots$$

are a sequence of meromorphic functions in D with

$$\lim_{k\to\infty} f_k(p) = f_{N(p)-1}(p) =: F(p)$$

for all $p \in Cl(T)$.

Let $D_0 = D$, $m \geq 0$, and $0 < |q| < 1$. Then

$$\begin{aligned} \lim_{k\to\infty} c_k &= \lim_{k\to\infty} \frac{q(\beta - \alpha q^{2k-2})(\alpha - \beta q^{2k-2})}{[1 + q^{2k} - \alpha\beta q(1 + q^{2k-2})][1 + q^{2k-2} - \alpha\beta q(1 + q^{2k-4})]} \\ &= \frac{\alpha\beta q}{(1 - \alpha\beta q)^2}, \end{aligned} \tag{5.85}$$

and the convergence is locally uniform with respect to a, b, and q. The identity follows by Remark 1.2.4 if $\alpha\beta q/(1-\alpha\beta q)^2 \notin (-\infty, -1/4]$. Since $f(z) = z/(1-z)^2$ is the Koebe function that maps the unit disk onto $\mathbb{C} \setminus (-\infty, -1/4]$, this condition is equivalent to $|\alpha\beta q| < 1$.

Note that $f(z)$ also maps $\{z \in \mathbb{C} : |z| > 1\}$ onto $\mathbb{C} \setminus (-\infty, -1/4]$. In this case, $|\alpha\beta q| > 1$. Let $a = 1/\alpha$ and $b = 1/\beta$. Then $|ab/q| < 1$, and an equivalence transformation converts the continued fraction of Theorem 5.3.1 into

$$\frac{q(a-b)}{q(1+q^0)-ab(1+q^2)+}\,\frac{q(a-bq^2)(b-aq^2)}{q(1+q^2)-ab(1+q^4)+}\,\frac{q(a-bq^4)(b-aq^4)}{q(1+q^4)-ab(1+q^6)+\cdots}. \tag{5.86}$$

By Theorem 5.2.1, this continued fraction converges to $(\alpha P' - \beta Q')/(P' + Q')$, where $P' = (q^3/\alpha^2; q^4)_\infty (q/\beta^2; q^4)_\infty$ and $Q' = (q/\alpha^2; q^4)_\infty (q^3/\beta^2; q^4)_\infty$. Since this is not in general equal to $(P - Q)/(\alpha P + \beta Q)$, Theorem 5.3.1 is not true for $|\alpha\beta q| > 1$. □

5.4 Remarks regarding higher order q-analogues

There are two main difficulties that arise when attempting to use the methods of this thesis to obtain q-analogues of identities with three variables, such as Entries 34, 36, 39, and Theorem 3.1.1. The first difficulty is that, while the three-term recurrences of the ${}_3F_2$ hypergeometric functions use $x = 1$ for all three terms, many of the recurrences of ${}_3\phi_2$ basic hypergeometric functions have two terms containing the same value for x while the third term has a different value. As an example, the hypergeometric recurrence

$$\begin{aligned} {}_3F_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; 1\right) &= \frac{(b+c-d+1)(1-d)}{(b-d+1)(c-d+1)} {}_3F_2\left(\begin{matrix} a-1,\, b,\, c \\ d-1,\, e \end{matrix}; 1\right) \\ &\quad + \frac{bc(d+e-a-b-c-1)}{e(b-d+1)(c-d+1)} {}_3F_2\left(\begin{matrix} a,\, b+1,\, c+1 \\ d,\, e+1 \end{matrix}; 1\right) \end{aligned} \tag{5.87}$$

has the q-analogue

$$\begin{aligned} {}_3\phi_2\left(\begin{matrix} a,\, b,\, c \\ d,\, e \end{matrix}; q; \frac{de}{abc}\right) &= \frac{(1-bcq/d)(1-q/d)}{(1-bq/d)(1-cq/d)} {}_3\phi_2\left(\begin{matrix} a/q,\, b,\, c \\ d/q,\, e \end{matrix}; q; \frac{de}{abc}\right) \\ &\quad + \frac{q(1-b)(1-c)(1-de/(abcq))}{d(1-e)(1-bq/d)(1-cq/d)} {}_3\phi_2\left(\begin{matrix} a,\, bq,\, cq \\ d,\, eq \end{matrix}; q; \frac{de}{abcq}\right). \end{aligned} \tag{5.88}$$

This feature makes such recurrences ill-suited to the iterative division process used to produce a continued fraction like the one found in Lemma 2.1.4. There are some contiguous relations of ${}_3\phi_2$ basic hypergeometric functions with $x = de/(abc)$ for all three terms, so it may be that this difficulty can be overcome.

The second difficulty has to do with the evaluation of the basic hypergeometric functions. There is not a general q-analogue of Dixon's Identity for use in evaluating a ${}_3\phi_2$ basic hypergeometric function. There is a partial q-analogue due to F.H. Jackson and restated by W.N. Bailey in [3] as

$$ {}_3\phi_2\left(\begin{matrix} a,\, b,\, c \\ aq/b,\, aq/c \end{matrix}; q; \frac{q^2\sqrt{a}}{bc}\right) = \frac{(q;q)_{2n}(b;q)_n(c;q)_n(bc;q)_{2n}}{(q;q)_n(b;q)_{2n}(c;q)_{2n}(bc;q)_n}, \tag{5.89} $$

when $a = q^{-2n}$ and $|q^2\sqrt{a}/(bc)| < 1$, and there is a general q-analogue, given as Equation (2.7.2) in [6], that uses a ${}_4\phi_3$ basic hypergeometric function. This second q-analogue,

$$ {}_4\phi_3\left(\begin{matrix} a,\, -q\sqrt{a},\, b,\, c \\ -\sqrt{a},\, aq/b,\, aq/c \end{matrix}; q; \frac{q\sqrt{a}}{bc}\right) = \frac{(aq;q)_\infty, (aq/(bc);q)_\infty, (q\sqrt{a}/b;q)_\infty, (q\sqrt{a}/c;q)_\infty}{(aq/b;q)_\infty, (aq/c;q)_\infty, (q\sqrt{a};q)_\infty, (q\sqrt{a}/(bc);q)_\infty}, \tag{5.90} $$

requires that $|q\sqrt{a}/(bc)| < 1$. It is not apparent that either of these q-analogues would be useful in finding the desired continued fraction identities. However, given that Masson's proof of Entry 40 in [13] uses ${}_9F_8$ hypergeometric functions and his proof with D.P. Gupta of Watson's q-analogue of Entry 40 in [8] uses ${}_{10}\phi_9$ basic hypergeometric functions, it may be that (5.90) could be used in finding q-analogues of Entries 34, 36, and 39.

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