IDEALS OF POWERS OF LINEAR FORMS

BY

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DISSERTATION

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Abstract

This thesis addresses two closely related problems about ideals of powers of linear forms.

In the first chapter, we analyze a problem from spline theory, namely to compute the dimension of the vector space of trivariate splines on a special class of tetrahedral complexes, using ideals of powers of linear forms. By Macaulay's inverse system, this class of ideals is closely related to ideals of fat points.

In the second chapter, we approach a conjecture of Postnikov and Shapiro concerning the minimal free resolutions of a class of ideals of powers of linear forms in n variables which are constructed from complete graphs on $n+1$ vertices. This statement was also conjectured by Schenck in the special case of $n = 3$. We provide two different approaches to his conjecture. We prove the conjecture of Postnikov and Shapiro under the additional condition that certain modules are free.

To Father and Mother.

Acknowledgments

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Chapter 1

Applications of ideals of powers of linear forms to spline theory

In this chapter, we compute the dimension of the vector space of trivariate splines on a special class of tetrahedral complexes. Our main result is stated in §1.3, after an introduction to spline theory in §1.1 and a review of previous work in §1.2. The proof of the main theorem covers the rest of the chapter. The main tools of proof are given in §1.4 to §1.6. In §1.7, we provide examples of computation using these tools. In §1.8, we state again our main theorem and compare our results with results in the literature.

1.1 A short introduction to spline theory from the algebraic point of view

In mathematics it is often useful to approximate a function f on a region by a "simpler" function. A natural way to do this is to divide the region into simplices, and then approximate f on each simplex by a polynomial function. A polynomial function on each simplex is smooth inside that region, but the polynomial functions on two adjacent simplices should agree on the boundary to make the function f continuous on the whole region. We may even require certain smoothness properties of these functions at the boundary of two regions. Therefore, we are faced with the problem of constructing and analyzing these **piecewise polynomial** or **spline** functions with a specified degree of smoothness on subdivisions of regions in $\mathbb{R}^n$.

Two-variable functions of this sort are frequently used in computer-aided design to specify the shapes of curved surfaces, and the degree of smoothness attainable in some specified class of piecewise polynomial functions is an important design consideration. Uni- and multivariate splines are also used to interpolate values or approximate other functions in numerical analysis, most notably in the **finite element method** for deriving approximate solutions to partial differential equations. Here we give a simple example to illustrate this idea.

1.1.1 One-dimensional spline theory

Consider $a < c < b$ on $\mathbb{R}$. A piecewise polynomial function is a function f such that

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in [a, c] \\ f_2(x) & \text{if } x \in [c, b], \end{cases} \tag{1.1.1}$$

where f_1, f_2 are polynomials in $\mathbb{R}[x]$.

Observe that f is continuous if and only if $f_1(c) = f_2(c)$, or more generally, f is C^k if and only if

$$f_1^{(j)}(c) = f_2^{(j)}(c), \text{ for } 0 \leq j \leq k.$$

In that case, $f_1(x) - f_2(x)$ is divisible by $(x-c)^{k+1}$. Let $\langle g_1, \cdots, g_n \rangle$ denote the ideal generated by $g_1, \cdots, g_n$. Then

$$f_1 - f_2 \in \langle (x-c)^{k+1} \rangle. \tag{1.1.2}$$

Therefore, we can represent a C^r spline function on $[a, c] \cup [c, b]$ by a pair (f_1, f_2) satisfying the above condition (1.1.2). It is clear that the set of such spline functions form a vector subspace of $\mathbb{R}[x]^2$ under the usual componentwise addition and scalar multiplication. If we restrict the degree of each component to be at most k, and denote the resulting vector spaces of splines as V_k^r, we get a finite-dimensional space and a natural question is to determine its dimension. The answer is given by the following simple formula:

$$\dim V_k^r = \begin{cases} k+1 & \text{if } r+1 > k \\ 2k - r + 1 & \text{if } r+1 \leq k. \end{cases}$$

This theory of spline functions on the intervals of $\mathbb{R}$ is easy. But the generalization of spline functions to higher dimensional regions is more difficult. In the next subsection, we give precise definitions and some examples.

1.1.2 Definitions and examples

Definition 1.1.1 (underlying complexes). *1. A **polytope** is the convex hull of a finite set in $\mathbb{R}^n$.*

*2. A **polyhedral complex** $\Delta \subset \mathbb{R}^n$ is a finite collection of polytopes such that the faces of each element* *of Δ are elements of Δ. The k-dimensional elements of a complex Δ are called k-cells.*

3. *A polyhedral complex $\Delta \subset \mathbb{R}^n$ is called **pure** n-dimensional if every maximal element of Δ(with respective to inclusion) is an n-dimensional polyhedron.*

4. *Two n-dimensional polytopes in a complex Δ are **adjacent** if they intersect along a common face of dimension $n-1$.*

5. *Δ is **hereditary** if for every $\tau \in \Delta$, any two n-dimensional polytopes σ, σ' that contain τ can be connected by a sequence $\sigma = \sigma_1, \sigma_2, \cdots, \sigma_m = \sigma'$ in Δ such that each σ_i is n-dimensional, each σ_i contains τ, and σ_i, σ_{i+1} are adjacent for each i.*

Having defined the underlying region where the spline functions are supported, we can define these functions. In the following, we assume the polyhedral complex Δ is pure n-dimensional in $\mathbb{R}^n$. Let $\sigma_1, \cdots, \sigma_m$ be a given, fixed, ordering of the n-cells in Δ, and let $R = \bigcup_{i=1}^m \sigma_i$.

Definition 1.1.2 (Splines)**.** 1. *For each $r \geq 0$, we denote by $C^r(\Delta)$ the collection of C^r functions f on R such that for every $\delta \in \Delta$ (including those of dimension $< n$), the restriction $f|_\delta$ is a polynomial function in $\mathbb{R}[x_1, \cdots, x_n]$.*

2. *$C_k^r(\Delta)$ is the subset of $f \in C^r(\Delta)$ such that the restriction of f to each cell in Δ is a polynomial function of degree at most k.*

Now we ask the central question in spline theory.

Question: what is the dimension of the vector space $C_k^r(\Delta)$, in terms of k, r and the geometry of R?

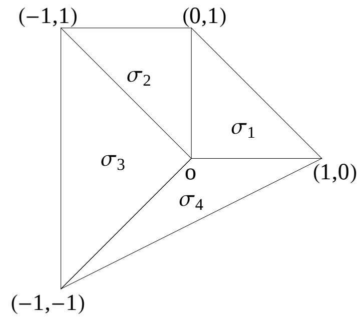
Here is a simple example in two dimensions.

Example 1.1.3. The plane region R is divided into four triangles, labeled as σ_1, σ_2, σ_3, σ_4. A spline $f \in C^r(R)$ is given by a 4-tuple (f_1, f_2, f_3, f_4) with f_i a polynomial in x, y on σ_i, for each $i = 1, 2, 3, 4$.

The intersection of $\sigma_1 \cap \sigma_2$ is the interval defined by $x = 0$. Therefore, similar to equation (1.1.2), we have

$$f_1 - f_2 \in \langle x^{r+1} \rangle, \text{ equivalently, } f_1 - f_2 = a_1 x^{r+1}, \tag{1.1.3}$$

for some polynomial $a_1 \in \mathbb{R}[x, y]$.



Similarly, $\sigma_2 \cap \sigma_3$ is the interval defined by $x+y=0$, $\sigma_3 \cap \sigma_4$ is defined by $x-y=0$ and $\sigma_4 \cap \sigma_1$ is defined by $y=0$. So we have the following equations

$$f_1 - f_2 = a_1 x^{r+1} \tag{1.1.4}$$

$$f_2 - f_3 = a_2 (x+y)^{r+1} \tag{1.1.5}$$

$$f_3 - f_4 = a_3 (x-y)^{r+1} \tag{1.1.6}$$

$$f_4 - f_1 = a_4 y^{r+1} \tag{1.1.7}$$

for some polynomials $a_1, a_2, a_3, a_4 \in \mathbb{R}[x,y]$.

Adding these four equations gives

$$0 = a_1 x^{r+1} + a_2(x+y)^{r+1} + a_3(x-y)^{r+1} + a_4 y^{r+1}. \tag{1.1.8}$$

This algebraic relation is called a *syzygy* in $\langle x^{r+1}, (x+y)^{r+1}, (x-y)^{r+1}, y^{r+1}\rangle$. We refer to Chapter 2 Section 1 for a more detailed introduction to syzygies. Therefore, a spline function f gives rise to a syzygy in the 4 polynomials, which are powers of linear forms; each linear form defines the faces of intersection of the 2-dimensional facets of the complex. Conversely, given such a syzygy, namely a 4-tuple (a_1, a_2, a_3, a_4) satisfying the equation (1.1.8), we can solve say, f_1, f_2, f_3 in terms of f_4 and then give a formula for the dimension of $C^r_k(R)$.

The upshot of this argument is, knowing all the syzygies enables us to find the dimension of $C^r_k(R)$. Understanding syzygies of ideals is an important question in commutative algebra. Here, we see one application of algebra to spline theory.

1.2 Previous work on splines

There have been two different approaches to splines. One is analytic, using Bernstein-Bezier coordinates. The other is algebraic, which we follow in this thesis.

In the planar case, Alfeld and Schumaker [AS90] use Bezier-Bernstein techniques to give an explicit formula for the dimension of $C_k^r(\Delta)$ when $k \geq 3r+1$. In [Bil88], Billera constructed a complex of modules where the spline module $C^r(\Delta)$ appeared as the top homology. Combining this tool with a vanishing result of Whiteley [Whi91] allowed him to prove a conjecture of Strang [Str73] on $\dim C_k^1(\Delta)$, for generic complex Δ (that is, complexes where all 2-cells are triangles whose edges are in sufficiently general position).

In [Sch97],[SS97], Schenck and Stillman introduced a chain complex different from that used by Billera. The top homology of their complex also gives the spline module and the lower homologies have nicer properties. Using this tool, Geramita and Schenck [GS97] determined the dimension of planar (mixed) splines in sufficiently high degrees. Another interesting aspect of [GS97] is the use of inverse system relating ideals in $\mathbb{R}[x,y]$ generated by powers of homogeneous linear forms and ideals of fatpoints in $\mathbb{P}^1$.

In the case of trivariate splines, Alfeld, Schumaker and Whiteley [ASW93] determined the dimension of C^1 generic tetrahedral splines for degree $d \geq 8$. But for $r > 1$, there is no general formula known. In [Alf96], [AS08] Alfeld and Schumaker gave upper and lower bounds for $\dim C_k^r(\Delta)$.

It is natural to first consider some simple tetrahedral complexes, as a first step in understanding splines on general tetrahedral complexes. In this chapter, for a tetrahedral complex Δ_v which consists of several tetrahedra sharing a single interior vertex v, we generalize the approach of Geramita and Schenck [GS97] and find a lower bound for the $\dim C_k^r(\Delta)$, $r = 1, 2$. See §1.8 for a precise statement.

1.3 Statement of the result

For a tetrahedral complex $\Delta = \Delta_v$, denote the number of tetrahedra by f_3, the number of 2-dimensional interior faces passing through v by f_2, the number of interior edges with $h_e = 2$, $h_e = 3$, and $h_e \geq 4$, respectively by $f_{1,2}$, $f_{1,3}$, and $f_{1,4}$. Recall that h_e is the number of distinct hyperplanes incident to e. Let f_1 be the number of interior edges, so $f_1 = f_{1,2} + f_{1,3} + f_{1,4}$.

In the following, the notation $\binom{a}{k}$ denotes the binomial coefficients, with the standard convention that $\binom{a}{k} = 0$, if $a < k$.

Theorem 1.3.1. *The dimension of* $C^1(\Delta)_k$*, the vector space of splines of smoothness* $r = 1$ *of degree exactly* k*, is given as follows:*

$$\dim C^1(\Delta)_k = h_{2,k} + C_k,$$

where

$$h_{2,k} = \dim H_2(\mathcal{R}/\mathcal{J})_k,$$

$$\begin{aligned} C_k =& f_3\binom{k+2}{2} - f_2\Big[\binom{k+2}{2} - \binom{k}{2}\Big] \\ &+ f_{1,2}\Big[\binom{k+2}{2} - 2\binom{k}{2} + \binom{k-2}{2}\Big] \\ &+ (f_{1,3} + f_{1,4})\Big[\binom{k+2}{2} - 3\binom{k}{2} + 2\binom{k-1}{2}\Big] \\ &- \dim(R/J(v))_k, \end{aligned}$$

and $\dim(R/J(v))_k$ *is given by equations* (1.5.4),(1.5.5) *and* (1.5.6)*, and explicitly computed using the method of §1.6.*

Proposition 1.3.2. $\dim(R/J(v))_k$ *in the following cases is given by*

k	0	1	2	≥ 3
4 hyperplanes	1	3	2	0
5 hyperplanes(Δ_1)	1	3	1	0
5 hyperplanes(Δ_2)	1	3	2	0
Clough-Tocher	1	3	0	0

Theorem 1.3.3. *The dimension of* $C^2(\Delta)_k$*, the vector space of splines of smoothness* $r = 2$ *of degree exactly* k*, is given as follows.*

$$\dim C^2(\Delta)_k = h_{2,k} + D_k,$$

where

$$h_{2,k} = \dim H_2(\mathcal{R}/\mathcal{J})_k,$$

$$
\begin{aligned}
D_k =& f_3\binom{k+2}{2} - f_2\Big[\binom{k+2}{2} - \binom{k-1}{2}\Big] \\
&+ f_{1,2}\Big[\binom{k+2}{2} - 2\binom{k-1}{2} + \binom{k-4}{2}\Big] \\
&+ f_{1,3}\Big[\binom{k+2}{2} - 3\binom{k-1}{2} + \binom{k-2}{2} + \binom{k-3}{2}\Big] \\
&+ f_{1,4}\Big[\binom{k+2}{2} - 4\binom{k-1}{2} + 3\binom{k-2}{2}\Big] \\
&- \dim(R/J(v))_k,
\end{aligned}
$$

and $\dim(R/J(v))_k$ *is explicitly computed using the method of* §*1.6.*

Proposition 1.3.4. $\dim(R/J(v))_k$ *in the following cases are given by*

k	*0*	*1*	*2*	*3*	*4*	*5*	≥ 6
4 hyperplanes	*1*	*3*	*6*	*6*	*4*	*1*	*0*
5 hyperplanes(Δ_1)	*1*	*3*	*6*	*5*	*3*	*0*	*0*
5 hyperplanes(Δ_2)	*1*	*3*	*6*	*5*	*2*	*0*	*0*
Clough-Tocher	*1*	*3*	*6*	*4*	*1*	*0*	*0*

Our goal in this chapter is to prove the above result. In the following sections, we develop necessary tools to analyze the tetrahedral splines. In §1.4, we define spline complexes, following [Sch97]. To understand each component of the spline complex, we were led to analyze certain ideals of powers of linear forms in §1.5; All the components are easy to deal with except one, for which we have to use tools from both commutative algebra and algebraic geometry. In §1.6, §1.7, we completely determine the last component and give examples.

1.4 Spline Complexes

Let $R = \mathbb{R}[x, y, z]$ be fixed throughout this chapter. Our tetrahedral complex Δ_v, which we call a *Cell*, consists of several tetrahedra sharing a single interior vertex v. Following Schenck [Sch97], we define the spline complex for $C^r(\Delta_v)$, for any $r \geq 0$.

In general, for a tetrahedral complex Δ, $C^r(\Delta)$ is not a graded module over R and it is convenient to have a graded module to compute the dimension of splines for each degree. Denote by $\hat{\Delta}$ the simplicial complex obtained by embedding the simplicial complex $\Delta \subset \mathbb{R}^3$ in the plane $\{w = 1\} \subset \mathbb{R}^4$ and forming the cone with the origin. Then the set of splines (of all degrees) on $\hat{\Delta}$ is a graded module $C^r(\hat{\Delta})$ over a polynomial ring $S = \mathbb{R}[x, y, z, w]$ and $C^r(\hat{\Delta})_k = C^r_k(\Delta)$.

In our situation, we don't need to do the above cone construction and $C^r(\Delta)$ is still a graded module over R and $C^r(\Delta)_k$ will be the vector space of splines of smoothness r of degree exactly k, since there is a single interior vertex v for our tetrahedral complex Δ_v and we can put the vertex v at the origin $O = (0, 0, 0) \in \mathbb{R}^3$, so every linear form defining a hyperplane passing through v will be homogeneous.

Let $\Delta = \Delta_v$ in the rest of the paper, unless otherwise stated.

Fix an integer $r \geq 0$. Define a complex of ideals of $\mathcal{J}$ on Δ by

$$\begin{aligned} J(\sigma) &= 0 && \text{for } \sigma \in \Delta_3, \\ J(\tau) &= \langle {l_\tau}^{r+1} \rangle && \text{for } \tau \in \Delta_2^0, \\ J(e) &= \langle {l_\tau}^{r+1} \rangle_{e \in \tau} && \text{for } e \in \Delta_1^0, \\ J(v) &= \langle {l_\tau}^{r+1} \rangle_{v \in \tau} && \text{for } v \in \Delta_0^0. \end{aligned}$$

Here Δ_i^0 are the i-dimensional interior faces of Δ and we consider all the tetrahedra Δ_3 as interior. l_τ is the homogeneous linear form in R defining the affine hull of τ. We denote h_e and h_v as the number of hyperplanes incident to e and v respectively. Then $J(e)$ is an ideal generated by h_e powers of linear forms, and similarly $J(v)$ is generated by h_v powers of linear forms.

We also define the constant complex $\mathcal{R}$ on Δ by $\mathcal{R}(\sigma) = R$ for each face $\sigma \in \Delta$ with the boundary map ∂_i the usual simplicial boundary map. We get the following quotient complex $\mathcal{R}/\mathcal{J}$:

$$0 \to \sum_{\sigma \in \Delta_3} R \xrightarrow{\partial_3} \sum_{\tau \in \Delta_2^0} R/J(\tau) \xrightarrow{\partial_2} \sum_{e \in \Delta_1^0} R/J(e) \xrightarrow{\partial_1} R/J(v) \to 0. \tag{1.4.1}$$

Lemma 1.4.1. *$H_1(\mathcal{R}/\mathcal{J}) = H_0(\mathcal{R}/\mathcal{J}) = 0$, and $H_2(\mathcal{R}/\mathcal{J})$ is Artinian.*

Proof. If we form the cone $\hat{\Delta}$, and define the constant complex $\mathcal{S}$ on Δ by $\mathcal{S}(\Delta) = S$ for each face $\sigma \in \Delta$, we get the quotient complex $\mathcal{S}/\mathcal{J}$, see [Sch97]:

$$0 \to \sum_{\sigma \in \Delta_3} S \xrightarrow{\partial_3} \sum_{\tau \in \Delta_2^0} S/J(\tau) \xrightarrow{\partial_2} \sum_{e \in \Delta_1^0} S/J(e) \xrightarrow{\partial_1} S/J(v) \to 0.$$

Since

$$\mathcal{S}/\mathcal{J} = \mathcal{R}/\mathcal{J} \otimes_R R[w],$$

we have

$$H_1(\mathcal{S}/\mathcal{J}) = H_1(\mathcal{R}/\mathcal{J}) \otimes_R R[w],$$
$$H_2(\mathcal{S}/\mathcal{J}) = H_2(\mathcal{R}/\mathcal{J}) \otimes_R R[w].$$

By Lemma 3.1 in [Sch97], $\dim H_2(\mathcal{S}/\mathcal{J}) \leq 1$, so we have

$$\dim H_2(\mathcal{R}/\mathcal{J}) \leq 0.$$

Similarly, $\dim H_1(\mathcal{S}/\mathcal{J}) \leq 0$ implies that

$$H_1(\mathcal{R}/\mathcal{J}) = H_0(\mathcal{R}/\mathcal{J}) = 0.$$

□

1.5 Dimension of graded components of the modules

It is well known that

$$\dim R_k = \binom{k+2}{2}. \tag{1.5.1}$$

Since $J(\tau)$ is a principal ideal generated by an element of degree $r+1$, we also have

$$\dim(R/J(\tau))_k = \binom{k+2}{2} - \binom{k-r+1}{2}. \tag{1.5.2}$$

1.5.1 The case $r = 1$

To compute $\dim(R/J(e))_k$, we use the minimal free resolution of the ideal $J(e)$.

Lemma 1.5.1. *The minimal free resolution of $J(e)$ is given by*

$$0 \to R(-4) \to R(-2)^2 \to R \to R/J(e) \to 0 \qquad \text{if } h_e = 2,$$
$$0 \to R(-3)^2 \to R(-2)^3 \to R \to R/J(e) \to 0 \qquad \text{if } h_e \geq 3.$$

So we get

$$\dim(R/J(e))_k = \begin{cases} \binom{k+2}{2} - 2\binom{k}{2} + \binom{k-2}{2} & \text{if } h_e = 2, \\ \binom{k+2}{2} - 3\binom{k}{2} + 2\binom{k-1}{2} & \text{if } h_e \geq 3. \end{cases} \tag{1.5.3}$$

Proof. If $h_e = 2$, then $J(e)$ is a complete intersection, generated by two quadratics. If $h_e \geq 3$, then $J(e) = \langle l_1^2, l_1 l_2, l_2^2 \rangle$, the result follows. □

Similarly, we can analyse the ideal $J(v)$, which is generated by squares of the linear forms which define the hyperplanes passing through v. Since the dimension of quadratic forms in R is 6, we only need to consider the case $h_v \leq 6$. If $h_v \geq 6$, then $J(v) = \langle x^2, y^2, z^2, xy, xz, yz \rangle$, so

$$\dim(R/J(v))_k = \begin{cases} 1 & \text{if } k = 0 \\ 3 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \end{cases} \tag{1.5.4}$$

This case is actually the Clough-Tocher complex in Example 1.7.8.

At the other extreme, if $h_v = 3$, then

$$J(v) = \langle x^2, y^2, z^2 \rangle,$$

and therefore,

k	0	1	2	3	≥ 4
$\dim(R/J(v))_k$	1	3	3	1	0

We are thus left with the case $h_v = 4$, or 5. If $h_v = 4$, suppose the four hyperplanes passing through v are defined by l_1, l_2, l_3, l_4, so the ideal $J(v) = \langle l_1^2, l_2^2, l_3^2, l_4^2 \rangle$. After a change of variables,

$$J(v) = \langle x^2, y^2, z^2, l^2 \rangle,$$

for some linear form l in x, y, z. This ideal is an example of almost complete intersection, whose Hilbert series are given by Iarrobino (Lemma C of [Iar97]), giving

$$\dim(R/J(v))_k = \begin{cases} 1 & \text{if } k = 0 \\ 3 & \text{if } k = 1 \\ 2 & \text{if } k = 2 \\ 0 & \text{if } k \geq 3 \end{cases} \tag{1.5.5}$$

For $h_v = 5$, there are more variations, depending on the five linear forms defining the hyperplanes passing through v. After a change of variables, we may assume the linear forms are given by $x, y, z, l_1(x, y, z), l_2(x, y, z)$. If the linear forms l_1, l_2 only involve two variables, say x, y, (see Example 1.7.3), then

$$J(v) = \langle x^2, y^2, xy, z^2 \rangle,$$

and the $\dim(R/J(v))_k$ is the same as Equation (1.5.5).

In the other cases, it is harder to analyze the ideal $J(v)$, though we can still compute a Grobner basis and find the dimension as given by

$$\dim(R/J(v))_k = \begin{cases} 1 & \text{if } k = 0 \\ 3 & \text{if } k = 1 \\ 1 & \text{if } k = 2 \\ 0 & \text{if } k \geq 3 \end{cases} \tag{1.5.6}$$

We can also get the above formulas of $\dim(R/J(v))_k$ using fatpoints as in §1.6.

1.5.2 The case $r = 2$.

Lemma 1.5.2. *The minimal free resolution of $J(e)$ is given by*

$$0 \to R(-6) \to R(-3)^2 \to R \to R/J(e) \to 0 \qquad \textit{if } h_e = 2,$$
$$0 \to R(-4) \oplus R(-5) \to R(-3)^3 \to R \to R/J(e) \to 0 \qquad \textit{if } h_e = 3,$$
$$0 \to R(-4)^3 \to R(-3)^4 \to R \to R/J(e) \to 0 \qquad \textit{if } h_e \geq 4.$$

So we get

$$\dim(R/J(e))_k = \begin{cases} \binom{k+2}{2} - 2\binom{k-1}{2} + \binom{k-4}{2} & \text{if } h_e = 2, \\ \binom{k+2}{2} - 3\binom{k-1}{2} + \binom{k-2}{2} + \binom{k-3}{2} & \text{if } h_e = 3, \\ \binom{k+2}{2} - 4\binom{k-1}{2} + 3\binom{k-2}{2} & \text{if } h_e \geq 4. \end{cases} \tag{1.5.7}$$

Proof. The ideal $J(e)$ is of codimension 2 in R, so we can apply the Hilbert-Burch Theorem [Eis95]. There are 3 cases:

Case 1: $h_e = 2$. It is similar to the case $r = 1$, but $J(e)$ is a complete intersection of two cubics.

Case 2: $h_e = 3$. Suppose the linear forms are given by l_1, l_2 and $l_3 = al_1 + bl_2$, then it is not hard to see the linear syzygy of l_1^3, l_2^3, l_3^3 is given by

$$-a^3(al_1 + 2bl_2)l_1^3 + b^3(2al_1 + bl_2)l_2^3 + (al_1 - bl_2)l_3^3 = 0,$$

and the quadratic syzygy is given by

$$(a^3l_2^2)l_1^3 + (2a^2bl_1^2 + 2ab^2l_1l_2 + b^3l_2^2)l_2^3 + (-l_2^3)l_3^3 = 0.$$

Then the minimal free resolution of $J(e)$ is given by

$$0 \to R(-4) \oplus R(-5) \xrightarrow{\varphi} R(-3)^3 \xrightarrow{\langle l_1^3, l_2^3, l_3^3 \rangle} R \to R/J(e) \to 0,$$

where

$$\varphi = \begin{bmatrix} -a^3(al_1 + 2bl_2) & a^3l_2^2 \\ b^3(2al_1 + bl_2) & 2a^2bl_1^2 + 2ab^2l_1l_2 + b^3l_2^2 \\ al_1 - bl_2 & -l_2^3 \end{bmatrix}.$$

Case 3: $h_e \geq 4$. Suppose the hyperplanes incident to e are given by $l_1, l_2, \cdots, l_s$, where $l_i = a_il_1 + b_il_2$ for $i \geq 3$, then it is easy to see the ideal $J(e) = \langle l_1^3, l_1^2l_2, l_1l_2^2, l_2^3 \rangle$, so the minimal free resolution of $J(e)$ is given by

$$0 \to R(-4)^3 \xrightarrow{\psi} R(-3)^4 \xrightarrow{\langle l_1^3, l_1^2l_2, l_1l_2^2, l_2^3 \rangle} R \to R/J(e) \to 0,$$

where

$$\psi = \begin{bmatrix} -l_2 & 0 & 0 \\ l_1 & -l_2 & 0 \\ 0 & l_1 & -l_2 \\ 0 & 0 & l_1 \end{bmatrix}.$$

□

As in the case above, though the number of hyperplanes passing through v may be big, the dimension of $R/J(v)$ only depends on the ideal $J(v)$. Since the dimension of cubic forms in R is 10, we only need to consider the case $h_v \leq 10$.

Example 1.5.3. If $h_v \geq 10$, then $J(v) = \langle x, y, z \rangle^3$ and

k	0	1	2	≥ 3
$\dim(R/J(v))_k$	1	3	6	0

At the other extreme, if $h_v = 3$, then $J(v) = \langle x^3, y^3, z^3 \rangle$, so

k	0	1	2	3	4	5	6	≥ 7
$\dim(R/J(v))_k$	1	3	6	7	6	3	1	0

We are thus left to consider the possibilities for $h_v \in \{4, 5, .., 9\}$. We use the inverse system dictionary to translate this question into one about the Hilbert function of h_v fatpoints on $\mathbb{P}^2$. There are two distinct cases.

Case 1: $h_v \in \{4, \cdots, 8\}$. In this case, we can give a complete answer to the dimension of $(R/J(v))_k$ for each degree k.

Case 2: $h_v = 9$. Then there are two cases depending on whether the cone of numerically effective classes of divisors on the surface obtained by blowup $\mathbb{P}^2$ at the 9 points is finitely generated or not. If the cone is finitely generated, then Harbourne's algorithm, which we will give below [1.6.2], still works and enables us to compute the Hilbert function of fatpoints, thus $\dim(R/J(v))_k$, for each k. However, if the cone is not finitely generated, it is a famous open problem in algebraic geometry (see Miranda's survey article [Mir99]) to determine the Hilbert function of fatpoints, and therefore difficult to compute $\dim(R/J(v))_k$.

1.6 Review of Inverse System and Fatpoints on $\mathbb{P}^2$

1.6.1 Inverse system

In [EI95], Emsalem and Iarrobino proved there is a close connection between ideals generated by powers of linear forms and ideals of fatpoints. We use their results in the special case of ideals generated by powers of linear forms in three variables and ideals of fatpoints in $\mathbb{P}^2$. See [Ger96], [GS97] for more information.

Let $p_1, \cdots, p_n \in \mathbb{P}^2$ be a set of distinct points,

$$p_i = [p_{i1} : p_{i2} : p_{i3}],$$
$$I(p_i) = \wp_i \subseteq R' = k[x', y', z'].$$

A fat point ideal is an ideal of the form

$$F = \bigcap_{i=1}^{n} \wp_i^{\alpha_i+1} \subset R'. \tag{1.6.1}$$

We define

$$L_{p_i} = p_{i1}x + p_{i2}y + p_{i3}z \in R, \text{ for } 1 \leq i \leq n. \tag{1.6.2}$$

Define an action of R' on R by partial differentiation:

$$p(x', y', z') \cdot q(x, y, z) = p(\partial/\partial x, \partial/\partial y, \partial/\partial z)q(x, y, z). \tag{1.6.3}$$

Since F is a submodule of R', it acts on R. The set of elements annihilated by the action of F is denoted by F^{-1}.

Theorem 1.6.1 (Emsalem and Iarrobino [EI95])**.** *Let F be an ideal of fatpoints*

$$F = \bigcap_{i=1}^{n} \wp_i^{\alpha_i+1},$$

then

$$(F^{-1})_j = \begin{cases} R_j & \textit{for } j \leq \max\{\alpha_i\}, \\ L_{p_1}^{j-\alpha_1} R_{\alpha_1} + \cdots, L_{p_n}^{j-\alpha_n} R_{\alpha_n} & \textit{for } j \geq \max\{\alpha_i + 1\}. \end{cases} \tag{1.6.4}$$

and

$$\dim_k(F^{-1})_j = \dim_k(R/F)_j.$$

Corollary 1.6.2. *In the case $r = 1, 2$, let*

$$F = \wp_1^{j-r} \cap \cdots \cap \wp_n^{j-r}$$

be an ideal of fatpoints on $\mathbb{P}^2$. Then $(F^{-1})_j = \langle L_{p_1}^{r+1}, \cdots, L_{p_n}^{r+1} \rangle_j$, and

$$\dim(R/J(v))_j = \begin{cases} \binom{j+2}{2} & \textit{for } 0 \leq j \leq r, \\ \dim F_j & \textit{for } j \geq r+1. \end{cases} \tag{1.6.5}$$

Therefore, to obtain the dimension of $(R/J(v))_k$, for each k, it is necessary to consider a corresponding ideal of fatpoints on $\mathbb{P}^2$.

1.6.2 Blowing up points in $\mathbb{P}^2$

Here we will use some facts about rational surfaces obtained by blowing-up n points $p_1, \cdots, p_n$ on $\mathbb{P}^2$, see Hartshorne [Har77]. We follow Harbourne [GHM09] and only state what is needed in this thesis.

There is a well-known correspondence between the graded pieces of an ideal of fat points $F \subset R$ and the global sections of a line bundle on the surface X which is the blowup of $\mathbb{P}^2$ at the points. Let E_i be the class of the exceptional divisor over the point p_i, and L the pullback of a line on $\mathbb{P}^2$. For the fatpoint ideal F in corollary 1.6.2, define

$$D_j = jL - (j-r)(E_1 + \cdots + E_n). \tag{1.6.6}$$

Then $\dim J_j = h^0(D_j)$, thus we have

$$\dim(R/J(v))_j = \begin{cases} \binom{j+2}{2} & \text{for } 0 \leq j \leq r, \\ h^0(D_j) & \text{for } j \geq r+1. \end{cases} \tag{1.6.7}$$

Remark 1.6.3. This equation tells us that $\dim(R/J(v))_j$ only depends on the divisor D_j, which only depends on the configuration of the fatpoints, and thus only depends on the geometry of the hyperplanes passing through v, See §1.7 for examples.

On X, the divisor class group $Cl(X)$ is a free abelian group with basis $L, E_1, \cdots, E_n$ which has the intersection product:

$$L^2 = -E_i^2 = 1, \qquad L.E_i = E_j.E_i = 0, \qquad \text{for } j \neq i. \tag{1.6.8}$$

The canonical class of X is

$$K_X = -3L + E_1 + \cdots + E_n.$$

We also define

$$A_n = (n-2)L - K_X.$$

A prime divisor is the class of a reduced irreducible curve on X and an effective divisor is a nonnegative integer combination of prime divisors. We denote the set of effective divisors by $EFF(X)$. A divisor whose intersection product with every effective divisor is ≥ 0 is called *numerically effective*(nef). We define $Neg(X)$ as the classes of prime divisors C with $C^2 < 0$. In [GHM09] Proposition 3.1 and 4.1, $Neg(X)$ is explicitly determined, which is the main point for the following algorithm of Geramita, Harbourne, and Migliore to compute $h^0(F)$ for any divisor F on X. To determine $Neg(X)$, we first define a few classes of divisors on X.

1. $\mathcal{B}_r = \{E_1, \cdots, E_r\}$;
2. $\mathcal{L}_r = \{L - E_{i_1} - \cdots - E_{i_j} | 2 \leq j, 0 < i_1 < \cdots < i_j \leq r\}$;
3. $\mathcal{Q}_r = \{2L - E_{i_1} - \cdots - E_{i_j} | 5 \leq j \leq r\}$;
4. $\mathcal{C}_r = \{3L - 2E_{i_1} - E_{i_2} - \cdots - E_{i_j} | 7 \leq j \leq 8, j \leq r\}$;
5. $\mathcal{M}_8 = \{4L - 2E_{i_1} - 2E_{i_2} - 2E_{i_3} - E_{i_4} - \cdots - E_{i_8}, 5L - 2E_{i_1} - 2E_{i_2} - \cdots - 2E_{i_6} - E_{i_7} - E_{i_8}, 6L - 2E_{i_1} - 2E_{i_2} - \cdots - 2E_{i_8}\}$.

Let $\mathcal{N}_r = \mathcal{B}_r \cup \mathcal{L}_r \cup \mathcal{Q}_r \cup \mathcal{C}_r \cup \mathcal{M}_8$. Let X be obtained by blowing up $2 \leq r \leq 8$ distinct points of $\mathbb{P}^2$, then

$$Neg(X) \subset \mathcal{N}_r.$$

and

$$Neg(X) = neg(X) \cup \{C \in \mathcal{N}_r | C^2 = -1, C \cdot D \geq 0, \text{ for all } D \in neg(X)\},$$

where $neg(X)$ is the subset of $Neg(X)$ of classes of those C with $C^2 = C \cdot C < -1$.

Remark 1.6.4. In any given case, we can list the five classes of divisors on X and $Neg(X)$ is the union of the classes C with $C^2 < -1$ and the classes C' with $C'^2 = -1$ and $C' \cdot D \geq 0$ for all $D \in neg(X)$. Each class in $\mathcal{L}_r$ is the pullback of a line passing through the points $p_{i_1}, \cdots, p_{i_j}$ if they are on a line; similarly, each class in $\mathcal{Q}_r$ is the pullback of a conic passing through the points $p_{i_1}, \cdots, p_{i_j}$ if they are on a conic; and so on. The computation of C^2 and $C \cdot D$ just uses the intersection product, see Equation (1.6.8). See also examples in §1.7.

Once we have determined $Neg(X)$, we can use the following algorithm due to Geramita, Harbourne, and Migliore [GHM09] to compute $h^0(F)$ for any class F on X.

Algorithm :

Start with $H = F$, $N = 0$.

If $H.C < 0$ for some $C \in Neg(X)$, replace H by $H - C$ and replace N by $N + C$. Eventually either $H.A_n < 0$ or $H.C \geq 0$ for all $C \in Neg(X)$.

In the first case, F is not effective, and $h^0(F) = 0$.

In the latter case, H is nef and effective and we have a Zariski decomposition

$$F = H + N,$$

with

$$h^0(F) = h^0(H) = (H^2 - H.K_X)/2 + 1.$$

Remark 1.6.5. The above algorithm is based on Bezout Theorem. See Miranda [Mir99] for an elementary exposition.

1.7 Examples of Fatpoint Computation

In this section, we will apply the above algorithm to compute $\dim(R/J(v))_j$, depending on the number of hyperplanes h_v passing through v, where $h_v \in \{4, 5, 6, 7, 8\}$. We mainly consider the case $r = 2$ and indicate the similar computation for $r = 1$ in remarks. We demonstrate the computation with examples of tetrahedral complexes constructed from the standard octahedron Δ by perturbing a vertex to get different numbers of hyperplanes passing through O. A key point is to determine $Neg(X)$ in each case, where X as above, is the blowup of $\mathbb{P}^2$ at the fatpoints corresponding to the linear forms defining h_v hyperplanes. For concreteness, we give the coordinates of the vertices of Δ as $O = (0,0,0)$, $P_1 = (10,0,0)$, $P_2 = (0,10,0)$, $P_3 = (-10,0,0)$, $P_4 = (0,-10,0)$, $P_5 = (0,0,10)$, $P_6 = (0,0,-10)$. As noted in Remark 1.6.3, the result does not depend on the actual coordinates.

Example 1.7.1. [4 hyperplanes]

By perturbing one vertex along one of the edges, we get an example with 4 hyperplanes. For example, move P_1 along the edge P_1P_2 to get $P_1' = (7,3,0)$. Then there are 3 hyperplanes passing through the interior edge OP_5 with defining equations and the corresponding 3 points in $\mathbb{P}^2$ as follows.

$$l_1 = x \longleftrightarrow Q_1 = [1:0:0],$$
$$l_2 = y \longleftrightarrow Q_2 = [0:1:0],$$
$$l_3 = 3x - 7y \longleftrightarrow Q_3 = [3:-7:0].$$

The points Q_1, Q_2, Q_3 are collinear. The other hyperplane defined by

$$l_4 = z \longleftrightarrow Q_4 = [0:0:1].$$

Q_4 is not collinear with the other 3 points. So on the surface X, the divisor

$$C_1 = L - E_1 - E_2 - E_3 \in Neg(X),$$

where L is the pullback of a line on $\mathbb{P}^2$ and E_i is the exceptional divisor corresponding to Q_i for $i = \{1,2,3,4\}$. In fact,

$$Neg(X) = \{C_1, L - E_1 - E_4, L - E_2 - E_4, L - E_3 - E_4, E_1, E_2, E_3, E_4\}.$$

Define D_j as in equation (1.6.6),

$$D_j = jL - (j-2)(E_1 + E_2 + E_3 + E_4).$$

Remark 1.7.2. In this example, $\mathcal{N}_r = \mathcal{B}_r \cup \mathcal{L}_r$. It seems that we should include $L - E_1 - E_2$, $L - E_1 - E_3$,

$L-E_2-E_3$ in $Neg(X)$. However, these classes are not in $Neg(X)$, because they are not prime. For example,

$$L - E_1 - E_2 = (L - E_1 - E_2 - E_3) + E_3$$

is a sum of two prime divisors. The class $L - E_1 - E_2 - E_3$ is prime because the points Q_1, Q_2, Q_3 are collinear. Moreover, $neg(X) = \emptyset$.

Let's just show that $h^0(D_4) = 4$ as a sample of computation, using the intersection product. First,

$$\begin{aligned} D_4 &= 4L - 2(E_1 + E_2 + E_3 + E_4), \\ D_4.C_1 &= 4L^2 + 2E_1^2 + 2E_2^2 + 2E_3^2 \\ &= 4 - 2 - 2 - 2 = -2 < 0. \end{aligned}$$

So we take

$$D_4' = D_4 - C_1 = 3L - E_1 - E_2 - E_3 - 2E_4.$$

It is easy to check that

$$D_4' \cdot C \geq 0, \text{ for any } C \in Neg(X),$$

therefore, D_4' is nef and effective. So the Zariski decomposition of D_4 is

$$D_4 = D_4' + C_1.$$

Using the intersection product (1.6.8) again, we have

$$\begin{aligned} D_4'^2 &= (3L)^2 + E_1^2 + E_2^2 + E_4^2 + (2E_3)^2 \\ &= 9 - 1 - 1 - 1 - 4 \\ &= 2 \end{aligned}$$

Similarly,

$$\begin{aligned} K_X &= -3L + E_1 + E_2 + E_3 + E_4, \\ D_4'.K_X &= -(3L)^2 - E_1^2 - E_2^2 - E_4^2 - 2E_3^2 \\ &= -9 + 1 + 1 + 1 + 2 \\ &= -4 \end{aligned}$$

So we get

$$h^0(D_4) = h^0(D_4') = (D_4'^2 - D_4'.K_X)/2 + 1 = 4.$$

A similar computation shows the Zariski decomposition of D_5 is

$$D_5 = D_5' + 2C_1,$$

where

$$D_5' = 3L - E_1 - E_2 - E_3 - 3E_4,$$

and

$$h^0(D_5) = h^0(D_5') = 1.$$

Summarizing, we have

$$\dim(R/J(v))_j = h^0(D_j) = \begin{cases} 6 & \text{for } j = 3 \\ 4 & \text{for } j = 4 \\ 1 & \text{for } j = 5 \\ 0 & \text{for } j \geq 6 \end{cases} \tag{1.7.1}$$

Example 1.7.3. [5 hyperplanes: Δ_1]

By perturbing P_2, P_3 on the plane $z = 0$, there are 4 hyperplanes passing through the interior edge OP_5(or OP_6), so there are 4 corresponding points Q_1, Q_2, Q_3, Q_4 on $\mathbb{P}^2$ which lie on a line l.

There is another point $Q_5 = [0 : 0 : 1]$ corresponding to the plane $z = 0$, not lying on l. On the surface

X from blowup of the $Q_i's$, as above, E_i corresponds to Q_i, for $1 \leq i \leq 5$, the divisor class

$$C_1 = L - E_1 - E_2 - E_3 - E_4 \in Neg(X).$$

In fact,

$$Neg(X) = \{C_1, L - E_i - E_5, E_i, E_5, i \in \{1, 2, 3, 4\}\}.$$

We also have

$$D_j = jL - (j-2)(E_1 + E_2 + E_3 + E_4 + E_5).$$

We analyse the case $j = 4$ in details, since it is similar for any j. First,

$$\begin{aligned} D_4 &= 4L - 2(E_1 + E_2 + E_3 + E_4 + E_5), \\ D_4.C_1 &= 4L^2 + 2E_1^2 + 2E_2^2 + 2E_3^2 + 2E_4^2 \\ &= 4 - 2 - 2 - 2 - 2 \\ &= -4 < 0. \end{aligned}$$

So we take

$$D_4' = D_4 - C_1 = 3L - E_1 - E_2 - E_3 - E_4 - 2E_5.$$

Moreover,

$$\begin{aligned} D_4'.C_1 &= 3L^2 + E_1^2 + E_2^2 + E_3^2 + E_4^2 \\ &= 3 - 1 - 1 - 1 - 1 = -1 < 0. \end{aligned}$$

So we subtract C_1 from D_4' to get

$$D_4'' = D_4' - C_1 = 2L - 2E_5.$$

Now, we can check

$$D_4''.C_1 = 2L^2 = 2 > 0.$$

In fact, $D''_4.C \geq 0$ for any $C \in Neg(X)$. Therefore, we have the Zariski Decomposation of D_4 as

$$D_4 = D''_4 + 2C_1.$$

A similar computation will show that,

$$D''^2_4 = 0, \qquad D''_4.K_X = -4.$$

So we get

$$h^0(D_4) = h^0(D''_4) = 3.$$

Summarizing, we have

$$\dim(R/J(v))_j = h^0(D_j) = \begin{cases} 5 & \text{for } j = 3 \\ 3 & \text{for } j = 4 \\ 0 & \text{for } j \geq 5 \end{cases} \tag{1.7.2}$$

Remark 1.7.4. We have given the formula of $\dim(R/J(v))_k$ in Equation (1.5.5) for the case $r = 1$, by applying a result of Iarrobino [Iar97]. Here we reprove that formula using a similar computation as above. Since $r = 1$, the divisor D_j is given by

$$D_j = jL - (j-1)(E_1 + \cdots + E_5).$$

By Corollary 1.6.2, we just need to compute $\dim(R/J(v))_j$, or equivalently $h^0(D_j)$ for $j \geq 2$. For $j = 2$, $D_2 = 2L - (E_1 + \cdots + E_5)$. Since

$$D_2 \cdot C_1 = 2L^2 + E_1^2 + \cdots + E_4^2 = 2 - 4 = -2 < 0,$$

we get $D'_2 = D_2 - C_1 = L - E_5$, which is effective. Since

$$D'^2_2 = 0, \text{ and } D'_2 \cdot K_X = -3 + 1 = -2,$$

we get

$$h^0(D_4) = h^0(D_4') = \frac{0-(-2)}{2} + 1 = 2.$$

The computation for $j > 2$ is completely similar.

Example 1.7.5. [5 hyperplanes: Δ_2]

By perturbing one vertex along the interior of a face, we can get another example of 5 hyperplanes. For example, if we perturb $P_5 = (0,0,10)$ to $P_5' = (1,1,8)$, then there are 3 hyperplanes passing through the interior edge OP_1(or OP_3) with defining equations and the corresponding 3 points in $\mathbb{P}^2$ as follows.

$$l_1 = z \longleftrightarrow Q_1 = [0:0:1],$$
$$l_2 = y \longleftrightarrow Q_2 = [0:1:0],$$
$$l_3 = 8y - z \longleftrightarrow Q_3 = [0:8:-1].$$

The points Q_1, Q_2, Q_3 are collinear in $\mathbb{P}^2$. Similarly, through the interior edge OP_2(or OP_4), there are 3 hyperplanes

$$l_1 = z \longleftrightarrow Q_1 = [0:0:1],$$
$$l_4 = x \longleftrightarrow Q_4 = [1:0:0],$$
$$l_5 = 8x - z \longleftrightarrow Q_5 = [8:0:-1].$$

Similarly, Q_1, Q_4, Q_5 are collinear and Q_1 is the intersection of the two lines.

So, on the surface X, the two divisors

$$C_1 = L - E_1 - E_2 - E_3,$$
$$C_2 = L - E_1 - E_4 - E_5,$$

are in $Neg(X)$, where E_i is the exceptional divisor corresponding to Q_i for $i = \{1,2,3,4,5\}$. In this case, $Neg(X)$ given by

$$\{C_1, C_2, L - E_2 - E_4, L - E_2 - E_5, L - E_3 - E_4, L - E_3 - E_5, E_i, i = \{1,2,3,4,5\}\}.$$

We also have

$$D_j = jL - (j-2)(E_1 + E_2 + E_3 + E_4 + E_5).$$

It is easy to check that D_3 is nef, and a similar computation shows

$$D_3^2 = 4, \qquad D_3.K_X = -4.$$

So

$$h^0(D_3) = 5.$$

For

$$D_4 = 4L - 2(E_1 + E_2 + E_3 + E_4 + E_5),$$

we have

$$\begin{aligned} D_4.C_1 &= 4L^2 + 2E_1^2 + 2E_2^2 + 2E_3^2 \\ &= 4 - 2 - 2 - 2 < 0. \end{aligned}$$

So we take

$$D_4' = D_4 - C_1 = 3L - E_1 - E_2 - E_3 - 2E_4 - 2E_5.$$

Since

$$\begin{aligned} D_4'.C_2 &= 3L^2 + E_1^2 + 2E_4^2 + 2E_5^2 \\ &= 3 - 1 - 2 - 2 < 0. \end{aligned}$$

So we take

$$D_4'' = D_4' - C_2 = 2L - E_2 - E_3 - E_4 - E_5.$$

It is easy to check D_4'' is nef, so we get the Zariski decomposition

$$D_4 = D_4'' + C_1 + C_2.$$

A similar computation will show that,

$$D_4''^2 = 0, \qquad D_4''.K_X = -2.$$

Thus,

$$h^0(D_4) = h^0(D_4') = 2.$$

For $j \geq 5$, D_j is not effective, so $h^0(D_j) = 0$.

Summarizing, we have

$$\dim(R/J(v))_j = h^0(D_j) = \begin{cases} 5 & \text{for } j = 3 \\ 2 & \text{for } j = 4 \\ 0 & \text{for } j \geq 5 \end{cases} \tag{1.7.3}$$

Remark 1.7.6. Comparing example 1.7.3 and example 1.7.5, the $\dim(R/J(v))_j$ differ at $j = 4$, even though in both examples, $J(v)$ is an ideal generated by 5 powers of linear forms in x, y, z.

Remark 1.7.7. In Equation (1.5.6), we have given a formula of $\dim(R/J(v))_k$ in the case $r = 1$. Here we prove that formula using the same computation. For $k = 2$, we consider the divisor $D_2 = 2L-(E_1+\cdots+E_5)$. Now

$$D_2 \cdot C_1 = 2 - 3 = -1 < 0,$$

so we get $D_2' = D_2 - C_1 = L - E_4 - E_5$. It is easy to check that $D_2' \cdot D \geq 0$, for all $D \in Neg(X)$, so D_2' is *nef*. Since

$$D_2'^2 = -1, \text{ and } D_2' \cdot K_X = -3 + 2 = -1,$$

we get

$$h^0(D_2) = h^0(D_2') = \frac{-1-(-1)}{2} + 1 = 1.$$

Example 1.7.8. [6 hyperplanes: *Clough-Tocher*(CT)]

This tetrahedral complex CT is constructed by putting an interior point O, which we put at the origin $(0, 0, 0)$, in the tetrahedron and decomposing the tetrahedron into four tetrahedra.

Through each interior edge of CT, there are 3 different hyperplanes, each corresponding to a point in $\mathbb{P}^2$. So we have 4 lines in $\mathbb{P}^2$, with each line corresponding to an interior edge of CT, and on each line, there are exactly 3 points. Moreover, each point is the intersection of two lines. For example, through the interior edge OP_1, we have the planes OP_1P_2,OP_1P_3 and OP_1P_4, each corresponding to a point, say Q_1,Q_2 and Q_3 in $\mathbb{P}^2$. Similarly, around OP_2,OP_3 and OP_4, we have the following corresponding points.

OP_2	OP_3	OP_4
$OP_2P_1 \longleftrightarrow Q_1$	$OP_3P_1 \longleftrightarrow Q_4$	$OP_4P_1 \longleftrightarrow Q_5$
$OP_2P_3 \longleftrightarrow Q_4$	$OP_3P_2 \longleftrightarrow Q_2$	$OP_4P_2 \longleftrightarrow Q_3$
$OP_2P_4 \longleftrightarrow Q_5$	$OP_3P_4 \longleftrightarrow Q_6$	$OP_4P_3 \longleftrightarrow Q_6$

The configuration of the 6 points on $\mathbb{P}^2$ is type 10 in the table of [GHM09].

So on the surface X obtained from the blowup the 6 points, we have the following class of divisors in $Neg(X)$.

$$C_1 = L - E_1 - E_2 - E_3, \qquad C_2 = L - E_1 - E_4 - E_5,$$
$$C_3 = L - E_2 - E_4 - E_6, \qquad C_4 = L - E_3 - E_5 - E_6,$$

with E_i as the exceptional divisor from blowup of Q_i, for $1 \leq i \leq 6$.

In this case, D_3 is nef, with

$$D_3^2 = 3, \qquad D_3.K_X = -3.$$

So $h^0(D_3) = 4$. As for D_4, the Zariski decomposition is

$$D_4 = 0 + C_1 + C_2 + C_3 + C_4.$$

So $h^0(D_4) = h^0(0) = 1$. Summarizing, we have

$$\dim(R/J(v))_j = h^0(D_j) = \begin{cases} 4 & \text{for } j = 3 \\ 1 & \text{for } j = 4 \\ 0 & \text{for } j \geq 5 \end{cases} \tag{1.7.4}$$

1.8 Main Result

1.8.1 Theorems

Now we have computed the dimension for each component of the complex $\mathcal{R}/\mathcal{J}$. Putting the results together, we get our main result.

For a tetrahedral complex $\Delta = \Delta_v$, denote the number of tetrahedra by f_3, the number of 2-dimensional interior faces passing through v by f_2, the number of interior edges with $h_e = 2$, $h_e = 3$, and $h_e \geq 4$, respectively by $f_{1,2}$, $f_{1,3}$, and $f_{1,4}$. Recall that h_e is the number of hyperplanes incident to e. Let f_1 be the number of interior edges, so $f_1 = f_{1,2} + f_{1,3} + f_{1,4}$.

Theorem 1.8.1. *The dimension of $C^1(\Delta)_k$, the vector space of splines of smoothness $r = 1$ of degree exactly k, is given as follows.*

$$\dim C^1(\Delta)_k = h_{2,k} + C_k,$$

where

$$h_{2,k} = \dim H_2(\mathcal{R}/\mathcal{J})_k,$$

$$\begin{aligned} C_k =& f_3\binom{k+2}{2} - f_2\Big[\binom{k+2}{2} - \binom{k}{2}\Big] \\ &+ f_{1,2}\Big[\binom{k+2}{2} - 2\binom{k}{2} + \binom{k-2}{2}\Big] \\ &+ (f_{1,3}+f_{1,4})\Big[\binom{k+2}{2} - 3\binom{k}{2} + 2\binom{k-1}{2}\Big] \\ &- \dim(R/J(v))_k, \end{aligned}$$

and $\dim(R/J(v))_k$ *is given by Equations* (1.5.4),(1.5.5) *and* (1.5.6)*, and explicitly computed using the method of §1.6.*

Theorem 1.8.2. *The dimension of $C^2(\Delta)_k$, the vector space of splines of smoothness $r = 2$ of degree exactly k, is given as follows.*

$$\dim C^2(\Delta)_k = h_{2,k} + D_k,$$

where

$$h_{2,k} = \dim H_2(\mathcal{R}/\mathcal{J})_k,$$

$$
\begin{aligned}
D_k =& f_3\binom{k+2}{2} - f_2\Big[\binom{k+2}{2} - \binom{k-1}{2}\Big] \\
&+ f_{1,2}\Big[\binom{k+2}{2} - 2\binom{k-1}{2} + \binom{k-4}{2}\Big] \\
&+ f_{1,3}\Big[\binom{k+2}{2} - 3\binom{k-1}{2} + \binom{k-2}{2} + \binom{k-3}{2}\Big] \\
&+ f_{1,4}\Big[\binom{k+2}{2} - 4\binom{k-1}{2} + 3\binom{k-2}{2}\Big] \\
&- \dim(R/J(v))_k,
\end{aligned}
$$

and $\dim(R/J(v))_k$ *is explicitly computed using the method of §1.6.*

In the above theorems,

$$\binom{a}{2} = 0, \text{ if } a < 2.$$

of Theorems 1.8.1, 1.8.2. The Euler characteristic equation applied to the complex $\mathcal{R}/\mathcal{J}$ is

$$\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J}).$$

Since $C^2(\Delta) \simeq H_3(\mathcal{R}/\mathcal{J})$, this fact implies that

$$\dim C^2(\Delta)_k = \dim \sum_{i=0}^{3}(-1)^i \bigoplus_{\beta\in\Delta^0_{3-i}} (R/J(\beta))_k + \dim \sum_{i=0}^{2}(-1)^i H_{2-i}(\mathcal{R}/\mathcal{J})_k$$

By equations (1.5.3), (1.5.7), we get $\dim(R/J(\tau))_k$ and $\dim(R/J(e))_k$. By Lemma 1.4.1, $H_1(\mathcal{R}/\mathcal{J}) = H_0(\mathcal{R}/\mathcal{J}) = 0$. Also $H_2(\mathcal{R}/\mathcal{J})$ is Artinian, so its k-th graded component vanishes when $k \gg 0$. □

Corollary 1.8.3. $\dim C^1(\Delta)_k \geq C_k$, *and* $\dim C^2(\Delta)_k \geq D_k$.

Remark 1.8.4. The two complexes $\mathcal{R}/\mathcal{J}$ for $r = 1$ and $r = 2$ are different, so are the modules $H_2(\mathcal{R}/\mathcal{J})$.

Corollary 1.8.5. *The dimension of* $C^1_d(\Delta)$*, the vector space of splines of smoothness* $r = 1$ *of degree at most* d*, is bounded below as*

$$
\begin{aligned}
\dim C^1_d(\Delta) \geq& (f_3 - f_2 + f_1)\binom{d+3}{3} + (f_2 - 2f_{1,2} - 3f_{1,3} - 3f_{1,4})\binom{d+1}{3} \\
&+ 2(f_{1,3} + f_{1,4})\binom{d}{3} + f_{1,2}\binom{d-1}{3} - \sum_{k=0}^{d}\dim(R/J(v))_k.
\end{aligned}
\tag{1.8.1}
$$

For $d \geq 4$, the inequality simplifies to

$$\begin{aligned}\dim C_d^1(\Delta) \geq & \frac{f_3}{6}d^3 + (f_3 - f_2)d^2 + (\frac{11}{6}f_3 - 2f_2 + 3f_1 + f_{1,2})d \\ & + (f_3 - f_2 + f_{1,3} + f_{1,4}) - \sum_{k=0}^{d} \dim(R/J(v))_k.\end{aligned} \tag{1.8.2}$$

Corollary 1.8.6. *The dimension of $C_d^2(\Delta)$, the vector space of splines of smoothness $r = 2$ of degree at most d, is bounded below as*

$$\begin{aligned}\dim C_d^2(\Delta) \geq & (f_3 - f_2 + f_1)\binom{d+3}{3} + (f_2 - 2f_{1,2} - 3f_{1,3} - 4f_{1,4})\binom{d}{3} \\ & + (f_{1,3} + 3f_{1,4})\binom{d-1}{3} + f_{1,3}\binom{d-2}{3} + f_{1,2}\binom{d-3}{3} \\ & - \sum_{k=0}^{d} \dim(R/J(v))_k.\end{aligned} \tag{1.8.3}$$

For $d \geq 6$, the inequality simplifies to

$$\begin{aligned}\dim C_d^2(\Delta) \geq & \frac{f_3}{6}d^3 + (f_3 - \frac{3}{2}f_2)d^2 + (\frac{11}{6}f_3 - \frac{3}{2}f_2 + 6f_1 + 3f_{1,2} + f_{1,3})d \\ & + (f_3 - f_2 - 9f_{1,2} - 4f_{1,3} - 2f_{1,4}) - \sum_{k=0}^{d} \dim(R/J(v))_k.\end{aligned} \tag{1.8.4}$$

For the extremal cases of exactly 3 or ≥ 10 hyperplanes, we work out $\dim(R/J(v))_k$ in Example 1.5.3. Here we put our results on the above examples of 4, 5 or 6 hyperplanes in one place for the readers' convenience. We don't claim these are all the cases of 4, 5 or 6 hyperplanes. Our point is to illustrate the computation of $\dim(R/J(\tau))_k$ by the algorithm. All the remaining cases are similar but more complicated.

Proposition 1.8.7. *In the case $r = 2$, $\dim(R/J(v))_k$ for the following cases are given by*

k	*0*	*1*	*2*	*3*	*4*	*5*	≥ 6
4 hyperplanes	*1*	*3*	*6*	*6*	*4*	*1*	*0*
5 hyperplanes(Δ_1)	*1*	*3*	*6*	*5*	*3*	*0*	*0*
5 hyperplanes(Δ_2)	*1*	*3*	*6*	*5*	*2*	*0*	*0*
Clough-Tocher	*1*	*3*	*6*	*4*	*1*	*0*	*0*

1.8.2 Comparison and Examples

In the case $r = 1$, Alfeld, Schumaker and Whiteley [ASW93] Theorem 54 also give a lower bound on $\dim C_d^1(\Delta)$

$$\dim C_d^1(\Delta) \geq \frac{d(d-1)(d-5)}{6}T + 3(d-1)V_I + d(d-1)V_B + 1 + 5d - 2d^2, \text{ for } d \geq 3,$$

where T, V_B, V_I are the number of tetrahedra, boundary vertices, and interior vertices, respectively.

In the setting of our paper, $V_I = 1$. Using the relation $V_B = 2f_3 - f_2 + 2$, their bound is given by

$$\dim C_d^1(\Delta) \geq \frac{f_3}{6}d^3 + (f_3 - f_2)d^2 + (-\frac{7}{6}f_3 + f_2 + 6)d - 2. \tag{1.8.5}$$

Compare our bound in Equation (1.8.2) with their bound, the difference is

$$f_{1,2}d + (f_3 - f_2 + f_{1,3} + f_{1,4}) + 2 - \sum_{k=0}^{d} \dim(R/J(v))_k. \tag{1.8.6}$$

It is clear that our bound is better if $f_{1,2} > 0$. If $f_{1,2} = 0$, the difference is only

$$-\sum_{k=2}^{d} \dim(R/J(v))_k.$$

For a tetrahedral partition Δ of a simply connected polygonal region $D \subset \mathbb{R}^3$ and $d > r$, Lau [Lau06] proved that, a lower bound of $C_d^r(\Delta)$ is given by

$$\begin{aligned}\dim C_d^r(\Delta) \geq &\binom{d+3}{3} + f_2\binom{d-r+2}{3} \\ &- f_1[\binom{d+3}{3} - \binom{r+3}{3} - (d-r)\binom{r+2}{2}] + \delta,\end{aligned} \tag{1.8.7}$$

where

$$\delta = \sum_{k=1}^{f_1}\sum_{l=1}^{d-r}\sum_{j=1}^{l}(r + 1 + j - je_{k*})_+,$$

and e_{k*} is the number of interior faces attached to the interior edge e_k $(k = 1, 2, \cdots, f_1)$ which lie on different planes. Here, $(x)_+ = x$, if $x > 0$. Otherwise, $(x)_+ = 0$.

The leading term of his formula is only

$$\frac{1-f_1+f_2}{6}d^3 = \frac{f_3-1}{6}d^3,$$

thus is weaker than our bounds, especially when d is large.

Example 1.8.8. For the example of 4 hyperplanes, we have

$$f_3 = 8, f_2 = 12, f_{1,2} = 4, f_{1,3} = 2, f_{1,4} = 0.$$

For the two interior edges OP_5 and OP_6, there are three hyperplanes passing through each edge. For the other four edges OP_1, OP_2, OP_3, OP_4, only two hyperplanes passing through each edge. The formula above gives the following lower bound for $\dim C_d^1(\Delta)$

d	0	1	2	3
Bound	1	4	12	30

and

$$\dim C_d^1(\Delta) \geq 4/3d^3 - 4d^2 + 38/3d - 8, \text{ for } d \geq 4.$$

In this case, the bound is actually exact and $H_2(\mathcal{R}/\mathcal{J}) = 0$.

In the case $r = 2$, we get the lower bound for $\dim C_d^2(\Delta)$ as

d	0	1	2	3	4
Bound	1	4	10	22	44

and

$$\dim C_d^2(\Delta) \geq 4/3d^3 - 10d^2 + 140/3d - 69, \text{ for } d \geq 6.$$

In this case, the bound is actually also exact and therefore $H_2(\mathcal{R}/\mathcal{J}) = 0$.

Example 1.8.9. For the Clough-Tocher, we have

$$f_3 = 4, f_2 = 6, f_{1,2} = f_{1,4} = 0, f_{1,3} = 4.$$

The lower bound for $\dim C^2_d(\Delta)$ is given by

$$\dim C^2_d(\Delta) \geq 2\binom{d+3}{3} - 6\binom{d}{3} + 4\binom{d-1}{3} + 4\binom{d-2}{3} - \sum_{k=0}^{d} \dim(R/J(v))_k.$$

The right hand side is equal to $1, 4, 10, 20, 35$ for $d = 0, 1, 2, 3, 4$ and

$$2/3d^3 - 5d^2 + 79/3d - 33, \text{ for } d \geq 5.$$

For example, when $d = 17$, this formula gives 2245, agrees with the computation in [ASS92].

Similarly, the lower bound for $\dim C^1_d(\Delta)$ is given by

$$\dim C^1_d(\Delta) \geq 2\binom{d+3}{3} - 6\binom{d+1}{3} + 8\binom{d}{3} - \sum_{k=0}^{d} \dim(R/J(v))_k.$$

Example 1.8.10. For the example of 5 hyperplanes, Example 1.7.3, we have

$$f_3 = 8, f_2 = 12, f_{1,2} = 4, f_{1,3} = 0, f_{1,4} = 2.$$

For the example of 5 hyperplanes, Example 1.7.5, we have

$$f_3 = 8, f_2 = 12, f_{1,2} = 1, f_{1,3} = 4, f_{1,4} = 1.$$

We can also get lower bounds in the same way as in Examples 1.8.8 and 1.8.9.

Remark 1.8.11. Using Macaulay2, we found that $C^2(\Delta)$ is a free module over R for the above examples of 4 hyperplanes and Clough-Tocher. By Schenck's Theorem in [Sch97], this observation implies $H_2(\mathcal{R}/\mathcal{J}) = 0$ and $\dim C^2(\Delta)_k = C_k$, so our bound in corollary 1.8.3 is tight. In any case, corollary 1.8.3 agrees with Macaulay2's output, thus provids strong supports of our theorem.

Remark 1.8.12. To compute the homology $H_2(\mathcal{R}/\mathcal{J})$, one way is to program in the appropriate maps and have Macaulay 2 compute the homology.

Remark 1.8.13. For any given tetrahedral complex Δ_v, we can find the configuration of the fatpoints corresponding to the hyperplanes passing through v. The classification of all configurations of fatpoints up to 8 points is given in [GHM09], though some configurations do not correspond to a tetrahedral complex Δ_v.

Chapter 2

The Postnikov-Shapiro Conjecture

In this chapter, we give one approach to the Postnikov-Shapiro conjecture, including Schenck's conjecture as a special case, about the minimal free resolutions of a particular class of ideals of powers of linear forms. The conjecture is stated in §2.2.6. In §2.1, we give a short introduction to syzygies and resolutions of general homogeneous ideals. In §2.2, we give an overview on how the particular class of ideals arise. Our analysis of Schenck's conjecture, is given in §2.5 to §2.12, after introducing the basic tool of the Hilbert-Burch Theorem. In this part of analysis, we provide two different approaches to Schenck's conjecture. One of our main results is Theorem 2.8.2 in §2.8. In §2.13, we provide a proof of Postnikov-Shapiro conjecture in the case of four variables under an additional hypothesis. In §2.14, we gave an inductive proof of Postnikov-Shapiro conjecture for any number of variables, under additional hypotheses.

2.1 A short introduction to syzygies

In algebraic geometry, the ideal-variety correspondence enables us to study the geometric properties of a projective variety $X \subset \mathbb{P}^n$ from its homogeneous ideal I. Fix a field $\mathbb{K}$ of characteristic 0 throughout this chapter. The homogeneous coordinate ring of $\mathbb{P}^n$ is $S = \mathbb{K}[x_0, \cdots, x_n]$. It turns out that to study ideals effectively we also need to study graded modules over S; the primary example is the homogeneous coordinate ring of X, given by $S(X) = S/I$.

Let $M = \oplus_{d \in \mathbb{Z}} M_d$ be a finitely generated graded S-module with d-th graded component M_d. Because M is finitely generated, each M_d is a finite-dimensional vector space, and the *Hilbert function* of M is defined by

$$H_M(d) = \dim_{\mathbb{K}} M_d.$$

The Hilbert series of M is the generating function, defined by

$$HS_M(t) = \sum_{d \geq 0} H_M(d) t^d.$$

Hilbert had the idea of computing $H_M(d)$ by comparing M with free modules, using a *free resolution.* For any graded module M, denote by $M(a)$ the module M shifted by a,

$$M(a)_d = M_{a+d}$$

For instance, the free S-module of rank 1 generated by an element of degree a is $S(-a)$. Given homogeneous elements $m_i \in M$ of degree a_i that generate M as an S-module, we may define a degree-preserving map from the graded free module $F_0 = \oplus_i S(-a_i)$ onto M by sending the i-th generator to m_i. Let $M_1 \subset F_0$ be the kernel of this map. By the Hilbert Basis Theorem, M_1 is also a finitely generated module. The elements of M_1 are called *syzygies* on the generators m_i. Choosing finitely many homogeneous syzygies that generate M_1, we may define a map from a graded free module F_1 to F_0 with image M_1. Continuing in this way we construct a sequence of maps of graded free modules, called a *graded free resolution* of M :

$$\cdots \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0.$$

Then *Hilbert Syzygy Theorem* says that every finitely generated graded S-module has a finite graded resolution of length at most $n+1$. We say the above resolution is *minimal* if for each $l \geq 0$, all the nonzero entries of the matrix of ϕ_l are in the maximal ideal of S. Then a finitely graded module M has a unique minimal resolution up to isomorphism.

If the graded S-module M has a finite free resolution

$$\mathbb{F} : 0 \to F_m \xrightarrow{\phi_m} F_{m-1} \to \cdots F_1 \xrightarrow{\phi_1} F_0$$

with each F_i a finitely generated free module $F_i = \oplus_j S(-a_{i,j})$, then

$$H_M(d) = \sum_{i=0}^{m} (-1)^i \sum_j \binom{n + d - a_{i,j}}{n}.$$

There is a unique polynomial HP_M, called the *Hilbert polynomial*, such that

$$H_M(d) = HP_M(d)$$

when d is sufficiently large. The Hilbert polynomial contains interesting geometric information. For example,

1. The degree of the Hilbert polynomial $HP_{S(X)}$ is the dimension of the variety X.
2. If $\dim X = d$, then the degree of X, defined as the number of points where X intersects a generic $(n-d)$-dimensional linear subspace of $\mathbb{P}^n$, is $d!$ times the leading coefficient of $HP_{S(X)}$.

Since the Hilbert function of M is determined by the $a_{i,j}$ in the minimal free resolution of M, they are finer invariants than the Hilbert function. We use a compact way to display them, called a *Betti diagram.* If

$$F_i = \oplus_j S(-j)^{\beta_{i,j}};$$

that is, F_i requires $\beta_{i,j}$ minimal generators of degree j, then the Betti diagram of $\mathbb{F}$ is

	0	1	$\cdots$	s
i	$\beta_{0,i}$	$\beta_{0,i+1}$	$\cdots$	$\beta_{s,i+s}$
$i+1$	$\beta_{0,i+1}$	$\beta_{0,i+2}$	$\cdots$	$\beta_{s,i+1+s}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
j	$\beta_{0,j}$	$\beta_{0,j+1}$	$\cdots$	$\beta_{s,j+s}$

Example 2.1.1. [Three points in $\mathbb{P}^2$] There are two cases, depending on whether or not the points are collinear. First, if they are not collinear, we may take them to be the points $X = \{[0,0,1],[0,1,0],[1,0,0]\}$. It is easy to see the ideal of X is

$$I = \langle x, y\rangle \cap \langle x, z\rangle \cap \langle y, z\rangle,$$

which is generated by the quadratic monomials

$$F_1 = xy, F_2 = xz, \text{ and } F_3 = yz.$$

The relations among these three are easy to find; they are given by

$$-z \cdot F_1 + y \cdot F_2 = 0$$
$$-y \cdot F_2 + x \cdot F_3 = 0.$$

Therefore, the minimal free resolution of $S(X)$ is

$$0 \to S(-3)^2 \xrightarrow{\phi_1} S(-2)^3 \xrightarrow{\phi_0} S \to S(X) \to 0,$$

where

$$\phi_0 = \begin{bmatrix} xy & xz & yz \end{bmatrix}.$$

and

$$\phi_1 = \begin{bmatrix} -z & 0 \\ y & -y \\ 0 & x \end{bmatrix}.$$

The Betti diagram is

Total	1	3	2
0	1	-	-
1	-	3	2

From this resolution, we get the Hilbert function of $S(X)$ as

$$H_X(m) = \binom{m+2}{2} - 3\binom{m}{2} + 2\binom{m-1}{2},$$

which for $m \geq 1$ is

$$\frac{(m+1)(m+2) - 3m(m-1) + 2(m-1)(m-2)}{2} = 3.$$

Therefore, the Hilbert polynomial of X is the constant 3.

Now suppose the three points are collinear. For example, if the points are $[0,0,1],[0,1,0],[0,1,1]$, then the ideal of X is generated by

$$F_1 = x, F_2 = yz(y-z).$$

There is only the trivial relation

$$F_1 \cdot F_2 - F_2 \cdot F_1 = 0.$$

The minimal free resolution is given by

$$0 \to S(-4) \xrightarrow{\phi_1} S(-3) \oplus S(-1) \xrightarrow{\phi_0} S \to S(X) \to 0,$$

where

$$\phi_0 = \begin{bmatrix} x & yz(y-z) \end{bmatrix}.$$

and

$$\phi_1 = \begin{bmatrix} -yz(y-z) \\ x \end{bmatrix}.$$

The Betti diagram is

Total	1	2	1
0	1	1	-
1	-	-	-
2	-	1	1

It is easy to obtain the Hilbert function of $S(X)$ from the above resolution and conclude that the Hilbert polynomial is also the constant 3. Note the Betti diagrams of these two ideals are different.

So far, we introduced the minimal free resolutions of ideals and gave a simple example. In general, it is difficult to compute the minimal free resolution of a given ideal. One particular class of ideals for which the minimal free resolution is known are ideals generated by monomials. In this thesis, our primary object of study is the class of ideals generated by powers of linear forms. In the next section, we discuss the work of Postnikov-Shapiro, where the particular ideals generated by powers of linear forms arise.

2.2 How these ideals of powers of linear forms arise

2.2.1 Motivation

Let $Fl_n = SL(n, \mathbb{C})/B$ be the manifold of complete flags in $\mathbb{C}^n$. The manifold Fl_n comes equipped with a flag of tautological vector bundles $E_0 \subset E_1 \subset \cdots E_n$ and associated sequence of line bundles $L_i = E_i/E_{i-1}$,

$i = 1, 2, \cdots, n$. The L_i possess natural Hermitian structures induced from the standard Hermian metric $\sum z_i \bar{z}_i$ on $\mathbb{C}^n$. For $i = 1, 2, \cdots, n$, we denote by w_i the curvature form of the Hermitian line bundle L_i, which represents the Chern class $c_1(L_i)$ in the 2-dimensional cohomology of Fl_n. B. Shapiro and M. Shapiro [SS98] investigated the ring B_n generated by the forms $w_1, \cdots, w_n$. As an additive group, B_n is a free abelian group. The ring B_n is graded: $B_n = B_n^0 \oplus B_n^1 \oplus B_n^2 \oplus \cdots$. The component B_n^k consists of the $2k$-dimensional forms. The cohomology ring $H^*(Fl_n, \mathbb{Z})$ is a quotient of B_n, since the former is generated by the Chern clases $c_1(L_i)$.

Let $\hat{J}_n$ be the ideal in the polynomial ring $R = \mathbb{K}[x_1, \cdots, x_n]$ generated by the $2^n - 1$ polynomials of the form

$$\hat{p}_I = (x_{i_1} + \cdots + x_{i_r})^{r(n-r)+1},$$

where $I = \{i_1, \cdots, i_r\}$ is any nonempty subset of $\{1, 2, \cdots, n\}$. In [SS98], Shapiro-Shapiro proved

Theorem 2.2.1. *The ring B_n is canonically isomorphic, as a graded ring, to the quotient $R/\hat{J}_n$, The isomorphism is given by sending the generators w_i of B_n to the corresponding x_i.*

2.2.2 Work of Postnikov-Shapiro-Shapiro

In [PSS], Postnikov-Shapiro-Shapiro proved the conjecture in [SS98] on the dimension of the ring B_n by relating the ideal $\hat{J}_n$ to another ideal $\hat{I}_n$.

In the polynomial ring R, they define the ideal $\hat{I}_n$ generated by the monomials $\hat{m}_I$, given by

$$\hat{m}_I = (x_{i_1} \cdots x_{i_r})^{n-r} x_{i_1},$$

where $I = \{i_1 < \cdots < i_r\}$, ranges over nonempty subsets of $\{1, \cdots, n\}$. Then define $A_n = R/\hat{I}_n$.

A non-negative integer sequence $b = (b_1, \cdots, b_n)$ is called an *almost parking function* of size n if the monomial $x^b = x_1^{b_1} \cdots x_n^{b_n}$ does not belong to the ideal $\hat{I}_n$.

A forest is a graph without cycles. For a forest F on the vertices $0, \cdots, n$, an inversion is a pair of vertices labeled i and j such that $i > j$ and the vertex i belong to the path in F that joins the vertex j with the minimal vertex in its connected component.

Recall that for a graded algebra $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$, with $A_0 = \mathbb{K}$, the Hilbert series of A is the formal power series in q given by

$$HS(A) = \sum_{k \geq 0} q^k \dim_{\mathbb{K}} A_k.$$

Postnikov-Shapiro-Shapiro proved the following

Theorem 2.2.2. *[PSS] The algebras A_n and B_n have the same Hilbert series. The dimension of these algebras is equal to the number of forests on $n+1$ vertices. Moreover, the dimension $\dim_{\mathbb{K}} A_n^k = \dim_{\mathbb{K}} B_n^k$ of the k-th graded components of the algebras A_n and B_n is equal to*

1. *the number of almost parking functions b of size n such that $\sum_{i=1}^n b_i = k$;*

2. *the number of forests on $n+1$ vertices with $\binom{n}{2} - k$ inversions.*

The images of the monomials x^b, where b ranges over almost parking functions of size n, form linear bases in both algebras A_n and B_n.

2.2.3 Postnikov-Shapiro's generalization

The algebras in the previous subsection are associated to the complete graph K_{n+1} on $n+1$ vertices. Postnikov-Shapiro in [PS04] generalized these algebras to A_G and B_G associated to any graph G. They generalized parking functions to G-parking functions and the Theorem [PSS] to the setting of A_G and B_G. In this subsection, we first define G-parking functions and the algebras A_G and B_G, then state Postnikov-Shapiro's theorem and conjecture. Along the way, we will also discuss a special case of the conjecture, Schenck's conjecture.

A parking function of size n is a sequence $b = (b_1, \cdots, b_n)$ of non-negative integers such that its increasing rearrangement $c_1 \leq \cdots \leq c_n$ satisfies $c_i < i$. A famous formula of Cayley says the number of trees on $n+1$ labeled vertices equals $(n+1)^{n-1}$ and also equals the number of parking functions of size n. Postnikov-Shapiro [PS04] defined G-parking functions for any graph G, and Gabrielov [Gab93] proved that the number of G-parking functions equals the number of spanning trees of G, which specializes to Cayley's formula when $G = K_{n+1}$, the complete graph on $n+1$ vertices.

Let G be an undirected graph on the set of vertices $[n+1] := \{0, 1, \cdots, n\}$. Let a_{ij} be the number of edges connecting vertex i to vertex j. For a subset S in $\{1, \cdots, n\}$ and a vertex $i \in S$, let

$$d_S(i) = \sum_{j \notin S} a_{ij},$$

the number of edges from the vertex i to a vertex outside of the subset S. A sequence $b = (b_1, \cdots, b_n)$ of non-negative integers is a *G-parking function* if, for any nonempty subset $S \subseteq \{1, \cdots, n\}$, there exists $i \in S$ such that $b_i < d_S(i)$.

Postnikov-Shapiro construct two ideals I_G and J_G in the polynomial ring $R = \mathbb{K}[x_1, \cdots, x_n]$ as follows.

Definition 2.2.3. *The monomial ideal $I_G = \langle m_S \rangle$ is generated by*

$$m_S = \prod_{i \in S} x_i^{d_S(i)},$$

for all nonempty subsets $S \subseteq \{1, 2, \cdots, n\}$.

Let

$$D_S = \sum_{i \in S, j \notin S} a_{ij} = \sum_{i \in S} d_I(i)$$

be the total number of edges that join some vertex in S with a vertex outside of S. The ideal $J_G = \langle p_S \rangle$ is generated by

$$p_S = \big(\sum_{i \in S} x_i\big)^{D_S},$$

for all nonempty subsets $S \subseteq \{1, 2, \cdots, n\}$. Let $A_G = R/I_G$ and $B_G = R/J_G$.

It is easy to see that a non-negative integer sequence $b = (b_1, \cdots, b_n)$ is a G-parking function if and only if the monomial $x^b = x_1^{b_1} \cdots x_n^{b_n}$ is nonvanishing in the algebra A_G. Thus the monomials x^b, where b ranges over G-parking functions, form a basis of the algebra A_G.

Theorem 2.2.4. *[PS04] The monomials x^b, where b ranges over G-parking functions, form a linear basis of the algebra B_G. Thus, their Hilbert series are equal:*

$$HS(A_G) = HS(B_G). \tag{2.2.1}$$

Both of these algebras are finite-dimensional as linear spaces over $\mathbb{K}$, and

$$\dim_{\mathbb{K}} A_G = \dim_{\mathbb{K}} B_G = N_G,$$

where N_G is the number of spanning trees of the graph G.

Example 2.2.5. Consider the graph $G = K^{l,k}_{n+1}$, the complete graph on the vertices $\{0, 1, \cdots, n\}$ with the edges $e_{i,j}$, $i, j \neq 0$, of multiplicity k and the edges $e_{0,i}$ of multiplicity l, where l and k are two fixed nonnegative integers. For a nonempty subset $I = \{i_1, \cdots, i_r\} \subseteq \{1, \cdots, n\}$, we have

$$d_I(r) = \phi(r) = l + k(n - r). \tag{2.2.2}$$

Then

$$m_I = (x_{i_1} \cdots x_{i_r})^{\phi(r)},$$
$$p_I = (x_{i_1} + \cdots + x_{i_r})^{r\phi(r)}.$$

The two ideals associated to G are given by

$$I_\phi = \langle m_I \rangle, \qquad J_\phi = \langle p_I \rangle, \tag{2.2.3}$$

where I runs through all nonempty subsets of $\{1, \cdots, n\}$. See Equations (2.11.1) and (2.11.2) below for examples when $n = 3$.

2.2.4 The conjectures

Postnikov-Shapiro [PS04] showed $HS(R/I_\phi) = HS(R/J_\phi)$ as a corollary of Theorem 2.2.4. They also gave the following minimal free resolution of R/I_ϕ

$$... \longrightarrow C_3 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 = R \longrightarrow R/I_\phi \longrightarrow 0, \tag{2.2.4}$$

with

$$C_i = \bigoplus_{l_1, l_2, \cdots, l_i} R(-d(l_1, \cdots, l_i))^{\binom{n}{l_1, \cdots, l_i}}, \tag{2.2.5}$$

where the direct sum is over $l_1, \cdots, l_i \geq 1$ such that $l_1 + \cdots + l_i \leq n$,

$$d(l_1, \cdots, l_i) = l_1\phi(l_1) + l_2\phi(l_1 + l_2) + \cdots + l_i\phi(l_1 + \cdots + l_i),$$

and

$$\binom{n}{l_1, \cdots, l_i} = \frac{n!}{l_1! \cdots l_i!(n - l_1 - \cdots - l_i)!}$$

is the multinomial coefficient. This resolution means that the graded Betti numbers of I_ϕ are given by

$$b_{i,d(l_1,\cdots,l_i)} = \binom{n}{l_1, \cdots, l_i}.$$

Moreover, the i-th Betti number is given by

$$b_i(I_\phi) = i!S(n+1, i+1),$$

where $S(n+1, i+1)$ is the Stirling number of the second kind, i.e., the number of partitions of the set $\{0, 1, \cdots, n\}$ into $i+1$ nonempty subsets. In fact, they found the minimal free resolution of $I_\rho = \langle m_S \rangle$, generated by

$$m_S = (x_{i_1} \cdots x_{i_r})^{\rho_r},$$

for all nonempty subsets $S = \{i_1, \cdots, i_r\} \subseteq \{1, \cdots, n\}$, where $\rho_1 > \cdots > \rho_n > 0$, $\rho_i \in \mathbb{N}$, $1 \leq i \leq n$.

Conjecture 2.2.6. *[PS04] The graded Betti numbers of J_ϕ are also given by* (2.2.5)*.*

More generally, they conjecture

Conjecture 2.2.7. *[PS04] For any graph G, I_G and J_G have the same graded Betti numbers.*

In the special case $n = 3$, the two ideals are given by

$$I_\phi = \langle x^{l+2k}, y^{l+2k}, z^{l+2k}, (xy)^{l+k}, (xz)^{l+k}, (yz)^{l+k}, (xyz)^l \rangle, \tag{2.2.6}$$

$$J_\phi = \langle x^{l+2k}, y^{l+2k}, z^{l+2k}, (x+y)^{2l+2k}, (x+z)^{2l+2k}, (y+z)^{2l+2k}, (x+y+z)^{3l} \rangle, \tag{2.2.7}$$

where we use x, y, z instead of x_1, x_2, x_3 to simplify the notation. Schenck [Sch04] used the Inverse System of Macaulay, as we discussed in Chapter 1.8.2, to compute the Hilbert series of R/J_ϕ. He proved that the Hilbert series of R/J_ϕ is equal to that of R/I_ϕ. He also conjectured

Conjecture 2.2.8. *For $n = 3$, the minimal free resolution of J_ϕ is:*

$$0 \to R(-\sum_{i=1}^{3} \phi(i))^6 \to \begin{array}{c} R(-2\phi(2)-\phi(3))^6 \\ \oplus \\ R(-\phi(1)-\phi(2))^6 \end{array} \to \begin{array}{c} R(-\phi(1))^3 \\ \oplus \\ R(-2\phi(2))^3 \\ \oplus \\ R(-3\phi(3)) \end{array} \to R \to R/J_\phi \to 0.$$

The main focus of this chapter is on Conjecture 2.2.8. We also prove conditionally Conjecture 2.2.6. We show that the minimal free resolution of J_ϕ is obtained from gluing the minimal free resolutions of all its subideals, which are naturally associated to the partitions of $[n+1]$. To show the glued complex is exact, we use two theorems of Buchsbaum-Eisenbud. First, we use the Buchsbaum-Eisenbud Theorem on the factorizations of complementary minors in the complexes of minimal free resolutions of the subideals. Second, we apply the Eisenbud-Buchsbaum's Criterion of exactness to show the glued complex is exact, under the condition that certain module is free. In a certain sense, the proof illustrates a remark of Buchsbaum [Buc00], "One could get information about modules of finite homological dimension by transferring information from the 'tail' of its resolution to its 'head'."

We stress that most work on ideals generated by powers of linear forms uses Macaulay Inverse Systems, see [Ger96],[EI95] and [Iar97] for example, to translate into questions about fatpoints. However, that approach seems not sufficient to deal with the free resolutions of these ideals generated by powers of linear forms.

In the literature, there have been some work on G-parking function ideals, see [MSW] and [MS] for example. Their G-parking function ideals are monomial ideals and lattice ideals, while our ideals are generated by powers of linear forms, which are completely different.

The structure of the following sections is as follows. In §2.3, we give more examples of free resolutions, including free resolutions of monomial ideals. In §2.4 we state the Hilbert-Burch resolution of Cohen-Macaulay ideals of codimension 2. In §2.5, we explicitly compute the Hilbert-Burch resolution for ideals of codimension 2 generated by powers of linear forms. In §2.6, we explicitly syzygies of the ideal J_ϕ in conjecture 2.2.8 using the methods of §2.5. In §2.7, we show the constructed syzygies are minimal. In §2.8 and §2.9, we show the constructed syzygies generate the syzygies of J_ϕ, except one degree. In §2.10, we give a different approach to show the constructed syzygies generate the syzygies of J_ϕ. In §2.11, we prove

Conjecture 2.2.8 conditionally using Buchsbaum-Eisenbud's Criterion of exactness. The last three sections §2.12, §2.13 and §2.14 provide conditional proof of Conjecture 2.2.6.

2.3 Free resolutions of monomial ideals

In this section, we give more examples of resolutions. Here we follow the presentation of Eisenbud [Eis95].

A finite *simplicial complex* Δ is a finite set N, called the *vertices* of Δ, and a collection F of subsets of N, called the *faces* of Δ, such that if $A \in F$ is a face and $B \subset A$ then B is also in F. Maximal faces are called *facets.* A *simplex* is a simplicial complex in which every subset of N is a face.

We say that Δ is *labeled* by monomials of R if there is a monomial of R associated to each vertex of Δ. We then label each face A of Δ by the least common multiple of the labels of the vertices in A. We write m_A for the monomial that is the label of A. By convention the label of the empty face is $m_\emptyset = 1$. We also denote Δ_m for the subcomplex consisting of those faces of Δ whose labels divide m.

Let Δ be an oriented labeled simplicial complex, and write $I \subset S$ for the ideal generated by the monomials $m_j = x^{\alpha_j}$ labeling the vertices of Δ. We will associate to Δ a graded complex of free R-modules

$$\mathbb{F}(\Delta, R) : ... \to F_i \xrightarrow{\delta} F_{i-1} \to \cdots \xrightarrow{\delta} F_0 = R,$$

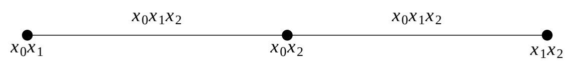
where F_i is the free R-module whose basis consists of the set of faces of Δ having i elements. The differential δ is given by

$$\delta(A) = \sum_{n \in A} (-1)^{pos(n,A)} \frac{m_A}{m_{A \backslash n}} (A \backslash n),$$

where $pos(n, A)$, the position of vertex n in A, is the number of elements preceding n in the ordering of A, and $A \backslash n$ denotes the face obtained from A by removing n.

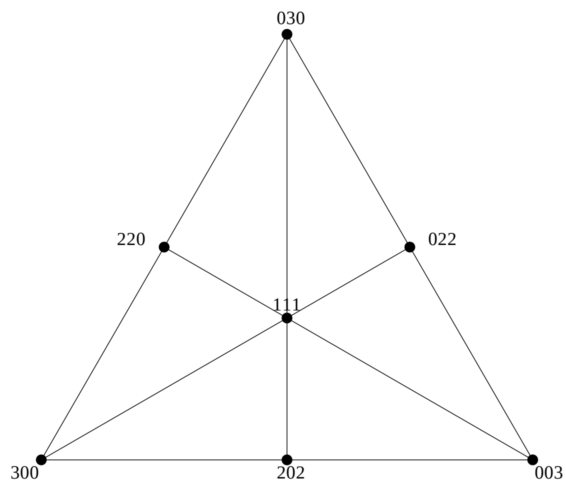
Theorem 2.3.1 (Bayer, Peeva, and Sturmfels)**.** *Let Δ be a simplicial complex labeled by the monomials $\{m_1, \cdots, m_t\} \in S$, and let $I = \langle m_1, \cdots, m_t \rangle \subset R$ be the ideal in R generated by the vertex labels. The complex $\mathbb{F}(\Delta, R)$ is a free resolution of R/I if and only if the reduced simplicial homology $H_i(\Delta_m, \mathbb{K})$ vanishes for every monomial m and every $i \geq 0$. Moreover, $\mathbb{F}(\Delta, R)$ is a minimal complex if and only if $m_A \neq m_{A'}$ for every proper subface A' of a face A.*

Example 2.3.2. We turn to our example of three points on $\mathbb{P}^2$, see 2.1.1. We can label the simplicial complex as follows:



The distinct subcomplexes of the form Δ_m are the empty complex Δ_1, the complexes $\Delta_{x_0x_1}$, $\Delta_{x_0x_2}$, $\Delta_{x_1x_2}$, each of which consists of a single point, and the complex $\Delta_{x_0x_1x_2}$ itself. As each of these is contractible, they have no higher reduced homology, and we see that the complex is the minimal free resolution of $R/\langle x_0x_1, x_0x_2, x_1x_2\rangle$.

Example 2.3.3. We consider the minimal free resolution of the ideal I_{K_4} associated to the complete graph K_4. In this case the simplicial complex Δ is the barycentric subdivision of a triangle. The following figure shows the complex Δ with the vertices marked by exponent vectors of the generators of I_{K_4}.



The Betti numbers $(1, 7, 12, 6)$ of the ideal I_{K_4}, which are also the numbers of i-dimensional faces of Δ, for $i = 0, 1, 2$, can be expressed in terms of the Stirling numbers. The graded Betti numbers of this ideal are indicated in the following minimal free resolution:

$$0 \to R(-6)^6 \to R(-5)^{12} \to R(-3)^4 \oplus R(-4)^3 \to R \to R/I \to 0.$$

Similarly, a minimal free resolution of the ideal $I_{K_{n+1}}$ associated with the complete graph K_{n+1} is given by the complex corresponding to the simplicial complex Δ, which is the barycentric subdivision of the $(n-1)$-dimensional simplex.

In the above Example 2.1.1, we have seen the explicit minimal free resolutions of two ideals. Both ideals are examples of the class of ideals of Cohen-Macaulay codimension two, for which the resolutions are given by the Hilbert-Burch Theorem.

2.4 Another class of free resolutions: Hilbert-Burch

The following theorem is of fundamental importance to this thesis. It describes the minimal free resolutions of an important class of ideals whose projective dimension $pd(R/I) = 2$.

Theorem 2.4.1 (Hilbert-Burch)**.** *Suppose that an ideal I in $R = \mathbb{K}[x_1, \cdots, x_n]$ has a free resolution of the form*

$$0 \to R^{m-1} \xrightarrow{A} R^m \xrightarrow{B} R \to R/I \to 0$$

for some m. Then there exists a nonzero element $g \in R$ such that $B = (g\tilde{f_1}, \cdots, g\tilde{f_m})$, where $\tilde{f_i}$ is the determinant of the $(m-1) \times (m-1)$ submatrix of A obtained by deleting row i. If k is algebraically closed and $V(I)$ has dimension $n-2$, then we may take $g = 1$.

Example 2.4.2. Let's compute the minimal free resolution of the ideal $I = \langle x^3, y^3, (x+y)^3 \rangle$ by hand. In this case, the ideal I is minimally generated by the three polynomials, so $m = 3$. If we take B as

$$B = \left[x^3, y^3, (x+y)^3\right],$$

then A is a 3×2 matrix, such that

$$BA = 0.$$

Note that,

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3. \tag{2.4.1}$$

So

$$x(x+y)^3 = x^3(x+3y) + xy^3 + 3x^2y^2, \tag{2.4.2}$$

$$y(x+y)^3 = x^3y + y^3(y+3x) + 3x^2y^2. \tag{2.4.3}$$

From Equations (2.4.2), (2.4.3), we get one syzygy of the ideal I,

$$(x+2y)x^3 + (2x+y)y^3 + (-x+y)(x+y)^3 = 0. \tag{2.4.4}$$

In other words, we have obtained one column of the matrix A, as

$$s_1 = \begin{bmatrix} x+2y \\ 2x+y \\ -x+y \end{bmatrix}.$$

We can also get a second syzygy easily by multiplying both sides of Equation (2.4.3) by y and rewriting the equation as

$$y^2x^3 + (3x^2+3xy+y^2)y^3 + (-y^2)(x+y)^3 = 0. \tag{2.4.5}$$

So we have obtained another column of A as

$$s_2 = \begin{bmatrix} y^2 \\ 3x^2+3xy+y^2 \\ -y^2 \end{bmatrix}.$$

In fact, the above two syzygies are independent and generates all the syzygies of the ideal I. First, the second syzygy is not a multiple of the first syzygy, in other words, s_2 is not a multiple of s_1, which is clear, for example, by noticing that $-x+y$ does not divide y^2. Second, these two syzygies generate all the syzygies of the ideal I. For example, there is another syzygy obtained by multiplying both sides of Equation (2.4.2) by x to get

$$(x^2+3xy+3y^2)x^3 + x^2y^3 + (-x^2)(x+y)^3 = 0. \tag{2.4.6}$$

In vector notation, we have

$$s_3 = \begin{bmatrix} x^2+3xy+3y^2 \\ x^2 \\ -x^2 \end{bmatrix}.$$

We can check that $s_3 = (x+y)s_1 + s_2$.

Now that we have the matrix A given by

$$A = \begin{bmatrix} x+2y & y^2 \\ 2x+y & 3x^2+3xy+y^2 \\ -x+y & -y^2 \end{bmatrix},$$

we can see that

$$\det \begin{bmatrix} x+2y & y^2 \\ -x+y & -y^2 \end{bmatrix} = -3y^3,$$

$$\det \begin{bmatrix} x+2y & y^2 \\ -x+y & -y^2 \end{bmatrix} = -3x^3,$$

$$\det \begin{bmatrix} x+2y & y^2 \\ 2x+y & 3x^2+3xy+y^2 \end{bmatrix} = -3(x+y)^3.$$

The same method of constructing syzygies can be applied to ideals of the form $I = \langle x^a, y^b, (x+y)^c \rangle$. However, it would be very messy if we compute the syzygies in the same way. In the following section, we give a more systematic approach to this class of ideals.

2.5 Explicit Computation of the Hilbert-Burch resolution

Given an ideal of the form $I = \langle x^a, y^b, (x+y)^c \rangle$ in the polynomial ring $R = k[x, y]$, we would like to write down explicitly its minimal free resolutions. By a change of variables, we have

$$\begin{aligned} I &= \langle x^a, y^b, (x+y)^c \rangle \\ &= \langle (x+y)^a, y^b, x^c \rangle \\ &= \langle x^a, (x+y)^b, y^c \rangle \end{aligned}$$

where the second equality follows by the change of variables

$$x := x+y, y := -y$$

and the last equality follows by the change of variables

$$x := -x, y := x + y.$$

So we can assume that $a \leq b \leq c$ from the start without loss of generality.

Also notice that if $c \geq a + b - 1$, then the generator $(x + y)^c$ is redundant and $I = \langle x^a, y^b \rangle$, so there is only the trivial syzygy

$$x^a y^b - y^b x^a = 0.$$

Therefore we focus on the case where $c \leq a + b - 2$ in the following. Set $r = \lfloor \frac{a+b-c}{2} \rfloor$ and $s = \lceil \frac{a+b-c}{2} \rceil$. Then the resolution of I is given the Hilbert-Burch resolution of the form

$$0 \to R(-c-r) \oplus R(-c-r) \xrightarrow{\phi_0} R(-a) \oplus R(-b) \oplus R(-c) \to I \to 0,$$

where

$$\phi_0 = \begin{bmatrix} A & D \\ B & E \\ C & F \end{bmatrix}$$

is a matrix of forms. In terms of the entries of the matrix ϕ_0 we may write the ideal $J = \langle C, F \rangle$ as

$$J = \langle x^a, y^b \rangle : (x + y)^c \tag{2.5.1}$$

where $\deg C = r$, $\deg F = s$, and $r + s = a + b - c$.

In the following we focus on the case where $r = s = \frac{a+b-c}{2}$, because it is sufficient for our later use and also avoids the unnecessary complication in the argument. The goal is to construct explicitly the matrix ϕ_0.

Suppose $f \in J_d$ is given by

$$f = \sum_{i+j=d} a_{i,j} x^i y^j,$$

then

$$
\begin{aligned}
f(x+y)^c &= \sum_{i+j=d} a_{i,j} x^i y^j \sum_{m+n=c} \binom{c}{m} x^m y^n \\
&= \sum_{i+j=d} \sum_{m+n=c} \binom{c}{m} a_{i,j} x^{m+i} y^{n+j} \\
&= \sum_{u+v=c+d} \left(\sum_{m+i=u} a_{i,j} \binom{c}{m} \right) x^u y^v .
\end{aligned}
$$

The last expression is in the ideal $\langle x^a, y^b \rangle$ if and only if the nonzero coefficients only occur when $u \geq a$ or $v \geq b$. Equivalently, if $u < a$ and $v < b$, the coefficients of $x^u y^v$ must be zero. Therefore, we have the following system of equations in the coefficients $a_{i,d-i}$, $i = 0, 1, \cdots, d$.

$$
\sum_{m+i=u} a_{i,d-i} \binom{c}{m} = 0, \text{ for } u = c+d-b+1, \cdots, a-1. \tag{2.5.2}
$$

Let $e = c + d - b$, then we can write these conditions in the following form

$$
\begin{bmatrix}
\binom{c}{e+1} & \binom{c}{e} & \cdots & \binom{c}{e-d+1} \\
\binom{c}{e+2} & \binom{c}{e+1} & \cdots & \binom{c}{e-d+2} \\
\cdots & \cdots & & \\
\binom{c}{a-1} & \binom{c}{a-2} & \cdots & \binom{c}{a-d-1}
\end{bmatrix}
\begin{bmatrix}
a_{0,d} \\ a_{1,d-1} \\ \vdots \\ a_{d,0}
\end{bmatrix} = 0. \tag{2.5.3}
$$

Later, we will show the above matrix of size $(a+b-c-d-1) \times (d+1)$ is of full rank. So if

$$
a + b - c - d - 1 - (d+1) \geq 0,
$$

equivalently, if $d < r = \frac{a+b-c}{2}$, there is no solution to the above equation (2.5.3). When $d = r$, there is a 2-dimensional solution and correspondingly two polynomials $f \in J_r$. This fact is consistent with the that J is generated by the two polynomials C and F of degree r. The matrix equation 2.5.3 just shows the condition on the coefficients of a polynomial of degree r to be in J.

Taking any solution to equation (2.5.3), we get a polynomial $f \in J_r$. So $f(x+y)^c = gx^a + hy^b$ for some polynomials g and h. Thus we have obtained a syzygy of the ideal I as

$$(g, h, -f)^t.$$

Since there is a 2-dimensional solution to equation (2.5.3), we have generated two linearly independent syzygies of the ideal I, which is the matrix ϕ_0 we are looking for. In the following section, we will give examples to demonstrate this idea more clearly.

Now we start our analysis of Schenck's conjecture (2.2.8).

2.6 Construction of syzygies

Consider the following six subideals J_i of J_ϕ, $i = 0, 1, 2, 3, 4, 5$,

$$\begin{aligned}
J_0 &= \langle x^{l+2k}, y^{l+2k}, (x+y)^{2l+2k} \rangle, \\
J_1 &= \langle x^{l+2k}, z^{l+2k}, (x+z)^{2l+2k} \rangle, \\
J_2 &= \langle y^{l+2k}, z^{l+2k}, (y+z)^{2l+2k} \rangle, \\
J_3 &= \langle x^{l+2k}, (y+z)^{2l+2k}, (x+y+z)^{3l} \rangle, \\
J_4 &= \langle y^{l+2k}, (x+z)^{2l+2k}, (x+y+z)^{3l} \rangle, \\
J_5 &= \langle z^{l+2k}, (x+y)^{2l+2k}, (x+y+z)^{3l} \rangle.
\end{aligned}$$

In the polynomial ring $R = k[x, y]$, the ideal

$$J_0 = \langle x^{l+2k}, y^{l+2k}, (x+y)^{2l+2k} \rangle$$

is codimension two and has the following minimal free resolution by the Hilbert-Burch Theorem [EI95],

$$0 \longrightarrow R(-2l-3k)^2 \xrightarrow{\phi_0} R(-l-2k)^2 \oplus R(-2l-2k) \to R \to R/J_0 \to 0,$$

where

$$\phi_0 = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}.$$

Here a_1, a_2, b_1, b_2 are polynomials of degree $l+k$, c_1, c_2 are polynomials of degree k. The entries of ϕ_0 satisfy the following equations:

$$\begin{aligned} b_1c_2 - b_2c_1 &= \alpha_0 x^{l+2k}, \\ a_1c_2 - a_2c_1 &= -\alpha_0 y^{l+2k}, \\ a_1b_2 - a_2b_1 &= \alpha_0 (x+y)^{2l+2k}, \end{aligned} \tag{2.6.1}$$

for some nonzero constant α_0. Moreover, we have

$$\langle c_1, c_2 \rangle = \langle x^{l+2k}, y^{l+2k} \rangle : (x+y)^{2l+2k}. \tag{2.6.2}$$

The two syzygies of the ideal J_0, given by $(a_1, b_1, c_1)^t$ and $(a_2, b_2, c_2)^t$, are of degree $2l+3k$. They can be naturally extended to syzygies of the ideal J_ϕ as follows,

$$\begin{aligned} s_1 &= (a_1, b_1, 0, c_1, 0, 0, 0)^t, \\ s_2 &= (a_2, b_2, 0, c_2, 0, 0, 0)^t. \end{aligned}$$

Inspired by the Therefore we have obtained two first syzygies of degree $2l+3k$ of the ideal J_ϕ.

Here, we apply the results of the previous section (2.5) to explicitly construct the matrix ϕ_0. In this case, the matrix (2.5.3) condition for

$$f = \sum_{i+j=k} a_{i,j} x^i y^j \in \langle c_1, c_2 \rangle$$

is

$$\begin{bmatrix} \binom{2l+2k}{l+k+1} & \binom{2l+2k}{l+k} & \cdots & \binom{2l+2k}{l+1} \\ \binom{2l+2k}{l+k+2} & \binom{2l+2k}{l+k+1} & \cdots & \binom{2l+2k}{l+2} \\ \cdots & \cdots & & \\ \binom{2l+2k}{l+2k-1} & \binom{2l+2k}{l+2k-2} & \cdots & \binom{2l+2k}{l+k-1} \end{bmatrix} \begin{bmatrix} a_{0,k} \\ a_{1,k-1} \\ \vdots \\ a_{k,0} \end{bmatrix} = 0. \tag{2.6.3}$$

There is a 2-dimensional solution to this equation. Given a solution $v = (a_{0,k}, \cdots, a_{k,0})$, we have a syzygy

$(g, h, -f)^t$, where

$$g = \sum_{u=0}^{l+k} \left(\sum_{m+i=u+l+2k} a_{i,j} \binom{2l+2k}{m} \right) x^u y^{l+k-u}$$
$$h = \sum_{u=0}^{l+k} \left(\sum_{m+i=u} a_{i,j} \binom{2l+2k}{m} \right) x^u y^{l+k-u}.$$

Taking one solution v_1 with $a_{k,0} = 0$, then we have y divides f. Moreover, the coefficient of x^{l+k} in g is

$$\sum_{m+i=2l+3k} a_{i,k-i} \binom{2l+2k}{m} = a_{k,0} \binom{2l+2k}{2l+2k} = 0.$$

Therefore y divides both f and g in this syzygy.

Taking another solution v_2 to the above equation which satisfies

$$\sum_{m+i=l+k} a_{i,k-i} \binom{2l+2k}{m} = 0.$$

Equivalently, v_2 is (up to scaling) the unique solution to the matrix obtained by adding one row

$$\left[\binom{2l+2k}{l+k}, \binom{2l+2k}{l+k-1}, \dots, \binom{2l+2k}{l} \right]$$

on top of the matrix (2.6.3) above. Corresponding to this solution v_2, we have another syzygy

$$(G, H, -F)^t,$$

where y divides H.

Lemma 2.6.1. *The Hilbert-Burch matrix*

$$\phi_0 = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

in the resolution of the ideal $\langle x^{l+2k}, y^{l+2k}, (x+y)^{2l+2k} \rangle$ has the property that y divides a_1, c_1, b_2.

The other two ideals J_1, J_2 have completely similar minimal free resolutions, with the matrix of the first differential given by

$$\phi_1 = \begin{bmatrix} d_1 & d_2 \\ e_1 & e_2 \\ f_1 & f_2 \end{bmatrix}, \ \phi_2 = \begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \\ k_1 & k_2 \end{bmatrix}.$$

The entries of ϕ_1, ϕ_2 satisfy the following equations:

$$\begin{aligned} e_1 f_2 - e_2 f_1 &= \alpha_1 x^{l+2k}, & h_1 k_2 - h_2 k_1 &= \alpha_2 y^{l+2k}, \\ d_1 f_2 - d_2 f_1 &= -\alpha_1 z^{l+2k}, & g_1 k_2 - g_2 k_1 &= -\alpha_2 z^{l+2k}, \\ d_1 e_2 - d_2 e_1 &= \alpha_1 (x+z)^{2l+2k}, & g_1 h_2 - g_2 h_1 &= \alpha_2 (y+z)^{2l+2k}, \end{aligned} \tag{2.6.4}$$

for some nonzero constant α_1, α_2.

We also have

$$\langle d_1, d_2 \rangle = \langle z^{l+2k}, (x+z)^{2l+2k} \rangle : x^{l+2k}. \tag{2.6.5}$$

The two syzygies of J_1 and those of J_2 can also be extended to syzygies of the ideal J_ϕ, given by

$$s_3 = (d_1, 0, e_1, 0, f_1, 0, 0)^t,$$

$$s_4 = (d_2, 0, e_2, 0, f_2, 0, 0)^t,$$

$$s_5 = (0, g_1, h_1, 0, 0, k_1, 0)^t,$$

$$s_6 = (0, g_2, h_2, 0, 0, k_2, 0)^t.$$

Therefore, we have constructed six first syzygies of degree $2l + 3k$ from the ideals J_0, J_1, J_2. In §2.7, we show these syzygies are independent.

To construct six first syzygies of degree $3l + 2k$, we consider the subideals J_3, J_4, and J_5. For example, the ideal

$$J_3 = \langle x^{l+2k}, (y+z)^{2l+2k}, (x+y+z)^{3l} \rangle$$

is essentially an ideal in two variables $x, y+z$ and has a Hilbert-Burch resolution,

$$0 \longrightarrow R(-3l-2k)^2 \xrightarrow{\phi_3} R(-l-2k) \oplus R(-2l-2k) \oplus R(-3l) \to R \to R/J_3 \to 0,$$

where the matrix of differential is given by

$$\phi_3 = \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \\ C_1 & C_2 \end{bmatrix}.$$

Here A_1, A_2 are polynomials in $x, y+z$ of degree $2l$, B_1, B_2 are of degree l, C_1, C_2 are of degree $2k$. Similarly, the entries of ϕ_3 satisfy the equations:

$$\begin{aligned} B_1C_2 - B_2C_1 &= \beta_0 x^{l+2k}, \\ A_1C_2 - A_2C_1 &= -\beta_0 (y+z)^{2l+2k}, \\ A_1B_2 - A_2B_1 &= \beta_0 (x+y+z)^{3l}, \end{aligned} \tag{2.6.6}$$

for some nonzero constant β_0. Moreover, we have

$$\langle A_1, A_2 \rangle = \langle (y+z)^{2l+2k}, (x+y+z)^{3l} \rangle : x^{l+2k}. \tag{2.6.7}$$

The two syzygies of the ideal J_3, given by $(A_1, B_1, C_1)^t$ and $(A_2, B_2, C_2)^t$ are of degree $3l+2k$. They can also be extended to syzygies of the ideal J_ϕ as follows

$$\begin{aligned} s_7 &= (A_1, 0, 0, 0, 0, B_1, C_1)^t, \\ s_8 &= (A_2, 0, 0, 0, 0, B_2, C_2)^t. \end{aligned}$$

The ideals J_4, J_5 have completely similar minimal free resolutions with their matrices of first differentials given by

$$\phi_4 = \begin{bmatrix} D_1 & D_2 \\ E_1 & E_2 \\ F_1 & F_2 \end{bmatrix}, \ \phi_5 = \begin{bmatrix} G_1 & G_2 \\ H_1 & H_2 \\ K_1 & K_2 \end{bmatrix}.$$

Here $D_1, D_2, E_1, E_2, F_1, F_2$ are polynomials in $y, x+z$ and $G_1, G_2, H_1, H_2, K_1, K_2$ are polynomials in $z, x+y$. They satisfy equations similar to Equations (2.6.6). For latter use in §2.8, we have

$$\langle H_1, H_2 \rangle = \langle z^{l+2k}, (x+y)^{2l+2k} \rangle : (x+y+z)^{3l}. \tag{2.6.8}$$

The two syzygies of J_4 and those of J_5 are of degree $3l+2k$, too. They can be extended to syzygies of the ideal J_ϕ, given by

$$s_9 = (0, D_1, 0, 0, E_1, 0, F_1)^t,$$
$$s_{10} = (0, D_2, 0, 0, E_2, 0, F_2)^t,$$
$$s_{11} = (0, 0, G_1, H_1, 0, 0, K_1)^t,$$
$$s_{12} = (0, 0, G_2, H_2, 0, 0, K_2)^t.$$

Therefore, we have constructed six first syzygies of degree $3l+2k$.

Remark 2.6.2. As we have done for the ideal J_0, we can apply the same method to construct the Hilbert-Burch matrix ϕ_5 for the ideal J_5. In this case, we have z divides H_1, K_1, and G_2. We omit the tedious details.

2.7 Constructed Syzygies are Minimal

Now we show the syzygies of degree $2l+3k$ and $3l+2k$ constructed above are minimal generators of the first syzygies of the ideal J_ϕ; and there are no other first syzygies of degree at most $\max(2l+3k, 3l+2k)$. For that purpose, we make use of the structure of the Betti diagram and the Hilbert series of J_ϕ. We divide our analysis into three cases, depending on l, k.

Case 1: $l = k$. This case is trivial, since $2l+3k = 3l+2k$, the constructed syzygies are of the same degree.

Case 2: $k < l$. So $2l+3k < 3l+2k$. In this case, the six first syzygies of degree $2l+3k$ must be minimal and it is impossible to have first syzygies of degree less than $2l+3k$, since there are no second syzygies of the same degree to cancel those first syzygies.

Now we show it is also impossible to have first syzygies of degree s, $s = 2l+3k+1, \cdots, 3l+2k-1$. Starting with $s = 2l+3k+1$, suppose there are k_s first syzygies of degree s, there must be k_s second syzygies of degree s of J_ϕ, since there is no term t^s in the numerator of the Hilbert series of R/J_ϕ. Those potential second syzygies of degree s must be the syzygies of the six syzygies of degree $2l+3k$. However, there is no such syzygy of degree $s < 3l+2k$ by the following lemma.

Lemma 2.7.1. *The degree of the syzygies of the six syzygies of degree $2l+3k$ is at least $3l+6k$.*

Proof. The six syzygies of degree $2l+3k$ are the columns of the following matrix

$$A = \begin{bmatrix} a_1 & a_2 & d_1 & d_2 & 0 & 0 \\ b_1 & b_2 & 0 & 0 & g_1 & g_2 \\ 0 & 0 & e_1 & e_2 & h_1 & h_2 \\ c_1 & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_1 & f_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_1 & k_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The syzygies of these six syzygies are just the column vectors $v = (v_1, v_2, v_3, v_4, v_5, v_6)^t$, where each component v_i is a homogeneous polynomial in x, y, z, such that

$$Av = 0.$$

Writing explicitly, we have

$$\begin{cases} a_1v_1 + a_2v_2 + d_1v_3 + d_2v_4 = 0 \\ b_1v_1 + b_2v_2 + g_1v_5 + g_2v_6 = 0 \\ e_1v_3 + e_2v_4 + h_1v_5 + h_2v_6 = 0 \\ c_1v_1 + c_2v_2 = 0 \\ f_1v_3 + f_2v_4 = 0 \\ k_1v_5 + k_2v_6 = 0 \end{cases} \tag{2.7.1}$$

Since c_1, c_2 are co-prime from Equation (2.6.1), the fourth equation implies that

$$(v_1, v_2) = p_1(-c_2, c_1),$$

for some polynomial p_1. Similarly, we have

$$(v_3, v_4) = p_2(-f_2, f_1),$$

$$(v_5, v_6) = p_3(-k_2, k_1),$$

for some polynomial p_2, p_3 from the fifth and the last equation, respectively.

Substitute the $v_1, v_2, v_3, v_4, v_5, v_6$ into the first three equations, we get

$$\begin{cases} p_1(-a_1c_2 + a_2c_1) + p_2(-d_1f_2 + d_2f_1) = 0 \\ p_1(-b_1c_2 + b_2c_1) + p_3(-g_1k_2 + g_2k_1) = 0 \\ p_2(-e_1f_2 + e_2f_1) + p_3(-h_1k_2 + h_2k_1) = 0 \end{cases} \tag{2.7.2}$$

By equations (2.6.1), (2.6.4), and (2.6.6), the above three equations are

$$\begin{cases} p_1(\alpha_0 y^{l+2k}) + p_2(\alpha_1 z^{l+2k}) = 0 \\ p_1(-\alpha_0 x^{l+2k}) + p_3(\alpha_2 z^{l+2k}) = 0 \\ p_2(-\alpha_1 x^{l+2k}) + p_3(-\alpha_2 y^{l+2k}) = 0 \end{cases} \tag{2.7.3}$$

The only solution to these equations is

$$p_1 = c\alpha_1\alpha_2 z^{l+2k}, p_2 = -c\alpha_0\alpha_2 y^{l+2k}, \text{ and } p_3 = c\alpha_0\alpha_1 x^{l+2k}, \tag{2.7.4}$$

for some nonzero polynomial c, possibly constant.

Therefore, the only nonzero syzygies of the six syzygies of degree $2l + 3k$ are

$$v = (-c_2p_1, c_1p_1, -f_2p_2, f_1p_2, -k_2p_3, k_1p_3)^t,$$

with p_1, p_2, p_3 given in Equation (2.7.4). Since

$$\deg p_1 = \deg p_2 = \deg p_3 \geq l + 2k, \tag{2.7.5}$$

$$\deg c_i = \deg f_i = \deg k_i = k, \text{ for } i = 1, 2. \tag{2.7.6}$$

Each component of v is of degree at least $l + 3k$. Since the six syzygies are of degree $2l + 3k$, the degree of the syzygies of the six syzygies is at least $3l + 6k$. □

Therefore there are no first syzygies of degree s where $2l + 3k < s < 3l + 2k$. Again, by the Hilbert series, the six syzygies of degree $3l + 2k$ must be minimal.

Case 3:$k > l$. The analysis is similar to the case $k < l$. In this case, $3l + 2k < 2l + 3k$. There are no first syzygies of degree less than $3l+2k$ and the six first syzygies of degree $3l+2k$ are minimal.

There are also no first syzygies of degree s such that $3l+2k < s < 2l+3k$. If there were, then there would be second syzygies of the syzygies of degree $3l+2k$ which are the columns of the matrix

$$B = \begin{bmatrix} A_1 & A_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_1 & D_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & G_1 & G_2 \\ 0 & 0 & 0 & 0 & H_1 & H_2 \\ 0 & 0 & E_1 & E_2 & 0 & 0 \\ B_1 & B_2 & 0 & 0 & 0 & 0 \\ C_1 & C_2 & F_1 & F_2 & K_1 & K_2 \end{bmatrix}.$$

The syzygies of the six syzygies are the vectors $w = (w_1, w_2, w_3, w_4, w_5, w_6)^t$ such that $Bw = 0$.

Lemma 2.7.2. *The only solution to the equation $Bw = 0$ is $w = 0$.*

Proof. Writting the equation $Bw = 0$ explicitly, we have

$$\begin{cases} A_1w_1 + A_2w_2 = 0 \\ D_1w_3 + D_2w_4 = 0 \\ G_1w_5 + G_2w_6 = 0 \\ H_1w_5 + H_2w_6 = 0 \\ E_1w_3 + E_2w_4 = 0 \\ B_1w_1 + B_2w_2 = 0 \\ C_1w_1 + C_2w_2 + F_1w_3 + F_2w_4 + K_1w_5 + K_2w_6 = 0 \end{cases} \tag{2.7.7}$$

The first and the sixth equation together imply that $w_1 = w_2 = 0$, since

$$\det \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix} = \beta_0(x+y+z)^{3l},$$

by Equation (2.6.6). Similarly, the second and the fifth equation imply that $w_3 = w_4 = 0$, since

$$\det \begin{bmatrix} D_1 & D_2 \\ E_1 & E_2 \end{bmatrix} = \beta_1 (x+y+z)^{3l}.$$

We also have $w_5 = w_6 = 0$ from the third and the fourth equation. □

The above proof shows that the constructed syzygies of degree $2l+3k$ and $3l+2k$ are minimal.

2.8 No higher degree first syzygies

In this section, we aim to show the syzygies constructed above generate all the first syzygies of the ideal J_ϕ, by proving that there are no generators of first syzygies of degree bigger than $\max(2l+3k, 3l+2k)$. The argument is similar to showing that there are no other first syzygies of degrees at most $\max(2l+3k, 3l+2k)$. Because of the Hilbert series, we show that there are no second syzygies of J_ϕ of degree bigger than $\max(2l+3k, 3l+2k)$, except those of degree $3l+3k$.

Since the ideal J_ϕ is Artinian, its regularity is equal to the maximum degree d such that $(R/J_\phi)_d \neq 0$, which is equal to the highest exponent in the Hilbert series of J_ϕ. We see that the regularity of R/J_ϕ is $3l+3k-3$, or equivalently, the regularity of J_ϕ is $3l+3k-2$. Since the regularity is obtained at the last step of the minimal free resolution, the maximum degree of the second syzygies of J_ϕ is $3l+3k$. Our goal is to show that there are no second syzygies of degree strictly smaller than $3l+3k$. For that purpose, we consider the syzygies of the six syzygies of degree $2l+3k$ and the six syzygies of degree $3l+2k$.

We define the matrix

$$\Phi = A|B = \begin{bmatrix} a_1 & a_2 & d_1 & d_2 & 0 & 0 & A_1 & A_2 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & 0 & 0 & g_1 & g_2 & 0 & 0 & D_1 & D_2 & 0 & 0 \\ 0 & 0 & e_1 & e_2 & h_1 & h_2 & 0 & 0 & 0 & 0 & G_1 & G_2 \\ c_1 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_1 & H_2 \\ 0 & 0 & f_1 & f_2 & 0 & 0 & 0 & 0 & E_1 & E_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_1 & k_2 & B_1 & B_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_1 & C_2 & F_1 & F_2 & K_1 & K_2 \end{bmatrix}.$$

A syzygy of the six syzygies of degree $2l+3k$ and six syzygies of degree $3l+2k$ is a vector

$$U = (v_1, v_2, \cdots, v_6, w_1, \cdots, w_6)^t,$$

where each component v_i, w_i is a homogeneous polynomial of x, y, z such that

$$\Phi U = 0.$$

Expanding $\Phi U = 0$, one of the equations we get is

$$c_1 v_1 + c_2 v_2 + H_1 w_5 + H_2 w_6 = 0,$$

which is equivalent to

$$c_1 v_1 + c_2 v_2 = -H_1 w_5 - H_2 w_6 = f,$$

for some element $f \in R$. Therefore, we must have

$$f \in I := \langle c_1, c_2 \rangle \cap \langle H_1, H_2 \rangle.$$

Just prior to equation (2.3), we have shown that

$$\deg c_1 = \deg c_2 = k,$$
$$\deg H_1 = \deg H_2 = l.$$

If $f \neq 0$, the degree of the syzygy corresponding to U is

$$2l + 2k + \deg f.$$

In the following, we show that $HF(I,d) = 0$ for $d < l+k-1$. Therefore, there is no syzygy of degree smaller than $2l+3k-1$.

Theorem 2.8.1. *For $d < l+k-1$, $HF(I,d) = 0$.*

As a consequence of this theorem, we get the best partial result on Schenck's conjecture as follows:

Theorem 2.8.2. *The syzygies constructed in §2.7 generate all the first syzygies, except possibly of degree* $3l + 3k - 1$.

2.9 The proof

This section is devoted to the proof of Theorem 2.8.1 in the last section. Recall that, we have

$$\begin{aligned}\langle c_1, c_2\rangle &= \langle x^{l+2k}, y^{l+2k}\rangle : (x+y)^{2l+2k},\\ \langle H_1, H_2\rangle &= \langle z^{l+2k}, (x+y)^{2l+2k}\rangle : (x+y+z)^{3l},\\ I &= \langle c_1, c_2\rangle \cap \langle H_1, H_2\rangle.\end{aligned}$$

To compute the Hilbert function $HF(I, d)$, we make an invertible change of variables

$$\begin{aligned}x &= z + y\\ y &= -y\\ z &= x.\end{aligned}$$

Then the ideal $\langle c_1, c_2\rangle$ becomes

$$L_1 = \langle y^{l+2k}, (y+z)^{l+2k}\rangle : z^{2l+2k},$$

and the ideal $\langle H_1, H_2\rangle$ becomes

$$L_2 = \langle x^{l+2k}, (x+z)^{3l}\rangle : z^{2l+2k}.$$

Therefore,

$$HF(I, d) = HF(L_1 \cap L_2, d).$$

Definition 2.9.1. *Let* $I \in k[x_0, \cdots, x_n]$ *be a homogeneous ideal and let* $>$ *be a monomial order on* $k[x_0, \cdots, x_n]$. *Then the initial ideal of* I *with respect to* $>$, *denoted* $in_>(I)$, *is the monomial ideal generated by the leading terms of all elements* $f \in I$.

We also use $in(I)$ if there is no confusion about the monomial order $>$. An important fact is the following theorem of Macaulay, see [CLO05].

Lemma 2.9.2. *An ideal and its initial ideal has the same Hilbert function.*

The point of introducing the above change of variables is that we can compute the initial ideals of L_1 and L_2 easily. Once we have computed $in(L_1)$ and $in(L_2)$, we use the following simple observation

$$in(L_1 \cap L_2) \subseteq in(L_1) \cap in(L_2), \tag{2.9.1}$$

and thus

$$HF(L_1 \cap L_2, d) \leq HF(in(L_1) \cap in(L_2), d), \text{ for any } d \geq 0.$$

The conclusion follows since $HF(in(L_1) \cap in(L_2), d) = 0$ for $d < l + k - 1$.

Proposition 2.9.3. *In the polynomial ring $R = k[x, y, z]$, with respect to the standard lexicographic order $x > y > z$, the initial ideal of L_1 is*

$$K_1 = \langle x^k, x^{k-1}z, \cdots, x^{k-i}z^{2i-1}, \cdots, z^{2k-1} \rangle,$$

and that of L_2 is

$$K_2 = \langle y^l, y^{l-1}z, \cdots, x^{l-i}z^{2i-1}, \cdots, z^{2l-1} \rangle.$$

Proof. These are the special cases of the following Lemma 2.9.4. The ideal L_1 is just the case where $p = q = l + 2k$, $r = 2l + 2k$ and the ideal L_2 is the case where $p = l + 2k$, $q = 3l$ and $r = 2l + 2k$. □

Notice that each of L_1 and L_2 is an ideal in two variables and there is no loss in considering ideals in two variables with general exponents. Therefore, in the polynomial ring $S = k[s, t]$, we consider the ideal

$$J = \langle s^p, (s + t)^q \rangle : t^r,$$

where p, q, r are positive integers. To avoid considering degenerate cases where the ideal $\langle s^p, t^r, (s + t)^q \rangle$ is generated by two polynomials, we assume that

$$p+q-r \geq 2$$
$$p+r-q \geq 2$$
$$q+r-p \geq 2$$

Lemma 2.9.4. *Given p, q, r positive integers satisfying the above conditions, set*

$$a = \lfloor \frac{p+q-r}{2} \rfloor, b = \lceil \frac{p+q-r}{2} \rceil.$$

With respect to the lexicographic order $s > t$ in $S = k[s,t]$, the initial ideal of

$$J = \langle s^p, (s+t)^q \rangle : t^r$$

is given by

$$H(a,b) = \langle s^a, s^{a-1}t^{b-a+1}, s^{a-2}t^{b-a+3}, \cdots, s^{a-i}t^{b-a+2i-1}, \cdots, t^{a+b-1} \rangle.$$

Proof. The main idea of the proof is to construct polynomials in J with the leading terms given in the lemma and then show they generate a monomial ideal with the same Hilbert function as J. Since J is minimally generated at degrees a and b, we assume the polynomials in J have degree at least a.

Suppose $P \in J_d$, where $d \geq a$, then there are homogeneous polynomials f and Q such that

$$Pt^r = f(s+t)^q + Qs^p,$$

where

$$\deg f = d + r - q, \deg Q = d + r - p.$$

Suppose

$$P = \sum_{k+l=d} p_{k,l} s^k t^l,$$
$$f = \sum_{i+j=d+r-q} a_{i,j} s^i t^j,$$

Then

$$\begin{aligned} f(s+t)^q &= \sum_{i+j=d+r-q} a_{i,j} s^i t^j \sum_{m+n=q} \binom{q}{m} s^m t^n \\ &= \sum_{i+j=d+r-q} \sum_{m+n=q} \binom{q}{m} a_{i,j} s^{m+i} t^{n+j} \\ &= \sum_{u+v=d+r} s^u t^v \left(\sum_{m+i=u} \binom{q}{m} a_{i,j} \right). \end{aligned}$$

Since

$$f(s+t)^q = Pt^r - Qs^p \in \langle s^p, t^r \rangle,$$

the last expression is in the ideal $\langle s^p, t^r \rangle$ if and only if the nonzero coefficients only occur when $u \geq p$ or $v \geq r$. Equivalently, if $u < p$ and $v < r$, the coefficients of $s^u t^v$ must be zero. Since

$$v = d + r - u < r \Leftrightarrow d < u.$$

If $d \geq p - 1$, then there is no u and v such that $u < p$ and $v < r$. In other words, there is no condition on the coefficients $a_{i,j}$. In this case,

$$P = \sum_{u+v=d} s^u t^v \left(\sum_{m+i=u} \binom{q}{m} a_{i,j} \right).$$

If $d < p - 1$, then for $d < u < p$,

$$\sum_{m+i=u} \binom{q}{m} a_{i,j} = 0.$$

We fit these together into the following matrix condition on the coefficients of f,

$$\begin{bmatrix} \binom{q}{d+1} & \binom{q}{d} & \cdots & \binom{q}{q-r+1} \\ \binom{q}{d+2} & \binom{q}{d+1} & \cdots & \binom{q}{q-r+2} \\ \cdots & \cdots & & \\ \binom{q}{p-1} & \binom{q}{p-2} & \cdots & \binom{q}{p+q-r-d-1} \end{bmatrix} \begin{bmatrix} a_{0,d+r-q} \\ a_{1,d+r-q-1} \\ \vdots \\ a_{d+r-q,0} \end{bmatrix} = 0. \tag{2.9.2}$$

Denote the matrix above as $M(p,q,r,d)$, whose entries are given by

$$M(p,q,r,d)_{i,j} = \binom{q}{d+1+i-j},$$

for $i = 0, 1, \cdots, p-d-2$, $j = 0, 1, \cdots, d+r-q$.

Moreover,

$$f(s+t)^q = \left(\sum_{u=0}^{d} + \sum_{u=p}^{d+r}\right)\left(\sum_{m+i=u}\binom{q}{m}a_{i,j}\right)s^u t^{d+r-u}.$$

The first summand above gives the polynomial P as

$$P = \sum_{u+v=d} s^u t^v \left(\sum_{m+i=u}\binom{q}{m}a_{i,j}\right),$$

thus we have

$$p_{k,d-k} = \sum_{m+i=k}\binom{q}{m}a_{i,d+r-q-i},\ k = 0, 1, \cdots, d$$

This is encoded by the following $(d+1)\times(d+r-q+1)$ exchange matrix, which we denote by $E(p,q,r,d)$.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ q & 1 & 0 & \cdots & 0 \\ \binom{q}{2} & q & 1 & \cdots & 0 \\ \cdots & & \cdots & & \\ \binom{q}{d} & \binom{q}{d-1} & & \cdots & \binom{q}{q-r-1} \end{bmatrix} \begin{bmatrix} a_{0,d+r-q} \\ a_{1,d+r-q-1} \\ \vdots \\ a_{d+r-q,0} \end{bmatrix} = \begin{bmatrix} p_{0,d} \\ p_{1,d-1} \\ \vdots \\ p_{d,0} \end{bmatrix}. \tag{2.9.3}$$

$E(p,q,r,d)$ has entries

$$E(p,q,r,d)_{i,j} = \binom{q}{i-j},$$

for $i = 0, 1, \cdots, d$ and $j = 0, 1, \cdots, d+r-q$. With this choice of indexing, row i of $E(p,q,r,d)$ corresponds to coefficient $p_{i,d-i}$. There is a similar matrix relating the coefficients of f and Q, but we will not need it. The following lemma is fundamental for understanding $M(p,q,r,d)$ and $E(p,q,r,d)$.

Lemma 2.9.5. *Let $t \geq h_0 \geq h_1 \cdots \geq h_w > 0$ be a sequence of integers, and let N be a square matrix with* *entries*

$$N_{i,j} = \binom{t}{h_j + i - j}$$

for $i = 0, 1, \cdots, w,$ *and* $j = 0, 1, \cdots, w.$ *Then* N *has nonzero determinant.*

Proof. This observation is made in §3.1 of [MT13], where it is noted that determinants of such matrices play a role in the representation theory of the special linear group $SL(V)$, where V is a t-dimensional vector space. In particular, if $\lambda = \mu'$, the conjugate partition to μ, then $\det N(\mu)$ is the dimension of the Weyl module $S_\lambda V$, which is a nontrivial irreducible representation of $GL(V)$. More explicitly, $\det N(\mu) = s_\lambda(1, 1, \cdots, 1)$, where $s_\lambda(x_1, x_2, \cdots, x_t)$ is the Schur polynomial in t variables of the partition $\lambda = \mu'$. In particular, $N(\mu)$ has a nonzero determinant. See [Eis95], §6.1 and Appendix A.1 for more details. ☐

Now we construct polynomials in the ideal J with specific leading terms by applying Lemma 2.9.5.

For $j = 1, 2, \cdots, a$, define the square matrix

$$N_j = \begin{bmatrix} \binom{q}{a-j} & \binom{q}{a-j-1} & \cdots & \binom{q}{2a-2j-p+1} \\ \binom{q}{a-j+1} & \binom{q}{a-j} & \cdots & \binom{q}{2a-2j-p+2} \\ \cdots & & \cdots & \\ \binom{q}{p-1} & \binom{q}{p-2} & \cdots & \binom{q}{a-j} \end{bmatrix}. \tag{2.9.4}$$

Applying Lemma 2.9.5 with $h_0 = \cdots = h_{b+j+r-q} = a-j$, we see that $\det N_j \neq 0$, so $\ker N_j = 0$. Moreover, taking the submatrix N_j' of N_j by removing its first row, then $\ker N_j'$ is 1-dimensional, say generated by

$$v = \begin{bmatrix} a_{0,b+j+r-q-1} \\ a_{1,b+j+r-q-2} \\ \vdots \\ a_{b+j+r-q-1,0} \end{bmatrix}.$$

Then $N_j \cdot v$ has all components equal to 0 except the first one, which is $p_{a-j,b-a+2j-1}$. Therefore we have found a polynomial P in the ideal J with leading term $s^{a-j}t^{b-a+2j-1}$, for each $j = 1, 2, \cdots, a$. Therefore, we have shown that $H(a, b) \subseteq in(J)$.

To establish $H(a, b) = in(J)$, it suffices to show that $HF(R/H(a, b), d) = HF(R/J, d)$, for all $d \geq 0$.

Since J is a complete intersection generated in degrees a and b, its minimal free resolution is given by

$$0 \to R(-a-b) \to R(-a) \oplus R(-b) \to R \to R/J \to 0.$$

Therefore

$$HF(R/J, d) = \binom{d+1}{1} - \binom{d+1-a}{1} - \binom{d+1-b}{1} + \binom{d+1-a-b}{1}.$$

More explicitly,

$$HF(R/J, d) = \begin{cases} d+1 & \text{if } 0 \le d < a \\ a+b-1-d & \text{if } a \le d < a+b-1 \\ 0 & \text{if } a+b-1 \le d \end{cases}$$

It is clear that

$$HF(R/H(a,b), d) = \begin{cases} d+1 & \text{if } 0 \le d < a \\ 0 & \text{if } a+b-1 \le d \end{cases}$$

In degree $d = b+i-1$, $i = 1, 2, \cdots, a$, the monomials $s^j t^{b+i-1-j}$ for $0 \le j < a-i$ are outside of $H(a,b)$, so

$$HF(R/H(a,b), b+i-1) = a-i = (a+b-1)-(b+i-1).$$

If $d = a$ and $a = b$, the monomials $s^j t^{a-j}$ for $0 \le j \le a-2$ are outside of $H(a,b)$, so

$$HF(R/H(a,b), a) = a-1 = b-1.$$

If $d = a$ and $b = a+1$, the monomials $s^j t^{a-j}$ for $0 \le j \le a-1$ are outside of $H(a,b)$, so

$$HF(R/H(a,b), a) = a = b-1.$$

Hence $HF(R/H(a,b), d) = HF(R/J, d)$, for all $d \ge 0$ and we are done. □

Completion of the proof:

So we have computed the initial ideal K_1 of L_1 and K_2 of L_2. As outlined before Lemma 2.9.2, we just need to compute $K_1 \cap K_2$ and show $(K_1 \cap K_2)_d = 0$, for $d < l + k - 1$.

If $m = x^p y^q z^{d-p-q} \in (K_1 \cap K_2)_d$, where $d < l + k - 1$. We divide into two cases.

Case 1: $p < k$. Since $m \in K_1$, we have

$$d - p - q \geq 2(k - p) - 1 \Leftrightarrow d + p - q \geq 2k - 1.$$

Also since $m \in K_2$, we have

$$d - p - q \geq 2(l - q) - 1 \Leftrightarrow d - p + q \geq 2l - 1.$$

Adding these two inequalities, we get

$$d \geq l + k - 1,$$

contradiction to our assumption.

Case 2: $p \geq k$. Then we must have

$$0 \leq q \leq d - p \leq d - k < l - 1,$$

therefore, $d - p + q < 2l - 1$.

Also since $m \in K_2$, we have

$$d - p - q \geq 2(l - q) - 1 \Leftrightarrow d - p + q \geq 2l - 1,$$

A contradiction. So $(K_1 \cap K_2)_d = 0$, for $d < l + k - 1$. Thus

$$HF(I, d) = HF(L_1 \cap L_2, d) = 0, \text{ for } d < l + k - 1.$$

2.10 A different approach: reduction to two variables

In this section, we use a different approach to analyze the second syzygies of the 7×12 matrix M in three variables x, y, z. Recall that

$$\Phi = \begin{bmatrix} a_1 & a_2 & d_1 & d_2 & 0 & 0 & A_1 & A_2 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & 0 & 0 & g_1 & g_2 & 0 & 0 & D_1 & D_2 & 0 & 0 \\ 0 & 0 & e_1 & e_2 & h_1 & h_2 & 0 & 0 & 0 & 0 & G_1 & G_2 \\ c_1 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_1 & H_2 \\ 0 & 0 & f_1 & f_2 & 0 & 0 & 0 & 0 & E_1 & E_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_1 & k_2 & B_1 & B_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_1 & C_2 & F_1 & F_2 & K_1 & K_2 \end{bmatrix}.$$

A second syzygy is a vector

$$U = (v_1, v_2, \cdots, v_6, w_1, \cdots, w_6)^t,$$

where each component v_i, w_i is a homogeneous polynomial of x, y, z such that

$$\Phi U = 0.$$

The degree of this syzygy is

$$2l + 3k + \deg v_i = 3l + 2k + \deg w_i, \text{ for any } v_i \neq 0, w_i \neq 0.$$

Our goal is to show there is no nonzero second syzygy of degree smaller than $3l + 3k$.

Suppose $U = (v_1, v_2, \cdots, v_6, w_1, \cdots, w_6)^t$ is a syzygy of Φ. We can write

$$U = z^j (\tilde{v_1}, \tilde{v_2}, \cdots, \tilde{v_6}, \tilde{w_1}, \cdots \tilde{w_6})^t = z^j \tilde{U},$$

where z^j is the highest power of z dividing all the entries of U. The equation $MU = 0$ is equivalent to

$$M\tilde{U} = 0. \tag{2.10.1}$$

As syzygies, $\deg U = j + \deg \tilde{U}$. Moreover, the syzygy $\tilde{U}$ has at least one entry, say $\tilde{v_1}$, not divisible by z. Setting $z = 0$ in the Equation (2.10.1) and denote the resulting matrix and vector as M' and $\tilde{U}'$, we get

$$M'\tilde{U}' = 0.$$

Since $\tilde{U}' \neq 0$, $\deg(U') = \deg \tilde{U}'$, and

$$\deg U = j + \deg \tilde{U} = j + \deg \tilde{U}' \geq \deg \tilde{U}'.$$

In other words, the degree of a nonzero syzygy U of M must be bigger or equal to any nonzero syzygy of M'. Thus we can focus on the nonzero syzygies of the matrix M', which is a matrix in two variables x and y, thus easier to analyze. To write down explicitly the matrix M', we use our results of §2.6, where we describe the syzygies of the subideals, especially Lemma 2.6.1 shows the divisibility on the entries of the syzygies. Since the first two columns of the matrix M are obtained from syzygies of the ideal $J_1 = \langle x^{l+2k}, y^{l+2k}, (x+y)^{2l+3k}\rangle$, which does not involve the variable z. So setting $z = 0$ in M has no effects on these two columns. We get the following matrix M',

$$M' = \begin{bmatrix} a_1 & a_2 & 0 & x^{l+k} & 0 & 0 & A_1' & A_2' & 0 & 0 & 0 & 0 \\ b_1 & b_2 & 0 & 0 & 0 & y^{l+k} & 0 & 0 & D_1' & D_2' & 0 & 0 \\ 0 & 0 & x^{l+k} & 0 & y^{l+k} & 0 & 0 & 0 & 0 & 0 & (x+y)^{2l} & 0 \\ c_1 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (x+y)^{l} \\ 0 & 0 & 0 & x^k & 0 & 0 & 0 & 0 & E_1' & E_2' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y^k & B_1' & B_2' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_1' & C_2' & F_1' & F_2' & 0 & (x+y)^{2k} \end{bmatrix}.$$

If $U' = (v_1', v_2', \cdots, v_6', w_1', \cdots w_6')^t$ is a syzygy of M', i.e., $M'U' = 0$. Then we have the following equations obtained from the first, fourth and sixth row of M',

$$a_1v_1' + a_2v_2' + x^{l+k}v_4' + A_1'w_1' + A_2'w_2' = 0 \tag{2.10.2a}$$

$$c_1v_1' + c_2v_2' + (x+y)^l w_6' = 0 \tag{2.10.2b}$$

$$y^k v_6' + B_1'w_1' + B_2'w_2' = 0 \tag{2.10.2c}$$

Recall that we have the equations,

$$x^{l+2k}a_1 + b_1y^{l+2k} + c_1(x+y)^{2l+2k} = 0$$
$$x^{l+2k}a_2 + b_2y^{l+2k} + c_2(x+y)^{2l+2k} = 0$$

Therefore,

$$(c_1v_1' + c_2v_2')(x+y)^{2l+2k} = -(x^{l+2k}(a_1v_1' + a_2v_2') + y^{l+2k}(b_1v_1' + b_2v_2'))$$

Substituting into the above Equation (2.10.2b), we get

$$(x+y)^{3l+2k}v_6' = x^{l+2k}(a_1v_1' + a_2v_2') + y^{l+2k}(b_1v_1' + b_2v_2').$$

Therefore,

$$a_1v_1' + a_2v_2' \in \langle y^{l+2k}, (x+y)^{3l+2k}\rangle : x^{l+2k}.$$

We denote this ideal by K_1.

Similarly, we have

$$x^{l+2k}A_1 + B_1(y+z)^{2l+2k} + C_1(x+y+z)^{3l} = 0,$$
$$x^{l+2k}A_2 + B_2(y+z)^{2l+2k} + C_2(x+y+z)^{3l} = 0.$$

So setting $z = 0$, we have

$$x^{l+2k}A_1' + B_1'y^{2l+2k} + C_1'(x+y)^{3l} = 0$$
$$x^{l+2k}A_2' + B_2'y^{2l+2k} + C_2'(x+y)^{3l} = 0$$

Substituting into the above Equation (2.10.2c), we get

$$y^{2l+3k}v_6' = x^{l+2k}(A_1'w_1' + A_2'w_2') + (x+y)^{3l}(C_1'w_1' + C_2'w_2').$$

Therefore,

$$A_1'w_1' + A_2'w_2' \in \langle y^{2l+3k}, (x+y)^{3l}\rangle : x^{l+2k}.$$

We denote this ideal by K_2.

Back to the first equation (2.10.2c)

$$a_1 v_1' + a_2 v_2' + x^{l+k} v_4' + A_1' w_1' + A_2' w_2' = 0,$$

which is equivalent to

$$a_1 v_1' + a_2 v_2' + A_1' w_1' + A_2' w_2' = -x^{l+k} v_4'.$$

Notice that the left hand of this equation is an element in the ideal $K_1 + K_2$, while the right hand is an element in the ideal generated by x^{l+k}. So this equation means that there is a nonzero element in the ideal $(K_1 + K_2) \cap \langle x^{l+k} \rangle$, which we denote as K.

Our goal is to show that there is no nonzero element in K of degree $2l + k - 1$.

Example 2.10.1. $[l = k = 2]$ In this case, the ideal $\langle x^6, y^6, (x+y)^{10} \rangle$ has the following Hilbert-Burch matrix

$$M_1 = \begin{bmatrix} x^4y + 10x^3y^2 + 45x^2y^3 + 120xy^4 + 210y^5 & 6x^5 + 55x^4y + 220x^3y^2 + 495x^2y^3 + 660xy^4 + 462y^5 \\ 252x^5 + 210x^4y + 120x^3y^2 + 45x^2y^3 + 10xy^4 + y^5 & -330x^4y - 330x^3y^2 - 165x^2y^3 - 44xy^4 - 5y^5 \\ -y & -6x + 5y \end{bmatrix}$$

which can be constructed as in §2.5. So the ideal K_1 is generated by the top row entries of M_1, denoted as f_1, f_2. In the same way, the ideal $\langle x^6, y^{10}, (x+y)^6 \rangle$ has the Hilbert-Burch matrix

$$M_2 = \begin{bmatrix} 14x^4y + 70x^3y^2 + 135x^2y^3 + 120xy^4 + 42y^5 & 126x^5 + 756x^4y + 1855x^3y^2 + 2340x^2y^3 + 1530xy^4 + 420y^5 \\ 2x + y & 9x + 4y \\ -14x^4y + 14x^3y^2 - 9x^2y^3 + 4xy^4 - y^5 & -126x^5 + 35x^3y^2 - 30x^2y^3 + 15xy^4 - 4y^5 \end{bmatrix},$$

then the ideal K_2 is generated by the top row entries of M_2, denoted as f_3, f_4. So $K_1 + K_2$ is generated by f_1, f_2, f_3, f_4, each of degree 5. For this example, our goal is to show there is no nonzero element in $K_1 + K_2$ of degree 5, which is divisible by x^4. A general element in $K_1 + K_2$ of degree 5 is of the form

$$f = c_1 f_1 + c_2 f_2 + c_3 f_3 + c_4 f_4,$$

where c_i's are scales. This polynomial f is divisible by x^4 if and only if all the coefficients of $x^3y^2, x^2y^3, xy^4, y^5$

in f are zero. So in the end, we get the system of equations in c_1, c_2, c_3, c_4

$$\begin{bmatrix} 1855 & 70 & 220 & 10 \\ 2340 & 135 & 495 & 45 \\ 1533 & 120 & 660 & 120 \\ 420 & 42 & 462 & 210 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0$$

The matrix above has a nonzero determinant and thus the only solution is

$$c_1 = c_2 = c_3 = c_4 = 0$$

Therefore, there is no nonzero element in $K_1 + K_2$ of degree 5 divisible by x^4. By our argument above, we have proved Schenck's conjecture 2.2.8 in this case. For any given l and k, the same argument leads to show an $(l+k) \times (l+k)$ matrix has a nonzero determinant. This matrix is constructed from the entries of the top row of the Hilbert-Burch matrix of the ideals $\langle x^{l+2k}, y^{l+2k}, (x+y)^{3l+2k} \rangle$ and $\langle x^{l+2k}, y^{2l+3k}, (x+y)^{3l} \rangle$. However, we have not been able to evaluate the determinant obtained this way in general.

The above example is interesting because its determinant and the determinant of all its square submatrices are positive; hence it is an example of a totally positive matrix. See [FZ00] for a survey.

2.11 Proof of Conjecture 2.2.8 under an extra hypothesis

Recall that we have the two ideals

$$I_\phi = \langle x^{l+2k}, y^{l+2k}, z^{l+2k}, (xy)^{l+k}, (xz)^{l+k}, (yz)^{l+k}, (xyz)^{l} \rangle, \tag{2.11.1}$$

$$J_\phi = \langle x^{l+2k}, y^{l+2k}, z^{l+2k}, (x+y)^{2l+2k}, (x+z)^{2l+2k}, (y+z)^{2l+2k}, (x+y+z)^{3l} \rangle. \tag{2.11.2}$$

In construction of the syzygies of J_ϕ, we have used its several subideals in (essentially) two variables.

$$J_0 = \langle x^{l+2k}, y^{l+2k}, (x+y)^{2l+2k}\rangle, \qquad I_0 = \langle x^{l+2k}, y^{l+2k}, (xy)^{l+k}\rangle, \tag{2.11.3}$$

$$J_1 = \langle x^{l+2k}, z^{l+2k}, (x+z)^{2l+2k}\rangle, \qquad I_1 = \langle x^{l+2k}, z^{l+2k}, (xz)^{l+k}\rangle, \tag{2.11.4}$$

$$J_2 = \langle y^{l+2k}, z^{l+2k}, (y+z)^{2l+2k}\rangle, \qquad I_2 = \langle y^{l+2k}, z^{l+2k}, (yz)^{l+k}\rangle, \tag{2.11.5}$$

$$J_3 = \langle x^{l+2k}, (y+z)^{2l+2k}, (x+y+z)^{3l}\rangle, \qquad I_3 = \langle x^{l+2k}, (yz)^{l+k}, (xyz)^{l}\rangle, \tag{2.11.6}$$

$$J_4 = \langle y^{l+2k}, (x+z)^{2l+2k}, (x+y+z)^{3l}\rangle, \qquad I_4 = \langle y^{l+2k}, (xz)^{l+k}, (xyz)^{l}\rangle, \tag{2.11.7}$$

$$J_5 = \langle z^{l+2k}, (x+y)^{2l+2k}, (x+y+z)^{3l}\rangle, \qquad I_5 = \langle z^{l+2k}, (xy)^{l+k}, (xyz)^{l}\rangle. \tag{2.11.8}$$

Here we index these subideals in a way easy to generalize to similar ideals in more variables.

Because of the structure of the generators of the ideals J_ϕ and I_ϕ, it is convenient to view each pair of sub-ideals $\{J_i, I_i\}$ as constructed from a partition $\Pi_i = \{N_0, N_1, N_2\}$ of $\{0,1,2,3\}$ with $0 \in N_0$. For each nonempty subset $S \subseteq \{N_1, N_2\}$, there is a generator p_S and m_S of J_i and I_i respectively. For example, $\{J_0, I_0\}$ is constructed from the partition $\{03, 1, 2\}$; $\{J_1, I_1\}$ from the partition $\{02, 1, 3\}$ and $\{J_2, I_2\}$ from the partition $\{01, 2, 3\}$.

The other three pairs of ideals $\{J_3, I_3\}$, $\{J_4, I_4\}$ and $\{J_5, I_5\}$ are constructed from the partition $\{0, 1, 23\}$, $\{0, 2, 13\}$ and $\{0, 3, 12\}$, respectively. The ideals J_ϕ and I_ϕ can be considered as constructed from the partition of $\{0,1,2,3\}$ into four nonempty subsets, i.e., $\{0,1,2,3\}$ and S runs through all nonempty subsets of $\{1,2,3\}$. If we consider the partitions of $\{0,1,2,3\}$ into two nonempty subsets $\{N_0, N_1\}$ with $0 \in N_0$, we just get the corresponding principal sub-ideal $\langle p_{N_1}\rangle$, $\langle m_{N_1}\rangle$ of J_ϕ and I_ϕ. For example, for $\{0, 123\}$, we get $\langle (x+y+z)^{3l}\rangle$ and $\langle (xyz)^l\rangle$ respectively.

As we have seen before, the following 7×12 matrix encodes all the syzygies of J_ϕ we constructed from its subideals.

$$\Phi = \begin{bmatrix} a_1 & a_2 & d_1 & d_2 & 0 & 0 & l_1 & l_2 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & 0 & 0 & g_1 & g_2 & 0 & 0 & p_1 & p_2 & 0 & 0 \\ 0 & 0 & e_1 & e_2 & h_1 & h_2 & 0 & 0 & 0 & 0 & s_1 & s_2 \\ c_1 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 & t_2 \\ 0 & 0 & f_1 & f_2 & 0 & 0 & 0 & 0 & q_1 & q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_1 & k_2 & m_1 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_1 & n_2 & r_1 & r_2 & u_1 & u_2 \end{bmatrix}.$$

Since

$$m_x = \det \begin{bmatrix} b_1 & b_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_2 & 0 & 0 \\ c_1 & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_1 & f_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_1 & m_2 \\ 0 & 0 & 0 & 0 & n_1 & n_2 \end{bmatrix} = \alpha_0\alpha_1\beta_0 x^{3l+6k},$$

$$m_y = \det \begin{bmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_1 & h_2 & 0 & 0 \\ c_1 & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_1 & q_2 \\ 0 & 0 & k_1 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_1 & r_2 \end{bmatrix} = -\alpha_0\alpha_2\beta_1 y^{3l+6k},$$

and

$$m_z = \det \begin{bmatrix} d_1 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_1 & g_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_1 & t_2 \\ f_1 & f_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_1 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_1 & u_2 \end{bmatrix} = -\alpha_1\alpha_2\beta_2 z^{3l+6k},$$

these three nonzero minors of Φ provide us a regular sequence of length three. Since the 7×12 matrix Φ is of rank six, its kernel is of rank six. In 2.12.5, we show that the length of a maximal regular sequence of the ideal of 6×6 minors is also three.

Remark 2.11.1. Here the 6×6 matrix to compute m_x is a block matrix after a permutation of the rows, consisting of three 2×2 blocks. Each block is taken from the second and third row of the matrix in the Hilbert-Burch resolution of the ideals J_0, J_1, J_3, i.e., all the sub-ideals of J_ϕ with a generator x^{l+2k}. To visualize this fact easily, notice that the partitions corresponding to J_0, J_1, J_3 are given by

$$\{03, 1, 2\}, \{02, 1, 3\}, \{0, 1, 23\}.$$

Each partition has a subset given by $\{1\}$, which is assigned the variable x.

Similarly, the 6×6 matrix to compute m_y consists of three 2×2 blocks. Each block is taken from the first and third row of the matrix in the resolution of the ideals J_0, J_2, J_4, which are all the sub-ideals of J_ϕ with a generator y^{l+2k}. These subideals correspond to the partitions

$$\{03, 1, 2\}, \{01, 2, 3\}, \{0, 2, 13\},$$

Each of which has a subset given by $\{2\}$, which is assigned the variable y. Similar patterns apply to m_z. It is clear that the argument may be generalized to ideals in more variables and applied in the proof of Conjecture 2.2.7 in §4.

Recall the Buchsbaum-Eisenbud Criterion for exactness:

Theorem 2.11.2. *[BE73] A complex of free modules*

$$F_* : 0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0$$

over Noetherian ring R is exact if and only if

$$\mathrm{rank}(\phi_{i+1}) + \mathrm{rank}(\phi_i) = \mathrm{rank}(F_i), \text{ and } \mathrm{depth}(I(\phi_i)) \geq i \text{ for every } i,$$

where $I(\phi_i)$ is the ideal generated by the minors of order $\mathrm{rank}(\phi_i)$, *and* $\mathrm{depth}(I(\phi_i))$ *is the length of a maximal regular sequence of homogeneous elements of positive degree.*

Theorem 2.11.3. *Let Φ be the matrix constructed in §2.8. If the kernel of Φ is a free module, then the minimal free resolution of the ideal*

$$J_\phi = \langle x^{l+2k}, y^{l+2k}, z^{l+2k}, (x+y)^{2l+2k}, (x+z)^{2l+2k}, (y+z)^{2l+2k}, (x+y+z)^{3l} \rangle$$

is given by

$$0 \to R(-3l-3k)^6 \xrightarrow{d_3} \begin{array}{c} R(-2l-3k)^6 \\ \oplus \\ R(-3l-2k)^6 \end{array} \xrightarrow{\Phi} \begin{array}{c} R(-l-2k)^3 \\ \oplus \\ R(-2l-2k)^3 \\ \oplus \\ R(-3l) \end{array} \xrightarrow{\Psi} R \to R/J_\phi \to 0.$$

Proof. Since $F_3 = \ker d_3$, the complex of free modules is exact at the last step. It is exact at the first step. So we only need to show it is exact in the middle. This statement is equivalent to depth$\Phi = 2$ and rank$\Phi = 6$, which we showed on the previous page.

Since the Hilbert series of R/J_ϕ is given by

$$\frac{1 - 3t^{l+2k} - 3t^{2l+2k} - t^{3l} + 6t^{2l+3k} + 6t^{3l+2k} - 6t^{3l+3k}}{(1-t)^3}.$$

We conclude that $F_3 \cong R(-3l-3k)^6$ and the theorem is proved. □

2.12 Complementary factorization of minors

In the complex in Theorem (2.11.3), the differential $d_2 : F_2 \to F_1$ is represented explicitly by the matrix Φ and F_3 is defined as the kernel of Φ. Now we explore the structure of the differential $d_3 : F_3 \to F_2$. We have only shown that d_3 is represented by a 12×6 matrix, say by Δ. In this section, we show how to choose three nonzero 6×6 minors, which form a regular sequence of length three, even though we don't know an explicit form of Δ. This is important, since it makes possible to apply the Buchsbaum-Eisenbud criterion to show exactness of the complexes we construct later for the minimal free resolution of the ideals J_ϕ in more variables. For that, we will use the following theorem of Buchsbaum-Eisenbud, Theorem 3.1 in [BE74]. Here we follow the presentation in [EN73]. First, some preliminaries:

Definition 2.12.1. *Given a matrix $C = (c_{ij})$ over a commutative ring T with a nonzero identity element. We say C factorizes completely if there exists $a_i, b_j \in T$, such that $c_{ij} = a_i b_j$, for all i, j. If C is a row matrix, i.e., a matrix with only one row, we take $a_i = 1$ and $b_j = c_{ij}$, and call this factorization its canonical complete factorization.*

Definition 2.12.2. *Suppose C is a matrix of $p \times q$, and $0 \leq \mu \leq \min\{p, q\}$. The minors $C^{\mu}_{J,K}$, where $J = \{j_1, j_2, \cdots, j_\mu\}$ such that $1 \leq j_1 < j_2 < \cdots < j_\mu \leq p$ and $K = \{k_1, k_2, \cdots, k_\mu\}$ such that $1 \leq k_1 < k_2 < \cdots < k_\mu \leq q$, produce an $\binom{p}{\mu} \times \binom{q}{\mu}$ matrix, denoted by $C^{(\mu)}$.*

Definition 2.12.3. *Given two matrices A and B of size $p \times q$ and $q \times t$, such that $AB = 0$. Let $\mu \geq 0$, $\nu \geq 0$ be integers such that $\mu + \nu = q$ and assume that each entry of the matrix $A^{(\mu)}$ and $B^{(\nu)}$ factorizes completely, say*

$$A^{(\mu)}_{J,K} = u_J v_K,$$
$$B^{(\nu)}_{M,N} = \omega_M z_N.$$

These two factorization is said to be complementary if

$$\omega_M = sgn(M, M') v_{M'},$$

where M' denotes the complement of M in $[q] := \{1, 2, \cdots, q\}$, and $sgn(M, M')$ is the sign of the permutation $\{M, M'\}$ of $[q]$.

For a complex of free T-modules

$$F_* : 0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} F_{n-2} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0.$$

such that $\mathrm{rank}(F_k) = \mu_k$, for $1 \leq k \leq n$, suppose that for each k a basis for F_k has been chosen. The matrix of φ_k with respect to the chosen bases for F_k and F_{k-1} will be denoted by A_k.

Theorem 2.12.4 (Theorem 3 [EN73])**.** *Suppose F_* is exact. Then there is a unique way in which the matrices $A^{(\mu_k)}_k$ can simultaneously be factored completely so that (a) the factorization of the row matrix $A^{(\mu_n)}_n$ is the canonical factorization, and (b) for $1 \leq k \leq n-1$, the factorization of $A^{(\mu_{k+1})}_{k+1}$ and $A^{(\mu_k)}_k$ are complementary.*

Theorem 2.12.4 provides us important information about the maximal minors of two consecutive differentials in an exact sequence. We apply the theorem to the complex in Theorem 2.11.3.

Example 2.12.5. In this case, $\Delta^{(6)}_{M,N}$ is the canonical factorization, i.e, $N = [6] := \{1, 2, 3, 4, 5, 6\}$, and is complementary to $\Phi^{(6)}_{J,K}$. So $M = K'$. We have shown three minors m_x, m_y, m_z by choosing the columns of

Φ as

$$K_x = \{0, 1, 2, 3, 6, 7\},$$
$$K_y = \{0, 1, 4, 5, 8, 9\},$$
$$K_z = \{2, 3, 4, 5, 10, 11\}.$$

Taking M to be complement of these $K's$:

$$M_x = \{4, 5, 8, 9, 10, 11\},$$
$$M_y = \{2, 3, 6, 7, 10, 11\},$$
$$M_z = \{0, 1, 6, 7, 8, 9\}.$$

Then

$$\Delta^{(6)}_{M_x,[6]} = sgn(M_x, K_x)v_{K_x},$$
$$\Delta^{(6)}_{M_y,[6]} = sgn(M_y, K_y)v_{K_y},$$
$$\Delta^{(6)}_{M_z,[6]} = sgn(M_z, K_z)v_{K_z}.$$

Since

$$v_{K_x} \text{ divides } m_x = \alpha_0\alpha_1\beta_0 x^{3l+6k},$$

$\Delta^{(6)}_{M_x,[6]}$ is also a constant multiple of power of x. Similarly $\Delta^{(6)}_{M_y,[6]}$ and $\Delta^{(6)}_{M_z,[6]}$ are constant multiples of power of y and z respectively. Therefore, we have found three nonzero minors of Δ, which give a regular sequence of length three.

Here we make another observation on how the above rows M_x, M_y and M_z are chosen. For example, we have chosen the rows M_x as the complement of those columns of syzygies of the ideals J_0, J_1, J_3, each of which has x^{l+2k} as a generator. Therefore, the rows M_x correspond exactly to the union of columns of syzygies of the other ideals J_2, J_4, J_5, none of which has x^{l+2k} as a generator. In terms of partitions, these ideals correspond to the partitions without the subset $\{1\}$, namely

$$\{01, 2, 3\}, \{0, 2, 13\}, \{0, 3, 12\}.$$

The conclusion is, to have a 6×6 minor of Δ as a nonzero constant multiple of power of x, we take the rows of Δ by the six columns of syzygies of the ideals corresponding to the partitions without the subset $\{1\}$.

Similarly, to have a 6×6 minor of Δ as a nonzero constant multiple of power of y, or z, we take the rows of Δ by the six columns of syzygies of the ideals corresponding to the partitions without the subset $\{2\}$, or $\{3\}$.

We apply this observation in the proof of Conjecture 2.2.6 for $n = 4$ in §4.

2.13 Proof of Conjecture 2.2.6 for $n = 4$ under an extra hypothesis

In this section, we prove Conjecture 2.2.6 under an extra hypothesis for the case $n = 4$ to illustrate the ideas of proofs in the general case. As we have done in §2.6, we first construct the minimal free resolutions of its sub-ideals and show they have the same graded Betti numbers as the corresponding monomial sub-ideals of I_ϕ. In §2.13.1, we list the subideals J_Π, I_Π of J_ϕ, I_ϕ corresponding to the partitions Π of $\{0,1,2,3,4\}$ and show the equality of their Hilbert series. In §2.13.2, we show J_Π and I_Π have the same graded Betti numbers. In §2.13.3, we construct a complex from the complexes of minimal free resolutions of the $J'_\Pi s$ and show it is exact, which gives the minimal free resolution of J_ϕ.

2.13.1 Partitions and subideals

As we have seen in §2.11 for the case $n = 3$, the first syzygies of J_ϕ are constructed by combining the first syzygies of the sub-ideals J_i associated to the partitions of $\{0,1,2,3\}$ into three nonempty subsets Π_i, for $0 \le i \le 5$. For $n = 4$, we need to consider the sub-ideals associated to the partitions of $\{0,1,2,3,4\}$ into four nonempty subsets. There are the following ten sub-ideals of J_ϕ (there are also corresponding ten monomial sub-ideals of I_ϕ).

$$\begin{aligned} \text{type I}: &\{J_{04,1,2,3}, J_{03,1,2,4}, J_{02,1,3,4}, J_{01,2,3,4}\}, \\ \text{type II}: &\{J_{0,1,2,34}, J_{0,1,3,24}, J_{0,2,3,14}, J_{0,1,4,23}, J_{0,2,4,13}, J_{0,3,4,12}\}. \end{aligned} \tag{2.13.1}$$

By the permutation symmetry, the ideals in each type are isomorphic, so we only need to consider one sub-ideal from each type. For example,

$$\begin{aligned} J_{0,1,2,34} =& \langle x_1^{\phi(1)}, x_2^{\phi(1)}, (x_3+x_4)^{2\phi(2)}, (x_1+x_2)^{2\phi(2)}, \\ & (x_1+x_3+x_4)^{3\phi(3)}, (x_2+x_3+x_4)^{3\phi(3)}, (x_1+x_2+x_3+x_4)^{4\phi(4)}\rangle. \\ I_{0,1,2,34} =& \langle x_1^{\phi(1)}, x_2^{\phi(1)}, (x_3x_4)^{\phi(2)}, (x_1x_2)^{\phi(2)}, (x_1x_3x_4)^{\phi(3)}, (x_2x_3x_4)^{\phi(3)}, (x_1x_2x_3x_4)^{\phi(4)}\rangle. \end{aligned}$$

Similar to the case $n = 3$, the ideals $J_{0,1,2,34}$ and $I_{0,1,2,34}$ have six pairs of subideals corresponding to the partitions

$$\{0, 12, 34\}, \{0, 2, 134\}, \{0, 1, 234\}, \{01, 2, 34\}, \{02, 1, 34\}, \{034, 1, 2\}. \tag{2.13.2}$$

For example,

$$\begin{aligned} J_{01,2,34} &= \langle x_2^{\phi(1)}, (x_3+x_4)^{2\phi(2)}, (x_2+x_3+x_4)^{3\phi(3)}\rangle, \\ I_{01,2,34} &= \langle x_2^{\phi(1)}, (x_3x_4)^{\phi(2)}, (x_2x_3x_4)^{\phi(3)}\rangle. \end{aligned}$$

Notice that, $J_{01,2,34}$ and $I_{01,2,34}$ are ideals in essentially two variables $\{x_2, x_3 + x_4\}$ and $\{x_2, x_3x_4\}$ respectively. Therefore, it is easy to find their minimal free resolutions. There are 25 partitions of $\{0, 1, 2, 3, 4\}$ into three nonempty subsets in total. Correspondingly there are 25 pairs of sub-ideals of J_ϕ and I_ϕ. Some ideals are the sub-ideals of more than one partition of $\{0, 1, 2, 3, 4\}$ to four nonempty subsets. For example, $J_{1,2,034}$ is a sub-ideal of both $J_{1,2,3,04}$ and $J_{0,1,2,34}$.

It is clear that the ideal $J_{0,1,2,34}$ is an ideal in essentially 3-variables $x_1, x_2, x_3 + x_4$ and the ideal $I_{0,1,2,34}$ is also an ideal in essentially 3-variables x_1, x_2, x_3x_4. Let $R = \mathbb{K}[x_1, x_2, x_3, x_4]$ and $R' = \mathbb{K}[x_1, x_2, x_3]$. Since x_4 is nonzerodivisor of R, we have

$$R/J_{0,1,2,34} \cong (R/x_4)/(J_{0,1,2,34}/x_4) = R'/J',$$

where

$$J' = \langle x_1^{\phi(1)}, x_2^{\phi(1)}, x_3^{2\phi(2)}, (x_1+x_2)^{2\phi(2)}, (x_1+x_3)^{3\phi(3)}, (x_2+x_3)^{3\phi(3)}, (x_1+x_2+x_3)^{4\phi(4)}\rangle.$$

Similarly, $x_3 - x_4$ is nonzerodivisor of R, we have

$$R/I_{0,1,2,34} \cong (R/(x_3 - x_4))/(I_{0,1,2,34}/(x_3 - x_4)) = R'/I',$$

where

$$I' = \langle x_1^{\phi(1)}, x_2^{\phi(1)}, x_3^{2\phi(2)}, (x_1x_2)^{\phi(2)}, (x_1x_3^2)^{\phi(3)}, (x_2x_3^2)^{\phi(3)}, (x_1x_2x_3^2)^{\phi(4)} \rangle.$$

Now the Hilbert series of R'/J' and R'/I' are the same by Postnikov-Shapiro [PS04], since The ideals J' and I' can be constructed from the same graph G'.

The Hilbert series of the sub-ideals $J_{01,2,34}$ and $I_{01,2,34}$ are also the same, considered as ideals in two variables, since both can be constructed from a graph on $\{0, 1, 2\}$. Other pairs of sub-ideals corresponding to partitions of $\{0, 1, 2, 3, 4\}$ into three nonempty subsets also have the same Hilbert series.

2.13.2 Minimal Free Resolution of the subideals

It is easy to find the minimal free resolution of the ideal $I_{0,1,2,34}$ as

$$0 \to F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} R \to R/I_{0,1,2,34} \to 0, \tag{2.13.3}$$

where

$$\begin{aligned} F_1 &= R(-\phi(1))^2 \oplus R(-2\phi(2))^2 \oplus R(-3\phi(3))^2 \oplus R(-4\phi(4)), \\ F_2 &= R(-d(1,1))^2 \oplus R(-d(1,2))^4 \oplus R(-d(1,3))^4 \oplus R(-d(2,2))^2, \\ F_3 &= R(-d(1,1,2))^6. \end{aligned} \tag{2.13.4}$$

Recall that

$$d(l_1, \cdots, l_k) = l_1\phi(l_1) + l_2\phi(l_1 + l_2) + \cdots + l_k\phi(l_1 + \cdots + l_k).$$

Proposition 2.13.1. *The ideal $J_{0,1,2,34}$ has the same graded Betti numbers as $I_{0,1,2,34}$, if the kernel of the matrix d_2 in (2.13.6) is a free module.*

Proof. We show the minimal free resolution of the ideal $J_{0,1,2,34}$ is constructed similarly as Theorem 2.11.3.

The first syzygies of $J_{0,1,2,34}$ are obtained by combining the first syzygies from the six sub-ideals of $J_{0,1,2,34}$ corresponding to partitions in (2.13.2). For example, the minimal free resolution of $J_{01,2,34}$ is given by

$$0 \longrightarrow R(-d(1,2))^2 \to \begin{array}{c} R(-\phi(1)) \\ \oplus \\ R(-2\phi(2)) \\ \oplus \\ R(-3\phi(3)) \end{array} \to R \to R/J_{01,2,34} \to 0, \tag{2.13.5}$$

which has the same graded Betti numbers as $I_{01,2,34}$. This resolution provides us two first syzygies of degree $2d(1,2)$. Similarly, we get another two first syzygies of degree $2d(1,2)$ from $I_{02,1,34}$; two syzygies of degree $2d(1,1)$ from $J_{034,1,2}$ and two of degree $2d(2,2)$ from $J_{0,12,34}$; each of $J_{0,2,134}$ and $J_{0,1,234}$ provides us two first syzygies of degree $2d(1,3)$. Notice the arguments of the function $d(l_1,l_2)$ are exactly the sizes of the subsets in the partition.

Similar to the proof of Conjecture 2.2.8, we can construct a complex by combining the six complexes of the form (2.13.5), given by

$$0 \to K_3 \xrightarrow{d_3} K_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} R \to R/J_{0,1,2,34} \to 0, \tag{2.13.6}$$

Assume that the kernel of d_2 is a free module, denoted by K_3. Therefore (2.13.6) is a complex of free modules.

Now K_2 and F_2 are isomorphic; hence K_3 and F_3 also are, since

$$HS(R/I_{0,1,2,34}) = HS(R/J_{0,1,2,34}).$$

We conclude that the minimal free resolution of $J_{0,1,2,34}$ is given by the complex in (2.13.6) and has the same graded Betti numbers as $I_{0,1,2,34}$. □

A similar analysis will show that the minimal free resolution of $J_{01,2,3,4}$ is given by a similar complex as (2.13.6) and has the same graded Betti numbers as $I_{01,2,3,4}$, corresponding to the partition $\{01,2,3,4\}$ of type II.

2.13.3 Exactness of the constructed complex

Inspired by the construction of the differentials in our approach to Schenck's conjecture, we give the following

Definition 2.13.2. *Suppose an ideal I in $R = k[x_1, \cdots, x_n]$ is a sum of m subideals I_i, $i = 1, 2, \cdots, m$. and the minimal free resolution of I_i is given by*

$$F_i : 0 \xrightarrow{d^i_{n-1}} F^i_{n-2} \cdots \xrightarrow{d^i_3} F^i_2 \xrightarrow{d^i_2} F^i_1 \xrightarrow{d^i_1} F^i_0 \to I_i \to 0.$$

The S-sum of these complexes is a new complex,

$$F : 0 \to F_{n-1} \to F_{n-2} \cdots \xrightarrow{D_3} F_2 \xrightarrow{D_2} F_1 \xrightarrow{D_1} F_0 \to I \to 0,$$

where, for $1 \leq i \leq n-2$, the matrix of differential D_i is obtained from the union of columns of all the d_i extended by 0's. We take F_{n-1} as the kernel of D_{n-2}.

Example 2.13.3. *For any $l, k \geq 1$, consider the monomial ideal I in $k[x, y, z]$ defined by*

$$I = \langle x^{l+2k}, y^{l+2k}, z^{l+2k}, (xy)^{l+k}, (xz)^{l+k}, (yz)^{l+k}, (xyz)^l \rangle.$$

Postnikov-Shapiro found the minimal free resolution of I as follows, see (2.2.4) *for the special case of* $n = 3$.

$$0 \to R(-3l-3k)^6 \xrightarrow{d_3} \begin{matrix} R(-3l-2k)^6 \\ \oplus \\ R(-2l-3k)^6 \end{matrix} \xrightarrow{d_2} \begin{matrix} R(-l-2k)^3 \\ \oplus \\ R(-2l-2k)^3 \\ \oplus \\ R(-3l) \end{matrix} \xrightarrow{d_1} R \to R/J_\phi \to 0.$$

where the matrix of differentials (up to a permutation of columns) can be written as
$d_1 = [x^{l+2k}, y^{l+2k}, z^{l+2k}, (xy)^{l+k}, (xz)^{l+k}, (yz)^{l+k}, (xyz)^l]$,

$$d_2 = \begin{bmatrix} -y^{l+k} & 0 & -z^{l+k} & 0 & 0 & 0 & -(yz)^l & 0 & 0 & 0 & 0 & 0 \\ 0 & -x^{l+k} & 0 & 0 & -z^{l+k} & 0 & 0 & 0 & -(xz)^l & 0 & 0 & 0 \\ 0 & 0 & 0 & -x^{l+k} & 0 & -y^{l+k} & 0 & 0 & 0 & 0 & -(xy)^k & 0 \\ x^k & y^k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -z^l \\ 0 & 0 & x^k & z^k & 0 & 0 & 0 & 0 & 0 & -y^l & 0 & 0 \\ 0 & 0 & 0 & 0 & y^k & z^k & 0 & -x^l & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^{2k} & (yz)^k & y^{2k} & (xz)^k & z^{2k} & (xy)^k \end{bmatrix}.$$

Here, the twelve columns of the matrix d_2 is obtained from the union of columns of all the differentials of its six subideals I_i in (2.11.3), extended by 0's.

Remark 2.13.4. In general, the S-sum complex will not give the minimal resolution of the ideal.

Theorem 2.13.5. *Fix any $l, k \geq 1$. Assume that G_4 in the complex (2.13.7) is a free module. Then the minimal free resolution of the ideal*

$$J_\phi = \langle (x_{i_1} + \cdots + x_{i_r})^{r\phi(r)} \rangle,$$

where $\{i_1, \cdots, i_r\}$ runs through all nonempty subsets of $\{1,2,3,4\}$ and $\phi(r) = l + k(4-r)$, is

$$0 \to G_4 \to G_3 \xrightarrow{D_3} G_2 \xrightarrow{D_2} G_1 \xrightarrow{D_1} R \to R/J_\phi \to 0.$$

Here the modules G_i are given by

$$G_1 = R(-l-3k)^4 \oplus R(-2l-4k)^6 \oplus R(-3l-3k)^4 \oplus R(-4l),$$
$$G_2 = R(-2l-5k)^{12} \oplus R(-3l-5k)^{24} \oplus R(-4l-3k)^8 \oplus R(-4l-4k)^6,$$
$$G_3 = R(-3l-6k)^{24} \oplus R(-4l-5k)^{36},$$
$$G_4 = R(-4l-6k)^{36}.$$

Proof. By taking the S-sum of the ten complexes of resolutions of the subideals in equation (2.13.1) together, we get

$$0 \to G_4 \to G_3 \xrightarrow{D_3} G_2 \xrightarrow{D_2} G_1 \xrightarrow{D_1} R \to R/J_\phi \to 0, \tag{2.13.7}$$

where G_4 is the kernel of D_3, and

$$G_1 = R(-d(1))^4 \oplus R(-d(2))^6 \oplus R(-d(3))^4 \oplus R(-d(4)),$$
$$G_2 = R(-d(1,1))^{12} \oplus R(-d(1,2))^{24} \oplus R(-d(1,3))^8 \oplus R(-d(2,2))^6,$$
$$G_3 = R(-d(1,1,1))^{24} \oplus R(-d(1,1,2))^{36}.$$

We can apply the Buchsbaum-Eisenbud criterion (2.11.2) to show the complex (2.13.7) is exact. D_1 is clearly of rank one, and the generators $x_1^{\phi(1)}$, $x_2^{\phi(1)}$, $x_3^{\phi(1)}$ and $x_4^{\phi(1)}$ are a regular sequence of length four.

Next, we show D_2 is of rank 14 and there is also a regular sequence of length four of its ideal of minors of maximal order, by choosing appropriate 14×14 minors, in a way similar to our choice of the 6×6 minors in the proof of Theorem 2.11.3, see Remark 2.11.1.

We choose the 14×14 minor as the block matrix consisting of seven 2×2 blocks, each block is from the the second and third row of the 3×2 matrices in the resolutions of the sub-ideals with $x_1^{\phi(1)}$ as a generator, which corresponds to the partitions with a subset $\{1\}$,

$$\{023, 1, 4\}, \qquad \{034, 1, 2\}, \qquad \{024, 1, 3\},$$
$$\{02, 1, 34\}, \qquad \{03, 1, 24\}, \qquad \{04, 1, 23\}, \qquad \{0, 1, 234\}.$$

The determinant of each block is a nonzero constant multiple of $x_1^{\phi(1)}$, so the 14×14 minor is a nonzero constant multiple of $x_1^{7\phi(1)}$. Similarly, we can choose the other three 14×14 minors given by a nonzero constant multiple of $x_2^{7\phi(1)}$, $x_3^{7\phi(1)}$, and $x_4^{7\phi(1)}$.

To show D_3 is of rank 36, we use the argument given at the end of §3. We can choose a 36×36 sub-matrix, consisting of six 6×6 blocks, whose determinant is a power of x_1. Notice that there are six sub-ideals of J_ϕ in essentially three variables which have $x_1^{\phi(1)}$ as a generator, corresponding to the partitions with one subset $\{1\}$,

$$\{04, 1, 2, 3\}, \qquad \{03, 1, 2, 4\}, \qquad \{02, 1, 3, 4\},$$
$$\{0, 1, 2, 34\}, \qquad \{0, 1, 3, 24\}, \qquad \{0, 1, 4, 23\}.$$

For each of these six sub-ideals, the differential d_3 of its minimal free resolution is represented by a 12×6

matrix. We can choose a 6×6 sub-matrix whose determinant is a power of x_1. For example, we choose a 6×6 sub-matrix by choosing the six rows as the six columns of syzygies of the three sub-ideals of $J_{1,2,3,04}$ which don't have $x_1^{\phi(1)}$ as a generator, which corresponds to the partitions

$$\{04, 12, 3\}, \{04, 13, 2\}, \{014, 2, 3\}. \tag{2.13.8}$$

This argument shows the rank of D_3 is 36, and the ideal of its minors of order 36 has a regular sequence of length four.

Now, the assumption that G_4 is free, combined with the equality of the Hilbert series

$$HS(R/J_\phi) = HS(R/I_\phi),$$

and the graded Betti numbers of I_ϕ agree with those of J_ϕ on G_3, G_2 and G_1, we conclude

$$G_4 \cong R(-d(1,1,1,1))^{24}.$$

Applying the Buchsbaum-Eisenbud criterion 2.11.2 again, we see the complex (2.13.7) is exact, which completes the proof of the Conjecture 2.2.6 for $n = 4$. □

2.14 Proof of Conjecture 2.2.6 for general n under an extra hypothesis

For an arbitrary $n \in \mathbb{N}$, in $R = \mathbb{K}[x_1, x_2, \cdots, x_n]$ we have the two ideals

$$I_\phi = \langle x_1^{\phi(1)}, \cdots, x_n^{\phi(1)}, (x_1x_2)^{\phi(2)}, \cdots, (x_{i_1} \cdots x_{i_r})^{\phi(r)}, \cdots \rangle, \text{ and}$$
$$J_\phi = \langle x_1^{\phi(1)}, \cdots, x_n^{\phi(1)}, (x_1 + x_2)^{2\phi(2)}, \cdots, (x_{i_1} + \cdots + x_{i_r})^{r\phi(r)}, \cdots \rangle,$$

where $\{i_1, \cdots, i_r\}$ runs through all nonempty subsets of $\{1, 2, \cdots, n\}$. Both are constructed from the graph $K_{n+1}^{l,k}$ on the vertices $[n+1] = \{0, 1, \cdots, n\}$.

We want to construct a minimal free resolution of J_ϕ by generalizing the method in the cases $n = 3, 4$. So we have to find the minimal free resolutions of suitably chosen sub-ideals associated to partitions of $[n+1]$.

For $1 \leq p \leq n$, consider a partition of $[n+1]$ into $p+1$ nonempty subsets,

$$\Pi_p = \{N_0, N_1, \cdots, N_p\}.$$

By convention, we always assume $0 \in N_0$.

There are sub-ideals of J_ϕ and I_ϕ associated to the partition Π_p, which we denote by J_{Π_p} and I_{Π_p}, generated by p_S and m_S for all nonempty subsets $S \subseteq \{N_1, \cdots, N_p\}$, respectively.

For two partitions $P = \{P_0, \cdots, P_r\}$ and $Q = \{Q_0, \cdots, Q_s\}$, where $r < s$, we say Q is a *refinement* of P, denoted by $P \prec Q$, if each P_i is a union of some $Q_j's$. This definition defines a poset structure on all partitions of $[n+1]$ into nonempty subsets.

Notice that if $P \prec Q$, then J_P is a sub-ideal of J_Q and similarly I_P is a sub-ideal of I_Q. The unique partition of $[n+1]$ into $n+1$ subsets, with $N_i = \{i\}$, for $0 \leq i \leq n$, is a refinement of any other partition, corresponding to the ideals I_ϕ, J_ϕ themselves. Therefore, it suffices to find the minimal free resolution of the ideals J_{Π_p}, for any $1 \leq p \leq n$.

In the following, we fix a partition $\Pi_p = \{N_0, N_1, \cdots, N_p\}$ with $|N_i| = n_i$, for $1 \leq i \leq p$. We aim to construct the minimal free resolution of J_{Π_p} and show it has the same graded Betti numbers as I_{Π_p}. If $N_i = \{j_1, j_2, \cdots, j_{n_i}\}$ for $1 \leq i \leq p$, the ideal J_{Π_p} is essentially an ideal in p-variables $x_{N_i} := x_{j_1} + x_{j_2} + \cdots + x_{j_{n_i}}$, for $1 \leq i \leq p$. Similarly, the ideal I_{Π_p} is also essentially an ideal in p-variables $y_{N_i} := x_{j_1} x_{j_2} \cdots x_{j_{n_i}}$, for $1 \leq i \leq p$. As we have done in the case $n = 4$, we can reduce the analysis of the ideals J_{Π_p} and I_{Π_p} to ideals in p variables, renamed as $x_1, x_2, \cdots, x_p$ to simplify the notation. Then we have

$$\begin{aligned} J_{\Pi_p} = \langle & x_1^{n_1\phi(n_1)}, x_2^{n_2\phi(n_2)} \cdots, x_p^{n_p\phi(n_p)}, (x_1+x_2)^{(n_1+n_2)\phi(n_1+n_2)}, \cdots, \\ & (x_i+x_j)^{(n_i+n_j)\phi(n_i+n_j)}, \cdots, (x_1+x_2+x_3)^{(n_1+n_2+n_3)\phi(n_1+n_2+n_3)}, \cdots, \\ & (x_1+x_2+\cdots+x_p)^{(n_1+n_2+\cdots+n_p)\phi(n_1+n_2+\cdots+n_p)}\rangle, \\ I_{\Pi_p} = \langle & x_1^{n_1\phi(n_1)}, \cdots, x_p^{n_p\phi(n_p)}, (x_1^{n_1}x_2^{n_2})^{\phi(n_1+n_2)}, \cdots, (x_i^{n_i}x_j^{n_j})^{\phi(n_i+n_j)}, \cdots, \\ & (x_1^{n_1}x_2^{n_2}x_3^{n_3})^{\phi(n_1+n_2+n_3)}, \cdots, (x_1^{n_1}x_2^{n_2}\cdots x_p^{n_p})^{\phi(n_1+n_2+\cdots+n_p)}\rangle. \end{aligned} \tag{2.14.1}$$

Lemma 2.14.1. *For any partition Π_p of $[n+1]$, the Hilbert series of R/J_{Π_p} and R/I_{Π_p} are equal.*

Proof. Both ideals can be constructed from the same graph G_p on $\{0, 1, 2, \cdots, p\}$. For example, the multiplicity of the edges connecting i and 0 is $n_i\phi(n_1 + n_2 + \cdots + n_p)$, for each i. □

Lemma 2.14.2. *The k-th total Betti numbers of I_{Π_p} is given by*

$$\beta_k(I_{\Pi_p}) = k!S(p+1, k+1).$$

Proof. This lemma follows from Corollary 6.8 in [PS04], since the ideal I_{Π_p} is a strictly monotone monomial ideal in the sense of Postnikov-Shapiro [PS04] and we take all nonempty subsets in $\{N_1, \cdots, N_p\}$ to construct I_{Π_p}. □

Recall that,

$$d(l_1, l_2, \cdots, l_k) = l_1\phi(l_1) + l_2\phi(l_1 + l_2) + \cdots + l_k\phi(l_1 + \cdots + l_k).$$

Theorem 2.14.3. *For the ideal J_{Π_p} corresponding to $\Pi_p = \{N_0, N_1, \cdots, N_p\}$ with $|N_i| = n_i$, for $1 \leq i \leq p$, form the following complex*

$$F_* : 0 \longrightarrow F_p \xrightarrow{d_p} F_{p-1} \xrightarrow{d_{p-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} R \to R/J_{\Pi_p} \to 0, \tag{2.14.2}$$

which is obtained by combining the complexes of minimal free resolutions of the subideals of J_{Π_p} corresponding to all partitions $\Pi_i = \{A_0, A_1, \cdots, A_i\}$ such that $\Pi_i \prec \Pi_p$, and $l_i = |A_i|$ is the cardinality of A_i. Therefore,

$$F_i = \bigoplus_{\Pi_i \prec \Pi_p} R(-d(l_1, l_2, \cdots, l_i))^{i!}, \tag{2.14.3}$$

and the differential d_i is obtained from combining the $d_i^{'}$ of the minimal free resolution of these sub-ideals correspondent to Π_i. F_p is isomorphic to the kernel of d_{p-1}. ***Assume F_p is a free module,*** *then the complex (2.14.3) is exact and thus gives the minimal free resolution of J_{Π_p} with*

$$F_p \cong R(-d(n_1, n_2, \cdots, n_p))^{p!}.$$

Moreover, all the graded Betti numbers of I_{Π_p} are the same as J_{Π_p}.

Proof. The proof is by induction. For $p = 1$, it is clear. For $p = 2$, both J_{Π_p} and I_{Π_p} are ideals in essentially two variables, their minimal free resolutions are given by the Hilbert-Burch resolution and they have the same graded Betti numbers.

Suppose we have proved the theorem for all ideals J_{Π_p} corresponding to partitions Π_p of $[n+1]$ into p

nonempty subsets. We now consider ideals corresponding to partitions into $p+1$ nonempty subsets.

Pick one partition $\Pi_p = \{N_0, N_1, \cdots, N_p\}$ with $|N_i| = n_i \geq 1$, for $1 \leq i \leq p$. We have the pair of ideals $\{J_{\Pi_p}, I_{\Pi_p}\}$ and by Lemma 2.14.1,

$$HS(R/J_{\Pi_p}) = HS(R/I_{\Pi_p}).$$

If we take the union of two subsets from Π_p as one subset, we get partitions of $[n+1]$ into p nonempty subsets, which has Π_p as the refinement. The number of such partitions is $t = \binom{p}{2}$. We denote these partitions as Π_p^i, $1 \leq i \leq t$. Correspondingly, there are t sub-ideals of J_{Π_p}, I_{Π_p} in essentially $p-1$ variables.

By induction, we have the minimal free resolutions for each of these sub-ideals of the form

$$E_* : 0 \longrightarrow E_{p-1} \xrightarrow{d_{p-1}} E_{p-2} \xrightarrow{d_{p-2}} \cdots E_1 \xrightarrow{d_1} R \xrightarrow{d_0} R/J_{\Pi_p^i} \to 0.$$

By combining these t complexes together and taking F_p as the kernel of D_{p-1} which is obtained from combining the d_{p-1}, we get the complex

$$F_* : 0 \to F_p \xrightarrow{D_p} F_{p-1} \xrightarrow{D_{p-1}} \cdots F_1 \xrightarrow{D_1} R \xrightarrow{D_0} R/J_{\Pi_p} \to 0.$$

We assume that F_p is a free module.

Next, we show the complex F_* is exact. To see this, we show for $1 \leq i \leq p$, the D_i has the maximal rank and there is a minor of D_i equal to a power of $x_1, x_2, \cdots, x_p$. By symmetry, we just show a minor of D_i equal to a power of x_1 for $1 \leq i \leq p$. Therefore, the ideal of maximal minors of D_i always has a regular sequence of length p.

D_1 is given by the generators of J_{Π_p}, so of rank 1 and $x_1^{n_1\phi(n_1)}$ is a generator.

F_2 is the union of first syzygies of all sub-ideals of J_{Π_p} in essentially 2-variables, and the matrix D_2 is obtained from the union of $d_2's$. Consider those sub-ideals with a generator $x_1^{n_1\phi(n_1)}$, corresponding to partitions Π_2 of $[n+1]$ into three nonempty subsets such that $\Pi_2 \prec \Pi_p$ and Π_2 has N_1 as a subset. These partitions are equivalent to partitions of $\{N_0, N_2, \cdots, N_p\}$ into 2 nonempty subsets, in total $S(p, 2)$.

Then the rank of D_2 is $2S(p, 2)$, by choosing the union of the 2×2 sub-matrices of the 3×2 matrices, where we take the second and third row of the syzygies of these $S(p, 2)$ sub-ideals. Each minor is a constant multiple of $x_1^{S(p,2)n_1\phi(n_1)}$. So the rank is the maximal possible, since $\text{rank}(G_1) = S(p+1,2)$ and

$$2S(p,2) + 1 = 2^p - 1 = S(p+1,2),$$

which is $\text{rank}(D_2) + \text{rank}(D_1) = \text{rank}(F_1)$.

By construction, D_k is obtained by combining the matrix of $d_k' s$ of the sub-ideals corresponding to partitions Π_k of $[n+1]$ into $k+1$ nonempty subsets, such that $\Pi_k \prec \Pi_p$, say $\{M_0, M_1, \cdots, M_k\}$ with $|M_i| = m_i$, for $1 \le i \le k$, with each one contributing to E_k by

$$R(-d(m_1, m_2, \cdots, m_k))^{k!},$$

by induction. Moreover, d_k is represented by a matrix of size

$$(k-1)!S(p,k) \times k!.$$

Now we show the rank of D_k is $k!S(p,k)$. Since the maximal minors of d_k have the canonical factorization, we choose the columns as the union of the $k!$ columns for the sub-ideals with a generator $x_1^{n_1\phi(n_1)}$, corresponding to those Π_k with a subset equal to N_1, thus in total $S(p,k)$.

For each d_k, we choose the rows as complements to the union of columns in d_{k-1}, which arise from the $(k-1)$-th syzygies of all the sub-ideals of J_{Π_k} in $(k-1)$-variables, as suggested by Buchsbaum-Eisenbud's Theorem 2.12.4. The total number of columns is $(k-1)!S(p+1,k)$, and the number of complementary columns is

$$(k-1)!S(p+1,k) - (k-1)!S(p,k-1) = k!S(p,k),$$

which is $\text{rank}(F_{k-1}) - \text{rank}(D_{k-1}) = k!S(p,k)$.

In terms of the partitions, the maximal minor of D_k, for $1 \le k \le p$, corresponds to the partitions $\Pi_k = \{M_0, M_1, \cdots, M_k\} \prec \Pi_p$ with $M_1 = N_1$ as fixed. These correspond to the partitions of $\{N_0, N_2, \cdots, N_p\}$ into k nonempty subsets, thus the number is $S(p,k)$.

The $k!S(p,k) \times k!S(p,k)$ minor of D_k is a power of x_1. Therefore, each D_k has the maximal rank $k!S(p,k)$. The ideal of maximal minors also gives a regular sequence of length p. Therefore, we proved the exactness of the glued complex.

Since the Hilbert series of R/J_{Π_p} is equal to that of R/I_{Π_p}, we have

$$F_p \cong S(-d(n_1, n_2, \cdots, n_p))^{p!}.$$

Thus the complex F_* gives a minimal free resolution of J_{Π_p}. □

Corollary 2.14.4. *Taking $p = n$, so $\Pi_p = \{0, 1, \cdots, n\}$ and $J_{\Pi_p} = J_\phi$, the Postnikov-Shapiro ideal of powers of linear forms in n variables. Then the constructed complex (2.14.2) gives a minimal free resolution of J_ϕ, assuming that F_n is a free module.*

Remark 2.14.5. It is tempting to construct a basis for the F_i at each step and thus give a very explicit description of the differentials in the resolution, but we have not been able to do so.

References

[Alf96] Peter Alfeld, *Upper and lower bounds on the dimension of multivariate spline spaces*, SIAM J. Numer. Anal. **33** (1996), no. 2, 571–588.

[AS90] Peter Alfeld and Larry L. Schumaker, *On the dimension of bivariate spline spaces of smoothness r and degree $d = 3r + 1$*, Numer. Math. **57** (1990), no. 6-7, 651–661.

[AS08] ———, *Bounds on the dimensions of trivariate spline spaces*, Adv. Comput. Math. **29** (2008), no. 4, 315–335.

[ASS92] Peter Alfeld, Larry L. Schumaker, and Maritza Sirvent, *On dimension and existence of local bases for multivariate spline spaces*, J. Approx. Theory **70** (1992), no. 2, 243–264.

[ASW93] Peter Alfeld, Larry L. Schumaker, and Walter Whiteley, *The generic dimension of the space of C^1 splines of degree $d \geq 8$ on tetrahedral decompositions*, SIAM J. Numer. Anal. **30** (1993), no. 3, 889–920.

[BE73] David A. Buchsbaum and David Eisenbud, *What makes a complex exact?*, J. Algebra **25** (1973), 259–268.

[BE74] ———, *Some structure theorems for finite free resolutions*, Advances in Math. **12** (1974), 84–139.

[Bil88] Louis J. Billera, *Homology of smooth splines: generic triangulations and a conjecture of Strang*, Trans. Amer. Math. Soc. **310** (1988), no. 1, 325–340.

[Buc00] David A. Buchsbaum, *On commutative algebra and characteristic-free representation theory*, J. Pure Appl. Algebra **152** (2000), no. 1-3, 41–48, Commutative algebra, homological algebra and representation theory (Catania/Genoa/Rome, 1998).

[CLO05] David A. Cox, John Little, and Donal O'Shea, *Using algebraic geometry*, second ed., Graduate Texts in Mathematics, vol. 185, Springer, New York, 2005.

[EI95] J. Emsalem and A. Iarrobino, *Inverse system of a symbolic power. I*, J. Algebra **174** (1995), no. 3, 1080–1090.

[Eis95] David Eisenbud, *Commutative algebra, with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.

[EN73] J. A. Eagon and D. G. Northcott, *On the Buchsbaum-Eisenbud theory of finite free resolutions*, J. Reine Angew. Math. **262/263** (1973), 205–219, Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday.

[FZ00] Sergey Fomin and Andrei Zelevinsky, *Total positivity: tests and parametrizations*, Math. Intelligencer **22** (2000), no. 1, 23–33.

[Gab93] Andrei Gabrielov, *Abelian avalanches and Tutte polynomials*, Phys. A **195** (1993), no. 1-2, 253–274.

[Ger96] Anthony V. Geramita, *Inverse systems of fat points: Waring's problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals*, The Curves Seminar at Queen's, Vol. X (Kingston, ON, 1995), Queen's Papers in Pure and Appl. Math., vol. 102, Queen's Univ., Kingston, ON, 1996, pp. 2–114.

[GHM09] A. V. Geramita, Brian Harbourne, and Juan Migliore, *Classifying Hilbert functions of fat point subschemes in* $\mathbb{P}^2$, Collect. Math. **60** (2009), no. 2, 159–192.

[GS97] A. V. Geramita and H. Schenck, *Fat points, inverse systems and piecewise polynomial functions*, The Curves Seminar at Queen's, Vol. XI (Kingston, ON, 1997), Queen's Papers in Pure and Appl. Math., vol. 105, Queen's Univ., Kingston, ON, 1997, pp. 98–116.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[Iar97] A. Iarrobino, *Inverse system of a symbolic power. III. Thin algebras and fat points*, Compositio Math. **108** (1997), no. 3, 319–356.

[Lau06] Wai Wah Lau, *A lower bound for the dimension of trivariate spline spaces*, Constr. Approx. **23** (2006), no. 1, 23–31.

[Mir99] Rick Miranda, *Linear systems of plane curves*, Notices Amer. Math. Soc. **46** (1999), no. 2, 192–201.

[MS] Fatemeh Mohammadi and Farbod Shokrieh, *Divisors on graphs, connected flags, and syzygies*, International Mathematics Research Notices(to appear).

[MSW] Madhusudan Manjunath, Frank-Olaf Schreyer, and John Wilmes, *Minimal free resolutions of g-parking function ideal and the toppling ideal*, Transactions of the American Mathematical Society (to appear).

[MT13] Ján Mináč and Stefan O. Tohăneanu, *From spline approximation to Roth's equation and Schur functors*, Manuscripta Math. **142** (2013), no. 1-2, 101–126.

[PS04] Alexander Postnikov and Boris Shapiro, *Trees, parking functions, syzygies, and deformations of monomial ideals*, Trans. Amer. Math. Soc. **356** (2004), no. 8, 3109–3142 (electronic).

[PSS] Alexander Postnikov, Boris Shapiro, and Michael Shapiro, *Chern forms on flag manifolds and forests*.

[Sch97] Hal Schenck, *A spectral sequence for splines*, Adv. in Appl. Math. **19** (1997), no. 2, 183–199.

[Sch04] Henry K. Schenck, *Linear systems on a special rational surface*, Math. Res. Lett. **11** (2004), no. 5-6, 697–713.

[SS97] Hal Schenck and Mike Stillman, *Local cohomology of bivariate splines*, J. Pure Appl. Algebra **117/118** (1997), 535–548, Algorithms for algebra (Eindhoven, 1996).

[SS98] Boris Shapiro and Mikhail Shapiro, *On ring generated by Chern* 2*-forms on* SL_n/B, C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), no. 1, 75–80.

[Str73] Gilbert Strang, *Piecewise polynomials and the finite element method*, Bull. Amer. Math. Soc. **79** (1973), 1128–1137.

[Whi91] Walter Whiteley, *A matrix for splines*, Progress in approximation theory, Academic Press, Boston, MA, 1991, pp. 821–828.