THE DUAL BOUNDARY ELEMENT METHOD APPLIED TO SEMI-INFINITE SURFACE CRACKS IN TWO-DIMENSIONS

BY

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THESIS

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Abstract

The goal of his thesis is to evaluate the stress intensity factors (SIF) for surface cracks in semi-infinite solids via the DBEM. We first detail the derivations of the boundary element and dual boundary element methods. The main difficulty with these methods are integrations of the singular integrals which cannot be accuartely evaluated via standard Gauss quadrature schemes. An analytical integration scheme has therefore been implemented to overcome this difficulty. The SIFs are recovered as a post-processing step using the crack tip opening displacement method. A computer code using FORTRAN.90 is developed to verify the effectiveness of the DBEM and its application to fracture problems.

To Mom and Dad

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1 Introduction

The boundary element method (BEM) is a numerical technique that is used to solve potential and elasticity problems. It is similar to the finite element method (FEM) in the sense that nodal values (displacements, tractions, temperature, etc.) are computed over a mesh; however, one of the main advantages of the BEM is that discretization is performed only on the boundary [1, 2]. Discretizing the boundary reduces the dimensionality of the problem by one, thus reducing the number of unknowns to be solved, and making it more computationally efficient than the FEM [3].

Boundary element method formulations can be characterized as direct or indirect [4]. The work done by the author utilizes the direct (collocation point) method, and therefore, the indirect method is not discussed. The direct method uses the fundamental solutions of a point load in an infinite domain, i.e. Kelvin's solution, applied to Betti's Reciprocal work theorem to generate Somigliana's identity [5].

Somigliana's identity, which relates the displacement of an internal point to the displacement and traction over the boundary, is manipulated to relate the displacements at a boundary point to the traction and displacement over the remainder of the boundary via the boundary integral equation (BIE). Discretization of the BIE allows for the numerical solution of the unknown boundary displacement and traction.

In the last 30 years, there has been much interest in applying the BEM to solve problems involving cracked domains, i.e. in fracture mechanics [6]. The direct BEM cannot be used to analyze cracked domains because the coincidental nodes on the mating crack faces give rise to a set of linearly dependent equations. This dilemma is avoided by applying the dual boundary element method (DBEM) [7] wherein a second set of equations derived from the BIE are enforced on one of the cracked faces, which results in a linearly independent system of equations.

The presence of a crack in a linear elastic material generates a stress singularity in the vicinity of the crack tip, which can be characterized by several different quantities such as stress intensity factors (SIF), the J-integral, and the Energy Release Rate (ERR) [8]. These quantities are dependent on crack length, material, geometry and loading. The focus here is to investigate the SIFs and ERR for two-dimensional mixed mode fracture problems, which can be accomplished as a post processing step using the response obtained from the DBEM [5, 7, 9, 10].

2 Boundary Element Formulation

2.1 Elasticity

Before deriving the BIE used in the BEM, the governing elasticity equations are presented. Their derivations are provided in [11]. A linear elastic homogeneous isotropic body Ω is loaded with the surface traction $\mathbf{t}^p$ on its boundary Γ^t and fixed on the complimentary boundary Γ^u such that $\partial\Omega = \Gamma^t \cup \Gamma^u$ and $\Gamma^t \cap \Gamma^u = \emptyset$, cf. Figure 1. Our goal is to compute the displacement $\mathbf{u}$ of the body Ω under this loading.

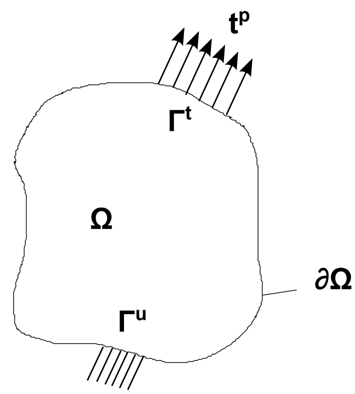


Figure 1: Elastic boundary value problem.

The body has an elasticity tensor

$$\mathbb{C} = 2\,\mu\,\mathbb{S} + \lambda\,\mathbf{I} \otimes \mathbf{I} \ , \tag{1}$$

where μ and λ are Lame's constants. The Poisson ratio ν is related to these constants such that

$$\begin{aligned} \nu &= \frac{\lambda}{2\,(\lambda + \mu)} \ , \\ \lambda &= \frac{2\mu\,\nu}{1 - 2\,\nu} \ . \end{aligned} \tag{2}$$

The gradient of the displacement $D\mathbf{u}$, relates the displacement to the stress tensor $\mathbf{T}$ via the elasticity tensor as

$$\begin{aligned} \mathbf{T} &= \mathbb{C}[D\mathbf{u}] \\ &= \mu\,(D\mathbf{u} + (D\mathbf{u})^T) + \lambda\,\mathrm{tr}(D\mathbf{u})\,\mathbf{I} \ , \\ T_{ij} &= \mu\,(u_{i,j} + u_{j,i}) + \lambda\,u_{k,k}\,\delta_{ij} \ . \end{aligned} \tag{3}$$

The traction vector $\mathbf{t}^{\mathbf{n}}$ acting on the surface with the unit normal vector $\mathbf{n}$ is subsequently defined as

$$\mathbf{t}^{\mathbf{n}} = \mathbf{T}\mathbf{n} \ . \tag{4}$$

By the conservation of linear momentum, i.e. equilibrium for our static case, the following partial differential equation must be satisfied for the displacement $\mathbf{u}$

$$\begin{aligned} \mathrm{div}\mathbf{T} + \mathbf{b} &= \mathbf{0} \ , \\ \mathbf{T} &= \mathbb{C}[D\mathbf{u}] \ , \\ \mathbf{u} &= 0 \ \text{ on } \ \Gamma^u \ , \\ \mathbf{T}\mathbf{n} &= \mathbf{t}^p \ \text{ on } \ \Gamma^t \ . \end{aligned} \tag{5}$$

Substituting 3 into 5, applying the relationships described in 2, yields the Navier-Cauchy equation

$$\Delta\mathbf{u} + \frac{1}{1-2\,\nu}\,\nabla\mathrm{div}\mathbf{u} + \frac{1}{\mu}\mathbf{b} = \mathbf{0} \ . \tag{6}$$

In what follows, we consider the displacement due to a concentrated load $\mathbf{l}_y$ applied at point $\mathbf{y}$, i.e.

$$\mathbf{b}(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{y})\mathbf{l}_y \ . \tag{7}$$

Instead of solving this difficult problem, we approximate the concentrated load as a distributed load over a small circle $B(\mathbf{y}, \epsilon)$ of radius ϵ centered at $\mathbf{y}$ such that

$$\mathbf{b}(\mathbf{x}) = \alpha(\mathbf{x} - \mathbf{y})\mathbf{l}_y \ , \tag{8}$$

where

$$\alpha(\mathbf{x}) = \begin{cases} e^{\left(\frac{-1}{\epsilon^2 - |\mathbf{x}|^2}\right)} & \text{for } \mathbf{x} \in B(0, \epsilon) \ , \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

and hence $\mathbf{b}$ is only nonzero in the ball surrounding $\mathbf{y}$ and the net load is as desired, i.e.

$$\mathbf{l}_y = \int_{\Omega} \mathbf{b}(\mathbf{x}) \, dV = \int_{B(\mathbf{y},\epsilon)} \alpha(\mathbf{x} - \mathbf{y}) \, \mathbf{l}_y \, dv \ . \tag{10}$$

However, now we have a smooth $\mathbf{b}$ and thusly we can use the soon to be described analytical methods to obtain the $\mathbf{u}$ that solves 6 under the loading of 8.

2.2 Potential Derivations

We wish to represent the solution to $\mathbf{u}$ in 6 using potentials. This can be done using Boussinesq-Papkovitch-Neuber solution in [12], cf. Equation A.16

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\psi}(\mathbf{x}) - \frac{1}{4\,(1 - \nu)} \nabla((\mathbf{x} - \mathbf{y}) \cdot \boldsymbol{\psi}(\mathbf{x}) + \phi(\mathbf{x})) \ , \tag{11}$$

where $\boldsymbol{\psi}$ and ϕ satisfy the following

$$\begin{aligned} \Delta\boldsymbol{\psi}(\mathbf{x}) &= -\frac{1}{\mu} \mathbf{b}(\mathbf{x}) \ , \\ \Delta\phi(\mathbf{x}) &= \frac{1}{\mu} (\mathbf{x} - \mathbf{y}) \cdot \mathbf{b}(\mathbf{x}) \ . \end{aligned} \tag{12}$$

We see that the solution to 11 while enforcing the conditions outlined in 12 does indeed satisfy the Navier-Cauchy 6,

$$
\begin{aligned}
\Delta \mathbf{u} + \frac{1}{1-2\,\nu}\,\nabla \mathrm{div}\mathbf{u} + \frac{1}{\mu}\mathbf{b} &= \\
\Delta \boldsymbol{\psi} + \frac{1}{1-2\,\nu}\,\nabla \mathrm{div}\boldsymbol{\psi} - \frac{1}{4\,(1-\nu)}\,\Big\{ \Delta\nabla((\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\psi}+\phi) + & \\
\frac{1}{1-2\,\nu}\,\nabla \mathrm{div}\nabla((\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\psi}+\phi)\Big\} + \frac{1}{\mu}\mathbf{b} &= \\
\Delta \boldsymbol{\psi} + \frac{1}{1-2\,\nu}\,\nabla \mathrm{div}\boldsymbol{\psi} - \frac{1}{4\,(1-\nu)}\,\Big\{ \Delta\nabla((\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\psi}+\phi) & \\
+\frac{1}{1-2\,\nu}\,\nabla \Delta((\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\psi}+\phi)\Big\} + \frac{1}{\mu}\mathbf{b} &= \\
\Delta \boldsymbol{\psi} + \frac{1}{1-2\,\nu}\,\nabla \mathrm{div}\boldsymbol{\psi} - \frac{1}{2\,(1-2\,\nu)}\,[\nabla\Delta((\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\psi}+\phi)] + \frac{1}{\mu}\mathbf{b} &= \\
\Delta \boldsymbol{\psi} + \frac{1}{1-2\,\nu}\,\nabla \mathrm{div}\boldsymbol{\psi} - \frac{1}{2\,(1-2\,\nu)}\,[\nabla((\mathbf{x}-\mathbf{y})\cdot\Delta\boldsymbol{\psi} + 2\,\mathrm{div}\boldsymbol{\psi} + \Delta\phi)] + \frac{1}{\mu}\mathbf{b} &= \\
\underbrace{\Delta \boldsymbol{\psi}}_{=-\frac{1}{\mu}\,\mathbf{b}} + \frac{1}{1-2\,\nu}\,\nabla \mathrm{div}\boldsymbol{\psi} - \frac{1}{2\,(1-2\,\nu)}\left[\nabla(\underbrace{(\mathbf{x}-\mathbf{y})\cdot\Delta\boldsymbol{\psi}}_{=-\Delta\phi} + \Delta\phi + 2\,\mathrm{div}\boldsymbol{\psi})\right] + \frac{1}{\mu}\mathbf{b} &= \\
-\frac{1}{\mu}\mathbf{b} + \underbrace{\frac{1}{1-2\,\nu}\,\nabla \mathrm{div}\boldsymbol{\psi} - \frac{1}{2\,(1-2\,\nu)}\,[\nabla(2\,\mathrm{div}\boldsymbol{\psi})]}_{=\mathbf{0}} + \frac{1}{\mu}\mathbf{b} &= \\
-\frac{1}{\mu}\mathbf{b} + \frac{1}{\mu}\mathbf{b} &= \mathbf{0}\ . \qquad (13)
\end{aligned}
$$

So now the problem of finding $\mathbf{u}$ becomes that of finding a $\boldsymbol{\psi}(\mathbf{x})$ and $\phi(\mathbf{x})$ that satisfy 12. Therefore, we consider the logarithmic potential in 2-D, cf. [13] pg. 19,

$$
\psi(\mathbf{x}) = -\frac{1}{2\,\pi}\int_{\Omega} \phi(\mathbf{y}) \ln \frac{1}{|\mathbf{x}-\mathbf{y}|}\, dv_y \ , \qquad (14)
$$

where notably, for $\phi \in C^N(\Omega)$ with $N > 1$, we necessarily have $\psi \in C^{N+1}(\Omega)$, and the identity

$$
\Delta\psi(\mathbf{x}) = \phi(\mathbf{x}) \ . \qquad (15)
$$

Using 15, we define the potentials in 12 as

$$\begin{aligned}\boldsymbol{\psi}(\mathbf{x}) &= \frac{1}{2\,\pi\,\mu}\int_{B(\mathbf{y},\epsilon)} \mathbf{b}(\mathbf{y}) \ln \frac{1}{|\mathbf{x}-\mathbf{y}|}\, dv_y \ , \\ \phi(\mathbf{x}) &= -\frac{1}{2\,\pi\,\mu}\int_{B(\mathbf{y},\epsilon)} (\mathbf{x}-\mathbf{y})\cdot\mathbf{b}(\mathbf{y}) \ln \frac{1}{|\mathbf{x}-\mathbf{y}|}\, dv_y \ , \end{aligned} \tag{16}$$

where $\mathbf{b}$ is defined in 8.

Recall that by letting the radius $\epsilon \to 0$ in 8, we recover the concentrated load $\mathbf{b}$ of 7. And thus, the limit as $\epsilon \to 0$ in 16 yields the $\boldsymbol{\psi}$ and ϕ from which we obtain the displacement $\mathbf{u}$ of 11 corresponding to the $\mathbf{b}$ of 7. These limits render

$$\begin{aligned}\boldsymbol{\psi}(\mathbf{x}) &= \frac{1}{2\,\pi\,\mu}\mathbf{l}_y \ln \frac{1}{|\mathbf{x}-\mathbf{y}|} \ , \\ \phi(\mathbf{x}) &= 0 \ , \end{aligned} \tag{17}$$

for all $\mathbf{x} \in \Omega/\mathbf{y}$. Details of this derivation are provided in [14].

Now, upon defining

$$\begin{aligned}\mathbf{r}_y(\mathbf{x}) &= \mathbf{x}-\mathbf{y} \ , \\ r_y(\mathbf{x}) &= |\mathbf{r}_y(\mathbf{x})| \ , \\ \hat{\mathbf{r}}_y(\mathbf{x}) &= \frac{1}{r_y(\mathbf{x})}\mathbf{r}_y(\mathbf{x}) \ , \end{aligned} \tag{18}$$

and using

$$\begin{aligned} D\mathbf{r}_y(\mathbf{x}) &= \mathbf{I} \ , \\ \nabla r_y(\mathbf{x}) &= \frac{1}{r_y(\mathbf{x})}\,\mathbf{r}_y(\mathbf{x}) \\ &= \hat{\mathbf{r}}_y(\mathbf{x}) \ , \\ D\hat{\mathbf{r}}_y(\mathbf{x}) &= -\frac{1}{r_y^2(\mathbf{x})}\,\mathbf{r}_y(\mathbf{x})\otimes\nabla r_y(\mathbf{x}) + \frac{1}{r_y(\mathbf{x})}\,D\mathbf{r}_y(\mathbf{x}) \\ &= \frac{1}{r_y(\mathbf{x})}\left(\mathbf{I}-\hat{\mathbf{r}}_y(\mathbf{x})\otimes\hat{\mathbf{r}}_y(\mathbf{x})\right) \ , \end{aligned} \tag{19}$$

we obtain

$$
\begin{aligned}
\mathbf{u}_{\mathbf{l}_y}(\mathbf{x}) &= \boldsymbol{\psi}(\mathbf{x}) - \frac{1}{4\,(1-\nu)}\nabla((\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\psi}(\mathbf{x}) + \phi(\mathbf{x}))\\
&= \frac{1}{2\,\pi\,\mu}\left[\ln\,\frac{1}{r_y(\mathbf{x})}\mathbf{l}_y - \frac{1}{4\,(1-\nu)}\nabla(\mathbf{r}_y(\mathbf{x})\cdot\mathbf{l}_y\,\ln\,\frac{1}{r_y(\mathbf{x})})\right]\\
&= \frac{1}{2\,\pi\,\mu}\left[\ln\,\frac{1}{r_y(\mathbf{x})}\mathbf{l}_y - \frac{1}{4\,(1-\nu)}\left([D\mathbf{r}_y(\mathbf{x})]^T\mathbf{l}_y\,\ln\,\frac{1}{r_y(\mathbf{x})} + [D(\mathbf{l}_y\,\ln\,\frac{1}{r_y(\mathbf{x})})]^T\mathbf{r}_y(\mathbf{x})\right)\right]\\
&= \frac{1}{2\,\pi\,\mu}\left[\ln\,\frac{1}{r_y(\mathbf{x})}\mathbf{l}_y - \frac{1}{4\,(1-\nu)}\left(\mathbf{I}\,\ln\,\frac{1}{r_y(\mathbf{x})}\mathbf{l}_y + \left[\mathbf{l}_y\otimes\nabla\left(\ln\frac{1}{r_y(\mathbf{x})}\right)\right]^T\mathbf{r}_y(\mathbf{x})\right)\right]\\
&= \frac{1}{2\,\pi\,\mu}\left[\ln\,\frac{1}{r_y(\mathbf{x})}\mathbf{l}_y - \frac{1}{4\,(1-\nu)}\left(\ln\,\frac{1}{r_y(\mathbf{x})}\mathbf{l}_y + \left[\nabla\left(\ln\,\frac{1}{r_y(\mathbf{x})}\right)\otimes\mathbf{l}_y\right]\mathbf{r}_y(\mathbf{x})\right)\right]\\
&= \frac{1}{2\,\pi\,\mu}\left[\ln\,\frac{1}{r_y(\mathbf{x})}\mathbf{l}_y - \frac{1}{4\,(1-\nu)}\left(\ln\,\frac{1}{r_y(\mathbf{x})}\mathbf{l}_y + \left[-\frac{1}{r_y(\mathbf{x})}\nabla r_y(\mathbf{x})\otimes\mathbf{l}_y\right]\mathbf{r}_y(\mathbf{x})\right)\right]\\
&= \frac{1}{2\,\pi\,\mu}\left[\ln\,\frac{1}{r_y(\mathbf{x})}\mathbf{l}_y - \frac{1}{4\,(1-\nu)}\left(\ln\,\frac{1}{r_y(\mathbf{x})}\mathbf{l}_y + \left[-\frac{1}{r_y(\mathbf{x})}\hat{\mathbf{r}}_y(\mathbf{x})\otimes\mathbf{l}_y\right]\mathbf{r}_y(\mathbf{x})\right)\right]\\
&= \frac{1}{2\,\pi\,\mu}\left[\ln\,\frac{1}{r_y(\mathbf{x})}\mathbf{l}_y - \frac{1}{4\,(1-\nu)}\left(\ln\,\frac{1}{r_y(\mathbf{x})}\mathbf{l}_y + \left[-\frac{1}{r_y(\mathbf{x})}\hat{\mathbf{r}}_y(\mathbf{x})\otimes\mathbf{r}_y(\mathbf{x})\right]\mathbf{l}_y\right)\right]\\
&= \frac{1}{2\,\pi\,\mu}\left[\frac{3-4\,\nu}{4\,(1-\nu)}\,\ln\,\frac{1}{r_y(\mathbf{x})}\,\mathbf{I} + \frac{1}{4\,(1-\nu)}\,\hat{\mathbf{r}}_y(\mathbf{x})\otimes\hat{\mathbf{r}}_y(\mathbf{x})\right]\,\mathbf{l}_y\\
&= \frac{1}{8\,\pi\,\mu\,(1-\nu)}\left[(3-4\,\nu)\,\ln\,\frac{1}{r_y(\mathbf{x})}\,\mathbf{I} + \hat{\mathbf{r}}_y(\mathbf{x})\otimes\hat{\mathbf{r}}_y(\mathbf{x})\right]\mathbf{l}_y\\
&= \mathcal{U}_y(\mathbf{x})\,\mathbf{l}_y\ ,
\end{aligned}
\tag{20}
$$

where

$$
\mathcal{U}_y(\mathbf{x}) = \frac{C}{\mu}\left[(3-4\,\nu)\,\ln\,\frac{1}{r_y(\mathbf{x})}\,\mathbf{I} + \hat{\mathbf{r}}_y(\mathbf{x})\otimes\hat{\mathbf{r}}_y(\mathbf{x})\right]\ , \tag{21}
$$

with

$$
C = \frac{1}{8\,\pi\,(1-\nu)}\ . \tag{22}
$$

The displacement potential $\mathcal{U}_y$ as seen above, can be used to find the displacement at $\mathbf{x}$ corresponding to the concentrated load $\mathbf{l}_y$ applied at $\mathbf{y}$. The subscript y emphasizes the definition of r_y, cf. 18.

It is also of interest, as will be seen in the following sections, to obtain the traction potential i.e. a 20 like mapping giving the traction on the surface element at $\mathbf{x}$ with normal $\mathbf{n}(\mathbf{x})$ corresponding to a concentrated load $\mathbf{l}_y$ applied at $\mathbf{y}$. To obtain this potential we first obtain the displacement gradient which follows from expanding 20 as

$$\mathbf{u}_{\mathbf{l}_y}(\mathbf{x}) \;=\; \frac{C}{\mu}\left[(3-4\,\nu)\ln\frac{1}{r_y(\mathbf{x})}\,\mathbf{l}_y + (\hat{\mathbf{r}}_y(\mathbf{x})\cdot\mathbf{l}_y)\,\hat{\mathbf{r}}_y(\mathbf{x})\right]\;. \tag{23}$$

And, since the derivative at $\mathbf{x}$, i.e. $D\mathbf{u}(\mathbf{x})$ acting on the increment $\mathbf{a}$, i.e. $D\mathbf{u}(\mathbf{x})\,[\mathbf{a}]$ is a vector because by definition $D\mathbf{u}(\mathbf{x})\,\mathbf{a} = \mathbf{u}(\mathbf{x}+\epsilon\,\mathbf{a}) - \mathbf{u}(\mathbf{x}) - o(|\mathbf{a}|)$ which is a vector, we have

$$\begin{aligned}
D\mathbf{u}_{\mathbf{l}_y}[\mathbf{a}] &= \frac{C}{\mu}\left(-(3-4\,\nu)\,\frac{1}{r_y}\,(\nabla r_y\cdot\mathbf{a})\mathbf{l}_y + (D\hat{\mathbf{r}}_y[\mathbf{a}]\cdot\mathbf{l}_y)\,\hat{\mathbf{r}}_y + (\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,D\hat{\mathbf{r}}_y[\mathbf{a}]\right)\\
&= \frac{C}{\mu}\left(-(3-4\,\nu)\,\frac{1}{r_y}\,(\mathbf{l}_y\otimes\nabla r_y)\,\mathbf{a} + (\mathbf{a}\cdot(D\hat{\mathbf{r}}_y)^T\mathbf{l}_y)\,\hat{\mathbf{r}}_y + (\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,D\hat{\mathbf{r}}_y[\mathbf{a}]\right)\\
&= \frac{C}{\mu}\left(-(3-4\,\nu)\,\frac{1}{r_y}\,(\mathbf{l}_y\otimes\nabla r_y) + (\hat{\mathbf{r}}_y\otimes(D\hat{\mathbf{r}}_y)^T\mathbf{l}_y)\; + (\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,D\hat{\mathbf{r}}_y\right)\,\mathbf{a}\;,
\end{aligned} \tag{24}$$

where the argument $\mathbf{x}$ has been suppressed for conciseness. By factoring out the $\mathbf{a}$, we obtain the desired expression for $D\mathbf{u}_{\mathbf{l}_y}$ as well as its transpose and trace, i.e.

$$\begin{aligned}
D\mathbf{u}_{\mathbf{l}_y} &= \frac{C}{\mu\,r_y}\,\left(-(3-4\,\nu)\,(\mathbf{l}_y\otimes\hat{\mathbf{r}}_y) + \hat{\mathbf{r}}_y\otimes((\mathbf{I}-\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y)\,\mathbf{l}_y) + (\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,(\mathbf{I}-\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y)\right)\\
&= \frac{C}{\mu\,r_y}\,\left(-(3-4\,\nu)\,(\mathbf{l}_y\otimes\hat{\mathbf{r}}_y) + \hat{\mathbf{r}}_y\otimes\mathbf{l}_y - (\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y + (\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\mathbf{I} - (\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y\right)\\
&= \frac{C}{\mu\,r_y}\,\left(-(3-4\,\nu)\,(\mathbf{l}_y\otimes\hat{\mathbf{r}}_y) + \hat{\mathbf{r}}_y\otimes\mathbf{l}_y - 2\,(\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y + (\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\mathbf{I}\right)\;,\\
(D\mathbf{u}_{\mathbf{l}_y})^T &= \frac{C}{\mu\,r_y}\,\left(-(3-4\,\nu)\,(\hat{\mathbf{r}}_y\otimes\mathbf{l}_y) + \mathbf{l}_y\otimes\hat{\mathbf{r}}_y - 2\,(\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y + (\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\mathbf{I}\right)\;,\\
\mathrm{tr}D\mathbf{u}_{\mathbf{l}_y} &= \frac{C}{\mu\,r_y}\,\left(-(3-4\,\nu)+1-2+2\right)\,\hat{\mathbf{r}}_y\cdot\mathbf{l}_y\\
&= \frac{-2\,C\,(1-2\,\nu))}{\mu\,r_y}\,\hat{\mathbf{r}}_y\cdot\mathbf{l}\;.
\end{aligned} \tag{25}$$

Having the displacement gradient, we next define the stress potential, i.e., a 20 like mapping giving the stress at $\mathbf{x}$ due to the point load $\mathbf{l}_y$ applied at $\mathbf{y}$. Combining 3 and 25 yields

$$
\begin{aligned}
\mathbf{T}_{\mathbf{l}_y} &= \mu\,(D\mathbf{u}_{\mathbf{l}_y} + (D\mathbf{u}_{\mathbf{l}_y})^T) + \lambda\,\mathrm{tr}(D\mathbf{u}_{\mathbf{l}_y})\,\mathbf{I} \\
&= \frac{C}{r_y}\,\Big[(-(3-4\,\nu)+1)(\mathbf{l}_y\otimes\hat{\mathbf{r}}_y + \hat{\mathbf{r}}_y\otimes\mathbf{l}_y) - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y + 2\,(\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\mathbf{I} \\
&\quad + \frac{-2\,\lambda\,(1-2\,\nu)}{\mu}\,(\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\mathbf{I}\Big] \\
&= \frac{C}{r_y}\,\Big[-2\,(1-2\,\nu)(\mathbf{l}_y\otimes\hat{\mathbf{r}}_y + \hat{\mathbf{r}}_y\otimes\mathbf{l}_y) - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y + \frac{2\,\mu - 2\,\lambda\,(1-2\,\nu))}{\mu}\,(\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\mathbf{I}\Big] \\
&= \frac{C}{r_y}\,[-2\,(1-2\,\nu)(\mathbf{l}_y\otimes\hat{\mathbf{r}}_y + \hat{\mathbf{r}}_y\otimes\mathbf{l}_y) - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y + 2\,(1-2\,\nu))\,(\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\mathbf{I}] \\
&= \frac{C}{r_y}\,[-2\,(1-2\,\nu)(\mathbf{l}_y\otimes\hat{\mathbf{r}}_y + \hat{\mathbf{r}}_y\otimes\mathbf{l}_y - (\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\mathbf{I}) - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y] \\
&= \Sigma_y\,\mathbf{l}_y \quad , \qquad (26)
\end{aligned}
$$

where we used 2 and define the stress potential Σ_y as

$$
\Sigma_y = -\frac{2\,C}{r_y}\,[(1-2\,\nu)(\mathbb{T}_3\,\hat{\mathbf{r}}_y\otimes\mathbf{I} + \hat{\mathbf{r}}_y\otimes\mathbf{I} - \mathbf{I}\otimes\hat{\mathbf{r}}_y) + 2\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y] \quad , \qquad (27)
$$

with the transpose $\mathbb{T}_3$ defined such that $\mathbb{T}_3[\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}] = \mathbf{b}\otimes\mathbf{c}\otimes\mathbf{a}$. Finally, the traction potential is obtained using the stress of 27 acting on the normal $\mathbf{n}(\mathbf{x})$, i.e.

$$
\begin{aligned}
\mathbf{t}^{\mathbf{n}_x}_{\mathbf{l}_y} &= (\Sigma_y\,\mathbf{l}_y)\,\mathbf{n} \\
&= \frac{C}{r_y}\,[-2\,(1-2\,\nu)(\mathbf{l}_y\otimes\hat{\mathbf{r}}_y + \hat{\mathbf{r}}_y\otimes\mathbf{l}_y - (\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\mathbf{I}) - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y]\,\mathbf{n} \\
&= \frac{C}{r_y}\,[-2\,(1-2\,\nu)((\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{l}_y + (\mathbf{l}\cdot\mathbf{n})\,\hat{\mathbf{r}}_y - (\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,\mathbf{n}) - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{l}_y)\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y] \\
&= \frac{C}{r_y}\,[-2\,(1-2\,\nu)((\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{I} + \hat{\mathbf{r}}_y\otimes\mathbf{n} - \mathbf{n}\otimes\hat{\mathbf{r}}_y) - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y]\,\mathbf{l}_y \\
&= \mathcal{T}^x_y\,\mathbf{l}_y \quad , \qquad (28)
\end{aligned}
$$

where

$$
\mathcal{T}^x_y \;=\; -\frac{2\,C}{r_y}\,[(1-2\,\nu)((\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{I} + \hat{\mathbf{r}}_y\otimes\mathbf{n} - \mathbf{n}\otimes\hat{\mathbf{r}}_y) + 2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y] \quad . \qquad (29)
$$

The superscript x emphasizes the normal definition, i.e. $\mathbf{n}(\mathbf{x})$.

Using the reciprocal theorem we now relate the responses of our original problem 5 (under the assumption of zero body force) to our point load problem, giving us Somigliana's Identity

$$\int_{\partial\Omega} \mathbf{u}_{\mathbf{l}_y} \cdot \mathbf{t}^{\mathbf{n}}\, ds = \int_{\Omega} \mathbf{u} \cdot \delta(\mathbf{x}-\mathbf{y})\, \mathbf{l}_y\, dv + \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{t}^{\mathbf{n}}_{\mathbf{l}_y}\, dS \ ,$$
$$\int_{\partial\Omega} \mathcal{U}_y\, \mathbf{l}_y \cdot \mathbf{t}^{\mathbf{n}}\, ds = \mathbf{u}(\mathbf{y}) \cdot \mathbf{l}_y + \int_{\partial\Omega} \mathbf{u} \cdot \mathcal{T}^x_y\, \mathbf{l}_y\, dS \ , \tag{30}$$

where we use 20 and 28. Note that on the left hand side $\mathbf{t}^{\mathbf{n}} = \mathbf{t}^p$ on Γ^t and on the right hand side $\mathbf{u} = 0$ on Γ^u. Using the symmetry of $\mathcal{U}_y$ in 30 yields

$$\begin{aligned} \mathbf{u}(\mathbf{y}) \cdot \mathbf{l}_y &= \int_{\partial\Omega} \left(\mathcal{U}_y\, \mathbf{l}_y \cdot \mathbf{t}^{\mathbf{n}} - \mathbf{u} \cdot \mathcal{T}^x_y\, \mathbf{l}_y \right) dS \\ &= \int_{\partial\Omega} \left(\mathcal{U}_y \mathbf{t}^{\mathbf{n}} - [\mathcal{T}^x_y]^T \mathbf{u} \right) dS \cdot \mathbf{l}_y \ , \end{aligned} \tag{31}$$

whence, cf. Tuhkuri 1,

$$\mathbf{u}(\mathbf{y}) = \int_{\partial\Omega} \left(\mathcal{U}_y(\mathbf{x}) \mathbf{t}^{\mathbf{n}}(\mathbf{x}) - [\mathcal{T}^x_y(\mathbf{x})]^T \mathbf{u}(\mathbf{x}) \right) dS \ . \tag{32}$$

The above BIE 32 relates the displacement at $\mathbf{y} \in \Omega$ to the displacement and traction on the boundary $\partial\Omega$. To compute displacement $\mathbf{u}(\mathbf{y})$ for $\mathbf{y} \in \partial\Omega$, i.e. for $\mathbf{y}$ on the boundary, we apply the domain extension technique [2, 5, 15].

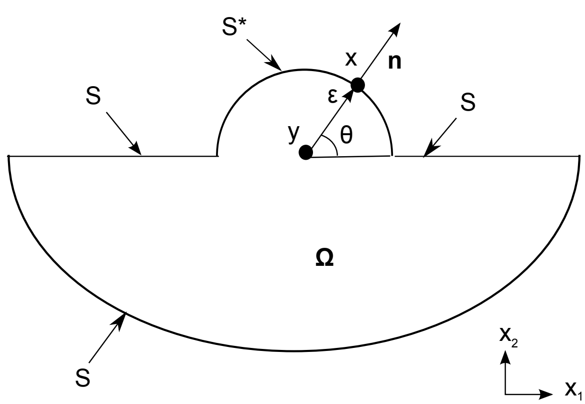


Figure 2: Domain extension for $\mathbf{y} \rightarrow \mathbf{y} \in \partial\Omega$.

Referencing Figure 2, we let the boundary $\partial\Omega$ in 32 become $S^A = S + S^*$, where S^* is the semi-circle of radius ϵ centered at point $\mathbf{y}$ and S is the remainder of the boundary i.e. $S = \partial\Omega / B(\mathbf{y}, \epsilon)$. In the limit $\epsilon \rightarrow 0$, we have $S^* \rightarrow 0$, $S^A \rightarrow \partial\Omega$ and $\mathbf{y} \rightarrow \mathbf{y} \in \partial\Omega$ as desired. Expressing 32 for the given boundary S^A yields

$$\int_{S_A} \left(\mathcal{U}_y(\mathbf{x})\mathbf{t}^{\mathbf{n}}(\mathbf{x}) - [\mathcal{T}_y^x(\mathbf{x})]^T\mathbf{u}(\mathbf{x})\right) \, dS = \lim_{\epsilon\to 0}\int_S \mathcal{U}_y(\mathbf{x})\mathbf{t}^{\mathbf{n}}(\mathbf{x})\, dS - \lim_{\epsilon\to 0}\int_S [\mathcal{T}_y^x(\mathbf{x})]^T\mathbf{u}(\mathbf{x})\, dS + \lim_{\epsilon\to 0}\int_{S^*} \mathcal{U}_y(\mathbf{x})\mathbf{t}^{\mathbf{n}}(\mathbf{x})\, dS - \lim_{\epsilon\to 0}\int_{S^*} [\mathcal{T}_y^x(\mathbf{x})]^T\mathbf{u}(\mathbf{x})\, dS \ . \quad (33)$$

The first two integrals on the right hand side of 33 contain weak and strong singularities, respectively. Their evaluation is addressed in the Discretization and Integration Section. The last two integrals are evaluated using cylindrical coordinate systems whereupon we write

$$\begin{aligned} \mathbf{r}_y(\epsilon,\theta) &= \epsilon\cos\theta\,\mathbf{e}_1 + \epsilon\sin\theta\,\mathbf{e}_2 \ , \\ r_y(\epsilon,\theta) &= \epsilon \ , \\ \mathbf{n}(\epsilon,\theta) &= \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2 \ , \\ \hat{\mathbf{r}}_y(\epsilon,\theta) &= \mathbf{n} \ , \end{aligned} \quad (34)$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the standard basis vectors. Noting that ϵ is constant over the region of integration, the third integral in 33 reduces to

$$\begin{aligned} \lim_{\epsilon\to 0}\int_{S^*} \mathcal{U}_y(\mathbf{x})\, dS &= \lim_{\epsilon\to 0}\int_{S^*} \frac{C}{\mu}\left[(3-4\,\nu)\,\ln\frac{1}{\epsilon}\mathbf{I} + \hat{\mathbf{r}}_y(\epsilon,\theta)\otimes\hat{\mathbf{r}}_y(\epsilon,\theta)\right]\epsilon d\theta \quad (35) \\ &= \lim_{\epsilon\to 0}\int_0^\pi \frac{C}{\mu}(3-4\,\nu)\,\ln\frac{1}{\epsilon}\mathbf{I}\,\epsilon\, d\theta + \lim_{\epsilon\to 0}\int_0^\pi \frac{C}{\mu}\hat{\mathbf{r}}_y(\epsilon,\theta)\otimes\hat{\mathbf{r}}_y(\epsilon,\theta)\,\epsilon\, d\theta \\ &= \lim_{\epsilon\to 0}\epsilon\ln\frac{1}{\epsilon}\mathbf{I}\int_0^\pi \frac{C}{\mu}(3-4\,\nu)\, d\theta + 0 \\ &= \lim_{\epsilon\to 0}\epsilon\,\mathbf{I}\int_0^\pi \frac{C}{\mu}(3-4\,\nu)\, d\theta \\ &= 0 \ , \end{aligned}$$

where we used L'Hospital's rule to obtain $\lim_{\epsilon\to 0}\frac{-\ln\epsilon}{\frac{1}{\epsilon}} = \lim_{\epsilon\to 0}\epsilon$. We next look at each component of the last integral in 33 and note that on the surface S^* $\mathcal{T}_y^x(\mathbf{x})$ is symmetric since $\mathbf{n}(\mathbf{x}) = \hat{\mathbf{r}}_y(\mathbf{x})$. Substituting 34 into 29 yields

$$\lim_{\epsilon\to 0}\int_{S^*}[\mathcal{T}_y^x(\mathbf{x})]_{11}\,dS = \lim_{\epsilon\to 0}\int_0^\pi \frac{-1}{4\epsilon\pi(1-\nu)}\left[(1-2\,\nu)+2\,\cos^2(\theta)\right]\epsilon\,d\theta \tag{36}$$
$$= \frac{-1}{4\pi(1-\nu)}\int_0^\pi\left[(1-2\,\nu)+2\,\cos^2(\theta)\right]\,d\theta$$
$$= \frac{-1}{4\pi(1-\nu)}[(1-2\,\nu)\theta+\theta+\sin(\theta)\cos(\theta)]_0^\pi$$
$$= \frac{-2\pi(1-\nu)}{4\pi(1-\nu)}$$
$$= -\frac{1}{2}\ ,$$

$$\lim_{\epsilon\to 0}\int_{S^*}[\mathcal{T}_y^x(\mathbf{x})]_{22}\,dS = \lim_{\epsilon\to 0}\int_0^\pi \frac{-1}{4\epsilon\pi(1-\nu)}\left[(1-2\,\nu)+2\,\sin^2(\theta)\right]\epsilon\,d\theta \tag{37}$$
$$= \frac{-1}{4\pi(1-\nu)}\int_0^\pi\left[(1-2\,\nu)+2\,\sin^2(\theta)\right]\,d\theta$$
$$= \frac{-1}{4\pi(1-\nu)}[(1-2\,\nu)\theta+\theta-\sin(\theta)\cos(\theta)]_0^\pi$$
$$= \frac{-2\pi(1-\nu)}{4\pi(1-\nu)}$$
$$= -\frac{1}{2}\ ,$$

$$\lim_{\epsilon\to 0}\int_{S^*}[\mathcal{T}_y^x(\mathbf{x})]_{12}\,dS = \lim_{\epsilon\to 0}\int_{S^*}[\mathcal{T}_y^x(\mathbf{x})]_{21}\,dS \tag{38}$$
$$= \lim_{\epsilon\to 0}\int_0^\pi \frac{-1}{4\epsilon\pi(1-\nu)}\left[2\,\sin(\theta)\cos(\theta)\right]\epsilon\,d\theta$$
$$= \frac{-1}{4\pi(1-\nu)}\int_0^\pi\left[2\,\sin(\theta)\cos(\theta)\right]\,d\theta$$
$$= \frac{-1}{4\pi(1-\nu)}[-\frac{1}{2}\cos(2\theta)]_0^\pi$$
$$= 0\ .$$

Thus, we see that

$$\lim_{\epsilon\to 0}\int_{S^*}[\mathcal{T}_y^x(\mathbf{x})]^T\,dS = -\frac{1}{2}\,\mathbf{I}\ . \tag{39}$$

We now combine the results of 33 and Soligiama's identity 32 to express the displacement at the boundary point $\mathbf{y}$

$$\frac{1}{2}\,\mathbf{u}(\mathbf{y}) = \int_{\partial\Omega}\mathcal{U}_y(\mathbf{x})\mathbf{t}^{\mathbf{n}}(\mathbf{x})\,dS - \fint_{\partial\Omega}[\mathcal{T}_y^x(\mathbf{x})]^T\,\mathbf{u}(\mathbf{x})\,dS\ , \tag{40}$$

where we use $\lim_{\epsilon\to 0} S = \partial\Omega$. In 36-38 we have assumed $\mathbf{y}$ is a regular point of $\partial\Omega$ (i.e. smooth surface). If not, then the $(0, \pi)$ limits of integration need to be appropriately modified to accommodate e.g. corner points.

The Cauchy Principal Value (CPV) integral is defined as

$$\fint_a^b f(x)\, dx = \lim_{\epsilon\to 0} \left[\int_a^{\lambda-\epsilon} f(x)\, dx + \int_{\lambda+\epsilon}^b f(x)\, dx \right] , \tag{41}$$

cf. [16] 1.2, i.e. the finite part of the improper integral with the singularity at $\lambda \in (a, b)$. In reference to 40, the integrand $[\mathcal{T}_y^x(\mathbf{x})]^T\, \mathbf{u}(\mathbf{x})$ becomes singular when $\mathbf{x} \to \mathbf{y}$ and hence the need for the CPV.

3 Dual Boundary Element Formulation

3.1 Fundamental Solution Derivatives

We wish to solve the problem illustrated in Figure 3, where the far field traction $\mathbf{t}^\infty$ is applied to the half-space with a crack on its free surface. Let S_c^+ and S_c^- be the upper and lower crack faces respectively, and S' be the remaining boundary such that the entire boundary $\partial\Omega$ is given by $\partial\Omega = S_c^+ \cup S_c^- \cup S'$. [1]

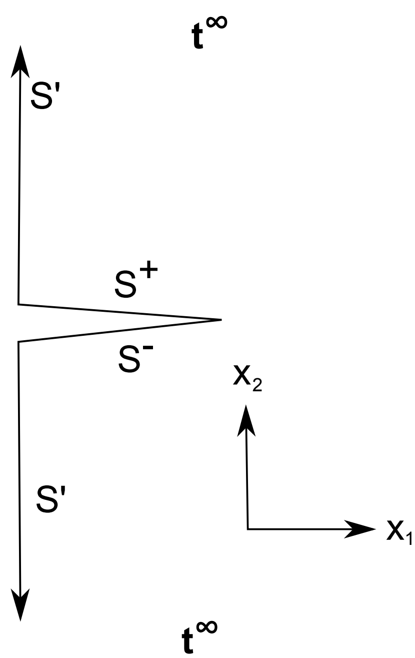


Figure 3: Semi-infinite edge crack.

As stated previously, the discretization of the BIE over the two crack faces S^+ and S^- leads to a system of linearly dependent equations. It was first shown by [7] that taking the gradient of the displacement at the crack face $\mathbf{y}$, i.e. $D[\mathbf{u}(\mathbf{y})]$[2] and applying 3 and 4 we obtain the so-called traction integral equation.

[1]There is a gap drawn in Figure 3 to depict the two surfaces, even though physically the two crack faces S^+ and S^- are coincidental.

[2]We evaluate this derivative with respect to $\mathbf{x}$ and using the fact that $\mathbf{r}_y(\mathbf{x}) = \mathbf{x} - \mathbf{y}$, negate our results to obtain the desired derivatives in 42.

$$\begin{aligned}
\frac{1}{2}\mathbb{C}[D\mathbf{u}(\mathbf{y})]\,\mathbf{n}(\mathbf{y}) &= \left[\int_{\partial\Omega}\mathbb{C}[D(\mathcal{U}_y(\mathbf{x}))]\mathbf{t}^{\mathbf{n}}(\mathbf{x})\,dS - \fint_{\partial\Omega}\mathbb{C}[D(\mathcal{T}_y^x(\mathbf{x}))^T]\,\mathbf{u}(\mathbf{x})\,dS\right]\mathbf{n}(\mathbf{y}) \qquad (42)\\
\frac{1}{2}\mathbf{t}^{\mathbf{n}}(\mathbf{y}) &= -\left[\fint_{\partial\Omega}\Sigma_y(\mathbf{x})\mathbf{t}^{\mathbf{n}}(\mathbf{x})\,dS - \fint_{\partial\Omega}\mathcal{Z}_y^x(\mathbf{x})\,\mathbf{u}(\mathbf{x})\,dS\right]\mathbf{n}(\mathbf{y})\\
&= -\left[\fint_{\partial\Omega}\mathcal{D}_y^y(\mathbf{x})\mathbf{t}^{\mathbf{n}}(\mathbf{x})\,dS - \fint_{\partial\Omega}\mathcal{S}_y^{xy}(\mathbf{x})\,\mathbf{u}(\mathbf{x})\,dS\right]\ .
\end{aligned}$$

In the above, Σ_y and $\mathcal{Z}_y^x$ are third-order tensors whereas $\mathcal{D}_y^y$ and $\mathcal{S}_y^{xy}$ are second-order tensors which incorporate the normal $\mathbf{n}(\mathbf{y})$. Notice that Σ_y has already been computed in the derivation of the traction potential $\mathcal{T}_y^x$, cf. 27. Therefore, only the gradient $D(\mathcal{T}_y^x(\mathbf{x}))^T$ needs to be computed. This derivation is obtained by taking the derivative of $(\mathcal{T}_y^x(\mathbf{x}))^T\,\mathbf{b}$ acting on the increment $\mathbf{a}$, i.e.

$$
\begin{aligned}
D\left[(\mathcal{T}_y^x)^T\,\mathbf{b}\right]\,\mathbf{a} \;\; &= \;\; -\frac{C}{r_y^2}\,\Big[-2\,(1-2\,\nu)((\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}-(\mathbf{n}\cdot\mathbf{b})\,\hat{\mathbf{r}}_y+(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n})- \\
&\quad 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\Big]\Big(\nabla r_y\cdot\mathbf{a}\Big)+\frac{C}{r_y}\,\Big[-2\,(1-2\,\nu)((D\hat{\mathbf{r}}_y[\mathbf{a}]\cdot\mathbf{n})\,\mathbf{b}- \\
&\quad (\mathbf{n}\cdot\mathbf{b})\,D\hat{\mathbf{r}}_y[\mathbf{a}]+(D\hat{\mathbf{r}}_y[\mathbf{a}]\cdot\mathbf{b})\,\mathbf{n})-4\,(D\hat{\mathbf{r}}_y[\mathbf{a}]\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y- \\
&\quad 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(D\hat{\mathbf{r}}_y[\mathbf{a}]\cdot\mathbf{b})\,\hat{\mathbf{r}}_y-4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,D\hat{\mathbf{r}}_y[\mathbf{a}]\,\Big] \\
&= \;\; -\frac{C}{r_y^2}\,\Big[-2\,(1-2\,\nu)((\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}-(\mathbf{n}\cdot\mathbf{b})\,\hat{\mathbf{r}}_y+(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n})- \\
&\quad 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y]\,(\nabla r_y\cdot\mathbf{a})+\frac{C}{r_y}\,\big[-2\,(1-2\,\nu)((\mathbf{a}\cdot(D\hat{\mathbf{r}}_y)^T\mathbf{n})\,\mathbf{b}- \\
&\quad (\mathbf{n}\cdot\mathbf{b})\,D\hat{\mathbf{r}}_y[\mathbf{a}]+(\mathbf{a}\cdot(D\hat{\mathbf{r}}_y)^T\mathbf{b})\,\mathbf{n})-4\,(\mathbf{a}\cdot(D\hat{\mathbf{r}}_y)^T\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y- \\
&\quad 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\mathbf{a}\cdot(D\hat{\mathbf{r}}_y)^T\mathbf{b})\,\hat{\mathbf{r}}_y-4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,D\hat{\mathbf{r}}_y[\mathbf{a}]\,\Big] \\
&= \;\; -\frac{C}{r_y^2}\,\Big[-2\,(1-2\,\nu)((\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}-(\mathbf{n}\cdot\mathbf{b})\,\hat{\mathbf{r}}_y+(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n})- \\
&\quad 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}]\otimes\nabla r_y+\frac{C}{r_y}\,\big[-2\,(1-2\,\nu)(\mathbf{b}\otimes(D\hat{\mathbf{r}}_y)^T\mathbf{n} \\
&\quad -(\mathbf{n}\cdot\mathbf{b})\,D\hat{\mathbf{r}}_y+\mathbf{n}\otimes(D\hat{\mathbf{r}}_y)^T\mathbf{b})-4\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes(D\hat{\mathbf{r}}_y)^T\mathbf{n}- \\
&\quad 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes(D\hat{\mathbf{r}}_y)^T\mathbf{b}-4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,D\hat{\mathbf{r}}_y]\,\Big]\,\mathbf{a} \\
&= \;\; -\frac{C}{r_y^2}\,\Big[-2\,(1-2\,\nu)((\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}-(\mathbf{n}\cdot\mathbf{b})\,\hat{\mathbf{r}}_y+(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n})- \\
&\quad 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y]\otimes\nabla r_y+ \\
&\quad \frac{C}{r_y}\,[-2\,(1-2\,\nu)(\mathbf{b}\otimes\mathbf{n}-(\mathbf{n}\cdot\mathbf{b})\,\mathbf{I}+\mathbf{n}\otimes\mathbf{b})- \\
&\quad 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\mathbf{n}-4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes\mathbf{b}-4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{I}]\;D\hat{\mathbf{r}}_y\,\Big]\,\mathbf{a} \\
&= \;\; \frac{C}{r_y^2}\,\Big\{2\,(1-2\,\nu)((\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}\otimes\hat{\mathbf{r}}_y-(\mathbf{n}\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y+(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n}\otimes\hat{\mathbf{r}}_y)+ \\
&\quad 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y+[-2\,(1-2\,\nu)(\mathbf{b}\otimes\mathbf{n}-(\mathbf{n}\cdot\mathbf{b})\,\mathbf{I}+\mathbf{n}\otimes\mathbf{b})- \\
&\quad 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\mathbf{n}-4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes\mathbf{b}-4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{I}]\;(\mathbf{I}-\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y)\Big\}\,\mathbf{a}
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{C}{r_y^2} \Big\{ 2\,(1-2\,\nu)((\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}\otimes\hat{\mathbf{r}}_y - (\mathbf{n}\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y + (\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n}\otimes\hat{\mathbf{r}}_y) + \\
& 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y + [-2\,(1-2\,\nu)(\mathbf{b}\otimes\mathbf{n} - (\mathbf{n}\cdot\mathbf{b})\,\mathbf{I} + \mathbf{n}\otimes\mathbf{b}) - \\
& 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\mathbf{n} - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes\mathbf{b} - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{I}] - \\
& [-2\,(1-2\,\nu)(\mathbf{b}\otimes\mathbf{n} - (\mathbf{n}\cdot\mathbf{b})\,\mathbf{I} + \mathbf{n}\otimes\mathbf{b}) - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\mathbf{n} - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes\mathbf{b} - \\
& 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{I}]\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y \Big\}\,\mathbf{a}
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{C}{r_y^2} \Big\{ 2\,(1-2\,\nu)((\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}\otimes\hat{\mathbf{r}}_y - (\mathbf{n}\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y + (\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n}\otimes\hat{\mathbf{r}}_y) + \\
& 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y + [-2\,(1-2\,\nu)(\mathbf{b}\otimes\mathbf{n} - (\mathbf{n}\cdot\mathbf{b})\,\mathbf{I} + \mathbf{n}\otimes\mathbf{b}) - \\
& 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\mathbf{n} - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes\mathbf{b} - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{I}] - \\
& [-2\,(1-2\,\nu)((\mathbf{n}\cdot\hat{\mathbf{r}}_y)\,\mathbf{b}\otimes\hat{\mathbf{r}}_y - (\mathbf{n}\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y + (\mathbf{b}\cdot\hat{\mathbf{r}}_y)\mathbf{n}\otimes\hat{\mathbf{r}}_y) - \\
& 4\,(\mathbf{n}\cdot\hat{\mathbf{r}}_y)\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y - 4\,(\mathbf{b}\cdot\hat{\mathbf{r}}_y)\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y] \Big\}\,\mathbf{a}\ . \quad (43)
\end{aligned}
$$

And, therefore, we have

$$
\begin{aligned}
D\left[(\mathcal{T}_y^x)^T\,\mathbf{b}\right] \; = \; & \frac{C}{r_y^2} \Big\{ 2\,(1-2\,\nu)(2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}\otimes\hat{\mathbf{r}}_y - 2\,(\mathbf{n}\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}} \\
& +2\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n}\otimes\hat{\mathbf{r}}_y - \mathbf{b}\otimes\mathbf{n} + (\mathbf{n}\cdot\mathbf{b})\,\mathbf{I} - \mathbf{n}\otimes\mathbf{b}) + \\
& 16\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\mathbf{n} - \\
& 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes\mathbf{b} - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{I} \Big\}\ ,
\end{aligned}
$$

$$
\begin{aligned}
(D\left[(\mathcal{T}_y^x)^T\,\mathbf{b}\right])^T \; = \; & \frac{C}{r_y^2} \Big\{ 2\,(1-2\,\nu)(2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes\mathbf{b} - 2\,(\mathbf{n}\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}} \\
& +2\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\mathbf{n} - \mathbf{b}\otimes\mathbf{n} + (\mathbf{n}\cdot\mathbf{b})\,\mathbf{I} - \mathbf{n}\otimes\mathbf{b}) + \\
& 16\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n}\otimes\hat{\mathbf{r}}_y - \\
& 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}\otimes\hat{\mathbf{r}}_y - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{I} \Big\}\ ,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr} D\left[(\mathcal{T}_y^x)^T\,\mathbf{b}\right] &= \frac{C}{r_y^2}\,\Big\{2\,(1-2\,\nu)(2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}\cdot\hat{\mathbf{r}}_y - 2\,(\mathbf{n}\cdot\mathbf{b}) \\
&\quad +2\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n}\cdot\hat{\mathbf{r}}_y - \mathbf{b}\cdot\mathbf{n} + 2\,(\mathbf{n}\cdot\mathbf{b}) - \mathbf{n}\cdot\mathbf{b}) + \\
&\quad 16\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b}) - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n}\cdot\hat{\mathbf{r}}_y - \\
&\quad 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}\cdot\hat{\mathbf{r}}_y - 8\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\Big\} \\
&= \frac{C}{r_y^2}\,\Big\{2\,(1-2\,\nu)(4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}\cdot\hat{\mathbf{r}}_y - 2\,(\mathbf{n}\cdot\mathbf{b}))\,\Big\}\;. \qquad (44)
\end{aligned}
$$

Now we apply 3 and factor out the vector $\mathbf{b}$ to obtain $\mathcal{Z}_y^x$, i.e.

$$
\begin{aligned}
\mathcal{Z}_y^x(\mathbf{x})\,\mathbf{b} &= \mathbb{C}[D(\mathcal{T}_y^x(\mathbf{x}))^T\mathbf{b}] \\
&= \mu\,(D\left[(\mathcal{T}_y^x)^T\,\mathbf{b}\right] + (D\left[(\mathcal{T}_y^x)^T\,\mathbf{b}\right])^T) + \lambda\,\mathrm{tr} D\left[(\mathcal{T}_y^x)^T\,\mathbf{b}\right] \\
&= \frac{\mu\,C}{r_y^2}\,\{2\,(1-2\,\nu)(2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}\otimes\hat{\mathbf{r}}_y + 2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes\mathbf{b} - \\
&\quad 4\,(\mathbf{n}\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y + 2\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n}\otimes\hat{\mathbf{r}}_y + 2\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\mathbf{n} - \\
&\quad 2\,\mathbf{b}\otimes\mathbf{n} + 2\,(\mathbf{n}\cdot\mathbf{b})\,\mathbf{I} - 2\,\mathbf{n}\otimes\mathbf{b}) + 32\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y - \\
&\quad 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n}\otimes\hat{\mathbf{r}}_y - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\mathbf{n} - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}\otimes\hat{\mathbf{r}}_y - \\
&\quad 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes\mathbf{b} - 8\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{I}\} + \\
&\quad \frac{\lambda\,c}{r_y^2}\,\{4\,(1-2\,\nu)(2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}\cdot\hat{\mathbf{r}}_y - (\mathbf{n}\cdot\mathbf{b}))\}\,\mathbf{I} \\
&= \frac{\mu\,C}{r_y^2}\,\{2\,(1-2\,\nu)(2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}\otimes\hat{\mathbf{r}}_y + 2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes\mathbf{b} - \\
&\quad 4\,(\mathbf{n}\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}} + 2\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n}\otimes\hat{\mathbf{r}}_y + 2\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\mathbf{n} - \\
&\quad 2\,\mathbf{b}\otimes\mathbf{n} + 2\,(\mathbf{n}\cdot\mathbf{b})\,\mathbf{I} - 2\,\mathbf{n}\otimes\mathbf{b}) + 32\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y - \\
&\quad 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n}\otimes\hat{\mathbf{r}}_y - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\mathbf{n} - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}\otimes\hat{\mathbf{r}}_y - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\hat{\mathbf{r}}_y\otimes\mathbf{b} - \\
&\quad 8\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{I} + 8\,\nu\,(2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\mathbf{b}\cdot\hat{\mathbf{r}}_y - (\mathbf{n}\cdot\mathbf{b}))\,\mathbf{I}\} \\
&= \frac{4\,\mu\,C}{r_y^2}\,\{(\hat{\mathbf{r}}_y\cdot\mathbf{n})\,\left((1-2\,\nu)(\mathbf{b}\otimes\hat{\mathbf{r}}_y + \hat{\mathbf{r}}_y\otimes\mathbf{b}) + 8\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y - \right. \\
&\quad \mathbf{b}\otimes\hat{\mathbf{r}}_y - \hat{\mathbf{r}}_y\otimes\mathbf{b} - 2\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{I} + 4\,\nu\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{I}) + \\
&\quad (1-2\,\nu)(-2\,(\mathbf{n}\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y + (\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n}\otimes\hat{\mathbf{r}}_y + (\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\mathbf{n} - \mathbf{b}\otimes\mathbf{n} + \\
&\quad (\mathbf{n}\cdot\mathbf{b})\,\mathbf{I} - \mathbf{n}\otimes\mathbf{b}) - (\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{n}\otimes\hat{\mathbf{r}}_y - (\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\mathbf{n} - 2\,\nu\,(\mathbf{n}\cdot\mathbf{b})\,\mathbf{I}\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4\,\mu\,C}{r_y^2}\ \{(\hat{\mathbf{r}}_y\cdot\mathbf{n})\ (-2\,\nu\,(\mathbf{b}\otimes\hat{\mathbf{r}}_y+\hat{\mathbf{r}}_y\otimes\mathbf{b})+8\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y- \\
&\quad 2\,(1-2\,\nu)\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,\mathbf{I})+(1-2\,\nu)(-2\,(\mathbf{n}\cdot\mathbf{b})\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y-\mathbf{b}\otimes\mathbf{n}-\mathbf{n}\otimes\mathbf{b})- \\
&\quad 2\,\nu\,(\hat{\mathbf{r}}_y\cdot\mathbf{b})\,(\mathbf{n}\otimes\hat{\mathbf{r}}_y+\hat{\mathbf{r}}_y\otimes\mathbf{n})+(1-4\,\nu)\,(\mathbf{n}\cdot\mathbf{b})\,\mathbf{I}\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4\,\mu\,C}{r_y^2}\ \{2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\ [-\nu\,(\mathbb{T}_3\,\hat{\mathbf{r}}_y\otimes\mathbf{I}+\hat{\mathbf{r}}_y\otimes\mathbf{I})+4\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y- \\
&\quad (1-2\,\nu)\,\mathbf{I}\otimes\hat{\mathbf{r}}_y]-(1-2\,\nu)(2\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y\otimes\mathbf{n}+\mathbb{T}_3\,\mathbf{n}\otimes\mathbf{I}+\mathbf{n}\otimes\mathbf{I}) \\
&\quad -2\,\nu\,(\mathbf{n}\otimes\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y+\hat{\mathbf{r}}_y\otimes\mathbf{n}\otimes\hat{\mathbf{r}}_y)+(1-4\,\nu)\,\mathbf{I}\otimes\mathbf{n}\}\ \mathbf{b}\ . \qquad (45)
\end{aligned}
$$

Thus, we have the third-order tensors that appear in the traction boundary integral equation

$$
\begin{aligned}
\Sigma_y(\mathbf{x}) &= -\frac{2\,C}{r_y}\ [(1-2\,\nu)(\mathbb{T}_3\,\hat{\mathbf{r}}_y\otimes\mathbf{I}+\hat{\mathbf{r}}_y\otimes\mathbf{I}-\mathbf{I}\otimes\hat{\mathbf{r}}_y)+2\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y]\ , \\
\mathcal{Z}_y^x(\mathbf{x}) &= -\frac{4\,\mu\,C}{r_y^2}\ \{2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\ [\nu\,(\mathbb{T}_3\,\hat{\mathbf{r}}_y\otimes\mathbf{I}+\hat{\mathbf{r}}_y\otimes\mathbf{I})-4\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y+(1-2\,\nu)\,\mathbf{I}\otimes\hat{\mathbf{r}}_y]+ \\
&\quad (1-2\,\nu)(2\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y\otimes\mathbf{n}+\mathbb{T}_3\,\mathbf{n}\otimes\mathbf{I}+\mathbf{n}\otimes\mathbf{I})+ \\
&\quad 2\,\nu\,(\mathbf{n}\otimes\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y+\hat{\mathbf{r}}_y\otimes\mathbf{n}\otimes\hat{\mathbf{r}}_y)-(1-4\,\nu)\,\mathbf{I}\otimes\mathbf{n}\}\ . \qquad (46)
\end{aligned}
$$

3.2 Reduction to Second-Order Tensor

We wish to represent the traction BIE with second-order tensors acting on $\mathbf{t}^{\mathbf{n}}(\mathbf{x})$ and $\mathbf{u}(\mathbf{x})$. Since the third-order tensor $\mathcal{Z}_y^x(\mathbf{x})$ is defined in terms of $\mathbf{n}(\mathbf{x})$ for notational clarity we define $\mathbf{m}(\mathbf{y}) = \mathbf{n}(\mathbf{y})$. Thus, upon suppressing the arguments $\mathbf{x}$ and $\mathbf{y}$ for conciseness we have

$$\begin{aligned}
\int_{\partial\Omega} \Sigma_y(\mathbf{x})\,\mathbf{t^n}(\mathbf{x})\,dS\,\mathbf{n}(\mathbf{y}) &= \int_{\partial\Omega} \Bigg[-\frac{2\,C}{r_y}\;(1-2\,\nu)\,[\mathbb{T}_3\,\hat{\mathbf{r}}_y \otimes \mathbf{I} + \hat{\mathbf{r}}_y \otimes \mathbf{I} - \mathbf{I} \otimes \hat{\mathbf{r}}_y] \\
&\qquad + 2\,\hat{\mathbf{r}}_y \otimes \hat{\mathbf{r}}_y \otimes \hat{\mathbf{r}}_y \mathbf{t^n}\,dS \Bigg]\mathbf{m} \\
&= \int_{\partial\Omega} \Bigg[-\frac{2\,C}{r_y}\;[\,(1-2\,\nu)\,[\mathbb{T}_3\,\hat{\mathbf{r}}_y \otimes \mathbf{t^n} + \hat{\mathbf{r}}_y \otimes \mathbf{t^n} - (\hat{\mathbf{r}}_y \cdot \mathbf{t^n})\mathbf{I}] \\
&\qquad + 2\,(\hat{\mathbf{r}}_y \cdot \mathbf{t^n})\hat{\mathbf{r}}_y \otimes \hat{\mathbf{r}}_y]\,dS \Bigg]\mathbf{m} \\
&= \int_{\partial\Omega} -\frac{2\,C}{r_y}\;[\,(1-2\,\nu)\,[(\mathbf{t^n} \otimes \hat{\mathbf{r}}_y)\mathbf{m} + (\hat{\mathbf{r}}_y \otimes \mathbf{t^n})\mathbf{m} - (\hat{\mathbf{r}}_y \cdot \mathbf{t^n})\mathbf{m}] \\
&\qquad + 2\,(\hat{\mathbf{r}}_y \cdot \mathbf{t^n})(\hat{\mathbf{r}}_y \otimes \hat{\mathbf{r}}_y)\mathbf{m}]\;dS \\
&= \int_{\partial\Omega} -\frac{2\,C}{r_y}\;[\,(1-2\,\nu)\,[(\hat{\mathbf{r}}_y \cdot \mathbf{m})\mathbf{t^n} + (\mathbf{t^n} \cdot \mathbf{m})\hat{\mathbf{r}}_y - (\hat{\mathbf{r}}_y \cdot \mathbf{t^n})\mathbf{m}] \\
&\qquad + 2\,(\hat{\mathbf{r}}_y \cdot \mathbf{t^n})(\hat{\mathbf{r}}_y \cdot \mathbf{m})\hat{\mathbf{r}}_y]\;dS \\
&= \int_{\partial\Omega} -\frac{2\,C}{r_y}\;[\,(1-2\,\nu)\,[(\hat{\mathbf{r}}_y \cdot \mathbf{m})\mathbf{I}\,\mathbf{t^n} + (\hat{\mathbf{r}}_y \otimes \mathbf{m})\mathbf{t^n} - (\mathbf{m} \otimes \hat{\mathbf{r}}_y)\mathbf{t^n}] \\
&\qquad + 2\,(\hat{\mathbf{r}}_y \cdot \mathbf{m})(\hat{\mathbf{r}}_y \otimes \hat{\mathbf{r}}_y)\mathbf{t^n}]\;dS \\
&= \int_{\partial\Omega} -\frac{2\,C}{r_y}\;[\,(1-2\,\nu)\,[(\hat{\mathbf{r}}_y \cdot \mathbf{m})\mathbf{I} + (\hat{\mathbf{r}}_y \otimes \mathbf{m}) - (\mathbf{m} \otimes \hat{\mathbf{r}}_y)] \\
&\qquad + 2\,(\hat{\mathbf{r}}_y \cdot \mathbf{m})(\hat{\mathbf{r}}_y \otimes \hat{\mathbf{r}}_y)]\,\mathbf{t^n}\,dS \\
&= \int_{\partial\Omega} \mathcal{D}_y^y(\mathbf{x})\mathbf{t}(\mathbf{x})\,dS\;\;. \qquad (47)
\end{aligned}$$

Similarly we have for $\mathcal{Z}_y^x$ acting on $\mathbf{u}(\mathbf{x})$

$$
\begin{aligned}
\int_{\partial\Omega} \mathcal{Z}_y^x\,\mathbf{u}(\mathbf{x})\,dS\,\mathbf{n}(\mathbf{y}) \;=\;& \int_{\partial\Omega} \frac{-4\,\mu\,C}{r_y^2}\;\{2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\;[\nu\,(\mathbb{T}\,\hat{\mathbf{r}}_y\otimes\mathbf{I}+\hat{\mathbf{r}}_y\otimes\mathbf{I})-4\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y \\
& +(1-2\,\nu)\,\mathbf{I}\otimes\hat{\mathbf{r}}_y)]+(1-2\,\nu)(2\,\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y\otimes\mathbf{n}+\mathbb{T}_3\,\mathbf{n}\otimes\mathbf{I}+\mathbf{n}\otimes\mathbf{I}) \\
& +2\,\nu\,(\mathbf{n}\otimes\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y+\hat{\mathbf{r}}_y\otimes\mathbf{n}\otimes\hat{\mathbf{r}}_y)-(1-4\,\nu)\,\mathbf{I}\otimes\mathbf{n}\}\;\mathbf{u}\,dS\,\mathbf{m} \\
\\
=\;& \int_{\partial\Omega} \frac{-4\,\mu\,C}{r_y^2}\;\{2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\;[\nu\,(\mathbb{T}\,\hat{\mathbf{r}}_y\otimes\mathbf{u}+\hat{\mathbf{r}}_y\otimes\mathbf{u})-4\,(\hat{\mathbf{r}}_y\cdot\mathbf{u})\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y \\
& +(1-2\,\nu)\,(\hat{\mathbf{r}}_y\cdot\mathbf{u})\mathbf{I})]+(1-2\,\nu)(2\,(\mathbf{n}\cdot\mathbf{u})\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y+\mathbb{T}_3\,\mathbf{n}\otimes\mathbf{u}+\mathbf{n}\otimes\mathbf{u}) \\
& +2\,\nu\,((\hat{\mathbf{r}}_y\cdot\mathbf{u})\mathbf{n}\otimes\hat{\mathbf{r}}_y+(\hat{\mathbf{r}}_y\cdot\mathbf{u})\hat{\mathbf{r}}_y\otimes\mathbf{n})-(1-4\,\nu)\,(\mathbf{n}\cdot\mathbf{u})\mathbf{I}\}\;\mathbf{m}\,dS \\
\\
=\;& \int_{\partial\Omega} \frac{-4\,\mu\,C}{r_y^2}\;\{2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\;[\nu\,((\mathbf{u}\otimes\hat{\mathbf{r}}_y)\mathbf{m}+(\hat{\mathbf{r}}_y\otimes\mathbf{u})\mathbf{m})-4\,(\hat{\mathbf{r}}_y\cdot\mathbf{u}) \\
& \times(\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y)\mathbf{m}+(1-2\,\nu)\,(\hat{\mathbf{r}}_y\cdot\mathbf{u})\mathbf{m})]+(1-2\,\nu)(2\,(\mathbf{n}\cdot\mathbf{u})(\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y)\mathbf{m} \\
& +(\mathbf{u}\otimes\mathbf{n})\mathbf{m}+(\mathbf{n}\otimes\mathbf{u})\mathbf{m})+\;2\,\nu\,((\hat{\mathbf{r}}_y\cdot\mathbf{u})(\mathbf{n}\otimes\hat{\mathbf{r}}_y)\mathbf{m} \\
& +(\hat{\mathbf{r}}_y\cdot\mathbf{u})(\hat{\mathbf{r}}_y\otimes\mathbf{n})\mathbf{m})-(1-4\,\nu)\,(\mathbf{n}\cdot\mathbf{u})\mathbf{m}\}\;dS \\
\\
=\;& \int_{\partial\Omega} \frac{-4\,\mu\,C}{r_y^2}\;\{2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\;[\nu\,(\hat{\mathbf{r}}_y\cdot\mathbf{m})\mathbf{u}+(\mathbf{u}\cdot\mathbf{m})\hat{\mathbf{r}}_y)-4\,(\hat{\mathbf{r}}_y\cdot\mathbf{u})(\hat{\mathbf{r}}_y\cdot\mathbf{m})\hat{\mathbf{r}}_y \\
& \qquad +(1-2\,\nu)\,(\hat{\mathbf{r}}_y\cdot\mathbf{u})\mathbf{m})]+(1-2\,\nu)(2\,(\mathbf{n}\cdot\mathbf{u})(\hat{\mathbf{r}}_y\cdot\mathbf{m})\hat{\mathbf{r}}_y \\
& \qquad +(\mathbf{n}\cdot\mathbf{m})\mathbf{u}+(\mathbf{u}\cdot\mathbf{m})\mathbf{n})+\;2\,\nu\,((\hat{\mathbf{r}}_y\cdot\mathbf{u})(\hat{\mathbf{r}}_y\cdot\mathbf{m})\mathbf{n} \\
& \qquad +(\hat{\mathbf{r}}_y\cdot\mathbf{u})(\mathbf{n}\cdot\mathbf{m})\hat{\mathbf{r}}_y)-(1-4\,\nu)\,(\mathbf{n}\cdot\mathbf{u})\mathbf{m}\}\;dS \\
\\
=\;& \int_{\partial\Omega} \frac{-4\,\mu\,C}{r_y^2}\;\{2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\;[\nu\,(\hat{\mathbf{r}}_y\cdot\mathbf{m})\mathbf{u}+(\hat{\mathbf{r}}_y\otimes\mathbf{m})\mathbf{u})-4\,(\hat{\mathbf{r}}_y\cdot\mathbf{m})(\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y)\mathbf{u} \\
& \qquad +(1-2\,\nu)\,(\mathbf{m}\otimes\hat{\mathbf{r}}_y)\mathbf{u})]+(1-2\,\nu)(2\,(\hat{\mathbf{r}}_y\cdot\mathbf{m})(\hat{\mathbf{r}}_y\otimes\mathbf{n})\mathbf{u} \\
& \qquad +(\mathbf{n}\cdot\mathbf{m})\mathbf{u}+(\mathbf{n}\otimes\mathbf{m})\mathbf{u})+\;2\,\nu\,((\hat{\mathbf{r}}_y\cdot\mathbf{m})(\mathbf{n}\otimes\hat{\mathbf{r}}_y)\mathbf{u} \\
& \qquad +(\mathbf{n}\cdot\mathbf{m})(\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y)\mathbf{u})-(1-4\,\nu)\,(\mathbf{m}\otimes\mathbf{n})\mathbf{u}\}\;dS \\
\\
=\;& \int_{\partial\Omega} \frac{-4\,\mu\,C}{r_y^2}\;\{2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\;[\nu\,((\hat{\mathbf{r}}_y\cdot\mathbf{m})\mathbf{I}+\hat{\mathbf{r}}_y\otimes\mathbf{m})-4\,(\hat{\mathbf{r}}_y\cdot\mathbf{m})\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y \\
& \qquad +(1-2\,\nu)\,\mathbf{m}\otimes\hat{\mathbf{r}}_y]+(1-2\,\nu)(2\,(\hat{\mathbf{r}}_y\cdot\mathbf{m})\hat{\mathbf{r}}_y\otimes\mathbf{n} \\
& \qquad +(\mathbf{n}\cdot\mathbf{m})\mathbf{I}+\mathbf{n}\otimes\mathbf{m})+\;2\,\nu\,((\hat{\mathbf{r}}_y\cdot\mathbf{m})\mathbf{n}\otimes\hat{\mathbf{r}}_y \\
& \qquad +(\mathbf{n}\cdot\mathbf{m})\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y)-(1-4\,\nu)\,\mathbf{m}\otimes\mathbf{n}\}\,\mathbf{u}\,dS
\end{aligned}
\tag{48}
$$

$$= \int_{\partial\Omega} \mathcal{S}_y^{xy}\, \mathbf{u}(\mathbf{x})\, dS \ , \tag{49}$$

where the superscript xy emphasizes the fact that both $\mathbf{n}(\mathbf{x})$ and $\mathbf{m}(\mathbf{y})$ appear in the expression. Summarizing we have the second-order tensors used in the DBEM

$$\begin{aligned}
\mathcal{D}_y^y(\mathbf{x}) &= -\frac{2\,C}{r_y}\ \left(\,(1-2\,\nu)\,[(\hat{\mathbf{r}}_y\cdot\mathbf{m})\mathbf{I} + (\hat{\mathbf{r}}_y\otimes\mathbf{m}) - (\mathbf{m}\otimes\hat{\mathbf{r}}_y)] + 2\,(\hat{\mathbf{r}}_y\cdot\mathbf{m})(\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y)\right) \ , \\
\mathcal{S}_y^{xy}(\mathbf{x}) &= -\frac{4\,\mu\,C}{r_y^2}\ \{2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\ [\nu\,((\hat{\mathbf{r}}_y\cdot\mathbf{m})\mathbf{I} + \hat{\mathbf{r}}_y\otimes\mathbf{m}) - 4\,(\hat{\mathbf{r}}_y\cdot\mathbf{m})\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y + (1-2\,\nu)\,\mathbf{m}\otimes\hat{\mathbf{r}}_y] \\
&\quad +(1-2\,\nu)(2\,(\hat{\mathbf{r}}_y\cdot\mathbf{m})\hat{\mathbf{r}}_y\otimes\mathbf{n} + (\mathbf{n}\cdot\mathbf{m})\mathbf{I} + \mathbf{n}\otimes\mathbf{m}) \\
&\quad +2\,\nu\,((\hat{\mathbf{r}}_y\cdot\mathbf{m})\mathbf{n}\otimes\hat{\mathbf{r}}_y + (\mathbf{n}\cdot\mathbf{m})\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y) - (1-4\,\nu)\,\mathbf{m}\otimes\mathbf{n}\} \ .
\end{aligned} \tag{50}$$

In the DBEM we discretize the displacement equation 40 on the boundaries $S' \cup S^+$ and the traction equation 42 on the boundary S^-, i.e.

$$\begin{aligned}
&\tfrac{1}{2}\mathbf{u}(\mathbf{y}) = \int_{\partial\Omega}\mathcal{U}_y(\mathbf{x})\mathbf{t}^{\mathbf{n}}(\mathbf{x})\,dS - ⨍_{\partial\Omega}[\mathcal{T}_y^x(\mathbf{x})]^T\,\mathbf{u}(\mathbf{x})\,dS \quad \text{if } \mathbf{y}\in S^{'}\cup S^{+} \ , \\
&\tfrac{1}{2}\mathbf{t}^{\mathbf{n}}(\mathbf{y}) = ⨍_{\partial\Omega}\mathcal{D}_y^y(\mathbf{x})\mathbf{t}^{\mathbf{n}}(\mathbf{x})\,dS - ⨎_{\partial\Omega}\mathcal{S}_y^{xy}(\mathbf{x})\,\mathbf{u}(\mathbf{x})\,dS \quad \text{if } \mathbf{y}\in S^{-} \ ,
\end{aligned} \tag{51}$$

where the $⨎$ denotes the hypersingular integral often referred to as the Hadamard value, which is related to the CPV such that $⨎_a^b\ f(x,y)\,dx = \frac{d}{dy}⨍_a^b f(x,y)\,dx$. Special care must be taken to evaluate these integrals as will be seen in the following section.

4 Discretization and Integration

4.1 Discretization of Finite Boundary

The BEM is similar to the FEM, wherein a global stiffness matrix $[K]$ and load vector $\{f\}$, assembled from the element domains, relates the applied loads to the unknown node displacements $\{u\}$, i.e. $[K]\{u\} = \{f\}$. In the BEM the global matrix equation takes the form

$$[G]\{u\} = [H]\{t\} = \{f\} \ , \tag{52}$$

where the column vectors $\{u\}$ and $\{t\}$ are the displacements and tractions, respectively, at the *collocation* points $\mathbf{y}$. In most cases, at any point $\mathbf{y}$, either $\mathbf{u}$ or $\mathbf{t}^{\mathbf{n}}$ is known, thus there are unknown variables on both sides of 52. However, in our application $\mathbf{t}^{\mathbf{n}} = \mathbf{t}^p$ is prescribed over the entire boundary so $[H]\{t\} = \{f\}$ is known. As in the FEM, the matrices $[G]$ and $[H]$ are assembled from their element counterparts.

To begin the discretization, the geometry $\partial\Omega$ is replaced by a mesh of 3-noded elements. With $\mathbf{X}^I$ being the geometric coordinates of node I in a given element e we have the position vector (in element e)

$$\begin{aligned} \mathbf{x} &= \mathbf{f}(\xi) \\ &= \sum_{I=1}^{3} N_I^*(\xi)\,\mathbf{X}^I \ , \end{aligned} \tag{53}$$

where $N_I^*(\xi)$ are the continuous shape functions, cf. [5] 2.125

$$\begin{aligned} N_1^*(\xi) &= -\frac{1}{2}\,\xi\,(1-\xi) \ , \\ N_2^*(\xi) &= \frac{1}{2}\,\xi\,(1+\xi) \ , \\ N_3^*(\xi) &= (1-\ \xi)\ (1+\ \xi) \ . \end{aligned} \tag{54}$$

The Jacobian $J(\xi)$ of the transformation $\mathbf{f}(\xi)$ is given by

$$J(\xi) = \left|\frac{\partial \mathbf{x}}{\partial \xi}\right| = \left|\sum_{I=1}^{3} \frac{\partial N_I^*(\xi)}{d\xi}\,\mathbf{X}^I\right| \ . \tag{55}$$

Note that $J(\xi) = \frac{L}{2}$ for a straight element of length L with *equally* spaced nodes in e (i.e. $\frac{X^2 - X^1}{2} = X^3$), cf. Figure 4.

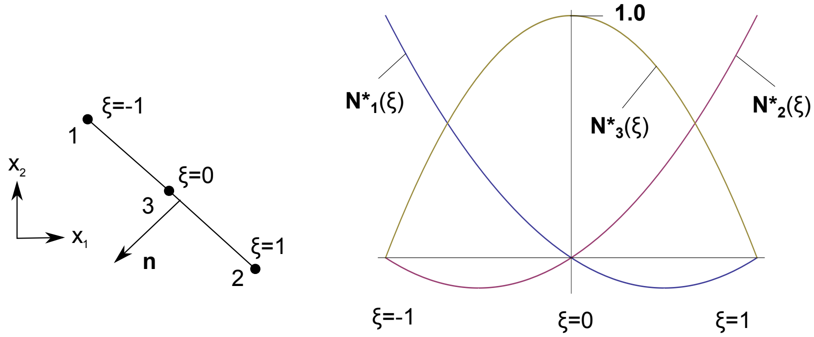


Figure 4: Continuous shape functions cf. 54.

The interpolation of $\mathbf{u}$ and $\mathbf{t^n}$ over the element are similar to that of $\mathbf{x}$; however, unlike the above interpolation of $\mathbf{x}$ which is continuous over $\partial\Omega$, $\mathbf{u}$ and $\mathbf{t^n}$ are not as they are interpolated via discontinuous shape functions. Although the interpolations of $\mathbf{u}$ and $\mathbf{t^n}$ are not continuous over $\partial\Omega$ they *are* continuous inside the element e. The rational for using the discontinuous interpolation is addressed in the sequel. By letting $\mathbf{x} = \mathbf{f}(\xi)$, the displacement in the element is interpolated as

$$\begin{aligned}
\mathbf{u}(\mathbf{f}(\xi)) &= \sum_{I=1}^{3} N_I(\xi)\,\mathbf{U}^I \\
&= \begin{bmatrix} \mathbf{U}^1 & \mathbf{U}^2 & \mathbf{U}^3 \end{bmatrix} \begin{Bmatrix} N_1(\xi) \\ N_2(\xi) \\ N_3(\xi) \end{Bmatrix} \\
&= \mathbf{U}\,\mathbf{N}(\xi) \\
&= (\mathbf{N}^T(\xi) \odot \mathbf{I})\,\mathrm{vec}\,(\mathbf{U}) \quad , \qquad (56)
\end{aligned}$$

where $\mathbf{U}^I$ is the collocation point displacement vector for collocation point I in element e and $\odot$ is the Kronecker product with the property vec$(\mathbf{AXB}) = (\mathbf{B}^T \odot \mathbf{A})vec(\mathbf{X})$ cf. [11] 6.113-6.115. Similarly, for the traction we have

$$
\begin{aligned}
\mathbf{t}^{\mathbf{n}}(\mathbf{f}(\xi)) &= \sum_{I=1}^{3} N_I(\xi)\,\mathbf{T}^I \\
&= \begin{bmatrix} \mathbf{T}^1 & \mathbf{T}^2 & \mathbf{T}^3 \end{bmatrix} \begin{Bmatrix} N_1(\xi) \\ N_2(\xi) \\ N_3(\xi) \end{Bmatrix} \\
&= \mathbf{T}\,\mathbf{N}(\xi) \\
&= (\mathbf{N}^T(\xi) \odot \mathbf{I})\,\mathrm{vec}\,(\mathbf{T}) \quad ,
\end{aligned} \tag{57}
$$

where $\mathbf{T}^I$ is the collocation point traction vector for collocation point I in element e. The discontinuous shape functions $N_I(\xi)$ are cf. [10] 36,

$$
\begin{aligned}
N_1(\xi) &= \xi \left(\frac{9}{8}\,\xi - \frac{3}{4} \right) \quad , \\
N_2(\xi) &= \xi \left(\frac{9}{8}\,\xi + \frac{3}{4} \right) \quad , \\
N_3(\xi) &= \left(1 - \frac{3}{2}\,\xi \right) \left(1 + \frac{3}{2}\,\xi \right) \quad ,
\end{aligned} \tag{58}
$$

which are illustrated in Figure 5. As seen in the figure, the three collocation points are located at $\mathbf{Y}^1 = \mathbf{f}(-\frac{2}{3})$, $\mathbf{Y}^2 = \mathbf{f}(+\frac{2}{3})$, $\mathbf{Y}^3 = \mathbf{f}(0)$, respectively.

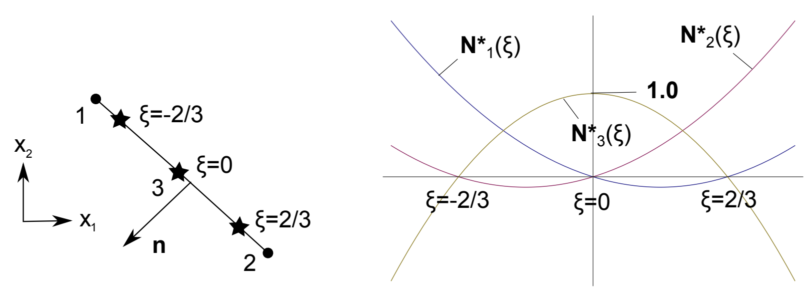


Figure 5: Discontinuous element collocation points.

We note that the fundamental solutions contain different singularities as $r_y \to 0$. The displacement potential $\mathcal{U}_y$ exhibits a weak singularity (i.e. $\mathcal{U}_y \propto \ln\left(\frac{1}{r_y}\right)$), the traction potential $\mathcal{T}_y^x$ and the displacement derivative potential $\mathcal{D}_y^y$ exhibit strong singularities (i.e. $\mathcal{T}_y^x$, $\mathcal{D}_y^y \propto \frac{1}{r_y}$), and the traction derivative potential $\mathcal{S}_y^{xy}$ exhibits a hyersingularity (i.e. $\mathcal{S}_y^{xy} \propto \frac{1}{r_y^2}$). In the following, we discuss the evaluation of the integrals in 51 over element e. Two cases are considered; when the collocation point $\mathbf{y}$ does not belong to the integration element e (i.e. $\mathbf{y} \notin e$) and when $\mathbf{y}$ belongs to the integration element e (i.e. $\mathbf{y} \in e$). Sections 4.2-4.5 discusses these integrations.

4.2 Weakly Singular Integrals - $\mathbf{y} \notin e$

The weakly singular kernel $\mathcal{U}_y$ is integrated using a standard Gaussian quadrature scheme. The Gauss-order to approximate the solution depends on the ratio between the radius r_y and the element length, L as shown by [17]. The smaller the ratio the higher the Gauss-order [17]. Thus, the first integral on the right hand side of 51 for a given element is approximated as

$$\begin{aligned} \int_e \mathcal{U}_y(\mathbf{x})\mathbf{t}^{\mathbf{n}}(\mathbf{x})\, dS &= \int_{-1}^{1} \mathcal{U}_y(\xi)(\mathbf{N}^T(\xi) \odot \mathbf{I})\,\mathrm{vec}\,(\mathbf{T})\, J(\xi) d\xi \\ &= \left[\sum_{i=1}^{m} \mathcal{U}_y(\xi_i)(\mathbf{N}^T(\xi_i) \odot \mathbf{I})\, J(\xi_i)\, w_i\, d\xi\right] \mathrm{vec}\,(\mathbf{T}) \\ &= \mathcal{B}_y^e\,\mathrm{vec}\,(\mathbf{T}) \quad , \end{aligned} \tag{59}$$

where $\mathcal{B}_y^e$ is a 2×6 displacement matrix whose rows correspond to the collocation point $\mathbf{Y}^I$ displacement equation and whose columns correspond to the three element nodes $\mathbf{X}^J$ of element e. In 59 ξ_i and w_i are the Gauss points and weights. Note that the dimension of $\mathcal{U}_y$ is 2×2, $(\mathbf{N}^T(\xi_i)\odot\mathbf{I})$ is 2×6, and $\mathrm{vec}\,(\mathbf{T})$ is 6×1 resulting a 2×1 vector $\mathcal{B}_y^e\,\mathrm{vec}(\mathbf{T})$, which aligns with the two displacement degrees of freedom at the collocation point $\mathbf{Y}^I$.

4.3 Singular and Hypersingular Integrals - $\mathbf{y} \notin e$

The singular and hypersingular integrals can also be evaluated using Gauss quadrature; however, for small ratios r_y/L, high-order quadratures or special integration schemes must be used to accurately approximate these integrals [5, 18]. It was shown by Zhang [19] that these integrals applied to discontinuous elements can be evaluated analytically without special treatment. To eliminate this potential source of error due to the quadrature, this thesis adopts the analytical integration procedure.

We wish to analytically integrate the singular and hypersingular integrals which, after the interpolation of 56 and 57, take the form of 59, i.e.

$$\int_e \mathcal{T}_y^x(\mathbf{x})\mathbf{t^n}(\mathbf{x})\,dS \;=\; \left[\int_{-1}^{1} \mathcal{T}_y^x(\mathbf{g}(\xi))(\mathbf{N}^T(\xi)\odot\mathbf{I})\,J(\xi)\,d\xi\right]\,\mathrm{vec}\,(\mathbf{U}) \;\;,$$
$$\int_e \mathcal{D}_y^y(\mathbf{x})\mathbf{t^n}(\mathbf{x})\,dS \;=\; \left[\int_{-1}^{1} \mathcal{D}_y^y(\mathbf{g}(\xi))(\mathbf{N}^T(\xi)\odot\mathbf{I})\,J(\xi)\,d\xi\right]\,\mathrm{vec}\,(\mathbf{T}) \;\;,$$
$$\int_e \mathcal{S}_y^x(\mathbf{x})\mathbf{t^n}(\mathbf{x})\,dS \;=\; \left[\int_{-1}^{1} \mathcal{S}_y^{xy}(\mathbf{g}(\xi))(\mathbf{N}^T(\xi)\odot\mathbf{I})\,J(\xi)\,d\xi\right]\,\mathrm{vec}\,(\mathbf{U}) \;\;. \tag{60}$$

For simplicity, we first integrate the potentials $\mathcal{T}_y^x, \mathcal{D}_y^y, \mathcal{S}_y^x$ and then integrate their products with the shape functions. To begin, assume the elements are straight and the nodes are equally spaced so that within each element

$$\begin{aligned} r_y &= \mathbf{x}-\mathbf{y} \\ &= \mathbf{f}(\xi)-\mathbf{y} \\ &= \mathbf{c}+\mathbf{d}\xi \;\;, \end{aligned} \tag{61}$$

where $\xi \in [-1, 1]$ and

$$\begin{aligned} \mathbf{c} &= \frac{1}{2}(\mathbf{X}^2+\mathbf{X}^1)-\mathbf{Y}^I \;\;, \\ \mathbf{d} &= \frac{1}{2}(\mathbf{X}^2-\mathbf{X}^1) \;\;. \end{aligned} \tag{62}$$

Using the above we obtain

$$r_y = (a\;\xi^2+2\;b\;\xi+c)^{\frac{1}{2}} \;\;, \tag{63}$$
$$\begin{aligned} \hat{\mathbf{r}}_y &= \frac{\mathbf{c}+\mathbf{d}\xi}{r_y} \;\;, \\ dS &= |D\mathbf{f}(\xi)|\,d\xi \\ &= \frac{L}{2}\,d\xi \;\;, \end{aligned} \tag{64}$$

where $a = \mathbf{d}\cdot\mathbf{d}$, $b = 2\mathbf{c}\cdot\mathbf{d}$, $c = \mathbf{c}\cdot\mathbf{c}$ [3], and L is the element length of e.

From the mapping depicted in Figure 6, we see that $\mathbf{c}$ represents the vector from the collocation point $\mathbf{Y}^I$ to the element e centroid. And since $\mathbf{X}^2$ and $\mathbf{X}^1$ are constants for the element, so are $\mathbf{c}$,

[3] It was noticed that there was an error in Zhang[6] that has been corrected here (i.e. the $\frac{L}{2}$ term in front of $\mathcal{D}_y^y$).

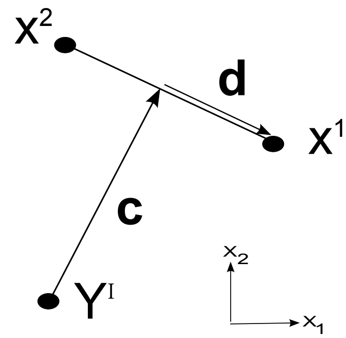


Figure 6: Analytical integration element for mapping of 61.

$\mathbf{d}$, a, b, and c. If the element is curved or nodes are not equally spaced, e.g. as in a quarter-point element, then the map of 61 would have to change to reflect, e.g. $\mathbf{x} = \sum_{I=1}^{3} N_I^*(\xi)\,\mathbf{X}^I - \mathbf{Y}$.

Neglecting the shape function contribution, the change of variable then renders for a given element e

$$\begin{aligned}
\int_e \left[\mathcal{T}_y^x(\mathbf{x})\right]^T dS &= \int_{-1}^{1} \left[\mathcal{T}_y^x(\mathbf{f}(\xi))\right]^T \frac{L}{2}\, d\xi \ , \\
\int_e \mathcal{D}_y^y(\mathbf{x})\, dS &= \int_{-1}^{1} \mathcal{D}_y^y(\mathbf{f}(\xi))\, \frac{L}{2}\, d\xi \ , \\
\int_e \mathcal{S}_y^{xy}(\mathbf{x})\, dS &= \int_{-1}^{1} \mathcal{S}_y^{xy}(\mathbf{f}(\xi))\, \frac{L}{2}\, d\xi \ .
\end{aligned} \tag{65}$$

Using the above, we expand51.1 and 51.2 and collect like terms. Integration over the first term in 29 gives

$$\int_{\partial\Omega} -\frac{2\,C}{r_y}\,(1-2\,\nu)\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\mathbf{I}\,da = \int_{-1}^{1} -\frac{2\,C}{a\xi^2+b\xi+c}\,(1-2\,\nu)\,[(\mathbf{c}+\mathbf{d}\xi)\cdot\mathbf{n}]\,\mathbf{I}\,\frac{L}{2}\,d\,\xi \tag{66}$$

$$\begin{aligned}
&= -2\,C\,(1-2\,\nu)\left[(\mathbf{c}\cdot\mathbf{m})\,\mathbf{I}\int_{-1}^{1}\frac{1}{a\xi^2+b\xi+c}\,\frac{L}{2}\,d\,\xi \right. \\
&\quad \left. +(\mathbf{d}\cdot\mathbf{n})\,\mathbf{I}\int_{-1}^{1}\frac{\xi}{a\xi^2+b\xi+c}\,\frac{L}{2}\,d\,\xi\right] \ .
\end{aligned} \tag{67}$$

For convenience let us define the following integrals as

$$\begin{aligned}\int_{-1}^{1} \frac{\xi^i}{a\xi^2 + b\xi + c} d\,\xi &= F_i \ , \\ \int_{-1}^{1} \frac{\xi^i}{(a\xi^2 + b\xi + c)^2} d\,\xi &= G_i \ .\end{aligned} \tag{68}$$

Thus, 66 reads

$$\int_{-1}^{1} -\frac{2\,C}{r_y}\ (1-2\,\nu)\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})\mathbf{I}\,\frac{L}{2}\,d\,\xi \ = \ -2\,C\,(1-2\,\nu)[(\mathbf{c}\cdot\mathbf{n})\,\mathbf{I}\,F_0 + (\mathbf{d}\cdot\mathbf{n})\,\mathbf{I}\,F_1] \ . \tag{69}$$

For the next terms we have

$$\begin{aligned}\int_{-1}^{1} -\frac{2\,C}{r_y}\ (1-2\,\nu)\,(\hat{\mathbf{r}}_y\otimes\mathbf{n})\,\frac{L}{2}\,d\,\xi &= -2\,C\,(1-2\,\nu)\,[(\mathbf{c}\otimes\mathbf{n})\,F_0 + (\mathbf{d}\otimes\mathbf{n})\,F_1] \ , \\ \int_{-1}^{1} \frac{2\,C}{r_y}\ (1-2\,\nu)\,(\mathbf{n}\otimes\hat{\mathbf{r}}_y)\,\frac{L}{2}\,d\,\xi &= 2\,C\,(1-2\,\nu)\,[(\mathbf{n}\otimes\mathbf{c})\,F_0 + (\mathbf{n}\otimes\mathbf{d})\,F_1] \ , \\ \int_{-1}^{1} -\frac{2\,C}{r_y}\,[2\,(\hat{\mathbf{r}}_y\cdot\mathbf{n})(\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y)]\,d\,\xi &= -4\,C((\mathbf{c}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{c})G_0 \\ &\quad +((\mathbf{d}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{c}) + (\mathbf{c}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{d}))G_1 \\ &\quad +((\mathbf{c}\cdot\mathbf{n})(\mathbf{d}\otimes\mathbf{d}) + (\mathbf{d}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c}))G_2 \\ &\quad +(\mathbf{d}\cdot\mathbf{n})(\mathbf{d}\otimes\mathbf{d})G_3) \quad .\end{aligned} \tag{70}$$

Combining all terms gives us

$$\begin{aligned}\int_{-1}^{1} \left[\mathcal{T}_y^x(\mathbf{f}(\xi))\right]^T\,\frac{L}{2}\,d\xi &= -L\,C\,[(1-2\,\nu)[[(\mathbf{c}\cdot\mathbf{n})\,\mathbf{I}\,+\mathbf{c}\otimes\mathbf{n}-\mathbf{n}\otimes\mathbf{c}]F_0 \\ &\quad +((\mathbf{d}\cdot\mathbf{n})\,\mathbf{I}\,+\mathbf{d}\otimes\mathbf{n}-\mathbf{n}\otimes\mathbf{d})F_1] \\ &\quad +2\,[(\mathbf{c}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{c})G_0 \\ &\quad +[(\mathbf{c}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c}) + (\mathbf{d}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{c})]G_1 \\ &\quad +[(\mathbf{c}\cdot\mathbf{n})(\mathbf{d}\otimes\mathbf{d}) + (\mathbf{d}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c})]G_2 + (\mathbf{d}\cdot\mathbf{n})(\mathbf{d}\otimes\mathbf{d})G_3]] \quad .\end{aligned} \tag{71}$$

The integral in 71 has yet to incorporate any of the shape functions of our quadratic element 58 which are dependent on ξ. Fortunately, since the shape functions are polynomials, the integrals described by F_i and G_i will increase by orders of 0, 1 and 2 in ξ. Specifically, the discontinuous shape functions given in 58 will change the F_i to

$$\begin{aligned}\int_{-1}^{1} \frac{\xi^i}{a\xi^2 + b\xi + c}\,N_1(\xi)\,d\,\xi &= \frac{9}{8}F_{i+2} - \frac{3}{4}F_{i+1} \ , \\ \int_{-1}^{1} \frac{\xi^i}{a\xi^2 + b\xi + c}\,N_2(\xi)\,d\,\xi &= \frac{9}{8}F_{i+2} + \frac{3}{4}F_{i+1} \ , \\ \int_{-1}^{1} \frac{\xi^i}{a\xi^2 + b\xi + c}\,N_3(\xi)\,d\,\xi &= F_i - \frac{9}{4}F_{i+2} \ ,\end{aligned} \tag{72}$$

and similarly

$$\begin{aligned}
\int_{-1}^{1} \frac{\xi^i}{(a\xi^2+b\xi+c)^2}\, N_1(\xi)\, d\,\xi &= \frac{9}{8}G_{i+2} - \frac{3}{4}G_{i+1} \;\;, \\
\int_{-1}^{1} \frac{\xi^i}{(a\xi^2+b\xi+c)^2}\, N_2(\xi)\, d\,\xi &= \frac{9}{8}G_{i+2} + \frac{3}{4}G_{i+1} \;\;, \\
\int_{-1}^{1} \frac{\xi^i}{(a\xi^2+b\xi+c)^2}\, N_3(\xi)\, d\,\xi &= G_i - \frac{9}{4}G_{i+2} \;\;.
\end{aligned} \tag{73}$$

Using the above, we consider the integrands on the right hand side of 60.1, viz for integrands containing N_1:

$$\begin{aligned}
\int_{-1}^{1} \left[\mathcal{T}_y^x(\mathbf{f}(\xi))\right]^T N_1(\xi)\,\frac{L}{2}\,d\,\xi \;\;=\;\; & -C\,L\Bigg[\Big(1-2\,\nu\Big)\Big[\Big[(\mathbf{c}\cdot\mathbf{n})\,\mathbf{I}\,+\mathbf{c}\otimes\mathbf{n}-\mathbf{n}\otimes\mathbf{c}\Big]\Big(\frac{9}{8}F_2-\frac{3}{4}F_1\Big) \\
& +\Big((\mathbf{d}\cdot\mathbf{n})\,\mathbf{I}\,+\mathbf{d}\otimes\mathbf{n}-\mathbf{n}\otimes\mathbf{d}\Big)\Big(\frac{9}{8}F_3-\frac{3}{4}F_2\Big)\Big] \\
& +2\,\Big[\Big((\mathbf{c}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{c})\Big)\Big(\frac{9}{8}G_2-\frac{3}{4}G_1\Big) \\
& +\Big[(\mathbf{c}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c})+(\mathbf{d}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{c})\Big]\Big(\frac{9}{8}G_3-\frac{3}{4}G_2\Big) \\
& +\Big[(\mathbf{c}\cdot\mathbf{n})(\mathbf{d}\otimes\mathbf{d})+(\mathbf{d}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c})\Big]\Big(\frac{9}{8}G_4-\frac{3}{4}G_3\Big) \\
& +\Big[(\mathbf{d}\cdot\mathbf{n})(\mathbf{d}\otimes\mathbf{d})\Big]\Big(\frac{9}{8}G_5-\frac{3}{4}G_4\Big)\Big]\Bigg] \;\;,
\end{aligned} \tag{74}$$

N_2:

$$\begin{aligned}
\int_{-1}^{1} \left[\mathcal{T}_y^x(\mathbf{f}(\xi))\right]^T N_1(\xi)\,\frac{L}{2}\,d\,\xi \;\;=\;\; & -C\,L\Bigg[\Big(1-2\,\nu\Big)\Big[\Big[(\mathbf{c}\cdot\mathbf{n})\,\mathbf{I}\,+\mathbf{c}\otimes\mathbf{n}-\mathbf{n}\otimes\mathbf{c}\Big]\Big(\frac{9}{8}F_2+\frac{3}{4}F_1\Big) \\
& +\Big((\mathbf{d}\cdot\mathbf{n})\,\mathbf{I}\,+\mathbf{d}\otimes\mathbf{n}-\mathbf{n}\otimes\mathbf{d}\Big)\Big(\frac{9}{8}F_3+\frac{3}{4}F_2\Big)\Big] \\
& +2\,\Big[\Big((\mathbf{c}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{c})\Big)\Big(\frac{9}{8}G_2+\frac{3}{4}G_1\Big) \\
& +\Big[(\mathbf{c}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c})+(\mathbf{d}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{c})\Big]\Big(\frac{9}{8}G_3+\frac{3}{4}G_2\Big) \\
& +\Big[(\mathbf{c}\cdot\mathbf{n})(\mathbf{d}\otimes\mathbf{d})+(\mathbf{d}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c})\Big]\Big(\frac{9}{8}G_4+\frac{3}{4}G_3\Big) \\
& +\Big[(\mathbf{d}\cdot\mathbf{n})(\mathbf{d}\otimes\mathbf{d})\Big]\Big(\frac{9}{8}G_5+\frac{3}{4}G_4\Big)\Big]\Bigg] \;\;,
\end{aligned} \tag{75}$$

and N_3:

$$\int_{-1}^{1} \left[\mathcal{T}_y^x(\mathbf{f}(\xi))\right]^T N_1(\xi)\,\frac{L}{2}\,d\,\xi \;=\; -C\,L\Bigg[\Big(1-2\,\nu\Big)\Big[\Big[(\mathbf{c}\cdot\mathbf{n})\,\mathbf{I}\,+\mathbf{c}\otimes\mathbf{n}-\mathbf{n}\otimes\mathbf{c}\Big]\Big(F_0-\frac{9}{4}F_2\Big)$$
$$+\Big((\mathbf{d}\cdot\mathbf{n})\,\mathbf{I}\,+\mathbf{d}\otimes\mathbf{n}-\mathbf{n}\otimes\mathbf{d}\Big)\Big(F_1-\frac{9}{4}F_2\Big)\Big]$$
$$+2\Big[\Big((\mathbf{c}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{c})\Big)\Big(G_0-\frac{9}{4}G_2\Big)$$
$$+\Big[(\mathbf{c}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c})+(\mathbf{d}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{c})\Big]\Big(G_1-\frac{9}{4}G_3\Big)$$
$$+\Big[(\mathbf{c}\cdot\mathbf{n})(\mathbf{d}\otimes\mathbf{d})+(\mathbf{d}\cdot\mathbf{n})(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c})\Big]\Big(G_2-\frac{9}{4}G_4\Big)$$
$$+\Big[(\mathbf{d}\cdot\mathbf{n})(\mathbf{d}\otimes\mathbf{d})\Big]\Big(G_3-\frac{9}{4}G_5\Big)\Big]\Bigg]\;. \tag{76}$$

The integrals of 60.2 can easily be determined by taking the transpose of the dyadic products in 74-76 and replacing $\mathbf{n}(\mathbf{x})$ with $\mathbf{m}(\mathbf{y})$ whereupon we see that $\mathcal{D}_y^y = \mathcal{T}_y^y$ (note the superscript y on $\mathcal{T}_y^y$). For sake of brevity, these integrals have not been included.

Finally, the hypersingular kernel given by 51.2, involves an r_y^2 term. Therefore, as will be seen in the following, it is necessary to evaluate integrals of the form

$$\int_{-1}^{1} \frac{\xi^i}{(a\xi^2+b\xi+c)^3}\,d\,\xi = K_i\;\;, \tag{77}$$

where, for convenience, the integrals will be labeled by K_i cf. 68. Also, to lessen the burden upon expansion after the change of variable, the following constants are defined

$$\begin{aligned} \mathbf{c}\cdot\mathbf{n} &= c_n\;\;, \\ \mathbf{c}\cdot\mathbf{m} &= c_m\;\;, \\ \mathbf{d}\cdot\mathbf{n} &= d_n\;\;, \\ \mathbf{d}\cdot\mathbf{m} &= d_m\;\;, \end{aligned} \tag{78}$$

such that

$$\hat{\mathbf{r}}_y\cdot\mathbf{n} = \frac{(\mathbf{c}+\mathbf{d}\xi)\cdot\mathbf{n}}{(a\xi^2+b\xi+c)^{\frac{1}{2}}} \;=\; \frac{c_n+d_n\xi}{(a\xi^2+b\xi+c)^{\frac{1}{2}}}\;\;,$$
$$\hat{\mathbf{r}}_y\cdot\mathbf{m} = \frac{(\mathbf{c}+\mathbf{d}\xi)\cdot\mathbf{m}}{(a\xi^2+b\xi+c)^{\frac{1}{2}}} \;=\; \frac{c_m+d_m\xi}{(a\xi^2+b\xi+c)^{\frac{1}{2}}}\;\;,$$

$$\int_{-1}^{1} (\hat{\mathbf{r}}_y \cdot \mathbf{n})(\hat{\mathbf{r}}_y \cdot \mathbf{m})\, d\xi \;=\; \int_{-1}^{1} \frac{((\mathbf{c}+\mathbf{d}\xi)\cdot \mathbf{m})((\mathbf{c}+\mathbf{d}\xi)\cdot \mathbf{n})}{a\xi^2+b\xi+c} d\xi$$

$$= \left[d_m d_n \xi^2 + (d_m c_n + c_m d_n)\xi + c_m c_n\right] G_0 \;\; . \tag{79}$$

Once again, the expansion of 60.3 goes term by term and initially we only integrate the kernels, i.e. we consider $\int_{-1}^{1} \mathcal{S}_y^{xy}(\mathbf{f}(\xi))\, J(\xi)\, d\xi$

Term 1:

$$\int_{-1}^{1} \frac{-4\mu C}{r_y^2}\, 2\,\nu\, (\hat{\mathbf{r}}_y \cdot \mathbf{n})(\hat{\mathbf{r}}_y \cdot \mathbf{m})\mathbf{I}\, \frac{L}{2}\, d\xi \;=\; -4\mu C L\nu\, (d_m d_n G_2 + (d_m c_n + c_m d_n) G_1 + c_m c_n G_0)\mathbf{I} \;\; , \tag{80}$$

Term 2:

$$\int_{-1}^{1} \frac{-4\mu C}{r_y^2}\, 2\,\nu(\hat{\mathbf{r}}_y\cdot\mathbf{n})(\hat{\mathbf{r}}_y\otimes\mathbf{m})\,\frac{L}{2}\,d\xi \;=\; -4\mu CL\nu\,[c_n(\mathbf{c}\otimes\mathbf{m})G_0 + (d_n(\mathbf{c}\otimes\mathbf{m}) + c_n\,(\mathbf{d}\otimes\mathbf{m}))G_1 + d_n\,(\mathbf{d}\otimes\mathbf{m})\,G_2] \quad , \tag{81}$$

Term 3:

$$\begin{aligned}\int_{-1}^{1} \frac{-4\mu C}{r_y^2}\,(-8(\hat{\mathbf{r}}_y\cdot\mathbf{n})(\hat{\mathbf{r}}_y\cdot\mathbf{m}))(\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y)\,\frac{L}{2}\,d\xi \;=\;& 16\mu\,CL[[c_nc_m\mathbf{c}\otimes\mathbf{c}]K_0 \\ &+[c_nc_m(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c}) \\ &+(c_nd_m+d_nc_m)\mathbf{c}\otimes\mathbf{c}]K_1 \\ &+[c_nc_m\mathbf{d}\otimes\mathbf{d}+(d_nc_m+c_nd_m)(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c}) \\ &+d_nd_m\mathbf{c}\otimes\mathbf{c}]K_2+[(\mathbf{d}\otimes\mathbf{d})(d_nc_m+c_nd_m) \\ &+d_nd_m(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c})]K_3 \\ &+d_md_n(\mathbf{d}\otimes\mathbf{d})K_4] \quad ,\end{aligned} \tag{82}$$

Term 4:

$$\begin{aligned}\int_{-1}^{1} \frac{-4\mu C}{r_y^2}\,(1-2\,\nu)(2\hat{\mathbf{r}}_y\cdot\mathbf{n})(\mathbf{m}\otimes\hat{\mathbf{r}}_y)\,\frac{L}{2}\,d\xi \;=\;& -4\mu C\,L\,(1-2\nu)[G_0(c_n\mathbf{m}\otimes\mathbf{c}) \\ &+G_1(d_n\mathbf{m}\otimes\mathbf{c}+c_n\mathbf{m}\otimes\mathbf{d}) \\ &+G_2d_n(\mathbf{m}\otimes\mathbf{d})] \quad ,\end{aligned} \tag{83}$$

Term 5:

$$\begin{aligned}\int_{-1}^{1} \frac{-4\mu C}{r_y^2}\,(1-2\,\nu)(2\,\hat{\mathbf{r}}_y\cdot\mathbf{m})(\hat{\mathbf{r}}_y\otimes\mathbf{n})\,\frac{L}{2}\,d\xi \;=\;& -4\mu CL\,(1-2\nu)(G_0[c_m\mathbf{c}\otimes\mathbf{n}] \\ &+G_1[d_m\mathbf{c}\otimes\mathbf{n}+c_m\mathbf{d}\otimes\mathbf{n}]+G_2d_m\mathbf{d}\otimes\mathbf{n}) \;,\end{aligned} \tag{84}$$

Term 6:

$$\int_{-1}^{1} \frac{-4\mu C}{r_y^2}\,(1-2\,\nu)(\mathbf{n}\cdot\mathbf{m})\mathbf{I}\,\frac{L}{2}\,d\xi \;=\; -2\mu CL(1-2\,\nu)(\mathbf{n}\cdot\mathbf{m})\mathbf{I}\,F_0 \;\;, \tag{85}$$

Term 7:

$$\int_{-1}^{1} \frac{-4\mu C}{r_y^2}(1-2\,\nu)(\mathbf{n}\otimes\mathbf{m})\,\frac{L}{2}\,d\xi \;=\; -2\mu CL(1-2\,\nu)(\mathbf{n}\otimes\mathbf{m})\,F_0 \;\;, \tag{86}$$

Term 8:

$$\int_{-1}^{1} \frac{-4\mu C}{r_y^2} 2\,\nu(\hat{\mathbf{r}}_y \cdot \mathbf{m})(\mathbf{n}\otimes\hat{\mathbf{r}}_y)\,\frac{L}{2}\,d\xi \;=\; -4\mu CL\,\nu\,[G_0 c_m \mathbf{n}\otimes\mathbf{c} + G_1(d_m\mathbf{n}\otimes\mathbf{c} + c_m\mathbf{n}\otimes\mathbf{d}) + G_2 d_m \mathbf{n}\otimes\mathbf{d}]\quad , \tag{87}$$

Term 9:

$$\int_{-1}^{1} \frac{-4\mu C}{r_y^2} 2\,\nu\,(\mathbf{n}\cdot\mathbf{m})(\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y)\,\frac{L}{2}\,d\xi \;=\; -4\mu CL\;\nu\,(\mathbf{n}\cdot\mathbf{m})[G_0\mathbf{c}\otimes\mathbf{c} + G_1(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c}) + G_2\mathbf{d}\otimes\mathbf{d}]\quad , \tag{88}$$

Term 10:

$$\int_{-1}^{1} \frac{-4\mu C}{r_y^2} 2\,(1-4\,\nu)(\mathbf{m}\otimes\mathbf{n})\,\frac{L}{2}\,d\xi \;=\; -4\mu CL\,(1-4\,\nu)(\mathbf{m}\otimes\mathbf{n})F_0\ \ . \tag{89}$$

Combining terms yields

$$\begin{aligned}\int_{-1}^{1} \mathcal{S}_y^{xy}(\mathbf{g}(\xi))\,\frac{L}{2}\,d\xi \;=\; & -2\mu\,CL\,[G_0[2\,\nu(c_m c_n\mathbf{I} + c_n(\,\mathbf{c}\otimes\mathbf{m} - 2\,\mathbf{m}\otimes\mathbf{c}) + c_m\,(\mathbf{n}\otimes\mathbf{c} - 2\mathbf{c}\otimes\mathbf{n}) \\ & +(\mathbf{n}\cdot\mathbf{m})\,\mathbf{c}\otimes\mathbf{c}) + 2\,(c_m\,\mathbf{c}\otimes\mathbf{n} + c_n\,\mathbf{m}\otimes\mathbf{c})] \\ & +G_1[2\,\nu\,((d_m c_n + c_m d_n)\mathbf{I} + (\mathbf{n}\cdot\mathbf{m})(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c}) \\ & +c_n(\mathbf{d}\otimes\mathbf{m} - 2\mathbf{m}\otimes\mathbf{d}) + c_m(\mathbf{n}\otimes\mathbf{d} - 2\,\mathbf{d}\otimes\mathbf{n}) \\ & +d_n(\mathbf{c}\otimes\mathbf{m} - 2\mathbf{m}\otimes\mathbf{c}) + d_m(\mathbf{n}\otimes\mathbf{c} - 2\,\mathbf{c}\otimes\mathbf{n})) \\ & +2(d_n\,(\mathbf{m}\otimes\mathbf{c}) + d_m\,(\mathbf{c}\otimes\mathbf{n}) + c_n\,(\mathbf{m}\otimes\mathbf{d}) + c_m\,(\mathbf{d}\otimes\mathbf{n}))] \\ & +G_2[2\,\nu\,(d_m d_n\,\mathbf{I} + d_n(\mathbf{d}\otimes\mathbf{m} - 2\,\mathbf{m}\otimes\mathbf{d}) + d_m(\mathbf{n}\otimes\mathbf{d} - 2\mathbf{d}\otimes\mathbf{n}) \\ & +(\mathbf{n}\cdot\mathbf{m})(\mathbf{d}\otimes\mathbf{d})) + 2(\,d_n\,\mathbf{m}\otimes\mathbf{d} + d_m\mathbf{d}\otimes\mathbf{n})] \\ & 8[[c_n c_m\,\mathbf{c}\otimes\mathbf{c}]K_0 + [c_n c_m(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c}) + (c_n d_m + d_n c_m)\mathbf{c}\otimes\mathbf{c}]K_1 \\ & +[c_n c_m\mathbf{d}\otimes\mathbf{d} + (d_n c_m + c_n d_m)(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c}) + d_n d_m\mathbf{c}\otimes\mathbf{c}]K_2 \\ & +[(\mathbf{d}\otimes\mathbf{d})(d_n c_m + c_n d_m) + d_n d_m(\mathbf{c}\otimes\mathbf{d}+\mathbf{d}\otimes\mathbf{c})]K_3 + K_4 d_m d_n\mathbf{d}\otimes\mathbf{d}] \\ & +((1-2\,\nu)(\mathbf{n}\otimes\mathbf{m} + \mathbf{n}\cdot\mathbf{m}) - (1-4\,\nu)(\mathbf{m}\otimes\mathbf{n}))\,F_0]\quad .\end{aligned} \tag{90}$$

Using an analogous recursion formula for K_i as for F_i and G_i in 72 and 73 yields for shape function

N_1:

$$\int_{-1}^{1} \mathcal{S}_y^{xy}(\mathbf{g}(\xi))\, N_1(\xi)\, \frac{L}{2}\, d\xi \;=\; -2\mu\, CL\, [(\frac{9}{8}G_2 - \frac{3}{4}G_1)[2\,\nu(c_m c_n - c_n\,\mathbf{m}\otimes\mathbf{c} - 2\,c_m\,\mathbf{c}\otimes\mathbf{n}$$
$$+(\mathbf{n}\cdot\mathbf{m})\,\mathbf{c}\otimes\mathbf{c}) + c_n\,(2\mathbf{c}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{c}) + 2\,c_m\,(\mathbf{n}\otimes\mathbf{c} + \mathbf{c}\otimes\mathbf{n})]$$
$$+(\frac{9}{8}G_3 - \frac{3}{4}G_2)[2\,\nu\,(d_m c_n + c_m d_n + \mathbf{c}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{c}$$
$$-2(d_m\,\mathbf{c}\otimes\mathbf{n} + c_m\,\mathbf{d}\otimes\mathbf{n} + d_n\,\mathbf{m}\otimes\mathbf{c} + c_n\,\mathbf{m}\otimes\mathbf{d}))$$
$$+2(d_n\,(\mathbf{c}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{c}) + d_m\,(\mathbf{n}\otimes\mathbf{c} + \mathbf{c}\otimes\mathbf{n})$$
$$+c_n\,(\mathbf{d}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{d}) + c_m\,(\mathbf{n}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{n}))]$$
$$+(\frac{9}{8}G_4 - \frac{3}{4}G_3)[2\,\nu\,(d_m d_n + \mathbf{d}\otimes\mathbf{d} - 2\,d_m\,\mathbf{d}\otimes\mathbf{n} - 2\,d_n\,\mathbf{m}\otimes\mathbf{d})$$
$$+2\,d_n\,(\mathbf{d}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{d}) + 2\,d_m\,(\mathbf{d}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{d})]$$
$$8[[c_n c_m \mathbf{c}\otimes\mathbf{c}](\frac{9}{8}K_2 - \frac{3}{4}K_1)$$
$$+[c_n c_m(\mathbf{c}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{c}) + (c_n d_m + d_n c_m)\mathbf{c}\otimes\mathbf{c}](\frac{9}{8}K_3 - \frac{3}{4}K_2)$$
$$+[c_n c_m \mathbf{d}\otimes\mathbf{d} + (d_n c_m + c_n d_m)(\mathbf{c}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{c}) + d_n d_m \mathbf{c}\otimes\mathbf{c}]$$
$$\times(\frac{9}{8}K_4 - \frac{3}{4}K_3) + [(\mathbf{d}\otimes\mathbf{d})(d_n c_m + c_n d_m) + d_n d_m(\mathbf{c}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{c})]$$
$$\times(\frac{9}{8}K_5 - \frac{3}{4}K_3)] + (1-2\,\nu)(\mathbf{n}\otimes\mathbf{m} + \mathbf{n}\cdot\mathbf{m})\,(\frac{9}{8}F_2 - \frac{3}{4}F_1)] \quad , \qquad (91)$$

N_2:

$$\int_{-1}^{1} \mathcal{S}_y^{xy}(\mathbf{g}(\xi))\, N_2(\xi)\, \frac{L}{2}\, d\xi \;=\; -2\mu\, CL\, [(\frac{9}{8}G_2 + \frac{3}{4}G_1)[2\,\nu(c_m c_n - c_n\,\mathbf{m}\otimes\mathbf{c} - 2\,c_m\,\mathbf{c}\otimes\mathbf{n}$$
$$+(\mathbf{n}\cdot\mathbf{m})\,\mathbf{c}\otimes\mathbf{c}) + c_n\,(2\mathbf{c}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{c}) + 2\,c_m\,(\mathbf{n}\otimes\mathbf{c} + \mathbf{c}\otimes\mathbf{n})]$$
$$+(\frac{9}{8}G_3 + \frac{3}{4}G_2)[2\,\nu\,(d_m c_n + c_m d_n + \mathbf{c}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{c}$$
$$-2(d_m\,\mathbf{c}\otimes\mathbf{n} + c_m\,\mathbf{d}\otimes\mathbf{n} + d_n\,\mathbf{m}\otimes\mathbf{c} + c_n\,\mathbf{m}\otimes\mathbf{d}))$$
$$+2(d_n\,(\mathbf{c}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{c}) + d_m\,(\mathbf{n}\otimes\mathbf{c} + \mathbf{c}\otimes\mathbf{n})$$
$$+c_n\,(\mathbf{d}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{d}) + c_m\,(\mathbf{n}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{n}))]$$
$$+(\frac{9}{8}G_4 + \frac{3}{4}G_3)[2\,\nu\,(d_m d_n + \mathbf{d}\otimes\mathbf{d} - 2\,d_m\,\mathbf{d}\otimes\mathbf{n} - 2\,d_n\,\mathbf{m}\otimes\mathbf{d})$$
$$+2\,d_n\,(\mathbf{d}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{d}) + 2\,d_m\,(\mathbf{d}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{d})]$$
$$8[[c_n c_m \mathbf{c}\otimes\mathbf{c}](\frac{9}{8}K_2 + \frac{3}{4}K_1)$$
$$+[c_n c_m(\mathbf{c}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{c}) + (c_n d_m + d_n c_m)\mathbf{c}\otimes\mathbf{c}](\frac{9}{8}K_3 + \frac{3}{4}K_2)$$
$$+[c_n c_m \mathbf{d}\otimes\mathbf{d} + (d_n c_m + c_n d_m)(\mathbf{c}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{c}) + d_n d_m \mathbf{c}\otimes\mathbf{c}]$$
$$\times(\frac{9}{8}K_4 + \frac{3}{4}K_3) + [(\mathbf{d}\otimes\mathbf{d})(d_n c_m + c_n d_m) + d_n d_m(\mathbf{c}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{c})]$$
$$\times(\frac{9}{8}K_5 + \frac{3}{4}K_3)] + (1-2\,\nu)(\mathbf{n}\otimes\mathbf{m} + \mathbf{n}\cdot\mathbf{m})\,(\frac{9}{8}F_2 + \frac{3}{4}F_1)] \quad , \qquad (92)$$

N_3:

$$\int_{-1}^{1} \mathcal{S}_y^{xy}(\mathbf{g}(\xi))\, N_3(\xi)\, \frac{L}{2}\, d\xi \;=\; -2\mu\, C L\, [(G_0 - \frac{9}{4} G_2)[2\,\nu (c_m c_n - c_n\, \mathbf{m} \otimes \mathbf{c} - 2\, c_m\, \mathbf{c} \otimes \mathbf{n}$$
$$+(\mathbf{n}\cdot\mathbf{m})\, \mathbf{c}\otimes\mathbf{c}) + c_n\, (2\mathbf{c}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{c}) + 2\, c_m\, (\mathbf{n}\otimes\mathbf{c} + \mathbf{c}\otimes\mathbf{n})]$$
$$+(G_1 - \frac{9}{4} G_3)[2\,\nu\, (d_m c_n + c_m d_n + \mathbf{c}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{c}$$
$$-2(d_m\, \mathbf{c}\otimes\mathbf{n} + c_m\, \mathbf{d}\otimes\mathbf{n} + d_n\, \mathbf{m}\otimes\mathbf{c} + c_n\, \mathbf{m}\otimes\mathbf{d}))$$
$$+2(d_n\, (\mathbf{c}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{c}) + d_m\, (\mathbf{n}\otimes\mathbf{c} + \mathbf{c}\otimes\mathbf{n})$$
$$+c_n\, (\mathbf{d}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{d}) + c_m\, (\mathbf{n}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{n}))]$$
$$+(G_2 - \frac{9}{4} G_4)[2\,\nu\, (d_m d_n + \mathbf{d}\otimes\mathbf{d} - 2\, d_m\, \mathbf{d}\otimes\mathbf{n} - 2\, d_n\, \mathbf{m}\otimes\mathbf{d})$$
$$+2\, d_n\, (\mathbf{d}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{d}) + 2\, d_m\, (\mathbf{d}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{d})]$$
$$8[[c_n c_m \mathbf{c}\otimes\mathbf{c}](K_0 - \frac{9}{4} K_2)$$
$$+[c_n c_m (\mathbf{c}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{c}) + (c_n d_m + d_n c_m)\mathbf{c}\otimes\mathbf{c}](K_1 - \frac{9}{4} K_3)$$
$$+[c_n c_m \mathbf{d}\otimes\mathbf{d} + (d_n c_m + c_n d_m)(\mathbf{c}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{c}) + d_n d_m \mathbf{c}\otimes\mathbf{c}]$$
$$\times(K_2 - \frac{9}{4} K_4) + [(\mathbf{d}\otimes\mathbf{d})(d_n c_m + c_n d_m) + d_n d_m (\mathbf{c}\otimes\mathbf{d} + \mathbf{d}\otimes\mathbf{c})]$$
$$\times(K_3 - \frac{9}{4} K_5)]$$
$$+(1 - 2\,\nu)(\mathbf{n}\otimes\mathbf{m} + \mathbf{n}\cdot\mathbf{m})\, (F_0 - \frac{9}{4} F_2)] \quad . \tag{93}$$

The following integrals F_i, G_i, and K_i appear in Zhang [19] cf. A16-A19, A23-A28, A30-A-36 and are verified in Abramowitz [22], cf. 3.3.12, 3.3.17, and 3.3.18.

$$F_0 = \begin{cases} \frac{2}{\sqrt{4ac-b^2}} \tan^{-1}\left(\frac{2a+b}{\sqrt{4ac-b^2}}\right) - \tan^{-1}\frac{-2a+b}{\sqrt{4ac-b^2}} & \text{for } (4ac > b^2) \\ \frac{2}{b-2a} - \frac{2}{b+2a} & \text{for } (4ac = b^2) \end{cases} ,$$

$$F_1 = \frac{1}{2a} \ln\frac{a+b+c}{a-b+c} - \frac{b}{2a} F_0 \quad ,$$
$$F_2 = \frac{2}{a} - \frac{c}{a} F_0 - \frac{b}{a} F_1 \quad ,$$
$$F_3 = -\frac{c}{a} F_1 - \frac{b}{a} F_2 \quad , \tag{94}$$

$$G_0 = \begin{cases} \frac{2a+b}{(4ac-b^2)(a+b+c)} - \frac{-2a+b}{(4ac-b^2)(a-b+c)} + \frac{2a}{4ac-b^2} F_0 & \text{for } (4ac > b^2) \\ \frac{8a}{3(b-2a)^3} - \frac{8a}{3(b+2a)^3} & \text{for } (4ac = b^2) \end{cases} ,$$

$$G_1 = \begin{cases} \frac{b+2c}{(b^2-4ac)(a+b+c)} - \frac{-b+2c}{(4ac-b^2)(a-b+c)} + \frac{b}{b^2-4ac} F_0 & \text{for } (4ac > b^2) \\ \left(\frac{8a}{3(2a+b)^3} - \frac{8a}{3(-2a+b)^3}\right) - \frac{2}{3}\left(\frac{1}{(2a+b)^2} - \frac{1}{(-2a+b)^2}\right) & \text{for } (4ac = b^2) \end{cases} ,$$

$$\begin{aligned} G_2 &= \frac{-1}{a(a-b+c)} - \frac{1}{a(a+b+c)} + \frac{c}{a}G_0 \quad , \\ G_3 &= \frac{1}{2a^2}\left(\ln\frac{a+b+c}{a-b+c} - 3abG_2 - (2ac+b^2)G_1 - bcG_0\right) \quad , \\ G_4 &= \frac{1}{a}\left(\frac{1}{a+b+c} + \frac{1}{a-b+c}\right) - \frac{2b}{a}G_3 - \frac{3c}{a}G_2 \quad , \\ G_5 &= \frac{1}{2a}\left(\frac{1}{a+b+c} - \frac{1}{a-b+c}\right) - \frac{3b}{2a}G_4 - \frac{2c}{a}G_3 \quad , \end{aligned} \tag{95}$$

$$K_0 = \begin{cases} \frac{1}{2(4ac-b^2)}\left[\frac{b+2a}{(a+b+c)^2} - \frac{b-2a}{(a-b+c)^2}\right] + \frac{6a}{2(4ac-b^2)} G_0 & \text{for } (4ac > b^2) \\ -\frac{32a^2}{5}\left[\frac{1}{(2a+b)^5} - \frac{1}{-2a+b)^5}\right] & \text{for } (4ac = b^2) \end{cases} ,$$

$$K_1 = \begin{cases} \frac{-1}{2(4ac-b^2)}\left[\frac{b+2c}{(a+b+c)^2} + \frac{b-2c}{(a-b+c)^2}\right] + \frac{-3b}{2(4ac-b^2)} G_0 & \text{for } (4ac > b^2) \\ -\frac{32a^2}{5}\left[\frac{1}{(2a+b)^5} + \frac{1}{-2a+b)^5}\right] - \frac{4a}{5}\left[\frac{1}{(2a+b)^4} - \frac{1}{(-2a+b)^4}\right] & \text{for } (4ac = b^2) \end{cases} ,$$

$$\begin{aligned} K_2 &= -\frac{1}{3a}\left[\frac{1}{(a+b+c)^2} + \frac{1}{(a-b+c)^2}\right] - \frac{b}{3a}K_1 + \frac{c}{3a}K_0 \quad , \\ K_3 &= -\frac{1}{2a}\left[\frac{1}{(a+b+c)^2} - \frac{1}{(a-b+c)^2}\right] + \frac{c}{a}K_1 \quad , \\ K_4 &= -\frac{1}{a}\left[\frac{1}{(a+b+c)^2} + \frac{1}{(a-b+c)^2}\right] + \frac{b}{a}K_3 + \frac{3c}{a}K_2 \quad , \\ K_5 &= \frac{1}{a}G_3 - \frac{b}{a}K_4 - \frac{c}{a}K_3 \quad , \\ K_6 &= \frac{1}{a}\left[\frac{1}{(a+b+c)} + \frac{1}{(a-b+c)}\right] - \frac{3b}{a}K_5 - \frac{5c}{a}K_4 \quad . \end{aligned} \tag{96}$$

4.4 Weak, Strong, and Hypersingular integration - $\mathbf{y} \in e$

When $\mathbf{y} \in e$ is considered singularities degrade the accuracy of the standard Gauss Quadrature [20]. In the standard BEM the weakly singular kernel $\mathcal{U}_y$ is integrated using subdivision techniques and the strongly singular kernel $\mathcal{T}_y^x$ is integrated using the rigid body motion technique [21]. In the latter, we consider a rigid body motion $\mathbf{u} = \mathbf{e}$, where for the two translations $\mathbf{e} = \mathbf{e}_1$ or $\mathbf{e} = \mathbf{e}_2$, we have $\mathbf{t}^{\mathbf{n}} = \mathbf{0}$ and hence the singular integrals of $\mathcal{T}_y^x$ and $\mathcal{S}_y^{xy}$ over e are found via 51, i.e.

$$\begin{aligned} \fint_{\partial\Omega\cap e} [\mathcal{T}_y^x(\mathbf{x})]^T \mathbf{e} &= -\frac{1}{2}\mathbf{e} + \int_{\partial\Omega/e} [\mathcal{T}_y^x(\mathbf{x})]^T \mathbf{e} \ , \\ \fint\!\!\!\!=_{\partial\Omega\cap e} \mathcal{S}_y^{xy}(\mathbf{x})\mathbf{e} &= -\int_{\partial\Omega/e} \mathcal{S}_y^{xy}(\mathbf{x})\mathbf{e} \ . \end{aligned} \tag{97}$$

Unfortunately, the rigid body motion technique cannot be used to evaluate the singular integral $\fint_{\partial\Omega\cap e} \mathcal{D}_y^y(\mathbf{x})\, \mathbf{t}^{\mathbf{n}}(\mathbf{x})$ in the DBEM.

It was first shown by Portela & Alibadi [7] that the use of discontinuous shape functions allow for the analytical evaluation of these singular and hypersingular integrals, which are often referred to as the Cauchy-Principal Value (CPV) and the Hardamard value for the singular traction potential, $\mathcal{T}_y^x$, and hypersingular potential $\mathcal{S}_y^{xy}$, respectively. As previously stated, the analytical technique which we present is only applicable to flat elements with evenly spaced nodes. In [7], it is extended to curved elements.

The paper by Portella & Alibadi [7] considers finite domain cracks, and in turn traction free crack faces. Therefore, [7] only provides results for the integrands containing $[\mathcal{T}_y^x]^T \mathbf{u}(\mathbf{x})$ and $\mathcal{S}_y^{xy}\mathbf{u}(\mathbf{x})$ as $\mathbf{t}^{\mathbf{n}} = \mathbf{0}$. However, we apply the traction to the crackfaces for our semi-infinite domain problems, thusly we must consider all kernels. Fortunately, the paper by Tuhkuri [10] gives these results for the remaining integrals, i.e. for the integrands containing $\mathcal{D}_y^y\, \mathbf{t}^{\mathbf{n}}(\mathbf{x})$ and $\mathcal{U}_y\, \mathbf{t}^{\mathbf{n}}(\mathbf{x})$. The procedure used in both papers is highlighted in what follows and corrections of typographical errors found in [10] are noted.

Looking first at the displacement potential $\mathcal{U}_y$, letting $\mathbf{x} = \mathbf{f}(\xi)$ and the collocation point $\mathbf{Y}^I = \mathbf{f}(\xi')$, and using the interpolation of the tractions in 56 gives

$$\begin{aligned}
\int_{\partial\Omega} \mathcal{U}_y(\mathbf{x})\,\mathbf{t}^{\mathbf{n}}(\mathbf{x})\,dS &= \int_{-1}^{1} \frac{C\,L}{2\mu} \left[\left[(3-4\,\nu)\,\ln\,\frac{1}{|\xi-\xi'|}\,\mathbf{I} + \hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y\right](\mathbf{N}^T(\xi)\odot\mathbf{I})\,d\xi\right]\mathrm{vec}(\mathbf{T}) \\
&= -\frac{(3-4\,\nu)C\,L}{2\mu}\int_{-1}^{1}\left[\mathbf{I}\left(\ln\,|\xi-\xi'|\,+\ln\frac{L}{2}\right)(\mathbf{N}^T(\xi)\odot\mathbf{I})\,d\xi\right]\mathrm{vec}(\mathbf{T}) \\
&\quad +\frac{C\,L}{2\mu}\int_{-1}^{1}\left[(\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y)\,(\mathbf{N}^T(\xi)\odot\mathbf{I})\,d\xi\right]\mathrm{vec}(\mathbf{T}) \\
&= -\frac{(3-4\,\nu)C\,L}{2\mu}\,\mathbf{I}\left(\int_{-1}^{1}\ln\,|\xi-\xi'|\,\mathbf{N}^T(\xi)\,d\xi\odot\mathbf{I}\right)\mathrm{vec}(\mathbf{T}) \\
&\quad -\frac{C\,L}{2\mu}\left[(3-4\nu)\ln\frac{L}{2}\,\mathbf{I}-\hat{\mathbf{r}}_y\otimes\hat{\mathbf{r}}_y\right]\left(\int_{-1}^{1}\mathbf{N}^T(\xi)\,d\xi\odot\mathbf{I}\right)\mathrm{vec}(\mathbf{T}) \quad , \qquad (98)
\end{aligned}$$

where the following relationships are used

$$\begin{aligned}
dS &= \frac{L}{2}d\xi \;\;, \\
r_y &= \frac{L}{2}|\xi-\xi'| \;\;, \qquad (99)
\end{aligned}$$

and hence

$$\begin{aligned}
\ln\frac{1}{r_y} &= \ln\frac{1}{\frac{L}{2}|\xi-\xi'|} \\
&= -\ln|\xi-\xi'| - \ln\frac{L}{2} \;\;. \qquad (100)
\end{aligned}$$

Note that $\hat{\mathbf{r}}_y(\mathbf{x})$ is constant for flat elements and thusly is taken out of the integral. Results for the first integral in 98 are obtained [4] using the `Mathematica` command
`Assuming[` ξ' `<1 &` ξ' `> -1, Integrate[f(`ξ`) , {`ξ`, -1, 1}]]`

$$\begin{aligned}
\int_{-1}^{1} N_1(\xi)\,\ln|\xi-\xi'|\,d\xi &= \frac{3}{8}\left[\ln(1-\xi'^2)-(\xi'^3-\xi'^2+1)\ln\frac{|1-\xi'|}{|1+\xi'|}-2(\xi'^2-\xi'+\frac{1}{3})\right] \;, \\
\int_{-1}^{1} N_2(\xi)\,\ln|\xi-\xi'|\,d\xi &= \frac{3}{8}\left[\ln(1-\xi'^2)-(\xi'^3+\xi'^2-1)\ln\frac{|1-\xi'|}{|1+\xi'|}-2(\xi'^2+\xi'+\frac{1}{3})\right] \;, \\
\int_{-1}^{1} N_3(\xi)\,\ln|\xi-\xi'|\,d\xi &= \frac{1}{4}\left[\ln(1-\xi'^2)+\xi'(3\xi'^2-4)\ln\frac{|1-\xi'|}{|1+\xi'|}-6(1-\xi'^2)\right] \;. \qquad (101)
\end{aligned}$$

The second integral in 98 is evaluated using the same `Mathematica` command

[4] where the error has been corrected in Tuhkuri [10], cf. 42.2.

$$\int_{-1}^{1} N_1(\xi)\, d\xi = \frac{3}{4} ,$$
$$\int_{-1}^{1} N_2(\xi)\, d\xi = \frac{3}{4} ,$$
$$\int_{-1}^{1} N_3(\xi)\, d\xi = \frac{1}{2} . \tag{102}$$

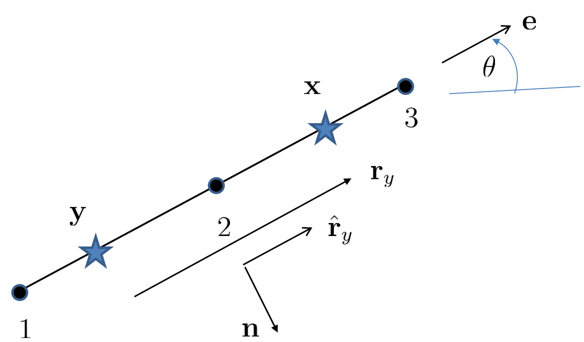


Figure 7: Discontinuous element for $\mathbf{y} \in e$.

As seen in Figure 7, $(\hat{\mathbf{r}}_y \cdot \mathbf{n}) = 0$ and thus the kernels $[\mathcal{T}_y^x(\mathbf{x})]^T$ and $\mathcal{D}_y^y(\mathbf{x})$ simplify to

$$\begin{aligned}
\fint_{\partial\Omega} [\mathcal{T}_y^x(\mathbf{x})]^T \, \mathbf{u}(\mathbf{x})\, dS &= -\int_{\partial\Omega} \frac{2\,C}{r_y} \, [(1-2\,\nu)((\hat{\mathbf{r}}_y \cdot \mathbf{n})\,\mathbf{I} + \hat{\mathbf{r}}_y \otimes \mathbf{n} - \mathbf{n} \otimes \hat{\mathbf{r}}_y) \\
&\quad +2\,(\hat{\mathbf{r}}_y \cdot \mathbf{n})\,\hat{\mathbf{r}}_y \otimes \hat{\mathbf{r}}_y]^T \, \mathbf{u}\, dS \\
&= \int_{\partial\Omega} \frac{2\,C}{r_y} \, [(1-2\,\nu)(\hat{\mathbf{r}}_y \otimes \mathbf{n} - \mathbf{n} \otimes \hat{\mathbf{r}}_y)] \, \mathbf{u}\, dS \\
&= \int_{\partial\Omega} \frac{2\,C}{r_y} \, (1-2\,\nu)(\mathbf{n} \times \hat{\mathbf{r}}_y) \times \mathbf{u}\, dS \\
&= \int_{\partial\Omega} \frac{2\,C}{r_y} \, (1-2\,\nu)\,(\mathbf{e} \cdot \hat{\mathbf{r}}_y)\,\mathbf{e}_3 \times \mathbf{u}\, dS \\
&= \int_{\partial\Omega} \frac{2\,C}{r_y} \, (1-2\,\nu)\,(\mathbf{e} \cdot \hat{\mathbf{r}}_y) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{u}\, dS \ ,
\end{aligned} \tag{103}$$

where the normal is assigned such that $\mathbf{n} = \mathbf{e} \times \mathbf{e}_3$ where $\mathbf{e}$ is the unit vector that is oriented along the element, cf. Figure 7. In this way, $\mathbf{n} \times \hat{\mathbf{r}}_y = (\mathbf{e} \cdot \hat{\mathbf{r}}_y)\, \mathbf{e}_3$. Once again, defining $\mathbf{x} = \mathbf{f}(\xi)$ and $\mathbf{Y}^I = \mathbf{f}(\xi')$, cf. [7] 15,

$$
\begin{aligned}
\int_{\partial\Omega} [\mathcal{T}_y^x(\mathbf{x})]^T\, \mathbf{u}(\mathbf{x})\, dS &= \int_{-1}^{1} \frac{2\,C}{|\xi - \xi'|}\, (1 - 2\,\nu)\, \mathrm{sign}(\xi - \xi') \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{u}\, d\xi \\
&= \int_{-1}^{1} \frac{2\,C}{\xi - \xi'}\, (1 - 2\,\nu) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{u}\, d\xi \\
&= 2\,C\,(1 - 2\,\nu) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \int_{-1}^{1} \frac{1}{\xi - \xi'}\, \mathbf{u}\, d\xi \ ,
\end{aligned} \tag{104}
$$

where the same relationships given by 99 are used along with the fact that $(\mathbf{e} \cdot \hat{\mathbf{r}}_y) = \mathrm{sign}(\xi - \xi')$ and $|x| = \mathrm{sign}(x)\, x$. Using the discretization described in 56 gives,

$$
\begin{aligned}
\int_{\partial\Omega} [\mathcal{T}_y^x(\mathbf{x})]^T\, \mathbf{u}(\mathbf{x})\, dS &= 2\,C\,(1 - 2\,\nu) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \int_{-1}^{1} \frac{1}{\xi - \xi'}\, (\mathbf{N}^T(\xi) \odot \mathbf{I})\, d\xi\ \mathrm{vec}\,(\mathbf{U}) \\
&= 2\,C\,(1 - 2\,\nu) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left(\int_{-1}^{1} \frac{\mathbf{N}^T(\xi)}{\xi - \xi'}\, d\xi \odot \mathbf{I} \right) \mathrm{vec}\,(\mathbf{U}) \ .
\end{aligned} \tag{105}
$$

By applying the Cauchy Principal Value, e.g. $⨍_{-1}^{1} \frac{N_i(\xi)}{\xi - \xi'}\, d\xi = \lim_{\epsilon \to 0^+} \left(\int_{-1}^{-\epsilon} \frac{N_i(\xi)}{\xi - \xi'}\, d\xi + \int_{\epsilon}^{1} \frac{N_i(\xi)}{\xi - \xi'}\, d\xi \right)$, the singular integral in 105 is evaluated via the `Mathematica` command

`Assume[` ξ' `<1 &` ξ' `>-1, Integrate[` $\mathbf{f}(\xi)$ `,` $\{\xi$ `, -1, 1` $\}$`,PrincipalValue -> True]`

$$
\begin{aligned}
⨍_{-1}^{1} \frac{N_1(\xi)}{\xi - \xi'}\, d\xi &= \frac{3}{8}(3\xi - 2) \left(\xi \log\left(\frac{2}{\xi + 1} - 1 \right) + 2 \right) = \frac{3}{8}(3\xi' - 2) \left(\xi' \log\left(\frac{1 - \xi'}{\xi' + 1} \right) + 2 \right), \\
⨍_{-1}^{1} \frac{N_2(\xi)}{\xi - \xi'}\, d\xi &= \frac{3}{8}(3\xi' + 2) \left(\xi' \log\left(\frac{2}{\xi' + 1} - 1 \right) + 2 \right) = \frac{3}{8}(3\xi' + 2) \left(\xi' \log\left(\frac{1 - \xi'}{\xi' + 1} \right) + 2 \right) \ , \\
⨍_{-1}^{1} \frac{N_3(\xi)}{\xi - \xi'}\, d\xi &= \left(1 - \frac{9(\xi')^2}{4} \right) \log\left(\frac{2}{\xi' + 1} - 1 \right) - \frac{9\xi'}{2} = \frac{1}{4}\left(4 - 9(\xi')^2\right) \log\left(\frac{1 - \xi'}{\xi' + 1} \right) - \frac{9\xi'}{2} \ ,
\end{aligned} \tag{106}
$$

cf. [7] 16-18. For these singular elements $\mathbf{m}(\mathbf{y}) = \mathbf{n}(\mathbf{x})$, hence $\mathcal{D}_y^y(\mathbf{x}) = \mathcal{T}_y^x(\mathbf{x})$. Rather than repeating the same procedure, only the result for $\mathcal{D}_y^y(\mathbf{x})$ is presented

$$
\int_S \mathcal{D}_y^y(\mathbf{x})\, \mathbf{t}^{\mathbf{n}}(\mathbf{x}) = 2\,C\,(1 - 2\,\nu) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \left(\int_{-1}^{1} \frac{\mathbf{N}^T(\xi)}{\xi - \xi'}\, d\xi \odot \mathbf{I} \right) \mathrm{vec}\,(\mathbf{T}) \ , \tag{107}
$$

where the integrals of the shape functions are provided in 106. Lastly, for the hypersingular kernel $\mathcal{S}_y^{xy}$, and again using $\hat{\mathbf{r}} \cdot \mathbf{n} = 0$, we obtain

$$\begin{aligned}
\int_{\partial\Omega} \mathcal{Z}_y^x(\mathbf{x})\,\mathbf{u}(\mathbf{x})\,dS\,\mathbf{n}(\mathbf{y}) &= \int_{\partial\Omega} \frac{-4\,\mu\,C}{r_y^2}\,\{(1-2\,\nu)(2\,\hat{\mathbf{r}}_y \otimes \hat{\mathbf{r}}_y \otimes \mathbf{n} + \mathbb{T}_3\,\mathbf{n} \otimes \mathbf{I} + \mathbf{n} \otimes \mathbf{I}) \\
&\quad +2\,\nu\,(\mathbf{n} \otimes \hat{\mathbf{r}}_y \otimes \hat{\mathbf{r}}_y + \hat{\mathbf{r}}_y \otimes \mathbf{n} \otimes \hat{\mathbf{r}}_y) - (1-4\,\nu)\,\mathbf{I} \otimes \mathbf{n}\}\;\mathbf{u}\,dS\,\mathbf{n} \\
&= -4\,\mu\,C \int_{\partial\Omega} \frac{1}{r_y^2}\,\{(1-2\,\nu)(2\,(\mathbf{u}\cdot\mathbf{n})\,\hat{\mathbf{r}}_y \otimes \hat{\mathbf{r}}_y + \mathbf{u} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{u}) \\
&\quad +2\,\nu\,((\mathbf{u}\cdot\hat{\mathbf{r}}_y)\,\mathbf{n} \otimes \hat{\mathbf{r}}_y + (\mathbf{u}\cdot\hat{\mathbf{r}}_y)\,\hat{\mathbf{r}}_y \otimes \mathbf{n}) - (1-4\,\nu)(\mathbf{u}\cdot\mathbf{n})\,\mathbf{I}\}\;dS\,\mathbf{n} \\
&= -4\,\mu\,C \int_{\partial\Omega} \frac{1}{r_y^2}\,\{(1-2\,\nu)(\mathbf{u} + (\mathbf{u}\cdot\mathbf{n})\,\mathbf{n}) + 2\,\nu\,(\mathbf{u}\cdot\hat{\mathbf{r}}_y)\,\hat{\mathbf{r}}_y \\
&\quad -(1-4\,\nu)(\mathbf{u}\cdot\mathbf{n})\,\mathbf{n}\}\;dS \\
&= -4\,\mu\,C \int_{\partial\Omega} \frac{1}{r_y^2}\,\{(1-2\,\nu)\mathbf{u} + 2\,\nu\,(\mathbf{u}\cdot\mathbf{n})\,\mathbf{n} + 2\,\nu\,(\mathbf{u}\cdot\hat{\mathbf{r}}_y)\,\hat{\mathbf{r}}_y\}\;dS \\
&= -4\,\mu\,C \int_{\partial\Omega} \frac{1}{r_y^2}\,\{(1-2\,\nu)\,\mathbf{I} + 2\,\nu\,\mathbf{n} \otimes \mathbf{n} + 2\,\nu\,\hat{\mathbf{r}}_y \otimes \hat{\mathbf{r}}_y\}\;\mathbf{u}\,dS \\
&= -4\,\mu\,C \int_{\partial\Omega} \frac{1}{r_y^2}\,\{(1-2\,\nu)\,\mathbf{I} + 2\,\nu\,\mathbf{I}\}\;\mathbf{u}\,dS \\
&= -4\,\mu\,C \int_{\partial\Omega} \frac{1}{r_y^2}\,\mathbf{I}\,\mathbf{u}\,dS \\
&= -4\,\mu\,C\,\mathbf{I} \int_{\partial\Omega} \frac{1}{r_y^2}\,\mathbf{u}\,dS\;, \qquad (108)
\end{aligned}$$

where we used the facts that $\mathbf{n} \cdot \mathbf{n} = 1$ and $\mathbf{n} \otimes \mathbf{n} + \hat{\mathbf{r}}_y \otimes \hat{\mathbf{r}}_y = \mathbf{I}$. Applying 99 gives

$$\begin{aligned}
\int_{\partial\Omega} \mathcal{Z}_y^x(\mathbf{x})\,\mathbf{u}(\mathbf{x})\,dS\,\mathbf{n}(\mathbf{y}) &= -4\,\mu\,C\,\mathbf{I} \int_{-1}^{1} \frac{2}{L\,(\xi-\xi')^2}\,\mathbf{u}\,d\xi \\
&= -\frac{8\,\mu\,C}{L}\,\mathbf{I} \int_{-1}^{1} \frac{1}{(\xi-\xi')^2}\,\mathbf{u}\,d\xi\;, \qquad (109)
\end{aligned}$$

and substituting the interpolation of 56 yields

$$\begin{aligned}
\int_{\partial\Omega} \mathcal{Z}_y^x(\mathbf{x})\,\mathbf{u}(\mathbf{x})\,dS\,\mathbf{n}(\mathbf{y}) &= -\frac{8\,\mu\,C}{L}\,\mathbf{I} \int_{-1}^{1} \frac{1}{(\xi-\xi')^2}\,(\mathbf{N}^T(\xi) \odot \mathbf{I})\,d\xi\,\mathrm{vec}\,(\mathbf{U}) \\
&= -\frac{8\,\mu\,C}{L}\left(\int_{-1}^{1} \frac{\mathbf{N}^T(\xi)}{(\xi-\xi')^2}\,d\xi \odot \mathbf{I}\right)\mathrm{vec}\,(\mathbf{U})\;, \qquad (110)
\end{aligned}$$

i.e., the desired expression. We evaluate the hypersingular integral in 110 using the Hadamard finite part, e.g. $\oint_{-1}^{1} \frac{N_i(\xi)}{(\xi-\xi')^2}\,d\xi = \frac{d}{d\xi'} \fint_{-1}^{1} \frac{N_i(\xi)}{\xi-\xi'}\,d\xi$. Integrating with `Mathematica` as above and subsequently differentiating, again with `Mathematica`, yields

$$
\begin{aligned}
\fint_{-1}^{1} \frac{N_1(\xi)}{(\xi-\xi')^2}\,d\xi &= \frac{3\left(6\xi^2+\left(3\xi^3-\xi^2-3\xi+1\right)\log\left(\frac{2}{\xi+1}-1\right)-2\xi-3\right)}{4\left(\xi^2-1\right)} \\
&= \frac{3\left(6\xi^2+(3\xi-1)(\xi^2-1)\log\left(\frac{1-\xi}{1+\xi}\right)-2\xi-3\right)}{4\left(\xi^2-1\right)} \ , \\
\fint_{-1}^{1} \frac{N_2(\xi)}{(\xi-\xi')^2}\,d\xi &= \frac{-18\xi^2-9\left(\xi^2-1\right)\xi\log\left(\frac{2}{\xi+1}-1\right)+13}{2\left(\xi^2-1\right)} \\
&= \frac{-18\xi^2-9\left(\xi^2-1\right)\xi\log\left(\frac{1-\xi}{1+\xi}\right)+13}{2\left(\xi^2-1\right)} \ , \\
\fint_{-1}^{1} \frac{N_3(\xi)}{(\xi-\xi')^2}\,d\xi &= \frac{3\left(6\xi^2+\left(3\xi^3+\xi^2-3\xi-1\right)\log\left(\frac{2}{\xi+1}-1\right)+2\xi-3\right)}{4\left(\xi^2-1\right)} \\
&= \frac{3\left(6\xi^2+(3\xi+1)(\xi^2-1)\log\left(\frac{1-\xi}{1+\xi}\right)+2\xi-3\right)}{4\left(\xi^2-1\right)} \ ,
\end{aligned}
\tag{111}
$$

cf. [7] 22-24.

4.5 Infinite Element Integration and Discretization

To model the infinite surface we define the mapping

$$
\begin{aligned}
\mathbf{x} &= \mathbf{h}(\xi) \\
&= \sum_{I=1}^{2} N_I^{\infty}(\xi)\,\mathbf{X}^{(I)} \ ,
\end{aligned}
\tag{112}
$$

where the infinite element basis functions are, cf. [23] 10,

$$
\begin{aligned}
N_1^{\infty}(\xi) &= -\frac{2\,\xi}{1-\xi} \ , \\
N_3^{\infty}(\xi) &= \frac{1+\xi}{1-\xi} \ .
\end{aligned}
\tag{113}
$$

As seen here, the infinite mapping requires only two element coordinates since the third node extends to infinity. Note, that as ξ is integrated over the interval [-1,1] in the local coordinate system, the shape functions $N_i \to \infty$ as $\xi \to +1$, thus creating the infinite surface that is desired. The Jacobian of the transformation is given by

$$
\begin{aligned}
|J_h(\xi)| &= \left| \sum_{I=1}^{2} \frac{\partial}{d\,\xi} N_I^{\infty}(\xi)\, \mathbf{X}^{(I)} \right| \\
&= \left| \frac{2}{(1-\xi)^2} \left(\mathbf{X}^3 - \mathbf{X}^1 \right) \right| \quad .
\end{aligned} \tag{114}
$$

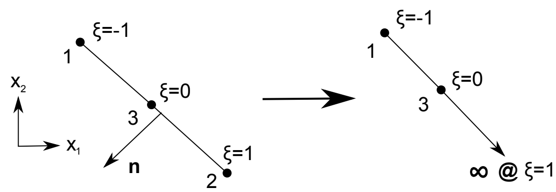


Figure 8: Infinite element mapping cf. 113.

Following [23, 24] we only need to interpolate the displacement and traction at the node in contact with the finite domain cf. [23] 13. Thusly the only collocation point $\mathbf{Y}^1$ is node 1, i.e. $\mathbf{Y}^1 = \mathbf{X}^1$ in the infinite element so that

$$
\begin{aligned}
\mathbf{u}(\mathbf{h}(\xi)) &= N_1^{u\infty}(\xi)\, \mathbf{U}^1 \quad , \\
\mathbf{t}(\mathbf{h}(\xi)) &= N_1^{t\infty}(\xi)\, \mathbf{T}^1 \quad ,
\end{aligned} \tag{115}
$$

where

$$
\begin{aligned}
N_1^{u\infty}(\xi) &= \frac{1}{2}\,(1-\xi) \quad , \\
N_1^{t\infty}(\xi) &= \frac{1}{4}\,(1-\xi)^2 \quad .
\end{aligned} \tag{116}
$$

Note that as $\xi \to 1$, i.e. $r \to \infty$, we have $N_1^{u\infty}(\xi)$, $N_1^{t\infty}(\xi) \to 0$ and, when $\xi \to -1$ we have $N_1^{u\infty}(\xi)$, $N_1^{t\infty}(\xi) \to 1$. We use the standard Gauss quadrature scheme for integration over the infinite element

$$
\begin{aligned}
\int_e \mathcal{U}_y(\mathbf{x})\mathbf{t}^{\mathbf{n}}(\mathbf{x})\,dS &= \left[\sum_{i=1}^{m} \mathcal{U}_y(\mathbf{h}(\xi))(\mathbf{N}^{t\infty}(\xi_i)\odot\mathbf{I})\,J_h(\xi_i)w_i\right]\mathrm{vec}\,(\mathbf{T}) \quad , \qquad (117)\\
\int_e \mathcal{T}_y^x(\mathbf{x})\mathbf{u}(\mathbf{x})\,dS &= \left[\sum_{i=1}^{m} \mathcal{T}_y^x(\mathbf{h}(\xi))(\mathbf{N}^{u\infty}(\xi_i)\odot\mathbf{I})\,J_h(\xi_i)w_i\right]\mathrm{vec}\,(\mathbf{U}) \quad ,\\
\int_e \mathcal{D}_y^y(\mathbf{x})\mathbf{t}^{\mathbf{n}}(\mathbf{x})\,dS &= \left[\sum_{i=1}^{m} \mathcal{D}_y^y(\mathbf{h}(\xi))(\mathbf{N}^{t\infty}(\xi_i)\odot\mathbf{I})\,J_h(\xi_i)w_i\right]\mathrm{vec}\,(\mathbf{T}) \quad ,\\
\int_e \mathcal{S}_y^{\mathbf{n}}(\mathbf{x})\mathbf{u}(\mathbf{x})\,dS &= \left[\sum_{i=1}^{m} \mathcal{S}_y^{xy}(\mathbf{h}(\xi))(\mathbf{N}^{u\infty}(\xi_i)\odot\mathbf{I})\,J_h(\xi_i)w_i\right]\mathrm{vec}\,(\mathbf{U}) \quad ,
\end{aligned}
$$

where $(\mathbf{N}^{(\cdot)\infty}(\xi_i)\odot\mathbf{I})$ is a 2×2 matrix and $\mathrm{vec}(\mathbf{U})$ and $\mathrm{vec}(\mathbf{T})$ are 2×1 vectors pertaining to collocation point $\mathbf{Y}^1 = \mathbf{X}^1$ in the infinite element.

5 Implementation and Examples

5.1 Assembly of Equations

Given the discretization explained previously, one can numerically solve for the collocation point unknowns, i.e. the displacement $\mathbf{u}(\mathbf{y})$. [5] A computer code using FORTRAN.90 has been developed and will be referenced in the following. Referring to Figure 9, we rearrange 51 to the following, cf. [5] 8.28-8.29,

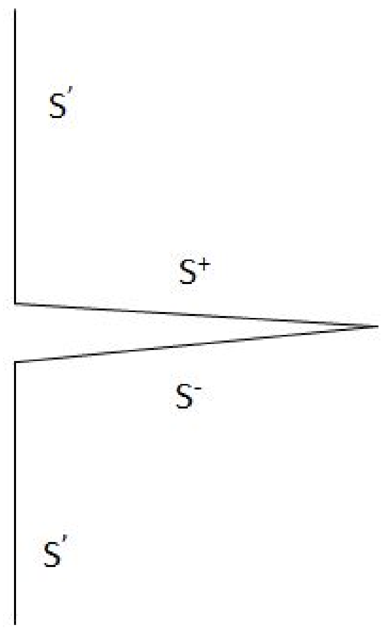


Figure 9: Semi-infinite edge crack.

$$
\begin{aligned}
&\tfrac{1}{2}\mathbf{u}(\mathbf{y}) + ⨍_{\partial\Omega}[\mathcal{T}_y^y(\mathbf{x})]^T\mathbf{u}(\mathbf{x})dS = \int_{\partial\Omega}\mathcal{U}_y(\mathbf{x})\mathbf{t}^{\mathbf{n}}(\mathbf{x})dS && \text{if } \mathbf{y} \in S' \ , \\
&\tfrac{1}{2}\mathbf{u}(\mathbf{y}^+) + \tfrac{1}{2}\mathbf{u}(\mathbf{y}^-) + ⨍_{\partial\Omega}[\mathcal{T}_y^y(\mathbf{x})]^T\mathbf{u}(\mathbf{x})dS = \int_{\partial\Omega}\mathcal{U}_y(\mathbf{x})\mathbf{t}^{\mathbf{n}}(\mathbf{x})dS && \text{if } \mathbf{y} \in S^+ \ , \qquad (118) \\
&⨎_{\partial\Omega}[\mathcal{S}_y^{xy}(\mathbf{x})]^T\mathbf{u}(\mathbf{x})dS = \tfrac{1}{2}\mathbf{t}^{\mathbf{n}}(\mathbf{y}^+) - \tfrac{1}{2}\mathbf{t}^{\mathbf{n}}(\mathbf{y}^-) + ⨍_{\partial\Omega}\mathcal{D}_y^y(\mathbf{x})\mathbf{t}^{\mathbf{n}}(\mathbf{x})dS && \text{if } \mathbf{y} \in S^- \ ,
\end{aligned}
$$

where the extra $\frac{1}{2}$ terms in 118.2 and 118.3 arise from the coincidental collocation points $\mathbf{y}^+$ and $\mathbf{y}^-$, i.e. we enforce equally the BEM and DBEM on both S^+ and S^-, respectively.

[5] Recall that this is a traction problem, i.e. the traction is prescribed on the entire surface

The global stiffness matrices $[G]$ and $[H]$, cf. 52, each contain 16 submatrices. To explain the submatrices, we let $S' = F^+$ for the free surface that extends to positive infinity, $S' = F^-$ for the free surface that extends to negative infinity and note that the computer code node numbering starts with the positive infinite element. In this way, we obtain the partitions

$$G = \begin{pmatrix} [\mathcal{T}^{F^+F^+}] & [\mathcal{T}^{F^+S^+}] & [\mathcal{T}^{F^+S^-}] & [\mathcal{T}^{F^+F^-}] \\ [\mathcal{T}^{S^+F^+}] & [\mathcal{T}^{S^+S^+}] & [\mathcal{T}^{S^+S^-}] & [\mathcal{T}^{S^+F^-}] \\ [\mathcal{S}^{S^-F^+}] & [\mathcal{S}^{S^-S^+}] & [\mathcal{S}^{S^-S^-}] & [\mathcal{S}^{S^-F^-}] \\ [\mathcal{T}^{F^-F^+}] & [\mathcal{T}^{F^-S^+}] & [\mathcal{T}^{F^-S^-}] & [\mathcal{T}^{F^-F^-}] \end{pmatrix},$$

$$H = \begin{pmatrix} [\mathcal{U}^{F^+F^+}] & [\mathcal{U}^{F^+S^+}] & [\mathcal{U}^{F^+S^-}] & [\mathcal{U}^{F^+F^-}] \\ [\mathcal{U}^{S^+F^+}] & [\mathcal{U}^{S^+S^+}] & [\mathcal{U}^{S^+S^-}] & [\mathcal{U}^{S^+F^-}] \\ [\mathcal{D}^{S^-F^+}] & [\mathcal{D}^{S^-S^+}] & [\mathcal{D}^{S^-S^-}] & [\mathcal{D}^{S^-F^-}] \\ [\mathcal{U}^{F^-F^+}] & [\mathcal{U}^{F^-S^+}] & [\mathcal{U}^{F^-S^-}] & [\mathcal{U}^{F^-F^-}] \end{pmatrix},$$

where each partition notes the kernel used with the first superscript pertaining to the collocation point, $\mathbf{Y}^I$, and the second to the nodal point, $\mathbf{X}^J$. For example, the second submatrix of G, i.e. $[\mathcal{T}^{F^+S^+}]$, corresponds to the collocation points $\mathbf{Y}^I$ on the free surface F^+ and the element integrals over the positive crack face S^+, via the $\mathcal{T}_y^x$ kernel. The diagonal of the submatrices with the same superscripts contain the singular and hypersingular calculations that were discussed previously.

It should be noted that the computer code solves for displacements element by element, as oppose to the FEM, which solves node by node. We denote the number of quadratic elements by N_e and the number of infinite elements by N_∞. The the number of collocation points per quadratic and infinite element are $N_{n/e} = 3$ and $N_{n/\infty} = 1$, and there are $N_{dof} = 2$ degrees of freedom per collocation point, hence the column vectors $\{u\}$ and $\{t\}$ have a length of $N = 6 \times N_e + 2 \times N_\infty$. And hence, $[G]$ and $[H]$ are $N \times N$ square matrices. The code uses a Gaussian elimination routine to solve for the unknown $\{u\}$.

5.2 Removal of Rigid Body Modes

The elastic boundary value problem cf. 5, does not have a unique solution when $\partial\Omega = \Gamma^t$ and $\Gamma^u = \emptyset$. Indeed, three rigid body modes exist and hence the matrix $[G]$ is singular. Blazquez [25] showed that, based on a priori knowledge of a problem, certain nodes can be constrained to remove the rigid body modes. The semi-infinite perpendicular crack could potentially be constrained beyond the crack tip in the $\mathbf{e}_1$-direction with a pseudo-element, and a symmetry condition could be used to constrain the $\mathbf{e}_2$-direction. However, for future work, we wish to analyze inclined surface cracks for a semi-infinite domain, where there is no symmetry. Therefore, we remove the rigid body modes using linear algebra principles [25, 26].

In two-dimensions, we consider the three rigid body modes; two translations (in the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions) and a rigid rotation about the $\mathbf{e}_3$ - axis, i.e. $\mathbf{e}_3 \times \mathbf{x}$. These modes form a basis which define the rigid body mode matrix $[M]$, cf. [25] 5,

$$[M] = [\mathbf{m}^1 \ \mathbf{m}^2 \ \mathbf{m}^3] \ , \tag{119}$$

where

$$\begin{aligned} \mathbf{m}^1 &= [1\ 0\ 1\ 0\ ...\ 1\ 0]^T \ , \\ \mathbf{m}^2 &= [0\ 1\ 0\ 1\ ...\ 0\ 1]^T \ , \\ \mathbf{m}^3 &= [-Y_2^1\ Y_1^1\ ... - Y_2^N\ Y_1^N]^T \ . \end{aligned} \tag{120}$$

Notably, we have $[G]\mathbf{m}^i = 0$, which means the vectors $\mathbf{m}^i$ are in the null space of $[G]$, i.e. $\mathbf{m}^i \in N_{sp}([G])$. Now the vectors $\mathbf{m}^i$ are linearly independent and the matrix $[H]$ is nonsingular, and hence the vectors $\mathbf{d}^i = [H]\mathbf{m}^i$ are linearly independent, cf. 23-24 in [26]. Using these vectors, we construct the matrix $[D] = [\mathbf{d}^1\ \mathbf{d}^2\ \mathbf{d}^3]$ and note that the $rank([D]) = 3$, and the range $\mathcal{R}([D])$ of $[D]$ is in the nullspace of $[G]$, since $[D]\{\alpha\} = [H][M]\{\alpha\}$. But the "load" $[M]\{\alpha\}$ is not in equilibrium, which implies no $\{u\}$ exists that can satisfy $[G]\{u\} = [D]\{\alpha\}$ for any 3×1 vector α, and in particular for the choice $\alpha = [M]^T\{u\}$. Thusly $[D][M]^T\{u\} = 0$ and we redefine our matrix $[G]$ in 52 as

$$[G] = [G] + [D][M]^T \ , \tag{121}$$

which is a nonsingular matrix that does not effect our desired result. Details of this proof can be found in [26], roughly, the range of $[D]$ which comprises $N_{sp}[G]$ is now appended to $[G]$ via the addition of the $[D][M]^T$ product.

5.3 Calculation of Stress Intensity Factors

We use the crack tip opening displacement (CTOD) method to determine the SIFs, K_I and K_{II} [5, 7]. Hence, K_I and K_{II} are determined as a post processing step after the computer code solves for the collocation points displacement. The analytical displacement field contain the SIFs, K_I and K_{II}, cf. [8] Table 2.2

$$
\begin{aligned}
u_x &= \begin{cases} \frac{K_I}{2\,\mu}\sqrt{\frac{r}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left[\kappa - 1 + 2\sin^2(\frac{\theta}{2})\right] & \text{for Mode I} \\ \frac{K_{II}}{2\,\mu}\sqrt{\frac{r}{2\pi}}\sin\left(\frac{\theta}{2}\right)\left[\kappa + 1 + 2\cos^2(\frac{\theta}{2})\right] & \text{for Mode II} \end{cases}, \\
u_y &= \begin{cases} \frac{K_I}{2\,\mu}\sqrt{\frac{r}{2\pi}}\sin\left(\frac{\theta}{2}\right)\left[\kappa + 1 - 2\cos^2(\frac{\theta}{2})\right] & \text{for Mode I} \\ -\frac{K_{II}}{2\,\mu}\sqrt{\frac{r}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left[\kappa - 1 - 2\sin^2(\frac{\theta}{2})\right] & \text{for Mode II} \end{cases},
\end{aligned}
\tag{122}
$$

where the cylindrical coordinate system, shown in Figure 10, is centered at the crack tip with $-\pi \leq \theta \leq \pi$. The solution for the displacement on the bottom and top crack faces can be determined for each mode when $\theta = -\pi$ and $\theta = \pi$, respectively, i.e.

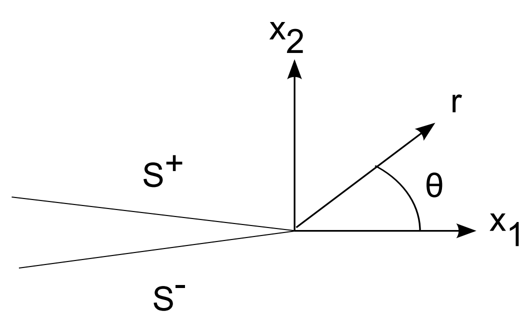


Figure 10: Cylindrical coordinate system, cf. 123.

$$u_x = \begin{cases} \theta = \pi \rightarrow 0 & \text{for Mode I} \\ \theta = -\pi \rightarrow 0 & \text{for Mode I} \\ \theta = \pi \rightarrow \frac{K_{II}}{2\,\mu}\sqrt{\frac{r}{2\pi}}\,[\kappa+1] & \text{for Mode II} \\ \theta = -\pi \rightarrow -\frac{K_{II}}{2\,\mu}\sqrt{\frac{r}{2\pi}}\,[\kappa+1] & \text{for Mode II} \;, \end{cases}$$

$$u_y = \begin{cases} \theta = \pi \rightarrow \frac{K_{I}}{2\,\mu}\sqrt{\frac{r}{2\pi}}\,[\kappa+1] & \text{for Mode I} \\ \theta = -\pi \rightarrow -\frac{K_{I}}{2\,\mu}\sqrt{\frac{r}{2\pi}}\,[\kappa+1] & \text{for Mode I} \\ \theta = \pi \rightarrow 0 & \text{for Mode II} \\ \theta = -\pi \rightarrow 0 & \text{for Mode II} \;, \end{cases} \tag{123}$$

where we used the fact that the bottom crack face will displace in both the negative $\mathbf{e}_1$ and $\mathbf{e}_2$ directions for the loading described in Figure 11 and $\kappa = 3 - 4\nu$ for plane strain. By defining $\Delta u_x = u_x(\pi) - u_x(-\pi)$ and $\Delta u_y = u_y(\pi) - u_y(-\pi)$, the CTOD can be obtained in terms of the SIFs, cf. [5] 8.71,

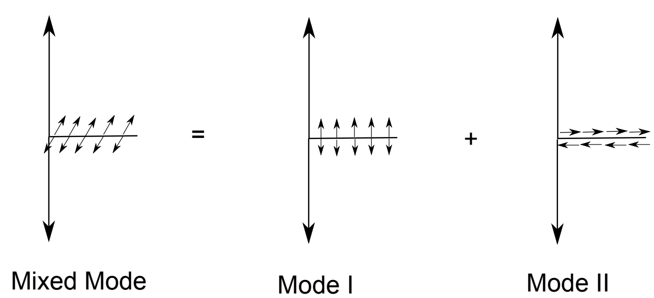


Figure 11: Superposition loading problem.

$$\begin{aligned} \Delta u_x &= \frac{K_{II}}{\mu}\sqrt{\frac{r}{2\pi}}\,[\kappa+1] \;, \\ \Delta u_y &= \frac{K_{I}}{\mu}\sqrt{\frac{r}{2\pi}}\,[\kappa+1] \;, \end{aligned} \tag{124}$$

and solving for K_I and K_{II} we have

$$K_I = \Delta u_y \frac{\mu}{\kappa+1}\sqrt{\frac{2\pi}{r}} \ ,$$
$$K_{II} = \Delta u_x \frac{\mu}{\kappa+1}\sqrt{\frac{2\pi}{r}} \ . \qquad (125)$$

The SIFs in 125 are determined by the CTOD $(\Delta u_y, \Delta u_x)$ for nodes and collocation points on the crack tip elements, cf. Figure 12. We use the numerical solution for the displacement $\{u\}$ at these two elements and calculate the SIFs at nodes and collocation points taking into account the distance r in 125 for the various node and collocation point coordinates, i.e.

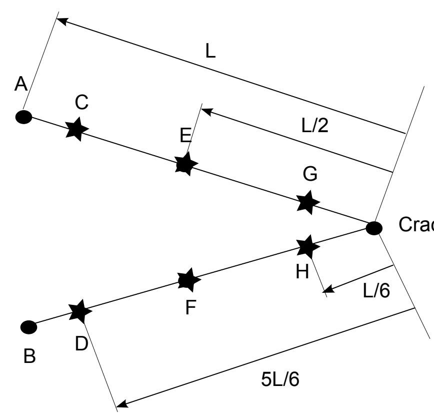


Figure 12: Crack tip elements used for SIF calculation.

$$K_i^{AB} = \Delta u_j \frac{\mu}{\kappa+1}\sqrt{\frac{2\pi}{L}} \ ,$$
$$K_i^{CD} = \Delta u_j \frac{\mu}{\kappa+1}\sqrt{\frac{12\pi}{5L}} \ ,$$
$$K_i^{EF} = \Delta u_j \frac{\mu}{\kappa+1}\sqrt{\frac{4\pi}{L}} \ ,$$
$$K_i^{GH} = \Delta u_j \frac{\mu}{\kappa+1}\sqrt{\frac{12\pi}{L}} \ , \qquad (126)$$

where, when $i = I$, $j = y$ and when $i = II$, $j = x$, i.e. K_I corresponds to Δu_y and K_{II} to Δu_x. We refer to the SIF calculations in 126 as *single node extraction*, where the displacements at the nodes A and B, i.e. $\mathbf{u}_a$ and $\mathbf{u}_b$, are interpolated from within the element using the displacements at the collocation points and the shape functions, cf. 58. Portela [7] and Alibadi [5] linearly extrapolated between the single node extractions in 126 using K_i^{AB} - K_i^{EF} and K_i^{CD} - K_i^{EF} to solve for the SIFs at the crack tip $r = 0$. Here we consider their extrapolation node selection and also three others using the single node extractions; K_i^{CD} - K_i^{GH}, K_i^{EF} - K_i^{GH}, and K_i^{AB} - K_i^{GH}. We extrapolate via

$$K(r_0) = K_{r_{k-1}} + \frac{r_0 - r_{k-1}}{r_k - r_{k-1}} \left(K_{r_k} - K_{r_{k-1}}\right) \quad . \tag{127}$$

Starting with the example in [5] 8.74, the midside nodes E-F and far left end nodes A-B gives

$$\begin{aligned} K_i &= K_i^{EF} + \frac{0 - \frac{L}{2}}{L - \frac{L}{2}} (K_i^{AB} - K_i^{EF}) \\ &= K_i^{EF} - (K_i^{AB} - K_i^{EF}) \\ &= 2\, K_i^{EF} - K_i^{AB} \quad , \end{aligned} \tag{128}$$

where $i = I, II$ corresponds to the respective modes and we equate $r_0 = 0$ for extrapolation to the crack tip. Extrapolation at the midside nodes E-F and the far left collocation points C-D yields, cf. [7] 33-34

$$\begin{aligned} K_i &= K_i^{EF} + \frac{0 - \frac{L}{2}}{\frac{5L}{6} - \frac{L}{2}} (K_i^{CD} - K_i^{EF}) \\ &= K_i^{EF} - \frac{3}{2}(K_i^{CD} - K_i^{EF}) \\ &= \frac{5}{2}\, K_i^{EF} - \frac{3}{2}\, K_i^{CD} \quad . \end{aligned} \tag{129}$$

The three proposed extrapolations K_i^{GH} - K_i^{EF}, K_i^{CD} - K_i^{GH}, K_i^{AB} - K_i^{GH} are similarly found as

$$\begin{aligned} K_i &= K_i^{GH} + \frac{0 - \frac{L}{6}}{\frac{L}{2} - \frac{L}{6}} (K_i^{EF} - K_i^{GH}) \\ &= K_i^{GH} - \frac{1}{2}(K_i^{EF} - K_i^{GH}) \\ &= \frac{3}{2}\, K_i^{GH} - \frac{1}{2}\, K_i^{EF} \quad , \end{aligned} \tag{130}$$

$$\begin{aligned} K_i &= K_i^{GH} + \frac{0 - \frac{L}{6}}{\frac{5L}{6} - \frac{L}{6}} (K_i^{CD} - K_i^{GH}) \\ &= K_i^{GH} - \frac{1}{4}(K_i^{CD} - K_i^{GH}) \\ &= \frac{5}{4}\, K_i^{GH} - \frac{1}{4}\, K_i^{CD} \quad , \end{aligned} \tag{131}$$

$$
\begin{aligned}
K_i &= K_i^{GH} + \frac{0 - \frac{L}{6}}{L - \frac{L}{6}} (K_i^{AB} - K_i^{GH}) \\
&= K_i^{GH} - \frac{1}{5} (K_i^{AB} - K_i^{GH}) \\
&= \frac{6}{5} K_i^{GH} - \frac{1}{5} K_i^{AB} \quad . \qquad (132)
\end{aligned}
$$

5.4 Examples

The computer code developed has been tested against the CTOD methods cf. 126, 128-132. The examples in this section are solved for pure Mode I and pure Mode II loading; however, using our assumptions of LEFM, allows us to use superposition, cf. Figure 11. to solve mixed mode problems.

Referring to Figure 13, we place the two infinite element nodes that are in contact with the finite domain at a distance h from the crack free surface intersection and the midside nodes at distances $2h$. Recall that the displacement and traction decay by factors of $\frac{1}{r}$ and $\frac{1}{r^2}$, respectively, as $r \to \infty$ from the crack tip. Positioning the infinite nodes as such automatically satisfies this decay, cf. [23] 11-15.

We compare the K_I and K_{II} based on the single node extraction methods cf. 126, the extraction methods seen in [5, 7] cf. 128-129, and the proposed extrapolation nodes considered cf. 130-132 to the analytical solutions, cf. [27] 1

$$
\begin{aligned}
K_I &= 1.1215 \, t_y \sqrt{\pi a} \quad , \\
K_{II} &= 1.1215 \, t_x \sqrt{\pi a} \quad , \qquad (133)
\end{aligned}
$$

for a given crack length of a.

The problem dimensions in Figure 13, the material properties and loading are described in Table 1. Error plots of K_I and K_{II} versus element length L on the crack face $S^+ \cup S^-$ and element length on the free surface L_f for values $L_f = 2a$, $L_f = a$ and $L_f = \frac{1}{4}a$ appear in Figures 14-19.

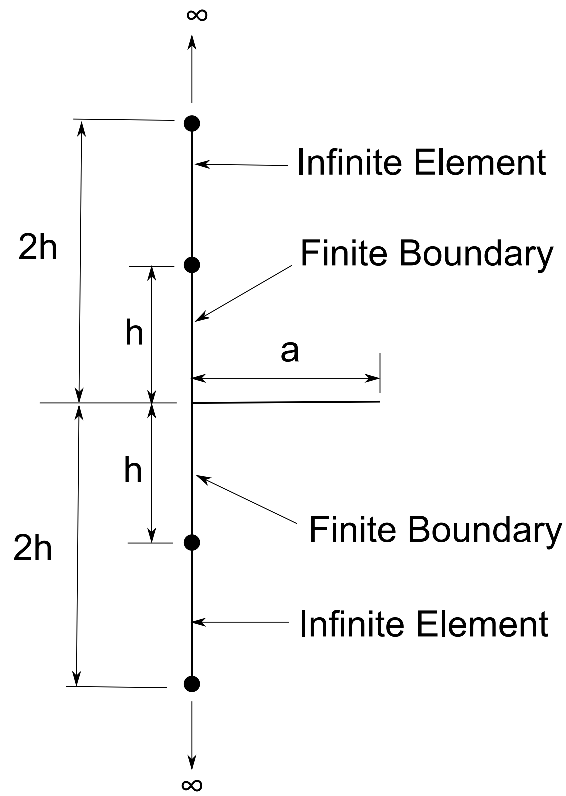
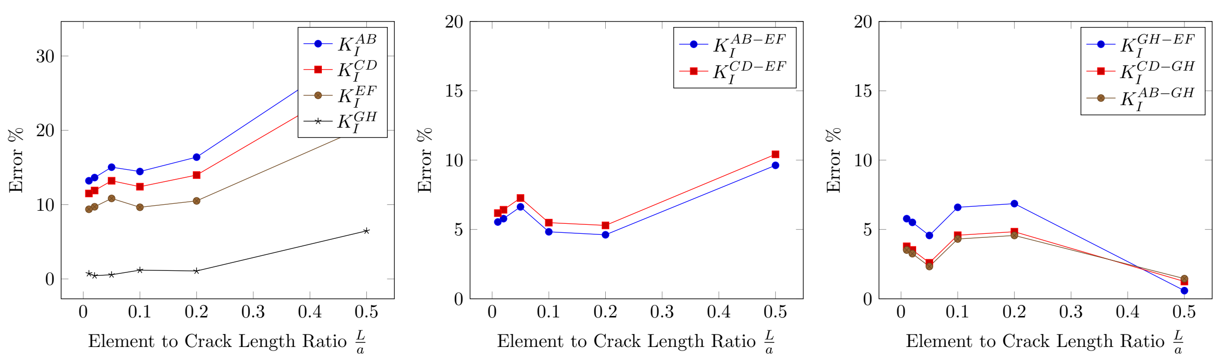


Figure 13: Problem dimensions.

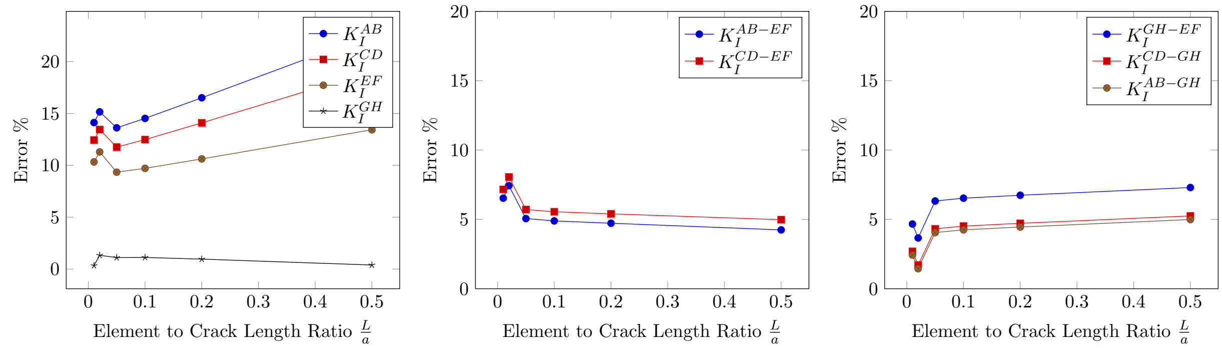


(a) Single Node Extraction cf. 126 (b) Extrapolation cf. 128-129 (c) Extrapolation cf. 130-132

Figure 14: K_I values for $L_f = 2a$ based on various extraction methods.

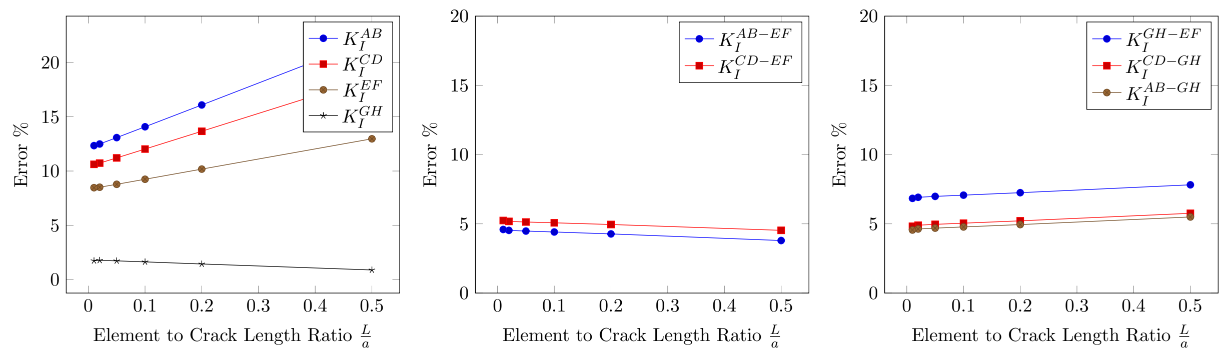
Parameter	Value
E	1 GPa
ν	0.3
t	100 kN
a	0.05m
h=25a	2.5m

Table 1: Problem dimensions and loading with material characteristics.



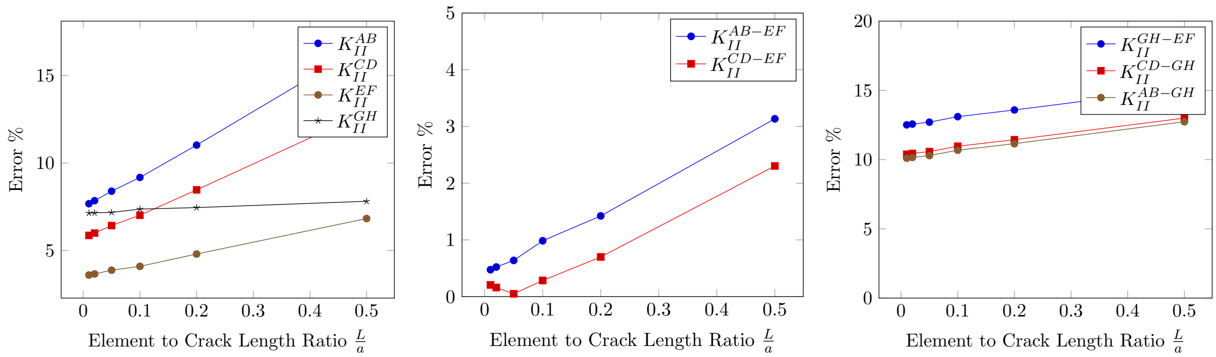
(a) Single Node Extraction cf. 126 (b) Extrapolation cf. 128-129 (c) Extrapolation cf. 130-132

Figure 15: K_I values for $L_f = a$ based on various extraction methods.



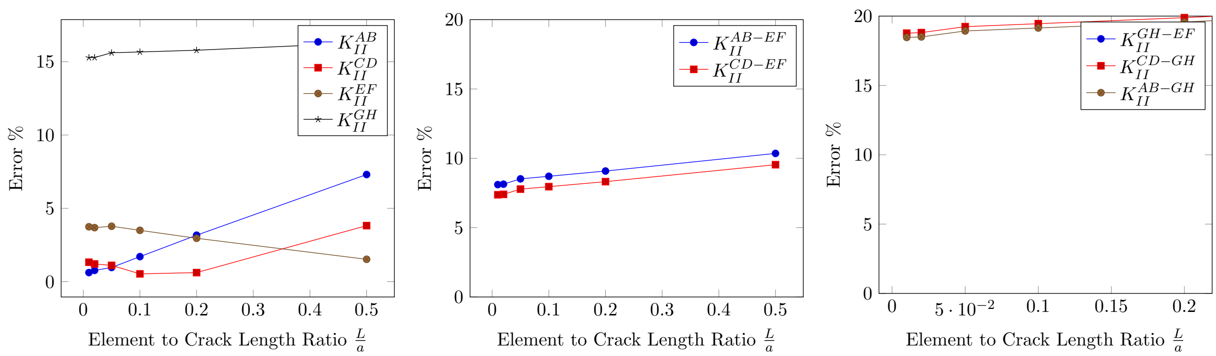
(a) Single Node Extraction cf. 126 (b) Extrapolation cf. 128-129 (c) Extrapolation cf. 130-132

Figure 16: K_I values for $L_f = \frac{1}{4}a$ based on various extraction methods.



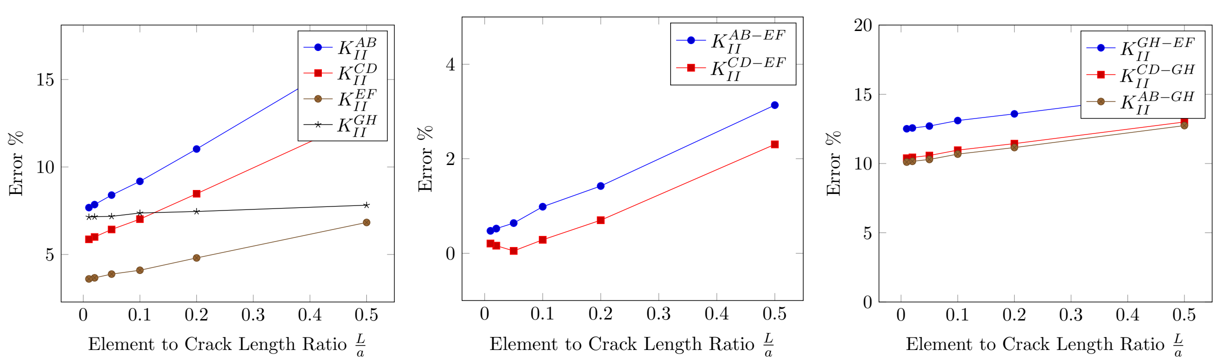
(a) Single Node Extraction cf. 126 (b) Extrapolation cf. 128-129 (c) Extrapolation cf. 130-132

Figure 17: K_{II} values for $L_f = 2a$ based on various extraction methods.



(a) Single Node Extraction cf. 126 (b) Extrapolation cf. 128-129 (c) Extrapolation 130-132

Figure 18: K_{II} values for $L_f = a$ based on various extraction methods.



(a) Single Node Extraction cf. 126 (b) Extrapolation 128-129 (c) Extrapolation 130-132

Figure 19: K_{II} values for $L_f = \frac{1}{4}a$ based on various extraction methods.

The single node extrapolation of the crack tip nodes (i.e. K_I^{G-H}) produced the best results for K_I regardless of element size on the free surface and with less than 2% for all L_f sizes. Hence, the linear extrapolation methods that utilize K_I^{G-H} have smaller errors than the extrapolations that do not.

Results for K_{II} produced little error with the linear extrapolations K_{II}^{AB-EF} and K_{II}^{CD-EF} proposed by Alibadi & Portella with $L_f = 2a$ and $L_f = \frac{1}{4}a$; however, the results for $L_f = a$ are some what troubling as they converge to an error of 7%. The single node extraction method for K_{II}^{E-F} give consistent results for all L_f values and converges to less than 4% error.

6 Conclusion and Future Work

The computer code developed based on *Sections 1-4* was able to produce results within 2% for each Mode of loading; however, some of the results lacked consistency. We expect mesh refinement on the free surface will yield a converged SIF. Mode I single node extrapolation of the crack tip collocation point G-H was consistent throughout the mesh refinement. However, this was not the case for the Mode II loading via Alibadi's and Portella's extrapolation methods.

The cause of this error needs to be investigated further. There are several possible sources of errors that should be considered. First, the integration of the element adjacent to the infinite element when $\mathbf{Y}^I$ belongs to the infinite element. The analytical integrations presented in *Section 4.3* are not applicable and therefore, a standard Gauss quadrature routine was used, which may lead to error upon mesh refinement. Second, integration over the infinite element when $\mathbf{Y}^I$ belongs to the infinite element. Beer [3] claims these integrals to bounded and therefore, a standard Gauss-quadrature should work. However, these integrations should be examined further. An additional thought is to use quarter-point elements at the crack tip.

Future work could also investigate other techniques such as the J-Integral method to calculate the SIFs, or shape sensitivity methods to calculate the ERR G. This is particularly important since the analysis of inclined cracks by the CTOD method is not applicable because $\Delta u_x(K_{II})$ and $\Delta u_y(K_I)$ become coupled, i.e. $\Delta u_x(K_I, K_{II})$ and $\Delta u_y(K_I, K_{II})$.

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