A SURFACE INTEGRAL EQUATION METHOD FOR DIELECTRICS
AT LOW FREQUENCIES

BY

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THESIS

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This thesis is dedicated to using surface integral equations to solve electromagnetic problems involved in integrated circuits. Since normally the sizes of the devices in this application are much smaller than the wavelength of the electromagnetic waves, special considerations are needed because of the low frequency breakdown.

The augmented technique, a useful remedy for low frequency breakdown of the electric field integral equation is introduced as the background of this thesis. This augmented electric field integral equation provides a simple solution for broadband electromagnetic simulation of perfect electric conductor structures. This thesis presented here exploits the augmented method for lossless and lossy dielectrics.

The use of the conventional Rao-Wilton-Glisson (RWG) basis function as basis and testing functions fails because of the testing issue. Instead, the Buffa-Christiansen (BC) basis function is proposed to overcome this difficulty. With the combined use of RWG and BC basis functions, a new formulation is developed achieving a good convergence and accuracy. For highly lossy medium, however, a new integration scheme and a simple, efficient strategy with a fast algorithm is adopted. After these treatments, the skin depth of current in the conductive medium can be accurately captured down to very low frequency.
To my parents and Haini
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CHAPTER 1
INTRODUCTION

Accurate and efficient solutions to many electromagnetic problems are generally required in academics and industry for predictions of electromagnetic performance of devices. Analytical solutions, however, can only be derived for geometries with high symmetry such as cylinders and spheres. With the rapid increase in computing power in the past decades, computational electromagnetics has become an important field of research. With a suitable computational electromagnetic algorithm, one can solve very complicated electromagnetic problems for arbitrary structures.

Among the many methods for solving electromagnetic problems, three are widely in use:

• Finite difference time domain (FDTD) method [1]. This is a very popular time domain method in electromagnetic simulations. The conventional method requires dividing the 3D domain into identical cubes and solving differential Maxwell’s equations in their difference forms. This method is simple and flexible but it is expensive due to the need of time marching and inaccurate due to the need of truncation of the 3D domain.

• Finite element method (FEM) [2]. This is also a widely used differential-based numerical method. FEM is a volumetric-mesh-based method and the whole domain requires discretization by 3D simplex. This method produces a sparse matrix, so the storage requirement of this algorithm is $O(N)$, where $N$ is the total number of unknowns in the simulation domain. Therefore, this method is very efficient and flexible for arbitrary structures and inhomogeneous medium. Again, truncation of the domain is needed for an open structure problem.

• Integral equation (IE) method [3]. There are two kinds of IEs, the volume integral equation (VIE), and the surface integral equation (SIE). They are both based on the integral form of Maxwell’s equation and utilization of Green’s function. The difference is that VIE needs volumetric mesh but SIE
requires only surface mesh. Since the field and source are related by Green’s function, this method produces a full matrix. However, a fast algorithm can be adopted with the integral equation method to reduce the computational cost dramatically.

Among all the methods, SIE is one of the most popular. The mesh required for this method is only on the surface of the structures. Therefore, the number of unknowns is much smaller than for other volumetric-mesh-based methods. Although the matrix generated with this method is usually dense, fast algorithms such as the multi-level fast multi-pole algorithm (MLFMA) at mid frequencies and the mixed-form FMA at low frequencies can be adopted. These algorithms reduce the complexity of this method from \( O(N^2) \) to \( O(N \log N) \).

The SIE can be categorized into three types: electric field integral equations (EFIE), magnetic field integral equation (MFIE) and a combination of the two. EFIE and MFIE are the most fundamental formulations for SIE. They are derived from the equivalence principle or the extinction theorem and the surface tangential electric and magnetic fields can be converted into unknowns with these two methods. However, they suffer from internal resonance. Then the combination of EFIE and MFIE is used to eliminate this. In this thesis, we will focus our study on EFIE rather than the other two methods.

1.1 Frequency Regimes of Electromagnetics

Generally, an electromagnetic problem can be classified into low frequency, mid frequency and high frequency regimes [4] depending on the scale difference of the wavelength \( \lambda \) and the size of the device \( S \).

- **Low frequency.** When \( \lambda < 0.1S \), the problem is classified as a low frequency problem. In this regime, the physics is very similar to static cases. Both electrostatic and magneto-static problems need to be considered in a full-wave electromagnetic solver. Also noteworthy is that the evanescent waves are important because the range of these waves is about \( \lambda \). Hence plane wave representations of the wave in this regime become inefficient.

- **Mid frequency.** When \( \lambda > 0.1S \) and \( \lambda < 400S \), the problem is classified as a mid frequency problem. In this regime, normally, both the evanescent
and propagating waves are important. This regime has been widely studied in the electromagnetic community.

- High frequency. When $\lambda > 400S$, the problem is classified as a high frequency problem. In this regime, the evanescent components become negligible and the ray approximation is usually applied to give an accurate result. It has many applications in optics.

1.2 Low Frequency Problems in Electromagnetics

The goal of this thesis is to provide the full-wave method for applications in integrated circuits. As the devices we are interested in become smaller, there is an urgent requirement for a stable and efficient electromagnetic solver at low frequency. When the size of the object is much smaller than the wavelength of the wave, for example $size < 0.1\lambda$, the problem is classified as a low frequency problem. While stable at mid frequencies, some methods, such as FEM and EFIE, suffer from low frequency breakdown.

This breakdown is due to the decoupling of electro-static and magneto-static physics at low frequency. The field produced by these two phenomena scales differently with respect to frequency. When solved with numerical methods, these two physical phenomena cannot be accurately captured simultaneously by using “naive” FEM or EFIE. Some new basis functions are found to remedy the breakdown, such as the tree-cotree basis [5] for FEM and the loop-tree basis [6] for EFIE. These kinds of methods are called quasi Helmholtz decompositions, where the electro-static and magneto-static physics are almost decoupled by frequency normalization. However, the accompanying numerical cost can be very large for complicated structures. The reason is that the searching of global loops can be very expensive for these structures.

Another method based on EFIE, called the augmented electric field integral equation (A-EFIE) eliminates the search of loops and trees. The computational cost is greatly reduced for low frequency problems. This method normalizes the EFIE by adding the current continuity equation into the formulation. Although the size of the problem becomes larger, the computational cost and memory usage does not increase much because the extra costs arise from the manipulations of some extremely sparse matrices. This method, when compared to loop-tree decomposition, is not as accurate when
the frequency is extremely low, but it is proved that this method is accurate down to very low frequency and it is sufficient for most applications.

1.3 Dielectrics and Conductors

The A-EFIE method has never before been applied to dielectrics. We can utilize the effectiveness of this method at low frequency and apply this to dielectrics to develop a broadband surface integral equation method for both the perfect electric conductor (PEC) and dielectrics.

One of the major challenges in this problem is to deal with the testing issue of the magnetic field operator or $\mathbf{K}$ operator. Three major conditions need to be satisfied to overcome this issue: first, the basis function is required to be divergence conforming. Second, the testing function should be in the dual space. Third, the testing function is a vector that is almost parallel to the field. These three conditions cannot all be satisfied if we stick with the traditional Rao-Wilton-Glisson (RWG) basis. However, more complicated basis functions, such as the Buffa-Christiansen (BC) basis [7] and Chen-Wilton basis [8], can be adapted in A-EFIE to solve the testing issue.

Another challenge is the convergence issue of this solver. Although the $\mathbf{K}$ operator is well-conditioned and converges very fast, the A-EFIE, as a saddle point problem, gives rise to very slow convergence for large problems. Motivated by A-EFIE for PEC, we can also find a diagonal-block matrix as the pre-conditioner and it accelerates the convergence greatly.

In many applications, the conductor can be approximated as PEC because the skin depth in the conductor is much smaller than the size. However, as the size of the structure gets smaller, it becomes comparable to the skin depth. In this situation, the electromagnetic field penetrates inside the conductor, and the PEC approximation is no longer valid. Another treatment is to use the impedance boundary condition (IBC), which establishes the relationship between the electric and magnetic currents. This method, although better than PEC approximation, is only accurate when the skin depth is smaller than the size of the structure.

Instead of using the approximations mentioned above, we can treat the conductors as lossy dielectrics. It can be proved easily from Maxwell’s equations that the conductor can be characterized as dielectrics with permittivity
as a function of conductivity and frequency. The larger the conductivity, the more lossy the dielectrics. However, due to the fast decaying as well as fast oscillating nature of Green’s function in a conductive environment, the traditional integration scheme for the lossless case is no longer suitable. We present an accurate and computationally efficient scheme that captures the lossy and oscillating nature of Green’s function. By replacing the traditional scheme with this one for the lossy medium, the skin depth effects can be captured accurately.

1.4 Organization of the Thesis

In this thesis, several extensions of A-EFIE are presented.

In Chapter 2, the previous work on A-EFIE is reviewed. This method remedies the low frequency breakdown. With the perturbation version of this method, it can be used for problems down to extremely low frequency.

In Chapter 3, A-EFIE is extended from PEC to dielectrics. The testing issues in Galerkin’s method are discussed and we propose the use of RWG basis for the electric current and the BC basis for the magnetic current as a remedy. With an appropriate pre-conditioner, this method can be used to solve lossless and lossy dielectric electromagnetic problems of a large number of unknowns accurately and efficiently.

In Chapter 4, conductive medium is treated as highly lossy dielectrics. A new integration scheme is proposed. By comparing the existing schemes with this new one, it is proved that this new integration method is more accurate and efficient in terms of treating conductive materials. Some numerical examples show that the skin depth is well captured inside the conductors and problems with a large number of unknowns can be solved with the mixed-form fast multi-pole algorithm.

In Chapter 5, a generalized impedance boundary condition (GIBC) is introduced. This GIBC concept eliminates the inaccuracy of the impedance boundary condition because the coupling of equivalent currents becomes more precise. Although it is more complicated, sparsity of the matrices formed can be fully utilized, thus greatly reducing the complexity of this algorithm.

Another work is presented at the end of this thesis in Chapter 6. The
visualization of modes inside the PEC and dielectric waveguides is shown and the guidance condition in optical fiber is also computed with the finite element method. This work is included here to complete the presentation of this thesis research.
CHAPTER 2
AUGMENTED ELECTRIC FIELD INTEGRAL EQUATION

The electric field integral equation (EFIE) breaks down at low frequency. This is due to the imbalance of the two terms (vector potential and scalar potential) in the EFIE ($\mathcal{L}$) operator. In this chapter, we summarize this low frequency breakdown and review an existing method, termed the augmented electric field integral equation (A-EFIE), to solve this problem.

2.1 Low Frequency Breakdown

2.1.1 EFIE Formulation

The reason for this low frequency breakdown of EFIE is well studied in [6],[9]. One can write the EFIE using the equivalence principle or extinction theorem [1] for a two-region (internal and external) problem as shown in Figure 2.1. Assuming the excitation source is in the external region:

$$\int_{S'} dS' \left[ i\omega \mu_{\text{ext}} \mathbf{G}_{\text{ext}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J} (\mathbf{r}') + \nabla' \times \mathbf{G}_{\text{ext}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{M} (\mathbf{r}') \right] = -\mathbf{E}_{\text{inc}} (\mathbf{r}), \; \mathbf{r} \in V_{\text{int}} \quad (2.1)$$

$$\int_{S'} dS' \left[ i\omega \mu_{\text{int}} \mathbf{G}_{\text{int}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J} (\mathbf{r}') + \nabla' \times \mathbf{G}_{\text{int}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{M} (\mathbf{r}') \right] = 0, \; \mathbf{r} \in V_{\text{ext}} \quad (2.2)$$

where the $\mathbf{J}$ and $\mathbf{M}$ are the equivalent electric and magnetic currents defined as:

$$\mathbf{J}(\mathbf{r}') = \hat{n} \times \mathbf{H}(\mathbf{r}'), \quad \mathbf{M}(\mathbf{r}') = -\hat{n} \times \mathbf{E}(\mathbf{r}') \quad (2.3)$$
Figure 2.1: A two-region problem with an excitation source in the external region.

$G(r, r')$ is the dyadic Green’s function, the subscript denotes the region, where $ext$ is for the external and $int$ for the internal.

$$G_i(r, r') = \left( I + \frac{\nabla \nabla}{k_i^2} \right) g_i(r, r') \quad (2.4)$$

where

$$g_i(r, r') = \frac{e^{ik_i|r-r'|}}{4\pi|r-r'|} \quad (2.5)$$

Conventionally, computational electromagnetics researchers write (2.1) and (2.2) in operator representations and force $r$ to approach the interface of the internal and external regions $S$:

$$L_{ext}(r, r') \cdot J(r') + K_{ext}(r, r') \cdot M(r') = -E_{inc}(r) \quad (2.6)$$

$$L_{int}(r, r') \cdot J(r') + K_{int}(r, r') \cdot M(r') = 0 \quad (2.7)$$

These two equations are the two famous EFIEs. We can use them to solve for equivalent currents $J$ and $M$.

When the internal region is the perfect electric conductor (PEC), the magnetic current $M$ is always zero because the tangential electric field is zero on the PEC surface. Then (2.7) is redundant and (2.6) becomes:

$$L_{ext}(r, r') \cdot J(r') = -E_{inc}(r) \quad (2.8)$$
This is the EFIE for PEC.

2.1.2 EFIE at Low Frequency

By expanding the $\mathcal{L}_{\text{ext}}$ with Green’s function $g(\mathbf{r}, \mathbf{r}')$, (2.8) can be further written as:

$$i\omega \mu \int_{S'} dS' g(\mathbf{r}, \mathbf{r}') J(\mathbf{r}') - \frac{1}{i\omega \epsilon} \int_{S'} dS' g(\mathbf{r}, \mathbf{r}') \nabla \cdot \mathbf{J}(\mathbf{r}') = -E_{\text{inc}} \quad (2.9)$$

Note that the subscript $\text{ext}$ is dropped for simpler notations.

When the frequency is very low ($\omega \to 0$), the first term ($O(\omega)$) is swamped by the second ($O(\omega^{-1})$) due to the finite precision of computer storage. Obviously there is a null-space in the second term in (2.9). Therefore, the solenoidal (divergence free) part of the current $\mathbf{J}$ can never be solved accurately due to the divergence operation.

Another more physical explanation is that the electrostatic and magnetostatic fields decoupled at zero frequency. This gives rise to the separation of solenoidal and irrotational (curl free) currents. Two of the Maxwell’s equations for electrostatic case are:

$$\nabla \times \mathbf{E} = 0 \quad (2.10)$$

$$\nabla \cdot \mathbf{D} = \rho = \lim_{\omega \to 0} \frac{\nabla \cdot \mathbf{J}}{i\omega} \quad (2.11)$$

The other two govern the magnetostatics:

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (2.12)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.13)$$

By Helmholtz decomposition, $\mathbf{J} = \mathbf{J}_{\text{irr}} + \mathbf{J}_{\text{sol}}$. We can find $\mathbf{J}_{\text{irr}}$ only contributes to the right-hand side of (2.11), while $\mathbf{J}_{\text{sol}}$ only contributes to the right-hand side of (2.12). Then (2.11) and (2.12) can be rewritten as:

$$\nabla \cdot \mathbf{D} = \lim_{\omega \to 0} \frac{\nabla \cdot \mathbf{J}_{\text{irr}}}{i\omega} \quad (2.14)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{sol}} \quad (2.15)$$
This shows the decoupling of two currents into two regimes.

Apparently, $\mathbf{J}_{\text{irr}} \sim O(\omega)$ as $\omega \to 0$, because $\rho$ must be finite. However, there are no such restrictions for $\mathbf{J}_{\text{sol}}$, the solenoidal contributions depend on the excitation sources and the structure of the problem. Due to this discrepant frequency dependence of solenoidal and irrotational components, we require a numerical method that captures both electrostatic and magnetostatic physics.

### 2.2 A-EFIE Formulation

Invoke the current continuity equation in the frequency domain:

$$\nabla' \cdot \mathbf{J}(\mathbf{r}') = i\omega \rho(\mathbf{r}')$$

(2.16)

Substituting (2.16) into (2.9), and noticing that $k_0 = \omega \sqrt{\varepsilon_0 \mu_0}$, $c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ and $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$, we have

$$ik_0 \eta_0 \mu_r \int_{S'} dS' g(\mathbf{r}, \mathbf{r}') \mathbf{J}(\mathbf{r}') - \frac{1}{\epsilon_r} c_0 \eta_0 \nabla \int_{S'} dS' g(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}') = -\mathbf{E}_{\text{inc}}$$

(2.17)

The equation (2.17) can be further normalized as:

$$\mu_r \int_{S'} dS' g(\mathbf{r}, \mathbf{r}') i k_0 \mathbf{J}(\mathbf{r}') - \frac{1}{\epsilon_r} \nabla \int_{S'} dS' g(\mathbf{r}, \mathbf{r}') c_0 \rho(\mathbf{r}') = -\eta_0^{-1} \mathbf{E}_{\text{inc}}$$

(2.18)

This equation, in which $ik_0 \mathbf{J}$ and $c_0 \rho$ are the new sets of unknowns, together with the normalized current continuity equation (2.19), overcomes the low frequency breakdown.

$$\nabla' \cdot ik_0 \mathbf{J}(\mathbf{r}') + k_0^2 c_0 \rho(\mathbf{r}') = 0$$

(2.19)

The introduction of (2.16) provides one degree of freedom for us to normalize EFIE for PEC as in (2.18). After this normalization, the contributions from the charge (from irrational current) and the total current are separated. The first term and the second term are now equally important in (2.19) as the frequency dependences are moved to the knowns. This avoids one term being swamped by the other and the null space no longer exists in the formulation.
By letting $\tilde{J} = i k_0 J$, $\tilde{\rho} = c_0 \rho$ and discretizing the surface into triangles, one can expand $\tilde{J}$ with the RWG basis function [10] $\Lambda(r')$:

$$\tilde{J}(r') = \sum a_n \Lambda(r')$$  \hspace{1cm} (2.20)

Then the basis function for $\tilde{\rho}$ can be easily obtained by taking the divergence of the RWG basis. This basis function is called the pulse basis function [11] $h(r')$:

$$\tilde{\rho}(r') = \sum b_n h(r')$$  \hspace{1cm} (2.21)

Testing (2.18) with the RWG basis function using the Galerkin method and discretizing (2.19), we can formulate a matrix equation: (more details can be found in [11])

$$\begin{bmatrix} \sqrt{V} & D^T \cdot P & \sqrt{I} \\ D & k_0^2 I \end{bmatrix} \cdot \begin{bmatrix} i k_0 J \\ c_0 \rho_r \end{bmatrix} = \begin{bmatrix} -\eta_0^{-1} b \\ 0 \end{bmatrix}$$  \hspace{1cm} (2.22)

where $I$ is the identity matrix,

$$[V]_{mn} = \mu_r \int_S dS \Lambda_m(r) \cdot \int_{S'} dS' g(r, r') \Lambda_n(r')$$  \hspace{1cm} (2.23)

$$[P]_{mn} = \epsilon_r^{-1} \int_S dS h_m(r) \int_{S'} dS' g(r, r') h_n(r')$$  \hspace{1cm} (2.24)

$$[D]_{mn} = \begin{cases} +1; \text{ triangle } m \text{ in positive RWG } n \\ -1; \text{ triangle } m \text{ in negative RWG } n \\ 0; \text{ otherwise} \end{cases}$$  \hspace{1cm} (2.25)

$$b_m = \int dS \Lambda(r) \cdot \mathbf{E}_{inc}(r)$$  \hspace{1cm} (2.26)

The matrix in (2.22) is rank-deficient due to the charge neutrality. For a closed object, the total charges should sum up to be zero. To apply this condition, we can remove one element in $c_0 \rho$ and a reduced unknown vector $c_0 \rho_r$ is formed. Then a new and full-rank matrix equation can be formulated:

$$\begin{bmatrix} \sqrt{V} & D^T \cdot P \cdot B \\ F \cdot D & k_0^2 I \end{bmatrix} \cdot \begin{bmatrix} i k_0 J \\ c_0 \rho_r \end{bmatrix} = \begin{bmatrix} -\eta_0^{-1} b \\ 0 \end{bmatrix}$$  \hspace{1cm} (2.27)
2.3 Pre-Conditioner for A-EFIE

To solve problems with a large number of unknowns, a suitable preconditioner is usually preferred for faster convergence. For A-EFIE, a block diagonal matrix is chosen as the preconditioning matrix and the right preconditioning scheme is chosen for residual preservation. To solve a nonsingular matrix equation:

\[ \mathbf{A} \cdot \mathbf{x} = \mathbf{y} \tag{2.28} \]

use the right pre-conditioner:

\[ \left( \mathbf{A} \cdot \mathbf{M}^{-1} \right) \cdot \left( \mathbf{M} \cdot \mathbf{x} \right) = \mathbf{y} \tag{2.29} \]

where the precondition matrix is:

\[ \mathbf{M} = \begin{bmatrix} \text{diag}(\mathbf{V}) & \mathbf{D}^T \cdot \text{diag}(\mathbf{P}) \cdot \mathbf{B} \\ \mathbf{F} \cdot \mathbf{D} & k_0^2 \mathbf{I}_r \end{bmatrix} \tag{2.30} \]

2.4 A-EFIE with Perturbation Method

Although A-EFIE overcomes the low frequency breakdown, it suffers from low frequency inaccuracy for plain wave and capacitor problems as the frequency decreases. This issue is extensively studied in [12].

It is pointed out that the charge can be solved accurately while the current loses accuracy at very low frequencies for the problems mentioned above. Noticing that the charge term only contains information about the irrotational current, then the term \( \mathbf{V} \cdot ik_0 \mathbf{J} \) determines all the contributions from the solenoidal current. We can easily find that the matrices \( \mathbf{V} \) and \( \mathbf{P} \) are frequency invariant in magnitude at low frequencies. Charge is also frequency independent. From

\[ \mathbf{V} \cdot ik_0 \mathbf{J} + \mathbf{D}^T \cdot \mathbf{P} \cdot c_0 \mathbf{\rho} = -\eta^{-1} \mathbf{b} \tag{2.31} \]

if \( \mathbf{J} \sim \omega^{-1} \) in magnitude, the two terms at the left-hand side of the above equations are well balanced. Then \( \mathbf{J} \) and \( \mathbf{\rho} \) are equally important and are to be solved accurately. However, it is not always the case.
To analyze the order of magnitude of $J$ with respect to frequency, we can invoke loop-tree decomposition; more details about this decomposition can be found [6] and [12]. The governing equation is

$$
\begin{bmatrix}
Z_{LL} & Z_{LC} \\
Z_{CL} & Z_{CC}
\end{bmatrix}
\begin{bmatrix}
J_L \\
J_C
\end{bmatrix}
= 
\begin{bmatrix}
b_L \\
b_C
\end{bmatrix}
$$

(2.32)

where $L$ refers to inductor physics, thus relating to the irrotational current. $C$ refers to capacitor physics, thus relating to the solenoidal current.

The order of magnitude in real and imaginary parts of the four matrices can be extracted by the Taylor expansion of Green’s function. They are:

$$
\begin{bmatrix}
Z_{LL}(\omega^2,\omega^1) & Z_{LC}(\omega^2,\omega^1) \\
Z_{CL}(\omega^2,\omega^1) & Z_{CC}(\omega^0,\omega^{-1})
\end{bmatrix}
$$

(2.33)

By examining the order of magnitude in $b_L$ and $b_C$, the magnitude of $J_L$ and $J_C$ can be derived.

For plain wave excitation: $b_L \sim (\omega^2,\omega^1), b_C \sim (\omega^0,\omega^1)$, then $J_L \sim (\omega^0,\omega^1)$ and $J \sim (\omega^0,\omega^1)$.

For capacitor problems: $b_L \sim (0,0), b_C \sim (\omega^0,0)$, then $J_L \sim (\omega^4,\omega^1), J_C \sim (\omega^4,\omega^1)$ and $J \sim (\omega^4,\omega^1)$.

For inductor problems: $b_L \sim (\omega^0,0), b_C \sim (\omega^0,0)$, then $J_L \sim (\omega^2,\omega^{-1}), J_C \sim (\omega^4,\omega^1)$ and $J \sim (\omega^2,\omega^{-1})$.

Apparently, only for inductor problems, the total current $J \sim \omega^{-1}$. For the other two cases, the current is too small and the precision will be lost due to the finite precision of computers.

As a remedy, a perturbation method based A-EFIE was proposed. The key step of this approach is to approximate the Green’s function in Taylor series: (letting $\delta = ik_0l$, where $l$ is length scale)

$$
g(r, r') \approx g^{(0)}(r, r') + \delta g^{(1)}(r, r') + \delta^2 g^{(2)}(r, r') = \frac{1}{4\pi R} \left[ 1 + \delta \frac{R}{l} + \delta^2 \frac{1}{2} \left( \frac{R}{l} \right)^2 \right]
$$

(2.34)

Then the matrices $\nabla$ and $\mathbf{P}$ can be written in a similar manner approximately:

$$
\nabla \approx \nabla^{(0)} + \delta \nabla^{(1)} + \delta^2 \nabla^{(2)}
$$

(2.35)
\[ \mathbf{P} \approx \mathbf{P}^{(0)} + \delta\mathbf{P}^{(1)} + \delta^2\mathbf{P}^{(2)} \]  

(2.36)

similarly, the excitation term \( \mathbf{b} \) and the unknowns \( \mathbf{J} \) and \( \rho_r \) can be approximated:

\[ \mathbf{b} \approx \mathbf{b}^{(0)} + \delta\mathbf{b}^{(1)} + \delta^2\mathbf{b}^{(2)} \]
\[ \mathbf{J} \approx \mathbf{J}^{(0)} + \delta\mathbf{J}^{(1)} + \delta^2\mathbf{J}^{(2)} \]
\[ \rho_r \approx \rho_r^{(0)} + \delta\rho_r^{(1)} + \delta^2\rho_r^{(2)} \]  

(2.37)

The error of the above approximation is bounded to \( O(\delta^3) \). Then using the basic perturbation method procedure, one can solve for the unknowns from lower order to higher order:

Letting

\[ \overline{\mathbf{A}}^{(0)} = \begin{bmatrix} \mathbf{V}^{(0)} & \mathbf{D}^T \cdot \mathbf{P}^{(0)} \cdot \mathbf{B} \\ \mathbf{F} \cdot \mathbf{D} & k_0^2 \mathbf{I} \end{bmatrix} \]  

(2.38)

the zeroth, first and second order of the unknowns can be solved:

\[ \overline{\mathbf{A}}^{(0)} \cdot \begin{bmatrix} ik_0 \mathbf{J}^{(0)} \\ c_0 \rho_r^{(0)} \end{bmatrix} = \begin{bmatrix} \mathbf{b}^{(0)} \\ 0 \end{bmatrix} \]  

(2.39)

\[ \overline{\mathbf{A}}^{(0)} \cdot \begin{bmatrix} ik_0 \mathbf{J}^{(1)} \\ c_0 \rho_r^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{b}^{(1)} - \mathbf{V}^{(1)} \cdot ik_0 \mathbf{J}^{(0)} - \mathbf{D}^T \cdot \mathbf{P}^{(1)} \cdot \mathbf{B} \cdot c_0 \rho_r^{(0)} \\ 0 \end{bmatrix} \]  

(2.40)

\[ \overline{\mathbf{A}}^{(0)} \cdot \begin{bmatrix} ik_0 \mathbf{J}^{(2)} \\ c_0 \rho_r^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{b}^{(2)} \\ 0 \end{bmatrix} - \begin{bmatrix} \mathbf{V}^{(2)} \cdot ik_0 \mathbf{J}^{(0)} + \mathbf{V}^{(1)} \cdot ik_0 \mathbf{J}^{(1)} \\ 0 \end{bmatrix} - \begin{bmatrix} \mathbf{D}^T \cdot \mathbf{P}^{(2)} \cdot \mathbf{B} \cdot c_0 \rho_r^{(0)} + \mathbf{D}^T \cdot \mathbf{P}^{(1)} \cdot \mathbf{B} \cdot c_0 \rho_r^{(1)} \\ -c_0 \rho_r^{(0)} \end{bmatrix} \]  

(2.41)

The solutions to current and charge can be solved to arbitrary order in this manner. Second-order perturbation approximation will be sufficient to solve various problems down to very low frequency. Many numerical examples of A-EFIE and the perturbation method can be found in [11] and [12].
CHAPTER 3
A SOLVER FOR DIELECTRICS USING A-EFIE

A few surface integral equation methods were established to solve dielectric problems. For example, electric field integral equations (EFIE) and magnetic field integral equation (MFIE) are the simplest. However, they suffer from internal resonance. As remedies, combined field integral equations (CFIE) [13], Poggio-Miller-Chang-Harrington-Wu-Tsai (PMCHWT) [14], [15] and Müller formulations [16] were proposed. But these solvers do not eliminate low frequency breakdown unless loop-tree or loop-star decomposition is used.

The loop-tree or loop-star decomposition is stable and accurate down to extremely low frequency. The cost arising from searching global loops is also extremely large, especially for complex structures. Therefore, it is not preferred for some applications. The A-EFIE method, which requires no loops and trees searching, will be more flexible and the complexity will be structure invariant.

Since A-EFIE is designed for low frequency or small-scale problems, the internal resonance frequency of structures we are interested in is much larger than the operating frequency. As a result, no internal resonance issues will be considered.

3.1 Formulation

The formulation of A-EFIE for dielectrics is simply to use the extinction theorem for both external and internal regions as in (2.1) and (2.2). The operator representations are then:

\[ L_{\text{ext}}(r, r') \cdot J(r') + K_{\text{ext}}(r, r') \cdot M(r') = -E_{\text{inc}}(r) \]  \hspace{1cm} (3.1)

\[ L_{\text{int}}(r, r') \cdot J(r') + K_{\text{int}}(r, r') \cdot M(r') = 0 \]  \hspace{1cm} (3.2)
Noticing that the lack of balance in the $L$ operator is the reason for breakdown at low frequency while the $K$ operator is free of low frequency breakdown, we can use the augment technique only for the $L$ operator. By discretizing the unknowns $J, M, \rho$ and testing the equations, together with the current continuity equation, a new matrix equation can be formed:

$$
\begin{bmatrix}
V_{\text{ext}} & K_{\text{ext}} & D^T \cdot \bar{P}_{\text{ext}} \cdot \bar{B} \\
V_{\text{int}} & K_{\text{int}} & D^T \cdot \bar{P}_{\text{int}} \cdot \bar{B} \\
\bar{F} \cdot \bar{D} & 0 & k_0^2 \bar{I}
\end{bmatrix}
\begin{bmatrix}
i k_0 J \\
\eta^{-1} M \\
c_0 \rho_r
\end{bmatrix}
= 
\begin{bmatrix}
-\eta_0^{-1} b \\
0 \\
0
\end{bmatrix}
$$

(3.3)

The matrix elements are:

$$
[V_j]_{mn} = \mu_r \int dS T_m(r) \cdot \int dS' g_j(r, r') \Lambda_n^J(r')
$$

(3.4)

$$
[K_j]_{mn} = \int dS T_m(r) \cdot \int dS' \nabla' g_j(r, r') \times \Lambda_n^M(r')
$$

(3.5)

$$
[P_j]_{mn} = \epsilon_r^{-1} \int dS \nabla \cdot T_m(r) \int dS' g_j(r, r') h_n(r')
$$

(3.6)

where the subscript $j$ refers to the region, which can be either $\text{int}$ or $\text{ext}$. $
\Lambda^J(r'), \Lambda^M(r')$ and $h(r')$ are the basis functions for $J, M$ and $\rho_r$ respectively. $T(r)$ is the testing function for (3.1) and (3.2). The matrices $\bar{D}, \bar{B}$ and $\bar{F}$ are of the same form as defined in Chapter 2.

### 3.2 Testing and Basis Functions

In this section, we will discuss the choices of the testing and basis functions based on function space analysis in electromagnetics [17], [18].

The fundamental physical reason for this function space analysis is that the energy is finite in the bounded domain $D$. So the electromagnetic field should be a square integrable in domain $D$. In other words, an electromagnetic field
\( \mathbf{v}(\mathbf{r}) \in L^2 \) such that
\[
\|\mathbf{v}(\mathbf{r})\|_{L^2(D)} = \left( \int_D d\mathbf{r} |\mathbf{v}(\mathbf{r})|^2 \right)^{\frac{1}{2}} < \infty \quad (3.8)
\]

Meanwhile, with Maxwell’s equations which relates the \( \mathbf{E} \) field and \( \mathbf{H} \) field, both \( \mathbf{E} \) and \( \nabla \times \mathbf{E} \) are integrable in the bounded domain. Then \( \mathbf{E} \) and \( \mathbf{H} \) should be within such a space:
\[
H(\text{curl}, D) := \{ \mathbf{v} | \mathbf{v} \in L^2(D), \nabla \times \mathbf{v} \in L^2(D) \} \quad (3.9)
\]

The definition of norm in this space is:
\[
\|\mathbf{v}\|_{\text{curl}, D} = \left( \|\mathbf{v}\|^2_{L^2(D)} + \|\nabla \times \mathbf{v}\|^2_{L^2(D)} \right)^{\frac{1}{2}} \quad (3.10)
\]

Similarly \( H(\text{div}, D) \) can be defined:
\[
H(\text{div}, D) := \{ \mathbf{v} | \mathbf{v} \in L^2(D), \nabla \cdot \mathbf{v} \in L^2(D) \} \quad (3.11)
\]

with norm:
\[
\|\mathbf{v}\|_{\text{div}, D} = \left( \|\mathbf{v}\|^2_{L^2(D)} + \|\nabla \cdot \mathbf{v}\|^2_{L^2(D)} \right)^{\frac{1}{2}} \quad (3.12)
\]

Apparently, electromagnetic fields lives in \( H(\text{curl}, D) \) while fluxes are in \( H(\text{div}, D) \). Noticing that all of the physical quantities are defined in a 3-D domain. Then what is the function space of these quantities at the boundary? It is proved by “trace theorems” [19] that “all boundary values of functions in \( H(D) \) form a space \( H^{\frac{1}{2}}(S) \)” [18]. Since \( \mathbf{J} = \hat{n} \times \mathbf{H} \) and \( \mathbf{M} = -\hat{n} \times \mathbf{E} \), and \( \mathbf{E}, \mathbf{H} \in H^{\frac{1}{2}}(\text{curl}, S) \), then \( \mathbf{J}, \mathbf{M} \in H^{\frac{1}{2}}(\text{div}, S) \).

According to the testing requirements, the testing function needs to be in the dual space of the functional space of the tested vector. The dual of \( H^{\frac{1}{2}}(\text{curl}, S) \) is \( H^{\frac{1}{2}}(\text{div}, S) \). Then for the A-EFIE, the testing function in \( H^{\frac{1}{2}}(\text{div}, S) \) is required.

In simpler words, the testing function should be in the dual space of the function space they are testing. For testing of vector equations, the direction of the testing function should also be aligned with the tested vector so that the testing is numerically representative. The basis function, on the other hand, should satisfy the conditions of the physical quantities and span the finite-dimensional space.
Figure 3.1: RWG divergence-conforming basis function on the triangle patches. \( T^+ \) and \( T^- \) are the positive charge and negative charge patches, \( V^+ \) and \( V^- \) are the vertices of the patches.

The electromagnetic currents, \( \mathbf{J} \) and \( \mathbf{M} \), are divergence-conforming. Then basis functions to represent them will also have this property. The equations in (3.3) are equations for electric field, which is in the curl-conforming space. Using its dual, divergence-conforming function as the testing function will be a good testing scheme.

The well-known Rao-Wilton-Glisson (RWG) function [10] is the divergence-conforming function of the lowest order as in Figure 3.1. The definition and the divergence of the normalized RWG basis functions are:

\[
\Lambda(r') = \begin{cases} 
\frac{1}{2A}(r' - V^+) & \text{if } r' \in T^+ \\
-\frac{1}{2A}(r' - V^-) & \text{if } r' \in T^- 
\end{cases}
\]

(3.13)

\[
\nabla \cdot \Lambda(r') = h(r') = \begin{cases} 
\frac{1}{A^+} & \text{if } r' \in T^+ \\
-\frac{1}{A^-} & \text{if } r' \in T^- 
\end{cases}
\]

(3.14)

Apparently, the RWG basis function is divergence-conforming, and it will be a good choice for both the testing function and the basis functions of \( \mathbf{J} \) and \( \mathbf{M} \). However, the condition number of the matrix in (3.3) is extremely large if we choose the RWG basis function as the testing and basis. The reason is that the magnetic field integral equation (MFIE) operator \( \mathcal{K} \) is ill-conditioned because the electric field generated by magnetic current is perpendicular to the direction of the testing function. In order to see that,
we can extract the residual of the $\mathcal{K}$ operator:

$$
\mathcal{K}_j(r, r') \cdot M(r') = \nabla \times \int_{S'} \mathcal{G}_j(r, r') \cdot M(r') = \int_{S'} \nabla g_j(r, r') \times M(r')
= \pm \frac{1}{2} \hat{n} \times M(r) + \int_{S'} \nabla g_j(r, r') \times M(r') \quad (3.15)
$$

Where “+” indicates the case when $j = \text{ext}$, “−” sign for the case of $j = \text{int}$. $\mathcal{G}$ is the symbol for the Cauchy principal value integral. We can write

$$
\tilde{\mathcal{K}}_j(r, r') \cdot M(r') = \int_{S'} \nabla g_j(r, r') \times M(r').
$$

Noticing that:

$$
\nabla g_i(r, r') = \frac{ik_j R - 1}{4\pi R^2} e^{ik_j R} \hat{R} \quad (3.16)
$$

If $S'$ is a planar surface, the principal value integral part has only a normal direction contribution and the first term in (3.15) contributes to the tangential component. Usually we use a tangential vector as the testing function, the second term will be filtered out, leaving only the first term. If we use the same function for the basis and testing, the first term vanishes after the integration since

$$
\Lambda(r) \cdot (\hat{n} \times \Lambda(r)) = 0 \quad (3.17)
$$

From the above analysis, we can see that the $\mathcal{K}$ operator has a residual part that generates a field with 90 degrees rotation from the source direction. Using the same function as the expansion basis function to test cannot produce a meaningful scalar number. As remedies, one can either change the testing or basis function. Since RWG works well for $\mathcal{V}$ and $\mathcal{P}$ operators for testing, we change the basis function to expand $M(r')$ so that the basis function is perpendicular to the RWG basis. Such basis functions are called dual basis functions.

Dual basis functions cannot be easily constructed on a triangle mesh unless a barycentric mesh [20] is established beforehand. This mesh can be obtained by finding the center of a triangle, and then connecting it to the nodes and edge centroids of this triangle, as shown in Figure 3.2. On this barycentric mesh, a dual basis function can be found, such as the Chen-Wilton basis function [8], [21], [22] and the Buffa-Christiansen (BC) basis function [7]. Or alternatively, a dual basis function can be found relatively easily in a quadrilateral mesh [23]. In Section 3.3, we will introduce the construction details and the properties of the BC basis function.
3.3 Buffa-Christiansen (BC) Basis

The BC basis function is constructed on barycentric mesh. By dividing the triangle in the original mesh into six small triangles, the BC basis function can be constructed as the superpositions of the RWG basis function of the small triangles. This basis function is divergence-conforming because it is a summation of the divergence-conforming RWG basis. It is also approximately perpendicular to the RWG basis function, hence making it suitable as a basis function for magnetic current $\mathbf{M}$. The direction and amplitude of this basis function is shown in Figure 3.3.

3.3.1 BC Basis on Closed Sections

When the reference edge (on which we define the BC basis function) has no vertices on the boundary (close section), the construction of this basis function is relatively simple. The BC basis function is a weighted summation
Figure 3.3: Buffa-Christiansen divergence-conforming, quasi curl-conforming basis function on barycentric mesh. The edge in red is the reference edge; the shaded region is the domain of the basis function for the reference edge. The arrows denote the directions and the amplitudes.

of a small RWG basis function of barycentric mesh, i.e.

\[ \mathbf{f}^{BC} = \sum_i c_i \mathbf{f}^{RWG}_b + \tilde{c}_i \mathbf{f}^{RWG}_b \]  

(3.18)

where \( \mathbf{f}^{BC} \) refers to the BC basis function defined on a reference edge, and \( c_i \) are the coefficients of the rightmost RWG basis function. \( \tilde{c}_i \) are those of the leftmost RWG basis function and \( \mathbf{f}^{RWG}_b \) are the RWG basis function of the barycentric edges.

The coefficients \( c_i \) and \( \tilde{c}_i \) are defined as:

\[ c_i = \frac{N_c - i}{2N_c}, \quad i = 1, 2, \ldots, 2N_c - 1 \]  

(3.19)

\[ \tilde{c}_i = \frac{\tilde{N}_c - i}{2\tilde{N}_c}, \quad i = 1, 2, \ldots, 2\tilde{N}_c - 1 \]  

(3.20)

where \( N_c \) and \( \tilde{N}_c \) are the numbers of original triangles on the rightmost and leftmost node respectively, as in Figure 3.4.
3.3.2 BC Basis on Open Sections

When the reference edge has one or two nodes on the boundary, this edge is on open section. The above method to construct the BC basis function is no longer valid and we need to make some modifications.

The definition of the BC basis function is still the same as (3.18). It is required that this summation gives rise to a solenoidal current. Hence, the coefficients need to change. The two cases when one node and two nodes on the boundary are shown in Figures 3.5 and 3.6. The determinations of coefficients for the cases are a little different. We will discuss both cases.

For one node on the boundary, as in Figure 3.5, we can assume that the node associated with the rightmost RWG basis function is on the boundary. Then the coefficients related to the leftmost RWG basis function are the same as those for a close section:

$$
\tilde{c}_i = \frac{\tilde{N}_e - i}{2\tilde{N}_e}, \quad i = 1, 2, \ldots, 2\tilde{N}_e - 1
$$

(3.21)

The coefficients related to the rightmost RWG basis function are determined
Figure 3.5: The case when one node of the reference edge is on the boundary. The dashed lines mean the boundary. The “+” and “−” refer to the signs of small RWG basis function, orange circles indicate the leftmost basis function, and the blue circles indicate the rightmost. The number is the $i$ in the coefficients $c_i$.

by:

$$c_0 = \frac{1}{2}$$  \hspace{1cm} (3.22)

$$c_i = \begin{cases} 
\frac{1-N_c}{N_c}, & i < N_{ref} \\
\frac{2-N_c}{2N_c}, & i = N_{ref} \\
\frac{1}{N_c}, & i > N_{ref}
\end{cases} \hspace{1cm} i = 1, 2, \cdots 2N_c + 1$$  \hspace{1cm} (3.23)

where $N_c$ is the number of original triangles on the rightmost node and $N_{ref}$ is the number of original edges associated with the boundary node.

For two nodes on the boundary, as shown in Figure 3.6, the definitions of the coefficients are similar to the open section discussed in this section.

$$c_0 = \frac{1}{2}$$  \hspace{1cm} (3.24)

$$\tilde{c}_0 = -\frac{1}{2}$$  \hspace{1cm} (3.25)
Figure 3.6: The case when two nodes of the reference edge are on the boundary. The dashed lines mean the boundary. The “+” and “−” refer to the signs of the small RWG basis function, orange circles indicate the leftmost basis function and the blue circles indicate the rightmost basis function. The number is the $i$ in the coefficients $c_i$.

$$c_i = \begin{cases} 
\frac{1-N_c}{N_c}, & i < N_{\text{ref}} \\
\frac{2-N_c}{2N_c}, & i = N_{\text{ref}} \\
\frac{1}{N_c}, & i > N_{\text{ref}} 
\end{cases} \quad i = 1, 2, \ldots, 2N_c + 1 \quad (3.26)$$

$$\tilde{c}_i = \begin{cases} 
\frac{1-\tilde{N}_c}{N_c}, & i < \tilde{N}_{\text{ref}} \\
\frac{2-\tilde{N}_c}{2N_c}, & i = \tilde{N}_{\text{ref}} \\
\frac{1}{\tilde{N}_c}, & i > \tilde{N}_{\text{ref}} 
\end{cases} \quad i = 1, 2, \ldots, 2\tilde{N}_c + 1 \quad (3.27)$$

where $\tilde{N}_c$ and $N_c$ are the numbers of original triangles on the leftmost and rightmost node. $\tilde{N}_{\text{ref}}$ and $N_{\text{ref}}$ are the number of original edges associated with the left and right nodes on the boundary.

With the definition of the BC basis function on closed and open sections, this basis function is well defined on the surface triangulated mesh.
3.4 Comparison of RWG and BC as Basis Functions

A comparison of direction and amplitude of the RWG and BC basis functions is shown in Figure 3.7. It can be observed that the RWG basis function is divergence conforming, while the BC basis function is a summation of the RWG basis function on barycentric mesh as defined in (3.18). Therefore, they are both divergence-conforming basis functions. However, as shown in Figure 3.7, the direction of the two basis functions are almost orthogonal to each other. Therefore, the BC basis will be suitable as the basis function for magnetic current $M$. Noticing that:

$$\mathcal{K}_j(r, r') \cdot M'(r') = \pm \frac{1}{2} \hat{n} \times M(r) + \int_{S'} \nabla' g_j(r, r') \times M(r')$$  \hspace{1cm} (3.28)

The second term vanishes when the testing function and the basis function are on the same triangle. We can easily find that the diagonal of the $\mathcal{K}$ operator is very small if the RWG basis function is used as the testing and basis function, which gives rise to an ill-conditioned matrix. However, the diagonal terms are large if RWG is used for testing and BC for the basis function. The $\mathcal{K}$ operator is well-conditioned. Table 3.1 shows the comparison of condition numbers of the $\mathcal{K}$ operator in these two basis schemes.
Table 3.1: A comparison of the condition numbers of the $\mathcal{K}$ operator with two basis functions. RWG is always used as the testing function.

<table>
<thead>
<tr>
<th>Basis Function</th>
<th>Condition Number @300 MHz</th>
<th>Condition Number @30 Hz</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>RWG basis</td>
<td>9.5E5</td>
<td>9.4E5</td>
<td>Ill-conditioned</td>
</tr>
<tr>
<td>BC basis</td>
<td>9.1</td>
<td>4.1</td>
<td>Well-conditioned</td>
</tr>
</tbody>
</table>

3.5 An A-EFIE Dielectric Solver

From the discussion above, we conclude that the suitable basis functions for $\mathbf{J}$ and $\mathbf{M}$ are the RWG and BC basis functions, while RWG can be used as the testing function for both $\mathcal{L}$ and $\mathcal{K}$ operators. The final formulation to solve the dielectric problem will be:

$$
\begin{bmatrix}
    \mathbf{V}_{\text{ext}} & \mathbf{K}_{\text{ext}} & \mathbf{D}^T \cdot \mathbf{P}_{\text{ext}} \cdot \mathbf{B} \\
    \mathbf{V}_{\text{int}} & \mathbf{K}_{\text{int}} & \mathbf{D}^T \cdot \mathbf{P}_{\text{int}} \cdot \mathbf{B} \\
    \mathbf{F} \cdot \mathbf{D} & 0 & k_0^2 \mathbf{I}
\end{bmatrix}
\begin{bmatrix}
    i k_0 \mathbf{J} \\
    \eta^{-1} \mathbf{M} \\
    c_0 \mathbf{\rho}_r
\end{bmatrix}
= \begin{bmatrix}
    -\eta_0^{-1} \mathbf{b} \\
    0 \\
    0
\end{bmatrix}
$$

(3.29)

The matrix elements are:

$$
[\mathbf{V}_j]_{mn} = \mu_r \int dS \mathbf{A}_m(r) \cdot \int dS' g_j(r, r') \mathbf{A}_n(r')
$$

(3.30)

$$
[\mathbf{K}_j]_{mn} = \int dS \mathbf{A}_m(r) \cdot \int dS' \nabla' g_j(r, r') \times \mathbf{\Gamma}_n(r')
$$

(3.31)

$$
[\mathbf{P}_j]_{mn} = \epsilon_r^{-1} \int dS h_n(r) \int dS' g_j(r, r') h_n(r')
$$

(3.32)

$$
g_j(r, r') = \frac{e^{ik_j|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}
$$

(3.33)

where $\mathbf{A}$ is the RWG basis function, $\mathbf{\Gamma}$ is the BC basis function, and $h$ is the pulse basis function or divergence of the RWG basis function. The $\mathcal{K}$ matrix element can be further written as a summation of two terms:

$$
[\mathbf{K}_j]_{mn} = \pm \frac{1}{2} \int dS \mathbf{A}_m \cdot (\hat{n} \times \mathbf{\Gamma}(r)) + [\overline{\mathbf{K}_j}]_{mn}
$$

(3.34)
where

\[
\left[ \tilde{K}_j \right]_{mn} = \int dS A_m \cdot \int_{S'} \nabla' g_j (r, r') \times \Gamma (r')
\]  

(3.35)

3.5.1 Large-Scale Problem

For problems that are multi-scale or have a large number of unknowns, a pre-conditioner is usually required for faster convergence when using iterative solvers. Motivated by the pre-conditioner designed for the A-EFIE PEC solver, we can easily construct a similar pre-conditioner matrix:

\[
\mathbf{M} = \begin{bmatrix}
\mathbf{V}_{\text{ext}} \text{diag} & \mathbf{K}_{\text{ext}} \text{diag} & \mathbf{D}^T \cdot \mathbf{P}_{\text{ext}} \text{diag} \cdot \mathbf{B} \\
\mathbf{V}_{\text{int}} \text{diag} & \mathbf{K}_{\text{int}} \text{diag} & \mathbf{D}^T \cdot \mathbf{P}_{\text{int}} \text{diag} \cdot \mathbf{B} \\
\mathbf{F} \cdot \mathbf{D} & 0 & k_0^2 \mathbf{I}
\end{bmatrix}
\]  

(3.36)

With this pre-conditioner, the converge rate is greatly accelerated. As an example, we use the GMRES solver with the restart number set to 30. Table 3.2 shows the number of iterations needed to satisfy a certain error tolerance. Apparently, the pre-conditioner reduces the iteration number by more than 25 times.

Table 3.2: Number of iterations needed for the A-EFIE dielectric solver to converge to a certain error tolerance using GMRES with and without a pre-conditioner.

<table>
<thead>
<tr>
<th>Error Tolerance</th>
<th>No Pre-Conditioner</th>
<th>With Pre-Conditioner</th>
</tr>
</thead>
<tbody>
<tr>
<td>1E-3</td>
<td>545</td>
<td>21</td>
</tr>
<tr>
<td>1E-6</td>
<td>&gt; 1000</td>
<td>42</td>
</tr>
<tr>
<td>1E-9</td>
<td>&gt; 1000</td>
<td>68</td>
</tr>
</tbody>
</table>

Meanwhile, for problems with a large number of unknowns, the method of moments requires \( O(N^2) \) computational complexity and \( O(N^2) \) memory cost. These costs can be reduced to \( O(N \log N) \) if the fast multipole algorithm (FMA) [24], [25] is applied. We use the mixed-form FMA to accelerate this solver, so that the leafy level box can be very small and multipoles are used to expand the wave. At a higher level when the box size becomes larger, they can be converted to plane wave expansion if needed. Then it will be more flexible to solve various frequency and multi-scale problems.
3.6 Numerical Examples

In this section, we validate this solver by comparing solutions with Mie series. For Mie series validation, two kinds of sources are used: plane wave and point source excitations. A sphere with radius of 1 unit is located at the origin. The relative permittivity of the material $\epsilon_r = 2$, the relative permeability $\mu_r = 1$ and the sphere is lossless. If the plane wave is the excitation, it is propagating to $-z$ with polarization in the $x$ direction. RCS and near field are both measured as outputs. If the point source is used, it is located at $(0, 0, 2 \times \text{unit})$ with polarization in the $z$ direction. Only near field is measured as the output.

Figure 3.8 shows the comparison of Mie series and RCS measurements of plane wave excitation. Figure 3.9 shows the near field measurements of plane wave excitation with the Mie series. Figure 3.10 shows the near field measurements of point source excitation compared with the Mie series.
Figure 3.9: The $r$ and $\theta$ components of near field scattered by a sphere with a radius of 1 m compared with the Mie series. The measurement radius is 2 m and the excitation source is the plane wave and the frequency is 100 MHz.

Figure 3.10: The $r$ and $\theta$ components of near field scattered by a sphere with a radius of 1 m compared with Mie series. The measurement radius is 2 m and the excitation source is the point source and the frequency is 100 MHz.
CHAPTER 4

A-EFIE FOR CONDUCTIVE MATERIALS

It is usually difficult to model highly conductive material using the full wave method. For the surface-based method, the impedance boundary condition is conventionally used. However, at lower frequency, this method is inaccurate. For the volume-based method, very high mesh density near the surface is required to capture the variations of field near the surface. Hence, the volume-based method is very inefficient and not favorable for such problems. One approach [26] is to use the finite difference method to replace the local surface impedance with global surface impedance (GSI). However, this method is only computational attractive for structures with a one-dimensional extension, such as a transmission line or a conductive bar. With A-EFIE, we can establish a rigorous global surface impedance boundary condition in a matrix form.

The A-EFIE for dielectrics is a general formulation for lossless and lossy dielectrics from low frequency to mid frequency. The external equation couples the equivalent electric and magnetic currents by the excitation source and the geometry of the objects with Green’s function. The internal equation couples the currents with Green’s function of the interior region.

The external Green’s function is generally slowly varying and the mesh will be able to capture the variation of the wave inside if the triangle size $h$ is smaller than or equal to $0.1\lambda$. The condition can be easily satisfied, especially at low frequency. As the relative permittivity of inside material becomes larger, the wave is oscillating faster. Then the mesh is required to be denser in order to capture the oscillation. If the material is lossless, the electric and magnetic currents are still coupled, even if they are far away from each other. However, for conductive material the situation is different.

The conductive material can be characterized by relative permittivity, permeability and conductivity. If the conductivity is very large, the wavenumber inside the conductor has large real and imaginary parts, which gives rise to
fast oscillating and fast decaying waves. The fast decaying property will make the coupling distance of the electric and magnetic current very small. If the conductivity is extremely large, they are locally coupled.

4.1 Characteristics of Conductive Material

Normally the existing materials are non-magnetic, so the relative permeability \( \mu_r \approx 1 \). The effective permittivity can be expressed as \( \varepsilon = \varepsilon' + i \frac{\sigma}{\omega} \) from \( \nabla \times \mathbf{H} = -i \omega \varepsilon \mathbf{E} + \sigma \mathbf{E} \). Then the effective relative permittivity is \( \varepsilon_r = \varepsilon'_r + i \frac{\sigma}{\omega \varepsilon_0} \). Then the wavenumber inside material is:

\[
k = \omega \sqrt{\varepsilon \mu} = k_0 \sqrt{\varepsilon_r} = k_0 \sqrt{\varepsilon'_r + \frac{i \sigma}{\omega \varepsilon_0}} \tag{4.1}
\]

where \( k_0 \) is the wavenumber in free space \( k_0 = \omega \sqrt{\varepsilon_0 \mu_0} \). When \( \varepsilon'_r \ll \frac{\sigma}{\omega \varepsilon_0} \),

\[
k = k' + ik'' \approx k_0 \sqrt{\frac{\sigma}{2 \omega \varepsilon_0}} + i k_0 \sqrt{\frac{\sigma}{2 \omega \varepsilon_0}} = \frac{1}{\delta_s} + i \frac{1}{\delta_s} \tag{4.2}
\]

where \( k' \approx k'' \), \( \delta_s \) is the skin depth of conductor with the expression as:

\[
\delta_s = \sqrt{\frac{2}{\omega \mu_0 \sigma}} \tag{4.3}
\]

When the conductivity is large or the frequency is high, the skin depth becomes very small, \( k' \) and \( k'' \) become very large.

The coupling of electric and magnetic currents \( \mathbf{J} \) and \( \mathbf{M} \) of the internal problem is through matrices \( \mathbf{V}_{int}, \mathbf{K}_{int} \) and \( \mathbf{P}_{int} \) with matrix elements defined in (3.30), (3.31) and (3.32). For highly lossy material, the Green’s function becomes fast oscillating and decaying. The far interaction of \( \mathbf{V}_{int}, \mathbf{K}_{int} \) and \( \mathbf{P}_{int} \) becomes extremely small due to the fast decaying. Then these matrices are highly sparse. By utilizing the sparsity of these matrices, the internal coupling of \( \mathbf{J} \) and \( \mathbf{M} \) can be further simplified. However, to capture the coupling correctly, the matrix elements should be evaluated accurately. The traditional method we have fails and we need to find a better method to do this.
4.2 Evaluation of Matrix Elements for Conductive Materials

The integration technique for (3.30), (3.31) and (3.32) has been discussed in many literatures [27], [28], [29]. Normally, the outer integration is computed with a few Gaussian quadrature points. More care is taken to evaluate the inner integrals:

\[ \int_{S'} dr' \frac{e^{ik|r-r'|}}{|r-r'|} \] (4.4)

\[ \int_{S'} dr' \frac{e^{ik|r-r'|}}{|r-r'|} (r - v) \] (4.5)

and

\[ \int_{S'} d\mathbf{r}' \mathbf{n} \cdot \frac{e^{ik|r-r'|}}{|r-r'|} (r - v) \] (4.6)

One approach to calculate the inner integral is called singularity subtractions: this subtracts the singular part and integrate analytically, then numerically computes the regular part.

By expressing Green’s function as:

\[ g(r, r') = \frac{e^{ik|r-r'|}}{4\pi |r-r'|} = \frac{1}{4\pi |r-r'|} + \frac{e^{ik|r-r'|} - 1}{4\pi |r-r'|} \] (4.7)

The integration of forms \( \int_{S'} \frac{1}{4\pi |r-r'|} dr' \) and \( \int_{S'} \frac{1}{4\pi |r-r'|} \Lambda(r') dr' \) can be integrated analytically while \( \int_{S'} \frac{e^{ik|r-r'|}}{4\pi |r-r'|} \Lambda(r') dr' \) and \( \int_{S'} \frac{e^{ik|r-r'|} - 1}{4\pi |r-r'|} \Lambda(r') dr' \) are considered to be regular because:

\[ \frac{e^{ik|r-r'|} - 1}{4\pi |r-r'|} = \frac{1}{4\pi} \left( ik + \frac{(ik)^2}{2!} |r-r'| + \cdots + \frac{(ik)^n}{n!} |r-r'|^{n-1} + \cdots \right) \] (4.8)

None of the terms at the right-hand side of (4.8) are singular. When \(|r-r'|\) is small, the Taylor’s series can be truncated to a few terms for simplifications. This technique is widely accepted and used in the computational electromagnetic community. It is apparent that the analytical form of singularity integration is accurate, while the numerical integration of the regular
part gives rise to errors, especially when the wavenumber $k$ is large and the triangle size $h$ is not small enough. That is why $h$ should be smaller than $0.1\lambda$ in order to capture the osculation of wave over the triangle. Otherwise, more quadrature points are needed for accurate numerical integration.

Another approach is given in [30], [31] and it is called singularity extraction. With this method, the integrands are not broken into two terms. Instead, the inner surface integral can be converted to line integrals by placing the triangle in polar coordinates. For example, one of the integrals is:

$$I_a = \int_{{S'}} \frac{e^{ik|r-r'|}}{|r-r'|} d\mathbf{r}' = \sum_{i=1}^{3} \int_{0}^{\theta_i^+} d\theta \int_{\rho(\theta)}^{\rho(0)} d\rho \frac{e^{ikR}}{R}$$  \hspace{1cm} (4.9)

Noticing that $R = \sqrt{\rho^2 + d^2}$, then $RdR = \rho d\rho$, the integral can be written as:

$$I_a = \sum_{i=1}^{3} \int_{\theta_i^-}^{\theta_i^+} d\theta \int_{0}^{R(\theta)} dRe^{ikR} = \sum_{i=1}^{3} \int_{\theta_i^-}^{\theta_i^+} d\theta \frac{e^{ikR(\theta)} - e^{ikd}}{ik}$$  \hspace{1cm} (4.10)

Then second term can be simply integrated $- \int_{\theta_i^-}^{\theta_i^+} d\theta \frac{e^{ikd}}{ik} = -\frac{e^{ikd}}{ik} (\theta_i^+ - \theta_i^-)$.

The first term is

$$\sum_{i=1}^{3} \int_{\theta_i^-}^{\theta_i^+} d\theta \frac{e^{ikR(\theta)}}{ik} = \frac{1}{ik} \sum_{i=1}^{3} \int_{x_i^-}^{x_i^+} dx \frac{h_i}{h_i^2 + x^2} \left( e^{ik\sqrt{d^2 + h_i^2 + x^2}} \right)$$  \hspace{1cm} (4.11)

where $x = h_i \tan \theta$. Then

$$I_a = \frac{1}{ik} \sum_{i=1}^{3} \int_{x_i^-}^{x_i^+} dx \frac{h_i}{h_i^2 + x^2} \left( e^{ik\sqrt{d^2 + h_i^2 + x^2}} \right) - \frac{e^{ikd}}{ik} \sum_{i=1}^{3} (\theta_i^+ - \theta_i^-)$$  \hspace{1cm} (4.12)

If this integral can be integrated analytically, the matrix elements are accurately evaluated. However, there is no closed-form solution to the above integral and it can only be computed numerically using quadrature points, i.e.

$$\int_{x_i^-}^{x_i^+} dx \frac{h_i}{h_i^2 + x^2} \left( e^{ik\sqrt{d^2 + h_i^2 + x^2}} \right) = \sum_{p=1}^{N} w_p \frac{h_i}{h_i^2 + x_p^2} \left( e^{ik\sqrt{d^2 + h_i^2 + x_p^2}} \right)$$  \hspace{1cm} (4.13)

where $x_p \in [x_i^-, x_i^+]$ is the quadrature point and $w_p$ is the weighting factor.
Figure 4.1: Parameters in singularity extraction integrals.

Other integrals related to the matrix elements are defined as:

\[ I_b = \int_{S'} d' e^{ik|\mathbf{r}' - \mathbf{r}|} (\mathbf{r}' - \mathbf{v}) \]  \hspace{1cm} (4.14)

\[ I_c = \int_{S'} (ik|\mathbf{r} - \mathbf{r}'| - 1) e^{ik|\mathbf{r}' - \mathbf{r}|} \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \]  \hspace{1cm} (4.15)

\[ I_d = \int_{S'} (ik|\mathbf{r} - \mathbf{r}'| - 1) e^{ik|\mathbf{r}' - \mathbf{r}|} \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r}_0 - \mathbf{v}) \]  \hspace{1cm} (4.16)

They can also be written in the forms similar to \( I_a \):

\[ I_b = \frac{1}{ik} \sum_{i=1}^{3} \hat{u}_i \int_{x_i}^{x_i^+} dx e^{ik\sqrt{d^2 + h_i^2 + x^2}} + (\mathbf{r}_0 - \mathbf{v}) I_a \]  \hspace{1cm} (4.17)

\[ I_c = \sum_{i=1}^{3} \int_{x_i^-}^{x_i^+} dx \frac{h_i}{h_i^2 + x^2} \left( \frac{e^{ik\sqrt{d^2 + h_i^2 + x^2}}}{\sqrt{d^2 + h_i^2 + x^2}} \right) - \frac{e^{ikd}}{d} \sum_{i=1}^{3} (\theta_i^+ - \theta_i^-) \]  \hspace{1cm} (4.18)

\[ I_d = \sum_{i=1}^{3} \hat{u}_i \int_{x_i}^{x_i^+} dx \frac{e^{ik\sqrt{d^2 + h_i^2 + x^2}}}{\sqrt{d^2 + h_i^2 + x^2}} \]  \hspace{1cm} (4.19)
Figure 4.2: Approximation method in singularity extraction integrals.

where \( \mathbf{r}_0 \) is the projection of \( \mathbf{r} \) on the plane of triangle \( S' \). \( d \) is the projection distance. \( \hat{\mathbf{u}}_i \) are the unit vectors perpendicular to the edges of the triangle and point away from \( \mathbf{r}_0 \). \( h_i \) gives the distance from \( \mathbf{r}_0 \) to the corresponding edge, as shown in Figure 4.1. The integral in \( \mathcal{K} \) operator is:

\[
I_e = - \int_{S'} e^{ik|\mathbf{r} - \mathbf{r}'|} \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}') \times (\mathbf{r}' - \mathbf{v})
\]

(4.20)

Noticing that \( \mathbf{r} - \mathbf{r}' = (\mathbf{r} - \mathbf{r}_0) + (\mathbf{r}_0 - \mathbf{r}') = -\mathbf{d} - \rho(\mathbf{r}') \) and \( \mathbf{r}' - \mathbf{v} = (\mathbf{r}' - \mathbf{r}_0) + (\mathbf{r}_0 - \mathbf{v}) = \rho(\mathbf{r}') + (\mathbf{r}_0 - \mathbf{v}) \). Then \( (\mathbf{r} - \mathbf{r}') \times (\mathbf{r}' - \mathbf{v}) = (\mathbf{r}_0 - \mathbf{v} - \mathbf{d}) \times \rho(\mathbf{r}') - \mathbf{d} \times (\mathbf{r}_0 - \mathbf{v}) \). Then

\[
I_e = -(\mathbf{r}_0 - \mathbf{v} - \mathbf{d}) \times I_d + \mathbf{d} \times (\mathbf{r}_0 - \mathbf{v}) I_c
\]

(4.21)

A very special case is when \( \mathbf{r} \) is on the source triangle \( S' \) and the real and imaginary parts of the wavenumber are much larger than the triangle size \( h \). The integrals above can be approximated. As shown in Figure 4.2, the shaded region has \( g(\mathbf{r}, \mathbf{r}') \approx 0 \). Then the integral over the triangle can be approximated as an integral over a circle.

\[
I_a = \int_{S'} e^{ik|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' = \int_{S'_{circle}} e^{ik|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' = \frac{2\pi i}{k}
\]

(4.22)
\[ I_b = \int_{S'} dr' \frac{e^{ik|r-r'|}}{|r-r'|}(r' - v) = \int_{S'_{circle}} dr' \frac{e^{ik|r-r'|}}{|r-r'|}(r' - v) = (r - v) \frac{2\pi i}{k} \] (4.23)

There is no need to calculate \( I_e \) since \( I_e \) has only a normal component and the testing function is purely tangential. Note that the matrix element of the \( K \) operator has a residual part \( \pm \frac{1}{2} \hat{n} \times \Gamma(r) \). It is very important that the formulation for A-EFIE of dielectrics is still of full rank, which will be discussed in Section 4.3.

The approximation equations (4.22) and (4.23) can be derived from (4.12) and (4.17).

When \( r \) is on the source triangle, \( d = 0 \). \( \sum_{i=1}^{3}(\theta_i^+ - \theta_i^-) = 2\pi \) while the first term in (4.22) is very small since \( dx \frac{h_i}{h_i^2 + x^2} \sim O(1) \) and \( e^{ik\sqrt{d^2 + h_i^2 + x^2}} \approx 0 \) if \( 3k \) is very large. Then \( I_a \approx \frac{2\pi i}{k} \), which is (4.22). Similarly, the first term in (4.17) is very small and \( I_b \approx (r - v) \frac{2\pi i}{k} \), which is (4.23).

When \( r \) is no longer on the source triangle, we can also derive from (4.12), (4.17) and (4.21) that the interaction is less important and the values of \( I_a \) and \( I_b \) are smaller.

When the projection point \( r_0 \) is on the source triangle, the second term in (4.12) becomes:

\[ -\frac{e^{ikd}}{ik} \sum_{i=1}^{3}(\theta_i^+ - \theta_i^-) = \frac{2\pi i}{k} e^{ikd} \] (4.24)

The integral in the first term, by using the stationary phase approximation becomes:

\[ \int_{h_i}^{x_i^+} dx \frac{h_i}{h_i^2 + x^2} \left( e^{ik\sqrt{d^2 + h_i^2 + x^2}} \right) \approx \frac{1}{h_i} e^{-k''\sqrt{d^2 + h_i^2}} \int_{x_i^-}^{x_i^+} dx e^{ik\sqrt{h_i^2 + d^2 + x^2}} \] (4.25)

Since \( \left| \int_{x_i^-}^{x_i^+} dx e^{ik\sqrt{h_i^2 + d^2 + x^2}} \right| \leq \int_{x_i^-}^{x_i^+} dx |e^{ik\sqrt{h_i^2 + d^2 + x^2}}| \leq \int_{x_i^-}^{x_i^+} dx = (x_i^+ - x_i^-) \), then:

\[ \left| \int_{x_i^-}^{x_i^+} dx \frac{h_i}{h_i^2 + x^2} \left( e^{ik\sqrt{d^2 + h_i^2 + x^2}} \right) \right| \leq \frac{x_i^+ - x_i^-}{h_i} e^{-k''\sqrt{d^2 + h_i^2}} \] (4.26)

if \( k' \) and \( k'' \) are very large, and the triangle size \( h_i \) is also large. The second term dominates in (4.12). The comparisons of the terms in \( I_b, I_c \) and \( I_d \) are very similar: the first term in (4.17) is of the order \( \frac{h_i}{k} e^{ik''\sqrt{d^2 + h_i^2}} \), the second
term is of the order $\frac{1}{h}$, where $h$ is the size of the triangle. In $I_c$, the second term dominates the first because of the larger exponential part in the first term. $I_d$ is of the same order of amplitude as the first term in $I_c$.

When $r_0$ is outside the triangle, the integrals become even smaller. It can be easily seen that

$$\sum_{i=1}^{3} (\theta_i^+ - \theta_i^-) = 0 \quad (4.27)$$

Then $I_a$, $I_b$, $I_c$ and $I_d$ are all of the order $e^{ik''\sqrt{d^2+h^2}}$ and become very small. Although small, these matrix elements relating triangles close to each other are required to be computed accurately in order to capture the coupling of $J, M$ on those triangles. As $k''$ gets smaller, i.e. the skin depth $\delta_s$ gets comparable to the triangle size $h$, the term $e^{ik''\sqrt{d^2+h^2}}$ is no longer very small. The first and the second term in (4.12) are equally important and are needed to be calculated accurately.

4.3 Comparison of Integration Schemes for Matrix Elements Evaluation

As shown Section 4.2, we have introduced three integration schemes to evaluate the matrix elements in $L, P$ and $K$ operators. The first one is the singularity subtraction. The subtracted regular part is computed numerically using Gaussian quadrature points on the triangle. The second approach is the singularity extraction or line integral. The surface integrals are converted to line integrals by coordinate transform. The third method is only valid for a special case when the observation point $r$ is on the source triangle $S'$. It is derived from the approximation that the integral over a triangle can be estimated as an integral over a circle.

To compare the three methods, we choose a triangle with a size and shape as shown in Figure 4.3. The triangle size $h = 1$ mm. The two points $A, B, C, D$ are used as the observation points $r$, where $B$ and $C$ are closer to one of the edges than the other two. The wavenumber $k = k' + ik''$ and $k' = k''$.

For the first method (we call the surface integral), seven quadrature points are used for regular part surface integration. For the second one (line integral), 10 quadrature points on each edge are used for the line integral, since
Figure 4.3: The triangle used for the integration scheme comparison. Two points A and B are used as the observation point \( r \) in this comparison.

The line integral is cheaper than the surface counterpart. For the third (circle approximation), the approximation equations (4.22) and (4.23) are used. The results computed from the three methods are compared to the exact solution, which is calculated using 1 million surface quadrature points using the first method. Figures 4.4 and 4.5 show the comparison of the error in percentage when the observation point \( r = A \). Figures 4.6 and 4.7 show the errors when \( r = B \).

Figure 4.4: A comparison of error of the integration \( I_a \) with the three different methods. The observation point is \( A \).

Apparently, when the observation point \( r \) get closer to the edge, the error of the line integral increases and more quadrature points are required to
reduce the error. This is because when \( h_i \) get smaller (as for \( B \)), the decay part in the first term of (4.12) and (4.17) gets larger in amplitude. Then the oscillation of the wave is more and more dominating. Thus more sampling points are required so that (4.13) converges to the correct value.

Also note that for small \( k'h \), the surface integral approach is more accurate than the line integral. This is because for small \( k'h \), the integrand is varying and non-zero over almost the whole range of \( x \in [x_i^-, x_i^+] \), while for large \( k'h \),
the integrand is almost all zero over \( x \in [x_i^-, x_i^+] \).

When the projection of the source point \( r \) is outside the triangle, as in the cases of \( C \) and \( D \) when \( r \) is close to the edges, numerical experiments show that more quadrature points are needed to minimize the error as in Figure 4.8. This is because the amplitude of the integral is much smaller than the case when \( r \) is inside, thus making the relative error larger. Among the three terms summation in \( I_a \), the term relating to the closest edge gives rise to the error. Then we can adaptively choose the number of quadrature points: 10 points for far edges and 50 for the near edge. The results are shown in Figure 4.9. A similar result for point \( D \) is shown in Figure 4.10.

From the above comparison, the line integral method is more effective and accurate than the seven-point surface integral approach. The reason is that the circle approximation and the correction terms are in the formula of line integral. By the coordinate transform, the lossy physics is better isolated and captured.

Based on the surface integral, another method, called adaptive surface Gaussian quadrature, can be applied. This is to divide the original triangle into a number of smaller ones so that the small triangle size satisfy \( h = 0.1\lambda \). However, this can be extremely expensive and the lossy physics is not captured in an elegant way.
Figure 4.8: A comparison of error in integration \( I_a \) using three different methods (ten-point line integral). The observation point is \( C \).

4.4 Requirements of Mesh Size

A natural question arises when you try to decide the size of the triangles of our mesh: how small the triangle should be in order that the matrix elements of the internal problem are accurately calculated? If the internal material is lossless dielectrics, conventionally, mesh size of \( h < 0.1\lambda \) is chosen, where \( \lambda \) is the wave length in the dielectrics. However, when the material is extremely lossy, skin depth \( \delta_s \) becomes very small and the internal wavenumber \( k = \frac{1}{\delta_s} + i\frac{1}{\delta_s} \) has very large real and imaginary parts, which means inside the material, the wave is fast oscillating as well as decaying. Does that mean the mesh size \( h \) is required to be smaller than \( 0.1\lambda \), which is extremely small?

To answer this question, we can first remind ourselves of current distribution on the PEC surface. To solve a PEC problem, the internal problem does not need to be solved because the electric field inside PEC is 0. When the mesh size \( h \) is smaller than \( 0.1\lambda \) of external material, the problem is solved accurately. We can observe that the current distribution is very uniform on a smooth surface and it only changes abruptly at the corners and the edges. When we change PEC to a non-perfect conductor gradually, there should only be very slight changes in the current distribution. Then for this non-perfect conductor, is it physically required that the mesh size should be very small? The answer is no. In a surface integral equation solver, a dense mesh is usually required when the fields (or corresponding equivalent cur-
Figure 4.9: A comparison of error in integration $I_a$ using two different methods (adaptively choose the number of quadrature points for line integral). The observation point is $C$.

...rents) change rapidly on the surface of a geometry. This usually corresponds to rapid varying of matrix elements in $\mathcal{L}$, $\mathcal{K}$ and $\mathcal{P}$ operators. Since the field does not change rapidly in conductors, a dense mesh is usually not required.

Mathematically, we can examine the Green’s function for conductors. The wavenumber $k$ inside is

$$k = \frac{1}{\delta_s} + i \frac{1}{\delta_s} \quad (4.28)$$

where $\delta_s$ is the skin depth and is very small for good conductors. The Green’s function is written as:

$$g(r, r) = \frac{e^{ikR}}{4\pi R} \quad (4.29)$$

Plugging in the $k$, the numerator becomes:

$$e^{ikR} = e^{\frac{R}{\delta_s} e^{-\frac{R}{\delta_s}}} \quad (4.30)$$

The first term represents the oscillating and the second term denotes the decaying. Apparently the wavelength of the oscillating part is $\frac{2\pi}{k'} = 2\pi \delta_s$, which is larger than the skin depth $\delta_s$. A plot showing the oscillating, decaying parts and the product of the two is in Figure 4.11. It is noted that the numerator of the Green’s function looks more like the decaying part as the fast oscillating part is filtered out. For triangle sizes much larger than the skin depth, only the contribution within the radius of $2\pi \delta_s$ is important.
Figure 4.10: A comparison of error in integration $I_a$ using two different methods (adaptively choose the number of quadrature points for line integral). The observation point is $D$.

in the Green’s function. Using Gaussian quadrature points on the triangle surface for integral evaluation does not capture the importance in this region, because the points are uniformly chosen on the triangle. However, the line integral approach mentioned above capture this, because the second term in (4.12) accounts for this contribution. Therefore, to solve the internal problem accurately, a dense mesh is not required, while a correct method for matrix elements evaluation is needed.

Figure 4.11: The real parts of the three terms related to Green’s function for lossy dielectrics.
4.5 A-EFIE for Conductors

The formulation of general dielectrics using A-EFIE is discussed in Chapter 3. For conductors, the formulation holds the same except that the matrix elements of the internal problem need more careful treatments. We will choose the line integral method to evaluate the matrix elements.

For problems with a large number of unknowns, it is not feasible to solve the internal problem with the method of moments because of the high expense. Instead, FMA needs to be applied. However, for very lossy materials, FMA breaks down because the plane wave physics no longer hold. A remedy is to make use of the sparsity of the matrices of the internal problem.

When the source point and the observation point are far from each other, we can observe no field in a very lossy environment. This gives rise to very sparse $L$, $P$ and $K$ operators. In FMA, it is equivalent to no interactions between boxes far away from each other. Then leaving sufficient numbers of buffer-boxes for near interaction will be a very good approximation.

Another question is: will the A-EFIE for dielectrics be rank-deficient if the internal material is very lossy? The answer is no. This formulation will be reduced to A-EFIE for PEC.

The dielectric formulation is:

$$
\begin{bmatrix}
V_{\text{ext}} & K_{\text{ext}} & D^T \cdot P_{\text{ext}} \cdot B \\
V_{\text{int}} & K_{\text{int}} & D^T \cdot P_{\text{int}} \cdot B \\
F \cdot D & 0 & k_0^2 I
\end{bmatrix}
\cdot
\begin{bmatrix}
\eta_0^{-1}b \\
\eta_0^{-1}M \\
c_0 \rho_r
\end{bmatrix} =
\begin{bmatrix}
-\eta_0^{-1}b \\
0 \\
0
\end{bmatrix}
$$

By invoking the circle approximation in (4.22) and (4.23), $k = k' + ik'' \to \infty$. Then the diagonal matrix elements of $V$ and $P$ become zero. Due to the extremely high loss, the off-diagonals are also zeros for these two matrices. Note that $K$ has a residual part $\pm \frac{1}{2} \hat{n} \times M$, and the principal value integrals are zeros, $K$ becomes a sparse and full-rank matrix. By solving the second equation in (4.31), we have $M = 0$. Remove the second equation and plug into the first, then (4.31) becomes:

$$
\begin{bmatrix}
V_{\text{ext}} & D^T \cdot P_{\text{ext}} \cdot B \\
F \cdot D & k_0^2 I
\end{bmatrix}
\cdot
\begin{bmatrix}
\eta_0^{-1}b \\
c_0 \rho_r
\end{bmatrix} =
\begin{bmatrix}
-\eta_0^{-1}b \\
0
\end{bmatrix}
$$
which is the A-EFIE for PEC and (4.31) is still a full-rank matrix.

Figure 4.12: Electric current of a 1 mm conductive sphere $\sigma = 5 \times 10^5$ S/m solved with A-EFIE for dielectrics at 3 GHz.

4.6 Numerical Examples

In this section, we will first show some results of conductive sphere scattering compared with the Mie series. Another simple structure: a cube is followed. Then we will explore the skin depth effects by examining the field inside conductors of a simple transmission line structure. Finally, a two-layer circuit with conductivity $\sigma = 5 \times 10^7$ S/m is simulated and the input admittances are found as a function of frequency.

A conductive sphere of $\sigma = 5 \times 10^5$ S/m is simulated with a plane wave incident. Figure 4.12 shows the electric current distribution on the sphere. The near field at $r = 2.5$ mm, $\phi = 0$ is plotted as a function of $\theta$ in Figure 4.13. The bi-static radar cross section (RCS) of $E$ and $H$ planes is shown in Figure 4.14.

The simulation results of a cube with the same conductivity is shown in Figures 4.15 and 4.16.

The cylindrical transmission line of $\sigma = 1 \times 10^6$ S/m as shown in Figure
4.17 is simulated. This structure is excited by a delta-gap source. The near field inside the line is computed. Since the electric current $J = \sigma E$ in the conductor, the electric field is proportional to the current. Figure 4.18 shows the normalized electric current inside the conductor. The simulation results show a good match of the skin depth equation $\delta_s = \sqrt{\frac{2}{\omega \mu_0 \sigma}}$.

Finally, consider a two-layer circuit as in Figure 4.19. The input port as shown is excited by a delta-gap source. The input admittance $Y_{11}$ is extracted. Figure 4.20 shows the frequency dependence of the real and imaginary parts of $Y_{11}$. The color indicates the amplitude of current in $dB$. 

Figure 4.13: Near field radiated by a 1 mm conductive sphere $\sigma = 5 \times 10^5$ S/m solved with A-EFIE for dielectrics at 3 GHz.
Figure 4.14: RCS of a 1 mm conductive sphere $\sigma = 5 \times 10^5$ S/m solved with A-EFIE for dielectrics at 3 GHz.

Figure 4.15: Tangential $E$ field on a conductive cube with $\sigma = 5 \times 10^5$ S/m solved with A-EFIE for dielectrics at 3 GHz.
Figure 4.16: RCS of a conductive cube with $\sigma = 5 \times 10^5$ S/m solved with A-EFIE for dielectrics at 3 GHz.

Figure 4.17: Tangential $E$ field on cylindrical transmission line solved with A-EFIE for dielectrics at 10 GHz.
Figure 4.18: The electric current over the cross section of transmission line at $x = 50 \, \mu m$ as frequency varies. The conductivity of the metal is $1 \times 10^6 \, S/m$. 
Figure 4.19: Tangential $E$ field on two-layer circuit solved with A-EFIE for dielectrics at 10 GHz.

Figure 4.20: The real and imaginary parts of input admittance versus frequency of the two-layer circuit.
CHAPTER 5

GENERALIZED IMPEDANCE BOUNDARY CONDITION FOR DIELECTRICS AND CONDUCTORS

In Chapters 3 and 4, we proposed an A-EFIE solver for dielectrics and conductive materials. With the same matrix equation, the solvers for the two kinds of the materials differ in the integration schemes to evaluate the matrix elements of the internal problem. For lossless dielectrics, the matrix representations of the operators are dense because the source and field are coupled through Green’s function. While for lossy dielectrics, especially conductors, the matrices will be sparse. A very special case is when the conductivity becomes very high, then the matrices $\mathbf{V}$, $\mathbf{K}$ and $\mathbf{P}$ become almost diagonal, giving rise to a very simple coupling relationship of $\mathbf{J}$ and $\mathbf{M}$, which we call impedance boundary condition (IBC) [32].

To generalize this idea about IBC, we can find the relationship between the equivalent magnetic and electric currents in the form of $\mathbf{M} = \mathbf{Z} \cdot \mathbf{J}$. The matrix $\mathbf{Z}$ is almost diagonal if the material is highly conductive. The denseness of this matrix will be larger as the conductivity decreases. To derive this generalized impedance boundary condition (GIBC), one can use the finite element method [33]. With the surface integral equation method, this GIBC also comes out naturally.

This GIBC is derived from the internal problem with no excitation source inside. It is location invariant and dependent on the geometry and the filling materials of the object. Hence, if a problem consists of multiple objects with the same shape and material, the GIBC can be shared and re-used.

5.1 Derivation of GIBC

The GIBC can be derived from the equation of the internal problem:

$$
\nabla \cdot \mathbf{J} + \mathbf{K} \cdot \mathbf{M} + \nabla \cdot \mathbf{P} \cdot \rho = 0
$$

(5.1)
This vector equation, written as the operator form, is valid for arbitrary observation point \( \mathbf{r} \) on the surface and the sources \( \mathbf{J} \) and \( \rho \) are functions of \( \mathbf{r}' \).

When the internal material is highly conductive, this equation is reduced to the impedance boundary condition \( \mathbf{M} = \eta \hat{n} \times \mathbf{J} \) and \( \eta = \sqrt{\frac{\omega \mu}{\sigma}} \). In this case,

\[
\mathcal{V}_{\text{int}} \cdot \mathbf{J} = \mu_r \frac{i}{2k} \mathbf{J} \tag{5.2}
\]

\[
\mathcal{K}_{\text{int}} \cdot \mathbf{M} = \frac{1}{2} \hat{n} \times \mathbf{M} \tag{5.3}
\]

\[
\nabla (\mathcal{P}_{\text{int}} \cdot \rho) = \nabla (\epsilon_r^{-1} \frac{i}{2k}) = 0 \tag{5.4}
\]

Then

\[
\frac{1}{2} \hat{n} \times \mathbf{M} = \mu_r \frac{k_0}{2k} \mathbf{J} \tag{5.5}
\]

That is:

\[
\mathbf{M} = \frac{\omega \mu}{i \sigma} \mathbf{J} \times \hat{n} \tag{5.6}
\]

In the above, we have used the circle approximation to evaluate the integrals related to operators \( \mathcal{V} \) and \( \mathcal{P} \). For \( \mathcal{K} \), only the residual term survives because the principal value integral vanishes due to the fast decay of Green’s function.

When the conductivity of internal material is relatively small, the circle approximation is no longer accurate and the principal value integral contributes to the value of amplitude of \( \mathcal{K} \). Then the impedance boundary condition is no longer a good approximation. But still, we can use the matrix formulation to solve problems of this kind, because the matrix form of this method describes the interaction of electric and magnetic current accurately in a more rigorous form.

Taking the second equation from (3.3), we have a matrix equation describing the internal problem:

\[
\nabla_{\text{int}} \cdot i k_0 \mathbf{J} + \mathbf{K}_{\text{int}} \cdot \eta_0^{-1} \mathbf{M} + \mathbf{D}^T \cdot \mathbf{P}_{\text{int}} \cdot \mathbf{c}_0 \mathbf{\rho}_r = 0 \tag{5.7}
\]

The matrix \( \mathbf{K} \) has a very small condition number, and we can then multiply its inverse \( \mathbf{K}^{-1} \) by the equation and get:

\[
\mathbf{M} = -\eta_0 \left( \mathbf{K}_{\text{int}}^{-1} \cdot \nabla_{\text{int}} \cdot i k_0 \mathbf{J} + \mathbf{K}_{\text{int}}^{-1} \cdot \mathbf{D}^T \cdot \mathbf{P}_{\text{int}} \cdot \mathbf{c}_0 \mathbf{\rho} \right) \tag{5.8}
\]
where we have used \( \mathbf{B} \cdot \mathbf{\rho}_r = \mathbf{\rho} \). Then we can have:

\[
M = \mathcal{G}_J \cdot \mathbf{J} + \mathcal{G}_\rho \cdot \mathbf{\rho} \tag{5.9}
\]

where

\[
\mathcal{G}_J = -i k_0 \eta_0 K^{-1}_{\text{int}} \cdot \mathbf{V}_{\text{int}} \tag{5.10}
\]

\[
\mathcal{G}_\rho = -c_0 \eta_0 K^{-1}_{\text{int}} \cdot \mathbf{D}^T \cdot \mathbf{P}_{\text{int}} \tag{5.11}
\]

Or we can using the current continuity equation to eliminate the charge and we have:

\[
\mathbf{M} = -\eta_0 \left( i k_0 K^{-1}_{\text{int}} \cdot \mathbf{V}_{\text{int}} - \frac{i}{k_0^2} K^{-1}_{\text{int}} \cdot \mathbf{D}^T \cdot \mathbf{P}_{\text{int}} \cdot \mathbf{D} \right) \mathbf{J} \tag{5.12}
\]

Then

\[
M = \mathcal{G} \cdot \mathbf{J} \tag{5.13}
\]

where

\[
\mathcal{G} = -i k_0 \eta_0 \left( K^{-1}_{\text{int}} \cdot \mathbf{V}_{\text{int}} - \frac{1}{k_0^2} K^{-1}_{\text{int}} \cdot \mathbf{D}^T \cdot \mathbf{P}_{\text{int}} \cdot \mathbf{D} \right) \tag{5.14}
\]

The above \( \mathcal{G}_J, \mathcal{G}_\rho \) and \( \mathcal{G} \) are called the generalized impedance boundary condition (GIBC).

### 5.2 GIBC Matrices

Since now the GIBC is written as a matrix form, we need to put in the basis and testing functions in order to evaluate the values of the matrix elements. However, for highly conductive cases of the matrix form, this is more complicated. The GIBC matrices as shown above are no longer diagonal even for highly conductive material. The reason is that at the point, the actual electric and magnetic currents are:

\[
\mathbf{J}(\mathbf{r}) = \sum_{i=1}^{3} a_i \Lambda(\mathbf{r}) \tag{5.15}
\]

\[
\mathbf{M}(\mathbf{r}) = \sum_{i}^{N} b_i \Gamma(\mathbf{r}) \tag{5.16}
\]
where the electric current at point \( r \) is a weighted summation of three RWG basis functions. The magnetic current is a weighted summation of a number of BC basis functions. Since both of them are defined on more than one triangle, the coupling becomes more complicated. However, this can be easily implemented numerically.

An obvious advantage of the GIBC matrices is that the object can be treated as a black box if the GIBC matrices on a defined mesh are obtained beforehand. These GIBC matrices, together with the mesh, can be used to couple with many other objects and excitations outside. The other advantage is that GIBC matrices can be re-used for identical objects. With these advantages, the use of GIBC will simplify the process of solving complicated problems.
The plot of field lines in the rectangular and circular waveguide was presented by [34]. It was the first time that many high-order modes were plotted. However, the methods of generating the plots was not given. In this report, we present a method to generate field lines in waveguides. Firstly we consider the shapes of field lines: electric field lines either start from the wall and terminate at the wall or form a self-closed contour. Magnetic field lines only form a self-close contour because magnetic charges do not exit. Then the method to find the centers of the contours is discussed. Using this method, the direction of the field and the magnitude of the field can be visualized with the line plots.

This method can be extended to the optical fibers with some modifications. The field lines in a dielectric waveguide must form a self-closed contour as there are no free charges in a dielectric material. The other difference is that there will be some evanescent modes outside the core region. These modes can also be found analytically and shown in the plots.

Two approaches are used to solve guidance conditions in the optical fiber. The first is to solve graphically. This approach helps us to understand the modes better, but is less efficient. The other way is full wave analysis: solve eigenvalues using the finite element method [35]. A combination of the nodal basis function and the vector basis function are used and both longitudinal and transverse components of fields are needed.

In the following part, we first describe the formulation of field line plots and show how to extend this method to dielectric waveguide. Then, the finite element method is formulated. Next, the plots of the field lines in two different waveguides, the graphic solution and the results of finite element method are presented.
6.1 Formulation

In this section, the formulations of three problems are presented. We first describe a way to plot field lines in the rectangular and circular waveguide. Then this approach is extended to the dielectric waveguide. Finally, a finite element method formulation of solving eigenvalue problems of the inhomogeneous waveguide is introduced.

6.1.1 Field Lines in the Rectangular and Circular Waveguide

To formulate the plots of electric and magnetic field lines in the rectangular and circular waveguides, the method of tracing directions of the fields is used inside the waveguides. Intuitively, for TE modes, the electric field lines are confined in the transverse plane. It either originates or terminates at the wall of the waveguide because of the free electric charges on the PEC surface, or forms a self-closed contour. Therefore, magnetic field lines will turn to the longitudinal direction where the transverse component vanishes, and then form a self-closed contour in 3D. For TM modes, the magnetic field lines are confined in the transverse plane. However, it always forms a self-closed contour as magnetic charges do not exit. The electric field line, thus, will have a pure longitudinal component when the field in the transverse plane vanishes.

With the knowledge of the above, we can formulate electric field lines of TE modes by starting from the wall of the waveguide. If there is more than 1 zero for \( H_z \) inside the waveguide, we need to consider the self-closed electric field line. This indicates that for the rectangular waveguide, only \( m \geq 2 \) and \( n \geq 2 \) needs to be considered. The number of zeros is \( (m - 1) \times (n - 1) \). For the circular waveguide, we need to consider the self-closed electric field line only when \( m \geq 2 \) and \( n \geq 1 \). The number of zeros is \( 2 \times n \times (m - 1) \). The centers of the electric field lines will be at these zeros. After that the magnetic field lines can be formulated easily by drawing perpendicular lines to the electric field lines.

Similarly, we can formulate magnetic field lines of TM modes. Since the magnetic field always terminates at itself, and the centers of the contours locate at the zeros of \( E_z \), for the rectangular waveguide, the number of zeros is \( m \times n \) and for the circular waveguide, the number of which is determined
by $2 \times m \times n$ if $n \neq 0$. When $n = 0$, the contours center at the origin (or the center of the waveguide). Then the electric field lines can be plotted as it is perpendicular to electric field lines.

The expressions for electric and magnetic field can be found in many textbooks [36] and lecture notes [37]. Here we show the electric and magnetic fields of TM modes and choose $\sin(n\phi)$ for $\phi$ variations:

$$E_z = E_0 J_n(k_\rho \rho) \sin(n\phi) e^{ik_z z}$$  \hspace{1cm} (6.1)

$$H_z = 0$$  \hspace{1cm} (6.2)

$$E_{\rho} = E_0 \frac{ik_z}{k_\rho} J'_n(k_\rho \rho) \sin(n\phi) e^{ik_z z}$$  \hspace{1cm} (6.3)

$$E_{\phi} = E_0 \frac{k_\rho}{\rho^2} J_n(k_\rho \rho) \cos(n\phi) e^{ik_z z}$$  \hspace{1cm} (6.4)

$$H_\rho = -H_0 \frac{\im \omega \epsilon}{\rho k_\rho^2} J_n(k_\rho \rho) \cos(n\phi) e^{ik_z z}$$  \hspace{1cm} (6.5)

$$H_{\phi} = H_0 \frac{\im \omega \epsilon}{k_\rho} J'_n(k_\rho \rho) \sin(n\phi) e^{ik_z z}$$  \hspace{1cm} (6.6)

where $n = 0, 1, 2, \ldots$.

We can start from $r = (\rho, \phi)$, where one of $H_\rho$ and $H_\phi$ is non-zero. After obtaining $H_\rho$ and $H_\phi$, we can determine the slope of the field line at $r$. By moving $r$ to $r'$ along the direction of $(H_\rho, H_\phi)$ by a very small distance (much smaller than the radius of the waveguide) and repeatedly doing so, a self-closed contour will be formed. It is thus guaranteed that the tangent line of the field line is pointing to the direction of the field. The other property of field lines, that the density of lines is proportional to the field magnitude, can also be realized. By choosing more starting points where the magnitude of the field is large, we can roughly say that the density indicates field strength.

### 6.1.2 Field Lines in Optical Fibers

To plot the field line in an optical fiber, the first step is to determine the guidance modes in the optical fibers. Two approaches are implemented.

The first approach is to solve guidance equations graphically. The equa-
tions are [37]:

$$\frac{J_{n+1}(k_{1\rho}a)}{k_{1\rho}aJ_n(k_{1\rho}a)} = \frac{1}{2}\left[ \left( \frac{\mu_2}{\mu_1} + \frac{\epsilon_2}{\epsilon_1} \right) \frac{K'_n(\alpha_2a)}{\alpha_2aK_n(\alpha_2a)} + \left( \frac{n}{(k_{1\rho}a)^2} - R \right) \right]$$  \hspace{1cm} (6.7)

for EH modes, and

$$\frac{J_{n-1}(k_{1\rho}a)}{k_{1\rho}aJ_n(k_{1\rho}a)} = -\frac{1}{2}\left[ \left( \frac{\mu_2}{\mu_1} + \frac{\epsilon_2}{\epsilon_1} \right) \frac{K'_n(\alpha_2a)}{\alpha_2aK_n(\alpha_2a)} + \left( \frac{n}{(k_{1\rho}a)^2} - R \right) \right]$$  \hspace{1cm} (6.8)

for HE modes. The expression of $R$ is:

$$R = \left[ \frac{1}{4}\left( \frac{\mu_2}{\mu_1} - \frac{\epsilon_2}{\epsilon_1} \right)^2 + \frac{n^2k_z^2}{k_1^2}\left( \frac{1}{k_{1\rho}^2a^2} + \frac{1}{\alpha_2^2a^2} \right)^2 \right]^{\frac{1}{2}}$$  \hspace{1cm} (6.9)

Solving the above equations gives $k_{1\rho}$, which satisfy $0 < k_{1\rho}^2 < k_1^2 - k_2^2 = V^2$. With the $k_{1\rho}$ of each mode, we can plot the field line in the core region using the equations:

$$E_{1\rho} = \frac{i}{k_{1\rho}^2} \left[ E_1 k_z k_{1\rho} J'_n(k_{1\rho}\rho) - H_1 \frac{n\omega \mu_1}{\rho} J_n(k_{1\rho}\rho) \right] \sin(n\phi)e^{ik_zz}$$ \hspace{1cm} (6.10)

$$E_{1\phi} = \frac{i}{k_{1\rho}^2} \left[ E_1 \frac{n k_z}{\rho} J_n(k_{1\rho}\rho) - H_1 \omega \mu_1 k_{1\rho} J'_n(k_{1\rho}\rho) \right] \cos(n\phi)e^{ik_zz}$$ \hspace{1cm} (6.11)

$$H_{1\rho} = \frac{i}{k_{1\rho}^2} \left[ -E_1 \frac{n\omega \epsilon_1}{\rho} J_n(k_{1\rho}\rho) + H_1 k_z k_{1\rho} J'_n(k_{1\rho}\rho) \right] \cos(n\phi)e^{ik_zz}$$ \hspace{1cm} (6.12)

$$H_{1\phi} = \frac{i}{k_{1\rho}^2} \left[ E_1 \omega \epsilon_1 k_{1\rho} J'_n(k_{1\rho}\rho) - H_1 \frac{n k_z}{\rho} J_n(k_{1\rho}\rho) \right] \sin(n\phi)e^{ik_zz}$$ \hspace{1cm} (6.13)

The field in the cladding region can be written down analogous to the above by replacing $k_{1\rho}$ and the Bessel function of the first kind by the Hankel function of the first kind. The Hankel function of the first kind will then be replaced by a modified Bessel function of the second kind as the variable is purely imaginary.

6.1.3 Finite Element Method in Solving Eigenvalues

The second approach to solve guidance conditions is to solve eigenvalue problems using the finite element method. This approach [35] uses a combination
of vector basis function and nodal basis function and it was implemented
during attendance of ECE540 in the spring semester of 2012. The procedure
is briefly outlined below.

For a general case of an inhomogeneous waveguide, we can arrive at the
Holmholzt equation:
\[ \nabla \times \left( \frac{1}{\mu_r} \nabla \times \mathbf{E} \right) - k_0^2 \epsilon_r \mathbf{E} = 0 \quad (6.14) \]
with the boundary condition of \( \hat{n} \times \mathbf{E} = 0 \) on \( \Gamma_1 \) and \( \hat{n} \times (\nabla \times \mathbf{E}) = 0 \) on
\( \Gamma_2 \). The expression of the electrical field propagating in the \( z \) direction can
be written as:
\[ \mathbf{E}(x,y,z) = \left[ \frac{1}{k_z} \mathbf{e}_t(x,y) - i \hat{z} \mathbf{e}_z(x,y) \right] e^{ik_z z} \quad (6.15) \]
Plug (6.15) into (6.14), and test the equation with the weighting function:
\[ \mathbf{W}(x,y,z) = \left[ \frac{1}{k_z} \mathbf{w}_t(x,y) - i \hat{z} \mathbf{w}_z(x,y) \right] e^{ik_z z} \quad (6.16) \]
and we arrive at
\[
\int \int_{\Omega} \left[ \frac{1}{\mu_r} (\nabla_t \times \mathbf{w}_t) \cdot (\nabla_t \times \mathbf{w}_t) - k_0 \epsilon_r \mathbf{w}_t \cdot \mathbf{e}_t \right] d\Omega \\
+ k_z^2 \int \int_{\Omega} \left[ \frac{1}{\mu_r} (\mathbf{w}_t + \nabla_t \mathbf{w}_z) \cdot (\mathbf{e}_t + \nabla_t \mathbf{e}_z) - k_0 \epsilon_r \mathbf{w}_z \cdot \mathbf{e}_z \right] d\Omega = 0 \quad (6.17)
\]
Considering that the whole domain is divided into small elements, we can
expand \( \mathbf{e}_t(x,y) \) and \( \mathbf{e}_z(x,y) \) as:
\[ \mathbf{e}_t(x,y) = \sum_{j=1}^{N_{edge}} \mathbf{N}_j(x,y) \mathbf{e}_{t,j} \quad \mathbf{e}_z(x,y) = \sum_{j=1}^{N} \mathbf{N}_j(x,y) \mathbf{e}_{z,j} \quad (6.18) \]
where \( N_{edge} \) and \( N \) denote the number of edges and nodes of the conducting
surface. By letting \( \mathbf{w}_i = \mathbf{N}_i \) and \( \mathbf{w}_z = \mathbf{N}_i \), the problem is transformed to a
generalized eigenvalue problem:
\[
\begin{bmatrix}
A_{tt} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{e}_t \\
\mathbf{e}_z
\end{bmatrix} = -k_z^2
\begin{bmatrix}
B_{tt} & B_{tz} \\
B_{zt} & B_{zz}
\end{bmatrix}
\begin{bmatrix}
\mathbf{e}_t \\
\mathbf{e}_z
\end{bmatrix} \quad (6.19)
\]
where

\[ A_{tt,ij} = \int \int_\Omega \left[ \frac{1}{\mu_r} (\nabla \times N_i) \cdot (\nabla \times N_j) - k_0 \epsilon_r \mathbf{N}_i \cdot \mathbf{N}_j \right] d\Omega \quad (6.20) \]

\[ B_{tt,ij} = \int \int_\Omega \frac{1}{\mu_r} \mathbf{N}_i \cdot \mathbf{N}_j d\Omega \quad (6.21) \]

\[ B_{tz,ij} = \int \int_\Omega \frac{1}{\mu_r} \mathbf{N}_i \cdot \nabla_t N_j d\Omega \quad (6.22) \]

\[ B_{zt,ij} = \int \int_\Omega \frac{1}{\mu_r} \nabla_t N_i \cdot \mathbf{N}_j d\Omega \quad (6.23) \]

\[ B_{zz,ij} = \int \int_\Omega \left[ \frac{1}{\mu_r} \nabla_t N_i \cdot \nabla_t N_j - k_0 \epsilon_r \mathbf{N}_i \cdot \mathbf{N}_j \right] d\Omega \quad (6.24) \]

After the assembly, we can then convert (6.19) into a simpler eigenvalue problem to obtain the eigenvalues:

\[ [A_{tt}] [\epsilon_t] = k_2^2 [B'_{tt}] [\epsilon_t] \quad (6.25) \]

where \([B'_{tt}] = [B_{tz}] [B_{zz}]^{-1} [B_{zt}] - [B_{tt}]\). Then by solving (6.25) for a given \(k_0\), we can calculate the propagation constant and field distribution of each mode. Within all the obtained propagation constants, only those \(k_z\) satisfying \(k_1 < k_z < k_2\) are the guidance modes in the optical fibers.

6.2 Numerical Results

The graphic solutions are shown in Figures 6.1, 6.2, 6.3, 6.4, where the radius \(a = 2\) mm, refractive index of core \(n_1 = 1.4567\), refractive index of cladding \(n_2 = 1.4380\) and angular frequency \(\omega = 3 \times 10^{12}\) rad/s. In Figures 6.1 and 6.2, when \(n = 0\), TE and TM fields are decoupled, and (6.10) - (6.13) reduce to (6.3) - (6.6) for TM modes. Fields for TE modes will also be reduced to the field of a hollow waveguide. Therefore, the transverse electric field lines will be similar to those of a hollow waveguide except that there will be evanescent modes in the cladding region. For \(EH_{1m}\) and \(HE_{1m}\) modes, we can see the solutions are \(k_{1,\rho} a = 4.105121\) for \(EH_{11}\) modes, and \(k_{1,\rho} a = 1.972200\) for \(HE_{11}\) and \(HE_{12}\) modes: \(k_{1,\rho} a = 4.327863\).

The diagram of dispersion curves in an optical fiber using the finite element
method is shown in Figure 6.5. It is proved that the results match the graphical solutions. The curves from left to right in Figure 6.5 correspond to $HE_{11}$ mode, $HE_{21}$ mode, $TE_{01}$ mode, $TM_{01}$ mode, etc.

Finally a few modes in the rectangular and circular waveguides and optical fibers are shown in Figures 6.6, 6.7, 6.8 and 6.9 respectively. The solid lines represent the electric field and the dash lines represent the magnetic field. Using the method discussed above, we can produce field plots of any order.

![Figure 6.1: Graphic solution of the TM mode in an optical fiber.](image1)

![Figure 6.2: Graphic solution of the TE mode in an optical fiber.](image2)
6.3 Conclusions

In this chapter, three problems are attempted and two problems are solved: plot of field lines in the rectangular and circular waveguides; use the finite element method to solve guidance condition. The method of plotting field lines in optical fibers is also demonstrated. In the finite element part, we took advantage of the vector basis to eliminate the presence of spurious solutions and ease the treatment of corners and edges. This method is validated by comparing the computational results with other published sources [36].
Figure 6.4: Graphic solution of the EH mode in an optical fiber.

Figure 6.5: Dispersion relation in an optical fiber, $n_1 = 1.4567$, $n_2 = 1.4380$; $n_1$, $n_2$ are the refractive index inside and outside.
Figure 6.6: Transverse field distribution of the first 18 modes in a rectangular waveguide.
Figure 6.7: Transverse field distribution of the first 15 modes in a circular waveguide.
Figure 6.8: Transverse field distribution of TE-like modes in an optical fiber. Inside the inner circle is the optical fiber; outside is the cladding region.
Figure 6.9: Transverse field distribution of the TM-like modes in an optical fiber. Inside the inner circle is the optical fiber; outside is the cladding region.
A powerful remedy for low frequency breakdown of the electric field integral equation, augmented electric field integral equation (A-EFIE), is presented as the introduction of this thesis. Many aspects of this algorithm are discussed to solve a PEC problem, including the formulation, pre-conditioning and a perturbation method.

A dielectric solver using A-EFIE is also formulated. The difficulties in testing of operators involved are discussed. To overcome this, more suitable basis functions are used. This formulation can be easily adapted to fast algorithms for both low and mid frequencies, such as the mixed-form fast multipole algorithm. The complexity of this method is then reduced to $O(N\log N)$. To solve more general dielectric problems, such as highly lossy dielectrics or conductors, another integration scheme is introduced and it excels the traditional methods by relative error comparisons. With this scheme for highly lossy dielectrics, the complexity is further reduced for the internal problem. A new concept, called generalized impedance boundary condition (GIBC), is then introduced so that a dielectric object can be treated as a blackbox with the GIBC operator as the matrix form.

Finally, another work related to waveguide is presented. The plots of electric and magnetic fields of different modes in waveguides are shown and the guidance conditions of optical fibers are calculated.
REFERENCES


