ABSTRACT

Assuming the "random ground" model, it is easy from direct considerations regarding frequency to calculate with considerable accuracy the spectra of ground clutter and moving targets.

The harmonics generated by an envelope detector are shown to have power which is always less than a few percent of the power in the fundamental.

The use of the measured harmonic content in determining the relative amplitude of moving targets and ground return is described.

Written by: C. W. Sherwin
In interpreting experimental observations in which clutter and signals are analysed in terms of frequency it is convenient to use a theory which works directly with frequency. The usual approach is a more indirect one which uses the auto-correlation function. The frequency method is usually avoided because of the unmanageable complexity caused by non-linear devices such as the envelope detector. However, the errors caused by the envelope detector are usually small, and frequency analysis has the advantage of being more intuitively understandable than is the approach through auto-correlation.

In calculating the clutter spectrum one divides the illuminated patch on the ground into regions, each with its own mean frequency. The "beating" of each region with each of the other regions produces the audio frequency clutter spectrum. A moving target signal "beats" with each frequency region in the ground patch producing the moving target signal.

Let \( \mathbf{V} \) be the velocity of the aircraft, and \( \gamma \) be the angle between \( \mathbf{V} \) and the direction of some ground target illuminated by the radar beam. The radial velocity of the target with respect to the aircraft is \( V \cos \gamma \). A cone of half angle \( \gamma \) with its axis along \( \mathbf{V} \) intersects level ground in a hyperbola \( x^2 + h^2 = (z \tan \gamma)^2 \), as shown in Fig. 1.

---

This paper presented in part at the MTI Symposium of the Office of Naval Research, February 28 & 29, 1952.
Fig. 1

Locus of points on plane which have Constant Doppler Shift

Aircraft velocity is $V$, Altitude = $h$, above plane

$X^2 + h^2 = (Z + h \tan \theta)^2$
All points on this hyperbola, since they lie on the cone of constant $\varphi$, have the same radial velocity toward the airplane. Therefore they all have the same doppler shift with respect to the transmitted frequency. For any given $V$ and $h$, one can draw a family of hyperbolas one for each value of $\varphi$. The radar beam illuminates a patch of ground, and these hyperbolic lines divide it into strips, each with its own average doppler frequency. The reflected power from any strip is proportional to the power illuminating it if one assumes the ground to be covered with small randomly distributed scatterers. The r.f. signals from each strip beat with the r.f. signals from each of the other strips causing the low frequency clutter noise observed at the second detector of the radar. One can consider the signal from each strip to be a band of noise passed by a narrow square filter, of the same width in frequency. A "band" or "line" of noise defined this way has the mean frequency of the filter, and a correlation time $= \frac{1}{\Delta f}$. Two noise lines beating with each other will produce a signal which is also noise, since it has a frequency spread and finite correlation time.

The simple theory described here replaces each strip, bounded by lines of constant $\varphi$, and by the radar pulse length, with a narrow band of noise whose power is calculated from the integral of the antenna pattern along the strip, and whose frequency is equal to the average frequency characteristic of the strip.
For the case where the slant range to the illuminated patch is several times the altitude of the aircraft, the lines of constant $f$ are very nearly radial, as shown in Fig. 2. The relative velocity, $\Delta V$ of any two points separated by $\Delta \gamma$ is $\Delta V = V \sin \gamma \Delta \gamma$. If $V$ is in miles per hour then the frequency difference $\Delta f$ between the two points is $\Delta f \approx 29 V \sin \gamma \Delta \gamma$ for x-band radar. The general shape of the clutter spectrum can be seen by supposing the illuminated patch in Fig. 2 to be divided into $n$ sections. Each section will give equal power since we are here assuming the beam to give uniform illumination over $\Delta \gamma$. If a square law detection system detects the beat frequency between each pair of noise lines, one calculates a triangular shaped audio frequency spectrum also shown in Fig. 2.

If $\delta f$ is the frequency difference between adjacent lines, it is clear that there are $(n-1)$ pairs contributing to the output audio frequency at $\delta f$, $(n-2)$ pairs at $2\delta f$, etc., and finally only one pair at frequency $(n-1)\delta f$. Since the different pairs contributing to any one frequency have only a short time coherence, one adds power to get the long time behavior, and hence the triangular clutter spectrum. For a rigorous analysis of this case see Lawson and Ulhenbeck.

A more realistic situation is shown in Fig. 3 where the antenna pattern is taken into account. Also, a moving target is assumed to be present in the illuminated patch. If it is not accelerating, it produces a sharply defined frequency, and in

---

Fig. 2 Ground clutter spectrum for low altitude aircraft. The radar beam is assumed to be uniform in azimuth over $\Delta \varphi$, and the ground is assumed to be uniformly reflecting.
Fig. 3 Frequency spectra of uniform ground clutter.  
3° beam, $\gamma = 20^\circ$, $V = 120$ mph
beating with each part of the ground clutter, shows accurately the shape of the clutter spectrum. Thus, moving targets act like carrier signals to isolate the actual clutter spectrum from the inter-clutter modulation. In Fig. 3 we assumed that the ground is uniformly covered with small scatterers, randomly distributed in range. If one large scattering unit is located in the illuminated patch, it will produce an extra large signal at the particular frequency characteristic of its azimuth.

The low frequency clutter spectrum can be calculated by the convolution of the spectrum revealed by the moving target, provided one is using a power detection system. Suppose, however, that the detection system produces harmonics, as is the case with the envelope detector. Even for this case the moving target signal should display the shape of the clutter spectrum with little distortion, since all harmonics are at very different frequencies from the fundamental moving target frequency, and do not contain much power. These harmonics, though small, can be observed, and as will be shown later, are useful in determining the relative size of the moving target signal compared to the clutter. Even when calculating the shape of the low frequency clutter spectrum due to the beating of the clutter with itself, the harmonic-producing effects of the envelope detector are not large, and can usually be neglected.

The potentialities are considerable for the use of frequency analysis to increase azimuth resolution for "off the ground track" viewing.
For example consider a coherent x-band radar in a 600 mph low altitude aircraft, looking off the ground track at an angle of 30 degrees. One samples and "box-cars" one pulse length, say 0.1 microseconds. If the radar beam is 20 degrees wide in azimuth, the relative radial velocity between targets at the edges of the beam is 3000 cps. If the frequency bins that can be analysed are 10 cps wide (i.e., a point target beating with the coherent transmitter in the non-accelerating aircraft will produce a signal which, when analysed, is 10 cps broad at half power). Thus there are 300 distinct frequencies in 20 degrees of azimuth. The angular resolution is \((20/300) = 1/15\) degree. At four miles the illuminated ground patch is effectively only 25 feet in azimuth and 50 feet in range. Resolution of this type might be very useful in certain problems, particularly the search for standing targets.

Unfortunately this system produces an error in azimuth whenever a target is moving since one actually measures only range and doppler frequency. The relation between doppler frequency and azimuth is known only if the relative radial motion of the target is known.*

We now turn to an analysis of the demodulation process in more detail, particularly the problem of harmonic content.

Suppose that two point targets at the same range are moving toward a fixed radar with radial velocities \(V_1\) and \(V_2\).

* This system is described in more detail in CSL report No. 19.
respectively. The difference in frequency is so small (the order of $10^3$ ops) compared to their mean frequency ($10^{10}$ cps) that for the duration of one pulse ($\sim 10^4$ cycles) the two reflections appear to have the same frequency. Thus they add coherently, the resultant amplitude and phase being given by simple vector addition as in Fig. 4a. Since pulses are viewed only once per repetition period, the relative phase of the two vectors of amplitude $A_1$ and $A_2$, changes by fixed amounts each time a reflection is received. The envelope of the resultant r.f. pulse, $A$, is given by means of trigonometric identities as,

$$A = \left[ A_1^2 + A_2^2 + 2A_1A_2 \cos (\omega_1 - \omega_2)t \right]^{1/2}$$

Note that $A^2$, i.e., the power level of the resultant pulse, has a purely sinusoidal variation at the beat frequency $(\omega_1 - \omega_2)$. This is the reason why a square law detector is so easy to use when calculating frequency spectra. Due to the presence of the square root, however, the envelope, $A$, contains all the harmonics of the beat frequency. These harmonics are very small when $A_2 \ll A_1$ (or vice versa). This is intuitively evident by imagining that $A_2$ is a small rotating vector, $A$ then varies in an almost purely sinusoidal manner at the frequency of rotation of $A_2$ with respect to $A_1$. However, if $A_2 = A_1$, $A$ is clearly not varying in a sinusoidal manner. However, the power present in the harmonics is quite small, as will be shown later.
Fig. 4  Vector addition of pulsed radar echoes from targets with relative radial velocity.
If one makes a Fourier series expansion of

\[
A = (A_1^2 + A_2^2)^{1/2} \sqrt{1 + a \cos t} \quad \omega = \omega_1 - \omega_2, \quad a = \frac{2A_1A_2}{A_1^2 + A_2^2}
\]

for the case where \( a = 1 \) (i.e., the two vectors \( A_1 \) and \( A_2 \) are equal in magnitude) the first harmonic has only \( 1/5 \)th the amplitude and \( 1/25 \)th the power of the fundamental.

Fourier analysis for several values of \( a \) were carried out by numerical integration, and the Maclaurin expansion was used for \( a < 1 \), yielding the curve in Fig. 5.* Thus, if one measures the ratio of fundamental to first harmonic, the relative amplitudes of the two contributing vectors can be inferred — providing, of course, that the only significant source of non-linearity in the system is in the envelope detector.

Since one is actually measuring the doppler beat signals (fundamental plus harmonics) by means of samples taken once each pulse recurrence period, one has to correct the observed signal amplitudes, by the factor \( \frac{x}{\sin x} \).

\( x = \pi tf \) where \( t = \) the pulse recurrence period, and \( f \) is the audio frequency being observed. This reconstructs the true values of the sinusoidal modulations of the pulse amplitude, (i.e., the actual modulation that would be observed with a very high pulse recurrence frequency).

* The author is indebted to Mr. W. C. Prothe for the calculation of the curve in Fig. 5.
Fig. 5. Harmonic content in an envelope detector.

$A_1$ and $A_2$ are defined in Fig. 4a.
Due to the curious effects of sampling, the harmonics can occur in unexpected places. For example, if the fundamental occurs at 700 cps and the pulse recurrence frequency is 1700 cps, one finds the first harmonic at 300 cps (i.e., 1400 cps below 1700 cps).

The experimental observations show that for moving targets on the ground track the measurement of the ratio of fundamental to the first harmonic yields ratios of $A_1/A_2$ which are in agreement with the direct measurements. For this system it was known that there were no other sources of non-linearity except the second detector.

We now consider the case of three moving targets returning signals at frequencies $\omega_1$, $\omega_2$, and $\omega_3$. The envelope $A$ of the sum of the three signals is,

\[
\frac{A}{\sqrt{A_1^2 + A_2^2 + A_3^2}} = 1 + \frac{2A_1A_2}{A_1^2 + A_2^2 + A_3^2} \cos (\omega_1 - \omega_2) t + \frac{2A_2A_3}{A_1^2 + A_2^2 + A_3^2} \cos (\omega_2 - \omega_3) t + \frac{2A_1A_3}{A_1^2 + A_2^2 + A_3^2} \cos (\omega_1 - \omega_3) t^{3/2}
\]

If any one of the three amplitudes $A_1$, $A_2$ or $A_3$ is several times larger than either of the others, the sum of the cosine terms (which we call $x$) is less than one, and the Maclaurin
expansion is fairly accurate,

\[
\frac{A}{(A_1^2 + A_2^2 + A_3^2)^{1/2}} = (1 + x)^{1/3} = 1 + \frac{x}{2} - \frac{1}{6} x^2 + \ldots \quad (3)
\]

There are three fundamental beat frequencies observed in the envelope voltage, \( A \).

\[
A = \left[ 1 + \frac{A_1 A_2}{A_1^2 + A_2^2 + A_3^2} \cos (\omega_1 - \omega_2)t + \frac{A_2 A_3}{A_1^2 + A_2^2 + A_3^2} \cos (\omega_2 - \omega_3)t \right]
\]

\[
+ \frac{A_1 A_3}{A_1^2 + A_2^2 + A_3^2} \cos (\omega_1 - \omega_3)t \right] \left( A_1^2 + A_2^2 + A_3^2 \right)^{1/2}
\]

The power in the audio signal voltage, \( A \), at the beat frequency \((\omega_1 - \omega_2)\) is

\[
\frac{(A_1 A_2)^2}{A_1^2 + A_2^2 + A_3^2}
\]

We define \( P_1 \equiv \frac{A_1^2}{2} \), etc., and \( P \equiv \frac{A_1^2 + A_2^2 + A_3^2}{2} \); then

the power in the audio signal at \((\omega_1 - \omega_2)\) is \( 2 \, P_1 P_2 / P \).

Thus in calculating the low frequency spectrum due to the different "lines" of the clutter noise beating among themselves, one calculates the convolution of the power spectrum of the original clutter, divided by the total power. This
process does not take account of the first harmonic terms caused by the $x^2$ term in (3), but these terms are small as can be seen from the following considerations.

Using the expansion (3) where $x$ is the sum of the cosine terms in (2) we calculate the ratio of fundamental to first harmonic power for two special cases:

<table>
<thead>
<tr>
<th>Harmonic Content in Envelope Detector</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

Thus, it is plausible that the harmonic content of the beat frequency produced by any pair of vectors is less than a few percent irrespective of the magnitude of the third vector.

All this is true for pure signals of constant amplitude, but what can be said in the case of lines of noise which act like pure tones only for a time shorter than their correlation time? Thus, $A_2$ and $A_3$ might be two lines of noise in the clutter spectrum, and $A_1$ could be all the other noise in the spectrum.
The broad band of noise represented by $A_1$ takes many different values during the correlation time of the narrow lines $A_2$ and $A_3$. However, $A_1$ tends to have values near its mean value, and is near zero infrequently. Thus we expect a maximum of only a few percent of the audio power in the harmonics. In any case, we have as yet found no experimental evidence that any pair of signals gives 1st harmonic power in excess of 4% of the power in the fundamental. The closest case we observed was one moving target whose directly measured modulation was 65% and which had 2\% of the audio power in the first harmonic.