RKLARKS C17 THE PROBLEM
OF TRANSMITTING RADAR DATA

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Copy 1 of 60 copies
Title Page
Numbered pages 7

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We consider an $N$ by $N$ square grid as illustrated below:

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N
2
1
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Such a grid contains $N^2$ elementary regions or "bins". Each bin is capable of two states: being occupied by an airplane, and not being so occupied. The problem is that of transmitting, in digital form, the total amount of information contained by such a grid at a given time, to another locale.

We shall assume that the channel which is employed is binary, that is, that it is capable of transmitting precisely two distinct symbols, which we may denote by 0 and 1. We further assume that the duration of a 0 is equal to the duration of a 1, and that it is as "convenient" to transmit a 0 as a 1 across the channel, at a given moment, regardless of

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1 An earlier version of this report contained some basic errors. Messrs. R.I. Balsizer, A.T. Nordsieck, and Nelson Wax participated in several discussions with the author of this report which were very clarifying.
the previous history of the channel.\footnote{In order to visualize a situation where this last assumption does not hold, we may think of the channel as a bowling alley down which we may "transmit" red balls or black balls. If our source of balls is limited (say that $1 + \varepsilon$ new balls are created for each ball we transmit), the color of the next ball we transmit may well be forced on us, i.e. balls of the other color may be unavailable at that instant.} If we choose our time units so that the time of transmission of a symbol is unity, then the number of messages which can be transmitted in time $n$ is $2^n$. Hence, the capacity of the channel in the sense of Shannon\footnote{The Mathematical Theory Of Communication, Claude E. Shannon and Warren Weaver, University of Illinois Press, Urbana, 1949.} is given by

$$C = \lim_{n \to \infty} \frac{\log_2 2^n}{n} = \log_2 2 = 1.$$ 

Let us assume that precisely $M$ planes occur in the grid and that no two planes ever occupy the same bin. Let us assume, in addition, that the planes are distributed "at random", i.e. that the probability of a given bin's being occupied is $\frac{M}{N^2}$.

We shall describe two methods by which the data may be transmitted across our channel. The first, is very simple and direct. It consists of transmitting an $N$ digit binary word, a digit for each bin. The digit associated with a given bin is $1$ if the bin is occupied, $0$ otherwise. The combination which consists of the source of information, the method by which the data is encoded, and the channel across which it is
transmitted is a communication system. Its efficiency may be measured by
the ratio \( H/C \). Here \( C \) is the channel capacity, as computed above, and \( H \)
is the entropy\(^3\) of our information source. Since \( C = 1 \), \( H/C = H \). It is
shown by Shannon\(^3\) that \( 0 \leq H/C \leq 1 \), and that, given any channel, and any
source of information, there exist methods by which the information may be
encoded yielding values of \( H/C \) as near 1 as we please.

We proceed to compute \( H \) for the system just described. We regard our
system as being capable of \( N^2 \) distinct states. When in the \( i \)-th state,
the digit which corresponds to the \( i \)-th bin (in some preassigned definite
order) is transmitted. Let \( H_i \) be the partial entropy of the system in the
\( i \)-th state. Then,

\[
H_i = - p_i(0) \log_2 p_i(0) - p_i(1) \log_2 p_i
\]

(1)

where \( p_i(0) \) is the probability that \( 0 \) is transmitted, \( p_i(1) \), the same
for \( 1 \). Then,

\[
H_i = -(1 - \frac{M}{N^2}) \log_2 (1 - \frac{M}{N^2}) = \frac{M}{N^2} \log_2 \frac{M}{N^2}
\]

Then the entropy, \( H \) of the entire system is given by

\[
H = \frac{1}{N^2} \sum_{i=1}^{N^2} H_i
\]

\[
= -(1 - \frac{M}{N^2}) \log_2 (1 - \frac{M}{N^2}) - \frac{M}{N^2} \log_2 \frac{M}{N^2}
\]
If we write $\theta = \frac{M}{N^2}$, then we have $H = -(1 - \theta) \log_2 (1 - \theta) - \theta \log_2 \theta$, so that the entropy of the system depends only on the ratio $\frac{M}{N^2}$. Since $C = 1$, the efficiency $H/C = H$. An examination of the graph of $H$ as a function of $\theta$, shows that the system is most efficient, and in fact has $H/C = 1$ when $\theta = \frac{1}{2}$, i.e. when $2M = N^2$. This agrees with what one would expect intuitively, since this is the case when exactly half of the bins are always occupied. However, the numbers which seem likely to be of practical interest (particularly as regards the program of the CSL) are of
the order of magnitude $M = N > 100$. Here, $\theta = \frac{M}{N^2} \approx 0.01$. A glance at the
graph of $H$ as a function of $\theta$, shows that $H$ (and hence $H/C$) is quite
small. Actually, $H \approx 7.3\%$.

An improved encoding procedure which naturally suggests itself (given
that $M \ll N^2$) is the following:

Transmit messages only for occupied bins. But, then it becomes neces­sary to transmit not just a "yes", but a specific label which refers to
the bin in question. E.g. the rectangular or polar coordinates of the bin
center may be transmitted. Let the bin labels be $S_1$, $S_2$, $\ldots$, $S_N$. A
message which contains all of the information present in our grid then has
the form: (1) $S_1 \ S_2 \ \ldots \ S_i \ \ldots \ S_M$. Actually, the mere act of transmit­
ting these $S$'s imposes a definite order on them. In practice this will be
the order in which the bins are being scanned. We may express this restric­
tion by the requirement

$$1 \leq i_1 < i_2 < \ldots < i_M \leq N^2.$$  

If we assume that the planes are distributed at random, then all sequences
of the form (1) are equally likely. All told there are $\binom{N^2}{M}$ such
sequences. Hence, the probability associated with each is $\frac{1}{\binom{N^2}{M}}$. Thus,
the entropy for such a system (taking the entire message as a symbol) is

$$\sum_{i=1}^{(N^2)} \left( \frac{1}{\binom{N^2}{M}} \log_2 \left( \frac{N^2}{M} \right) \right) = \log_2 \left( \frac{N^2}{M} \right).$$
However, we are taking as our basic symbols the binary digits 0, 1. Hence, this number must be divided by the number of binary digits (or bits) contained in such a message. This yields:

\[ H = H/C = \frac{\log_2 \left( \frac{N^2}{M} \right)}{\log_2 (N^2)^M} = \frac{\log \frac{N^2(M^2 - 1)}{M} \cdots \frac{N^2 - M + 1}{M}}{M \cdot 2M \log N} \]

Since we are assuming \( M \ll N^2 \) we may write:

\[ H/C = \frac{\log \left( \frac{N^2}{M} \right)}{2M \log N} = \frac{2M \log N - \log M}{2M \log N} = 1 - \frac{1}{2} \frac{\log M}{M \log N} \]

Replacing \( \log M \) by \( M \log N \), this yields \( H/C \approx 1 - \frac{1}{2} \frac{\log M}{\log N} \).

In particular, if \( M = N \), \( H/C = 50\% \), which shows a considerable improvement. As intuition would suggest, as the number of targets becomes very small compared to the number of bins, the efficiency of the system increases. For a perfect system, we must have: \( \log M \ll \log N \).

The efficiency of the system could be further improved by using the order in which the messages appear as a vehicle for the transmission of useful information.\(^4\)

\(^4\)This was remarked by A. T. Nordstiek. R. I. Hulsizer remarks that one way of accomplishing this is to transmit the difference in coordinates between targets rather than their coordinates with respect to the origin.