THE RELATION BETWEEN FREQUENCY RESPONSE AND RASTER CHARACTERISTICS IN THE SINUFLY COMPUTER

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Abstract

The heart of the Sinufly AMTI system is a storage tube computer which stores the radar video echos from a number of successive transmitted pulses and then performs a spectrum analysis of the moving target Doppler frequencies contained in the echos from each range interval. In this report the relation of the approximate spectrum analysis obtained as the output of the computer to the actual moving target Doppler spectrum is derived as a function of the resolution of the storage tube and of the amount of data which is stored.

If all the target hits occurring in a time interval \( t \) are stored, the computer will be able to distinguish between Doppler frequencies differing by \( 1/t \) cps. The moving target signals are therefore classified into "frequency bins" \( 1/t \) cycles wide.

The output spectrum of the computer is multiplied by a transfer characteristic extending from zero frequency to an upper limit determined by the width (and profile) of the lines of the raster in which the radar video is stored. The gain-band pass product given by the area under this transfer characteristic, expressed in units of \((\text{number of frequency bins} \times \text{relative amplitude in the output})\) gives the potential capacity of the computer; that is, the
effective number of distinguishable frequency bins which may be processed. This number is shown to be equal to the maximum number of completely resolved raster lines which can be written in the width of the storage surface. For our purposes two raster lines are regarded as completely resolved if they are separated by a dark line of the same width as the bright lines. In practice, Gaussian or triangular raster line profiles may be expected and about six-tenths of this potential capacity is available between the 3 db points of the transfer characteristic.

A second upper limit on the number of distinguishable Doppler frequencies which can be processed is set by the number of stored target hits; M stored hits serve to identify at most M/2 frequencies. If this number is less than that determined by the transfer characteristic of the computer the output will contain redundant information.
**Introduction:**

This report examines the relation of the frequency spectrum of the output of an idealized Sinufly storage tube computer to the audio modulation spectrum of the stored video signals. Particular attention is given to determining the capacity and resolution of the system as defined by the maximum number of different input modulation frequencies which will remain distinguishable in the output ("number of frequency bins") and the minimum frequency interval between two such distinguishable frequencies ("width of the frequency bins"). Noise introduced by the storage process is neglected.

**Preliminary:**

In the operation of the storage tube computer the video echos from successive transmitted pulses are stored as amplitude modulation of the charge deposited on the storage surface by the writing beam while it scans a raster of parallel, equi-spaced vertical lines. A single vertical

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sweep of the raster is assigned to the echo from each successive transmitted pulse.

When a number of echoes sufficient to fill the storage surface have been recorded, each range bin is read out by scanning the reading beam horizontally across the raster at the appropriate height.

Let us for the moment assume the reading beam has an infinitesimal diameter. The output signal as a function of the "x" deflection of the reading beam is then just a plot of the charge distribution along a cross section of the raster. Each individual range sweep is represented by a pulse, and the envelope of the pulses forms the doppler modulation envelope of the particular range bin. Fig. 2 illustrates such a plot for the case in which the range bin contains a single moving target and the modulation envelope is sinusoidal. In discussing the output signal it is convenient to regard it as a function of the "x" deflection of the beam rather than of time, so that the scanning velocity of the reading beam is not explicitly introduced. The output spectrum is thus the Fourier transform of the one-
dimensional charge distribution, $Q(x)$, sketched in Fig. 2.

The width of the raster, $L$, will be taken as the unit of length so that $x$, the horizontal coordinate of the reading beam, runs from $-1/2$ to $+1/2$, while all frequencies will be measured in terms of the number of whole cycles traced out across the width of the raster; this number will be defined as the frequency of the oscillation measured in "wave numbers". The new unit of frequency thus defined, the wave number, arises naturally because it permits the pattern of charge on the storage screen to be discussed independently of the scanning rates used in the writing and reading processes.

To demonstrate all the important features of the system it is sufficient to study the simple case illustrated by Fig. 1 where the modulation envelope contains a single frequency. For definiteness assume the raster to contain $M$ equi-spaced lines and the frequency of the
modulation envelope to be \( \mathbf{u} \) (wave numbers). In order to calculate the output spectrum we introduce the assumption that the charge distribution describing the cross section of a raster line has the same shape for every line, the only effect of the video modulation being to multiply the vertical scale*. Let the charge distribution be given by the function \( q(x) \) when the origin of \( x \) is chosen at the center of the line. We are now ready to calculate the output spectrum, but before proceeding we shall tabulate the terminology for easy reference.

**Terminology:**

\[ L = \text{length of raster}. \]  
\[ x = \text{horizontal coordinate on raster in units of } L. \]  
\[ M = 2m + 1 = \text{number of lines written in the raster}. \]  
\[ N = \text{maximum number of completely resolved lines which can be written on the storage surface}. \]  
\[ S = \text{number of resolvable frequency bins in the output of the computer}. \]

* This is really two assumptions:
  (a) The charge storage is linear
  (b) Deflection does not change the charge distribution in the beam.
for any reasonable definition of frequency resolution $S \approx N$.

$\nu = \text{frequency of the modulation envelope (wave numbers).}$

$q(x) = \text{function giving the charge distribution in the cross-section of a single raster line when the center of the line is chosen as the origin of } x.$

$Q(x) = \text{charge as a function of } x \text{ in the cross section of the complete raster (plotted in Fig. 2).}$

$G(\nu) = \text{output spectrum (amplitude)}$

$g(\nu) = \text{envelope of output spectrum.}$

$1 + \alpha \cos(2\pi \nu x + \psi) = \text{modulation envelope of the raster lines - i.e., the butterfly signal.}$

$\lambda = \text{modulation parameter; } \lambda = 1 \text{ corresponds to } 100\% \text{ modulation.}$

$\psi = \text{phase of the modulating signal, chosen with } x = 0 \text{ as origin.}$

Finally, the Fourier integral theorem will be used in the following form:

$$\text{If } Q(x) = \int_{-\infty}^{\infty} G(\nu) e^{2\pi i \nu x} d\nu$$

$$\text{then } G(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(x) e^{-2\pi i \nu x} dx$$

All spectra computed below will be amplitude spectra, which must be squared if power spectra are desired.
Fourier Analysis of the Output from a Single Range Bin:

The $M$ lines in the raster will be denoted by an index $n$ which runs through the integers $-m$ to $+m$. The centers of the adjacent lines are separated by the interval $\delta x = \frac{1}{M}$ and the center of the line denoted by $n = 0$ will be chosen as the origin of $x$. In an unmodulated raster the charge distribution in the cross section of this central line is given by the function $q(x)$; the $n$th line is centered at $x = \frac{n}{M}$ and has a charge distribution described by displacing the function $q(x)$ to give $q(x - \frac{n}{M})$. The total charge distribution $Q(x)$, across one range bin of the modulated raster is given by the sum of $M$ such displaced charge distributions, each multiplied by the value assumed by the modulating function at the center of the displaced line i.e., at $x = \frac{n}{M}$:

$$Q(x) = \sum_{-m}^{m} \left[ 1 + \alpha \cos \left( \frac{2\pi x n}{M} + \phi \right) \right] q(x - \frac{n}{M})$$

(1)

The spectrum of the output corresponding to this range bin is given by the Fourier transform of $Q(x)$; the amplitude $G(\nu)$, of the component of frequency $\nu$ is

$$G(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(x) e^{-2\pi i \nu x} \, dx$$

(2a)

$$= \frac{1}{2\pi} \sum_{n=-m}^{m} \left[ 1 + \alpha \cos \left( \frac{2\pi x n}{M} + \phi \right) \right] \int_{-\infty}^{\infty} e^{-2\pi i \nu x} q(x - \frac{n}{M}) \, dx$$

(2b)
where the sum and integral have been interchanged. The change of variables \( x = \frac{n}{M} \) = \( y \), \( dx = dy \), reduces each integral of the sum to the form

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2\pi i u x} q(x-M) dx = e^{-2\pi i u M} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2\pi i u y} q(y) dy
\]

\[
e^{-2\pi i \frac{2n}{M}} q(u) \quad \text{where} \quad q(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2\pi i u y} q(y) dy
\]

The integral \( q(u) \) is simply the Fourier transform of a single line of the raster. It is independent of \( n \) and may be factored out of the sum to give

\[
G(u) = \left\{ \sum_{-m}^{m} \left[ 1 + \cos \left( 2\pi \frac{-un}{M} + \psi \right) \right] e^{-2\pi i \frac{un}{M}} \right\} x(q(u))
\]

\[
= \left\{ \sum_{-m}^{m} e^{-2\pi i \frac{un}{M}} + \frac{1}{2} e^{i\psi} \sum_{-m}^{m} e^{2\pi i (\mu - \nu) \frac{a}{M}} + \frac{1}{2} e^{-i\psi} \sum_{-m}^{m} e^{-2\pi i (\mu + \nu) \frac{a}{M}} \right\} q(u)
\]
Each of these sums is a geometric series which may be summed by elementary means. After a little algebraic manipulation, one finds

$$G(\nu) = \left\{ \frac{\sin \pi u}{\sin \frac{\pi u}{M}} \right\} \left\{ e^{i\nu} \frac{\sin \pi (u+\lambda)}{\sin \frac{\pi (u+\lambda)}{M}} + \frac{e^{-i\nu}}{2} \frac{\sin \pi (u-\lambda)}{\sin \frac{\pi (u-\lambda)}{M}} \right\} g(\nu)$$

where

$$g(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2\pi iux} q(x) \, dx$$

$$G(\nu)$$ is the amplitude of the component of frequency $$\nu$$ in the output spectrum, while $$g(\nu)$$ is Fourier transform of the pulse resulting from scanning across a single line of the raster. Without any real loss in generality $$q(x)$$ may be taken to be an even function of $$x$$, in which case $$g(\nu)$$ is real and is given by the cosine transform of $$q(x)$$.

$$g(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos 2\pi \nu x \cdot q(x) \, dx$$

In the foregoing we have dealt with positive and negative frequencies on an equal footing, writing in equation (2a)

$$Q(x) = \int_{-\infty}^{\infty} \left\{ G(\nu) e^{2\pi iux} \right\} \, d\nu$$

$$Q(x)$$ measures the actual charge density in the raster and hence is of necessity a real function. $$G(\nu)$$ is thus the Fourier transform of a real function and must satisfy the identity $$G(\nu) = G^*(-\nu)$$, which permits equation 2a to be re-written in the form

$$Q(x) = 2 \int_{0}^{\infty} \text{Re} \left\{ G(\nu) e^{2\pi iux} \right\} \, d\nu$$
Inspection of this result shows that aside from the factor of 2, \( G(\omega) \) can be regarded as the complex amplitude of the A.C. component of frequency \( \omega \) in the output in the sense of ordinary A.C. circuit theory.

This calculation is seen to be equivalent to the calculation of the spectrum of a train of \( M \) "boxcarred" pulses if \( q(x) \) is interpreted as giving the shape of the individual pulses. If the modulation of the raster is neglected, the calculations may also be interpreted as being those leading to the Fraunhofer diffraction pattern of a grating composed of \( M \) equi-spaced identical slits; \( q(x) \) then corresponds to the distribution of illumination across each individual slit. These analogies will be useful in interpreting our results.

**Interpretation of the Spectrum:**

The terms of equation (3a) are readily recognized; \( M \) is the wave number of the basic raster and corresponds to the radar P.R.F. The first term \( \frac{\sin \frac{\pi \omega}{M}}{\sin \frac{\pi \omega}{M}} \), is that describing the diffraction pattern of a \( M \) grating with \( M \) slits as was mentioned above. It has principal maxima for frequencies which are integral multiples of \( M \) (i.e., \( \omega = nm \), where \( n \) is any positive integer) and therefore describes the peaks which occur in the "un-boxcarred" spectrum at each harmonic of the P.R.F. Similarly the B and C terms represent lines of the same shape with principle maxima at
nM + ω and nM - ω, respectively (i.e., symmetrically spaced about each harmonic of the P.R.F.) and are therefore the moving target side bands.

The whole spectrum is multiplied by the envelope g(ω) which is given by the Fourier transform of the pulse produced by reading across a single line of the raster. It is the exact analogue of the envelope arising when a train of pulses is "boxcarred," and may also be regarded as the transfer function of the computer in close analogy to network theory. This view may be illustrated as follows: For simplicity let us guard against the complications arising from an insufficient number of sampling pulses by supposing the raster lines to be so closely spaced as to be multiply overlapped. If we were to modulate this raster with doppler signals of constant percent modulation and successively higher frequencies we would find that g(ω) gives the relative percent modulation as a function of frequency for the resulting output signals. In this sense it is the ratio of the output to input amplitudes.

As mentioned above, except for the envelope g(ω) the spectrum would extend to infinite frequency, repeating at intervals equal to the P.R.F. frequency. How many of these repetitions actually appear in the output with finite amplitude obviously depends on the form of g(ω), as will be discussed later.

The whole spectrum is sketched schematically in Fig. 3.
Spectrum of the charge distribution of Fig. 2. The lines arising from terms A, B, C of equation (3a) are correspondingly labeled.

The positive and negative target sidebands (B and C terms) have equal absolute amplitudes and complementary phases which preserve the phase of the original modulation. This phase information is at present thrown away. Possibly it may be utilized (for example, to determine the sign of the radial velocity of the target), or possibly the two phase bins for every frequency bin may be traded for twice as many frequency bins.*

* By splitting $G(v)$ into real and imaginary components it can be readily shown that if the moving target sidebands associated with each harmonic of the P.R.F. are regarded as modulation sidebands with the P.R.F. harmonic as carrier, the in-phase component of the doppler modulation ($\cos 2\pi \nu \mu \alpha$) is preserved in the output as amplitude modulation, while the out-of-phase component ($\sin 2\pi \nu \mu \alpha$) appear as phase modulation.

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(1) The Definition and Width of a Resolvable Frequency Bin:

Each line in the above spectrum, whether due to a moving target or a P.R.F. harmonic, has the form

\[ \text{Power} \propto \frac{\sin^2 \frac{\pi \nu}{N}}{\sin^2 \frac{\pi \nu}{M} \nu} \]  

That is to say, a strictly monochromatic moving target frequency in the input signal will appear in the output as a narrow band of frequencies constituting a spectral line with the profile which is sketched in Fig. 4.

![Fig. 4](image)

To determine the width and number of resolvable frequency bins we note that the first nulls of the pattern lie at \( \Delta \nu = \pm 1 \). Using the Rayleigh criterion that two such lines are just resolved if the maximum of the first falls over the first null of the second, we see that each resolvable frequency bin is exactly one wave number wide. In other words, if the raster is modulated by two different sine waves of
equal amplitude, one having \( n \) complete cycles across the width of the raster and the other \( n + 1 \) cycles in the same length, the corresponding lines in the output spectrum could just be resolved. We thus lump the input moving target frequencies into bins according to whether they have 0 to 1, 1 to 2, 2 to 3, \ldots, complete cycles across the raster.

To evaluate the interference of a strong signal with weak signals in nearby frequency bins we note that for large \( M \) the line profile given by equation (6) is asymptotic to the form \( M^2 \left( \frac{\sin \pi \nu}{\pi \nu} \right)^2 \). The subsidiary maxima occur for values of \( \delta \nu \) approximately \( \pm 3/2, \pm 5/2, \ldots \), and have relative intensities as tabulated below.

<table>
<thead>
<tr>
<th>( \delta \nu )</th>
<th>Relative Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>Principal maximum</td>
<td>0</td>
</tr>
<tr>
<td>1st subsidiary maximum</td>
<td>1.43</td>
</tr>
<tr>
<td>2nd subsidiary maximum</td>
<td>2.46</td>
</tr>
<tr>
<td>3rd subsidiary maximum</td>
<td>3.47</td>
</tr>
</tbody>
</table>

(2) The Number of Output Frequency Bins

The computer is potentially capable of resolving as many different doppler frequencies in the input video signal as there are distinct frequency bins of the form defined above in the output pass band of the computer. In analogy to simple amplifier theory the pass band of the computer may be defined as the frequency interval between the
points at which the transfer function $g(\nu)$ falls off to some predetermined value -- for example, the interval between the 3 db points. We saw above that each bin occupies a band of frequencies one wave number wide; hence the number, $S$, of frequency bins is numerically equal to the pass band of the computer measured in wave numbers. Figure 5 is a sketch illustrating the manner in which the area under a certain transfer function might be divided into frequency bins.

![Fig. 5](image)

Illustrating the division of the area under a certain transfer function into frequency bins.

In this sketch the 3 db point falls at a frequency near eight wave numbers and the system might be said to possess eight frequency bins. However, we can afford to use frequencies which appear in the output with appreciably
reduced gains*, so this definition of the number of frequency bins is highly arbitrary: in a particular case it is better to determine the complete transfer characteristic. However, the gain-band pass product given by the area under the transfer characteristic is useful as a figure of merit for the storage tube. The relation between this product and the "resolution" of the tube is discussed in appendix 1.

So far the transfer characteristic which, as we have seen, determines the capacity of the computer has been discussed only in terms of its mathematical definition as the fourier transform of the output pulse resulting when the reading beam sweeps across an isolated raster line. For intuitive purposes it is far more useful to see how it is related to the experimentally determined resolution of the tube. Suppose that the lines forming the raster are of such a width that N white lines separated by N dark lines of equal width can be written in the width of the raster, as in Fig. 6.

* The loss in amplitude can easily be compensated, the deciding factor thus being the noise introduced by the computer itself. This noise might be expected to fall off at high frequencies according to a law similar to \( g(\Delta) \) itself since the resolution of the system for stored signals and for irregularities in the storage surface has a common limit set by the current distribution in the reading beam.
By our definition the resolution of the tube is \( N \) lines. It follows that the width at half maximum of each line is approximately \( \frac{L}{2N} \) and that to approximate the line profile sine waves having up to \( N \) cycles in the length \( L \) — that is, having frequencies up to \( N \) wave numbers — will be necessary. Hence, \( g(\nu) \) will be large for frequencies up to \( N \) and the number of frequency bins, \( S \), will be of the order of \( N^2 \). This result fits well with our earlier interpretation of \( g(\nu) \) as the transfer function of the computer: clearly raster lines of a width corresponding to the above definition of \( N \)-line resolution cannot be used very successfully to paint a sinusoidal charge distribution having more than \( N \) complete cycles across the face of the tube. The (effective)

* Experiment has shown \( N \) to be of the order of 100 for the Raytheon tube, QK 357; and 40 for the RCA Radechon C 73404.*
number of resolvable frequency bins that the tube will process is equal to the (effective) number of resolved cycles of a sine wave which can be written across the storage surface*.

(3) Interpretation in Terms of Real Time

To construct a computer having a capacity of $S$ frequency bins a tube with a resolution of at least $S$ lines is necessary. But this does not imply that it is necessary or desirable to use an $S$ line raster; so long as the storage process is linear the validity of our results is unchanged if the raster lines overlap. There is no a-priori limit to the number of pulses which may be recorded in one frame**.

Let the radar P.R.F. be $f_0$ cps and let the horizontal sweep speed used in writing the raster be such that the video echos from $M$ radar pulses are written in the width of the storage surface, possibly with overlap. The conversion factor between our private frequency unit, the

* This result is made more precise in appendix 1.

** Effects analogous to delay line cancellation may be obtained by recording alternate pulses with opposite signs. In such a system the amount of overlap of adjacent lines is obviously important.
wave number, and actual cycles per second in the input signal is now fixed since by hypothesis the basic raster frequency is $M$ wave numbers and $f_0$ cycles per second. One wave number thus stands for $f_0/M$ cps. Our previous results may now be tabulated in terms of the frequency of the input signals as measured in real time.

<table>
<thead>
<tr>
<th>Wave Numbers</th>
<th>CPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>$f_0$</td>
</tr>
<tr>
<td>$f_0/M$</td>
<td></td>
</tr>
<tr>
<td>$S \approx N$</td>
<td>$S/M \cdot f_0$</td>
</tr>
</tbody>
</table>

The number of bins in the output is equal to $S$ regardless of $M$. But a train of $M$ pulses can carry information concerning the amplitude of at most $M/2$ different Doppler frequencies, leading to a paradox if $M \ll N$. The resolution of the paradox lies in the fact that the output spectrum is periodic and repeats completely in each frequency interval equal to one half the P.R.F., or $M/2$ wave numbers. When $M \ll N$, $S$ the spectrum extends to cover several harmonics of the P.R.F., ($S/M$ in fact) and the information in the output bins is $2 \cdot S/M$ fold redundant. The actual information capacity is
thus in reality only \( \frac{M}{2} \) bins*. When excess resolution is available this redundancy might be used to discriminate against noise introduced by the storage process.

In the converse case, \( M \gg S \), the \( S \) bins become much narrower and cover a band of frequencies extending from zero to some fraction of the P.R.F. There is no redundancy and the capacity of the system is \( S \) frequencies.

Let us consider a set of concrete examples in which the operation of the system is varied by changing the length of time (and number of pulses) stored in one frame of the raster:

\[
P.R.F. = f_0 = 2000 \text{ cps}
\]

\[
N = 100 \text{ lines}
\]

<table>
<thead>
<tr>
<th>Frame Time</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>M = number of pulses stored</td>
<td>50</td>
<td>200</td>
<td>2000</td>
</tr>
<tr>
<td>Bin width, cps.</td>
<td>40</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>Upper frequency limit of bins</td>
<td>4000</td>
<td>1000</td>
<td>100</td>
</tr>
</tbody>
</table>

* In principle the information capacity is \( M \) bins, but these are assigned in pairs to \( \frac{M}{2} \) different Doppler modulation frequencies, the two bins for each frequency preserving the phase of the modulation.
Case B corresponds to the proposed operation of the computer.

(4) Examples of pass band characteristics associated with various idealized line profiles.

To build up a concrete picture of the shapes which the transfer characteristic of a real computer might have, this section will present a few examples calculated from raster line profiles of various degrees of realism.

Example A.

As the first example a highly artificial charge distribution will be assumed for the raster lines in order to obtain a particularly vivid illustration of the connection between the resolution of the tube and the capacity of the computer.

Assume the charge distribution to be of the form $\frac{\sin x}{x}$ and let two such lines be regarded as completely resolved if
they are separated by twice the distance given by the Rayleigh criterion - i.e., when the first nulls of the adjacent lines coincide. If $N$ is the maximum number of such resolved lines which may be written in the width of the raster, we have

\[
\text{Line profile} = q(x) = \frac{\sin \frac{2\pi Nx}{2\pi N}}{2\pi N}
\]

\[
\text{Transfer function} = g(\nu) = \begin{cases} 
\frac{1}{4\pi N} & \text{for } -N < \nu < N \\
0 & \text{otherwise}
\end{cases}
\]

In this case the pass band is rectangular and exactly $N$ wave numbers wide, giving $N$ frequency bins.

To facilitate comparison the line profiles assumed in these examples will be plotted together in Fig. 9, (page 29) while the resulting transfer functions will be plotted in Fig. 10, page 29. The curves marked "A" in the figures correspond to this example.

Example B

Throughout the discussion $q(x)$ has been called the charge distribution in a single raster line with the tacit assumption that the read out beam is of infinitesimal diameter. Strictly speaking, $q(x)$ is the function giving the output voltage as a function of $x$ as the reading beam is swept across the raster line. Now suppose the write-in and read-out beams
both have a uniform current density in a rectangular cross section. The charge distribution in the line will have a square wave profile while the read-out function will be composed of triangle waves.

If \( W \) is the width of both beams, the base of each triangle will be \( 2W \) wide. The tube will be said to have a resolution of \( N \) lines if \( N \) triangles can be written in with their bases just touching, as in Fig. 8b.

Line profile: 
\[
q(x) = \begin{cases} 
1 - 2N|x| & \text{for } -\frac{1}{2N} \leq x \leq \frac{1}{2N} \\
0 & \text{otherwise}
\end{cases}
\]

Transfer function: 
\[
g(\psi) = \frac{1}{4\pi N} \left( \frac{\sin \frac{\pi \psi}{2N}}{\pi \psi/2N} \right)^2
\]

The curves representing this line profile and its corresponding transfer function are labeled "B" in figures 9 and 10.
The first null in the spectrum occurs at $v = 2N$. However, $g(v)$ drops off to small values at appreciably lower frequencies, as tabulated below.

<table>
<thead>
<tr>
<th>$P/P_0$</th>
<th>$\pi v/2N$</th>
<th>$v$, wave numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-3db)</td>
<td>1/2</td>
<td>1.00 .636 N</td>
</tr>
<tr>
<td>(-6db)</td>
<td>1/4</td>
<td>1.39 .885 N</td>
</tr>
<tr>
<td>(-12db)</td>
<td>1/16</td>
<td>1.90 1.21 N</td>
</tr>
</tbody>
</table>

Example C

In the final and most realistic example the current distribution in both the reading and writing beams is assumed to be a gaussian function of radius.

Writing beam: current $\propto e^{-(r/R_1)^2}$

Reading beam: current $\propto e^{-(r/R_2)^2}$.

$R_1$, $R_2$, $r$ are all expressed in units of $L$, the raster width. A little calculation shows the shape of the read-out pulse to be

$$q(x) = e^{\frac{-x^2}{R_1^2 + R_2^2}} = e^{\frac{-x^2}{R^2}}$$

where $R^2 = R_1^2 + R_2^2$

No unique criterion for determining when two gaussian pulses are completely resolved recommends itself when the pulse shapes alone are considered. In the appendix it is
shown that a generally useful definition is as follows: Two pulses are completely resolved if in a plot of charge vs. distance the area in the trough separating the pulses is equal to the area under a single pulse. In this sense the number of resolved lines which can be written in the raster is

\[ N = \frac{1}{2\sqrt{\pi} R} \]

giving

\[ q(x) = e^{-\frac{1}{4\pi}(Nx)^2} \]

The separation of two such resolved lines is \( \delta x = \frac{1}{N} = 2\sqrt{\pi} R \); at the center of the minimum separating the lines, each contributes an amplitude of

\[ q(1/2N) = e^{-\pi} = .044 \text{ or } 4.4\% \]

and the relative amplitude at the minimum is

\[ \frac{2e^{-\pi}}{1 + 2e^{-2\pi}} = .087 \text{ or } 8.7\% \]

The envelope of the resulting spectrum is

\[ g(\nu) = \frac{R}{2\sqrt{\pi}} e^{-\left(\pi RV\nu\right)^2} = \frac{1}{4\pi N} e^{-\frac{\pi}{4}(\frac{\nu}{N})^2} \]

This function is plotted by curve C in figure 10 and representative values are tabulated below.
FIGURE 9

A. \( q(x) = \frac{\sin 2\pi N x}{2\pi N x} \)

B. \( q(x) = 1 - 2Nx, \quad \begin{cases} \frac{1}{2N} < x < \frac{1}{2N} \\ 0 & \text{otherwise} \end{cases} \)

C. \( q(x) = \exp(-4\pi (Nx)^2) \)

FIGURE 10

A. \( g(\nu) = \begin{cases} \frac{1}{4\pi N} & -N < \nu < N \\ 0 & \text{otherwise} \end{cases} \)

B. \( g(\nu) = \frac{1}{4\pi N} \left( \frac{\sin \pi \nu/2N}{\pi \nu/2N} \right)^2 \)

C. \( g(\nu) = \frac{1}{4\pi N} \exp\left(-\frac{\pi (\nu/N)^2}{4}\right) \)
Comparison of this table with that of the previous example shows that changing from a triangular to a gaussian line profile leads to a trivial change in the output bandwidth. This suggests that very similar results will be obtained for any charge distribution which is likely to be encountered.

Suggested Experimental Determination of the Pass Band

The statement that the number of resolvable frequency bins is equal to the maximum number of completely resolved lines which can be written on the storage surface is seen to be a rather rough approximation since the shape of the pass band is determined by the fourier transform of the charge distribution in an individual line of the raster. The numerical computation of this fourier transform from the observed line shape is tedious and it would seem desirable to determine the pass band directly.

This may be done very simply once a frequency analysis unit for the read out signals is in operation. A raster of 10 (or 20) lines having the same width as those to be used in the ultimate raster is painted on the storage surface, the lines being equi-spaced across the whole width of the storage
surface. The read out beam is swept orthogonally across the raster at a slow enough rate so that the highest harmonic of the sweep frequency that is expected in the output (for this tube about the 100th or 200th) is well within the pass band of the output system.

The frequency spectrum of the output will then be a raster of spikes spaced every 10 (or 20) frequency bins. The envelope of this raster will be the desired pass band of the storage tube computer.

![desired envelope](image)

**Fig. 11**

The stored raster is chosen to have 10 or more lines so that the subsidiary maxima which appear near each principle maximum of the frequency spectrum will not obscure the results.
Appendix A.

The purpose of this appendix is to derive a theorem relating the effective number of frequency bins, or rather, the gain-band pass product given by the area under the band pass characteristic, to the number of resolved lines which can be written on the raster.

First consider a diagram of charge on the storage surface versus distance, x, across the surface. In our original analysis we tacitly assumed that the storage surface had a sharply defined width and that within this width the effectiveness of the writing beam in depositing stored charge was constant. We now consider a slightly more general case in which the effectiveness of the writing process varies across the storage surface. Let $F(x)$ be the relative amount of charge deposited per unit time, by a unit writing current, at different points across the surface. The width of the storage surface is determined by the region in which $F(x)$ is finite; for example, our original assumption corresponded to the choice of a rectangular function for $F(x)$: $F(x) = \text{constant}$ for $-\frac{1}{2} < x < \frac{1}{2}$, and $F(x) = 0$ elsewhere. Further, let $f(x)$ be the charge distribution across a single raster line. We assume both $F(x)$ and $f(x)$ to be positive, real, and even functions of $x$.

The effective number of resolvable lines, $N$, may be taken as $1/2$ the ratio of the areas under the two functions.
when both are normalized to a maximum height of unity.
The factor $1/2$ arises from the fact that if two bright
lines are to be completely resolved there must be a dark
line between them*.

$$
N = \frac{1}{2} \frac{\int_{-\infty}^{\infty} F(x) \, dx}{F(0)} / \frac{\int_{-\infty}^{\infty} f(x) \, dx}{f(0)}
$$

Now consider the band pass characteristic of the
output. Let $g(\nu)$ be the band pass function and $G(\nu)$
be the functions giving the profile of a single line.

* It is reassuring to note that when this definition of
resolution is applied to the examples cited earlier
(Examples A and B, pp. 241-25) the resulting
criterion for "complete resolution" is identical
with the intuitive choice made previously from con­sideration of the shapes of the lines.

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The effective number of resolvable frequency bins may again be taken as $1/2$ the ratio of the areas under the normalized curves.

$$S = \frac{1}{2} \frac{\int_{-\infty}^{\infty} g(\nu) d\nu}{G(0)} / \frac{\int_{-\infty}^{\infty} G(\nu) d\nu}{G(0)}$$

Here the factor of $1/2$ arises from the fact that $g(\nu)$ is necessarily symmetrical about $\nu = 0$, whereas we are interested only in positive frequencies. Note that our requirement on the resolution of the frequency bins is only one half as stringent as that imposed on the raster lines.

In section A of this report it was shown that $g(\nu)$ is the cosine transform of $f(x)$.

$$g(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos 2\pi \nu x \ f(x) \ dx$$
Hence
\[ \frac{\sum_{-\infty}^{\infty} g(n) d\nu}{g(\omega)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) d\omega \]

Similarly, a slight extension of the previous work shows that
\[ G(\nu) = \sum_{\eta=-M'/2}^{M'/2} F(\frac{\eta}{M'}) e^{-2\pi i \nu \frac{\eta}{M'}} \]

where \( M' \) is the density of lines in the basic raster. To define a frequency bin we may let \( M' \to \infty \) so that in the limit the sum becomes an integral which is seen to be the Fourier transform of \( F(x) \). Then
\[ \frac{G(\nu)}{G(0)} = \int_{-\infty}^{\infty} F(x) e^{-2\pi i \nu x} dx \]
\[ \int_{-\infty}^{\infty} F(x) dx \]

and
\[ \frac{\sum_{-\infty}^{\infty} G(n) d\nu}{G(\nu)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) d\omega \]

It follows at once that \( N = S \) which is the desired result.

This theorem is not as strong as might be wished since
it does not follow from the restrictions on \( f(x) \), \( F(x) \), that \( g(\nu) \), \( G(\nu) \) are non-negative functions. It would be desirable to prove in addition that the main contribution to the integrals \( \int g(\nu) \, d\nu \), \( \int G(\nu) \, d\nu \) come from their principle maxima, the effect of any negative contributions being minor.