Detection of Pulsed Signals with A Narrow Band Filter, Detector, and Integrator

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DETECTION OF PULSED SIGNALS WITH A NARROW BAND FILTER, DETECTOR, AND INTEGRATOR

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Abstract

This paper is concerned with predicting the directly observable quantities of false alarm rate and signal detection probability for pulsed signals in continuous noise. The detection system consists of a band pass filter, a detector, an exponential integrator (low pass filter), a threshold detector, and an alarm circuit. The calculations are reduced to the elementary operation of adding $n$ independent samples to obtain approximately gaussian distributions from which detectability is predicted. Corrections are made for the deviations from the gaussian shape caused by finite $n$.

A general relationship is derived which gives the equivalent rate, $n_1$, of output of independent samples from a band of noise whose normalized auto-correlation function is $\rho(\tau)$. If one is integrating the noise for a time large compared to its correlation time

$$n_1 = \frac{1}{2 \int_0^\infty \rho(\tau) \, d\tau}$$

For rectified noise from a square bandpass filter of width $\Delta f$ $n_1 = \Delta f$, and from a single tuned circuit whose full width at half power is $\Delta f$, $n_1 = \pi \Delta f$. Also, for noise from a simple low pass filter of time constant $T_0$ (e.g. an R-C filter), $n_1 = 1/(2T_0)$.

The statistical characteristics of the output voltages of linear and square law detectors, connected to a band pass filter, are plotted in a form immediately useful for calculations.
I. Introduction

This paper is concerned with calculating the statistical behavior of a typical signal detection system consisting of a narrow band filter, a detector and an integrator. The integrator has a threshold detector and alarm circuit connected to its output. The alarm is supposed to sound only when a signal is detected. We are particularly concerned with calculating the false alarm rate (the number of alarms per second which occur when only noise is fed into the input), and also with the probability of detection of a signal of given power and duration, all for a given setting of the threshold.

Fig. 1 Typical Detection Circuit

We assume that the signal duration is approximately commensurate with the time constant $T_0$ of the integrator (such as an $R-C$ filter), which is in turn considerably larger than the time constant ($\sim \frac{1}{\Delta f}$) of the band pass filter. This system is suited to the detection of signals whose frequency is confined to a band of width $\Delta f$ but which have a duration of many times ($\sim \frac{1}{\Delta f}$).
The line of attack is as follows: For a given ratio of signal to noise power at the output of the band pass filter one calculates the mean $\bar{y}$ and the dispersion $\sigma_y^2$ of the voltage $y$ at the output of the detector. Of course $y$ is varying continuously but remains correlated for a time of the order of $1/\Delta f$ second. Thus approximately $\Delta f$ times per second $y$ reaches a new and independent value. More exactly, from the shape of the noise spectrum at the output of the detector one calculates the equivalent rate $n_1$ at which independent sames of $y$ are appearing at the output of the detector. The exponential integrator now adds (with exponential weighting) a total of $n_1 T$ samples of a signal plus noise of duration $T$, or an infinite number of noise samples (since the noise is assumed to be continuously present). The output $z$, of the integrator will have an approximately gaussian probability distribution since it is summing many ($\gtrsim 10$) independent samples. The noise, and the signal plus noise, each has its own gaussian distribution, and for any given threshold, one can calculate the false alarm probability and the signal detection probability. Finally, the false alarm rate can be calculated from the rate $n_1$ at which independent samples are coming out of the integrator. Again, $n_1$ can be calculated from the power spectrum at the output of the integrator.

This method of analysis is simple to apply since there are few important variations of filters, detectors, and integrators, and each system can be readily characterized by a few significant constants such as $n_1$, $T_0$ (the integration time.
constant of the output filter). The calculation then reduces itself to the elementary statistical problem of the addition of \( n \) independent samples of given mean and variance. The results give directly the false alarm rate and signal detection probability which are these items which are actually observed in a typical experiment where one is measuring the detection of short duration signals against a continuous noise background.

II. Statistical Characteristics of Narrow Band Noise after Detection

Rice* gives the moments of the probability distribution of the envelope, \( R \), of narrow band noise plus signal,

\[
R^n = (2\sigma^2)^{n/2} \int \left( \frac{n}{2} + 1 \right) \exp(-P^2/2\sigma^2) \text{I}_1\left(\frac{n}{2} + 1, 1, \frac{P^2}{2\sigma^2}\right)
\]  

(2-1)

For \( n = 1 \) we have the envelope, or "linear" detector. For \( n = 2 \), the square law detection etc. \( \sigma^2 \) is the total noise power transmitted by the filter. \( P \) is the amplitude of the sinusoidal signal transmitted by the filter, thus \( P^2/2\sigma^2 \) is the ratio of signal power to noise power at the output of the

* S. O. Rice, Bell System Tech. Jour. 23, 282-332 (1944) and 24, 46-156 (1945); and 27, pp. 109-157 (1948)
narrow band filter. \( _1F_1 \) is a confluent hypergeometric function and is plotted in Jahnke and Emde, for a limited range.

We shall consider first the square law detector. Thus if the detector output voltage is \( y \), we are interested in

\[
\overline{y} = \frac{R^2}{2} \quad \text{and} \quad \sigma_y^2 = \overline{y^2} - \overline{y}^2 = \frac{R^2}{4} - \left( \frac{R}{2} \right)^2
\]

for various values of signal to noise. Figs. 2a and 2b give these values for a range of practical interest in detection problems.

Second, we consider the linear detector. Again the output voltage is \( y \).

\[
\overline{y} = R \quad \text{and} \quad \sigma_y^2 = \overline{y^2} - \overline{y}^2 = \frac{R^2}{2} - \left( \frac{R}{2} \right)^2
\]

Fig. 2c gives plotted values of \( \overline{y} \) and \( \sigma_y \) over a range of values of input signal to noise power.

In many practical problems it is necessary to calculate the probability that the envelope, \( y \), of the (pure noise) output of a narrow band filter exceeds some threshold \( y_T \). Since the distribution function for \( y \) is \( \frac{y}{\sigma} \in \exp -\frac{y^2}{2\sigma^2} \), the probability that \( y \) exceeds \( y_T \), when independent samples of \( y \) are observed, is \( \in \exp -\frac{y_T^2}{2\sigma^2} \). If \( y \) is continuously observed, Rice (p. 127, 1948) gives the rate of \( N_R \) of positive slope crossings of the threshold \( y_T \) for an envelope detector on the output of a square band pass filter of width \( \Delta f \), as \( N_R = \Delta f \left( \frac{\pi}{6} \right)^{1/2} \left( \frac{y_T}{\sigma} \right) \exp -\frac{y_T^2}{2\sigma^2} \). These two false alarm probability curves are plotted on page 41 at the end of the report.


Also the values of the confluent hypergeometric functions are tabulated in the Report of the British Association for the Advancement of Science, 1927, p. 229 and 231 for \( \frac{P^2}{2\sigma^2} \) from 0 to 8.

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SQUARE LAW DETECTOR

\[ \bar{y} = R^2 = p^2 + 2 \sigma_x^2 \]

\[ \sigma_y = \sqrt{R^4 - (R^2)^2} \]

\( R = \text{ENVELOPE OF VOLTAGE AT OUTPUT OF BAND PASS FILTER} \)
SQUARE LAW DETECTOR

\[ y = R^2 = P^2 + 2\sigma^2 \]

\[ \sigma_y = \sqrt{R^2 - (R^2)^2} \]

ENVELOPE OF VOLTAGE AT OUTPUT OF BAND PASS FILTER

\[ \frac{P^2}{2\sigma^2} \text{ (SIGNAL/NOISE POWER AT OUTPUT OF FILTER)} \]
LINEAR DETECTOR

\[ \bar{y} = \bar{R} \]

\[ \sigma_y = \sqrt{\bar{R}^2 - (\bar{R})^2} \]

\[ R = \text{ENVELOPE OF VOLTAGE AT OUTPUT OF BAND PASS FILTER} \]

\[ y = 1.25 + 0.65 \frac{P^*}{2\sigma^*} \]

\[ \bar{y} \text{ and } \sigma_y \text{ in units of } \sigma \]

\[ P^*/2\sigma^* \text{ (SIGNAL/NOISE POWER AT OUTPUT OF FILTER)} \]
III. The Rate of Output of Independent Samples from the Detector

We first assume a linear integrator to be continuously summing the output $y$ of a detector for a time $T$. The voltage signal $y$ has a power spectrum of finite width and therefore many of the samples which the integrator is summing are strongly correlated. The continuous summation operation is equivalent to taking samples which are much more closely spaced than the correlation time which is approximately $\frac{1}{\Delta f}$, where $\Delta f$ is the bandwidth of the power spectrum.

First we calculate the improvement in signal to noise, for a given integration time $T$, which is effected by continuous integration of the correlated samples.

Second, we ask, "At what rate $n_1$ would independent samples have to be presented at the output of the detector to give the same signal to noise improvement in the same integration time $T$?"

We call $n_1$ the rate of output of independent samples from the detector, and from general considerations we know that $n_1$ is the order of $\Delta f$. The problem is to calculate it exactly.

Let the output voltage from the detector be $y$, which has a mean $\bar{y}$ and a dispersion $\sigma_y^2$. Let the output of the integrator be $z = \sum_{1}^{n_1} y_i$. 

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Instead of continuous integration we shall first assume that samples are taken $\Delta t$ seconds apart ($\Delta t \ll \text{correlation time of } y$). In $T$ seconds we sum $T/\Delta t$ samples each of mean value $\bar{y}$, so that

$$\bar{z} = \frac{T}{\Delta t} \bar{y} \quad (3-1)$$

Before calculating the dispersion $\sigma_z^2$ of $z$ we will first consider adding three variates $y_1$, $y_2$ and $y_3$, each with mean $\bar{y}$, and weighted as follows:

$$z = a_1 y_1 + a_2 y_2 + a_3 y_3$$

Thus,

$$\bar{z} = (a_1 + a_2 + a_3) \bar{y} \quad (3-2)$$

also the $y$'s have the same dispersion, so

$$\frac{(y_1 - \bar{y})^2}{\sigma_y^2} = \frac{(y_2 - \bar{y})^2}{\sigma_y^2} = \frac{(y_3 - \bar{y})^2}{\sigma_y^2} \quad (3-3)$$

However, they are correlated, such that the normalized auto-correlation functions

$$\rho_{12} = \frac{(y_1 - \bar{y})(y_2 - \bar{y})}{\sigma_y^2} \quad , \quad \rho_{13} = \frac{(y_1 - \bar{y})(y_3 - \bar{y})}{\sigma_y^2} \quad (3-4)$$

$$\rho_{23} = \frac{(y_2 - \bar{y})(y_3 - \bar{y})}{\sigma_y^2}$$

are not, in general, zero.

By definition,

$$\sigma_z^2 = \int \left[ a_1 (y_1 - \bar{y}) + a_2 (y_2 - \bar{y}) + a_3 (y_3 - \bar{y}) \right]^2 \, dy_1 \, dy_2 \, dy_3$$

$$= \sigma_y^2 \left[ a_1^2 \rho_{12} \rho_{13} \rho_{23} \right] \quad (3-5)$$
where the integration is over the whole range of \( y_1, y_2 \) and
\( y_3 \), where \( P(y_1, y_2, y_3, \rho_{12}, \rho_{13}, \rho_{23}) \) is the joint probability
that \( y_1, y_2 \) and \( y_3 \) will each lie their ranges \( y_1 \) and \( dy_1, y_2 + dy_3 \)
with the \( \rho \)'s given by (3-4). Squaring the bracket in (3-5),
we have three terms such as
\[
\int \int \int a_1^2 (y_1 - \bar{y})^2 P(y_1 y_2 y_3 \rho_{12} \rho_{13} \rho_{23}) dy_1 dy_2 dy_3 = a_1^2 \sigma_y^2
\]
also there are three terms such as
\[
\int \int \int 2a_1 a_2 (y_1 - \bar{y})(y_2 - \bar{y}) P(y_1 y_2 y_3 \rho_{12} \rho_{13} \rho_{23}) dy_1 dy_2 dy_3 = 2a_1 a_2 \sigma_y^2 \rho_{12}
\]
Thus for three correlated samples, equally weighted,
\[
z = y_1 + y_2 + y_3
\]
\[
\bar{z} = 3\bar{y}
\]
and
\[
\sigma_z^2 = 3\sigma_y^2 + 2\sigma_y^2 (\rho_{12} + \rho_{13} + \rho_{23})
\]
Now consider \( n \) samples of a correlated signal spaced \( \Delta t \)
seconds apart spread over the integration time \( T \), thus
\[
n = \frac{T}{\Delta t}
\]
There will be \( \binom{n-1}{2} \) pairs of samples spaced
\( \Delta t \) seconds apart, with correlation \( \rho(\Delta t) \). Thus will be
\( \binom{n-2}{2} \) pairs of samples spaced \( 2\Delta t \) seconds apart, with
correlation \( \rho(2\Delta t) \), etc. Thus
\[
\bar{z} = \frac{T}{\Delta t} \bar{y}
\]
\[
\sigma_z^2 = \frac{T}{\Delta t} \sigma_y^2 + 2\sigma_y^2 \left[ \left( \frac{T}{\Delta t} - 1 \right) \rho(\Delta t) + \left( \frac{T}{\Delta t} - 2 \right) \rho(2\Delta t) + \ldots \right.
\]
\[
\left. \ldots + \rho \left( \frac{T}{\Delta t} \cdot \Delta t \right) \right] + \ldots
\]
for \( \frac{T}{\Delta t} \gg 1 \) the first term is negligible, and

\[
\sigma_z^2 \approx \frac{2\sigma_y^2}{(\Delta t)^2} \left[ (T-\Delta t)\rho(\Delta t) + (T-2\Delta t)\rho(2\Delta t) + \ldots + \rho(T) \right] \Delta t
\]

The output signal to noise amplitude ratio is

\[
\frac{\bar{z}}{\sigma_z} = \frac{T}{\sqrt{2} \left[ (T-\Delta t)\rho(\Delta t) + \ldots + \rho(T) \right] \Delta t} \cdot \frac{\bar{y}}{\sigma_y}
\]

let \( \Delta t \to 0 \),

\[
\frac{\bar{z}}{\sigma_z} = \frac{T}{\sqrt{2} \int_0^T (T-t) \rho(t) \, dt} \cdot \frac{\bar{y}}{\sigma_y}
\]

and thus we have calculated the signal to noise improvement
effected by a linear integrator, which is continuously sampling
a correlated signal and summing for a time \( T \).

Now we ask, "At what rate \( n_1 \) would independent samples have
to be presented to the same integrator in order to effect the
same improvement in signal to noise?" We can write at once,

\[
\bar{z} = (n_1T) \bar{y} \quad \text{and} \quad \sigma_z^2 = (n_1T)\sigma_y^2
\]

thus

\[
\frac{\bar{z}}{\sigma_z} = \sqrt{n_1T} \cdot \frac{\bar{y}}{\sigma_y}
\]

(3-7)

*This result was obtained by J. P. Costas, "Periodic Sampling
of Stationary Time Series", Tech. Report 156, Research
Laboratory for Electronics, Massachusetts Institute of

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and by comparing (3-6) and (3-7) we have,

\[ n_1 = \frac{T}{2 \int_0^T \rho(t) dt} \quad (3-8) \]

Now, if we require that \( T \gg T \) times when \( \rho(t) \) becomes small (i.e., the sum includes many independent samples)

\[ n_1 = \frac{1}{2 \int_0^\infty \rho(t) dt} \quad (3-9) \]

Now, if \( G(f) \) is the power spectrum of the noise not including the d.c. term,

\[ \rho(t) = \frac{\int G(f) \cos 2\pi f t df}{\int G(f) df} \quad (3-10) \]

where \( \int G(f) df = \sigma^2 \).

If \( G_0(f) \) is the power spectrum at the output of the band pass filter, then the power spectrum at the output of the square law detector is,

\[ G(f) = \int_{-\infty}^{+\infty} G_0(f_1) G_0(f_1 + f) df \quad (3-11) \]

which spectrum is not zero at \( f = 0 \).

This analysis is valid only for a continuous power spectrum \( G(f) \) which is not zero at \( f = 0 \). If \( G(0) = 0 \), there is no power at d.c., and thus \( \bar{y} = 0 \). The average integrator output \( \bar{z} \) will be zero for both noise and signals plus noise. Thus no signal to noise improvement is possible with this method.

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Table 1 gives the results of the above calculations for several cases of interest.

Since the linear detector gives very nearly the same output power spectrum as does the square law detector, the two types of detector will have very nearly the same values of $n_1$.

We have considered only the summing of the detector output in a linear integrator. Actually an exponential integrator is often used in practice. An examination of the previous analysis shows that it is only necessary that the integrator be linear for a time which is the order of magnitude of the correlation time, a requirement which is always met if one is concerned with summing many independent samples.

So far we have calculated $n_1$ on the basis that only noise is present in the filter output. What does the presence of a coherent signal do to $n_1$? For the square filter, direct calculation shows that the presence of a coherent signal in any position and any intensity does not affect $n_1$. However a signal of amplitude $P$ in the center of the pass band of the single tuned circuit, gives

* Jack P. Ruina, private communication.
Table 1. Equivalent number of independent samples per second for several arrangements of filters and detectors. The filters are assumed to have white noise on the input.

<table>
<thead>
<tr>
<th>Type of Filter</th>
<th>Power Spectrum at output of Filter ( = G_0(f) )</th>
<th>Power Spectrum at Output of Square Law Detector (excluding d.c. term) ( = G(f) )</th>
<th>Normalized Auto correlation function of detector output. ( \phi(t) )</th>
<th>Equivalent Number of Independent Samples per second out of detector = ( n_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square band pass filter of width ( \Delta f ) ( G_0(f) = 1 ) from ( f = f_0 - \Delta f/2 ) to ( f = f_0 + \Delta f/2 ), zero elsewhere.</td>
<td>( G(f) = 1 - \frac{\Delta f}{2\pi} ) from ( f = 0 ) to ( f = \Delta f ), zero elsewhere</td>
<td>( 2 - \frac{1}{(2\pi\Delta f)^2} )</td>
<td>( \Delta f )</td>
<td></td>
</tr>
<tr>
<td>Single tuned filter ( Q \gg 1 ) full width at half power ( \Delta f = \frac{f_0}{Q} ) ( G_0(f) = \frac{R^2Q^2}{1 + 4Q^2(\frac{f-f_0}{f_0})^2} )</td>
<td>( G(f) = \frac{\pi R^4 Q^4 (\Delta f)^3}{4} ) ( \frac{1}{\pi R^4 Q^4 (\Delta f)^2 + \frac{1}{\Delta f^2}} ) ( \exp(-2\pi \Delta f t) )</td>
<td>( \pi \Delta f )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
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<th>Type of Filter</th>
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<th>Normalized Auto correlation function of detector output. ( \phi(t) )</th>
<th>Equivalent Number of Independent Samples per second out of detector = ( n_i ) at output of filter</th>
</tr>
</thead>
<tbody>
<tr>
<td>R-C Low Pass Filter ( G_0(f) = \frac{1}{1 + (2\pi fRC)^2} )</td>
<td>no detector</td>
<td>( \exp(-\frac{t}{RC}) )</td>
<td>( \frac{1}{2RC} )</td>
<td></td>
</tr>
<tr>
<td>Square, Low Pass Filter ( G_0(f) = 1 ) from ( f = 0 ) to ( f = f_1 ), zero elsewhere</td>
<td>no detector</td>
<td>( \sin(2\pi ft_1) ) ( \frac{1}{2\pi ft_1} )</td>
<td>( 2f_1 )</td>
<td></td>
</tr>
</tbody>
</table>
Thus if \( p^2/2\sigma^2 = 0.5 \), \( n_1 \) is smaller by 33\% than for the case of pure noise. For very large signal to noise, \( n_1 \rightarrow \frac{\pi}{2} \Delta f \) which is that value of \( n_1 \) for the equivalent square pass band of a single tuned circuit (see Section VIII).

In practice the signal at the output of the filter (before integration) is usually much less than the noise power and therefore has little effect upon \( n_1 \).

### IV. Integration Subsequent to Detection

Consider an exponential integrator whose time constant is \( T_0 \) and which is summing samples spaced \( t_1 \) seconds apart. The signal plus noise samples have a mean \( \overline{y}_{S+N} \), and a dispersion \( \sigma_{yS+N}^2 \). At the end of a signal of duration \( T \), the integrator has stored \( T/t_1 \) signal plus noise samples and an infinite number of noise samples of mean \( \overline{y}_N \) and dispersion \( \sigma_{yN}^2 \). These samples are all exponentially weighted so that even the infinite number of noise samples make only a finite contribution to the sum. This residue of noise samples is always present.
when the signal has finite duration.

Let \( z_{S+N} \) = the integrator output at the end of a signal of duration \( T \). Since \( t_i = \) the spacing of the independent samples, \( n_i = \frac{1}{t_i} = \) the rate at which independent samples are entering the integrator.

\[
\overline{z}_{S+N} = \overline{y}_{S+N} \left( 1 + e^{-t_1/T_0} + e^{-2t_1/T_0} + \ldots + e^{-(n-1)\frac{t_i}{T_0}} \right)
\]

Using the identity \( 1 + x + x^2 + \ldots x^{n-1} = \frac{1-x^n}{1-x} \), also let \( n = \frac{T}{t_i} \).

\[
\overline{z}_{S+N} = \overline{y}_{S+N} \left( 1 + e^{-T/T_0} \right) + \overline{y}_N e^{-T/T_0}
\]

Similarly,

\[
\sigma^2_{zS+N} = \sigma^2_{yS+N} \left( 1 + e^{-2t_1/T_0} + e^{-2\cdot2t_1/T_0} + \ldots e^{-2(n-1)t_i/T_0} \right)
\]

\[
\sigma^2_{yN} \left( e^{-2t_1/T_0} + e^{-2(n+1)t_i/T_0} + \ldots \right)
\]

\[
\sigma^2_{zS+N} = \sigma^2_{yS+N} \left( 1 - e^{-2T/T_0} \right) + \sigma^2_{yN} e^{-2T/T_0}
\]

(4-2)

When \( n_i T_0 \gg 1 \) there are many independent samples being summed and (4-1) and (4-2) are adequate to characterize the gaussian output probability distribution of the integrator.

Note that for signals of long duration \( (T \gg T_0) \), (4-1)
and (4-2) become

\[ \bar{z}_{S+N} = n_1 T_0 \bar{v}_{S+N} \]  \hspace{1cm} (4-3)

and

\[ \sigma^2_{z_{S+N}} = \frac{n_1 T_0}{2} \sigma^2_{v_{S+N}} \]  \hspace{1cm} (4-4)

The ratio of \( \bar{z}/\sigma_z \) is the same as for a linear integrator which sums for a time \( 2T_0 \), the "equivalent integration time". However, for finite \( T \) one needs to work directly with (4-1) and (4-2) in order to calculate the mean and the dispersion of the output of the exponential integrator.

Equations (4-3) and (4-4) also apply to noise if its duration is long compared to \( T_0 \). Thus the mean of the output distribution for noise input is

\[ \bar{z}_N = n_1 T_0 \bar{v}_N \]  \hspace{1cm} (4-5)

and the standard deviation of the noise alone distribution is

\[ \sigma_{z_N} = \sqrt{\frac{n_1 T_0}{2}} \sigma_{v_N} \]  \hspace{1cm} (4-6)

If a low false alarm rate is required, the alarm threshold must be several \( (K\sigma_z) \) standard deviations higher than \( \bar{z}_N \). Thus

\[ z_{\text{threshold}} = n_1 T_0 \bar{v}_N + K_N \sqrt{\frac{n_1 T_0}{2}} \sigma_{v_N} \]  \hspace{1cm} (4-7)

Finally if a signal is to be detected 50\% of the time, its mean must equal the threshold.

\[ z_{\text{threshold}} = \bar{z}_{S+N} = n_1 T_0 \left[ \frac{\bar{v}_{S+N}(1-e^{-T/T_0} + \bar{v}_N e^{-T/T_0})}{\bar{v}_{S+N}(1-e^{-T/T_0}) + \bar{v}_N e^{-T/T_0}} \right] \]  \hspace{1cm} (4-8)
Equating (4-7) and (4-8), we obtain 50 °/o signal detection if

\[(\bar{Y}_{S+N})_{50} \, \text{o/o} = \bar{Y}_N + K_N \frac{1}{\sqrt{2n_1 T_0}} \cdot \frac{1}{(1-\frac{e^{-T/T_0}}{T_0})} \cdot \sigma_{Y_N} \]

(4-9)

The type of detector determines \(\bar{Y}_N\) and \(\sigma_{Y_N}\) (see Figs. 1 and 2) and also the manner in which \(\bar{Y}_{S+N}\) and \(\sigma_{Y_{S+N}}\) vary with input signal to noise power. \(K_N\) is determined by the required false alarm rate, and \(n_1\) by the shape of the noise band (Table I in original report). Thus the value of \(\bar{Y}_{S+N}\) needed to attain 50 °/o detection can be calculated and then the corresponding \(\sigma_{Y_{S+N}}\) and input signal to noise power can be read off Figs. 1 and 2.

Since \((\bar{Y}_{S+N})_{50} \, \text{o/o}\) in equ. (4-9) does not involve knowledge of \(\sigma_{Y_{S+N}}\), it can be easily calculated. If however, one wishes a higher signal detectability one needs also to know \(\sigma_{Y_{S+N}}\). For example for 84 °/o signal detection one needs a value of \(\bar{Y}_{S+N}\) which is one unit of \(\sigma_{Y}\) by which \(\bar{Y}_{S+N}\) exceeds the threshold, then instead of equ. (4-9) we have

\[
\bar{Y}_{S+N} = \bar{Y}_N + \sqrt{2n_1 T_0} \cdot \frac{1}{(1-\frac{e^{-T/T_0}}{T_0})} \cdot \left[ K_N \sigma_{Y_N} + K_S \left\{ \frac{2}{\sigma_{Y_{S+N}}} \left(1-\frac{e^{-2T/T_0}}{T_0}\right) + \sigma_{Y_N}^2 e^{-2T/T_0} \right\}^{1/2} \right]^{1/2}
\]

(4-10)
In (4-10), \( \bar{y}_{S+N} \) and \( \sigma_{y_{S+N}} \) are both unknown. A quickly converging solution can be made however by first using (4-9) to get the value of \( \bar{y}_{S+N} \) needed for 50 \( \% \) detection and then using the value of \( \sigma_{y_{S+N}} \) thus determined in equ. (4-10), to give a second value of \( \bar{y}_{S+N} \) and also \( \sigma_{y_{S+N}} \). This last value of \( \sigma_{y_{S+N}} \) can again be substituted into (4-10) etc. The relative insensitivity of (4-10) to the value of \( \sigma_{y_{S+N}} \) makes this process converge rapidly. It is in fact practically necessary due to the lack of a simple analytical relationship between \( \sigma_{y_{S+N}} \) and \( \bar{y}_{S+N} \).

V. Correction for Finite Sample Size

Thus far we have assumed that the probability distribution for \( z \) is a gaussian error function,

\[
f(z) = \frac{1}{\sqrt{2\pi} \sigma_z} e^{-\left(\frac{z-\bar{y}}{\sigma_z}\right)^2 / 2\sigma_z^2}
\]

where \( \bar{z} = n_{1T_0} \bar{y} \) and \( \sigma_z = \sqrt{\frac{n_{1T_0}}{2}} \sigma_z \) \( \text{(5-2)} \)

We have noted that the ratio \( \bar{z}/\sigma_z \) is the same as if one had a simple linear integrator which sums for a time \( 2T_0 \). Thus, \( n = 2n_{1T_0} \) is a measure of the number of samples being summed, and the distribution \( f(z) \) approaches the exact gaussian form only when \( n \) is very large. When one is concerned with \( f(z) \) near its maximum the deviations from the gaussian form are not important, but when one calculates low false alarm rates the deviations of \( f(z) \) from a gaussian can become quite large.
Consider a linear integrator which sums n independent samples. The Edgeworth series* gives, through order $1/n$,

$$f(x) = \varphi(x) - \frac{1}{3! \sqrt{n}} (\gamma_1)^3 \frac{d^3}{dx^3} \varphi(x) + \frac{1}{4! \sqrt{n}} (\gamma_2)^2 \frac{d^4}{dx^4} \varphi(x)$$

$$+ \frac{10}{6! \sqrt{n}} (\gamma_1)^2 \frac{d^6}{dx^6} \varphi(x) + \ldots \quad (5-3)$$

where $x = \frac{Z - \overline{Z}}{\sqrt{n} \sigma_y}$

$$\sigma_z = \sqrt{n} \sigma_y \quad (5-4)$$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad (5-5)$$

Using Hermits polynomials,

$$\frac{d^3}{dx^3} \varphi(x) = \frac{d}{dx} \left[ \frac{1}{\sqrt{2\pi}} e^{-x^2/2} (x^2 - 1) \right] \quad (5-6)$$

$$\frac{d^4}{dx^4} \varphi(x) = \frac{d}{dx} \left[ \frac{1}{\sqrt{2\pi}} e^{-x^2/2} (x^3 - 3x) \right] \quad (5-7)$$

$$\frac{d^6}{dx^6} \varphi(x) = \frac{d}{dx} \left[ \frac{1}{\sqrt{2\pi}} e^{-x^2/2} (x^5 - 10x^3 + 15x) \right] \quad (5-8)$$

the "skewness", of the y distribution,

$$(\gamma_1)_y = \left( \frac{d^3}{dx^3} \right)_y = \frac{a_3 - 3m a_2 + 2m^2}{\sigma_y^3} \quad (5-9)$$

the "Coefficient of excess", of the y distribution,

$$(\gamma_2)_y = \left( \frac{d^4}{dx^4} - 3 \right)_y = \frac{a_4 - 2m a_3 + 6m^2 a_2 - 3m^4}{\sigma_y^4} - 3 \quad (5-10)$$

where the \( a_i \)'s are the moments of the \( y \) distribution \((m = a_1)\)
and are given by Rice for both the linear and square law
detector.

The false alarm probability involves only the area under
\( f(x) \) above some threshold \( X_T \) for the case that the input dis-
tribution in \( y \) is for noise alone.

\[
F.A. \text{ prob.} = \int_{X_T}^{\infty} f(x) \, dx \quad (5-11)
\]

The first term in the Edgeworth Series is just the ordinary
error function and the other terms can be simply integrated,

\[
F.A. \text{ Prob.} = \int_{X_T}^{\infty} \psi(x) dx + \frac{1}{\sqrt{2\pi}} e^{-x_T^2/2} \left[ \frac{(\sigma_1)_y(x_T^2-1)}{3! \sqrt{n}} + \frac{(\sigma_2)_y(x_T^3-3x_T)}{4! \sqrt{n}} \right] + \ldots \quad (5-13)
\]

Using Rice's equation for the moments, of the envelope detector,

\[
(\sigma_1)_y = 0.63
\]
\[
(\sigma_2)_y = 0.25
\]

For the Square Law Detector,

\[
(\sigma_1)_y = 2
\]
\[
(\sigma_2)_y = 12
\]

The two following graphs, Figs. 3 and 4, show how the
finite sample size progressively increases the false alarm
probability for large values of \( \sigma_x \). Of course the above
Figure 3. False alarm probability from integrated output of envelope or linear detector.

Edgeworth Series thru terms of order $n^{-1}$

$\gamma_1 = 0.63$

$\gamma_2 = 0.25$

$n =$ NO. of samples integrated
**Figure 4. False Alarm Probability**

From integrated output of square law detector

Edgeworth Series thru terms of order $n^{-1}$

$\gamma_1 = 2$
$\gamma_2 = 12$

$n =$ No. of samples integrated

False alarm probability

Threshold
discussion applies to the simple linear integrator, and we are basically concerned with an exponential integrator. However the effective number of samples being processed by an exponential integrator of time constant $T_0$ is $n = 2n_i T_0$. This estimate of $n$ will identify with adequate accuracy that curve which is most appropriate for calculating the false alarm probability.

The Rayleigh Distribution $R \sim e^{-R^2/2}$, from the envelope detector is more nearly gaussian in shape than is the exponential distribution from the square law detector. Thus it is reasonable to expect (as the graphs demonstrate) that the summed output of the envelope detector approaches the gaussian form much more rapidly than for the square law detector.

VI. The Rate of Output of Independent Samples From The Exponential Integrator

Using the method of section III we calculate that the exponential integrator of time constant $T_0$ will put out $n_1 = \frac{1}{2T_0}$ independent samples per second. (See Table 1). This method is concerned with the further summing of the integrator output samples.

Using the method of section IV we find by considering the operation of the exponential integrator on its input samples, that it is equivalent to a linear integrator which sums for a time $2T_0$ and then "clears" itself and starts over.
Such an equivalent integrator will certainly put out \( n_1 = \frac{1}{2T_0} \) independent samples per second.

Thus the two different approaches - one considering output samples and the other considering the input samples - both give the same result, \( n_1 = \frac{1}{2T_0} \).

We shall assume therefore without direct proof that the rate \( n_1 = \frac{1}{2T_0} \) also applies to the operation of the exponential integrator as far as a threshold detector is concerned. For example, if 1% of the area of the "noise-alone" probability distribution lies above the threshold, the false alarm rate = \( 10^{-2} \times \frac{1}{2T_0} \).

Similarly, if the signal has a duration which is several times \( 2T_0 \), it will have more than one chance of exceeding the threshold. The probability that at least one of \( n \) independent samples of signal plus noise will exceed the threshold is

\[
P = 1 - (1 - p)^n
\]

where \( p \) = probability that a single signal plus noise sample will exceed the threshold, and \((1 - p)^n\) is the probability that none of \( n \) independent samples (\( n = \text{signal duration}/2T_0 \)) will exceed the threshold.

VII. Measurement of Signal-to-Noise Improvement

A practical measure of the separation of signal and noise is the detectability \( D \)

\[
D = \frac{Z_{S+N} - Z_N}{\sigma_{Z_N}}
\]

(7-1)
This is particularly useful when the signal detection probability required is 0.5 in which case the threshold is set in the middle of the signal plus noise distribution so the dispersion of signal plus noise does not enter the calculations.

VIII. The Single Tuned Filter and the Equivalent Square Pass Filter

The equivalent square pass filter is defined as that filter which transmitts the same total noise power as the single tuned filter, but which has a constant transmission over its pass band equal to the transmission at the center frequency of the tuned circuit. (Assume white noise at the output.)
Using the approximate formula for the power spectrum at the output of the tuned filter, the total noise power transmitted is,

$$\sigma^2 = \int_{-\infty}^{\infty} G_0(f) \, df = \int_{-\infty}^{\infty} \frac{1}{1 + 4Q^2 \left(\frac{f-f_0}{f_0}\right)^2} \, df = \frac{\pi}{2} \frac{f_0}{Q} = \frac{\pi}{2} \Delta f$$

(8-1)

if the filter transmits one unit of noise power per cycle per second at the center frequency, $f_0$. Thus a square pass filter, approximately $\frac{\pi}{2} \Delta f$ cps wide will transmit the same total noise power as the tuned circuit.

The tuned filter and its equivalent square pass filter have approximately the same efficiency in detecting signals, considering the whole range, $f_0 \pm \frac{\pi \Delta f}{4}$.

For signals near $f_0$ the tuned filter is better because it puts out $\pi \Delta f$ independent samples per second compared to only $\frac{\pi}{2} \Delta f$ independent samples per second from the equivalent square filter. For signals near the edge of the band, the tuned filter is inferior since it attenuates the signal power by a factor of two or so, a handicap not completely overcome by its higher rate of output of independent samples. (For fixed integration time the rate of output of independent samples, $n_1$, improves the signal to noise by approximately $\sqrt{n_1}$.

The detail comparison of the tuned circuit and its equivalent square filter depends upon the nature of the detector, and also the region of input signal to noise power.
involved. Equ. (7-1) is useful however in determining the value of $\sigma^2$ needed in the calculation of input signal to noise power for the single tuned filter.

IX. An Example Related to Hearing

Consider a signal of duration $T = 1.0$ second, a square pass band of 60 cps, (thus $n_1 = 60$/second) a square law detector and an exponential integrator of time constant $T_0 = 0.5$ second.

Require that the false alarm rate be one in 200 seconds and since the filter reaches an independent value at a rate of $2T_0$/second the false alarm probability is $5 \times 10^{-3}$. Since the number of samples being processed is $2n_1 T_0 = 60$ we use the corresponding curve in Figure 4 and obtain the result that the threshold must be $2.9$ units of $\sigma_{Z_N}$ above the mean value of the noise $\overline{Z_N}$. From (4-9),

$$\overline{Y_{S+N}}_{50\%} = \overline{Y_N} + \frac{2.9}{\sqrt{60}} \frac{1}{(1-R^2)} \sigma_{Y_N} = 2\sigma^2(1 + \cdot42)$$

since from Fig. 2 both $\overline{Y_N}$ and $\sigma_{Y_N} = 2\sigma^2$. From Fig. 2 we find that $p^2/2\sigma^2 = \text{signal power/noise power at the output of the filter} = 0.42^2$.

Psycho acoustic measurements indicate that the human ear plus brain achieves the above specified detection state when the signal power has about .7 to 1.0 times the noise power in the critical band (which is approximately 60 cps wide at frequencies of a few hundred cps.). Thus it appears
that the filter-integrator system analyzed above should detect signals about 3 db below the signal level needed for the human ear plus brain.
$$p(z) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(z - \bar{z})^2}{2\sigma^2}}$$

Signal detection probability:

$$\text{PROB} = \int_{-\infty}^{z_T} p(z) \, dz$$