THE BIRTH AND DEATH OF TRACKS

Report R-49
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THE BIRTH AND DEATH OF TRACKS

Prepared by

Nelson Wax
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I. Introduction

In any automatic tracking (and control) system, the size of the computer's memory is of critical importance. The memory should be at least large enough to take care of the maximum number of actual tracks* which the system is to handle, but this number is certainly a lower bound, since noise will enter into the system, and some, perhaps many, noise tracks will be carried in the memory. In actual systems the number of noise reports per scan is often of the same order of magnitude, and frequently exceeds, the number of signal reports per scan. Thus the maximum number of real tracks may be a very crude estimate of the total number. A detailed examination of the processes of track formation (birth), track maintenance (life), and track scratching (death) appears to be necessary in order to obtain a reliable estimate of the total number of tracks, and thus of the size of the memory.

The purpose of this report is to present the results of a study of birth and death processes of tracks. It is shown that under certain conditions the noise track population reaches an equilibrium value, and that the equilibrium value is relatively small. It follows from these results that an automatic tracking system need not be saturated by noise, and that reasonable memory capacities can be sufficient for effective tracking. These statements will be made precise in later sections of this report.

Most of the assumptions which are made throughout the report will be listed and discussed in the next section. Sec. III contains a treatment

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*i.e.* tracks of genuine aircraft; the terms "real tracks" and "signal tracks" will also be used with the same meaning.
of birth processes; death processes are treated in Sec. IV. Some birth and death processes are studied in Sec. V, and a particular birth and death equation is singled out for study in Sec. VI. Signal to noise improvement is investigated in Sec. VII and the entire theory is re-cast in Sec. VIII. A summary of results is given in Sec. IX.

Frequent reference will be made to Nordsieck's proposal for an automatic tracking and control system. It was a consideration of his proposal which led to the present work.

II. General Considerations

We consider an automatic tracking system that consists of a network of radars, data processors at the radars, a set of communication links which transmit digitalized data obtained from the data processors, and an automatic digital tracking computer. The system is to track aircraft whose speeds are in the 100-600 mph range.

The essential purpose of the entire system is to furnish accurate up-to-date information concerning the tracks of real aircraft. A necessary consequence of this statement is that with limited time and computing facilities available the system must discriminate effectively between real tracks and noise tracks.

a) The Bin

The range, azimuth and height resolution of the radars set limits to the position in space to which an aircraft can be assigned. Further, the radars may have M.T.I., and hence limits to the velocity assignments may exist. The "region of uncertainty" in phase space will be termed a "radar bin". The process of digitalizing the radar data effectively
quantizes the phase space under surveillance. Whatever the radar, or quantizing bin sizes are, the computer will work with some characteristic region of phase space, termed a "bin", here assumed to be of constant size.

b) Report Rate

The functions of the radars, the data processors, and the communication links may be summarized by stating that the presence of an object in any bin is properly noted and reported to the computer, together with any other information deemed pertinent, such as the strength of the return. It is convenient, but not essential, to suppose that the reports arrive at a constant rate; thus in unit time, termed a "scan time", or "scan", a constant number of reports, \( N \), will be given the computer. If \( N_T \) be the total number of bins under surveillance, then a further assumption is that \( N \ll N_T \).

As an example, in Nordsieck's proposal the area under surveillance is roughly a square 500 miles on a side, the bin is a square 1 mile on a side and the maximum number of reports per scan is 1000. Thus \( N_T = 25 \times 10^4 \), and \( N = 1000 \).

c) Track Independence

The computer uses a report either by starting a new track, continuing an already existing track, or by ignoring the information. (These functions will be discussed more fully in later sections.) Cases of ambiguity can, and do, arise, namely where a report may be assigned to any one of a number of tracks which cross, or come close to each other. These ambiguities are ignored here. In effect, it is supposed that every track is independent of every other track, or, equivalently, that no track interacts with any other track.
It is true that under some circumstances the interaction of tracks may lead to complicated branching processes which require more memory than the original set of tracks; this would certainly be expected when \( N \) is of the same order as \( N_T \). If, however, the track density is low, and if, when ambiguity arises, a report is assigned to just one of the possible tracks to which it could be assigned*, then it seems likely that the probability of death for these tracks should be increased. If the death rate increases, then the memory requirements decrease. Hence one might expect that track interaction would decrease memory requirements, for the cases treated here.

d) Constant Association Region

Suppose one has a report which appears in a given bin, and which cannot be assigned to any previously existing track. (See Fig. 1) The report is assumed, further, to be of sufficient strength to be considered as initiating a track. (A point to be discussed later.) In the absence of any information concerning the velocity of the aircraft, any report which appears, during the next scan, in any bin within a proper region will be taken as a continuation of the track. The question arises, what is the size of the proper region? In order to get some feeling for the situation we choose some specific numbers, applicable to Nordsieck's report.

If the aircraft flies in a straight line, and has a maximum speed of 600 mph then in 15 seconds (taken as the scan time), the aircraft will have travelled 2.5 miles. The bin is taken as a 1 mile square. Thus, because of quantizing, and digitalizing errors and radar resolution, the next report of the aircraft can be expected to be some 2-\( 1/4 \) bins away, at most,

*This is the proposal made in R-35.
FIGURE 1. A REPRESENTATION OF THE POSSIBLE INITIAL REGION OF ASSOCIATION.

THE AIRCRAFT IS FIRST REPORTED IN THE CENTER BIN. THE NEXT REPORT MAY APPEAR IN ANY BIN WITHIN THE REGION SHOWN.
from the original bin. If one assumes that the minimum speed of the aircraft is, say, 125 mph, then the aircraft will have moved about 0.6 mile in a scan time, and it is possible that the second report will coincide with the first. The proper region is thus a region about as shown in Fig. 1, with the maximum lateral dimensions some 6-8 miles.

An estimate of the velocity is possible after the second report. Thus the area where the third report can be expected to occur is smaller than the area of the first proper region. A still more precise estimate of velocity after the third report leads to a smaller area, etc. The minimum area, after tracking has occurred for many scans, is not, however, one bin size, since the azimuthal resolution of most radars is usually worse than 1 mile at long range, and quantizing and digitalizing errors, as well as (random) accelerations of the aircraft itself limit the precision of the velocity determination. One would expect the minimum area to be 2-6 bin sizes in many cases.

The point of the above discussion is that the area of association is not usually considered to be a constant. In fact, when a report does not come in for an already established track then the association area is increased for the next scan. Hence the area of association increases and decreases.

A major assumption made here is that the area of association is a constant.

e) Constant Probability

The probability of obtaining a report in any given bin whether that of a signal or of noise, is a function of the bin's distance from the radar (if the bin is a region in 3-space), the methods of detection, and
a number of other factors. It is not, in general, a constant, but will be assumed to be one constant for noise, and another constant for a signal.

Assumptions (d) and (e) may be combined under one assumption, namely that the probability of obtaining an association has a constant value for noise, and another constant value for signals. If \( N_N \) is the number of noise reports per scan and \( N_T \) the total number of bins, then \( P_N \), the probability of obtaining a noise report in any bin, is a constant equal to \( N_N/N_T \). Similarly if \( P_S \) is the probability of obtaining a signal report in any bin, and \( N_S \) is the number of signal reports per scan, then \( P_S = N_S/N_T \) for all bins.

f) Two Bias Levels

The radar return of an aircraft is not of constant intensity, but will, because of slight changes in aspect, vary over many \( \text{db} \). Noise, as it appears in the output of the radar receiver, does not have the same probability distribution for the amplitude as does a signal, and hence the differences in probability distribution may be exploited to discriminate between signals and noise. The simplest test, aside from a detectability criterion (i.e. a method for deciding whether a signal is present or absent) is a bias criterion above the detectability criterion. Returns which, having passed the detectability criterion, exceed some critical amplitude, are said to be above the second bias level.

We can introduce the probabilities as follows:

\[ P_N \] = probability that if a noise return has passed the detectability criterion it will be above the second bias level.

\[ \Omega_N \] is the conditional probability of exceeding the second bias level if the report has exceeded the first bias level. \( \Omega_N \) can thus be
considered to be the fraction of noise reports which are above the second bias level.

Similarly, $Q_S$ can be interpreted as the fraction of signal reports which have exceeded the second bias level.

### III. Birth Processes

The initiation of a track is a somewhat arbitrary procedure, since it is a question of definition as to when a track "begins". We shall investigate three somewhat different birth processes in this report. In this section the simplest possible criterion will be used for birth, namely, that an unassociated report constitutes the beginning of a new track; the track appears immediately as a member of the population. In later sections (particularly when discussing the $f$- and $g$-schemes) a more complicated view is adopted. There a track is subject to a period of gestation before emerging as a bona fide member of the track population.

The assumption that tracks are independent [(e) in Sec. II] allows us to treat the noise and the signal tracks separately. We shall therefore drop the subscript notation (until we consider signal to noise improvement) and treat just one class of tracks. The numerical examples, and the discussion, usually will be confined to the noise case, however.

a) Single Bias Level

Using the assumptions made in the previous section, we say that a track is initiated by a report in any bin. If the next report is in an appropriate area, then the track will be continued, etc. Let $\alpha$ be the constant number of bins in which a report can appear that will serve as an area of association.
Let $\nu(n)$ be the average number of tracks carried by the computer at the end of scan $n$, $N$ be the (constant) number of noise reports per scan, and $P$ the (constant) probability of a report appearing in any bin.

Then our birth equation may be written as

$$
\nu(n) = \nu(n-1) + \left[ N - \alpha \rho \nu(n-1) \right], \quad (n \geq 1).
$$

(1)

Which, in words, is that the number of tracks at the end of scan $n$ is equal to the number of tracks at the end of scan $(n-1)$ plus the number of tracks born during the last scan. Those born are those unassociated, namely $N - \alpha \rho \nu(n-1)$.

Let $\beta = \alpha \rho$; $\beta$ is the probability of an association. Then Eq. (1) can be written as

$$
\nu(n) = (1-\beta) \nu(n-1) + N, \quad n \geq 1.
$$

(2)

We can consider that the system starts at scan 0, with $\nu(0) = N$. The general solution of Eq. (2) with $\nu(0) = N$ is given by

$$
\nu(n) = \frac{N}{\beta} \left\{ 1 - (1-\beta)^{n+1} \right\}.
$$

(3)

For $n \to \infty$, $\nu(n) \to \frac{N}{\alpha \frac{N}{N_0}} = \frac{N_0}{\alpha}$. Thus the limiting case implies that the entire region is covered with tracks (since every available region contains a track); under these circumstances our analysis breaks down. This result does have one important consequence, however: it establishes the existence of a finite upper bound for all other cases treated in this report.
b) Two Bias Level

A track will be initiated only when the report is sufficiently strong, i.e., exceeds the second bias level. An association will be made whenever a report comes in, i.e., exceeds the first bias level.

We then have, as our birth equation

\[ \nu(n) = \nu(n-1) + \Omega [N - \beta \nu(n-1)], \quad n \geq 1. \]  

(4)

The solution of Eq. (4) with \( \nu(0) = Q N \) is

\[ \nu(n) = \frac{N}{\beta} \left( 1 - (1 - \beta Q)^{n+1} \right). \]  

(5)

As before \( \nu(n) \rightarrow \frac{N}{\beta} = \frac{N}{\beta} \) as \( n \rightarrow \infty \). Thus the asymptotic value for \( \nu(n) \) is the same for Eq. (3) and Eq. (5), the limits being approached more slowly for Eq. (5), when \( \beta < 1 \), than for Eq. (3).

c) Continuous Birth Process

Let us consider a continuous process as an approximation to the discrete processes considered thus far. We let \( \mu(t) \) be the "number" of tracks at any time \( t \), \( Q \) be the "number" of reports per unit time, and \( c_2 \) the fraction of associated reports per unit time. We then have the analogue of the previous equations in

\[ \frac{d\mu(t)}{dt} = Q - c_2 \mu(t), \]  

(6)

and the initial condition \( \mu(t)|_{t=0} = \mu(0) \).

The solution of Eq. (6) subject to the initial condition is

\[ \mu(t) = \mu(0) e^{-c_2 t} + \frac{c_1}{c_2} \left[ 1 - e^{-c_2 t} \right]. \]  

(7)

The ratio \( \frac{c_1}{c_2} \) plays the same role, and has the same interpretation as \( \frac{N}{\beta} \).
IV. Death Processes

The criterion used for birth, namely the appearance of a sufficiently strong report, is a simple and perhaps even a "natural" one; the death processes considered in this report are analytically more complicated than the birth process. The additional complication is understandable since it is necessary to take into account the "age distribution" of the population in computing the death rate, a complication which does not arise, or at least can be ignored, when discussing birth.

Only a verbal discussion of death will be given in this section. The mathematical details are brought forth more clearly when considering birth and death processes, and this will be done in the next section.

Consider a track which has been carried in the computer for a long time, and which has had an association at every scan since birth. This track would certainly be judged to be a vigorous member of the population whatever criteria one uses. Now suppose that on the next scan the track does not have an association. One would hardly discard the track after one non-association. If, however, a long sequence of non-associations occur, then ultimately the track will be discarded. A question arises as to how long the sequence of non-associations should be, but this is relatively straightforward compared to other questions. Thus if \( m \) is the number of non-associations needed for death of a perfectly healthy track, how many non-associations are needed for a track which is not as vigorous as the first? What permutations of associations and non-associations lead to death, and what criteria are used to determine good choices?

Three different approaches will be presented in later sections. One is that a track can die if and only if it has suffered \( m \) successive non-
associations. Thus the history of the track, prior to the last \( m \) scans is neglected. We shall label this the \( m \)-scheme.

Another formalism for death will be investigated, too. In the later method, suggested by Nordsieck, a figure of merit, called the firmness is kept on each track. A track is scratched if the firmness falls below a given value. This method, called the \( f \)-scheme, has connections with the techniques of sequential analysis, and certain problems of first passage times in one-dimensional random walk theory. These connections will be exploited in a systematic fashion in a third method, the \( g \)-scheme.

V. Birth and Death Processes

The number of tracks at the end of \( r \) scans must be equal to the number at the end of \( r-1 \) scans, plus those that are born, minus those that die during the last scan. The birth process is that given in III (b).

a) The \( m \)-Scheme

We first introduce a notation which corresponds to the age distribution of the population.

Let \( \nu(l; i) \) be the number of tracks at the end of scan \( r \) which have suffered \( i \) successive non-associations. A track dies if \( i = m \).

We thus have

\[
\nu(l) = \sum_{i=0}^{m-1} \nu(l; i).
\]

Now the number of tracks with \( i \) non-associations, at the end of scan \( r \) is equal to that fraction of the number of tracks at the end of scan \( r-1 \) which had \( i-1 \) non-associations and which have again been non-associated. Or

\[
\nu(l; i) = (1-\beta) \nu(l-1; i-1).
\]
Hence

$$\nu(2; m) = (1 - \beta)^m \nu(2 - m; 0). \quad (10)$$

Our birth and death equation can now be written as

$$\nu(2) = \nu(2 - 1) + \Phi [N - \beta \nu(2 - 1)] - (1 - \beta)^m \nu(2 - m; 0). \quad (11)$$

Using Eq. (8), we can write Eq. (11) in the form

$$\sum_{i=0}^{m-1} \nu(2; i) = \Phi N + (1 - \beta \Phi) \sum_{i=0}^{m-1} \nu(2 - 1; i) - (1 - \beta)^m \nu(2 - m; 0) \quad (12)$$

or, using Eq. (9), as

$$\sum_{i=0}^{m-1} \nu(2 - i; 0) (1 - \beta)^i = \Phi N + (1 - \beta \Phi) \sum_{i=0}^{m-1} \nu(2 - i - 1) (1 - \beta)^i - (1 - \beta)^m \nu(2 - m; 0) \quad (13)$$

Let $\nu(2) = \nu(2; 0)$. We can then write Eq. (13) in the form

$$\sum_{i=0}^{m-1} (1 - \beta)^i \nu(2 - i) = \Phi N + (1 - \beta \Phi) \sum_{i=0}^{m-1} (1 - \beta)^i \nu(2 - i - 1) - (1 - \beta)^m \nu(2 - m) \quad (14)$$

Eq. (14) may be written more compactly as

$$\sum_{i=0}^{m} a_i \nu(2 - i) = \Phi N \quad (15)$$

where $a_0 = 1$, $a_i = -\beta (1 - \beta)^i (1 - \Phi)$, $1 \leq i \leq m$.

The homogeneous equation is thus

$$\sum_{i=0}^{m} a_i \nu(2 - i) = 0 \quad (16)$$
and, letting \( u(x-i) = x^{n-i} \), one has the auxiliary equation
\[
x^{m-n} - \beta (1-\alpha) \sum_{i=1}^{m-1} (1-\alpha) x^{m-i} = 0.
\]  

(17)

We note first that \( x = 1 \) is not a root of Eq. (17), for on substituting \( x = 1 \) in the equation one obtains
\[
-\beta (1-\alpha) \left[ \frac{1 - (1-\alpha)}{\beta} \right] = \phi (1-\alpha) (1-\beta) > 0.
\]  

(18)

Hence, if \( x_1, \ldots, x_m \) are the \( m \) roots of Eq. (17), then the general solution of Eq. (15) is given by
\[
u(x) = \frac{\phi N}{\phi + (1-\alpha)(1-\beta)} + \sum_{i=1}^{m} \omega_i x^{-2} (n \geq m),
\]  

(19)

where the coefficients \( \omega_i \) are given by the initial conditions.

One has, using Eq. (19) and the definition for \( v(x) \) that
\[
u(x) = \frac{\phi N \left[ 1 - (1-\alpha) \right]}{\beta \left[ \phi + (1-\alpha)(1-\beta) \right]} + \sum_{i=1}^{m} \xi_i x^{-2} (n \geq m),
\]  

(20)

where \( \xi_i = \frac{1 - (1-\alpha)}{\phi} \omega_i \).

The \( m \) values of \( \xi_i \) are determined from the set of \( m \) linear equations obtained by setting the \( v(x_i), (n = 0, 1, \ldots, m-1) \) given in Eq. (20), to the corresponding values given by the birth process, Eq. (5).

The quantity of greatest interest is the equilibrium population, namely
\[
\lim_{x \to \infty} \nu(x),
\]  

if it exists. The limit is independent of the \( \omega_i \), hence they will not be treated further.

That the limit does exist can be seen by noting that if \( x_1 \) be the largest real root of Eq. (17) then
\[
\sum_{i=1}^{m} \xi_i x^{2} \to \xi_1 x^2 \to 0
\]  

since \( 0 < x_1 < 1 \).
One has, therefore, from Eq. (20), that
\[
\lim_{n \to \infty} \nu(n) = \frac{\beta N \left[ \left( 1 - (1 - \beta)^m \right) \right]}{\beta \left[ \alpha + (1 - \alpha)(1 - \beta)^m \right]}. \tag{21}
\]
If \( \beta \ll 1 \), and if \((1 - \beta)^m(1 - \alpha) \ll 1 \), then one obtains the simple relationship
\[
\lim_{n \to \infty} \nu(n) = \frac{\beta \alpha}{m}. \tag{22}
\]

If one were interested just in keeping the noise track population small then, arguing either from Eq. (22) or directly, one would like \( m \) to be as small as possible. Small values of \( m \) will result in a high death rate for actual tracks, however; a rational choice of \( m \) should depend, therefore, on the statistics of real as well as noise tracks.

A variety of criteria may be used for choosing \( m \). We shall consider one in this section, a criterion adapted from the Neyman-Pearson methods in statistics.

It may be well to digress for a moment and point out that what is desired is an efficient statistical test which will allow one to decide that a track is signal or noise, early in the track's history, which will use additional reports in a re-examination of the original decision, and which will permit the deletion of a track quickly, at the end of the track's life-span.

The methods commonly used in statistics have not been designed with the above ends in view. A decision is reached concerning the presence or absence of a signal either after a fixed number of observations (Neymann-Pearson test) or after a run of varying length (Sequential test). In either case, however, additional observations are not used, once a decision is made.
The adaptation of the Neymann-Pearson methods follows.

Let $H_0$ be the hypothesis that a track is noise, and $H_1$, the alternative hypothesis that it is signal. Let $\mu_1$ be the probability of committing an error of the first kind (false alarm) by accepting $H_1$ when $H_0$ is in fact true, and let $\mu_2$ be the probability of an error of the second kind (miss) which occurs by accepting $H_0$ when $H_1$ is true.

**Determination of $m$**

The probability that a signal track is born, and then dies after the next $m$ time units, is given by $Q_s P_s (1-\beta_s)^m$. A simple way of determining $m$ is to let $m$ be the smallest integer, $m_0$, which satisfies the condition

$$Q_s P_s (1-\beta_s)^{m_0} \leq \mu_2. \quad (23)$$

The false alarm probability will be given by

$$\mu_1 = Q_s P_s \left[ 1 - (1-\beta_s)^{m_0} \right]. \quad (24)$$

A similar method for choosing $m$ could be used, by fixing $\mu_1; \mu_2$ would then be determined, once $m_0$ was found.

Other criteria, which depend on the long term behavior of a track, rather than the initial behavior, as given here, can be used, but are somewhat more complicated.

**b) The $f$-Scheme**

At first we shall discuss a particular method in detail, namely the one suggested by Nordsieck in R-35. The method is generalized, and a rationale given for it in the next section; the generalization is labelled the $g$-scheme.

The rules for the $f$-scheme are as follows:
When a report first comes in, which is not associated, and which exceeds the second bias level, then it is assigned a firmness 2. Every time a track is non-associated, the firmness is decreased by 1, until, at firmness 0 the track is scratched. Every time a track is associated the firmness, \( f \), is increased by 2 until either the firmness is 6 or 7. If \( f = 6 \), then a succeeding association increases the firmness by 1, if \( f = 7 \) a succeeding association does not alter the firmness.

It is clear that these rules may be interpreted as those for an unequal step one dimensional random walk with one absorbing and one reflecting barrier.

Let \( \nu(n; f) \) be the number of tracks at the end of scan \( n \) with firmness \( f \). We have, using the rules for the firmness, that

\[
\nu(n; f) = \nu(n-1; f-2) \beta + (1-\beta) \nu(n-1; f+1), \quad 3 \leq f \leq 6
\]  \tag{25}

\[
\nu(n; 2) = \nu(n-1; 3) (1-\beta) + \Phi \left[ N - \beta \nu(n-1) \right]
\]  \tag{26}

\[
\nu(n; 1) = (1-\beta) \nu(n-1; 2)
\]  \tag{27}

\[
\nu(n; 0) = (1-\beta) \nu(n-1; 1)
\]  \tag{28}
Also, by definition
\[ \nu(n) = \sum_{f=1}^{7} \nu(n; f). \] (30)

The system of equations (25-30) represents the birth and death process for the f-scheme. The system may be reduced to a single birth and death equation by summing on f. One obtains
\[ \nu(n) = \Phi [N - \beta \nu(n-1)] + \nu(n-1) - (1-\beta) \nu(n-2). \] (31)

The first term in the right hand member of Eq. (30) represents the tracks "born", the second term those that continue to live, and the last, those that die.

Eq. (31) may be written more compactly as
\[ \nu(n) = \Phi [N - \beta \nu(n-1)] + \Phi N - (1-\beta) \nu(n-1). \] (32)

The system of equations (25-30), or Eq. (32), together with the initial conditions, is a complete description of the birth and death process.

Before discussing the solutions, let us introduce the reduced variables
\[ n(r; f) = \nu(r; f)/N, \] i.e. \( m(r; f) \) as the number of tracks with firmness f at the end of scan r, per unit of reports in a scan.

Equations (25-30) become
\[ m(n; f) = \beta m(n-1; f-2) + (1-\beta) m(n-1; f+1), \quad \text{for } 3 \leq f \leq 6, \] (33)
\[ m(n; 2) = (1-\beta) m(n-1; 3) + \Phi [1-\beta] m(n-1). \] (34)

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Similarly, Eq. (32) becomes
\[ m(n) = (1 - \beta \Phi) m(n-1) + \Phi - (1 - \beta) m(n-1, 1). \]  
Eq. (39)

The initial conditions may be written as
\[ m(0, 2) = \Phi, \quad m(0, f) = 0 \quad f \geq 2. \]  
Eq. (40)

The complete solution of the system (33-38), or of Eq. (39) subject to the initial conditions (40) is difficult to obtain analytically. Numerical solutions have been obtained using the University of Illinois Digital Computer for \( \Phi = \frac{1}{2} \) and various values of \( \beta \); the solutions are plotted in Fig. 2.

The equilibrium or asymptotic values may be obtained analytically as follows. If equilibrium values exist, then for sufficiently large \( t, m(n, f) \)
FIG. 2 THE TOTAL NUMBERS OF TRACKS (INREDUCED UNITS), $n(r)$ VERSUS THE NUMBER OF SCANS, FOR VARIOUS VALUES OF $\beta$, WHEN $Q = 1/2$
is independent of $r$. Let

$$\lim_{z \to \alpha} \frac{m(z; \beta)}{z} = u_\alpha, \quad \sum_{\alpha=1}^{7} u_\alpha = v.$$  \hfill (4.4.1)

The $u_\alpha$ and hence $v$ are given as the solutions of the system of algebraic equations

$$u_1 = (\beta - 3) u_2,$$

$$u_2 = (\beta - 1) u_3 + \phi [1 - \beta v],$$

$$u_3 = \beta u_4 - 2 + (\beta - 1) u_5, \quad 3 \leq \alpha \leq 6 \hfill (4.4.2)

u_7 = \beta \frac{2}{\alpha} u_\alpha.$$ \hfill (4.4.3)

The system (4.4.2-4.4.5) is the limiting form of the system (3.3.38) when $z \to \alpha$ if the limits $u_\alpha (\alpha = 1, \ldots, 7)$ exist.

The $u_\alpha$ may be obtained using determinants. The details, while laborious, are straightforward, and will not be given here. The results are

$$u_\alpha = \frac{\Delta \phi / D}{(\beta - 1 \ldots 7)} \hfill (4.4.6)

where \quad \Delta_1 = \frac{(\beta - \beta)^2}{\beta}. \hfill (4.4.7)
\[ \Delta_2 = \frac{(1-\beta)^5}{\beta}, \]  

(48)

\[ \Delta_3 = (2-\beta)(1-\beta)^4, \]  

(49)

\[ \Delta_4 = (1-\beta)^3 + \beta (1-\beta)^4, \]  

(50)

\[ \Delta_5 = \beta (1-\beta)^2 (3-2\beta), \]  

(51)

\[ \Delta_6 = \beta (1-\beta)^2 + \beta^2 (1-\beta)(2-\beta)^2, \]  

(52)

\[ \Delta_7 = \beta^3 (2-\beta)^2 + 2\beta^2 (1-\beta)(2-\beta), \]  

(53)

and

\[ D = (1-\beta)^5 \left[ \frac{1+\beta}{\beta} + (1-\beta)^2 \frac{1-\beta(1-\beta)^2}{\beta} \right] + \beta (1-\beta) \frac{(1-\beta)^2}{\beta} - \beta (2-\beta)(1-\beta) \frac{1-\beta(1-\beta)^2}{\beta}. \]  

(54)
Two values of \( u \) have been computed, using the above equations (\( \alpha = \frac{1}{2} \) and \( \beta = \frac{1}{8}, \gamma = \frac{1}{8} \)), and found to agree with the corresponding asymptotic values, shown in Fig. 1, to within 5%.

Note that the \( u_0 \) are independent, as they should be, of the initial conditions imposed on the system; whatever the way in which the steady state values are approached, the values are given by Eq. (46-54).

A simple approximate expression for the total equilibrium track population may be obtained readily when \( \beta \ll 1 \). First, it can be noted that \( \Delta_1 \) and \( \Delta_2 \) are much larger than the remaining \( \Delta \)'s when \( \beta \ll 1 \). Secondly, the leading term in \( D \) is \( (\beta - \beta) \frac{1+\beta^2}{\beta^2} \). Hence

\[
\begin{align*}
\frac{\Delta_1 + \Delta_2}{D} &= \frac{(\beta - \beta) (2 - \beta)}{(\beta - \beta) 1 + \beta^2} = \frac{(2 - \beta) \Delta}{1 + \beta^2} \\
\Rightarrow \quad \frac{\Delta_1 + \Delta_2}{D} &= \frac{(2 - \beta) \Delta}{1 + \beta^2} \\
\text{or} \quad \frac{\Delta_1 + \Delta_2}{D} &= 2 \Delta
\end{align*}
\]

It is noteworthy but not surprising that when \( \beta \ll 1 \) the equilibrium track population is never more than a few times the number of reports per scan, whether one uses the m-scheme (Eq. 23) or the f-scheme (Eq. 57), and generally is considerably less than this number, when \( \Delta \) is small. For when \( \beta \) is very small, then tracks which are initiated will be kept only the minimum number of scans: in the m-scheme this is \( m \) (or \( m_0 \)), and in the f-scheme this is 2.
VI. Birth and Death - the g-Scheme

In the previous section the f-scheme was treated without giving any rationale for the method. Indeed, no a priori justification was advanced in R-35 for the procedure; the f-scheme was introduced as a reasonable method of "keeping tabs" on tracks.

The f-scheme is interpretable, however, as a special case of the methods of sequential analysis in statistics, since one has a running count, or index, of one's degree of confidence that the track is an actual track.

The basic notion in sequential analysis is indeed just the one where one continues observations until either there is some certainty that a "signal" is being observed, or that "noise" is under observation. A running index is kept on the sequence of observations. If the index falls below a given value then the sequence is put into the "noise" category, if the index exceeds another given value then the sequence is labelled "signal", and if the index remains intermediate between these two values then one is "uncertain", and continues the observations. (See A. Wald "Sequential Analysis" for a treatment of the theory).

It should be noted that once the index indicates that the sequence is "noise", or "signal", then the test terminates. The techniques may be adapted to our purposes, however, by selecting a portion of a track's development for study. The details are given in the following paragraphs.

The rules defining the f-scheme fixed three parameters of a sequential test: (a) the initial value of f (namely 2), (b) the ratio of the increase in f on association, to the decrease in f on non-association, (2:1), and (c) the maximum value of f, (7). Setting zero as the value for which a track is scratched involves no loss of generality.
A ready generalization of the f-scheme would use the following rules:

1) On initiation, a track is assigned the number $g_0$.

2) On each succeeding association the index $g$ is increased by $a$; succeeding non-associations each decrease the index by 1.

3) A track is scratched when $g = 0$.

4) A track is accepted as an actual track when $g = g_A$.

5) $g$ has a maximum value $g = g_M$.

The numbers $g_0$, $g_A$ and the ratio $a/1$ can be determined from the statistics of the behavior of signals and of noise, as will be done now.

The value of $g_M$ will also be determined.

Let $\mu_1$ and $\mu_2$ (both less than $\frac{1}{2}$) be given, where $\mu_1$ is the false alarm probability, and $\mu_2$ is the miss probability, as before.

Define $A_1$ and $A_2$ by means of the equations

$$A_1 = \frac{\mu_2}{1 - \mu_1}$$

$$A_2 = \frac{1 - \mu_2}{\mu_1}$$

One may determine $g_I$ from the condition that $g_I$ be the smallest integer satisfying

$$g_I = \log \left( \frac{Q_{\text{S}} P_3}{Q_{\text{N}} P_N} \cdot \frac{1}{A_1} \right)$$

Similarly $g_A$ is to be the smallest integer satisfying

$$g_A = \log \frac{A_2}{A_1} = \log \frac{(1 - \mu_1)(1 - \mu_2)}{\mu_1 \mu_2}$$

and $a$ the smallest integer satisfying

$$a = \log \frac{P_0}{P_0}$$

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In principle the maximum value of \( g \), \( g_M \), should be \( \infty \), or at least a very large number, since some (very rare) tracks may continue to form associations for long runs. One would like to keep the number of digits assigned to \( g \) small, however, in order to reduce the size of the computer. Again one is faced with some arbitrariness in the choice of a criterion.

The Neyman-Pearson method may be used to determine \( g_M \) in much the same way as the value of \( m_0 \) was obtained from inequality (23). Thus a real track having a maximum \( g \) value, \( g_M \), will be considered a tentative track if at least \( g_M - g_A + 1 \) successive non-associations occur. If \( \theta \) is assigned, then one may determine \( g_M \) from the condition that \( g_M \) be the smallest integer satisfying

\[
(-\beta_{\theta}) \sum_{g_A}^{g_M - g_A + 1} = \delta.
\]

(63)

Returning to a description of the \( g \)-scheme, we may write the system of difference equations

\[
\nu(n; g) = \beta \nu(n-1; g-a) + (-\beta) \nu(n-1; g+1)
\]

if \( g \neq g_A \), and \( a + 1 \leq g \leq g_M - 1 \),

\[
\nu(n; g_A) = \beta \nu(n-1; g_A-a) + (-\beta) \nu(n-1; g_A+1) + \theta \left[ N - \beta \nu(n-1) \right]
\]

(65)

if \( g_A > a \) if \( g_A = a \) then

\[
\nu(n; g_A) = (-\beta) \nu(n-1; g_A+1) + \theta \left[ N - \beta \nu(n-1) \right]
\]

(66)

also

\[
\nu(n; g_M) = \beta \sum_{g_A}^{g_M} \nu(n-1; g)
\]

(67)
Equations (68-69) may be written in reduced form as

\[ m(n; g) = \beta m(n-1; g-a) + (1-\beta)m(n-1; g+1) \]
\[ g = g = a + 1, a + 2, \ldots, a + m - 1 \]

\[ m(n; g_x) = \beta m(n-1; g_x - a_x) + (1-\beta)m(n-1; g_x + 1) + g \]
\[ g_x > a_x \]

If \( g_x = a \) then

\[ m(n; g_x) = (-\beta)m(n-1; g_x + 1) + \frac{g}{\beta}m(n-1) \]
\[ m(n; g_m) = \beta \sum_{g = g_m - a}^g m(n-1; g) \]

If \( g_x = a \) then

\[ m(n; g) = (-\beta)m(n-1; g+1) \]
\[ g = a \]

and, in general,

\[ m(n) = \sum_{g = 1}^{g_m} m(n; g) \]
A particular example of the above system has already been treated in the f-scheme.

VII. Signal to Noise Improvement

In the previous sections the major emphasis has been put on the total track population, mainly with the purpose of determining the memory requirements of the tracking computer. Another question of interest is the effectiveness with which the various birth and death processes discriminate between signal tracks, and noise tracks. In order to judge the effectiveness of the discrimination it is convenient to introduce a measure of performance, I, the improvement of signal to noise ratio for the track populations.

Let \( \nu_2^\prime (n) \) be the number of signal tracks which are considered to be real tracks after \( n \) scans, \( \nu_1^\prime (n) \) the number of noise tracks considered to be real tracks after \( n \) scans, \( N_2 \) the number of signal reports per scan, and \( N_1 \) the number of noise reports per scan. (The subscript 2 refers to a signal, 1 to noise.)

One defines the signal to noise improvement, I, by

\[
I = \lim_{n \to \infty} \frac{\nu_2^\prime (n)/\nu_1^\prime (n)}{N_2/N_1} - 1
\]

(77)

Note that in the definition of \( \nu_2^\prime (n) \), \( \nu_1^\prime (n) \), and hence of I, one considers only tracks which are accepted as being real.

a) m-Scheme

Here all tracks which have not suffered m successive non-associations are considered real, namely the total track population. One has, using Eq. (20), that
Eq. (78) becomes, when equilibrium populations exist,

\[ I = \frac{\phi_2}{\beta_2 [\phi_2 + (1 - \phi_2)(1 - \beta_2)]} + \sum_{i=1}^{m} 2 \delta_{0} x_{i}^{2} - 1. \]  

\[ \left( \frac{\phi_1}{\phi_2 + (1 - \phi_2)(1 - \beta_2)} \right) - 1. \]  

For the case where \( \beta_1 < 1, \beta_2 < 1 \) one has

\[ I = \frac{\phi_2}{\phi_1} \left( 1 - m \beta_2 (1 - \phi_2) \right) - 1. \]  

Using the numbers \( \phi_2 = \frac{1}{2}, \phi_1 = \frac{1}{6}, \beta_1 = \frac{1}{6}, \beta_2 = \frac{1}{6} \), \( m = 3 \), one finds that \( I = 3.2 \).

b) f-Scheme

The acceptance level has not been given in R-35 for this scheme.

If we choose \( f = 5 \) as the acceptance level (all \( f \) for which \( \frac{\phi_2}{\phi_1} \leq f \) ) for which tracks will be accepted as real tracks, then, using Eqs. (50-54), and changing the notation slightly one has

\[ I = \frac{D_1}{D_2} \left( \sum_{i=5}^{7} A_2 i / \sum_{i=5}^{7} D_1 i \right) - 1. \]  

When \( \beta_2 < 1, \beta_1 < 1 \)  Eq. (61) becomes

\[ I = \frac{\phi_2}{\phi_1} \left( 1 - \beta_2 \right)^{3} - 1. \]  

Using the same numbers as above, \( \phi_2 = \frac{1}{2}, \phi_1 = \frac{1}{6}, \beta_2 = \frac{1}{6}, \beta_1 = \frac{1}{6} \)  one finds that \( I = 3.6 \).

It is noteworthy that the signal to noise improvement factors are greater than 1 even though the equilibrium populations \( \phi_2 (\infty) \) and \( \phi_1 (\infty) \)
are of the same order of magnitude as the number of signal reports, and noise reports, per unit time, respectively. It is possible, therefore, to effect an improvement in signal to noise without imposing excessive memory requirements.

c) g-Scheme

The acceptance level is at $g_A$. One has, therefore, that

$$I = \lim_{n \to \infty} \left( \frac{\sum_{g=g_A}^{g_m} n_k (g, g') / \sum_{g=g_A}^{g_m} n_k (g, g')}{1} \right).$$

VIII. Markov Chain Treatment of Birth and Death

The birth and death process was formulated, in previous sections, in terms of systems of difference equations. In this section we shall describe the birth and death process using the terminology of Markov chains. Aside from the insight which another treatment may provide, the Markov chain development enables one to unify the theory in a general way, and to make approximations readily.

We start with some definitions. A given track at a fixed time may be considered to be in just one of a set of states $S_0, S_1, \ldots, S_m$. The subscript may denote $m$ minus the number of successive non-associations which the track has undergone, at the fixed time, if one is discussing the $m$-scheme, the various values of $f$, if the $f$-scheme is being treated, or the corresponding $g$-values, if the $g$-scheme is considered.

We consider a matrix of transition probabilities $\| P_{ij} \| = P$, where $P_{ij}$ is the probability that if a track is in state $S_i$ at a given scan, it will be in state $S_j$ at the end of the next scan. The matrix $P$ is, for the $m$-scheme, under the assumptions made heretofore
for the f-scheme $\mathcal{P} = \mathcal{P}_f$ is given by

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1-\beta & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1-\beta & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1-\beta & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1-\beta & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1-\beta
\end{bmatrix}
\]

and for the g-scheme one has, labelling $\mathcal{P}$ by $\mathcal{P}_g$

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1-\beta & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1-\beta & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1-\beta & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1-\beta & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1-\beta
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1-\beta & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1-\beta & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1-\beta & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1-\beta & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1-\beta
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1-\beta & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1-\beta & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1-\beta & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1-\beta & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1-\beta
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1-\beta & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1-\beta & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1-\beta & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1-\beta & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1-\beta
\end{bmatrix}
\]
The matrix $P$ is, in every case, one which has just a single element equal to 1 (the remaining elements of the row containing the 1 being necessarily zero). Now for all the $P$ matrices given above the only states which are recurrent are the (single) states where $P^{20} = I$ and the corresponding states are ergodic. Hence $S_0$ is ergodic, in the m as well as in the f- and the g-schemes. All other states are transient.

Let us define $\rho_j^{(a)}$ as the probability that a track starting in state $S_j$ will be in state $S_j'$ after exactly $r$ scans.

One has

$$||\rho_j^{(a)}|| = ||\rho_j||^2 = P_{j-j}^2$$

(87)

We are now in a position to formulate birth and death processes in terms of Markov chains.

First, it is to be noted that a track is "born" in state $S_0$ and it will not be scratched unless it gets into state $S_0$. One has, therefore, that the probability of a track born at the end of scan $s$ not being scratched by the end of scan $s$ is given by

$$\rho_j^{(a)} = \sum_{j=1}^{k} \rho_{j,j}^{(a)}$$

where $k$ is the total number of transient states.

Let $N(s)$ be the number of tracks born at the end of scan $s$. Then $N(s)$, the total track population, is given by

$$N(s) = \sum_{j=0}^{\infty} N^{(a)}(j) \left( \sum_{j=0}^{\infty} \rho_{j,j}^{(a)} \right)^n$$

(88)

with $\rho_j^{(a)} = 1 \forall j = 0$, and $\rho_j^{(a)} = 0 \forall j \neq 0$

Now the number of tracks born at the end of scan $i$, $N(i)$ depends linearly on the total track population at the end of the previous scan, hence

$$N(i) = A - B N^{(i-1)}$$

(89)
with \( N(0) = qN \). We need only remark that \( A = qN \), and \( B = \beta qN \) in order to complete the correspondence with previous work.

It follows, from Eq. (89) and Eq. (88) that

\[
\nu(n) = A \sum_{i=1}^{n} \sum_{j=1}^{(i-1)} \rho_{i,j} + B \sum_{i=1}^{n} \sum_{j=0}^{(i-1)} \sum_{k=1}^{j} \rho_{i,k} \rho_{k,j} + \sum_{i=1}^{n} N(0) \rho_{i,0} + \sum_{i=1}^{n} \sum_{j=1}^{i} \rho_{i,j}.
\]

(90)

We can introduce the reduced variable \( \tilde{\nu}(n) = \frac{N(n)}{A} \) and obtain, from Eq. (90), the expression

\[
\tilde{\nu}(n) = \sum_{i=1}^{n} \sum_{j=1}^{(i-1)} \rho_{i,j} - \beta \sum_{i=1}^{n} \sum_{j=0}^{(i-1)} \sum_{k=1}^{j} \rho_{i,k} \rho_{k,j} + \sum_{i=1}^{n} \rho_{i,0}.
\]

(91)

or we can write our fundamental difference equation in the form

\[
\tilde{\nu}(n) = \sum_{i=1}^{n} \sum_{j=1}^{(i-1)} \rho_{i,j} - \beta \sum_{i=1}^{n} \sum_{j=0}^{(i-1)} \rho_{i,j} \rho_{j+1} + \sum_{i=1}^{n} \rho_{i,0}.
\]

(92)

Furthermore we can obtain the \( \tilde{\nu}_j, j \) from the equations

\[
\tilde{\nu}_j = \sum_{i=1}^{n} \rho_{i,j} - \beta \sum_{i=1}^{n} \tilde{\nu}_{i-1} \rho_{i,j} + \tilde{\nu}_0 \rho_{0,j} \quad \forall j \in \mathbb{Z}.
\]

(93)

Thus far all that has been done is to re-cast our previous difference equations in a new form. The usefulness of the method becomes apparent when we attempt to obtain approximate solutions.

We remark, first, that \( \lim_{n \to \infty} \tilde{\nu}_j = 0 \) for all \( j \) which are transient.

Hence, using Eq. (92) we have, for \( n \gg 1 \)

\[
\tilde{\nu}(n) = 1 + \sum_{j=1}^{n} \left( \rho_{0,j} + \rho_{n,j} + \rho_{n-1,j} \right) - \beta \sum_{j=1}^{n} \tilde{\nu}_{j-1} \rho_{n+1,j} + \sum_{j=1}^{n} \rho_{n,0}.
\]

(94)

or

\[
\tilde{\nu}(n) + \beta \tilde{\nu}(n-1) + \delta \tilde{\nu}(n-2) = 0.
\]

(95)
where \( \sigma = \beta \sum_{i=1}^{k} P_{i,j}^{(0)} \), \( \delta = 1 + \frac{2}{k} \left( P_{i,j}^{(0)} + P_{i,j}^{(2)} \right) \).

The solution of Eq. (95) is given by

\[
m(n) = \mu + \sum_{j=1}^{n} \frac{P_{i,j}^{(2)} \delta}{\beta} \rightarrow \mu
\]

where \( \mu = \frac{\delta}{1 + \beta + \delta} \).

Therefore, the equilibrium population is

\[
m(\infty) = \frac{\delta}{1 + \beta + \delta}.
\]

The approximation can be extended to any order; one obtains as the mth approximation to the equilibrium population

\[
m(\infty) = \frac{\mu_{m}}{1 + \beta \mu_{m}}
\]

where \( \mu_{m} = \frac{1}{\delta} \sum_{j=0}^{m-1} P_{i,j}^{(0)} \).

Similarly the detailed age distribution at equilibrium may be obtained, from Eq. (93), using the above results.

One has

\[
m(i,j) = \sum_{i=0}^{m} P_{i,j}^{(i)} - \beta \sum_{i=0}^{m-1} \left( m(i-1) P_{i,j}^{(i-1)} + m(i) P_{i,j}^{(i)} \right)\]

or

\[
m(i,j) = \sum_{i=0}^{m} P_{i,j}^{(i)} - \beta \sum_{i=0}^{m-1} \left( m_{i,j}^{(i-1)} P_{i,j}^{(i-1)} + m_{i,j}^{(i)} P_{i,j}^{(i)} \right), \quad 1 \leq j \leq q
\]

The signal to noise improvement factor, \( I \), may be obtained immediately

\[
I = \left( \sum_{j=0}^{q} m(i,j) \right) / \left( \sum_{j=0}^{q} m(\infty,j) \right) - 1.
\]
A numerical computation of the equilibrium population has been performed using $\mathbf{Q}=\frac{1}{2}, \mathbf{P}=\frac{1}{16}$, various powers of the matrix $\mathbf{P}$, and Eq. (98). It was found that $m_2$ was in error by about 20%, $m_5$ by about 12%, and $m_5$ by about 8%. Thus the approximation is fairly accurate, in this case, even for small $s$.

Summary and Conclusions

Two quantities of interest, the equilibrium population, and the signal to noise improvement factor, have been obtained for some birth and death processes.

It is to be noted that the equilibrium population is of about the same order of magnitude as the number of reports per unit time if the total number of states is less than 10, and if, further, the probability of an association is small. The total memory requirements may therefore be kept to reasonably small values if an effective method is used for eliminating tracks.

In particular, if we assume that 500 real reports arrive per scan, and 500 noise reports arrive per scan (R-35) then the total track population carried by the sorting and tracking computer will be about 1000 tracks, or less, (since $Q_N$ will probably be much less than $\frac{3}{2}$). The estimates given in R-35 appear to be quite reasonable, therefore.

It is to be noted that the use of the firmness as a running index leads to a smaller number of tracks, and to larger signal to noise improvement factors than does the $m$-scheme ($m \geq 2$). The use of a running index, as suggested in R-35, also appears justified.
Footnotes

*(p. 11): "Average" and "expected" are to be interpreted as "mathematical expectation" throughout this paper. The terms "average" and "expected" will be omitted frequently.

*(p. 27): The assumption that $\alpha$ and $\beta$ are both less than 1/2 is made for convenience, and is not necessary. See Wald for further details.

*(p. 34): See Feller for details.
References

1) A. T. Nordsieck, "An Automatic Air Traffic Information and Control System."


