I. Introduction

This report is a summary of various studies which have been made in the last few months on the recognition of certain signals when they are embedded in noise. The main functions of these studies has been to familiarize some members of C.S.L. with the problems of signal detection, and in particular with various criteria used in making a decision that a signal is present or absent.

The recognition of a signal in a background of noise is dependent not only on the nature of the signal and of the noise, but also on the criteria which are used for deciding whether a signal is present, or absent. The signals and the associated noise are defined explicitly in the following paragraphs. Various detectability criteria are discussed in a general way in the remainder of this section; their analytical consequences are exhibited in detail in the next section.

Attention is confined in this report to the recognition of pulsed radar signals in a "white" noise background. The assumption that the noise in the output of the first detector of a radar receiver may be described by a Gaussian probability distribution for the
instantaneous amplitudes, and a constant power spectrum, as a function of frequency, seems to be a reasonable one to make. It has the fur-


the advantage of mathematical convenience.

Any received signal of a pulsed radar may be represented by a set of pulse trains. If the instantaneous amplitude and phase distributions of these pulse trains were known, then the signal would be specified completely. Most methods of detection ignore the instantaneous phases of the pulse trains, and further, respond to some mean value of the absolute values of the instantaneous amplitudes over an interval of time, i.e. they are, in effect, envelope detectors. At least one method, "coherent integration." has been developed which uses the instantaneous phases in the process of detection, with a considerable gain in sensitivity. The major disadvantage of coherent integration seems to be the complexity of the necessary equipment. Despite the feasibility, and advantages of coherent integration, the performance of a conventional radar receiver will be investigated; the receiver is to contain a local oscillator, a first detector (a mixer), an i.f. strip, a second detector, and a video strip. The radar signal will be approximated by a periodic function of time, whose period is the pulse repetition period. The major point to be discussed is, then.

2 Lawson and Uhlenbeck, op. cit.

+ This assumption implies that attention is confined to fixed targets. The detection of moving targets poses more complicated problems than those considered here, since one has to either "search" in frequency or else make arrangements for any frequency to give an indication.
the functions which the second detector and the video strip should perform in order that detection of a signal in noise be "best".

It has long been recognized that the detection of a signal embedded in noise presents a statistical problem, and, therefore, one cannot say with certainty when a signal is present or absent. The best that can be done is to maximize some probability by an appropriate choice of the way in which a decision is made.

The decision that a signal is, or is not present, can be in error in only one of two ways. A signal can be judged to be present, when it is in fact absent, or absent when it is truly present. The first kind of error will be termed a "false alarm", the second, a "miss".

The "best" methods used for deciding on the presence or absence of a signal are critically dependent on the relative importance assigned to "false alarm", or "misses". For example, if one were not penalized for false alarms and penalized heavily for "misses" then the best way of detecting a signal in noise would be to assert, always, that a signal was present. Similarly, if a premium were put on guessing the absence of a signal, and no rewards offered for guessing the presence of a signal, then the most profitable procedure would be to announce on all occasions that no signals were there.

The distinction between "misses" and "false alarms" as sources of error has been recognized for at least 25 years\(^3\), and techniques

for making the "best" choice have been developed under certain conditions. These will be mentioned in the next section. More recently, methods have been devised for treating cases where the relative weights of the "false alarms" and "misses" are fixed, but arbitrary.

It is believed that these newer methods should be investigated in a comprehensive treatment of the problem of detection of signals in noise. They are not considered here.

One final point should be mentioned: The a priori probability for the presence of a signal may be known. If it is, then the "best" criterion for detectability will in general differ from the "best" criterion when it is not known, as is shown below.

II The Detection of Independent Samples of Signal in Noise

A further idealization of the problem of detecting a radar signal embedded in noise is made here. Suppose that the envelope of the periodic function of time, which has been taken as an approximation to the radar signal, is sampled at a set of discrete intervals of time, with the time interval between successive samples chosen as the pulse repetition period, the samples being chosen as the maxima of the envelope during different pulse periods. It is assumed that these n samples are statistically independent, i.e., that there is no correlation between the sample value at one time, and the sample values
at adjacent sampling points, but that all \( n \) samples are representatives of signal plus noise, or all \( n \) are representative of the noise. The \( n \) samples are the experimental data on which the decision that a signal is present, or absent, is based.

As was mentioned in the introduction, a distinction should be drawn between cases where the a priori probability for the presence of a signal is and is not known.

A. Signal probability known — Theory of the Ideal Observer

The signal, thus far, has only been specified to the extent that it is a periodic function of time. Let it be further assumed that \( A \) is the given amplitude of the signal (phase information is discarded in this formulation), and \( p_0 \) be the known probability that the signal is present (for all \( n \) of these samples).

It is not known whether the \( n \) samples are representatives of signal plus noise, or of noise alone; one wishes to find the "best" method for making a decision.

One formulation is given by the Theory of the Ideal Observer\(^+\),

\[^{+}\] Lawson and Uhlenbeck p. 167.

as follows. If the signal is present, then the \( n \) observations are observations of signal plus noise for which the conditional probability density distribution is known, i.e., if \( r_1 \ r_2 \ldots \ r_n \) are the observed amplitudes, then

\[
p(S + N \mid r_1, \ldots, r_n) = p(S + N \mid r_1) \ldots p(S + N \mid r_n)
\]

\[^{++}\] By virtue of the independence assumption.
is the conditional probability density distribution, where \( S + N \) indicates the presence of a signal plus noise. If no signal is present, then

\[
p(N | r_1, \ldots, r_n) = p(N | r_1) \ldots p(N | r_n)
\]

is the conditional probability density distribution for the noise alone.

A set of \( n \) observations can be represented as a single point, with coordinates \((r_1, \ldots, r_n)\), in an \( n \) dimensional observation space. A criterion for making a decision is some division of the \( n \) dimensional observation space into two regions (\( n \) dimensional volumes) one volume representing the presence of a signal, the "on" region, and the other the absence of a signal, the "off" region.

Let \( P \) be the probability of making the correct decision, then

\[
P = p_o \left[ \frac{- \int_{on} p(S + N | r_1) \ldots p(S + N | r_n) \, dr_1 \ldots dr_n + (1 - p_o)}{(1 - p_o) \int_{off} p(N | r_1) \ldots p(N | r_n) \, dr_1 \ldots dr_n} \right]
\]

(2.1)

where the integrations are taken over the respective "on" and "off" regions. Note two features of Eq. (2.1) that were mentioned in the introduction: the a priori probability for the presence of a signal appears explicitly and the total probability of being right is the sum of the probabilities of guessing correctly when the signal is, and is not present. Thus "misses" and "false alarms" are given equal weight in this formulation.
The probability of making a correct decision will be a maximum if:

\[ \frac{\prod_{i=1}^{n} p(S + N | r_i)}{p(N | r_i)} = \frac{1 - p_0}{p_0} \]  \quad (2.2)

It is in the sense of making a decision with the maximum probability of guessing correctly that Eq. (2.2) is the "best" criterion.\footnote{Wald calls Eq. (2.2) a "Bayes solution."}

Eq. (2.2) can be applied directly to the probability distributions for the envelope in the i.f. strip of a radar receiver. One has

\[ p(S + N | v_1) = v_1 \exp \left[ -v_1^2 + \frac{\alpha^2}{2} \right] I_0 (\alpha v_1) \]  \quad (2.3)

where reduced variables \( v_1 = r_i / \text{root mean square noise amplitude} \), and \( \alpha = A / \text{r.m.s. noise amplitude} \). When \( \alpha = 0 \), Eq. (2.3) reduces to the case of noise alone, namely

\[ p(N | v_1) = v_1 \exp \left[ -v_1^2 / 2 \right] \]  \quad (2.4)

One obtains, on substituting Eqs. (2.3) and (2.4) into Eq. (2.2)

\[ e^{-n(\alpha^2/2)} \prod_{i=1}^{n} I_0 (\alpha v_1) = 1 - p_0 \quad \Rightarrow \quad \gamma \]  \quad (2.5)

or

\[ \prod_{i=1}^{n} I_0 (\alpha v_1) = \gamma e^{n(\alpha^2/2)} \]  \quad (2.6)

or

\[ \prod_{i=1}^{n} \log I_0 (\alpha v_1) = \log \gamma + n(\alpha^2/2) \]  \quad (2.7)
Now when the argument of the Bessel function is small, one has

\[ I_0 (\alpha v_1) = 1 + \frac{\alpha^2 v_1^2}{4} + \ldots \quad (2.8) \]

and

\[ \log I_0 (\alpha v_1) = \frac{\alpha^2 v_1^2}{4} + \ldots \quad (2.9) \]

The use of Eqs. (2.9) and (2.7) yields

\[ \frac{\alpha^2}{4} \sum_{i=1}^{n} v_i^2 = \log \gamma + n \frac{\alpha^2}{2} \quad (2.10) \]

or

\[ \sum_{i=1}^{n} v_i^2 - 2n = 4 \frac{\log \gamma}{\alpha^2} \quad (2.11) \]

It follows, from Eq. (2.11), that the surface in observation space which divides the "off" region from the "on" region is the surface of a sphere of radius \( 2^{1/2}(n + 2 \log \gamma / \alpha^2)^{1/2} \). Points within this sphere are taken to represent noise alone, those exterior to this sphere, signal plus noise.

If \( \alpha v_1 \gg 1 \), then

\[ I_0 (\alpha v_1) \sim e^{\alpha v_1} \quad (2.12) \]

and \( \log I_0 (\alpha v_1) \sim \alpha v_1 - 1/2 \log (2\pi \alpha v_1) = \alpha v_1 \quad (2.13) \)

Substituting (2.13) into (2.7), one obtains

\[ \sum_{i=1}^{n} v_i = \frac{n\alpha}{2} + \log \gamma / \alpha \quad (2.14) \]

Eq. (2.11), or some similar equation, has been used to justify the statement that the best possible second detector is a square law detector or energy integrator, whereas Eq. (2.14) has been
used to substantiate the assertion that an amplitude integrator is the best possible second detector. As can be seen from the above development, both statements are true, under suitable qualifications.

The choice of bias level, or, alternatively, the choice of surface which separates the "off" from the "on" region in observation space, determines the rates at which "false alarms" and "misses" occur. Thus, if $K_1$ is the bias level, and if the detectability criterion

$$n \sum_{i=1}^{N} v_i^2 = K \Omega 2n + \frac{4 \log \Delta}{\Omega}$$

(2.15)

is used, the false alarm rate will be greater, and the miss rate less than when $K_1 = 2n + \frac{4 \log \Delta}{\Omega}$, whereas the false alarm rate would be less, and the miss rate greater than that determined by Eq. (2.11) if $K > 2n + \frac{4 \log \Delta}{\Omega}$.

It should be remarked, in concluding this section, that $P$, as given by Eq. (2.1), or $p_{\text{max}}$, the value of $P$ when the "off" and "on" regions are determined by Eq. (2.2), has not been evaluated, although it can be calculated. Furthermore, if $p_0$, $\Delta$ and the bias level are known, then the probability of committing an error, whether it be a "miss" or a "false alarm" is determined completely.

B. 1. Signal probability unknown - signal to noise ratio known.

All of the assumptions made in section A, save for the one that $p_0$ is known, are taken to hold here.

It is a common procedure to reduce this case to the previous one by making some (usually tacit) assumption concerning the a priori probability, $p_0$. Such a procedure has often been
criticized severely as being untenable logically; it is the author's opinion that these criticisms are correct.

The situation to be studied now, can be described thusly. Given $n$ observations of the envelope of a signal with known amplitude, plus noise, or of noise alone. The a priori probability for the occurrence of a signal is unknown, but the conditional probabilities are known. In what way can these $n$ observations be used to decide whether a signal is present, or absent?

One criterion which can be used is the likelihood ratio test.  

Aside from the intuitive notion that this (and the similar maximum likelihood method to be discussed in the next section) is a reasonable criterion, there are various arguments which have been used to justify such a test as "best". See Cramer and Wald.

If one uses the ratio of the conditional joint probability densities of signal plus noise amplitude and of noise amplitude alone, which is called the likelihood ratio, then the likelihood ratio test takes the form

$$\frac{p(S+N|r_1 \ldots r_n)}{p(N|r_1 \ldots r_n)} = \frac{p(S+N|r_1) \cdot p(S+N|r_2) \ldots p(S+N|r_n)}{p(N|r_1) \cdot p(N|r_2) \ldots p(N|r_n)} > \lambda$$

(2.16)

where $\lambda$ is some arbitrarily chosen constant, $\lambda > 1$. If the observations $r_1 \ldots r_n$ satisfy the relation (2.16) then they are taken to be samples of signal plus noise, otherwise not. Note the formal similarity to Eq. (2.2). As before, one obtains the criterion

$$\sum_{i=1}^{n} I_0(\alpha r_i) > \lambda e^{\frac{2x^2}{\sigma^2}}$$

(2.17)
and equations just like Eqs. (2.11) and (2.14), with $\gamma$ replaced by $\lambda$.

It should be emphasized that while the likelihood ratio test and the Ideal Observer Theory yield analytical results which are formally identical, there is a conceptual difference between the two criteria. The constant $\lambda$, chosen arbitrarily, is a measure of the "degree of significance" which the observations must satisfy before they can be considered to be samples containing a signal, whereas $\gamma$ is a known quantity. Furthermore the probability of a "false alarm", and of a "miss" are separately determinate in the Ideal Observer Theory, while the error probability remains unknowable, even when $\lambda$ is fixed, in the likelihood ratio theory. The choice of $\lambda$ does, however, fix the ratio of "false alarm" to "miss" probabilities. The other remarks made in the previous section concerning the role of the bias level, $\lambda$, can be taken over in toto for $\lambda$, except that the ratio of "false alarm" to "miss" rates is set with a choice of $\lambda$.

B. 2. Signal probability unknown - signal to noise ratio unknown.

In detecting a radar target one often knows neither the probability of occurrence of the target nor the probability density distribution of amplitude of the received signal. This poses an additional complication not present in the previous discussions, and involves making an estimate of the signal amplitude before a decision concerning the presence or absence of a signal is reached.

The functional form of the conditional joint probability density of signal plus noise amplitude is assumed to be known; the
ratio of signal amplitude to r.m.s. noise amplitude is an unknown parameter. Otherwise, the assumptions are the same as in B.1. The method to be described is known as the method of maximum likelihood. Let \( L(S + N, r_1, \ldots, r_n) = p(S + N, r_1) \cdots p(S + N, r_n) \) or
\[
L(\alpha, v_1, \ldots, v_n) = \prod_{i=1}^{n} v_i \exp \left[ \frac{-v_i^2 + \alpha^2}{2} \right] I_0(\alpha v_i), \quad (2.18)
\]
in particular, where \( L \) is the likelihood function.

Note that when the observed values of \( r_1, \ldots, r_n \) (i.e., \( v_1, \ldots, v_n \)) are substituted into \( L(\alpha, v_1, \ldots, v_n) \) the likelihood is a function of the single unknown parameter \( \alpha \). Of the various choices that can be made for \( \alpha \), one (or perhaps more) will make \( L \) a maximum. Let this \( \alpha \) (assuming just one) be denoted by \( \alpha^* \); it can be obtained from the equation
\[
\frac{\partial \log L}{\partial \alpha} = 0, \quad (2.19)
\]
since \( \log L \) is strictly increasing with \( L \).

If Eq. (2.18) is substituted in Eq. (2.19) one obtains
\[
\sum_{i=1}^{n} \frac{\partial}{\partial \alpha} \log I_0(\alpha v_i) = n\alpha = 0 \quad (2.20)
\]
Since
\[
\frac{\partial}{\partial \alpha} \log I_0(\alpha v_i) = \frac{I_1(\alpha v_i)}{I_0(\alpha v_i)} \quad (2.21)
\]
one obtains \( \alpha^* \) as a root of the equation
\[
\alpha = \frac{1}{n} \sum_{i=1}^{n} \frac{I_1(\alpha v_i)}{I_0(\alpha v_i)} v_i \quad (2.22)
\]
Eq. (2.22) can be put in a more convenient form for small arguments, \( \propto v_1 \). One has

\[
I_1(x) = \frac{x}{2} \left[ 1 + \frac{x^2}{2} + \frac{x^4}{4} + \cdots \right],
\]

and

\[
I_0(x) = 1 + \frac{x^2}{4} + \cdots.
\]

One obtains, therefore

\[
\frac{I_1(x)}{I_0(x)} = \frac{x}{2} \left( 1 - \frac{x^2}{8} + \cdots \right). \tag{2.23}
\]

On substituting Eq. (2.23) into Eq. (2.22) one has, for \( \propto \neq 0 \),

\[
\frac{1}{2n} \sum_{i=1}^{n} v_i^2 \left[ 1 - \propto^2 \frac{v_i^2}{8} + \cdots \right] = 1, \tag{2.24}
\]

or

\[
\propto^2 = 8 \left[ \frac{\sum_{i=1}^{n} v_i^2}{\sum_{i=1}^{n} v_i^4} - 2n \right]. \tag{2.25}
\]

In order that \( \propto^{*} \), as determined by Eq. (2.25), be real

it is necessary that \( \sum_{i=1}^{n} v_i^2 \gtrsim 2n \). Again this result is similar

in form to those obtained in sections A, and B. 1.

Eq. (2.25), or a root of Eq. (2.22), yields an estimate

\( \propto^{*} \), of \( \propto \), namely the value of \( \propto \) which makes \( L \) a maximum. One

can now use some criterion which allows one to accept or reject

\( r_1 \), \( r_n \) as a set of samples of signal and noise.

One criterion for the acceptance of \( r_1 \), \( \cdots \), \( r_n \) as a set

of samples of signal in noise may be formulated as

\[
\propto^{*} > \propto_{c}, \tag{2.26}
\]
where $\lambda_0$ is some arbitrarily chosen constant. Condition (2.26) takes the form

$$\sum_{i=1}^{n} v_i^2 = 2n \gamma_1 c \frac{n}{\sum_{i=1}^{n} v_i^4}$$  \hspace{1cm} (2.27)

if Eq. (2.25) is used; $c$ is a constant ($=\sqrt{2}/8$).

Another criterion makes use of the likelihood ratio test. One requires that

$$\frac{L(\alpha^+, r_1, \ldots, r_n)}{L(0, r_1, \ldots, r_n)} > \lambda_1$$  \hspace{1cm} (2.28)

where $\lambda_1$ is some arbitrarily chosen constant. Condition (2.28) becomes, following condition (2.17),

$$\sum_{i=1}^{n} I(\alpha^+, v_i) > \lambda_1 e^\frac{n \alpha^2}{k}$$  \hspace{1cm} (2.29)

Upon some reduction, this may be written as

$$\left(\sum_{i=1}^{n} v_i^2 = 2n \right)^2 \log \frac{\mu_1}{2} \frac{n}{\sum_{i=1}^{n} v_i^4}$$  \hspace{1cm} (2.30)

Note that the conditions (2.26) and (2.28), or their more specialized forms (2.27) and (2.30), are not equivalent exactly.

III The Detection of Correlated Samples of Signal and Noise

The problems treated in II are somewhat artificial idealizations of the situations met with in practice. The "sample" was chosen to be the maximum value of the envelope during a pulse and this sample was taken as the only datum. In effect this approach ignores the
other possible measurements which could be made on a received pulse. The received pulse contains additional information concerning the presence or absence of a signal within the noise, other than its maximum value, and it is this additional information which one wishes to exploit.

Let \( f(t) \) represent the envelope of the received pulse in the i.f. strip of the receiver; it is a continuous function of time in the interval \( 0 \leq t \leq \tau \), where \( \tau \) is the pulse length. One can obtain, in addition to \( \max_0 f(t) \), which has been discussed, other values for \( f(t) \), and if these samples form a discrete set, which are independent, then the previous treatment remains essentially unchanged. The major consequence of using additional statistically independent samples is that the probability of making a correct decision is increased, because one bases a decision on a larger number of samples.

One might argue that in any time interval, however short, an infinite number of samples of the envelope exist, and it is possible, therefore, to use those samples to make the correct decision "with certainty". Aside from any computational difficulties which might arise in processing these data there are at least two objections to this proposal. The first is that the noise in the sampler sets an upper limit to the amplitude resolution of the samples. The second, and more important consideration here, is that there exists an inherent upper bound to the resolution of \( f(t) \) in time,
which arises because of the finite bandwidths of the i.f. and video strips. More precisely, if $F(t)$, (any function of time) contains no frequencies higher than $f_c$ cycles/sec, it is completely determined by giving its values at a series of points spaced $\leq 1/2 f_c$ seconds apart. Thus

$$F(t) = \sum_{l=-\infty}^{\infty} F\left(\frac{l}{2f_c}\right) \frac{\sin \pi \left(2f_c t - \frac{l}{f_c}\right)}{\pi \left(2f_c t - l\right)}, \quad (3.1)$$

where $l = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$. Furthermore, if $F(t)$ is a finite pulse train, as is $f(t)$, then a finite set of samples are sufficient to specify it. Let $2Tf_c = m$, where $f_c$ is the upper cut-off frequency of the i.f. strip, say then

$$f(t) = \sum_{l=0}^{m} f\left(\frac{l}{2f_c}\right) \frac{\sin \pi \left(2f_c t - \frac{l}{f_c}\right)}{\pi \left(2f_c t - l\right)}, \quad (3.2)$$

It follows, from Eq. (3.2), that if the accuracy of measurement of the samples is sufficiently high, it is pointless to obtain more than $m$ samples of $f(t)$.

An important complication arises when samples are taken $\sim \frac{1}{2f_c}$ seconds apart. Thus far it has been assumed that the samples were statistically independent, which was, essentially the assumption of no noise correlation for successive samples. The noise, in a circuit with upper cut-off frequency $f_c$, is, however, correlated for a time-interval $\sim \frac{1}{f_c}$ seconds. It becomes necessary, therefore to consider the correlation between samples.

In treating the case of correlated samples, it is assumed that a set of $n$ pulses is received, and that $m$ samples of each pulse are observed, the $m$ samples are correlated, but samples from different
pulses are statistically independent. Again, all m n samples are samples of signal and noise, or of noise alone.

The conditional joint probability density distribution for the amplitude of the envelope of the m n samples of signal and noise may be written as

\[
p(S + N | r_1, ..., r_m ) = p(S + N | r_1, ..., r_m ) p(S + N | r_{m+1}, ..., r_{2m}) ... p(S + N | r_{m(n-1) + 1}, ..., r_{mn})
\]

(3.3)

since successive pulse samples are independent, and similarly for the noise alone,

\[
p(N | r_1, ..., r_m ; ..., r_{m(n-1)} + 1, ..., r_{mn}) = p(N | r_1, ..., r_m ) p(N | r_{m} | r_{m(n-1)+1}, ..., r_{mn})
\]

(3.4)

The criteria used in section II may be applied directly to the above distributions. The development will be given for m = 2, using the method of maximum likelihood, and the likelihood ratio test.

The most laborious part of the calculation, that of obtaining the probability distributions, is given in the appendix, for m = 2. The results are, for m = 2,

\[
p(S + N | u_1, u_2, ..., u_{21-1}, u_{21}, ..., u_{n-1}, u_n) =
\]

\[
\left( \frac{1}{2^{n-1} n!} \right)^{n/2} \exp \left( -\frac{u_{21-1}^2 + u_{21}^2 + \beta^2 (1-2 \rho \gamma + \gamma 2)}{2} \right)
\]

(3.5)

where the new reduced variables \( u_j = \frac{r_j}{2W(1-\rho^2)} \), \( \beta = \sqrt{2W/(1-\rho^2)} \) have
been used, the signal is assumed of the form $A \cos \omega t$, $\rho$ is the normalized correlation function, $\Lambda_{01}$, $\Lambda_{11}$, $\Lambda_{21}$ are polynomials in $u_{21}$ and $u_{21}$, and $\gamma$ is a constant, defined in the appendix. Similarly

$$p(N|u_1, u_2, \ldots, u_{mn-1}, u_{mn}) = \frac{1}{(2\pi)^n} \prod_{i=1}^{n} u_{2i-1} u_{2i} \left[ \Lambda_{01} + \ldots \right] \exp \left[ -\left( u_{2i-1}^2 + u_{2i}^2 \right) \right]$$

The maximum likelihood estimate of $\beta$, $\beta^*$, is given as a root of

$$\frac{n}{i=1} \frac{1}{\beta} \log \left[ 1 + \frac{\Lambda_{11}}{\Lambda_{01}} \beta^2 + \frac{\Lambda_{21}}{\Lambda_{01}} \beta^4 + \ldots \right] = 2n \beta (1 - 2\rho \gamma + \gamma^2)$$

or, assuming $\frac{\Lambda_{11}}{\Lambda_{01}} \beta^2 + \frac{\Lambda_{21}}{\Lambda_{01}} \beta^4 < 1$, one may write

$$\frac{n}{i=1} \left( \frac{\Lambda_{11}}{\Lambda_{01}} + 2 \frac{\Lambda_{21}}{\Lambda_{01}} \beta^2 \right) = n (1 - 2\rho \gamma + \gamma^2).$$

Letting

$$\psi_0 = \frac{n}{i=1} \frac{\Lambda_{11}}{\Lambda_{01}}, \text{ and } 2 \frac{n}{i=1} \frac{\Lambda_{21}}{\Lambda_{01}} = \psi_1,$$

one obtains

$$\beta^* = \frac{n(1 - 2\rho \gamma + \gamma^2)}{\psi_1}, \psi_0.$$  

Note that the relatively simple form, Eq. (2.25) has to be replaced by the more complicated polynomial expressions of Eq. (3.9), when correlation is taken into account.

Again, the likelihood ratio test may be applied to the estimate to test for significance.
One has

\[
\sum_{i=1}^{n} \left[ 1 + \beta^{*2} \frac{\Omega_{11}}{\Omega_{01}} + \beta^{*4} \frac{\Omega_{21}}{\Omega_{01}} \right] \exp \left[ -\beta^{*2} (1-2 \psi \delta + \gamma^2) \right] > M^2
\]  

(3.10)

where \( M^2 \) is some arbitrarily chosen constant \( > 1 \). Upon expansion, and to the same degree of approximation as in deriving Eq. (3.9), one obtains the condition that \( \beta^{*} \) should satisfy

\[
\beta^{*2} \left\{ \frac{\beta^{*2}}{2} \psi \lambda + \psi \omega \right\} - n(1-2 \rho \delta + \gamma^2) \geq \log M^2  
\]  

(3.11)
APPENDIX

The calculation of the conditional joint probability distributions for signal and noise, and noise alone, where successive samples are correlated, is given here, when two observations are made per pulse \((m = 2)\), and these two observations are correlated.

One starts with the expressions given in Lawson and Uhlenbeck, pp. 152-155. The instantaneous signal and noise in the i.f. strip may be written as

\[
S(t) + N(t) = X(t) \cos 2\pi f_1 t + Y(t) \sin 2\pi f_1 t \quad \text{(A1)}
\]

where \(f_1\) is the i.f. carrier frequency, and

\[
\begin{align*}
X(t) &= a(t) + \phi_1(t) \\
Y(t) &= b(t) + \phi_2(t),
\end{align*}
\]

where \(a(t)\) and \(b(t)\) describe the shape of the received pulse when a signal is present; \(\phi_1\) and \(\phi_2\) are Fourier series representing the noise (see Eq. 10 p. 153 op. cit.). It was assumed before only that \(a(t)\) and \(b(t)\) were periodic, with period the pulse repetition time. Now it will be assumed that \(b(t) = 0\), and \(a(t) = A \cos \omega t\), which will serve to simplify the calculations\(^*\). The conditional joint probability

\[p(x_1, y_1, x_2, y_2) = \exp \left\{ - \frac{1}{2} \frac{1}{W(1 - \rho^2)} \left[ \frac{1}{4\pi^2 W(1 - \rho^2)} \left( (x_1 - a(t_1))^2 + y_1^2 + (x_2 - a(t_2))^2 + y_2^2 \right) - 2\rho \frac{1}{W(1 - \rho^2)} (x_1 - a(t_1))(x_2 - a(t_2)) + 2\rho (x_1 y_1 + y_2 x_2) \right] \right\},
\]

\[= \exp \left\{ - \frac{1}{2} \frac{1}{W(1 - \rho^2)} \left[ \frac{1}{4\pi^2 W(1 - \rho^2)} \left( (x_1 - a(t_1))^2 + y_1^2 + (x_2 - a(t_2))^2 + y_2^2 \right) - 2\rho \frac{1}{W(1 - \rho^2)} (x_1 - a(t_1))(x_2 - a(t_2)) + 2\rho (x_1 y_1 + y_2 x_2) \right] \right\},
\]

\[\text{(A3)}\]

\(^*\) These restrictions are not essential; they are made only for simplicity.
where \( \rho \) is the correlation function, and \( W \) the noise power. There is no loss in generality in taking \( t_1 \) such that \( a(t_1) = A \), and \( a(t_2) = \gamma A \), where \( \gamma \) is a constant \( \leq 1 \).

Now let
\[
\begin{align*}
x(t_1) &= r_1 \cos \theta_1, \\
y(t_1) &= r_1 \sin \theta_1, \\
x(t_2) &= r_2 \cos \theta_2, \\
y(t_2) &= r_2 \sin \theta_2.
\end{align*}
\]

The conditional joint probability distribution for the values \( r_1 \) and \( r_2 \) of the envelope is given by
\[
p(S + N \mid r_1, r_2) \, dr_1 \, dr_2 = \frac{r_1 \, r_2 \, dr_1 \, dr_2}{4 \pi^2 \cdot 2W(1 - \rho^2)} 
\]

\[
\int_0^{2\pi} \int_0^{2\pi} \exp \left\{ \frac{1}{2W(1 - \rho^2)} \left[ (r_1 \cos \theta_1 - A)^2 + r_1^2 \sin^2 \theta_1 + (r_2 \cos \theta_2 - \gamma A)^2 + r_2^2 \sin^2 \theta_2 \right] 
- 2\rho (r_1 \cos \theta_1 - A) (r_2 \cos \theta_2 - \gamma A) + r_1 r_2 \sin \theta_1 \sin \theta_2 \right\} \, d\theta_1 \, d\theta_2.
\]

Let \( \frac{1}{2W(1 - \rho^2)} = \mu^2 \). Then Eq. (A5) becomes
\[
p(S + N \mid r_1, r_2) = \frac{r_1 \, r_2 \, dr_1 \, dr_2 \, \mu^2 \exp \left\{ -\mu^2 (r_1^2 + r_2^2 + \gamma^2 (1 + \gamma^2 - 2\rho \gamma A^2)) \right\}}{2\pi^2} 
\]

\[
\int_0^{2\pi} \int_0^{2\pi} \exp \left\{ -\mu^2 \left( 2A(r_1 \cos \theta_1 + \gamma r_2 \cos \theta_2) - 2\rho r_1 r_2 \cos \left( \theta_1 - \frac{\theta_2}{2} \right) \right) + A(r_2 \cos \theta_2 + \gamma r_1 \cos \theta_1) \right\} \, d\theta_1 \, d\theta_2 \quad \text{(A6)}
\]
or, introducing a new notation for reduced variables \( r_i \)

\[
u_i \rightarrow \frac{A_{i/2}}{r_i} = p \left( |u_1| \sqrt{1+N} \right) u_1 u_2 = \frac{u_1 u_2}{r_i^2} \exp \left[ \left\{ \frac{u_1^2 + u_2^2 + r_i^2}{2} \right\} \left( 1 - \frac{2 \rho \sigma + \gamma^2}{2} \right) \right] \]

\[
\frac{2\pi}{2\pi} \exp \left[ -2 \rho \left[ u_1 u_2 \cos \theta_1 \theta_2 - \beta (u_2 \cos \theta_2 + \gamma u_1 \cos \theta_1) \right] \right] d\theta_1 d\theta_2
\]

The case of greatest interest is when the signal to noise ratio is not large. One can then expand the integrand in a power series, and integrate term by term. One obtains finally

\[
\int_0^{2\pi} \int_0^{2\pi} \exp \left[ -2 \rho \left[ u_1 u_2 \cos \theta_1 \theta_2 - \beta (u_2 \cos \theta_2 + \gamma u_1 \cos \theta_1) \right] \right] d\theta_1 d\theta_2 = \Omega_{01} + \Omega_{11} \beta^2 + \Omega_{21} \beta^4 + \cdots
\]

where

\[
\Omega_{01} = 4\pi + 2\rho^2 u_1^2 u_2^2 + \frac{3\pi \rho \gamma}{4} (u_1 u_2)^4,
\]

\[
\Omega_{11} = u_1^2 \left( 1 + 4 \gamma^2 \rho + 2 \gamma^2 \rho^2 \right) + u_2^2 (\gamma^2 + 4\gamma \rho + 2\rho^2)
\]

\[
+ u_1^2 u_2^2 \left[ 2 \rho^2 + \pi \left\{ \frac{3\rho^4}{2} + 3 \rho^4 \left( u_2^2 + \gamma u_1^2 \right) + 3 \gamma \rho^3 (u_1^2 + u_2^2) \right\} \right] + u_1^3 u_2 + 3\pi \rho^2 + u_1 u_2^3 \left[ 3\pi \gamma^2 \right].
\]
and

\[ \Omega_{21} = \pi u_1^4 \left( \frac{3}{4} \gamma + 3 \gamma^2 \rho^2 + 3 \gamma^3 \rho^3 \right) + u_2^4 \left( \frac{3}{4} \gamma + 3 \gamma^2 \rho^2 + \frac{3}{2} \gamma^2 \rho^2 + 3 \gamma \rho^3 \right) + u_1^2 u_2^2 \left( 3 \gamma^2 + 3 \gamma^3 \rho + 3 \rho^2 + 3 \gamma^2 \rho^2 + 3 \gamma \rho^3 + 6 \gamma \rho^3 + 2 \gamma^4 \rho^3 + 6 \rho^6 \right)^2. \]

One has, therefore

\[ p(s + N \mid u_1, u_2) = \frac{u_1 u_2}{2 \pi} \exp \left[ -\left( u_1^2 + u_2^2 + \beta^2 (1 - 2 \gamma \rho + \gamma^2) \right) \right] \exp \left[ -\left( \Omega_{11} \beta^2 + \Omega_{21} \beta^4 + \cdots \right) \right]. \]  

(A10)

The density distribution for the noise alone is obtained by setting \( \beta = 0 \), i.e.,

\[ p(N \mid u_1, u_2) = \frac{u_1 u_2}{2 \pi} \left[ \Omega_{11} + \cdots \right] \exp \left[ -(u_1^2 + u_2^2) \right]. \]  

(A11)

The general case (\( m \) arbitrary) proceeds along the same lines, but is even more involved since the correlation function is replaced by a correlation matrix. The calculation will not be given.