-REALIZABILITY OF FUNDAMENTAL 
CUT-SET MATRICES OF ORIENTED GRAPHS

Wataru Mayeda

REPORT R-143  JULY, 1962

COORDINATED SCIENCE LABORATORY 
UNIVERSITY OF ILLINOIS 
URBANA, ILLINOIS

Contract DA-36-039-SC-85122
D/A Sub-Task 3-99-01-002

The research reported in this document was made possible 
by support extended to the University of Illinois, Coordinated 
Science Laboratory, jointly by the Department of the Army 
(Signal Corps), Department of the Navy (Office of Naval 
Research), and the Department of the Air Force (Office of 
Scientific Research) under Signal Corps Contract 
DA-36-039-SC-85122.
Realizability of Fundamental Cut-set Matrices of Oriented Graphs
by
Wataru Mayeda
Coordinated Science Laboratory
University of Illinois
Urbana, Illinois

INTRODUCTION

As topology (linear graph theory) has been recognized to be a suitable tool to solve many problems in electrical networks, switching circuits, communication nets, etc., the necessary and sufficient conditions that a matrix be a fundamental cut-set (or circuit) matrix becomes one of the important problems in this field.

If the problem is to find whether a given matrix is a fundamental cut-set matrix of a non-oriented graph, there are four methods of testing such a matrix at present. This paper takes one of these methods and modifies it such that we can test whether a given matrix is a fundamental cut-set matrix of an oriented graph (where every entry in the matrix is +1, -1, or 0).

It is known that the theory of oriented graphs has more applications than that of non-oriented graphs. Also, in many cases, representation of systems by non-oriented graphs is a special case of representation of systems by oriented graphs. For example, topological representation of electrical networks and communication nets.

PRELIMINARY

In order to give a modified theorem in the paper "Necessary and Sufficient Conditions for Realizability of Cut-Set Matrices" so that we can use it to test whether a matrix is a fundamental cut-set matrix at an oriented graph, we will review definitions. Some of these definitions are modified so that it will fit to the problem in this paper.
Definition 1: H-submatrix of matrix \( N = [N_{11}] \) (where \( n_{ij} = N_{11} \) and \( n_{ij} = +1, 0 \)) with respect to row \( p \) is a matrix obtained from \( N \) by deleting row \( p \) and all columns which have non-zero elements at the intersection with row \( p \). For convenience, every row and column of H-submatrices and M-submatrices will be identified by the symbols which are used to identify the rows and columns of a given matrix \( N \) such that row \( p \) (column \( q \)) of H (or M)-submatrix is the row of H (or M) corresponding to row \( p \) (column \( q \)) of \( N \).

Definition 2: A pair of M-submatrices \( M_1 \) and \( M_2 \) of a matrix \( N \) with respect to row \( p \) where H-submatrix of \( N \) with respect to row \( p \) has the form

\[
H = \begin{bmatrix}
H_1 & 0 \\
0 & H_2
\end{bmatrix}
\]

is a pair of following submatrices of \( N \): (1) \( M_1 \) is obtained from \( N \) by deleting all rows and columns which belong to \( H_1 \) and (2) \( M_2 \) is obtained from \( N \) by deleting all rows and columns belonging to \( H_2 \). Notice that \( H_1 \) can be empty.

From a given matrix \( A \), we can obtain a pair of M-submatrices \( M_1 \) and \( M_2 \) have the form \( M \cup M_1 \), \( M_2 \) can be considered as a given matrix. Hence, if there exists a row in \( M \), which has not been used to obtain H-submatrix (to obtain M-submatrices), we can obtain M-submatrices \( M_a \) and \( M_b \) of \( M \), the collection \( (M_a, M_b, M_2) \) is also called a set of M-submatrices. Similarly, any of \( M_a \), \( M_b \), and \( M_2 \) can be considered as a given matrix. Hence if there exists a row of one of these matrices, say \( M_2 \), which has not been used to obtain M-submatrices before, we can obtain a pair of M-submatrices of \( M_2 \). Thus we can obtain another set of M-submatrices.

If a matrix \( M_p \) in the set of M-submatrices which has been obtained by the above process contains no rows which have not been used to form M-submatrices in the set, \( M_p \) is called a "minimum M-submatrix". If every matrix in the set of
M-submatrices which has been obtained by the above process is a minimum
M-submatrix, the set is called a "set of minimum M-submatrices".

TO TEST A MATRIX TO BE A FUNDAMENTAL CUT-SET
MATRIX OF AN ORIENTED GRAPH

The important theorem for testing whether a matrix is a fundamental
cut-set matrix of an oriented graph is given below. Even though this is the
modified theorem of a theorem in the paper "Necessary and Sufficient Condi-
tions for Realizability of Cut-Set Matrices", the proof becomes more compli-
cated than that of the original theorem.

Theorem 1: A Matrix $A = \begin{bmatrix} A_1 & U \end{bmatrix}$, where every entry is ± 1 or 0 and $U$ represents
a unit matrix is a fundamental cut-set matrix of an oriented graph if and only
if there exists a set of minimum M-submatrices obtained from $A$ such that every
matrix in the set becomes an incidence matrix of an oriented graph by multi-
plying (-1) to some row of the matrix.

Notice that a matrix $C = [C_{ij}]$ where $C_{ij} \equiv 1, 0$ is an incidence matrix
if and only if every column of $C$ has either at most one non-zero or two non-
zero with opposite signs.

The multiplication of -1 to some row of a M-submatrix is necessary because
assigning the sign of each branch in an oriented graph in an incidence set and
that of each branch in a cut-set are different. For example, suppose branches
$A$ in the graph in Fig. 1 is a branch in the tree corresponding to fundamental
cut-set matrix $A$. Then if we consider $\{a, b\}$ as an incidence set corresponding
to a row $v$ in an incidence $A$, the intersections of columns corresponding to $a$
and $b$ and row $v$ have -1 and 1 respectively. However, if we consider $\{a, b\}$ as a
cut-set in $A$, the intersections of columns $a$ and $b$ are now representing the cut-
set have 1 and -1 respectively. Hence when we form M-submatrices with respect
to row $p$ in $A = \begin{bmatrix} A_1 & U \end{bmatrix}$, the row $p$ in M-submatrices may not represent an incidence
set. However, if row \( p \) does not represent an incidence set, the multiplication by \(-1\) will make row \( p \) to represent an incidence set. Hence the proof of necessary part of the theorem is exactly the same as that of non-oriented case except that the multiplication of \( \pm 1 \).

Before proving the sufficient part of theorem 1, we will study the following two theorems:

**Theorem 2**: If a matrix \( M = [M_{11} U] \) where every entry of \( M_{11} \) is \( \pm 1 \) or 0 and \( U \) is a unit matrix is a fundamental cut-set matrix of oriented graph \( G \), then \( M \) is also a fundamental cut-set matrix of oriented graph \( \bar{G} \) which is obtained from \( G \) by reversing the orientation of every branch in \( G \).

**Proof**: Because of the definition of assigning the sign of elements in a row of \( M \) which represents a cut-set, the row of \( M \) does not change if the orientation of every branch in \( G \) is altered. Hence the theorem is true.

Suppose arrow \( p \) of \( M \) represents incidence set \( s \) but not a cut-set. Let \( s \) be consisted of branches \( e_1, e_2, \ldots, e_s \) which are incident at vertex \( p \) as shown in Fig. 2a. Then reversing the orientation of every branch in \( G \) as shown in Fig. 2b makes no longer row \( p \) to represent the incidence set \( s \) of branches which incident at vertex \( p \) because of the definition of assigning the sign of non-zero elements in row \( p \) corresponding to incidence set \( s \). However \((-1)\) times
row p in M will represent s in $\tilde{G}$. Theorem 2 only guarantees that row p represents a cut-set $S = (e_1, \ldots, e_i)$ in $\tilde{G}$.

**Theorem 3:** Let a pair of M-submatrices of a matrix $M_{i+j}$ with respect to row p be $M_i$ and $M_j$. Suppose there exist graphs $g_i$ and $g_j$ such that (1) the fundamental cut-set matrices of $g_i$ and $g_j$ are $M_i$ and $M_j$ respectively, (2) there exists vertex p in $g_i$ such that either row p in $M_i$ or (-1) times row p in $M_i$ represents an incidence set of branches which incident at vertex p and (3) there exists vertex p in $g_j$ such that either row p in $M_j$ or (-1) times row p in $M_j$ represents an incidence set of branches which incident vertex p. Then there exists a graph $g_{i+j}$ such that (a) $M_{i+j}$ is a fundamental cut-set matrix of $g_{i+j}$, (b) for every row q in $M_i$ except row p, which has the property that either row q or (-1) times row q represents an incidence set of branches which incident at vertex q in $g_i$, there exists row q in $M_{i+j}$ such that either row q or (-1) times row q in $M_{i+j}$ represents an incidence set of branches which incident at vertex q in $g_{i+j}$ and similarly, (c) for every row r in $M_j$ which has the property that either row r or (-1) times row r represents an incidence set of branches which incident at vertex r, there exists row r in $M_{i+j}$ such that either row r or (-1) times row r in $M_{i+j}$ represents an incidence set of branches which incident at vertex r in $g_{i+j}$.

We will prove theorem 3 by constructing the graph $g_{i+j}$ which satisfies a, b, and c. Since $M_i$ and $M_j$ are a pair of M-submatrices of $M_{i+j}$ with respect to row p, if and only if there exists a non-zero element at the intersection of row p and column e in $M_i$, there exists non-zero element at the intersection of row p and column e in $M_j$. Hence, if and only if a branch $e_q$ in $g_i$ is connected on vertex p, there exists branch $e_q$ connected on vertex p in $g_j$. Also the orientation of $e_q$ in $g_i$ with respect to vertex p is either the same as or opposite to the orientation of $e_q$ in $g_j$ with respect to vertex p.
If branch $e_q$ which is connected on vertex $p$ in $g_j$ has the same orientation with respect to $p$ as $e_q$ which is connected on vertex $p$ in $g_j$, we alter the orientation of all branches in $g_j$ to form graph $g_i$ so that the orientation of branch $e_q$ in $g_i$ with respect to vertex $p$ is opposite to the orientation of $e_q$ in $g_j$ with respect to $p$. $M_i$ is a cut-set matrix of $g_i$ by theorem 2. Also, it is clear that if either row $q$ or $(-1)$ times row $q$ represents an incidence set in $g_i$, either row $q$ or $(-1)$ times row $q$ represents an incidence set in $g_i$. Hence, the above operation will produce no alteration to the assumptions and results in theorem 3. If branch $e_q$ which is connected on vertex $p$ in $g_i$ has the opposite orientation as $e_q$ in $g_j$ with respect to vertex $p$, then we define that $g_i = g_j$. Now we construct $g_{i+j}$ whose cut-set matrix is $M_{i+j}$ from $g_i$ and $g_j$ as follows:

Let $g_i$ and $g_j$ be the graphs shown in Fig. 3a, where the cut-set corresponding to row $p$ of $M_i$ (and $M_j$) consists of branches $e_1$, $e_2$, ..., and $e_k$.

**Figure 3**

(a) Graph $g_i$ and $g_j$

(b) Graphs of $g_a + g_b$

(c) Graph $g_{i+j}$

Also let $e_w$ in $g_i$ be connected between vertices $v_w$ and $p$, and $e_w$ in $g_j$ be connected between vertices $u_w$ and $p$ for $w = 1, 2, ..., k$. (Fig. 3a).
(1) Remove all branches \( e_1, \ldots, e_k \) which are connected on vertex \( p \) in \( g_i \) and \( g_j \), as shown in Fig. 3b.

(2) Connect branch \( e_w \) between vertices \( v_w \) and \( u_w \) and the orientation of \( e_w \) is the orientation of \( e_w \) in \( g_i \) for \( w = 1, 2, \ldots, k \), i.e. if the orientation of \( e_w \) in \( g_i \) is away from \( v_w \), the orientation of \( e_w \) in the resultant graph is away from \( v_w \) and if the orientation of \( e_w \) in \( g_i \) is toward \( v_w \), the orientation of \( e_w \) in the resultant graph is toward \( v_w \), as shown in Fig. 3c.

Because of the first step of the above process, every branch in \( g_i \) other than \( e_1, e_2, \ldots, e_k \) does not be replaced in the resultant graph. Also, as far as the branches in \( g_i \) are concerned the second step of the above process only replaces the connection of \( e_w \) from vertex \( p \) to vertex \( u \) which is in \( g_j \) for \( w = 1, 2, \ldots, k \). Hence, if we coincide all vertices in the resultant graph which are also in \( g_j \), we can obtain \( g_i \). Likewise, if we coincide all vertices in the resultant graph which are also in \( g_i \), we can obtain \( g_j \). Because only cut-set \( (e_1, e_2, \ldots, e_k) \) is in both \( g_i \) and \( g_j \), \( M_{i+j} \) is the fundamental cut-set matrix of the resultant graph with respect to the tree consisting of the branches in the trees of \( g_i \) and \( g_j \) by which the fundamental cut-set matrices \( M_i \) and \( M_j \) have been obtained. Furthermore, if row \( r \) in \( M_j \) \((r \neq p)\) represents an incidence set in \( g_j \), row \( r \) in \( M_{i+j} \) represents an incidence set in the resultant graph.

If \((-1)\) times row \( r \) in \( M_j \) \((r \neq p)\) represents an incidence set in \( g_j \), \((-1)\) times row \( r \) represents an incidence set in the resultant graph. If \( g_i \) is identical with \( g_i \), the above property also holds for \( g_i \). Suppose \( g_i \) is obtained by reversing the orientations of all branches in \( g_i \), then if row \( q \) in \( M_i \) \((q \neq p)\) represents an incidence set in \( g_i \), \((-1)\) times row \( q \) represents an incidence set in \( g_i \), \((-1)\) times row \( q \) represents an incidence set in the resultant graph. Hence the resultant graph is \( g_{i+j} \), and theorem 3 is proved.
Now we will prove the sufficient part of theorem 1. Consider the process used to obtain set $S_v$ of minimum M-submatrices from a given matrix $A$. Let this process be $S_1, S_2, \ldots, S_v$ where $S_1 = (A)$ and $S_j$ ($j = 2, 3, \ldots, v$) is obtained from $S_{j-1}$ by using one matrix $M_d$ in $S_{j-1}$ to form a pair of M-submatrices $M_{d_1}$ and $M_{d_2}$ with respect to row $d$ which has not been used to form a pair of M-submatrices in $S_{j-k}$ (for $k = 1, 2, \ldots, j-k$) and replace $M_d$ by $M_{d_1}$ and $M_{d_2}$. Hence number of matrices in $S_j$ is one plus number of matrices in $S_{j-1}$.

Let distinct rows $p_2, p_3, \ldots, p_v$ be the sequence of rows which are in a given matrix $A$ such that row $p_i$ ($i = 2, 3, \ldots, v$) is used to obtain a pair of M-submatrices $M_{i_1}$ and $M_{i_2}$ in $S_i$ from a matrix in $S_{i-1}$ to form $S_i$ from $S_{i-1}$.

By the hypothesis of theorem 1, for each fundamental cut-set matrix $M$ in $S_v$, there exists an oriented graph $g$ such that either row $q$ or $-1$ times row $q$ represents an incidence set in $g$ for all rows in $M$. (Notice that an M-submatrix of a matrix $[C_{11}U]$ is of the form $[D_{11}U]$ where $U$ is a unit matrix). Hence, we can apply theorem 2 to a pair of M-submatrices $M_{v_1}$ and $M_{v_2}$ with respect to row $p_v$ and can prove that every matrix $M_j$ in $S_{v-1}$ is realizable as a fundamental cut-set matrix of an oriented graph $g_j$ such that either row $q$ or $-1$ times row $q$ of $M_j$ represents an incidence set in $g_j$ for all rows in $M_j$ except if row $q$ is row $p_v$.

If we can apply theorem 2 successively to a pair of M-submatrices $M_{i_1}$ and $M_{i_2}$ in $S_i$ which makes $S_i$ from $S_{i-1}$ for $j = v, v-1, \ldots, v-e$ ($e < v-2$), then we can prove that every matrix in $S_{i-1}$ is realizable as a fundamental cut-set matrix of an oriented graph $g$ such that either row $s$ or $-1$ times row $s$ of the matrix represents an incidence set in $g$ for all rows in the matrix except if row $s$ is one of rows $p_v, p_{v-1}, \ldots, p_{v-e}$. The requirements for using theorem 2 to a pair of realizable M-submatrices $M_{i_1}$ and $M_{i_2}$ with respect to row $p_1$ whose oriented graphs are $g_{i_1}$ and $g_{i_2}$ are that either row $p_1$ or $-1$ times row $p_1$ in $M_1$
represents an incidence set in $g_1$ and either row $p_1$ or $(-1)$ times row $p_i$ in $M_{1_2}$ represents an incidence set in $g_{1_2}$. Because rows $p_2, p_3, \ldots$ and $p_v$ are all different rows in a given matrix $A$, this requirement will be fulfilled for a pair of $M$-submatrices in $S_i$ by which $S_i$ is obtained from $S_{i-1}$ for $i = 2, 3, \ldots v$ if we apply it starting with a pair of $M$-submatrices in $S_v$ by which $S_v$ is obtained from $S_{v-1}$, then to a pair of $M$-submatrices in $S_{v-1}$ by which $S_{v-1}$ is formed from $S_{v-2}$, etc. Finally we can apply theorem 2 to a pair of $M$-submatrices in $S_2$ by which $S_2$ is obtained from $S_1 = (A)$ where $A$ is a given matrix which proves the sufficient part of theorem 1.

If $H$-submatrix $H_i$ of a matrix with respect to a row can be partitioned as

$$H_i = \begin{bmatrix} H_1 & H_2 & 0 \\ 0 & H_3 & \cdots \\ & & H_p \end{bmatrix}$$

then there are $2^{n-1}$ different pairs of $M$-submatrices of the matrix with respect to the row. Hence in general there will be many sets of minimum $M$-submatrices of a given matrix $A$. However, if one of these sets of minimum $M$-submatrices of $A = [A_{11}]$ where the entry in $A_{11}$ is $+1$, $-1$, or $0$ is satisfied the conditions in theorem 1, $A$ is realizable as a fundamental cut-set matrix of an oriented graph. In other words, unless all possible sets of minimum $M$-submatrices of $A$ are not satisfied the conditions in theorem 1, we cannot say that $A$ is not realizable as a fundamental cut-set matrix of an oriented graph. On the other hand, there may be a collection $U$ of sets of minimum $M$-submatrices of $A$ which has the property that if and only if a set $S$ in $U$ satisfies the conditions in theorem 1, any other set $S'$ in $U$ satisfies the conditions in theorem 1. Hence only one of sets in $U$ needs to be tested. The existence of such a collection can be shown as follows: If $A$ is realizable as $G_1$ in Fig. 4a with row $s$ in $A$
represents a cut-set $S = (e_1, e_2, \ldots, e_m, e_{21}, e_{22}, \ldots, e_{2m})$. Then $A$ can be realizable as $G_2$ in Figure 4b with either row $s$ or $(-1)$ times row $s$ represents an incidence set $S$ in $G_2$. In this case $H$-submatrix of $A$ with respect to row $s$ is of the form in Eq. 1 with $H_1, H_2 \neq \emptyset$ ($H_1 = \emptyset$ means $H_1$ consists of no row.) Then if the set of minimum $M$-submatrices of $A$ which is obtained by forming a pair of $M$-submatrices $M_a$ and $M_b$ of $A$ with respect to row $s$ by letting $H_1 = \emptyset$ and $H_2 = H$ (which is equivalent to saying that $M_a = A$ and $M_b$ is a one row matrix) satisfies the conditions in theorem 1, the set of minimum $M$-submatrices $M_c$ and $M_b$ of $A$ with respect to row $s$ by $H_1, H_2 \neq \emptyset$ also satisfies the conditions in theorem 1 and vice versa. This is also true when we take an $M$-submatrix $M$ in $S_{j-1}$ and form a pair of $M$-submatrices of $M$ to form set $S_j$ of $M$-submatrices. Thus, whenever $H$-submatrices of a matrix is of the form in Eq. 1 with $H_1, H_2 \neq \emptyset$ it is not necessary to form a pair of $M$-submatrices of the matrix with letting $H_1 = \emptyset$ and $H_2 = H$.

**Example of Using Theorem 1.** The following matrix is not realizable as a fundamental cut-set matrix of an oriented graph because of the following reasons:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

From $H$-submatrix of $A$ with respect to row $a$, which is

$$H = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
we obtain a pair of $M$-matrices $M_1$ and $M_2$ as

$$
M_1 = \begin{bmatrix}
1 & 3 & 4 & 5 \\
1 & 0 & 1 & 1 \\
0 & 1 & -1 & 1
\end{bmatrix}
$$

and

$$
M_2 = \begin{bmatrix}
1 & 2 & 4 & 5 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & -1
\end{bmatrix}
$$

Since $M_1$ (or $M_2$) is not realizable as an incidence matrix of an oriented graph by multiplying $(-1)$ to some rows in $M_1$ and since there is no other way of obtaining a pair of $M$-submatrices of $A$ except by letting $H_1 = \emptyset$ and $H_2 = H$, $A$ is not realizable as a fundamental cut-set matrix of an oriented graph. Notice that to obtain the set of minimum $M$-submatrices of $A$ we must obtain a pair of $M$-submatrices of $M_1$ with respect to row $c$ and a pair of $M$-submatrices of $M_2$ with respect to row $b$. It is clear that one of the above pair of $M$-submatrices of $M_1$ is the same as $M_1$ and the other is a single row matrix. Similarly, one of the pair of $M$-submatrices of $M_2$ with respect to row $b$ is the same as $M_2$ and
the other is a single row matrix. Hence, the set of minimum $M$-submatrix consists of $M_1$, $M_2$, and two single row matrices. Since a single row matrix always satisfies the conditions in theorem 1, it is only necessary to test whether $M_1$ and $M_2$ satisfy the conditions in theorem 1 to know whether $A$ can be a fundamental cut-set matrix. It is interesting to notice that if we replace all -1 by +1 in $A$, then $A$ is realizable as a fundamental cut-set matrix in both oriented and non-oriented graph.


3. L. Lofgreen, "Irredundant and Redundant Boolean Branch Networks", IRE-PGCT, CT-6 (Supple.), 158-175.


12. An incidence set is a collection of branches (sometimes called edge or elements) in a graph which are incident at a vertex.