OPTIMAL LINEAR SWITCHING FOR
SINGULAR LINEAR SYSTEMS

R.A. Rohrer
M. Sobral, Jr.

REPORT R-196 MARCH, 1964

COORDINATED SCIENCE LABORATORY
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS

Contract DA-36-039-AMC 02208(E)
DA Project 3A-99-25-004

The research reported in this document was made
possible by support extended to the University of
Illinois, Coordinated Science Laboratory, jointly
by the Department of the Army, Department of the
Navy (Office of Naval Research), and the Depart­
ment of the Air Force (Office of Scientific Research)
under Department of Army Contract DA-36-039-
AMC 02208(E).
ABSTRACT

The bounded-input control of a linear, time-invariant system with quadratic performance index is discussed from the standpoint of simple implementation. In particular the admissible control is constrained to be a linear function of the states fed back through a saturating amplifier. The resulting controlled system not only exhibits the usual maximum-effort made of operation, but also the terminal singular behavior where linear control is in evidence.
INTRODUCTION

I-1 Background

The optimal linear switching problem has been studied in some detail as an approximation to the true optimal control—sub-optimal control—for systems with quadratic performance indices [1,2,3,4]. The optimal relay control of Jen-Wei [1], although theoretically incorrect\(^1\), exhibited the so-called singular solutions (slippage modes) anticipating the later interest in depth in singular solutions [5,6]. These previous efforts lead naturally to the question: If a simple control scheme (viz., linear switching) is highly desirable, why not restrict the class of admissible controls to those satisfying this control scheme at the outset? The method for obtaining the terminal singular solution in the single-input case is at least conceptionally simple and computationally possible [5]; however, outside of the region of admissible linear control, the switching curve is difficult to obtain—and the switching is difficult to implement as well.

The proposed solution here might be called "sub-optimal" in that it is not the "best" that can be done when one has available an infinitude of resources. On the other hand it is truly optimal in the sense that the means of control is restricted at the outset with an eye to the eventual implementation (an implementation which is usually considered the practical compromise, anyway), and under this restriction the control is optimal. A distinct advantage of the linear switching obtained is that the control in no way depends on the system's initial conditions while still exhibiting the optimal singular behavior.
I-2 Formulation of the Problem

We consider a linear, time-invariant, single-input system described by the state equations

$$\dot{x} = Ax + bu$$  \hspace{1cm} (1a)

and the input constraint

$$|u| \leq 1.$$  \hspace{1cm} (1b)

Moreover, the control $u$ is to be given by the function

$$u = f(v)$$  \hspace{1cm} (2)

where

$$f(v) = v \text{ for } |v| \leq 1,$$  \hspace{1cm} (3a)

$$f(v) = +1 \text{ for } v > 1,$$  \hspace{1cm} (3b)

$$f(v) = -1 \text{ for } v < -1,$$  \hspace{1cm} (3c)

and the function $v(t)$ is constrained to be a linear combination of the state variables:

$$v(t) = c^T x(t) = \sum_{i=1}^{n} c_i x_i(t).$$  \hspace{1cm} (4)

The quadratic performance index is the functional

$$J(c) = \int_0^\infty x^T Q x \, dt,$$  \hspace{1cm} (5)

and the problem is to take the system from any initial condition to the origin while (5) is minimized. The problem can be restated as that of finding the vector $c$ in $\mathbb{E}^n$ (Euclidean n-space) which minimizes the functional (5) along the trajectories (1) and (2). Moreover, no magnitude constraints need be imposed on $c$; it may be any vector in $\mathbb{E}^n$. 

II. THE OPTIMAL CONTROL

The problem as formulated above is quite easily handled by means of the Maximum Principle of Pontryagin [7]. In order to apply the Maximum Principle, we first form the Hamiltonian:

\[ H(\psi, x, c) = \psi^T A x + \psi^T b f(c^T x) - x^T Q x, \]  

(6)

where the auxiliary vector \( \psi \) (the conjugate momentum) is the solution of

\[ \dot{\psi} = - \nabla_x H. \]  

(7)

Equation (7) in view of (6) becomes

\[ \dot{\psi} = -A^T \psi + 2 Q x - \psi^T b \nabla_x f(c^T x), \]  

(8a)

where the \( i \)th component of \( \nabla_x f(c^T x) \) is given by

\[ [\nabla_x f(c^T x)]_i = \frac{\partial f(c^T x)}{\partial x_i}; i = 1, 2, \ldots, n. \]  

(8b)

The Maximum Principle states that the vector \( c \) which minimizes the functional (5) must necessarily maximize the Hamiltonian (6); hence, at the maximum

\[ \frac{\partial H}{\partial c_i} = 0, i = 1, 2, \ldots, n. \]  

(9)

In view of (6) conditions (9) become

\[ \frac{\partial}{\partial c_i} [\psi^T b f(c^T x)] = 0, i = 1, 2, \ldots, n, \]  

(10a)

or

\[ \psi^T b \frac{\partial f(c^T x)}{\partial c_i} = 0, i = 1, 2, \ldots, n. \]  

(10b)
The derivatives indicated in (8b) and (10b), because of (3) and (4), can be rewritten

\[
\frac{\partial}{\partial x_i} f(c^T x) = \frac{\partial f(v)}{\partial v} \frac{\partial v}{\partial x_i}
\]  

(11a)

and

\[
\frac{\partial}{\partial c_i} f(c^T x) = \frac{\partial f(v)}{\partial v} \frac{\partial v}{\partial c_i},
\]  

(11b)

respectively. Moreover, from (3)

\[
\frac{\partial f(v)}{\partial v} = 1 \text{ for } |v| \leq 1
\]  

(12a)

and

\[
\frac{\partial f(v)}{\partial v} = 0 \text{ for } |v| > 1;
\]  

(12b)

furthermore, from (4)

\[
\frac{\partial v}{\partial x_i} = c_i
\]  

(13a)

and

\[
\frac{\partial v}{\partial c_i} = x_i.
\]  

(13b)

Consequently, conditions (10) become

\[
B^T x = 0 \text{ for } |v| \leq 1
\]  

(14a)

and

\[
0 = 0 \text{ for } |v| > 1;
\]  

(14b)

The optimal control can be obtained from any \( c \) such that for each state \( x \) in the state space

\[
|x^T| > 1,
\]  

(15)
which follows from (14b)—unless

\[ b^T \Psi = 0, \tag{16} \]

as in (14a). Condition (16) is that of the singular solution; a control \( u = f(v) \) which realizes (16) over a finite time interval is a singular control.

In Appendix B is a derivation for the singular surface on which (16) holds:

\[ \sum_{i=1}^{n} h_i x_i(t) = 0; \tag{17} \]

this linear equation describes an \((n-1)\)-dimensional hyperplane in the \(n\)-dimensional state space. Obviously, the system may satisfy (17) if the initial conditions satisfy the condition

\[ h^T x_0 = 0, \tag{18} \]

where

\[ h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}, \quad h_n = 1. \tag{19} \]

Moreover, the optimal singular control which maintains the system on the singular surface (when it is admissible) is given by

\[ u = f(v), \tag{20a} \]

\[ v = c^T x, \tag{20b} \]
and
\[ c = a - \hat{h} - k_h, \]  

where
\[ \hat{h} = \begin{bmatrix} 0 \\ h_1 \\ h_2 \\ \vdots \\ \vdots \\ h_{n-1} \end{bmatrix}, \]

is the negative of the transposed last row of the assumed companion form A-matrix and
\[ \frac{a_1 - c_1}{h_1}. \]

This control with \( k \) arbitrary (i.e., with \( c_1 \) arbitrary) causes the system trajectory to remain on the singular hyperplane (17) whenever
\[ |(a - \hat{h})^T x| < 1 \]  

and (18) is satisfied. If (22) is not satisfied, there is no control (among the admissible ones) which makes \( \frac{b^T \Psi}{b} = 0 \) over a finite time.
interval. Note, (20) determines all $c_i$'s except $c_1$; moreover, the control signal on the singular hyperplane is independent of the value of $c_1$. Letting

$$c_s = a - \hat{h}$$  \hspace{1cm} (23a)

and

$$c_{ns} = -kh,$$  \hspace{1cm} (23b)

we have

$$c = c_s + c_{ns};$$  \hspace{1cm} (24)

i.e., the control is a linear combination of singular and nonsingular controls. For all $x$ on the admissible singular hyperplane, i.e., for $x$ such that

$$h^T x = 0$$  \hspace{1cm} (25a)

and

$$|(a-h)^T x| < 1,$$  \hspace{1cm} (25b)

the optimal control is given by (23a):

$$u = c_s^T x = (a-h)^T x.$$  \hspace{1cm} (26)

For $x$ not satisfying (25) (both 25a and 25b),

$$\frac{\partial f(v)}{\partial v} = 0$$  \hspace{1cm} (27a)

or

$$|c^T x| > 1$$  \hspace{1cm} (27b)
must hold for (10) to be satisfied. Condition (27b) can be maintained (when appropriate) by taking 5

$$c_1 = -\infty .$$  \hspace{1cm} (28)

Of course, an infinite gain is difficult to obtain in practice, but from the simple implementation shown in Fig. 1, it is clear that a sufficiently large finite gain (+k) realizes the optimal control within any predetermined error. Moreover, limiting the gain (k) to that which is practically attainable assures that the optimal $c_1$ is chosen from a closed set.

The stability of the closed-loop system follows from the optimality of the control. The control given by (20c) is optimal; it thus provides a smaller value for the functional (5) than does the admissible control $u = 0$ (this is true for any $x_o$). But if the original system (1) is stable (this property is assumed here), the integral (5) is bounded for $u = 0$; it is therefore bounded for the optimal control—a clear indication of stability. The only problem, then, is to assure the optimality of the control (20c); this assurance in general involves an investigation of the second variation—unless strong physical arguments can be mustered—which is often quite difficult. An alternative verification of stability could be undertaken by employment of the most recently obtained sufficient conditions for the Lur's Problem [8].
Fig. 1. Implementation of optimal linear switching.
III. ILLUSTRATIVE EXAMPLE

Consider the system characterized by

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1 - 3x_2 + u
\end{align*}
\]

and

\[|u| \leq 1.\]

The performance criterion is to minimize the functional

\[
J = \int_0^\infty (x_1^2 + x_2^2) \, dt,
\]

while the system trajectory goes to the origin in a stable manner. The stable Euler equation is

\[x_1 + x_2 = 0,\]

and the feedback control which maintains it on the singular hyperplane (17) is

\[u = x_1 + 2x_2 - k(x_1 + x_2).\]

The system is stable for \(k > 0\) and optimal when \(k \to \infty\); the implementation of this control is shown in Fig. 2. In Fig. 3, the phase plane behavior of the system trajectory is given for similar initial conditions \((x_{20} = 0\) and \(x_{10} = 0.8, 1.2, 1.6, \text{ and } 2.0\)) and various values of gain \((k = 2, 20, \text{ and } 200)\). The linear region is indicated in each case as well as the shapes of the control curves \((u \text{ vs. } x_1)\) which provide the optimal. The pertinent feature which should be noticed is the extreme similarity between the curves obtained for \(k = 20\) and \(k = 200\), clearly indicating the sufficiency of a small, finite gain.
Fig. 2. Optimal control of second-order example.
Fig. 3. $x_2$ and $u$ verses $x_1$ for the second-order example with various values of gain: (a) $k = 2$. 
Fig. 3. $x_2$ and $u$ verses $x_1$ for the second-order example with various values of gain: (b) $k = 20$. 
Fig. 3. \( x_2 \) and \( u \) verses \( x_1 \) for the second-order example with various values of gain: (c) \( k = 200 \).
IV. EXTENSION—NONLINEAR, TIME-VARYING SYSTEMS

The extension of the above results to nonlinear, time-varying systems is quite easily undertaken. Under the assumption that the performance index is the same time-invariant quadratic form, we obtain the same stable Euler equation and the same simple expression for $\Psi$ as above. It is only in the implementation of the feedback control that the newly hypothesized qualities of the system enter the picture.

Suppose, for example, that we are provided with a single-input system characterized by an $n^{\text{th}}$ order nonlinear, time-varying differential equation:

$$x^{(n)} = - \sum_{i=1}^{n} a_i(x,t) x^{(i-1)} + u,$$  

(29)

where

$$
\begin{bmatrix}
x \\
x^{(1)} \\
x^{(2)} \\
\vdots \\
x^{(n-1)} \\
x^{(n)}
\end{bmatrix}
$$  

(30)

Moreover, the control $u$ is constrained, as in (2) - (4), and the performance criterion demands the minimization of the functional (5). The Euler equation and auxiliary vector $\Psi$ are exactly as found in Appendix B; the optimal control is given by

$$c = a(x,t) - \hat{h} - kh,$$  

(31)
where

\[
\mathbf{a}(x,t) = \begin{bmatrix}
a_1(x,t) \\
a_2(x,t) \\
\vdots \\
a_n(x,t)
\end{bmatrix}.
\]  

(32)

As before, the optimal is achieved by making the gain \((k)\) as large as possible (or desirable). The significant factor here is that the portion of the control \((c_{\text{ns}}^{T} x = kh^{T}x)\) which drives the system to the singular hyperplane (and, as such, defines the switching curve) is linear and time-invariant. Moreover, this portion of the control is solely dependent on the performance index; consequently, the closed-loop system should be relatively insensitive to plant parameter variations for large gain \((k)\).

V. CONCLUSION

The advantages of the optimal control scheme presented here are manifest, to name but a few:

1. Ease of implementation;
2. Indifference to initial conditions, i.e., the feedback control is the same regardless of starting point;
3. Applicability to optimal control of nonlinear, time-varying systems.

This form of control should enjoy wide application as it is the theoretical optimum tempered with practical constraints.
ACKNOWLEDGMENT

The authors are indebted to their colleagues at the Coordinated Science Laboratory, University of Illinois, and to Mr. J. J. Mele who implemented the analog computer simulation of the example of Section III.


APPENDIX A. SUB-OPTIMAL LINEAR SWITCHING

Consider the following problem.

Given the system equation

\[ \dot{x} = Ax + b^T u, \]  
(A.1)

find \( u \in U \) (\( U \) is defined as the set of all piecewise continuous functions \( u(t) \) such that

\[ |u(t)| \leq 1 \)  
(A.2)

such that the functional

\[ J(u) = \int_0^\infty x^T Q x \, dt \]  
(A.3)

is minimized. Under the assumption of complete controllability for the system (A.1), we can take without loss of generality [5]

\[ A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix}, \]  
(A.4a)

\[ b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \]  
(A.4b)
We assume that \( A \) (the uncontrolled system) is stable and that the matrix \( Q \) (A.3) is positive definite; therefore, the equation

\[
PA + A^T P = -2Q
\]

(A.5)

has a unique positive definite solution, \( P \). Let

\[
V(x) = x^T Px ;
\]

(A.6)

then, along the solution of (1)

\[
\frac{dV(x)}{dt} = 2x^T P \dot{x} = 2x^T PAx + 2x^T Pbu.
\]

(A.7)

Moreover, because of (A.5)

\[
\frac{dV(x)}{dt} = -2x^T Qx + 2x^T Pb u,
\]

(A.8)

which upon integration (0 to \( \infty \)) yields

\[
\int_0^\infty x^T Qx \ dt = -\frac{1}{2} \int_0^\infty \frac{dV}{dt} \ dt + \int_0^\infty x^T Pbu \ dt.
\]

(A.9)

Hence

\[
J(u) = \frac{1}{2} \int_0^\infty x^T Pbx_0 + S(u)
\]

(A.10)
(here we are assuming the controlled system to be stable), where

\[ S(u) = \int_0^\infty x^T P b u \, dt. \quad (A.11) \]

Clearly, the control \( u \) which minimizes \( J(u) \) is the same control which minimizes \( S(u) \).

By the Maximum Principle of Pontryagin [7], the optimal control \( u \) must maximize the Hamiltonian

\[ H(x, P, b, u) = 0 x \text{A} x + \Psi^T \Psi b u - x^T P b u. \quad (A.12) \]

Hence, when the system trajectory is not on a singular surface (defined by \( b^T (\Psi - P x) = 0 \)), the optimal control is given by

\[ u = \text{sign} [b^T (\Psi - P x)]. \quad (A.13) \]

From the definition of \( \Psi \),

\[ \frac{d\Psi}{dt} = -A^T \Psi + P b u; \quad (A.14) \]

hence,

\[ \Psi(t) = e^{-A t} \Psi_0 + \int_0^t e^{-A(t-\tau)} P b u(\tau) \, d\tau. \quad (A.15) \]

Jen Wei's [1] solution to the optimization problem is simply to take

\[ u = - \text{sign} b^T P x, \quad (A.16) \]

which would be correct if

\[ \Psi(t) = 0. \quad (A.17) \]
The control (A.16) is also Bass' [4] suggestion for a simple control scheme. The only difference is that Jen Wei considers the trajectory which keeps $b^T P x = 0$ to be optimal as well (he called it the slippage mode); he then finds the associated linear control--on the other hand, Bass simply lets $u = 0$ whenever $b^T P x = 0$. 
APPENDIX B. SINGULAR SURFACES

Without loss of generality we can put our assumed completely controllable system (1) into companion form [5]:

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_1 & -a_2 & -a_3 & \cdots & -a_n
\end{bmatrix}
x + \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix} u; \quad (B.1)
\]

we will assume that this is the case. It is obvious now that the maintenance of a singular solution over a finite time interval requires that the \(n\) component of \(\Psi\) and its derivatives be zero:

\[
\Psi_n = \Psi^{(1)}_n = \Psi^{(2)}_n = \ldots = 0 \quad (B.2)
\]

From (7), the expression for the first derivative of the \(i\)th component of \(\Psi\) is

\[
\dot{\Psi}_i = \frac{\partial x}{\partial x_i}, \quad (B.3)
\]

and the \(i\)th derivative is

\[
\frac{d^i}{dt^i} (-\Psi_i) = \frac{d^{i-1}}{dt^{i-1}} \frac{\partial x}{\partial x_i}. \quad (B.4)
\]

Consequently,

\[
\sum_{i=1}^{n} (-1)^i \frac{d^i}{dt^i} (-\Psi_i) = \sum_{i=1}^{n} (-1)^i \frac{d^{i-1}}{dt^{i-1}} \frac{\partial x}{\partial x_i}, \quad (B.5)
\]
or, in view of (6),
\[
\sum_{i=1}^{n} (-1)^i \frac{d_i}{dt} \frac{\partial (x^T Q x)}{\partial x_i} = \sum_{i=1}^{n} (-1)^i \frac{d_i}{dt} \left[ \frac{\partial}{\partial x_i} (\Psi^T A x + \Psi^T b f(v)) + \frac{d\Psi^T}{dt} \right] \tag{B.6}
\]

The singularity requirement (B.2) reduces (B.6) to
\[
\sum_{i=1}^{n} (-1)^i \frac{d_i}{dt} \frac{\partial (x^T Q x)}{\partial x_i} = \sum_{i=1}^{n} (-1)^i \frac{d_i}{dt} \left[ \frac{\partial}{\partial x_i} (\Psi^T A x) + \frac{d\Psi^T}{dt} \right] \tag{B.7}
\]

Moreover, because the system is in companion form (B.1), we find that
\[
\frac{\partial}{\partial x_1} (\Psi^T A x) = \frac{\partial}{\partial x} (x^{(i-1)}) (\Psi^T A x) = \Psi_n a_i + \Psi_{i-1} \tag{B.8}
\]

This result, when substituted into (B.7), yields
\[
\sum_{i=1}^{n} (-1)^i \frac{d_i}{dt} \frac{\partial (x^T Q x)}{\partial x_i} = \sum_{i=1}^{n} (-1)^i a_1 \Psi_{n} \tag{B.9}
\]

But, from (B.2), the sum on the right hand side is zero; consequently, the singular surfaces are given by the solutions of
\[
\sum_{i=1}^{n} (-1)^i \frac{d_i}{dt} \frac{\partial (x^T Q x)}{\partial x_i} = 0 \tag{B.10a}
\]
or
\[
\sum_{i=1}^{n} (-1)^i \frac{d_i}{dt} \frac{\partial (x^T Q x)}{\partial x} = 0 \tag{B.10b}
\]

which is the Euler equation and the fundamental necessary condition for an extremum of (5). We will not discuss sufficient conditions—in general they are more easily determined for a specific problem than for an overall class of problems.
It has been shown [10] that the roots of the characteristic polynomial associated with the Euler equation (B.10) occurs in quadrantal symmetry in the complex plane (this fact follows simply from the self-adjoint character of the Euler equation); consequently, we need only factor out the left half plane roots to obtain the stable singular solutions. The stable singular solutions satisfy the linear differential equation of order \((n-1)\)

\[
\sum_{i=1}^{n} h_i x^{(i-1)}(t) = 0, \quad (B.11)
\]

which describes an \((n-1)\)-dimensional hyperplane (the stable singular surface) in the \(n\)-dimensional state space.
1. For a short derivation and discussion of the results of Jen-Wei, see Appendix A.

2. The functional $J(c)$ is indicated as a function of the vector $c$ because the $c_i$'s are the only control variables, which follows from the problem formulation, (1) - (4).

3. The system is assumed completely controllable; hence the state equations can always be transformed to companion form (see [5] and Appendix B).

4. $h^T x = 0$ implies $u = (a - \hat{h})^T x$, which is the case on the singular surface.

5. The alternate solution, $c_1 = +\infty$, renders the system unstable.
DISTRIBUTION LIST AS OF JANUARY 29, 1964

1 Director
Air University Library
Maxwell Air Force Base, Alabama
Attn: CR-4803a

1 Redstone Scientific Information Center
U.S. Army Missile Command
Redstone Arsenal, Alabama

1 Electronics Research Laboratory
University of California
Berkeley 4, California

1 Hughes Aircraft Company
Florencce and Teale
Culver City, California
Attn: N. E. Devereux
Technical Document Center

3 Autonetics
9150 East Imperial Highway
Downey, California
Attn: Tech. Library, 3041-11

1 Dr. Arnold T. Nordsieck
General Motors Corporation
Defense Research Laboratories
6767 Hollister Avenue
Goleta, California

1 University of California
Lawrence Radiation Laboratory
P. O. Box 608
Livermore, California

1 Mr. Thomas L. Hartwick
Aerospace Corporation
P. O. Box 95085
Los Angeles 45, California

1 Lt. Colonel Willard Levin
Aerospace Corporation
P. O. Box 95085
Los Angeles 45, California

1 Professor Zorib Kaprielian
University of Southern California
University Park
Los Angeles 7, California

1 Sylvania Electronic Systems - West
Electronic Defense Laboratories
P. O. Box 201
Mountain View, California
Attn: Documents Center

1 Varian Associates
61 Hansen Way
Palo Alto, California
Attn: Dr. Ira Weissman

1 Huston Denslow
Library Supervisor
Jet Propulsion Laboratory
California Institute of Technology
Pasadena, California

1 Professor Nicholas George
California Institute of Technology
Electrical Engineering Department
Pasadena, California

1 Space Technology Labs., Inc.
One Space Park
Redondo Beach, California
Attn: Acquisitions Group
STL Technical Library

2 Commanding Officer and Director
U.S. Naval Electronics Laboratory
San Diego 52, California
Attn: Code 2800, C. S. Manning

1 Commanding Officer and Director
U.S. Navy Electronics Laboratory
San Diego 52, California
Attn: Library

1 Office of Naval Research Branch Office
1000 Geary Street
San Francisco, California

1 The RAND Corporation
1700 Main Street
Santa Monica, California
Attn: Library

1 Stanford Electronics Laboratories
Stanford University
Stanford, California
Attn: SEL Documents Librarian

1 Dr. L. F. Carter
Chief Scientist Air Force
Room 4E-324, Pentagon
Washington 25, D. C.

1 Mr. Robert L. Fest
Associate Director for Research
Research and Technology Division
AFMC
Boiling Air Force Base 25, D. C.

1 Captain Paul Johnson (USN-Ret)
National Aeronautics and Space Administration
1520 N Street, N. W.
Washington 25, D. C.

1 Major Edwin M. Myers
Headquarters USAF (AFRDM)
Washington 25, D. C.

1 Dr. James Ward
Office of Deputy Director
Research and Info
Department of Defense
Washington 25, D. C.

1 Dr. Alan T. Waterman
Director, National Science Foundation
Washington 25, D. C.

1 Mr. G. D. Watson
Defense Research Member
Canadian Joint Staff
2450 Massachusetts Ave., N. W.
Washington 8, D. C.

1 Mr. Arthur G. Winer
Chief Scientist
Air Force Systems Command
Andrews Air Force Base
Washington 25, D. C.

1 Director, Advanced Research Projects Agency
Washington 25, D. C.

1 Air Force Office of Scientific Branch
Directorate of Engineering Sciences
Washington 25, D. C.
Attn: Electronics Division

1 Director of Science and Technology
Headquarters, USAF
Washington 25, D. C.
Attn: AFRST-EL/GU

1 AFRST - SC
Headquarters, USAF
Washington 25, D. C.

1 Headquarters, R & T Division
Boiling Air Force Base
Washington 25, D. C.
Attn: RTHR

1 Headquarters, U. S. Army Material Command
Research Division, R & D Directorate
Washington 25, D. C.
Attn: Physics & Electronics Branch
Electronics Section

1 Commanding Officer
Diamond Ordnance Fuze Laboratories
Washington 25, D. C.
Attn: Librarian, Room 211, Bldg. 92

1 Operation Evaluation Group
Chief of Naval Operations (OP-03EO)
Department of Navy
Washington, D. C. 20330

1 Chief of Naval Operations (Code OP-O_NT)
Department of the Navy
Washington, D. C. 20330

1 Commanding Officer
U. S. Army Personnel Research Office
Washington 25, D. C.

1 Commanding Officer & Director
Code 142 Library
David W. Taylor Model Basin
Washington, D. C. 20077

1 Chief, Bureau of Ships (Code 686)
Department of the Navy
Washington, D. C. 20360

1 Chief, Bureau of Ships (Code 732)
Department of the Navy
Washington, D. C. 20360

1 Chief, Bureau of Naval Weapons
Technical Library, DIL-3
Department of the Navy
Washington, D. C. 20360

1 Director, (Code 5140)
U. S. Naval Research Laboratory
Washington, D. C. 20390

1 Chief of Naval Research (Code 437)
Department of the Navy
Washington, D. C. 20360

1 Dr. H. Wallace Sinaiko
Institute for Defense Analyses
Research & Engineering Support Division
1666 Connecticut Ave., N. W.
Washington 9, D. C.

1 Data Processing Systems Division
National Bureau of Standards
Room 239, Bldg. 10
Washington 25, D. C.
Attn: A. K. Smilow

1 National Bureau of Standards
Research Information Center & Advisory Service on Information Processing
Data Processing Systems Division
Washington 25, D. C.

1 Exchange and Gift Division
The Library of Congress
Washington 25, D. C.

1 NASA Headquarters
Office of Applications
400 Maryland Avenue, S. W.
Washington 25, D. G.
Attn: Mr. A. M. Greg Andrus
Code FC

1 APGJ (PGPJ)
Eglis Air Force Base
Florida

1 Martin Company
P. O. Box 5837
Orlando, Florida
Attn: Engineering Library
MP-30

1 Commanding Officer
Office of Naval Research, Chicago Branch
6th Floor, 230 North Michigan
Chicago 1, Illinois

1 Laboratories for Applied Sciences
University of Chicago
6220 South Drexel
Chicago 37, Illinois

1 Librarian
School of Electrical Engineering
Purdue University
Lafayette, Indiana