APPLICATION OF LINEAR GRAPHS TO ELECTRICAL NETWORKS, SWITCHING NETWORKS & COMMUNICATION NETS
W. Mayeda

REPORT R-203 APRIL, 1964

COORDINATED SCIENCE LABORATORY
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS

Contract DA-36-039-AMC 02208(E)
DA Project 3A-99-25-004

The research reported in this document was made possible by support extended to the University of Illinois, Coordinated Science Laboratory, jointly by the Department of the Army, Department of the Navy (Office of Naval Research), and the Department of the Air Force (Office of Scientific Research) under Department of Army Contract DA-36-039-AMC 02208(E).
# TABLE OF CONTENTS

| I.  | INTRODUCTION .......................................................... | 1 |
| II. | PRELIMINARY .......................................................... | 3 |
|     | 1. Linear Graphs .................................................... | 3 |
| III. | ANALYSIS OF PASSIVE NETWORKS BY LINEAR GRAPHS ............... | 19 |
|     | 2. Topological Formula for One Terminal-Pair Networks ...... | 20 |
|     | 3. Topological Formulas for Two-Terminal-pair Networks ..... | 23 |
|     | 4. Analysis of Passive Networks by Digital Computer ........ | 35 |
| IV. | ANALYSIS OF ACTIVE NETWORKS BY LINEAR GRAPHS ............... | 39 |
|     | 5. Linear Graphs of Active Networks ............................ | 41 |
|     | 6. Sign-Permutation and Principal Tree ....................... | 46 |
|     | 7. Topological Formulas for Open Circuit Functions of ... | 58 |
|     | Active Networks ................................................... |
|     | 8. Topological Formulas for Short Circuit Functions of ... | 63 |
|     | Active Networks ................................................... |
|     | 9. Analysis of Active Networks by Digital Computer and ... | 68 |
|     | Further Remarks ................................................... |
| V.  | APPLICATION OF LINEAR GRAPHS TO SWITCHING NETWORKS .......... | 72 |
|     | 10. Properties of Paths in a Non-Oriented Linear Graph and | 72 |
|     | S.C. Switching Functions ......................................... |
|     | 11. Realizability of Fundamental Cut Set Matrices of Non- | 78 |
|     | Oriented Linear Graphs ............................................ |
|     | 12. Cut Set Matrices to Incidence Matrices ................... | 90 |
13. Fundamental Cut Set Matrices of Oriented Linear Graphs ........................................ 93
14. Multi-Terminal S. C. Switching Networks ............................................... 101
15. On Multi-Contact Switching Networks and Remarks .......... 106

VI. APPLICATION OF LINEAR GRAPHS TO COMMUNICATION NETS ..................... 109
16. Lossless Non-Oriented Communication Nets ........................................ 109
17. Lossless Oriented Communication Nets ........................................ 118
18. Lossy Communication Nets .......................................................... 127

APPENDIX ............................................................................. 139
REFERENCES .......................................................................... 144
I. INTRODUCTION

Since Kirchhoff [1] and Maxwell [2] who introduced rules to analyze electrical networks by inspection, linear graphs have been applied not only to passive [3,4,5] and active [6,7,8] networks, but also to switching networks [9,10] and communication nets [11,12,13,14,15,16]. There are some difference in ways of using linear graphs in different areas. When a linear graph is used to synthesize a switching function, properties of paths and cut set matrices become important. On the other hand, in order to use a linear graph to study communication nets, existence of special cut sets, called saturated cut sets, becomes most important. Hence, it is reasonable to divide this paper into three parts and discuss application of linear graphs in electrical networks, in switching networks, and in communication nets separately.

Chapter 2 gives definitions and properties of linear graphs. The definitions of linear graphs used in this paper are somewhat different from those given by other people [17,18,19,20]. These new definitions will eliminate unclear operation on linear graphs such as deletion of edges from a linear graph.

Chapter 3 and 4 show how linear graphs are used in analysis of electrical networks. The proof for the topological formulas of transfer functions of passive networks without mutual couplings given here is the first formal and precise proof in this field. By the use of topological formulas, such a passive network can be analyzed by a digital computer. The flow chart given here is the first digital computer program to obtain non-numerical solutions of system problems [21,22].
Active networks can be represented by two linear graphs, and by doing so, analysis of such a network can be accomplished by inspection. This technique is called the "Coates-Mayeda" method. Furthermore, by the use of the above technique, an active network can be analyzed non-numerically by a digital computer [23] which is shown at the end of this chapter.

Chapter 5 discusses the application of linear graphs to switching networks by starting with exploration of the properties of paths in a linear graph [24]. Then the necessity of realizing a cut set matrix (or circuit matrix) is discussed.

Away from the switching theory, properties of a cut set matrix [25,26] and a non-singular matrix [27] which converts a cut set matrix to an incidence matrix are discussed. This leads to a realizability condition for a cut set matrix which has been used to synthesize a switching function by a computer. Tutte [28] also gives a necessary and sufficient condition for a cut set matrix, but it is very difficult to use in order to test a matrix to be a cut set matrix.

At the end of this chapter, properties of multi-ports switching functions are discussed, that is if $F_1, F_2, \ldots, F_k$ are the collection of switching functions of pairs of terminals in one switching network, then there exist definite relationships among these functions. These relationships give a method of realizing such a collection of switching functions.

Chapter 6 discusses a rather new field which is many-ports flow problems which is called the theory of communication nets. In a communication net, the maximum possible flow from a vertex $i$ to vertex $j$ is
given by \( t_{ij} \) called a terminal capacity. A terminal capacity matrix \( T \) is a matrix whose \((ij)\) entry gives the terminal capacity from vertex \( i \) to vertex \( j \). Then it is natural to ask whether a given matrix is a terminal capacity matrix or not. In the case of lossless non-oriented communication nets, a necessary and sufficient condition is known[11]. In the case of lossless oriented communication nets, necessary and sufficient conditions for a terminal capacity matrix have not been found. There are several necessary conditions and a sufficient conditions [14]. These are discussed in detail.

In the case of lossy communication nets [29] only necessary conditions have been found recently which are obtained by showing the existence of a saturated cut set in lossy communication nets. The existence of such a cut set gives the relationship between cut sets and the maximum flows which is more general than one given by Ford and Fulkerson [30].

The appendix gives all symbols which are used in this paper for convenience.

II. PRELIMINARY

There are several ways of defining linear graphs [17,18,19,20]. Here we introduce one which gives no difficulty to include parallel edges, isolated vertices, etc.

1. Linear Graphs

Let \( \Omega \) be a set of elements \( \xi_1, \xi_2, \ldots, \xi_k \) and \( G \) be a set of elements \( e_1, e_2, \ldots, e_n \). Then a linear graph is a class of two sets
G and \( \Omega \) represented by \( \{G, \Omega\}_f \) with a mapping function \( f \) which is many to one such that every \( e_p \) in \( G \)

\[
f(e_p) = (\xi_1, \xi_j) \in \Omega \times \Omega
\]

(1-1)

\( \xi \)'s are called vertices (sometimes named nodes or terminals). \( e \)'s are called edges (elements and branches are often used). \( \Omega \) is called a vertex-set and \( G \) is an edge-set.

Unless it is required to specify the mapping function \( f \), we will omit to give subscript \( f \) of \( \{G, \Omega\}_f \). If linear graphs \( \{G_p, \Omega_p\}_p \) (\( p=1,2 \)) have no subscripts, we assume that \( f_p \) of \( \{G_p, \Omega_p\}_p \) satisfies that for any edge \( e \) in both \( G_1 \) and \( G_2 \), \( f_1(e) = f_2(e) \).

Let \( f(e_p) = (\xi_1, \xi_j) \). Vertices \( \xi_1 \) and \( \xi_j \) are called end-points of edge \( e_p \). When \( \Omega \times \Omega \) is the set of non-ordered pairs of vertices, \( \{G, \Omega\} \) is said to be non-oriented or to be a non-oriented graph and edges in the graph are called non-oriented edges. When \( \Omega \times \Omega \) is the set of ordered pairs of vertices, then \( \{G, \Omega\} \) is said to be oriented or called an oriented graph and edges in the graph are called oriented edges. The orientation of edge \( e \) is from \( \xi_1 \) to \( \xi_2 \) where \( f(e) = (\xi_1, \xi_2) \). If \( f(e_p) = (\xi_1, \xi_j) \) and \( f(e_q) = (\xi_1, \xi_j) \) or \( f(e_q) = (\xi_j, \xi_1) \), then \( e_p \) and \( e_q \) are said to be in parallel. If \( f(e) = (\xi_1, \xi_1) \), then \( e \) is called a self-loop.

Linear graphs used in this paper are assumed to have no self-loops.

Definition 1-1: For a given \( \{G, \Omega\} \), \( \Omega(G) \) is the set of vertices each of which is an end-point of at least one edge in \( G \) and any edge \( e \in G \), \( f(e) \) is in \( \Omega(G) \times \Omega(G) \).
Definition 1-2: For a given \( \{G, \Omega\} \), and \( \Omega_1, \Omega_2 \subset \Omega \), \( G(\Omega_1 \times \Omega_2) \) is a set of all edges in G which satisfy that
\[
f(e) \in \Omega_1 \times \Omega_2
\]
Thus, \( G(\Omega_1 \times \Omega_2) \subset G \). When \( \Omega_1 = \Omega_2 \), we use \( G(\Omega_1) \) to represent \( G(\Omega_1 \times \Omega_2) \).

Example 1-1: Suppose \( \{G, \Omega\} \) is a linear graph in Fig. 1-1. Then \( \Omega(G) \equiv \{1, 2, 3\} \). Let \( \Omega_1 \equiv \{1, 2\} \) and \( \Omega_2 \equiv \{3\} \). Then \( G(\Omega_1) \equiv \{a\} \) and \( G(\Omega_1 \times \Omega_2) \equiv \{b, c\} \).

![Fig. 1-1. A Linear Graph \( \{G, \Omega\} \).](image)

Definition 1-3:
\[
\{G_1, \Omega_1\} \equiv \{G_2, \Omega_2\} \text{ means } G_1 \equiv G_2 \text{ and } \Omega_1 \equiv \Omega_2 \tag{1-2}
\]
\[
\{G_1, \Omega_1\} \supset \{G_2, \Omega_2\} \text{ means } G_1 \supset G_2 \text{ and } \Omega_1 \supset \Omega_2 \tag{1-3}
\]

Definition 1-4: \( \{G_1, \Omega_1\} \cap \{G_2, \Omega_2\} \equiv \emptyset \) means that \( G_1 \cap G_2 \equiv \emptyset \) and \( \Omega_1 \cap \Omega_2 \equiv \emptyset \).

Definition 1-5: Sets \( A_p \) (p=1,2,...,k) are said to be disjoint if
\[
A_{r_1} \cap A_{r_2} \cap \ldots \cap A_{r_s} \equiv \emptyset \text{ for all } r's \text{ where } 1 \leq r_1, r_2, \ldots, r_s \leq k, \text{ and } r_1 \neq r_2 \neq \ldots \neq r_s, 1 < r \leq k.
\]

Definition 1-6: \( \{G_2, \Omega_2\} \) is said to be a subgraph of \( \{G_1, \Omega_1\} \) expressed by \( \{G_2, \Omega_2\} \subset \{G_1, \Omega_1\} \) if
\[
G_2 \subset G_1 \text{ and } \Omega_2 \subset \Omega_1
\]
\( \{G_2, \Omega_2\} \) is said to be a proper subgraph of \( \{G_1, \Omega_1\} \) if \( \{G_2, \Omega_2\} \) is a subgraph of \( \{G_1, \Omega_1\} \) and either
\[ G_2 \not\subset G_1 \text{ or } \Omega_2 \not\subset \Omega_1 \]

Definition 1-7: Operation \( \odot \) between linear graphs is defined as
\[
\{G_1, \Omega_1\} \odot \{G_2, \Omega_2\} \equiv \{G_1 \odot G_2, \Omega_1 \odot \Omega_2\} \cup \Omega(G_1 \odot G_2) \quad (1-4)
\]
\( \odot \) can be any operation used in Set Theory. For example, \( \{G_1, \Omega_1\} \cap \{G_2, \Omega_2\} \equiv \{G_1 \cap G_2, \Omega_1 \cap \Omega_2\} \cup \Omega(G_1 \cap G_2) \). Notice that \( \{G_1, \Omega_1\} \cup \{G_2, \Omega_2\} \equiv \{G_1 \cup G_2, \Omega_1 \cup \Omega_2\} \cup \Omega(G_1 \cup G_2) \).  

Example 1-2: Suppose \( G_1 = \{a,b,c\}, \Omega_1 = \{1,2,3,4\}, G_2 = \{a,d\}, \) and \( \Omega_2 = \{a,d\} \), and \( \Omega_2 = \{1,2,4,5\} \) where \( f(a) = (1,2), \) \( f(b) = (1,3), \) \( f(c) = (3,4) \) and \( f(d) = (2,4) \). \( \{G_1, \Omega_1\} \) and \( \{G_2, \Omega_2\} \) are shown in Fig. 1-2a and b. Then \( \{G_1, \Omega_1\} \cap \{G_2, \Omega_2\} \equiv \{a,1,2,4\} \) as shown in Fig. 1-2c. \( \{G_1, \Omega_1\} \oplus \{G_2, \Omega_2\} \equiv \{b,c,d,1,2,3,4,5\} \) as shown in Fig. 1-2d.

Notice that \( \{G_1, \Omega_1\} \cap \{G_2, \Omega_2\} \) is a proper subgraph of \( \{G_1, \Omega_1\} \) but \( \{G_1, \Omega_1\} \oplus \{G_2, \Omega_2\} \) is a subgraph of neither \( \{G_1, \Omega_1\} \) nor \( \{G_2, \Omega_2\} \).
Definition 1-8: A linear graph \( \{G, \Omega\} \) is said to be separated if there exist subsets \( \Omega_1 \) and \( \Omega_2 \) such that

1. \( \Omega_1 \cap \Omega_2 = \emptyset \) \tag{1-5}
2. \( \Omega_1 \cup \Omega_2 = \Omega \) and \( \Omega_1 \cup \Omega_2 = \Omega \) \tag{1-6}
3. \( \{G(\Omega_1), \Omega_1\} \cup \{G(\Omega_2), \Omega_2\} = \{G, \Omega\} \) \tag{1-7}

Otherwise \( \{G, \Omega\} \) is said to be connected.

Example 1-3: Let \( G = \{a, b, c, d\} \) and \( \Omega = \{1, 2, 3, 4, 5\} \) as shown in Fig. 1-3. Then there exist \( \Omega_1 \) and \( \Omega_2 \) where \( \Omega_1 = \{1, 2, 3\} \) and \( \Omega_2 = \{4, 5\} \) which satisfy above conditions, that is \( \Omega_1 \cap \Omega_2 = \emptyset \) and \( \Omega_1 \cup \Omega_2 = \Omega \). Since \( G(\Omega_1) = \{a, b, c\} \) and \( G(\Omega_2) = \{d\} \), \( \{G(\Omega_1), \Omega_1\} \cup \{G(\Omega_2), \Omega_2\} = \{G, \Omega\} \). Thus \( \{G, \Omega\} \) is separated. On the other hand, \( \{G(\Omega_1), \Omega_2\} \) is connected.

![Fig. 1-3. \( \{G, \Omega\} \)](image)

Definition 1-9: For a given \( \{G, \Omega\} \), if there exist disjoint sets \( \Omega_1, \Omega_2, \ldots, \), and \( \Omega_k \) such that

1. \( \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k = \Omega \) \tag{1-8}
2. \( \{G(\Omega_1), \Omega_1\} \cup \{G(\Omega_2), \Omega_2 \cup \cdots \cup G(\Omega_k), \Omega_k\} = \{G, \Omega\} \) \tag{1-9}
3. every \( \{G(\Omega_p), \Omega_p\} \), \( p = 1, 2, \ldots, k \) is connected, which is called
a maximal connected subgraph, then we say that \( \rho \) of \( \{G, \Omega\} \) is \( k \), where \( \rho \) is the number of maximal connected subgraphs of \( \{G, \Omega\} \).

Definition 1-10: Suppose \( G \) consists of \( e \) edges and \( \Omega \) consists of \( v \) vertices. Then the rank of \( \{G, \Omega\} \) is \( v - \rho \) and the nullity of \( \{G, \Omega\} \) is \( e - v + \rho \).

Definition 1-11: The degree of a vertex \( \xi \) in \( \{G, \Omega\} \) is the number of edges in \( G \) whose end-point is \( \xi \).

Definition 1-12: A circuit is a connected linear graph \( \{G, \Omega\} \) which has the property that every vertex in \( \Omega \) is of degree two. Circuits \( \{G_1, \Omega_1\} \) and \( \{G_2, \Omega_2\} \) are said to be edge disjoint if \( G_1 \cap G_2 = \emptyset \).

Theorem 1-1: If every vertex in \( \Omega \) of a linear graph \( \{G, \Omega\} \) is of even degree, the graph is either a circuit or an edge disjoint union of circuits.

Proof: If every vertex in \( \Omega \) of a linear graph \( \{G, \Omega\} \) is of degree 2, it is a circuit by definition. If every vertex in \( \Omega \) of a linear graph \( \{G, \Omega\} \) is of even degree, we can obtain a sequence of edges starting with any edge in \( G \) as

\[ e_1, e_2, \ldots, e_k \]

such that vertex \( \xi_1 \) is the end-point of edges \( e_i \) and \( e_{i+1} \) for \( i = 1, 2, \ldots, k-1 \), \( \xi_0 \neq \xi_1 \) is the end-point of \( e_1 \), \( \xi_{k+1} \neq \xi_k \) is the end-point of \( e_k \), \( \xi_0, \xi_1, \xi_2, \ldots, \) and \( \xi_k \) are all distinct and \( \xi_{k+1} \) is one of \( \xi_0, \xi_1, \ldots, \) and \( \xi_{k-1} \). This is possible because every vertex in \( \Omega \) of \( \{G, \Omega\} \) is of even degree. Suppose \( \xi_{k+1} = \xi_p \) (\( 0 \leq p \leq k-1 \)). Then \( \{G_1, \Omega (G_1)\} \) is a circuit where \( G_1 = \{e_{p+1}, \ldots, e_k\} \). This means that if every vertex in \( \Omega \) of
\( \{G, \Omega\} \) is of even degree, there exists a subgraph \( \{G_1, \Omega(G_1)\} \) of 
\( \{G, \Omega\} \) which is a circuit. Also it is obvious that \( \{G-G_1, \Omega(G-G_1)\} \)
is of even degree if \( G-G_1 \equiv \emptyset \). Furthermore, \( G-G_1 \) consists of less
number of edges than \( G \).

Suppose the theorem is true for all linear graphs \( \{G_p, \Omega_p\} \)
which have the properties that every vertex in \( \Omega_q \) is of even
degree and \( G \) consists of \( n \) edges where \( n \leq \eta \). Then it is true for a
linear graph \( \{G_0, \Omega_0\} \) where every vertex in \( \Omega_0 \) is of even degree and
\( G_0 \) consists of \( \eta+1 \) edges. Q.E.D.

Theorem 1-2: Let \( \{G_1, \Omega_1\} \) and \( \{G_2, \Omega_2\} \) be either circuits or
disjoint unions of circuits. Then \( \{G_1, \Omega_1\} \oplus \{G_2, \Omega_2\} \) is either a
circuit or an edge disjoint union of circuits.

Proof: Consider a vertex \( \xi \) in \( \Omega_1 \oplus \Omega_2 \) where \( \Omega_1 \) and \( \Omega_2 \) are
vertex sets of linear graphs \( \{G_1, \Omega_1\} \) and \( \{G_2, \Omega_2\} \) respectively. Suppose
\( \xi \) is an end-point of an edge \( e \) in either \( G_1 \) or \( G_2 \). Then vertex \( \xi \) is in
\( \Omega(G_1 \oplus G_2) \) if edge \( e \) is in \( G_1 \oplus G_2 \). Suppose edge \( e \) is not in \( G_1 \oplus G_2 \),
then \( \xi \) is in \( \Omega_2 \) as well as in \( \Omega_1 \). Thus \( \xi \notin \Omega_1 \oplus \Omega_2 \) which contradicts
the assumption that \( \xi \in \Omega_1 \oplus \Omega_2 \). Thus if \( \xi \) is an end-point of edge \( e \)
in either \( G_1 \) or \( G_2 \), \( \xi \) is in \( \Omega(G_1 \oplus G_2) \). If \( \xi \) is not an end-point of
edge \( e \) in either \( G_1 \) or \( G_2 \), then the degree of vertex \( \xi \) is zero. Thus
we only need to show the degree of vertices in \( \Omega(G_1 \oplus G_2) \) is even in
order to prove the theorem. Since the degree of every vertex in
\( \Omega_1 \) of \( \{G_1, \Omega_1\} \) and that of vertices in \( \Omega_2 \) of \( \{G_2, \Omega_2\} \) is even, the degree
of every vertex in \( \Omega(G_1) \) of \( \{G_1, \Omega(G_1)\} \) and that of every vertex in
\( \Omega(G_2) \) of \( \{G_2, \Omega(G_2)\} \) is even. Hence degree of every vertex in
\( \Omega \left( G_1 \oplus G_2 \right) \) of \( \left\{ G_1, \Omega(G_1) \right\} \oplus \left\{ G_2, \Omega(G_2) \right\} \) is even. Thus by Theorem 1-1, the theorem is true. Q.E.D.

Definition 1-13: An edge sequence of a circuit \( \left\{ G, \Omega \right\} \) is the sequence of edges in \( G \) starting with an edge as
\[ e_1, e_2, \ldots, e_k \]
such that vertex \( z_1 \in \Omega \) is the end-point of \( e_1 \) and \( e_{i+1} \) for \( i=1,2,\ldots, \)
\( k-1 \), and \( z_k \) is the endpoint of \( e_1 \) and \( e_k \). The orientation of the circuit is chosen so that it agrees with the above edge sequence. Consider a circuit \( z = \{a,b,c\} \) of \( \left\{ G, \Omega \right\} \) in Figure 1-3. Then the edge sequence of \( z \) can be
\[ a, b, c. \]
So the orientation of circuit \( z \) agrees with the orientation of edge \( a \). However it disagrees with the orientation of edge \( c \).

Definition 1-14: A directed circuit is an oriented graph which is a circuit and satisfies that if the orientation of one edge in the circuit agrees with that of the circuit, the orientation of every edge in the circuit agrees with that of the circuit, or if the orientation of one edge disagrees with that of the circuit, then the orientation of every edge in the circuit disagrees with that of the circuit. For example, a circuit \( z = \{a,b,c\} \) of \( \left\{ G, \Omega \right\} \) in Figure 1-3 is not a directed circuit. However, if we change the orientation of edge \( c \) as \( f(c) = (3,1) \), the circuit becomes a directed circuit.
Definition 1-15: If every vertex in $\Omega$ of a connected linear graph $\{G, \Omega\}$ is of degree two except two vertices which are of degree one, then the graph is called a path. One of these two vertices which are of degree one is called the initial vertex and the other is called the final vertex of the path. The orientation of a path is defined to be from the initial vertex to the final vertex. A path from $i$ to $j$ means that a path whose initial vertex is $i$ and who final vertex is $j$.

Definition 1-16: An edge sequence of a path $\{G, \Omega\}$ from $p$ to $q$ is the sequence of edges in $G$

$$e_1, e_2, \ldots, e_k$$

such that vertex $\xi_i$ is the end-point of $e_i$ and $e_{i+1}$ for $i=1, 2, \ldots, k-1$, vertices $p$ and $\xi_1$ are the end-points of $e_1$ and vertices $q$ and $\xi_{k-1}$ are the end-points of $e_k$. The vertex sequence of a path $\{G, \Omega\}$ from $p$ to $q$ is a sequence

$$p, \xi_1, \xi_2, \ldots, \xi_{k-1}, q$$

of vertices in $\Omega$ corresponding to the edge sequence $e_1, e_2, \ldots, e_k$ of $\{G, \Omega\}$ such that $\xi_i$ is the endpoint of $e_i$ and $e_{i+1}$ for $i=1, 2, \ldots, k-1$.

Definition 1-17: A directed path from $p$ to $q$ is a path from $p$ to $q$ which is an oriented graph such that the orientation of every edge in the path agrees with that of the path.

Theorem 1-3: If every vertex in $\Omega$ of a graph $\{G, \Omega\}$ is of even degree except two which are of degree one, the graph is either a path of an edge disjoint union of a path and circuits.

The proof of this theorem is similar to that of Theorem 1-1.
Definition 1-18: \( \{G, \Omega\} \) is said to be non-separable if for any two edges in \( G \), there exists a circuit which contains these edges.

Definition 1-19: \( \{G, \Omega\} \) is a cut set of \( \{G, \Omega\} \) if

1. \( \Omega_1 \equiv \Omega(G_1) \)
2. The rank of \( \{G - G_1, \Omega\} \) is one less than that of \( \{G, \Omega\} \) and
3. There exist no subgraph \( \{G_2, \Omega_2\} \) of \( \{G_1, \Omega_1\} \) such that 1 and 2 hold.

It is convenient to define the orientation of a cut set which is in general arbitrary. However it is usually chosen such that it agrees with the orientation of an edge in the cut set.

Definition 1-20: Let \( \{G_1, \Omega_1\} \) be a cut set of a connected linear graph \( \{G, \Omega\} \). Then \( \{G', \Omega\} \) where \( G' \equiv G - G_1 \) has the property that there exists disjoint vertex sets \( \Omega_a \) and \( \Omega_b \) such that (1) \( \Omega_a \cup \Omega_b \equiv \Omega \) and (2) \( \{G'_a, \Omega_a\} \) and \( \{G'_b, \Omega_b\} \) are the maximally connected subgraphs of \( \{G', \Omega\} \). Cut set \( \{G_1, \Omega_1\} \) is said to separate vertex 1 and 2 if \( i \in \Omega_a \) and \( j \in \Omega_b \). \( \Omega_a \) and \( \Omega_b \) are called the corresponding vertex sets of the cut set, and \( G'_a \) and \( G'_b \) are called the corresponding maximal connected subgraphs with respect to the cut set.

Example 1-4: Let \( \{G, \Omega\} \) be the linear graph in Fig. 1-4, where \( \Omega \equiv \{1,2,3,4\} \) and \( G \equiv \{a,b,c,d,e\} \). Let \( G_1 \) be

\[
G_1 \equiv \{a,c,e\}
\]

Then \( \{G_1, \Omega(G_1)\} \) is a cut set and \( \Omega_a \equiv \{1,3\} \), \( \Omega_b \equiv \{2,4\} \) gives \( G'_a \equiv \{b\} \) and \( G'_b \equiv \{d\} \) where \( G' \equiv G - G_1 \). We can say that this cut set separates vertices 1 and 2 because \( 1 \in \Omega_a \) and \( 2 \in \Omega_b \). \( \Omega_a \) and \( \Omega_b \) are the
corresponding vertex sets of the cut set, and $G'(\Omega_a)$ and $G'(\Omega_b)$ are the corresponding maximal connected subgraphs with respect to the cut set.

![Diagram](image)

Fig. 1-4. \(\{G,\Omega\}\)

**Definition 1-21:** A vertex $\zeta \in \Omega$ of $\{G,\Omega\}$ is said to be an isolated vertex if there exists no edge in $G$ whose end-point is $\zeta$.

Notice that $\{\Omega - \Omega(G)\}$ is a set of isolated vertices.

**Definition 1-22:** Suppose $\{G_1,\Omega_1\}$ is a cut set of $\{G,\Omega\}$.

If $\{G-G_1,\Omega\}$ has an isolated vertex $\zeta$ which is not an isolated vertex in $\{G,\Omega\}$, then $\{G_1,\Omega_1\}$ is called an "incidence set" and $\zeta$ is called the corresponding vertex of the incidence set.

**Definition 1-23:** A connected linear graph $\{G,\Omega(G)\}$ is said to have the tree-property if there exists no subgraph $\{G_1,\Omega(G_1)\}$ of $\{G,\Omega(G)\}$ which is a circuit.

**Definition 1-24:** A linear graph $\{G,\Omega\}$ is said to have the n-tree-property if there exists disjoint vertex-sets $\Omega_1, \Omega_2, \ldots, \Omega_n$ such that

1. $\Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_n = \Omega$
2. $G(\Omega_1) \cup G(\Omega_2) \cup \ldots \cup G(\Omega_n) = G$ and
3. each $\{G(\Omega_i), \Omega_i\}$ has the tree property for $i=1,2,\ldots,n$.

$\{G(\Omega_1), \Omega_1\}, \{G(\Omega_2), \Omega_2\}, \ldots, \{G(\Omega_n), \Omega_n\}$ are called n maximally connected subgraphs of $\{G,\Omega\}$.

**Definition 1-25:** A tree of linear graph $\{G,\Omega\}$ is a subgraph
The set $\{G, \Omega\}$ of subgraphs of $G$ such that $\{G, \Omega\}$ has the tree-property. A n-tree of $\{G, \Omega\}$ is a subgraph $\{G, \Omega\}$ of $\{G, \Omega\}$ such that $\{G, \Omega\}$ has the n-tree-property.

Example 1-5: For a linear graph $\{G, \Omega\}$ in Fig. 1-5, let $G_1 = \{a, b, c, e\}$, $G_2 = \{a, b, c, e, f, g\}$, $G_3 = \{a, b, d\}$, and $G_4 = \{a, b, d, f, g\}$. Then $\{G_1, \Omega(G_1)\}$ is a circuit, $\{G_2, \Omega(G_2)\}$ is an edge disjoint union of circuits, $\{G_3, \Omega(G_3)\}$ is a path from vertices 2 and 5, and $\{G_4, \Omega(G_4)\}$ is an edge disjoint union of a path and a circuit. Let $G_5 = \{a, b, c, f\}$, and $G_6 = \{a, c\}$. Then $\{G_5, \Omega\}$ is a tree and $\{G_6, \Omega\}$ is a 3-tree.

![Fig. 1-5. A Graph {G, \Omega}](image)

For convenience, we define the following.

**Definition 1-26:** For a given $\{G, \Omega\}$, if

$$\Omega \equiv \Omega(G)$$

we can just give G to represent a linear graph, that is, a linear graph $G$ means $\{G, \Omega(G)\}$. Connected linear graphs, cut sets, circuits and paths are in this category. For example, a path $P$ from $i$ to $j$ which consists of edges $e_1, e_2, \ldots, e_k$ is a path $\{P, \Omega(P)\}$ where $P \equiv \{e_1, e_2, \ldots, e_k\}$. Consider $\Omega(G_1) - \Omega(G_2)$ which is the set of vertices in $\Omega(G_1)$ but not in $\Omega(G_2)$. Let $G_3$ be the set of edges in $G_1 \cup G_2$ such that at least one of the end points of every edge in $G_3$ is in $\Omega(G_1) - \Omega(G_2)$. It is clear that
Consider $\Omega(G_1) \cup \Omega(G_2)$ which is the set of vertices in either $\Omega(G_1)$ or $\Omega(G_2)$ but not both. Let $G_3$ be the set of edges in $G_1 \cup G_2$ such that at least one of the end points of every edge in $G_3$ is in $\Omega(G_1) \cup \Omega(G_2)$.

Then it is clear that $G_3 \subset G_1 \cup G_2$. Thus

$$\Omega(G_1) \cup \Omega(G_2) \subset \Omega(G_1 \cup G_2)$$

Similarly we have

$$\Omega(G_1) \cup \Omega(G_2) \subset \Omega(G_1 \cup G_2)$$

Then from Definition 1-7, if $\otimes$ is any one of "-", "\cup", and "\oplus",

$$\{G_1, \Omega(G_1)\} \otimes \{G_2, \Omega(G_2)\} = \{G_1 \otimes G_2, \Omega(G_1 \otimes G_2)\}$$

Thus we can consider linear graphs $G_1$ and $G_2$ as sets and operate them. The resultant set $G_1 \otimes G_2$ can now be considered as a linear graph which by definition $\{G_1 \otimes G_2, \Omega(G_1 \otimes G_2)\}$ agrees with the resultant graph obtained by considering $G_1$ and $G_2$ as linear graphs and operated them as $\{G_1, \Omega(G_1)\} \otimes \{G_2, \Omega(G_2)\}$ by Definition 1-7 if $\otimes$ is one of "\oplus", "\cup", and "-".

Definition 1-27: A collection of cut sets $S_1, S_2, \ldots, S_{v-1}$ of a connected linear graph $G$ is said to be a collection of fundamental cut sets if there exists subgraph $G_1$ of $G$ which has the tree-property such that every $S_j$ contains exactly one edge in $G_1$ and for every edge in $G_1$ there exists one cut set $S_j$ which contains the edge where $1 \leq j \leq v-1$. 
Definition 1-28: A collection of circuits $c_1, c_2, \ldots, c_{e-v+1}$ is said to be a collection of fundamental circuits of a connected linear graph $G$ which consists of $e$ edges and $v$ vertices if there exists subgraph $G_1$ of $G$ which has the tree-property such that every $c_j$ contains exactly one edge in $\{G - G_1\}$ and for every edge in $\{G - G_1\}$ there exists exactly one circuit $c_j$ which contains the edge where $1 \leq j \leq e-v+1$.

Let $c_1, c_2, \ldots, c_k$ be all possible circuits and edge disjoint unions of circuits in an oriented linear graph $\{G, \Omega\}$. A matrix $B_a = [b_{ij}]$ is defined as

$$b_{ij} = \begin{cases} 
1 & \text{if edge } j \text{ is in circuit } i \text{ and their orientations coincide} \\
-1 & \text{if edge } j \text{ is in circuit } i \text{ and their orientations oppose} \\
0 & \text{otherwise}
\end{cases}$$

(1-10)

In the case of a non-oriented graph, $b_{ij}$ is either 1 or 0 depending on whether edge $j$ is in circuit $i$.

Matrix $B$ which consists of independent rows of $B_a$ is usually called a circuit matrix and if $B$ is of the form $[U B_{12}]$ where $U$ is a unit matrix, it is called a fundamental circuit matrix.

Let $S_1, S_2, \ldots, S_k$ be all possible cut sets and edge disjoint unions of cut sets in an oriented linear graph $\{G, \Omega\}$. Then $Q_a = [q_{ij}]$ is defined as
When \( \{G,\Omega\} \) is non-oriented, \( q^j \) is either 1 or 0 depending on whether edge \( j \) is in \( S_i \). Matrix \( Q \) which consists of independent rows of \( Q_a \) is called a cut set matrix and if \( Q \) is of the form \([Q_{11} U]\) where \( U \) is a unit matrix, then \( Q \) is called a fundamental cut set matrix. Let \( A_a \) be a submatrix of \( Q_a \) which consists of rows representing \( \nu \) incidence sets where \( \nu \) is the number of vertices in \( \Omega \) of \( \{G,\Omega\} \). Matrix \( A \) called an incidence matrix is obtained from \( A_a \) by deleting \( \rho \) rows such that remaining rows are independent where \( \rho \) is the number of maximal connected subgraphs of \( \{G,\Omega\} \).

Suppose \( \rho=1 \). Let \( S_p \) be the row which has been removed from \( A_a \) to obtain \( A \). Then the corresponding vertex of \( S_p \) is called a "reference vertex." It is known that every column of \( A_a \) contains exactly 1 and one -1. Thus every column of \( A \) contains at most two non-zeros. Furthermore if a column contains two non-zeros, one is 1 and the other is -1. If a column consists of one non-zero, it is either 1 or -1. In the case of a non-oriented linear graph, every column of \( A \) contains at most two 1's. The rank of \( Q_a \) is the rank of \( \{G,\Omega\} \) and the rank of \( B_a \) is the nullity of \( \{G,\Omega\} \).

Definition 1-29: A weighted linear graph is a class of sets \( \omega_1, \omega_2, \ldots, \omega_k \) and \( \Omega \) such that \( \{G,\Omega\} \) is a linear graph and there is a
1-1 mapping $f_p$ from $\omega_p$ for $p=1,2,\ldots,k$ onto $G$,

$$f_p(\omega_p) = G \quad (1-12)$$

$w_r \in \omega_p$ is called the weight of edge $e_r$ if

$$f_p(e_r) = w_r \quad (1-13)$$

Because of Eq. 1-12, we can use $w_1$ or $G$ to represent a weighted linear graph as $\{\omega_1, \Omega\}$ which means $\{\omega_1, \Omega\}$. Furthermore, if $\Omega = \Omega(G)$, then we can just use $w_1$ or $G$ to represent a linear graph. For example, an electrical network consisting of admittances $y_p$ ($p=1,2,\ldots,k$) as shown in Fig. 1-5 can be considered as a weighted linear graph $\omega$ where $\omega = \{y_1, y_2, \ldots, y_k\}$.

![Fig. 1-5. An Electrical Network](image_url)

We can give weight $\xi_{\omega_1}$ to every vertex in $G_1$ by defining sets $\omega_{1, \xi}$, each of which has one to one correspondence with $\Omega$. However, we will not discuss such a weighted linear graph in this paper.

When a linear graph is used to represent a system, it is often necessary to modify the linear graph such as coinciding vertices, shorting edges. Thus we define the following:

Definition 1-25: Let $\xi_1$ and $\xi_2$ be in $\Omega$ of a linear graph $\{G, \Omega\}_f$. Then $f_{\xi_1} \equiv \xi_2$ is a mapping function defined by
\[ f_{\zeta_1=\zeta_2} (e) = (\zeta_p, \zeta_1) \text{ if } f(e) = (\zeta_p, \zeta_2) \]  
(1-14)

\[ f_{\zeta_1=\zeta_2} (e) = (\zeta_1, \zeta_p) \text{ if } f(e) = (\zeta_2, \zeta_p) \]  
(1-15)

and

\[ f_{\zeta_1=\zeta_2} (e) = (\zeta_r, \zeta_s) = f(e) \text{ if } r, s \neq 2. \]  
(1-16)

Let \( G_1 \) be the set of edges which have \( \zeta_1 \) and \( \zeta_2 \) as their two end points.

Then

\[ \{ G, \Omega \} \equiv \{ G - G_1, \Omega - \{ \zeta_2 \} \} \]  
(1-17)

which is called a resultant linear graph obtained from \( \{ G, \Omega \} \) by coinciding vertices \( \zeta_1 \) and \( \zeta_2 \).

Definition 1-30: Let edge \( e \in G \) of \( \{ G, \Omega \} \). Also let \( f(e) = (\zeta_1, \zeta_2) \) or \( f(e) = (\zeta_2, \zeta_1) \). Then the resultant linear graph obtained by coinciding \( \zeta_1 \) and \( \zeta_2 \) in \( \{ G, \Omega \} \) is called the resultant linear graph obtained by shorting edge \( e \).

Definition: 1-31: Let \( f' \) be a mapping function such that \( f(e) = f'(e) \) for all edge \( e \) except \( e_0 \) in \( G \) and \( f'(e_0) \neq f'(e_0) \in \Omega \times \Omega \). Then \( \{ G, \Omega \} \) is called the resultant graph obtained from \( \{ G, \Omega \} \) by relocation of \( e_0 \) to the location \( f'(e_0) \).

III. ANALYSIS OF PASSIVE NETWORKS BY LINEAR GRAPHS

The analysis of passive electrical networks by the use of linear graphs was started by Kirchhoff [1] (1847) for mesh equations and by Maxwell [2] (1872) for node equations. This section gives the proof of a number of topological formulas [4] which have been supported mainly
by heuristic reasoning. Especially the precise proof of the topological formula for transfer functions is given. Furthermore, new topological formulas for short circuit functions, voltage ratio and current ratio are presented. Also a technique to analyze a passive network by a digital computer [21,22] is given.

2. Topological Formula for One Terminal-Pair Networks

A passive network without mutual couplings can be considered as a weighted oriented linear graph \( \{\omega, y, \omega_i, G, \Omega\} \) in which \( \Omega \) consists of \( v \) vertices, \( G \) consists of \( e \) edges and \( (y_{tu}, v_{tu}, i_{tu}) \) is the weight of each edge \( e \) where \( f(e) = (t,u) \), \( y_{tu} \in \omega_y \), \( v_{tu} \in \omega_v \), and \( i_{tu} \in \omega_i \) such that \( y_{tu}, v_{tu} \) and \( i_{tu} \) satisfy the equation

\[
y_{tu} v_{tu} = i_{tu} \tag{2-1}
\]

where \( y_{tu} \) is an edge admittance, \( v_{tu} \) is the edge voltage of the edge with the positive side at \( t \), and \( i_{tu} \) is the edge current of the edge which flows from \( t \) to \( u \).

Kirchhoff's current law can be expressed [19] as

\[
A I_e = J_n \tag{2-2}
\]

or

\[
A Y V_e = J_n \tag{2-3}
\]

where \( J_n \) is the column of current drivers, \( I_e \) is the column of edge currents, \( V_e \) is the column of edge voltages, \( Y \) is the diagonal matrix of edge admittances and \( A \) is an incidence matrix.
Since
\[ V_e = A^t V_n \] (node transformation) \hspace{1cm} (2-4)

where \( V_n \) is the column of node voltages with respect to the reference vertex and \( A^t \) is the transpose of \( A \), Eq. (2-3) can be written as
\[ Y V_{n-n} = J_{n} \] \hspace{1cm} (2-5)

where
\[ Y_n = AYA^t \] \hspace{1cm} (2-6)

By Binet-Chuchy Theorem [32],
\[ \Delta_n = |AYA^t| = \sum_{e} \text{product of corresponding majors of } [AY] \text{ and } A^t \] \hspace{1cm} (2-7)

Because \( Y \) is a diagonal matrix, a major of \([AY]\) consisting of columns \( e_1, e_2, \ldots \), and \( e_{v-1} \) can be written as
\[ \text{A major of } [AY] = y_{e_1}y_{e_2}\cdots y_{e_{v-1}} \] (Major of \( A \)) \hspace{1cm} (2-8)

where \( y_{e_i} \) is the admittance at \((e_1, e_i)\) entry in \( Y \). Thus, Eq. (2-7) can be expressed as
\[ \Delta_n = \sum_{e} y_{e_1}y_{e_2}\cdots y_{e_{v-1}} \] (Major of \( A \))^2 \hspace{1cm} (2-9)

There are three known fundamental properties of \( A \) [19] which are

1. The rank of \( A \) is \( v-1 \) where \( A \) is of order \((v-1, e)\),
2. Every non-singular submatrix of order \( v-1 \) corresponds to a tree of the linear graph and conversely, every square submatrix corresponding to a tree is non-singular,
3. The determinant of any square submatrix of \( A \) is 1, -1, or 0.
From these properties, we can say that $\Delta_n$ is the sum of products of admittances for every set of $v-1$ edges constituting a tree of the linear graph, that is,

$$\Delta_n = \sum_{e} \text{tree admittance product}$$  \hspace{1cm} (2-10)

It is convenient to introduce a notation for sums of tree admittance product. The node determinant expressed in terms of the edge admittances $y$'s is a homogeneous polynomial of degree $v-1$ in these variables and is linear in any one $y$. Such a polynomial expression for a node determinant is known as the node discriminant [31] and is symbolized as $V$. With this symbol, Eq. (2-10) can be written as

$$\Delta_n = V$$  \hspace{1cm} (2-11)

Before obtaining a topological formula for a cofactor of $[AYA^t]$, we will define a type of 2-tree as follows.

Definition 2-1: $T_{t_1t_2\ldots t_m;u_1u_2\ldots u_n}$ is a 2-tree which has

the property that one of the two maximal connected subgraphs contains

the vertices $t_1,t_2,\ldots, t_m$, and $t_m$, and the other maximal connected subgraph contains vertices $u_1,u_2,\ldots, u_n$.

A cofactor $\Delta_{pp}$ of $Y_n$ can be written as

$$\Delta_{pp} = \left| A^{-p} Y A^t \right|$$  \hspace{1cm} (2-12)

where $A^{-p}$ is the submatrix of $A$ which is obtained by deleting row $p$. By noticing that $A^{-p}$ is an incidence matrix of linear graph $\{G_1,\Omega_1\}$ of linear $\{G,\Omega\}$ by coinciding vertex $p$ and the reference vertex, where the incidence
matrix of $\{G, \Omega\}$ is $A$, we can see that a non-zero major of $A_{-p}$ must be a 2-tree of the type $T_{2,p,o}$ (where $o$ is the reference vertex.) Thus

$$\Delta_{pp} = \sum 2\text{-tree } T_{2,p,o} \text{ admittance product} \quad (2-13)$$

For convenience, we use the symbol $W_{p,o}$ to represent the sum of all possible 2-tree admittance product of 2-trees which are of the type $T_{2,p,o}$. Hence,

$$\Delta_{pp} = W_{p,o} \quad (2-14)$$

Hence, the topological formula for a driving point admittance function $Y_{p,o}$ between vertices $p$ and $o$ is

$$Y_{p,o} = V/W_{p,o} \quad (2-15)$$

The properties of $W_{p,o}$ of a weighted linear graph $\{\omega_y, \omega_v, \omega_1, G, \Omega\}$ are as follows:

1. $W_{p,o} = W_{o,p} \quad (2-16)$

2. $W_{p,o} = W_{pq,o} + W_{p,qo}$ \quad (2-17)
   
   where $q$ is a vertex in the linear graph.

This is easily seen from Eq. (2-13) and Definition 2-1.

3. Topological Formulas for Two-Terminal-pair Networks

A cofactor $\Delta_{pq}$ of $Y_n$ can be written as

$$\Delta_{pq} = (-1)^{p+q} \left| A_{-p} Y_{A^t} -q \right| \quad (3-1)$$

which can be expressed as
\[ \Delta_{pq} = (-1)^{p+q} \sum \text{product of corresponding majors of } [A_pY] \text{ and } A_q^t \]  

(3-2)

In order that a major of \( A_p \) to be non-zero, it must represent a 2-tree of the type \( T_{pq,o} \) where \( o \) is the reference vertex. Similarly, a non-zero major of \( A_q \) must correspond to a 2-tree of the type \( T_{pq,o} \).

Thus in order that the corresponding majors of \( A_p \) and \( A_q^t \) are non-zero at the same time, the majors must represent 2-tree of the type \( T_{pq,o} \). Hence, Eq. (3-2) can be written as

\[ \Delta_{pq} = \sum \epsilon_r \text{ 2-tree } T_{pq,o}^r \text{ admittance product } \]  

(3-3)

where \( \epsilon_r \) is either +1 or -1.

In order to establish \( \epsilon_r \), the submatrix consisting of the columns corresponding to the edges of a 2-tree of the type \( T_{pq,o} \) is selected from \( A \). This submatrix is of order \( v-1,v-2 \). Deleting row \( p \) from this matrix and calculating the determinant provides the major of \( A_p \). If row \( q \) is deleted from this matrix and the determinant is taken, the value of the corresponding major of \( A_q^t \) is obtained.

Each such 2-tree necessarily contains a path between \( p \) and \( q \). Let the edge sequence of this path from \( p \) to \( q \) be

\[ e_{r_1}, e_{r_2}, \ldots, e_{r_k} \]

In the chosen submatrix of \( A \), the columns \( r_p \) corresponding to these edges \( e_{r_j} \) (\( j=1,2,\ldots,k \)) will have the following structure. Column \( r_1 \) will have non-zero entry in row \( p \). Columns \( r_1 \) and \( r_2 \) will have non-zero entries in the same row which is different from row \( p \). Columns \( r_2 \) and \( r_3 \) will have non-zero entries in another row, etc. Finally, column \( r_k \) has a non-zero entry in row \( q \).
Two columns which have non-zero entries in the same row can be called adjacent, since these correspond to adjacent edges in the linear graph whose incidence matrix is $A$. Then in the sequence of columns $r_1, r_2, \ldots, r_k$, only successive columns are adjacent. Using these results the submatrix of $A$ is reduced to one in which column $r_1$ has non-zero entries in rows $p$ and $q$ and zeros in the other rows. This reduction is achieved by means of column operations only, so that the majors of $A_{-p}$ and $A_{-q}$ are left invariant under these operations.

Let column $r_1$ have a 1 in row $p$: The case where this entry is -1 is the same and will not be considered. Column $r_1$ has a -1 in another row, say row $u$, and column $r_2$ has a +1 entry in row $u$. We add +1 times column $r_2$ to column $r_1$ by which the resultant matrix has the property that column $r_1$ has a 1 in row $p$, a -1 in row $u$ and zeros in all other rows. Using columns $r_1$ and $r_3$, if the common row entries have the same sign, subtract column $r_3$ from column $r_1$, but if they have opposite signs, add. After this the -1 in column $r_1$ is moved to a row in which column $r_4$ has a non-zero entry. After repeated application of this procedure, the -1 finally moves to a row in which column $r_k$ has a non-zero entry in row $q$. This gives a matrix in which column $r_1$ has a 1 in row $p$, a -1 in row $q$ and zeros in all other rows. Let this final matrix be $A_d$.

Without the loss of generality, let row $p$ and row $q$ be $p^{th}$ and $q^{th}$ rows of $A$. Also let $p>q$. Consider the major of $A_{-p}$ which is obtained by deleting row $p$ from $A_d$, and taking the determinant. This major is expanded by column $r_1$ which has only one non-zero entry, -1 in
the qth row. Let the determinant of the matrix obtained by deleting rows p and q and column r₁ from Aₖ be denoted as D. Then

\[
\text{Major of } A_{-p} = (-1)^{r_1+q} D = (-1)^{r_1+q+1} D
\]  

(3-4)

The major of A⁻₉ is obtained by deleting row q from Aₖ and taking determinant. Column r₁ of this major has 1 at p-th row (because p>q) and zeros in all other rows. Expand it by column r₁ gives

\[
\text{Major of } A_{-q} = (-1)^{r_1+p-1} D
\]  

(3-5)

Therefore, the product of the two corresponding majors of [A⁻₉ Y] and A⁻₉ is given by

\[
(-1)^{2r_1+p+q} D^2 (2\text{-tree } T_{2pq,o} \text{ admittance product})
\]

or

\[
(-1)^{p+q} (2\text{-tree } T_{2pq,o} \text{ admittance product})
\]

Because D is +1 as it is selected from the incidence matrix A. The p and q are independent of the major selected from A⁻₉ and A⁻₉, so

\[
\left| A_{-p} YA_{-q} \right| = (-1)^{p+q} \Sigma \text{2-tree } T_{2pq,o} \text{ admittance product}
\]  

(3-6)

Thus

\[
\Delta_{pq} = \Sigma \text{2-tree } T_{2pq,o} \text{ admittance product}
\]  

(3-7)

In other words, εᵣ in Eq. (3-3) is +1. For convenience, Wpq,₀ is defined as

\[
W_{pq,₀} = \Sigma \text{2-tree } T_{2pq,o} \text{ admittance product}
\]  

(3-8)
This result will be enough to give the topological formula for a common ground two terminal-pair open-circuit transfer admittance function $J_{oq} / V_{po}$ (see Fig. 3-1) as

$$J_{oq} / V_{po} = V / W_{pq,o}$$ (3-9)

![Fig. 3-1. Common Grounded Network](image)

Since $W_{pq,o}$ can be written as

$$W_{pq,o} = W_{pq,ro} + W_{pqr,o}$$ (3-10)

where $r$ is a vertex in a linear graph whose incidence matrix is $A$, we can write

$$\Delta_{pq} - \Delta_{pr} = W_{pq,o} - W_{pr,o} = W_{pq,ro} - W_{pr,qr}$$ (3-11)
Thus the open circuit transfer admittance function $J_{rq}/V_{po}$ of the electrical network without mutual couplings in Fig. 3-2 can be expressed as

$$J_{rq}/V_{po} = \frac{V}{W_{pq,ro} - W_{pr,qo}}$$  \hspace{1cm} (3-12)

Example 3-1: $J_2/V_1$ of the network in Fig. 3-3 can be obtained by the following way:

All possible trees are

$$(y_1, y_2, y_3), (y_1, y_2, y_4), (y_1, y_3, y_4), \text{ and } (y_2, y_3, y_4)$$

Thus

$$V = y_1y_2y_3 + y_1y_2y_4 + y_1y_3y_4 + y_2y_3y_4$$

All possible 2-trees of the type $T_{ab,cd}$ are just $(y_1, y_4)$. Thus

$$W_{ab,cd} = y_1y_4$$

Similarly, $(y_2, y_3)$ is the only 2-tree of the type $T_{ad,bc}$, thus,

$$W_{ad,bc} = y_2y_3$$
Hence

\[
J_2/N_1 = \frac{y_1 y_2 y_3 + y_1 y_2 y_4 + y_1 y_3 y_4 + y_2 y_3 y_4}{y_1 y_4 - y_2 y_3}
\]

Two terminal-pair networks are more often described independently of the load, which is connected between two vertices \( q \) and \( r \), by means of the coefficient matrix of the system of equations

\[
\begin{bmatrix}
J_{op}
\end{bmatrix} = \begin{bmatrix}
Y_{s11} & Y_{s12} \\
Y_{s12} & Y_{s22}
\end{bmatrix} \begin{bmatrix}
V_{po} \\
V_{qr}
\end{bmatrix}
\]

The functions \( Y_{s1j} \) of this matrix are known as short circuit admittance functions. The general node equations of the network in Fig. 3-2 can be written as

\[
Y_{nn} V_n = J_n
\]

where

\[
V_n = \begin{bmatrix}
V_{po} \\
V_{qo} \\
V_{ro} \\
\vdots \\
V_{to}
\end{bmatrix} \quad \text{and} \quad J_n = \begin{bmatrix}
J_{op} \\
J_{oq} \\
\vdots \\
0
\end{bmatrix}
\]

From Eq. (3-15), we have

\[
\begin{bmatrix}
V_{po} \\
V_{qo} \\
V_{ro} \\
\vdots \\
V_{to}
\end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix}
\Delta pp & \Delta pq & \Delta pr & \cdots & \Delta pt \\
\Delta pq & \Delta \Delta & \Delta qr & \cdots & \Delta qt \\
\Delta pr & \Delta qr & \Delta \Delta & \cdots & \Delta rt \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\end{bmatrix} \begin{bmatrix}
J_{op} \\
J_{oq} \\
\vdots \\
0
\end{bmatrix}
\]

(3-16)
Simplifying Eq. (3-16), we have

\[
\begin{bmatrix}
V_{po} \\
V_{qo} \\
V_{ro}
\end{bmatrix}
= \frac{1}{\Delta}
\begin{bmatrix}
\Delta_{pp} & \Delta_{pq} & \Delta_{pr} \\
\Delta_{pq} & \Delta_{qq} & \Delta_{qr} \\
\Delta_{pr} & \Delta_{qr} & \Delta_{rr}
\end{bmatrix}
\begin{bmatrix}
J_{op} \\
J_{oq} \\
-J_{rq}
\end{bmatrix}
\] (3-17)

Since \( V_{qr} = V_{qo} - V_{ro} \), the Eq. (3-17) becomes

\[
\begin{bmatrix}
V_{po} \\
V_{qr}
\end{bmatrix}
= \frac{1}{\Delta}
\begin{bmatrix}
\Delta_{pp} & \Delta_{pq} - \Delta_{pr} \\
\Delta_{pq} - \Delta_{pr} & \Delta_{qq} + \Delta_{qr} - 2\Delta_{qr}
\end{bmatrix}
\begin{bmatrix}
J_{op} \\
J_{rq}
\end{bmatrix}
\] (3-18)

The determinant of the coefficient matrix above is

\[
\frac{1}{\Delta^2} \left\{ \begin{array}{c}
\Delta_{pp} \Delta_{qq} + \Delta_{pp} \Delta_{rr} - 2\Delta_{pp} \Delta_{qr} - \Delta_{pp}^2 - \Delta_{pq}^2 + 2\Delta_{pq} \Delta_{pr} \\
\end{array} \right\} =
\frac{1}{\Delta^2} \left\{ (\Delta_{pp} \Delta_{qq} - \Delta_{pq}^2) + (\Delta_{pp} \Delta_{rr} - \Delta_{pr}^2) - 2(\Delta_{pp} \Delta_{qr} - \Delta_{pr} \Delta_{pq}) \right\}
\] (3-19)

Using the determinantal identity [32]

\[
\Delta_{ab} \Delta_{cd} - \Delta_{ad} \Delta_{cb} = \Delta_{ab,cd}
\] (3-20)

The above expression will reduce to

\[
\frac{1}{\Delta^2} \left\{ \begin{array}{c}
\Delta_{pp} \Delta_{ppqr} + \Delta_{ppqr} - 2\Delta_{ppqr} \\
\end{array} \right\}
\]

Thus, the solution for \( J_{op} \) and \( J_{rq} \) of Eq. (3-18) is

\[
\begin{bmatrix}
J_{op} \\
J_{rq}
\end{bmatrix}
= \frac{1}{\Delta_{pp} + \Delta_{ppqr} - 2\Delta_{ppqr}}
\begin{bmatrix}
\Delta_{qq} + \Delta_{rr} - 2\Delta_{qr} \Delta_{pr} - \Delta_{pq} \\
\Delta_{pr} - \Delta_{pq} + \Delta_{pp}
\end{bmatrix}
\begin{bmatrix}
V_{po} \\
V_{qr}
\end{bmatrix}
\] (3-21)
Definition 3-1: $T_{3_{ab,c,df}}$ is a 3-tree which satisfies the property that one of the three maximal connected subgraphs contains vertices $a$ and $b$, another contains vertex $c$ and the other contains the vertices $d$ and $f$.

It is clear that

$$\Delta_{ppab} = (-1)^{a+b} \left| A_{p-a} Y A_{-b}^t \right|$$

(3-22)

where $A_{p-a}$ is the submatrix of $A$ obtained by deleting rows $p$ and $a$. Similarly, $A_{p-b}$ is the submatrix of $A$ obtained by deleting rows $p$ and $b$. Recall that $A_p$ is the incidence matrix of a linear graph $G'$ obtained from the linear graph $G$, whose incidence matrix is $A$, by coinciding vertex $p$ and the reference vertex. Thus a non-zero major of $A_{p-a}$ is corresponding to a subgraph of $G$ such that coinciding vertex $p$ and the reference vertex gives 2-tree of the type $T_{3_{2_{a,0}}}$ of $G'$. Thus a non-zero major of $A_{p-a}$ is a 3-tree of the type $T_{3_{p,a,0}}$. Hence, modifying Eq. (3-22), we have

$$\Delta_{ppab} = \sum 3\text{-tree } T_{3_{p,ab,0}} \text{ admittance product}$$

(3-23)

For convenience, we define $U_{p,ab,0}$ as

$$U_{p,ab,0} = \sum 3\text{-tree } T_{3_{p,ab,0}} \text{ admittance product}$$

(3-24)

Likewise,

$$\Delta_{ppqq} = U_{p,q,0}$$

(3-25)

$$\Delta_{pprr} = U_{p,r,0}$$

(3-26)
Similar to $W$, $U_{a,b,c}$ has the property that

$$U_{a,b,c} = U_{af,b,c} + U_{a,bf,c} + U_{a,b,cf} \quad (3-28)$$

Hence

$$\Delta_{ppqr} = U_{p,qr,o} \quad (3-27)$$

$$\Delta_{ppqq} + \Delta_{pprr} - 2\Delta_{ppqr} = U_{p,q,o} + U_{p,r,o} - 2U_{p,qr,o}$$

$$= U_{pr,q,o} + U_{p,q,ro} + U_{pq,r,o} + U_{p,r,ro} \quad (3-29)$$

For convenience, $\Sigma U$ is used to represent

$$\Sigma U = U_{pr,q,o} + U_{p,q,ro} + U_{pq,r,o} + U_{p,r,ro} \quad (3-30)$$

Since $\Delta_q + \Delta_r - 2\Delta_r$ can be expressed as

$$\Delta_{qq} + \Delta_{rr} - 2\Delta_{qr} = W_{q,o} + W_{r,o} - 2W_{qr,o}$$

$$= W_{q,ro} + W_{r,ro} = W_{q,r} \quad (3-31)$$

Eq. (3-21) can be written as

$$[J_{op}] = \frac{1}{\Sigma U} \begin{bmatrix} W_{q,r} & W_{pr,ro} - W_{pq,ro} & \end{bmatrix} \begin{bmatrix} V_{po} \\ V_{pr} \end{bmatrix} \quad (3-32)$$

which is the topological formulas for the short circuit admittance functions of a passive network without mutual couplings. Similarly, Eq. (3-18) can be written as
which is the topological formulas for the open circuit impedance functions for a passive network without mutual couplings.

Example 3-2: The short circuit transfer function \( \frac{J_2}{V_1} \bigg|_{V_2=0} \) of the network in Fig. 3-3 can be obtained as follows:

By obtaining all possible 3-trees of the type \( T_{ad,b,c} \) from the linear graph in Fig. 3-3 we have

\[
U_{ad,b,c} = y_2
\]

Similarly,

\[
U_{a,b,dc} = y_4
\]

\[
U_{ab,d,c} = y_1
\]

and

\[
U_{a,d,bc} = y_3
\]

Hence,

\[
U = y_1 + y_2 + y_3 + y_4
\]

Thus with \( W_{ad,bc} \) which has been obtained in the previous example, we have

\[
\frac{J_2}{V_1} \bigg|_{V_2=0} = \frac{y_2y_3 - y_1y_4}{y_1 + y_2 + y_3 + y_4}
\]

When \( J_{rq} \) of a two terminal-pair network in Fig. 3-2 is zero, the ratio \( \frac{V_{qr}}{V_{po}} \) is defined as the voltage ratio transfer function. Under \( J_{rq} = 0 \),
we have

\[ \frac{V_{pq}}{J_{op}} \bigg|_{j_{rq}=0} = W_{p,q} \sqrt{V} \] (3-34)

and

\[ \frac{V_{qr}}{J_{op}} \bigg|_{j_{qr}=0} = \left( W_{pq,ro} - W_{pr,qr} \right) / V \] (3-35)

Thus the topological formula for the voltage ratio transfer function is

\[ \frac{V_{qr}}{V_{po}} \bigg|_{j_{qr}=0} = \left( W_{pq,ro} - W_{pr,qr} \right) / W_{p,q} \] (3-36)

When \( V_{qr} \) of a two terminal-pair network in Fig. 3-2 is zero, the ratio \( J_{qr}/J_{op} \) is defined as the current ratio transfer function. Under \( V_{qr} = 0 \), we have

\[ \frac{J_{op}}{V_{po}} \bigg|_{v_{qr}=0} = W_{q,r} \sqrt{U} \] (3-37)

and

\[ \frac{J_{rq}}{V_{po}} \bigg|_{v_{qr}=0} = \left( W_{pr,qr} - W_{pq,ro} \right) / \Sigma U \] (3-38)

Thus the topological formula for a current ratio transfer function is

\[ \frac{J_{rq}}{J_{op}} \bigg|_{v_{qr}=0} = \left( W_{pr,qr} - W_{pq,ro} \right) / W_{q,r} \] (3-39)

We define that \( W_{ab,cd} = 0 \) and \( W_{ad,bc} = W_{ad,b} \) if \( b = c \). Similarly, \( U_{abc,d,fg} = U_{ab,d,fg} \) and \( U_{ab,cd,fg} = 0 \) if \( b = c \). Then it can be seen that we can use topological formulas for two terminal-pair networks without common ground to those with common ground.

4. Analysis of Passive Networks by Digital Computer

The previous section indicates that to analyze a passive network which contains no mutual couplings, it is only necessary to find all possible trees, particularly 2-trees and 3-trees of the weighted linear graph corresponding to the network. If there exists a simple procedure to test whether a set of edges is a tree, 2-tree or 3-tree, we can analyze passive networks by a digital computer.

Suppose $G_{v-1}$ is a tree of a linear graph $G$ where $\Omega(G)$ consists of $v$ vertices. Then it is clear from Definition 1-25 that there exists at least one vertex $x_{v-1}$ in $\Omega(G_{v-1})$ which is of degree one. Let $e_{v-1}$ be the edge in $G_{v-1}$ whose end-point is $x_{v-1}$. Linear graph $G_{v-2} \equiv G_{v-1} \oplus G_{r_1}$ where $G_{r_1} \equiv \{e_{v-1}\}$ has the tree-property (Definition 1-23). Thus, there exists at least one vertex $x_{v-2}$ in $\Omega(G_{v-2})$ which is of degree one in $G_{v-2}$. Let $e_{v-2}$ be the edge in $G_{v-2}$ whose end-point is $x_{v-2}$. Linear graph $G_{v-3} \equiv G_{v-2} \oplus G_{r_2}$ where $G_{r_2} \equiv \{e_{v-2}\}$ has the tree-property. Thus, there exists at least one vertex in $\Omega(G_{v-3})$ which is of degree one in $G_{v-3}$ and so on. Thus we can obtain a sequence of linear graphs

$$G_{v-1}, G_{v-2}, G_{v-3}, \ldots, G_1$$

where $G_1$ consists of one edge. These linear graphs have the properties that $G_{v-1} \supset G_{v-2} \supset G_{v-3} \supset \ldots \supset G_1$ and all of these have the tree-property.

We can reverse the above process as follows: Let $E$ be a set of $v-1$ edges. Then choose any edge in $E$, say $e_1$, and define $G_1 \equiv \{e_1\}$. Select an edge in $E$ which is not in $G_1$ and form linear graph $G_2$ with the
edge in $G_1$ such that $G_2$ has the tree-property. Suppose we form $G_1, G_2, \ldots, G_k$ ($k \leq v-1$) successively such that $G_p$ ($p=1, 2, \ldots, k$) consists of $p$ edges, has the tree property and $G_1 \subseteq G_2 \subseteq \ldots \subseteq G_k$. Then we can form $G_{k+1}$ by selecting an edge in $E$ which is not in $G_k$ and define $G_{k+1}$ to be the set consisting of the selected edge and all edges in $G_k$ such that $G_{k+1}$ has the tree property. The existence of such an edge can be shown if $\{E, \Omega(E)\}$ has the tree-property as follows: Let $\xi_1 \in \Omega(G_k)$ and $\xi_2 \in \Omega(E)$ which is not in $\Omega(G_k)$. Then there exists a path from $\xi_1$ to $\xi_2$ in $\{E, \Omega(E)\}$. In the edge sequence of the path from $\xi_1$ to $\xi_2$, there must be an edge in $E$ but not in $G_k$ such that all leading edges in the sequence are in $G_k$. The $G_{k+1}$ consists of this and all edges in $G_k$ obviously have the tree-property. Thus we can obtain a sequence of linear graphs $G_1, G_2, \ldots, G_{v-1}$ which satisfy that $G_1 \subseteq G_2 \subseteq \ldots \subseteq G_{v-1}$ and every linear graph in this sequence has the tree-property. Notice that the edge which is in $G_{k+1}$ but not in $G_k$ has the property that one of two end-points of the edge is in $\Omega(G_k)$ and the other end-point is in $\Omega(E)$ but not in $\Omega(G_k)$. It is very easy to do the above process by a digital computer. Thus a computer program of obtaining all possible trees in a linear graph consists of (1) forming all possible sets of $v-1$ edges, (2) testing each of these sets by the above process, and (3) if the set is a tree, either store it or print the result with some modification if necessary (for example, print the result as a function of admittances.) The flow chart of this process is given in Fig. 4-1 where linear graph $G$ is given by giving $f(e)$ for all $e \in G$. Also the relative address (integers 1, 2, \ldots, $e$) is assigned.
to every edge in \( G \) for convenience. Two tables, one of which represents \( G_k \) and the other corresponds to \( E \), are used where \( r_p \) and \( c_r \) are used to represent the specific entry in these tables as well as the number in the entry. This program gives all possible trees. Hence we can obtain \( V \) by a digital computer.

From the previous section, we know that \( T_{ab,cd} \) is a 2-tree in which one of the two maximal connected subgraphs contains vertices \( a \) and \( b \) and the other contains vertices \( c \) and \( d \). Hence, if we add one of edges \( e_1, e_2, e_3 \) and \( e_4 \) where \( f(e_1) = (a,c), f(e_2) = (a,d), f(e_3) = (b,c) \) and \( f(e_4) = (b,d) \) to a 2-tree of the type \( T_{ab,cd} \), the resultant graph will have the tree property. In other words, suppose \( G_{v-2} \) is a subgraph of \( G \) which contains \( v-2 \) edges where \( \Omega(G) \) consists of \( v \) vertices. Also suppose the linear graph which consists of all edges in \( G_{v-2} \) and edge \( e_0 \) where \( f(e_0) = (p,o) \), has the tree-property, then \( G_{v-2} \) is a 2-tree of the type \( T_{p,o} \). Thus by modifying the flow chart in Fig. 4-1, we can obtain \( W \). In addition to \( G_{v-2} \) being a 2-tree of the type \( T_{ab,cd} \), if the linear graph, which consists of all edges in \( G_{v-2} \) and edge \( e_1 \) where \( f(e_1) = (q,r) \), has the tree-property, then \( G_{v-2} \) is a 2-tree of either \( T_{pq,ro} \) or \( T_{pr,qo} \). Suppose \( G_{v-2} \) is a 2-tree of either \( T_{pq,ro} \) or \( T_{pr,qo} \). Then it is clear that the linear graph \( G_{v-2} \cup \{ e_2 \} \) where \( f(e_2) = (p,r) \) has the tree-property if \( G_{v-2} \) is a 2-tree of the type \( T_{pq,ro} \) and does not have the tree-property if \( G_{v-2} \) is a 2-tree of the type \( T_{pr,qo} \). Thus we can obtain \( W_{pq,ro} - W_{pr,qo} \) by a digital computer easily.

Suppose \( G_{v-3} \) consists of \( v-3 \) edges. If the linear graph \( G_{v-3} \cup \{ e_0, e_1 \} \) where \( f(e_0) = (p,o) \) and \( f(e_1) = (q,r) \) has the tree-property,
Fig. 4-1. Flow Chart for Obtaining All Possible Trees of Passive Networks
then \( G \) is a 3-tree of one of the types \( T_{3, p, q, r, o} \), \( T_{3, p, q, r, o} \), \( T_{3, p, r, q, o} \) and \( T_{3, p, r, q, o} \). The reason is that any one of these 3-trees consists of three maximal connected subgraphs such that there are no paths between \( p \) and \( o \), and \( q \) and \( r \). Thus adding edges \( e_0 \) and \( e_1 \) does not produce circuits. Hence \( G \cup \{e_0, e_1\} \) has the tree-property. Thus we can obtain

\[
U = U_{p, q, r, o} + U_{p, q, r, o} + U_{p, r, q, o} + U_{p, r, q, o}
\]

by the use of a digital computer. The flow chart which is the modification of the previous flow chart is in Fig. 4-2 by which we can obtain \( V, W_{p, q, r, o}, W_{p, q, r, o}, W_{p, q, r, o} \) and \( U \). In other words, network functions which are discussed in the previous sections can be obtained by the use of a digital computer. Notice that instead of printing trees, 2-trees and 3-trees of \( G \), we can express the network functions as polynomials in \( s \) with the coefficients being \( R, L, \) and \( C \) by giving \( 1/R_t, C_t s \) and \( L_t/s \) as \( y_t \) for every edge admittance in a network. Also if we give numerical values to \( R_t, C_t \) and \( L_t \), we can easily obtain network functions expressed by polynomials in \( s \) with constant coefficients.

IV. ANALYSIS OF ACTIVE NETWORKS BY LINEAR GRAPHS

Analysis of active networks and networks with mutual couplings by the use of linear graphs [6,7,8] given here is called "Coates-Mayeda" method [19]. This method uses two linear graphs to represent a network, and the topological formulas for active networks are similar to those for passive networks which have been discussed in the previous chapter. Also by the use of this method, active networks can be analyzed non-numerically by a digital computer [23].
Fig. 4-2. Flow Chart for Analysis of Passive Networks
5. Linear Graphs of Active Networks

The previous sections give topological formulas for passive networks without mutual couplings. It is clear that these formulas hold even if we change $y_t$ to $-y_t$ in $Y$. In other words, as long as every element in a network is $y_t$ which may be a negative quantity which satisfy

$$y_t v_t = i_t$$  \hfill (5-1)

where $v_t$ is the edge voltage and $i_t$ is the edge current of edge $e_t$ whose edge admittance is $y_t$, the topological formulas given in the previous sections can be used to analyze the network.

In this section, a network may contain transadmittance $y_{ij,mn}$ which satisfies

$$y_{ij,mn} v_{mn} = i_{ij}$$  \hfill (5-2)

where $v_{mn}$ is the voltage across vertices $m$ and $n$ with the positive side being at $m$, and $i_{ij}$ is the current through edge, connected between $i$ and $j$, from $i$ in the network. Since the locations of edge voltage and the corresponding edge current are different, it is reasonable to use two linear graphs to represent a network such that one represents the locations of all edge voltages and the other represents the locations of all edge currents in the network. Thus, we define the following:

Definition 5-1: An electrical network consisting of elements of the type $y_{ij,mn}$ which satisfies Eq. (5-2) will be represented by two linear graphs one of which is $\{\omega_y, \omega_v, G, \Omega\}_v$ called a voltage graph and the other is $\{\omega_y, \omega_i, G, \Omega\}_i$ called a current graph which satisfy that for
any $e \in G$, edge weights $y_{ij} \in \omega_y$, $v_{mn} \in \omega_v$, and $i_j \in \omega_i$ satisfy Eq. 5-2 where $f_v(e) = (m,n)$, $f_i(e) = (i,j)$ and $i,j,m,n \in \Omega$. $\omega_v$ is the set of edge voltages, $\omega_i$ is the set of edge currents and $\omega_y$ is the set of edge admittances. An edge in a voltage graph is called a voltage edge and an edge in a current graph is a current edge.

By giving the symbol "v" inside of the arrow of an edge (which indicates the orientation of the edge) to indicate that the edge is a voltage edge and the symbol "i" inside of the arrow of an edge to indicate that the edge is a current edge, we can superimpose these two linear graphs to represent an electrical network as shown in Fig. 5-2b. In such a graph, one edge can be used to represent both voltage and current edges if edge admittance of these edges is $y_{ij,i_j}$. In other words, the linear graph in Fig. 5-1a is a short hand representation of the graph in Fig. 5-1b. For convenience the resultant graph by superimposing voltage graph $\{G_v, \Omega\}_{f_v}$ and current graph $\{G_i, \Omega\}_{f_i}$ is denoted by $G$.

Example 5-1: If we only consider the transconductance $g$ of the tube, the graph $G$ corresponding to the network in Fig. 5-2a is one shown in Fig. 5-2b. The orientation of these edges is arbitrary except $g$. 

![Fig. 5-1](image-url)
The orientations of $g$ are fixed in the relative sense, that is, if the orientation of current edge $g$ is given, then the orientation of voltage edge $g$ is fixed, and conversely if the orientation of voltage edge $g$ is chosen arbitrarily, then the orientation of the current edge $g$ will be fixed. Notice that in the voltage and current graphs of $G$ in Fig. 5-2c and d, symbols "v" and "i" in arrows are omitted because all edges in the voltage graph are voltage edges and all edges in the current graph are current edges.

For convenience, we use $G_v$ to represent the voltage graph $\{\omega_v, \omega_v, G_v, \Omega\}$ and $G_i$ to represent the current graph $\{\omega_i, \omega_i, G_i, \Omega\}$.
Let $A_v$ and $A_i$ be the incidence matrices of $G_v$ and $G_i$ respectively such that column $p$ of $A_v$ and $A_i$ represent edge $e_p$ in $G$ for all $p$ and vertex $o \in \Omega$ is the reference vertex in both $G_v$ and $G_i$. Kirchhoff's current law gives

$$A_i I_e = J_n \quad (5-3)$$

where $I_e$ is the column of edge currents and $J_n$ is the column of external currents each of which is from the reference vertex. The node transformation is

$$V_e = A_v^t V_n \quad (5-4)$$

where $V_e$ is the column of edge voltages and $V_n$ is the column of node voltages with respect to the reference vertex. Since each element in a network satisfies Eq. (5-2), we have

$$Y V_e = I_e \quad (5-5)$$

where $Y$ is a diagonal matrix in which each diagonal entry gives an admittance in a network. Combining Eqs. (5-3), (5-4), and (5-5) gives

$$Y V_n = J_n \quad (5-6)$$

where

$$Y_n = A_i Y A_v^t \quad (5-7)$$

By Binet-Cauchy's Theorem [32], the determinant of $Y_n$ can be written as

$$\Delta_n = |A_i Y A_v^t| = \sum \text{corresponding majors of } [A_i Y] \text{ and } A_v^t \quad (5-8)$$
Suppose columns \( c_1, c_2, \ldots, c_{v-1} \) of \( A_1 \) form a major of \( A_1 \).

Then it is clear that the major of \([A_1 Y]\) can be expressed as

\[
\text{Major of } [A_1 Y] = y_{c_1}, y_{c_2}, \ldots, y_{c_{v-1}} \quad \text{(major of } A_1) \tag{5-9}\]

Thus \( \Delta_n \) can be written as

\[
\Delta_n = \sum (c) \ y_{c_1}, y_{c_2}, \ldots, y_{c_{v-1}} \quad \text{(major of } A_1) \quad \text{(corresponding major of } A^t_v) \tag{5-10}\]

Definition 5-2: Suppose \( G_1 \subseteq G \) where \( G_1 \) consists of \( v-1 \) edges. If \( \{G_1, \Omega\}_f \) is a tree in \( G \) and \( \{G_1, \Omega\}_v \) is a tree in \( G_1 \), then we say that \( G_1^v \) is a common tree (another name is a "complete tree").

Example 5-2: A common tree \( G_1 \) of a graph \( G \) in Fig. 5-2b consists of \( g, y_f, \) and \( y_1 \) because \( \{G_1, \Omega\}_f \) is a tree in \( G \) and \( \{G_1, \Omega\}_v \) is a tree in \( G_1 \) as shown in Fig. 5-3a and b.

![Fig. 5-3.](image)

In order that a major of \( A_1 \) to be non-zero, the set of edges corresponding to the columns of the major must be a tree in \( G_1 \).

Similarly, a major of \( A_v \) is non-zero if and only if the set of edges
corresponding to the columns of the major is a tree of $G_v$. Thus

Eq. (5-8) can be written as

$$\Delta_n = \left| A_i^t Y_{A_i^{t}} \right| = \sum \epsilon_r \text{ common tree admittance product}$$

(5-10)

where $\epsilon_r$ is either +1 or -1 which is from (major of $A_i$) - (major of $A_v$) and these majors correspond to a common tree.

6. Sign-Permutation and Principal Tree

Consider a tree $T$ in linear graph $G$. Notice that a tree $T$ means $\{T, \Omega(T)\}$. We define the principal vertex of an edge in $T$ as follows:

Definition 6-1: The principal vertex of an edge in $T$ is one of the two end-points of the edge from which a path to the reference vertex will contain the edge.

Notice that there exists only one path between any two vertices in a tree. Thus for every edge in $T$ there is exactly one principal vertex and every vertex except the reference vertex is the principal vertex of an edge in $T$. Now we will define a sign-permutation which is related to $\epsilon_r$ in Eq. (5-10) as follows:

Definition 6-2: Let $T$ consisting of $y_1, y_2, \ldots, y_{v-1}$ be a common tree of a graph $G$ so that $\{T, \Omega\}_{f_i}$ and $\{T, \Omega\}_{f_v}$ are the trees of $G_i$ and $G_v$ respectively. The sign-permutation of $T$ is a two row i and v arrangement of $y$'s such that
\[ r_{iq} = \begin{cases} 
  y_p & \text{if the principal vertex of } y_p \text{ is } q \text{ in } \{ T, \Omega \}_f \text{ and the orientation of } y_p \text{ is away from } q \\
  -y_p & \text{if the principal vertex of } y_p \text{ is } q \text{ in } \{ T, \Omega \}_f \text{ and the orientation of } y_p \text{ is toward } q. 
\end{cases} \tag{6-1} \]

Similarly,
\[ r_{vq} = \begin{cases} 
  y_p & \text{if the principal vertex of } y_p \text{ is } q \text{ in } \{ T, \Omega \}_f \text{ and the orientation of } y_p \text{ is away from } q \vspace{1em} \\
  -y_p & \text{if the principal vertex of } y_p \text{ is } q \text{ in } \{ T, \Omega \}_f \text{ and the orientation of } y_p \text{ is toward } q. 
\end{cases} \tag{6-2} \]

where \(1 \leq p \leq v-1\).

Definition 6-3: The value of a sign-permutation is
\[ (-1)^{p+m} N_p N_m \]
where \(N_p\) is the number of permutations necessary in order that row \(i\) and row \(v\) become identical under neglecting minus sign superscripts and \(N_m\) is the number of minus sign superscripts in the sign-permutation.

Example 6-1: The sign-permutation of the common tree given in Example 5-2 with \(0\) as the reference vertex is
\[
\begin{array}{ccc}
1 & 2 & 3 \\
\hline
i & y_f^- & y_L^- & g^- \\
v & y_f^- & y_L^- & g^- \\
\end{array}
\]
The value of the above sign-permutation is +1.

Definition 6-4: A principal tree is a tree in which every edge is connected to the reference vertex. In other words, a principal tree is a star with the reference vertex as a center.

We are going to show that any tree T whose incidence matrix being $A_T$ can be changed to a principal tree such that the determinant of an incidence matrix of the principal tree is the same as the determinant of $A_T$. It is clear that in T there exists at least one vertex $\zeta$ other than the reference vertex on which only one edge is connected, that is, there exists $\zeta \in \Omega(T)$ of $\{T, \Omega(T)\}$ such that only one edge in T has $\zeta$ as its end-point. Let this edge be $y_c$ whose end-points are $\zeta$ and $\zeta'$. Also let column c in $A_T$ be the column corresponding to $y_c$. Suppose $\zeta'$ is not the reference vertex. Then $(\zeta, c)$ entry of column c is +1, $(\zeta', c)$ entry of column c is -1 and all other entries in the column are zero. Also since this edge is only the edge which is connected to $\zeta$, all entries except $(\zeta, c)$ entry of row $\zeta$ (the row corresponding to vertex $\zeta$) are zero. Thus adding row $\zeta$ to row $\zeta'$, which does not change the determinant of $A_T$, makes only one non-zero entries in column c. However this operation does not change any other entries in the matrix. Let the resultant matrix be $A_{T_1}$. The linear graph $T_1$ whose incidence matrix is $A_{T_1}$ is the same as T except $y_c$ which is now connected between $\zeta$ and the reference vertex. Notice that the orientation of $y_c$ with respect to $\zeta$ in T is the same as that in $T_1$. Consider linear graph $T_1 - \{y_c\}$ which obviously has the tree-property. The incidence matrix of $T_1 - \{y_c\}$ with the same reference vertex can be obtained from
by deleting column c and row £, Hence we can use the above process to $T_{1}$, and so on until all edges in T are connected to the reference vertex, which is a principal tree. It can be seen that the incidence matrix of a principal tree has the property that there is only one non-zero entry in every row and column. Furthermore, if $(p, q)$ entry is non-zero, then vertex p corresponding to row p is the principal vertex of the edge $y_{q}$ corresponding to column q.

Consider a common tree $T_{i}$ that is $\{T_{i}, \Omega\}_{f_{i}}$ is a tree of $G_{i}$ and $\{T_{i}, \Omega\}_{f_{v}}$ is a tree of $G_{v}$. Let $A_{T_{1}}$ and $A_{T_{v}}$ be the incidence matrices of $\{T_{i}, \Omega\}_{f_{i}}$ and $\{T_{i}, \Omega\}_{f_{v}}$ with the same reference vertex respectively. Then from the above discussion, we can obtain

\[
\begin{vmatrix}
A_{T_{1}} \\
A_{T_{v}}
\end{vmatrix} = \begin{vmatrix}
A_{T_{1}} \\
A_{T_{v}}
\end{vmatrix} = \begin{vmatrix}
A_{T_{1}} \\
A_{T_{v}}
\end{vmatrix}
\]

(6-3)

where $A_{T_{1}}$ and $A_{T_{v}}$ are the incidence matrices of the principal trees corresponding to $\{T_{i}, \Omega\}_{f_{i}}$ and $\{T_{i}, \Omega\}_{f_{v}}$ respectively. It is obvious that $A_{T_{1}}$ and $A_{T_{v}}$ are either +1 or -1. Hence it is only necessary to know whether the signs of $A_{T_{1}}$ and $A_{T_{v}}$ are the same or not in order to obtain $A_{T_{1}}$ and $A_{T_{v}}$.

Since every row and column in $A_{T_{1}}$ and $A_{T_{v}}$ has only one non-zero, which is either +1 or -1, we will first make every non-zero entry to +1 by multiplying -1 to the rows which have -1. Suppose ($\zeta, y$) entry in $A_{T_{1}}$ is -1. Then in $\{T_{i}, \Omega\}_{f_{i}}$, the orientation of $y$ is toward $\zeta$. Conversely, the orientation of $y$ is toward $\zeta$ in $\{T_{i}, \Omega\}_{f_{v}}$ if ($\zeta, y$) entry in $A_{T_{v}}$ is -1. Thus the number of rows to which -1 will be multiplied in order that all non-zero entries in $A_{T_{1}}$ and $A_{T_{v}}$ become +1 is equal to $N_{m}$.
in Definition 6-3. Thus we have

\[
\begin{vmatrix}
A_{T_1} & A_{T_v}
\end{vmatrix} = (-1)^m \begin{vmatrix}
A_{T_1} & A_{T_v}
\end{vmatrix}
\]

(6-4)

where \(A_{T_1}\) and \(A_{T_v}\) are obtained from \(A_{T_1}\) and \(A_{T_v}\) by replacing all -1 to +1. Now it is only necessary to interchange columns of \(A_{T_v}\) such that locations of the non-zero entries of \(A_{T_v}\) become identical with those of \(A_{T_1}\). Recall that there is only one non-zero in each row and column of these matrices. The number of interchange of columns necessary to make these matrices identical is obviously equal to \(N_p\).

Thus we have finally

\[
\begin{vmatrix}
A_{T_1} & A_{T_v}
\end{vmatrix} = (-1)^{N+N_p} m
\]

(6-5)

Therefore, \(\varepsilon_r\) in Eq. (5-10) is equal to the value of the sign-permutation of a corresponding common tree, or

\[
\Delta_n = \begin{vmatrix}
A_{T_{v_1}}^T
\end{vmatrix} = \Sigma \text{(value of sign-permutation of } T_c) \cdot (\text{common tree } T_c \text{ admittance product}) = V
\]

(6-6)

where we use the symbol \(V\) to represent the sum of the product of all possible common tree admittance product with the value of the corresponding sign-permutation for convenience.

Example 6-2: \(\Delta_n\) of the network in Fig. 5-2a can be obtained by first finding all possible common trees and evaluating sign-permutations of these trees as shown in Table 6-1, then by using Eq. (6-6) as

\[
V = y_{G_{iG}} + y_{G_{iG}}y_{G_{kG}} + y_{G_{iG}}y_{L} + y_{G_{kG}} + y_{G_{kG}}y_{L} + y_{G_{iG}}y_{L}
\]
<table>
<thead>
<tr>
<th>Common Tree</th>
<th>Sign-permutation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_G y_f y_g$</td>
<td>$\begin{bmatrix} y_G &amp; y_f &amp; g^- \ y_G &amp; y_f &amp; g^- \end{bmatrix} = 1$</td>
</tr>
<tr>
<td>$y_G y_f y_k$</td>
<td>$\begin{bmatrix} y_G &amp; y_f &amp; y_k^- \ y_G &amp; y_f &amp; y_k^- \end{bmatrix} = 1$</td>
</tr>
<tr>
<td>$y_G y_L$</td>
<td>$\begin{bmatrix} y_G &amp; y_L &amp; g^- \ y_G &amp; y_L &amp; g^- \end{bmatrix} = 1$</td>
</tr>
<tr>
<td>$y_G y_k y_L$</td>
<td>$\begin{bmatrix} y_G &amp; y_L &amp; y_k^- \ y_G &amp; y_L &amp; y_k^- \end{bmatrix} = 1$</td>
</tr>
<tr>
<td>$y_f g y_k$</td>
<td>$\begin{bmatrix} y_f &amp; g &amp; y_k^- \ g &amp; y_f &amp; y_k^- \end{bmatrix} = 1$</td>
</tr>
<tr>
<td>$y_f g y_L$</td>
<td>$\begin{bmatrix} y_f &amp; y_L &amp; g^- \ y_f &amp; y_L &amp; g^- \end{bmatrix} = 1$</td>
</tr>
<tr>
<td>$y_f y_k y_L$</td>
<td>$\begin{bmatrix} y_f &amp; y_L &amp; y_k^- \ y_f &amp; y_L &amp; y_k^- \end{bmatrix} = 1$</td>
</tr>
</tbody>
</table>

In some cases, it is easier to obtain the values of sign-permutations from the linear graphs rather than calculating them. For example, if a common tree consisting of passive elements excluding mutual couplings, the value of the sign-permutation is obviously +1.

The method of obtaining the values of sign-permutations from linear graphs can be done by the use of the following rules.

Let $\{T_i, \Omega_i\}$ and $\{T_v, \Omega_v\}$ be the trees in $G_i$ and $G_v$ respectively. Also let $A_{T_i}$ and $A_{T_v}$ be the incidence matrices of these trees with respect to the same reference vertex.
Rule 1: If \( \{T, \Omega\}_{f_1} \) and \( \{T, \Omega\}_{f_2} \) are identical when the orientations of edges are disregarded, then the value of sign-permutation is \((-1)^m\) where \(N\) is the number of edges whose orientations disagree.

Example 6-3: \( \{T, \Omega\}_{f_1} \) and \( \{T, \Omega\}_{f_2} \) in Fig. 6-1 are identical when we disregard the orientations of edges. Thus the value of the sign-permutation of this common tree is \((-1)^m = -1\) by Rule 1.

![Fig. 6-1. Complete Tree T](image)

This rule is easily proven from the definition of a sign-permutation.

Rule 2: Let \( y \) be in \( T \) which is connected between \( \xi_p \) and \( \xi_q \) in \( \{T, \Omega\}_{f_1} \) (or in \( \{T, \Omega\}_{f_2} \)). We can relocate \( y \) to the location between \( \xi_p' \) and \( \xi_q \) without changing the value of the sign-permutation of this common tree if the path from \( \xi_p \) to \( \xi_p' \) does not contain \( y \) and if the orientation of \( y \) with respect to \( \xi_q \) does not change.

Example 6-4: \( y_1 \) in \( \{T, \Omega\}_{f_1} \) in Fig. 6-1 can be relocated as shown in Fig. 6-2a and b by Rule 2.
This rule can be generalized as follows:

Rule 2': Let $\Omega_1$ and $\Omega_2$ be the corresponding vertex sets of the cut set which consists only of edge $y$. Also let $f_1(y) = (\zeta_p, \zeta_q)$ [or $f_v(y) = (\zeta_p, \zeta_q)$] and $\zeta_p \in \Omega_1$. Then the sign-permutation of the common tree $T$ is invariant by relocating $y$ from $\zeta_r$ to $\zeta_t$ where $\zeta_r \in \Omega_1$ and $\zeta_t \in \Omega_2$.

Example 6-5: Consider $y_0$ as a cut set $s$ in $\{T, \Omega\}_{f_1}$ in Fig. 6-3a then $\Omega_1$ and $\Omega_2$ are the corresponding vertex sets of $s$ where $\Omega_1 = \{3, 4, 5, \zeta_q\}$. Then we can place $y_0$ from one vertex in $\Omega_1$ to a vertex in $\Omega_2$ to obtain $\{T, \Omega\}_{f_1}$. Suppose we place $y_0$ from 1 to 5 as shown in Fig. 6-3b. Then by rule 2', the value of the common tree corresponding to $\{T, \Omega\}_{f_1}$ and $\{T, \Omega\}_{f_v}$ is the same as that of the common tree corresponding to $\{T, \Omega\}_{f_1}$ and $\{T, \Omega\}_{f_v}$. 

(a) Relocation of $y_1$ to o 
(b) Relocation of $y_1$ to 3 

Fig. 6-2. Relocation of edge
Fig. 6-3. Relocating an edge in a common tree.

The proof of Rule 2 can be given as follows: Suppose the path from $\zeta_p$ to $\zeta_p'$ in $\{T, \Omega\}_{f_1}$ consists of one edge $y'$. Then changing $\{T, \Omega\}_{f_1}$ by relocating $y$ to the location between $\zeta_p'$ and $\zeta_q$ alters two entries in the sign-permutation if $\zeta_p$ is the principal vertex of $y$. However if $\zeta_p$ is not the principal vertex of $y$, the relocation of $y$ according to Rule 2 will produce no change in the sign-permutation. Thus it is only necessary
to consider the case when $\xi_p$ is the principal vertex of $y$. In this case, $\xi_p'$ is the principal vertex of $y'$. The relocation of $y$ will cause to interchange the principal vertices of $y$ and $y'$. The reason is that in $\{T,\Omega\}_f^1$, the path $P_1$ from $\xi_p$ to the reference vertex must contain $y$ by definition of principal vertices. Also the path $P_2$ from $\xi_p'$ to the reference vertex must contain $y'$, which means that $P_2$ contains $y$. Because $\{T,\Omega\}_f^1$ is a tree, $P_2 \subseteq P_1$ or $P_2 = P_1 \cup \{y'\}$. Now we obtain $\{T,\Omega\}_f^1$ by replacing $y$ between $\xi_p'$ and $\xi_q$. In this linear graph, the path between $\xi_p$ and $\xi_q$ consists of $y$ and $y'$. Thus the path between $\xi_p$ and the reference vertex contains $y$ and $y'$. Thus the principal vertices of $y$ and $y'$ are interchanged.

The orientation of $y$ with respect to $\xi_q$ is invariant by this relocation. Thus the orientation of $y$ with respect to its principal vertex is not altered by this operation. On the other hand, the orientation of $y'$ with respect to its principal vertex in $\{T,\Omega\}_f^1$ is different from that in $\{T,\Omega\}_f^1$. In other words, such a relocation of $y$ is equivalent to interchanging $y$ and $y'$ in row $i$ of the sign-permutation and removing or adding a minus sign superscript to $y'$ depending on whether $y'$ has a minus sign superscript. Thus the relocation of $y$ in this case change $N_m$ by one and $N_p$ by one which makes the value of the sign-permutation invariant. Rule 2 is equivalent to the successive applications of above process. Also Rule 2' is equivalent to the successive application of Rule 2. The same argument will hold for relocation of edges in $\{T,\Omega\}_f^v$. 
Rule 3: If only a pair of identifiers $y_p$ and $y_q$ in $\{T, \Omega\}_{f_1}$ (or $\{T, \Omega\}_{f_v}$) are interchanged, the value of the sign-permutation will be altered.

Example 6-6: If we interchange identifiers $y_1$ and $y_2$ in $\{T, \Omega\}_{f_1}$ in Fig. 6-3a as shown in Fig. 6-4, the value of the sign-permutation of the common tree corresponding to this resultant linear graph and $\{T, \Omega\}_{f_v}$ in Fig. 6-3a is the negative of the value of the sign-permutation of the common tree corresponding to $\{T, \Omega\}_{f_1}$ and $\{T, \Omega\}_{f_v}$ in Fig. 6-3a.

![Fig. 6-4. A Tree Obtained by Interchanging Identifiers](image-url)

The proof of Rule 3 can easily be seen because this is equivalent directly to interchange $y_1$ and $y_2$ in row i (or row v) of the sign-permutation of a common tree.

Rules 1 and 2 (or 2') are enough to obtain the value of a sign-permutation of a common tree without using the sign-permutation. However in some cases, we can obtain the sign of a common tree simpler by using Rule 3 along with the other two rules.

Example 6-7: The value $\epsilon$ of the sign-permutation of a common tree in Fig. 6-5 can be obtained by the following process:
(1) Relocate $y_1$ and $y_4$ in $\{T, \Omega\}_{f_1}$ by Rule 2 as shown in Fig. 6-6a.

(2) Interchange identifiers of $y_1$ and $y_3$, and $y_2$ and $y_4$ by Rule 3 as shown in Fig. 6-6b. Because we interchange two pairs of identifiers, the value of the sign-permutation is unchanged.

(3) Because $\{T, \Omega\}_{f_1}$ in Fig. 6-6b and $\{T, \Omega\}_{f_1}$ in Fig. 6-5 are identical when the orientations are disregarded, Rule 1 gives $\varepsilon = -1$.

Fig. 6-5. A Common Tree

(a) Modified Tree by Rule 2

(b) Modified Tree by Rule 3

Fig. 6-6. A Modification of a Tree
Consider a network \( N \) in Fig. 7-1a to which a passive element \( y_{po} \) (which is a shorthand notation of \( y_{po,p0} \)) is connected from vertices \( p \) to \( o \). The graph \( G \) consisting of \( \{G, \Omega\}_f \) and \( \{G, \Omega\}_f \) is given in Fig. 7-1b. Notice that \( G \) consists of edges corresponding to all elements in \( N \) and \( y_{po} \). Since there is no external source, Eq. (5-6) becomes

\[
Y_n V_n = 0 \quad (7-1)
\]

In order that Eq. (7-1) has non-trivial solutions, the determinant of \( Y_n \) must be vanished. Hence

\[
\Delta = \left| Y_n \right| = \left| A_i Y A_v^t \right| = 0 \quad (7-2)
\]

The previous section shows that the determinant of \( A_i Y A_v^t \) is equal to

\[
\Delta = \sum_{(d)} \varepsilon_d \text{ common tree } T_d \text{ admittance product} \quad (7-3)
\]

Since \( \Delta \) is a homogeneous polynomial of degree \( v-1 \) in \( y \)'s and is linear in any one \( y \), Eq. (7-3) can be written as
\[ \Delta = \sum_{d} \epsilon_d \text{ common tree } T_d \text{ admittance product which contains } y + \sum_{d'} \epsilon_{d'} \text{ common tree } T_{d'} \text{ admittance product which does not contain } y_{po} \]

= 0 \quad (7-4)

Since the second part of Eq. (7-4) is the determinant of the admittance matrix of a network N itself, it is V. We define \( W_{p,o} \) as

\[ W_{p,o} = \frac{1}{y_{po}} \sum_{d'} \epsilon_{d'} \text{ common tree } T_{d'} \text{ admittance product which contains } y_{po} \quad (7-5) \]

Then Eq. (7-4) can be written as

\[ \Delta = V + y_{po} W_{p,o} = 0 \quad (7-6) \]

or

\[ -y_{po} = V/W_{p,o} \quad (7-7) \]

It is clear that \(-y_{po}\) can be represented as shown in Fig. 7-2 by the definition of transadmittances. Thus \(-y_{po}\) is the definition of the open circuit driving point admittance \( Y_{po} \) of a network N. Thus, the topological formula for \( Y_{po} \) is equal to

\[ Y_{po} = V/W_{p,o} \quad (7-8) \]
which is exactly the same form as that for passive networks.

Consider a network $N$ in Fig. 7-3 to which transadmittance $y_{po,qr}$ is connected.

Similar to the previous case, the determinant $\Delta$ of $A_y Y_A^t$ of this network with $y_{po,qr}$ can be written as

\[
\Delta = \sum_{(d)} \epsilon_d \text{ common tree } T_d \text{ admittance product which contains } y_{po,qr} + \sum_{(d')} \epsilon_{d'} \text{ common tree } T_{d'} \text{ admittance product which does not contain } y_{po,qr} = 0
\] (7-9)
By defining $W_{p,o;q,r}$ as

$$W_{p,o;q,r} = \frac{1}{y_{po,qr}} \Sigma_{d'} \epsilon_{d'} \text{ common tree } T_{d'} \text{ admittance product which contains } y_{po,qr} \tag{7-10}$$

We have from Eq. (7-9)

$$-y_{po,qr} = V/W_{p,o;q,r} \tag{7-11}$$

Since $-y_{po,qr}$ is the definition of an open circuit transfer admittance function $Y_{po,qr}$ we have the topological formula of $Y_{po,qr}$ as

$$Y_{po,qr} = V/W_{p,o;q,r} \tag{7-12}$$

The matrix form of network functions which is similar to that of passive networks in Eq. (3-33) is

$$\begin{bmatrix} V_{po} \\ V_{qr} \end{bmatrix} = \frac{1}{V} \begin{bmatrix} W_{p,o} & W_{q,r;p,o} \\ W_{p,o;q,r} & W_{q,r} \end{bmatrix} \begin{bmatrix} J_{op} \\ J_{rq} \end{bmatrix} \tag{7-13}$$

Example 7-1: $W_{1,0;2,0}$ of graph in Fig. 5-2b can be obtained by adding $y_{1,0;2,0}$ as shown in Fig. 7-4a whose voltage and current graphs are shown in Fig. 7-4b and c.
By using rules given in the previous section, we can see easily that the values of the sign-permutations of the first two common trees are +1; the value of the sign-permutation of the last common tree is -1. Thus $W_{1,0;2,0}$ is equal to

$$W_{1,0;2,0} = y_{fg} + y_{fy_k} - y_{kg}$$

Suppose all elements in the network $N$ in Fig. 7-5a are passive elements excluding mutual couplings. Then $G_i$ and $G_v$ corresponding to the network $N$ with $y_{po,qo}$ differ only the location of $y_{po,qo}$ as shown in Fig. 7-5b and c.
Then it is easily seen that any common tree $T$ which contains $y_{p_0,q_0}$ must consist of $y_{p_0,q_0}$ and 2-tree of $G_N$ of the type $T_{2pq,o}$. It is known that 2-tree of the type $T_{2pq,o}$ consists of two maximal connected subgraphs $G(\Omega_1)$ and $G(\Omega_2)$ where vertices $p$ and $q$ are in $\Omega_1$ and vertex $o$ is in $\Omega_2$. $\Omega_1$ and $\Omega_2$ are disjoint and $\Omega_1 \cup \Omega_2 = \Omega$. Thus by Rule 2, $y_{p_0,q_0}$ in $\{T,\Omega\}_{f_1}$ can be relocated to the location from vertices $q$ to $o$. By doing so, the resultant graph is identical to $\{T,\Omega\}_{f_v}$ including the orientations of all edges. Thus the value of the sign-permutation of the common tree $T$ is $+1$. Hence $W_{p_0,o;q_0,o}$ is equal to

$$W_{p_0,o;q_0,o} = W_{pq,o}$$  \hspace{1cm} (7-14)

which is a proof of $|A_{-p} Y A_{-q}^t|$ in Eq. (3-8). Similar proof can be given for $W_{pq,o;ro} = W_{pr,o}$ in Eq. (3-11).

8. Topological Formulas for Short Circuit Functions of Active Networks

Let $N_s$ be the network obtained from a network $N$ by coinciding vertices $p$ and $o$ as shown in Fig. 8-1a whose graph is given in Fig. 8-1b.

Fig. 8-1. An Active Network with Vertices $p$ and $o$ being Coincided
Let $N$ consist of $v$ vertices. Then a common tree $T$ in the graph $G_s$ in Fig. 8-1b consists of $v-2$ edges. Since vertices $p$ and $o$ are coincided to obtain $G_s$, $\{T, \Omega(T)\}_f$ is a 2-tree of the type $T_2$ in $\{G, \Omega\}_f$ and $\{T, \Omega(T)\}_f_v$ is a 2-tree if the type $T_2$ in $\{G, \Omega\}_f_v$ where $\{G, \Omega\}_f$ and $\{G, \Omega\}_f_v$ are the voltage and the current graphs of $G$ corresponding to $N$. Thus

$$V \text{ of the graph } G_s \text{ corresponding to } N_s = W_{p,o} \text{ of the graph } G \text{ corresponding to } N$$

(8-1)

This means that coinciding vertices $p$ and $o$ of a network are equivalent to deleting the $p^{th}$ row and the $p^{th}$ column of the admittance matrix $A_i^{t}A_v$ of network $N$, where the $p^{th}$ rows of $A_i$ and $A_v$ correspond to vertex $p$ in $G_i$ and $G_v$ of $G$ respectively.

From the node basis equations of network $N_s$ in Fig. 8-1a, the following equation can be obtained.

$$\begin{bmatrix} V_{to} \\ V_{qr} \end{bmatrix} = \frac{1}{\Delta'} \begin{bmatrix} \Delta_{tt} & \Delta_{qt} - \Delta_{rt} \\ \Delta_{tq} - \Delta_{tr} & \Delta_{tt} + \Delta_{qr} - \Delta_{rq} \end{bmatrix} \begin{bmatrix} J_{to} \\ J_{qr} \end{bmatrix}$$

(8-2)

Since $\Delta'$ is the determinant of an admittance matrix of $N$, $\Delta'$ is equal to the cofactor $\Delta_{pp}$ of an admittance matrix $A_i^{t}A_v$ of $N$ with vertex $o$ being the reference vertex which is equal to $W_{p,o}$ of $N$. Similarly $\Delta_{qq} + \Delta_{rr} - \Delta_{rq}$ in Eq. (8-2) must be equal to $\Delta_{ppqq} + \Delta_{pprr} - \Delta_{ppqr} - \Delta_{pprq}$ of $A_i^{t}A_v$ of $N$. Now we define $U_{p,o;q,r}$ as
\[ U_{p,o;q,r} = \frac{\Delta_{ppqq} + \Delta_{pprr} - \Delta_{ppqr} - \Delta_{pprq}}{2} \]  (8-3)

Since \( \Delta_{qq} + \Delta_{rr} - \Delta_{qr} - \Delta_{rq} \) is \( W \) of \( N \), \( U_{p,o;q,r} \) of \( N \) can be expressed as

\[ U_{p,o;q,r} = \frac{1}{y_{po}y_{qr}} \sum_{r \in \text{common tree } T} \text{product which contains both } y_{po} \text{ and } y_{qr} \]  (8-4)

where \( G_{y_{po}y_{qr}} \) is obtained from \( G \) by connecting \( y_{po} \) and \( y_{qr} \) between vertices \( p \) and \( o \) and vertices \( q \) and \( r \) respectively as shown in Fig. 8-2, and \( G \) is the graph corresponding to network \( N \).

\[ \text{(a)} \]

\[ \text{(b)} \]

**Fig. 8-2.** A Network with \( y_{po} \) and \( y_{qr} \)

Thus the short circuit admittance functions which are given by
\[
\begin{bmatrix}
J_{op} \\
J_{rq}
\end{bmatrix} = \frac{1}{\Delta ppq + \Delta pprr - \Delta ppqr - \Delta pprq} \begin{bmatrix}
\Delta qq + \Delta rr - \Delta qr - \Delta rq - (\Delta qp - \Delta rp) \\
- (\Delta pq - \Delta pr) \quad \Delta pp
\end{bmatrix} \begin{bmatrix}
V_p \\
V_q \\
V_r
\end{bmatrix}
\] (8-5)

can be expressed by topological formulas as

\[
\begin{bmatrix}
J_{op} \\
J_{rq}
\end{bmatrix} = \frac{1}{U_{p,q,r}} \begin{bmatrix}
W_{q,r} \\
-W_{q,r,p,o}
\end{bmatrix} \begin{bmatrix}
V_p \\
V_q \\
V_r
\end{bmatrix}
\] (8-6)

Example 8-1: In order to obtain \(U_{1,0;2,0}\) of \(G\) in Fig. 5-2b, we connect \(y_{10}\) and \(y_{20}\) as shown in Fig. 8-3a whose voltage and current graphs are given in Fig. 8-3b and c. Then from all the possible common trees which contain both \(y_{10}\) and \(y_{20}\) as

\((y_{10}, y_{20}, g)\) and \((y_{10}, y_{20}, y_k)\)

and by the use of Rules 1 and 2, we have

\(U_{1,0;2,0} = g + y_k\)
The voltage ratio and the current ratio transfer functions can be expressed as

\[
\frac{V_{qr}}{V_{po}} \bigg|_{J_{rq}=0} = \frac{W_{p,o;q,r}}{W_{p,o}} 
\]

(8-7)

\[
\frac{J_{rq}}{J_{op}} \bigg|_{V_{qr}=0} = -\frac{W_{p,o;q,r}}{W_{q,r}}
\]

(8-8)

Example 8-2: Open circuit driving point admittance function

\( Y_{po} \) and voltage ratio transfer function \( V_{po} \) of the network \( N \) in Fig. 8-4a can be obtained as follows:

For simplicity we consider transconductances \( g_1 \) and \( g_2 \) of these tubes and \( y_t \) only. The graph \( G \) of \( N \) is given in Fig. 8-4b and the voltage and the current graphs of \( G \) are given in Fig. 8-4c and d. All possible common trees are

\( (y_t; g_1) \) (\( y_t; g_2 \)) and \( (g_1; g_2) \)

By the use of Rules 1, 2, and 3 which give the signs of these common trees, we can obtain \( V \) as

\[ V = y{_t}g_1 + y_tg_2 - g_1g_2 \]

To obtain \( W_{p,o} \) we add edge \( y \) between vertices \( p \) and \( o \) as in Fig. 8-4e.

All possible common trees which contain \( y \) is \( (y; y_t) \) whose sign is obviously +1 because \( y \) and \( y_t \) are passive elements. Thus, \( Y_{po} \) is equal to

\[ Y_{po} = \frac{y_tg_1 + y_tg_2 - g_1g_2}{y_t} \]
In order to obtain $W_{p,q,r}$, we insert $y_{po,qr}$ as in Fig. 8-4f and obtain all possible common trees which contain $y_{po,qr}$. Those are $(y_{po,qr},y_t)$ and $(y_{po,qr},g_2)$. The sign of $(y_{po,qr},y_t)$ is +1 and that of $(y_{po,qr},g_2)$ is -1. Thus

$$W_{p,q,r} = y_t - g_2$$

or

$$V_{qo/V_{po}} = (y_t - g_2)/y_t.$$ 

9. Analysis of Active Networks by Digital Computer and Further Remarks

The method used here to analyze active networks by a digital computer is similar to that to analyze passive networks given in the previous chapter. The difference is that instead of finding trees we must find all possible common trees and the signs of these common trees.

Let $E$ consist of $v-1$ edges. First we test whether $E_{f_1}$ is a tree. If not, we know that $E$ is not a common tree. If so, we must test whether $E_{f_v}$ is a tree in $G_v$ which determines whether $E$ is a common tree or not. During the testing of $E$, we form the sign-permutation of $E$, hence at the end of testing $E$ for a common tree, we also have the value of the sign-permutation of $E$. The details of this procedure are as follows:

1. R-table in the flow chart gives $E_{f_1}$ and $E_{f_v}$.
2. We give the relative address 0, 1, 2, ..., $v-1$ to entries in R-table which are used to represent $y$ in the sign-permutation of a common tree.
(3) For convenience, the first entry in G-table is representing the reference vertex. C in G-table is zero if \( \Omega(E) \) of \( \{E,\Omega\}_{f_1} \) (or \( \{E,\Omega\}_{f_v} \)) does not contain vertex \( \xi \). C is the relative address of \( y \) if \( \xi \) is the principal vertex of \( y \) in \( \{E,\Omega\}_{f_1} \) (or \( \{E,\Omega\}_{f_v} \)).

When \( \{E,\Omega\}_{f_1} \) is found to be a tree, all entries except the first entry of G-table is row i of the sign-permutation of E without the minus sign superscripts. Thus we can compute the number \( N_{p_1} \) of interchange which is necessary in order that these entries in G-table become the natural order. Similarly, when \( \{E,\Omega\}_{f_v} \) is found to be a tree, all entries except the first entry of G-table is row v of the sign-permutation of E without the minus sign superscripts. From these entries, we can obtain number \( N_{p_2} \) of interchanges which are necessary to make them in natural order. Then, \((-1)^{p_1} N + N_{p_2} + N_{m} \) is the value of the sign-permutation of E if E is a common tree where \( N_{m} \) is the number of edges in \( \{E,\Omega\}_{f_1} \) and \( \{E,\Omega\}_{f_v} \) whose orientations are toward its principal vertex. The flow chart which is the modified flow chart given in Fig. 4-2 in order to analyze active networks is given in Fig. 9-1.

An important process for analyzing networks by a digital computer by the use of topological formulas is to obtain all possible trees in a linear graph. The method used here is to test all possible subgraphs of \( v-1 \) edges of a linear graph which consists of \( v \) vertices. Recently, several papers [33,34,35] are published which give techniques of obtaining trees without testing all possible subgraphs of \( v-1 \) edges. An interesting result about generation of all possible trees is as follows.
A tree transformation [36] is a transformation of a tree to another
tree of a linear graph by changing only one edge. Then R. C. Cummins [37]
shows that there exists a sequence of trees $T_1, T_2, \ldots, T_m$ such that these
are all possible trees in a linear graph and $T_{i+1}$ can be obtained from
$T_i$ by a tree transformation for $i=1, 2, \ldots, m-1$. No one has found a
way of choosing proper tree transformations such that the above sequence
can be obtained. If there is a process of selecting a proper tree trans­
formation which gives the sequence of distinct trees without testing,
whether the trees obtained by the tree transformation is different from
those obtained by previous tree transformations, the analysis of net­
works by a digital computer becomes very efficient. The present method
needs a rather long computation time when a network contains a large
number of elements such as forty elements.
Fig. 9-1. Flow Chart for Analysis of Active Networks
V. APPLICATION OF LINEAR GRAPHS TO SWITCHING NETWORKS

Though an s.c. (single contact) switching function is directly related to paths in a switching network, synthesis of s.c. switching functions by the use of linear graphs is impossible without having necessary and sufficient conditions for cut set (or circuit) matrices. Recently, several conditions have been found [25,26,28,38,39] and the use of these conditions—the synthesis of switching functions by a digital computer—becomes possible.

In this chapter, the properties of paths [24] and properties of s.c. switching functions are given first. Then a testing procedure for a cut set matrix [25,26] and related problems [27] will be discussed. Finally, properties of multi-terminal s.c. switching networks (unpublished) and multi-constant switching networks [24] are given.

10. Properties of Paths in a Non-oriented Linear Graph and S.C. Switching Functions

Let G be a non-separable non-oriented linear graph. Also let $C = \{c_0, c_1, c_2, \ldots, c_m\}$ be the class of all possible circuits, edge disjoint unions of circuits in G, and the empty set denoted by $\emptyset$. Then C is a group under the ring sum $\oplus$.

Let $P_{ij}$ be a path between distinct vertices i and j in G. We define a class $C^\oplus_{ij}$ of sets as

$$C^\oplus_{ij} = \{x | x = P_1 \oplus c_r, c_r \in C\}. \quad (10-1)$$

Let $P_2$ be a path between vertices i and j in G. Then it is clear that

$$P_1 \oplus P_2 = c_r \in C \quad (10-2)$$
or

$$P_1 \Theta c_r \equiv P_2.$$ (10-3)

Thus \(P_2 \in \mathcal{C}_{ij}^\varnothing\). This means that all possible paths between vertices \(i\) and \(j\) in \(G\) are in \(\mathcal{C}_{ij}^\varnothing\). Let \(z \in \mathcal{C}_{ij}^\varnothing\). Then \(z \notin \mathcal{C}\) because \(P_1 \Theta c_r \notin \mathcal{C}\).

Consider the class \(\mathcal{C}_{ij}^\varnothing \cup \mathcal{C}\) which is the class of all sets in \(\mathcal{C}_{ij}^\varnothing\) and in \(\mathcal{C}\). Then

$$z_1, z_2 \equiv \left\{ \begin{array}{ll}
\mathcal{C}_r & \text{if } c_r \in \mathcal{C} \\
\mathcal{P}_1 \Theta c_r & \text{else} \end{array} \right.$$ (10-4)

Thus

$$z_1 \Theta z_2 = z_2 \Theta z_1 \equiv \left\{ \begin{array}{ll}
\mathcal{P}_1 \Theta c_r \Theta \mathcal{P}_1 \Theta c_s \equiv c_t & \text{if } c_r, c_s \text{ and } c_t \in \mathcal{C} \\
\mathcal{P}_1 \Theta c_r \Theta c_s \equiv \mathcal{P}_1 \Theta c_t & \text{else} \end{array} \right.$$ (10-5)

Hence, \(z_1 \Theta z_2 \in \mathcal{C}_{ij}^\varnothing \cup \mathcal{C}\). Thus, \(\mathcal{C}_{ij}^\varnothing \cup \mathcal{C}\) is also a group under \(\Theta\).

Consider \(G_+e\) which is \(\{G \cup \{e\}, \Omega(G)\}_f\) and \(f(e) = (i, j)\). It is clear that any path \(P\) from vertex \(i\) to vertex \(j\) in \(G\) and edge \(e\) form a circuit in \(G_+e\). Let \(\mathcal{C}_{ij}^{\varnothing\cup\{e\}}\) be the class of sets obtained from \(\mathcal{C}_{ij}^\varnothing\) by adding \(e\) to every set in \(\mathcal{C}_{ij}^\varnothing\). Then \(\mathcal{C}_{ij}^{\varnothing\cup\{e\}} \cup \mathcal{C}\) is the class of all possible circuits, edge disjoint unions of circuits, and the empty set.

Thus \(u\) in \(\mathcal{C}_{ij}^\varnothing\) is either a path from vertex \(i\) to vertex \(j\) or edge disjoint union of a path from vertex \(i\) to vertex \(j\) and circuits.

Let \(u_1, u_2, u_3,\) and \(u_4\) be in \(\mathcal{C}_{ij}^\varnothing\). Then

$$u_1 \Theta u_2 \Theta u_3 \equiv u_4.$$ (10-6)

Let \(\mathcal{G}_{ij}\) be the class of all possible paths from vertex \(i\) to vertex \(j\) in \(G\). Then \(\mathcal{G}_{ij} \subseteq \mathcal{C}_{ij}^\varnothing\). Suppose \(P_1, P_2, P_3\), and \(P_4\) are in \(\mathcal{G}_{ij}\). Then
\[ P_1 \oplus P_2 \oplus P_3 = u \in C_{ij}. \quad (10-7) \]

Thus either
\[ P_1 \oplus P_2 \oplus P_3 = P_4 \]
or
\[ P_1 \oplus P_2 \oplus P_3 \supset P_4. \quad (10-8) \]

Definition 10-1: Let \( R \) be a class of sets \( r \)'s. Then \( (\text{Min } R) \) is defined as a subclass of \( R \) which satisfies that

1. If \( \phi \) is in \( R \), \( \phi \) is in \( (\text{Min } R) \);
2. For any \( r_p \neq \phi \in \{ R - (\text{Min } R) \} \), there exists \( r_s \neq \phi \) in \( (\text{Min } R) \) such that \( r_s \subseteq r_p \); and
3. For any \( r_s \neq \phi \) and \( r_t \neq \phi \) in \( (\text{Min } R) \), \( r_s \not\subset r_t \).

With the above definition, it can be seen that
\[ (\text{Min } C_{ij}) = \delta_{ij}. \quad (10-9) \]

Also,
\[ (\text{Min } C) = C_s \quad (10-10) \]

where \( C_s \) is the class of all circuits in \( G \) and the empty set.

Definition 10-2: Let \( R \) be a class of sets \( r \)'s and \( U \) a class of sets \( u \)'s. Then \( R \Theta S \) is defined as \( (\text{Min } W) \) where \( W \) is a class of distinct \( r \Theta u \) for all \( r \) and \( u \) in \( R \) and \( U \). For convenience, we define that
\[ R^n \equiv R^{n-1} \Theta R \quad \text{for } n > 1 \text{ where } R^1 = R. \quad (10-11) \]
With this definition, it is easily seen that

$$
\mathcal{F}^2_{ij} \equiv C_s \tag{10-12}
$$

$$
\mathcal{F}^3_{ij} \equiv \mathcal{F}_{ij} \odot C_s \equiv \mathcal{F}_{ij} \tag{10-13}
$$

$$
\mathcal{F}^4_{ij} \equiv (C_s)^2 \equiv C_s \tag{10-14}
$$

With binary field $\Lambda$ we define

$$
\beta C = \begin{cases} 
C & \text{if } \beta = 1 \\
\emptyset & \text{if } \beta = 0 
\end{cases} \tag{10-15}
$$

where $\beta \in \Lambda$ and $c_r$ and $\emptyset$ (empty set) are in $C$. Then $C$ and $\Lambda$ form a vector space. Let $G$ consist of $e$ edges and $\Omega(G)$ consist of $v$ vertices where $\{G\}$ is a connected linear graph. Then it is known that the dimension of this vector space is $e - v + 1$. Similarly, $C^s_{ij} \cup C$ and $\Lambda$ form a vector space. Since $C^s_{ij} \cup C \supset C$, and $P \in C^s_{ij}$ but $P \notin C$, the dimension of this space must be at least $e - v + 2$. Let $z_1$ and $z_2$ be in $C^s_{ij}$. Then we know that $z_1 \Theta z_2 \in C$. Thus the dimension of this vector space is no more than $e = v + 2$. Hence we conclude that the dimension of $C^s_{ij} \cup C$ with $C$ is $e - v + 2$. Since $G$ is non-separable, every $c \in C$ which is not an edge disjoint union of circuits can be expressed as $P_1 \Theta P_2$, where $P_1, P_2 \in \mathcal{F}_{ij}$. Thus a basis of $C^s_{ij} \cup C$ with $\Lambda$ can consist only of sets in $\mathcal{F}_{ij}$; that is, we can form $C^s_{ij} \cup C$ only by ring sum of paths between vertices $i$ and $j$ in $G$.

Let $N$ be an s.c. (single contact) switching network and $F$ be the s.c. switching function between vertices $i$ and $j$ of $N$. Then $N$ can be represented by a weighted non-oriented graph $\{w_s, G, \Omega\}$ where $w_s$ is the set of switching variables.
Definition 10-3: A set representation of a switching function $F$ is a class $\mathcal{L}$ of sets $\eta_k$ such that $\eta_k$ is the set of all switching variables in $f_k$ where $f_k$ is the product of switching variables in $F$ expressed as the sum of $f$'s as

$$F = \sum f_k$$  \hfill (10-16)

where every $f_k$ contains at least one switching variable.

Since $F$ is an s.c. switching function between vertices $i$ and $j$ of an s.c. switching network $N$, it is clear that

$$\mathcal{L} \subseteq \phi_{ij} \text{ of } G.$$  \hfill (10-17)

Thus a necessary condition that $F$ is an s.c. switching function is that (1) the ring sum of any odd number of $\eta$'s in $\mathcal{L}$ corresponding to $F$ is either

$$\eta_{k_1} \oplus \eta_{k_2} \oplus \ldots \oplus \eta_{k_m} \equiv \eta_n$$  \hfill (10-18)

or

$$\eta_{k_1} \oplus \eta_{k_2} \oplus \ldots \oplus \eta_{k_m} \supset \eta_n$$  \hfill (10-19)

where $\eta_{k_1}, \eta_{k_2}, \ldots, \eta_{k_m}$, and $\eta_n$ are in $\mathcal{L}$ (m odd) and (2) the ring sum of an even number of $\eta$'s in $\mathcal{L}$ is not a set in $\mathcal{L}$. Hence the following theorem can be obtained.

Theorem 10-1: A necessary condition that $F$ is an s.c. switching function is that the corresponding class $\mathcal{L}$ satisfies
(1) \( L_3 \cong L \); (10-20)

(2) Any \( \eta \in L \) does not contain \( z \in L^2 \).

Consider a linear graph \( G+y \) which is obtained from \( G \) by adding \( y \) between vertices \( i \) and \( j \) as shown in Figure 10-1; that is, \( G+y \) is \( \{ G \cup \{ y \}, \Omega(G) \}_f \) where \( f(y) = (i,j) \).

![Figure 10-1. A linear graph \( G+y \).](image)

Since \( y \) itself is a path between \( i \) and \( j \) in \( G+y \), \( \Phi_{ij+y} \) of \( G+y \) consists of \( \Phi_{ij} \) of \( G \) and \( \{ y \} \). Hence, if \( G+y \) is an s.c. switching network, \( L+y \) corresponding to the s.c. switching function between \( i \) and \( j \) of \( G+y \) consists of \( L \) corresponding to \( F \) between vertices \( i \) and \( j \) or \( G \) and \( \{ y \} \).

From Eq. (10-17) and (10-12), \( \text{Min} L+y \) is the class of all possible circuits in \( G+y \). Thus we can obtain a fundamental circuit matrix \( B = [U B_{12}] \) and a fundamental cut set matrix \( Q = [B^t_{12} U] \) of linear graph \( G+y \). Thus if \( L \) corresponding to a given s.c. switching function satisfies Theorem 10-1, and if a necessary and sufficient condition that \( Q \) (or \( B \)) which is obtained from \( L+y \) to be a cut set (or circuit) matrix is
satisfied, we can obtain a switching network \( G_{+y} \). Then deleting \( y \) from \( G_{+y} \) gives \( G \) and vertices \( i \) and \( j \). Testing for a cut set matrix will be discussed in the next section.

Example 10-1: Let \( F = x_1x_2 \cup x_1x_3 \cup x_2x_3 \). Then

\[
\mathcal{L} = \{x_1x_2, x_1x_3, x_2x_3\};
\]

\[
\mathcal{L}^2 = \{\emptyset, x_2x_3, x_1x_3, x_1x_2\} \notin \mathcal{L} \] which contradicts the condition 2 in Theorem 10-1.

Thus, \( F \) is not an s.c. switching function.

11. Realizability of Fundamental Cut Set Matrices of Non-Oriented Linear Graphs.

Suppose \( S = \{e_1, e_2, \ldots, e_k\} \) is a fundamental cut set in a connected linear graph \( \{G, \Omega\}_f \). Then it can be shown that there exists a pair of vertex sets \( \Omega_1 \) and \( \Omega_2 \) which are disjoint and \( \Omega_1 \cup \Omega_2 = \Omega \) such that every edge \( e_p \) in \( S \) has the properties that

\[
f(e_p) \in \Omega_1 \times \Omega_2 \tag{11-1}
\]

and there is no edge in \( \{G-S\} \) which satisfies Eq. (11-1). By deleting all edges in \( S \) from \( G \), we will obtain two maximally connected subgraphs \( G(\Omega_1) \) and \( G(\Omega_2) \) as shown in Figure 11-la. We construct two connected graphs \( \{G'_1, \Omega'_1\}_{f_1} \) and \( \{G'_2, \Omega'_2\}_{f_2} \) by the following procedure.
(a) Linear graph $G$ and cut set $S$.

(b) Inserting a vertex.

(c) Two resultant graphs.

Figure 11.1. Special way of obtaining two linear graphs from one.

(1) Insert a vertex at the middle of each edge in $S$. Then coalesce these vertices together as shown in Figure 11-1b.

(2) Split the new vertex as shown in Figure 11-1c. Notice that these two resultant graphs have edges $e_1, e_2, \ldots, e_k$ in common; that is, $G'_1 \cap G'_2 \subseteq S$. In other words, $\{G'_1, \Omega'_1\}_{f_1}$ and $\{G'_2, \Omega'_2\}_{f_2}$ have the following properties:

(a) $G'_1 \subseteq G(\Omega'_1) \cup S$, $G'_2 \subseteq G(\Omega'_2) \cup S$, $\Omega'_1 \subseteq \Omega'_2 \cup \{\zeta_0\}$, and $\Omega'_2 \subseteq \Omega'_2 \cup \{\zeta_0\}$; (11-2)

(b) For every edge $e$ in $G(\Omega'_1)$ and $G(\Omega'_2)$,

$$f_1(e) = f_2(e) = f(e);$$ (11-3)
(c) For every edge $e$ in $S$, $f_1(e)$ is either

$$f_1(e) = (\zeta_r, \zeta_o) \text{ if } f(e) = (\zeta_r, \zeta_s)$$

or

$$f_1(e) = (\zeta_o, \zeta_r) \text{ if } f(e) = (\zeta_s, \zeta_r)$$

where $\zeta_r \in \Omega_1$, $\zeta_s \in \Omega_2$, and $\zeta_o \notin \Omega$;

and

(d) For every edge $e$ in $S$, $f_2(e)$ is either

$$f_2(e) = (\zeta_o, \zeta_s) \text{ if } f(e) = (\zeta_r, \zeta_s)$$

or

$$f_2(e) = (\zeta_s, \zeta_o) \text{ if } f(e) = (\zeta_s, \zeta_r)$$

where $\zeta_r \in \Omega_1$, $\zeta_s \in \Omega_2$, and $\zeta_o \notin \Omega$.

An important property of these resultant graphs is that $S$, which is a cut set in $G$, is an incidence set in both of these graphs.

Let $Q$ be a fundamental cut set matrix of $G$. We are going to obtain fundamental cut set matrices of $\{G'_1, \Omega'_1\}_{f_1}$ and $\{G'_2, \Omega'_2\}_{f_2}$ directly from $Q$.

Definition 11-1: Suppose row $i$ of $Q$ represents cut set $S$. 

**H-submatrix** of $Q$ with respect to row $i$ denoted by $H_i$ is a submatrix of $Q$ obtained by removing every column which has non-zero at row $i$ and removing row $i$. 
It is clear that $H_1$ is a fundamental cut set matrix of linear graphs $G(\Omega_1) \cup G(\Omega_2)$. Hence, if there exists at least one edge in each $G(\Omega_1)$ and $G(\Omega_2)$, $H_1$ can be partitioned as

$$
H_1 = \begin{bmatrix}
H_{11} & 0 \\
0 & H_{12}
\end{bmatrix}
$$

(11-6)

where neither $H_{11}$ nor $H_{12}$ is empty.

It is true that even if $H_1$ is partitionable as in Eq. (11-6), $S$ corresponding to row $i$ can be an incidence set. For example, $S = \{e_1, e_2, \ldots, e_k\}$ of linear graph $G$ in Figure 11-2a is an incidence set. However, $H_1$ is partitionable as in Eq. (11-6) where row $i$ of a fundamental cut set matrix of $G$ represents $S$. The reason is that there exists a 2-isomorphic linear graph [19] $G'$ of $G$ as shown in Figure 11-2b in which $S$ is a fundamental cut set. Hence we can consider that whenever $H_1$ of $Q$ is partitionable as in Eq. (11-6) with non-empty $H_{11}$ and $H_{12}$, $S$ corresponding to row $i$ in $Q$ is a fundamental cut set which is not an incidence set.

(a) Linear Graph $G$  
(b) 2-isomorphic Graph of $G$

Figure 11-2. 2-isomorphic Graphs.
On the other hand, if $H_1$ cannot be partitioned as in Eq. (11-6) with non-empty $H_{i_1}$ and $H_{i_2}$, then $S$ corresponding to row $i$ of $Q$ is definitely an incidence set.

**Definition 11-2:** A pair of $M$-submatrices $M_{1}(i)$ and $M_{2}(i)$ of a fundamental cut set matrix $Q$ with respect to row $i$ is defined as follows:

Let $H_1$ of $Q$ with respect to row $i$ be partitioned as in Eq. (11-6). Then $M_{1}(i)$ is obtained from $Q$ by deleting all rows and columns which belong to $H_{i_2}$. $M_{2}(i)$ is obtained from $Q$ by deleting all rows and columns which belong to $H_{i_1}$. The symbol $i$ in the parenthesis of $M_{1}(i)$ and $M_{2}(i)$ indicates row $i$ which is used to obtain $H$-submatrix of $Q$ to obtain these $M$-submatrices. If either $H_{i_1}$ or $H_{i_2}$ is empty, say $H_{i_2}$, then $M_{1}(i) = Q$ and $M_{2}(i)$ is obtained from $Q$ by deleting all rows and columns which belong to $H_{i_1}$ by definition. Thus $M_{2}(i)$ is one row matrix which consists of the non-zero entries of row $i$ in $Q$.

By the above definition, it is clear that $M_{1}(i)$ is a fundamental cut set matrix of $\{G'_{1}, \Omega'_{1}\}_{f_{1}}$ and $M_{2}(i)$ is a fundamental cut set matrix of $\{G'_{2}, \Omega'_{2}\}_{f_{2}}$. The important properties of $M_{1}(i)$ and $M_{2}(i)$ are that row $i$ of $Q$ is in both $M_{1}(i)$ and $M_{2}(i)$. However, any other row of $Q$ is in either $M_{1}(i)$ or $M_{2}(i)$ but not in both. Also, it must be noticed that row $i$ of $M_{1}(i)$ represents incidence set in $\{G'_{1}, \Omega'_{1}\}_{f_{1}}$. Similarly, row $i$ of $M_{2}(i)$ represents an incidence set in $\{G'_{2}, \Omega'_{2}\}_{f_{2}}$.

Suppose there exist two non-oriented linear graphs $G'_{a}$ and $G'_{b}$ whose fundamental cut set matrices are $M_{1}(i)$ and $M_{2}(i)$ which are a pair of $M$-submatrices of $Q$ with respect to row $i$. Also suppose row $i$ of $M_{1}(i)$ represents an incidence set in $G'_{a}$ and row $i$ of $M_{2}(i)$ represents
an incidence set in \( G'_b \). Then we can construct a non-oriented linear graph \( G \) whose fundamental cut set matrix is \( Q \) by the following procedure.

Let \( S_i = \{e_1, e_2, \ldots, e_k\} \) be the incidence set corresponding to row \( i \) of \( M_1(i) \) and \( M_2(i) \) in \( G'_a \) and \( G'_b \) as shown in Figure 11-3a; that is,

\[ \{S_i, \Omega(S_i)\}_{f_1} \]

is an incidence set in \( \{G'_a, \Omega(G'_a)\}_{f_1} \) and \( \{S_i, \Omega(S_i)\}_{f_2} \) is an incidence set in \( \{G'_b, \Omega(G'_b)\}_{f_2} \). Also let \( f_1(e_j) = (\zeta_{aj}, \xi_{ao}) \) and \( f_2(e_j) = (\xi_{bj}, \zeta_{bo}) \) for \( j = 1, 2, \ldots, k \). First we remove all edges in \( S_i \) from \( G'_a \) and \( G'_b \). Then we connect \( e_j \) between \( \xi_{aj} \) and \( \xi_{bj} \) for \( j = 1, 2, \ldots, k \) as shown in Figure 11-3b. The resultant graph is \( \{G, \Omega\} \) where

\[ \Omega = G'_a \cup G'_b, \]

\[ f(e) = f_1(e) \] for every edge \( e \) in \( \{G'_a - S_i\} \), \[ f(e) = f_2(e) \] for every edge \( e \) in \( \{G'_b - S_i\} \), and \[ f(e) = (\zeta_{aj}, \xi_{bj}) \] for every edge \( e \) in \( S_i \). Hence, \( \{S_i, \Omega(S_i)\}_{f} \) is a cut set in the resultant graph. Also every cut set \( \{S_p, \Omega(S_p)\}_{f_1} \) in \( G'_a \) becomes a cut set \( \{S_p, \Omega(S_p)\}_{f} \) in \( G \). Similarly, every cut set \( \{S_q, \Omega(S_q)\}_{f_2} \) in \( G'_b \) is a cut set \( \{S_q, \Omega(S_q)\}_{f} \) in \( G \). Thus \( Q \) is a fundamental cut set matrix of \( G \). Therefore, the following theorem can be stated.

**Theorem 11-1:** If and only if \( M_1(i) \) and \( M_2(i) \) which are a pair of \( M \)-submatrices of \( Q = [Q_{11} \cup Q_{21}] \) with respect to row \( i \) are realizable as fundamental cut set matrices of non-oriented linear graphs \( G_1 \) and \( G_2 \), respectively, with row \( i \) of \( M_1(i) \) representing an incidence set in \( G_1 \) and row \( i \) of \( M_2(i) \) representing an incidence set in \( G_2 \), then there exists a non-oriented linear graph \( G \) such that

1. \( Q \) is a fundamental cut set matrix of \( G \);
2. if row \( r \) (\( r \neq i \)) of \( M_1(i) \) represents an incidence set in \( G_1 \), row \( r \) of \( Q \) represents an incidence set in \( G \); and
(3) if row $r$ of $M_2(i)$ represents an incidence set in $G_2$, row $r$ of $Q$ represents an incidence set in $G$ for all $r$ in $Q$.

(a) Two linear graphs

(b) Graph $G$

Figure 11-3. Formation of $G$.

Definition 11-3: $i_1, i_2, \ldots, i_m$ of $M(i_1 i_2 \ldots i_m)$ means that rows $i_1, i_2, \ldots, i_m$ of fundamental cut set matrix $M(i_1 i_2 \ldots i_m)$ must represent incidence sets.

Let $M(i)$ be one of a pair of $M$-submatrices of a fundamental cut set matrix. Hence $M(i) = [M_{11}(i) U]$. We can obtain a pair of $M$-submatrices of $M(i)$ with respect to row $j$ ($j \neq i$) because $M(i)$ is a fundamental cut set matrix of a non-oriented linear graph. These two submatrices of $M(i)$ will have the properties that one of these two contains row $i$ and row $j$ both of which must represent incidence sets, and the other contains row $j$ but not row $i$. One which contains both rows $i$ and $j$ will be indicated by $M(ij)$ and the other which contains only row $j$ will be indicated by $M(j)$. In general, we can obtain a pair of $M$-submatrices of $M(i_1 i_2 \ldots i_n)$ with respect to row $j$ which is not one of
Let one of these two submatrices contain rows $i_1, i_2, \ldots, i_m$ ($m \leq n$) and, of course, row $j$. We use $M(i_1, i_2, \ldots, i_m)$ to indicate this submatrix. It is obvious that rows $i_1, i_2, \ldots, i_m$ and $j$ must represent incidence sets because rows $i_1, i_2, \ldots, i_m$ in $M(i_1, i_2, \ldots, i_m)$ are incidence sets and row $j$ is the row used to obtain these $M$-submatrices. The other submatrix is clearly $M(i_{m+1}, \ldots, i_n)$ which indicates that rows $i_{m+1}, \ldots, i_n$ and $j$ must be incidence sets.

From a fundamental cut set matrix $Q$ of rank $v-1$, we form a sequence of sets of matrices

$$\{M_1\} \{M_2\} \{M_3\} \cdots \{M_v\}$$

and a sequence of sets of identifiers of rows in $Q$

$$\{R_1\} \{R_2\} \{R_3\} \cdots \{R_v\}$$

as follows: $\{M_1\} = \{Q\}$ and $\{R_1\} = \{\emptyset\}$. Let $\{M_p\} = \{M_1(i_1, i_2, \ldots, i_k), \ldots, M_p(j_1, j_2, \ldots, j_n)\}$ and $\{R_p\} = \{i_1, i_2, \ldots, i_k, \ldots, j_1, j_2, \ldots, j_n\}$. In order to obtain $\{M_{p+1}\}$ and $\{R_{p+1}\}$, we choose one matrix in $\{M_p\}$ which has a row $q$ where $q \notin \{R_p\}$. Then we replace this matrix in $\{M_p\}$ by the pair of $M$-submatrices of this matrix with respect to row $q$ to form $\{M_{p+1}\}$ and $\{R_{p+1}\}$ is obtained from $\{R_p\}$ by adding $q$ for all $p = 2, 3, \ldots, v$. Notice that $\{R_v\}$ contains $v-1$ identifiers of rows in $Q$ which means that there is no matrix in $\{M_v\}$ which has row $s$ such that $s \notin \{R_v\}$.

**Definition 11-4:** An $M$-submatrix $M(i_1, i_2, \ldots, i_k)$ is called a minimum $M$-submatrix if the matrix consists only of rows $i_1, i_2, \ldots, i_k$. 
With the above definition, \( \{M_v\} \) consists of minimum \( M \)-submatrices. Thus we call \( \{M_v\} \) a set of minimum \( M \)-submatrices of \( Q \).

Because a minimum \( M \)-submatrix \( M(i_1, i_2, \ldots, i_k) \) consists only of rows \( i_1, i_2, \ldots, i_k \) and by Definition 11-3, every row must represent an incidence set of a linear graph whose fundamental cut set matrix is \( M(i_1, i_2, \ldots, i_k) \).

Thus a minimum \( M \)-submatrix is an incidence matrix of a linear graph.

Suppose every matrix in \( \{M_v\} \) is an incidence matrix of a linear graph. Then we can prove by induction on \( \{M_v\} \) that by successive application of Theorem 11-1 a linear graph whose fundamental cut set matrix is \( Q \) can be obtained. It is obvious that if \( Q \) is a fundamental cut set matrix of a linear graph, we can obtain \( \{M_v\} \) in which every matrix is an incidence matrix of a linear graph. Suppose \( H_i \) of a fundamental cut set matrix \( Q \) with respect to row \( i \) can be partitioned as

\[
H_i = \begin{bmatrix}
H_{i1} & \circ & \circ \\
\circ & H_{i2} & \circ \\
\circ & \circ & \cdots \\
\circ & \circ & \cdots & H_{ik}
\end{bmatrix}
\]  

(11-7)

Then there are \( 2^{k-1} \) ways of partitioning \( H_i \) as in Eq. (11-6). Thus in general there may be more than one sequence of \( \{M_p\} \) (\( p = 1, 2, \ldots, v \)) for a given \( Q \). In other words, there may exist many sets of minimum \( M \)-submatrices of \( Q \) and even if only one of these sets of minimum \( M \)-submatrices of \( Q \) consists of incidence matrices of linear graphs, \( Q \) is realizable as a fundamental cut set matrix. Thus we have the following theorem.
Theorem 11-2: A matrix $Q\{q_{ij}\}$ whose entries are 1 or 0 is a fundamental cut set matrix of a non-oriented linear graph if and only if there exists a set of minimum M-submatrices of $Q$ all of which are realizable as incidence matrices of non-oriented linear graphs.

Example 11-1: Suppose the given matrix is

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
3 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
4 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

Matrix $H_1$, which is obtained by removing every column which has a 1 at row 1 and then by removing row 1, can be partitioned as

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
3 & 1 & 1 & 1 \\
4 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}
\]

Hence, the pair of M-submatrices, with respect to row 1, are

\[
M_1(1) = \begin{bmatrix}
a & b & c & f & g \\
1 & 1 & 1 & 1 & 0 \\
2 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\]

and

\[
M_2(1) = \begin{bmatrix}
a & b & c & d & e & f & h & i \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
3 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
4 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]
The pair of M-submatrices of $M_1(1)$, with respect to row 2, are

$$M_1(12) = M_1(1)(\operatorname{Min} M),$$

and

$$M_1(2) = 2[1 \ 1 \ 1](\operatorname{Min} M).$$

Notice that $H$ of $M_1(1)$, with respect to row 2, cannot be partitioned as shown in (11-6). By using row 3, the pair of M-submatrices of $M_2(1)$ are

$$M_2(13) = M_2(1),$$

and

$$M_2(3) = 3[1 \ 1 \ 1 \ 1](\operatorname{Min} M).$$

Finally, matrix $H$, which is obtained from $M_2(13)$ by removing every column which has a 1 at row 4 and by then removing row 4, can be partitioned as

$$H = \begin{bmatrix} a & f & d & h \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Hence, the pair of M-submatrices of $M_2(13)$, with respect to row 4, are

$$M_2(14) = \begin{bmatrix} a & b & c & e & f & i \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} (\operatorname{Min} M),$$

and
Since the set of minimum M-submatrices $M_1^{(12)}$, $M_1^{(2)}$, $M_2^{(3)}$, $M_2^{(14)}$, and $M_2^{(34)}$ satisfy the condition in Theorem 11-2, the given matrix is a fundamental cut set matrix. Also, one can form the graph from these minimum M-submatrices by the following procedure:

1. Form the graph whose fundamental cut set matrix is $M_2^{(13)}$ from the two graphs in Figure 11-4a and 11-4b, and whose incidence matrices are $M_2^{(14)}$ and $M_2^{(34)}$, respectively, as shown in Figure 11-4c;

2. Construct the graph, whose fundamental cut set matrix is the given matrix from the two graphs, one of which is the resultant graph obtained by step (1) and the other, the graph in Figure 11-4d whose incidence matrix is $M_1^{(1)}$, as shown in Figure 11-4e.

Figure 11-4. Construction of G.

For a given fundamental cut set matrix \( Q \) of a linear graph \( G \), it is known that there exists a non-singular matrix \( D \), such that \( DQ \) is an incidence matrix of \( G \). We are going to discuss a method to construct such a non-singular matrix.

From a fundamental cut set matrix \( Q \) of rank \( v-1 \), we can obtain a set \( \{M_v\} \) of minimum M-submatrices of \( Q \) which was discussed in the previous section. The properties of the set \( \{M_v\} \) are

1. There are exactly \( v \) matrices in \( \{M_v\} \);
2. Every row \( i \) in \( Q \) is in exactly two matrices in \( \{M_v\} \);
3. Each matrix in \( \{M_v\} \) is an incidence matrix of a non-oriented linear graph.

Let \( G_i \) be a non-oriented linear graph whose incidence matrix is \( M_i(j_1 \cdots) \) in which vertex \( p \) corresponds to row \( p \) of \( M_i(j_1 \cdots) \). Also, let \( r_i \) be the reference vertex in \( G_i \) for \( i = 1, 2, \ldots, v \). Let \( \{G_i\} \) be the collection of all \( G_i \) corresponding to all matrices \( M_i(j_1 \cdots) \) in \( \{M_v\} \). Then \( \{G_i\} \) has the following properties:

1. No linear graph \( G_j \) in \( \{G_i\} \) except \( G_p \) contains vertex \( r_p \);
2. For any vertex \( \zeta \) which is not a reference vertex in \( G_p \) \((1 \leq p \leq v)\), there exists exactly one linear graph \( G_q \) \((q \neq p)\) in \( \{G_i\} \) which contains vertex \( \zeta \).

These are directly from the properties of \( \{M_v\} \). From the previous section, we know that a linear graph \( Q \) can be obtained from \( \{G_i\} \), and from the process of obtaining \( G \) from \( \{G_i\} \), every vertex other than the reference vertices of linear graphs in \( \{G_i\} \) will be removed. Thus \( G \) consists only of reference vertices of the linear graphs in \( \{G_i\} \). In \( Q \),
there are rows corresponding to incidence sets of vertices in $G_i$ $(i = 1, 2, \ldots, v)$ other than reference vertex $r_i$. Notice that addition (Mod. 2) of all rows in $M_i(j_i \cdots)$ gives the row representing the incidence set of reference vertex $r_i$. Hence, by adding the rows in $Q$ which are also the rows in $M_i(j_i \cdots)$, we can obtain the row representing the incidence set of $r_i$. This is true for all vertices in $G$. Thus now it of the non-singular matrix $D$ must have the property that multiplying it to $Q$ gives the summation of all rows which are also in $M_i$. Let $D_r = [d_{ij}]$ be the matrix whose row represents $G_i$ in $\{G_i\}$ such that $d_{ij} = 1$ if $G_i$ contains row $j$ in $Q$ and $d_{ij} = 0$ otherwise. Then each row of $D_r Q$ represents an incidence set in $G$. Since $A$ consists of $v-1$ rows corresponding to $v-1$ incidence sets in $G$, removal of any row in $D_r$ gives a non-singular matrix $D$ which gives $DQ = A$.

Example 12-1: Let the given matrix be

$$
\begin{array}{cccccccccc}
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
2 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
4 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
5 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
$$

A set of minimum $M$-submatrices consists of the following six matrices:

$$
M_1 = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 0 \\
2 & 1 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}
$$
\[
M_2 = \begin{bmatrix}
2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
3 & 0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
M_3 = 3[1 1 1]
\]

\[
M_4 = \begin{bmatrix}
a & b & e & f & i \\
1 & 1 & 1 & 0 & 1 & 0 \\
4 & 1 & 1 & 1 & 0 & 1
\end{bmatrix}
\]

\[
M_5 = \begin{bmatrix}
a & b & e & i & j \\
4 & 1 & 1 & 1 & 1 & 0 \\
5 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\]

and

\[
M_6 = 5[1 1 1].
\]

Suppose we choose \(M_1, M_2, M_3, M_4,\) and \(M_6\) and let row 1 of \(D\) correspond to \(M_1\), row 2 to \(M_2\), etc.; \(D\) becomes

\[
D = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 1 & 1 & 0 & 0 & 0 \\
2 & 0 & 1 & 1 & 0 & 0 \\
3 & 0 & 0 & 1 & 0 & 0 \\
4 & 1 & 0 & 0 & 1 & 0 \\
5 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

and \(DQ\) becomes
which is an incidence matrix.


Oriented linear graphs will be useful in switching theory if the network contains unidirectional components. Even though we will not discuss such a network in this paper, properties of fundamental cut set matrices of oriented linear graphs will be discussed here. One reason is that it is a direct extension of the materials in the previous sections.

Let $Q_0$ be a fundamental cut set matrix of an oriented linear graph. Since every entry of $Q_0$ is one of +1, -1, and 0, every entry in a pair $M_1(i)$ and $M_2(i)$ of M-submatrices of $Q_0$ with respect to row $i$ is also +1, -1, or 0. Thus $M_1(i)$ and $M_2(i)$ are also fundamental cut set matrices of oriented linear graphs.

Consider a linear graph in Figure 13-1. Suppose edge $a$ is in the tree corresponding to fundamental cut set matrix $Q_0$. Then row $v$ corresponding to the cut set which contains edge $a$ has 1 at column $a$ and -1 at column $b$ where columns $a$ and $b$ correspond to edges $a$ and $b$, respectively. However, if we represent row $v$ as the incidence set of vertex $v$, row $v$ has -1 at column $a$ and 1 at column $b$. Hence, we must change the definition of M-submatrices for oriented linear graphs as follows:
Definition 13-1: $i_1, i_2, \ldots, i_k$ in the parenthesis of $M$-submatrix $M(i_1 i_2 \cdots i_k)$ indicate that either row $i_p$ or $-1$ times row $i_p$ represents incidence set of vertex $i_p$ for $p = 1, 2, \ldots, k$.

Lemma 13-1: If a matrix $M = [M_{i,j}]$ is a fundamental cut set matrix of oriented linear graph $\{G, \Omega\}_f$, then $M$ is also a fundamental cut set matrix of oriented linear graph $\{G, \Omega\}_f' = \{G, \Omega\}_f$, where $f_1(e) = (r, s)$ if $f(e) = (s, r)$ for every edge $e$ in $G$. ($\{G, \Omega\}_f$ is the graph obtained from $\{G, \Omega\}_f$ by reversing the orientation of every edge in $\{G, \Omega\}_f$.)

Proof: Because of the definition of each entry in a fundamental cut set matrix of an oriented linear graph, the rows of $M$ do not change by reversing the orientation of every edge in $G$.

Because the $-1$ times a row in $Q_1$ may be necessary in order that the row represents an incidence set, Theorem 11-1 must be modified as

Theorem 13-1: If and only if $M_1(i)$ and $M_2(i)$ which are a pair of $M$-submatrices of $Q_1$ with respect to row $i$ are realizable as fundamental cut set matrices of oriented linear graphs $G_1$ and $G_2$, respectively, with (1) either row $i$ or $-1$ times row $i$ of $M_1(i)$ representing an
incidence set in $G_1$ and (2) either row $i$ or $-1$ times row $i$ of $M_2(i)$ representing an incidence set in $G_2$, there exists an oriented linear graph $G$ such that

1. $Q_o$ is a fundamental cut set matrix of $G_o$;
2. if either row $r$ or $-1$ times row $r$ ($r \neq i$) of $M_1(i)$ represents an incidence set in $G_1$, then either row $r$ or $-1$ times row $r$ of $Q_o$ represents an incidence set in $G$;
3. if either row $s$ or $-1$ times row $s$ ($s \neq i$) of $M_2(i)$ represents an incidence set in $G_2$, then either row $s$ or $-1$ times row $s$ of $Q_o$ represents an incidence set in $G$.

The proof of the above theorem is as follows: Let $i$ be the vertex in $\Omega(G_1)$ and $\Omega(G_2)$ (of $\{G_1, \Omega(G_1)\}_{f_1}$ and $\{G_2, \Omega(G_2)\}_{f_2}$, respectively,) which corresponds to the incidence set $S = \{e_1, e_2, \ldots, e_k\}$ represented by either row $i$ or $-1$ times row $i$ of $M_1(i)$ and $M_2(i)$. We say that the orientations of edge $e_s$ ($1 \leq s \leq k$) in $G_1$ and $G_2$ agree if either

1. $f_1(e) = (i, t)$ and $f_2(e) = (u, i)$; or
2. $f_1(e) = (t, i)$ and $f_2(e) = (i, u)$.

Otherwise the orientations of $e$ in $G_1$ and $G_2$ are said to disagree. Since row $i$ of $M_1(i)$ and $M_2(i)$ are from row $i$ of $Q_o$, if the orientations of edge $e_s$ ($1 \leq s \leq k$) in $G_1$ and $G_2$ agree, the orientations of every edge $e_j$ for $j = 1, 2, \ldots, k$ in $G_1$ and $G_2$ agree. Also, if the orientations of edge $e_s$ ($1 \leq s \leq k$) in $G_1$ and $G_2$ disagree, the orientations of every edge $e_j$ for $j = 1, 2, \ldots, k$ in $G_1$ and $G_2$ disagree. When the orientations of $e_s$ in $G_1$ and $G_2$ disagree, we reverse the orientation of every edge in $G_2$ which does not alter $M_2(i)$ by Lemma 13-1. Hence we can always
obtain two oriented linear graphs whose fundamental cut set matrices are $M_1(i)$ and $M_2(i)$ in which row $i$ or $-1$ times row $i$ of $M_1(i)$ and $M_2(i)$ represents an incidence set in these two linear graphs and the orientations of every edge in $S$ in these two linear graphs agree. The process to obtain an oriented linear graph $G$ whose fundamental cut set matrix is $Q_0$ is similar to that for a non-oriented linear graph discussed previously.

For convenience, we define the following:

Definition 13-2: A matrix $M$ is said to be a modified incidence matrix of an oriented linear graph if either $M$ is an incidence matrix or $M$ becomes an incidence matrix of an oriented linear graph by multiplying $(-1)$ to some rows of $M$.

Now we can state a necessary and sufficient condition for a matrix to be a fundamental cut set matrix of an oriented linear graph which is similar to that for non-oriented linear graphs given by Theorem 11-2 as follows:

Theorem 13-2: A matrix $Q$ whose entries are $+1$, $-1$, or $0$ is a fundamental cut set matrix of an oriented linear graph if and only if there exists a set of minimum $M$-submatrices of $Q$ such that every matrix in the set is a modified incidence matrix of an oriented linear graph.

The proof is easily accomplished by the successive application of Theorem 13-1 and Lemma 13-1.

Example 13-1: The following matrix is not realizable as a fundamental cut set matrix of an oriented graph because of the following reasons.
From $H$-submatrix of $Q$ with respect to row $a$,

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
a & 1 & 1 & 1 & 0 \\
b & 1 & -1 & 0 & 1 \\
c & -1 & 1 & 0 & 0 & 1
\end{pmatrix}
\]

we obtain a pair of $M$-matrices $M_1$ and $M_2$,

\[
M_1 = \begin{pmatrix}
1 & 2 & 3 & 5 \\
a & 1 & 1 & 1 & 0 \\
c & -1 & 1 & 0 & 1
\end{pmatrix}
\]

and

\[
M_2 = \begin{pmatrix}
1 & 2 & 3 & 4 \\
a & 1 & 1 & 0 \\
b & 1 & -1 & 0 & 1
\end{pmatrix}
\]

Since $M_1$ (or $M_2$) is not realizable as an incidence matrix of an oriented graph by multiplying $(-1)$ to some rows in $M_1$ and since there is no other way of obtaining a pair of $M$-submatrices of $Q$ except by lettering $H_1 = \emptyset$ and $H_2 = H$, $Q$ is not realizable as a fundamental cut set matrix of an oriented graph. Notice that to obtain the set of minimum $M$-submatrices of $Q$, we must obtain a pair of $M$-submatrices of $M_1$ with respect to row $c$ and a pair of $M$-submatrices of $M_2$ with respect to row $b$. It is clear that one of the above pair of $M$-submatrices of $M_1$ is the same as $M_1$ and and the other is a single row matrix. Similarly, one of the pair of
M-submatrices of $M_2$ with respect to row $b$ is the same as $M_2$ and the other is a single row matrix. Hence, the set of minimum M-submatrix consists of $M_1$, $M_2$, and two single row matrices. Since a single row matrix always satisfies the conditions in Theorem 13-2, it is only necessary to test whether $M_1$ and $M_2$ satisfy the conditions in Theorem 13-2 to know whether $Q$ can be a fundamental cut set matrix. It is interesting to notice that if we replace all -1 to +1 in $Q$ then $Q$ is realizable as a fundamental cut set matrix in both oriented and non-oriented graphs.

Similar to a non-singular matrix $D$ in the previous section, we can obtain $\tilde{D}$ such that $\tilde{D}_o$ is a modified incidence matrix of an oriented linear graph whose fundamental cut set matrix is $Q_o$ by the following theorem.

**Theorem 13-3:** Let $\{M_1\}$ be a set of minimum M-submatrices for a fundamental cut set matrix $Q_o$ of rank $v-1$. From $\{M_1\}$ from a set $\{M'_1\}$ of incidence matrices, where $M'_1$ is obtained by multiplying some of the rows of $M_1$ by -1. Next, construct a non-singular matrix $\tilde{D}$ as follows:

1. the $v-1$ rows of $\tilde{D}$ correspond to $v-1$ of the $v$ matrices in $\{M'_1\}$, and
2. entry $d_{ij}$ of $\tilde{D} = [d_{ij}]$ is

\[
d_{ij} = \begin{cases} 
1 & \text{if the matrix } M'_i \text{ corresponding to the } i \text{th row of } \tilde{D} \text{ contains the } j \text{th row of } Q; \\
-1 & \text{if the matrix } M'_i \text{ corresponding to the } i \text{th row of } \tilde{D} \text{ contains the negative of the } j \text{th row of } Q; \\
0 & \text{otherwise.}
\end{cases}
\]
Then $\tilde{D}Q$ is a modified incidence matrix.

Example 13-2: If the given matrix is

$$
\begin{bmatrix}
1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
2 & -1 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\
Q & = & 3 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\
4 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
5 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

then a set of minimum $M$-submatrices consists of the six matrices

$$
\begin{align*}
M_1 &= \begin{bmatrix}
a & b & c & d & f & g \\
1 & -1 & 1 & 0 & 0 & 1 & 0 \\
2 & -1 & 1 & -1 & -1 & 0 & 1 \\
\end{bmatrix} \\
M_2 &= \begin{bmatrix}
a & b & c & d & g & h \\
2 & -1 & 1 & -1 & -1 & 1 & 0 \\
3 & 0 & -1 & 1 & 0 & 0 & 1 \\
\end{bmatrix} \\
M_3 &= \begin{bmatrix}
bc & h \\
3 & [-1 & 1 & 1] \\
\end{bmatrix} \\
M_4 &= \begin{bmatrix}
a & b & e & f & i \\
1 & -1 & 1 & 0 & 1 & 0 \\
4 & -1 & 1 & -1 & 0 & 1 \\
\end{bmatrix} \\
M_5 &= \begin{bmatrix}
a & b & e & i & j \\
4 & [-1 & 1 & -1 & 1 & 0 \\
5 & -1 & 0 & -1 & 0 & 1 \\
\end{bmatrix}
\end{align*}
$$
and

\[ M_6 = 5[-1 -1 1]. \]

Then set \( \{M'_1\} \) can be

\[
M'_1 = -2 \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & -1 \end{bmatrix} \]
\[
M'_2 = M_2
\]
\[
M'_3 = M_3
\]
\[
M'_4 = -1 \begin{bmatrix} -1 & -1 & 0 & -1 & 0 \end{bmatrix} \]
\[
M'_5 = -4 \begin{bmatrix} 1 & -1 & 1 & -1 & 0 \end{bmatrix} \]
\[
M'_6 = M_6.
\]

Suppose we choose row 1 of \( \widetilde{D} \) to correspond to \( M'_1 \), row 2 of \( \widetilde{D} \) to \( M'_2 \), row 3 to \( M'_3 \), row 4 to \( M'_4 \) and row 5 to \( M'_5 \). Then \( \widetilde{D} \) becomes

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
1 & -1 & 0 & 0 & 0 \\
2 & 0 & 1 & 1 & 0 \\
\end{array}
\]
\[
\widetilde{D} = \begin{bmatrix} 3 & 0 & 0 & 1 & 0 \\
4 & -1 & 0 & 0 & 1 \\
5 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}
\]
and \( \tilde{\mathbf{Q}} \) becomes

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \\
2 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\
\tilde{\mathbf{Q}} = 3 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 \\
5 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
\end{bmatrix}
\]

Multiplying row 3 by -1, the above matrix becomes an incidence matrix of an oriented linear graph which has \( \mathbf{Q} \) for fundamental cut set matrix.


In Section 10, we define \( \mathcal{P}_{ij} \equiv \mathcal{P}_{ji} \) as the class of all possible paths between vertices \( i \) and \( j \) in a non-oriented non-separable linear graph \( \mathcal{G} \). For distinct vertices \( t, u, \) and \( v \) in \( \Omega(\mathcal{G}) \), we have \( \mathcal{P}_{tu}, \mathcal{P}_{uv}, \) and \( \mathcal{P}_{vt} \) which have the following property.

**Theorem 14-1:** For any \( t, u, v \in \Omega(\mathcal{G}) \),

\[
\mathcal{P}_{tu} \oplus \mathcal{P}_{uv} \oplus \mathcal{P}_{vt} \in C_s \quad (14-1)
\]

where \( C_s \) is the class of all possible circuits in \( \mathcal{G} \) and the empty set.

**Proof:** Let \( P_{tu} \in \mathcal{P}_{tu}, P_{uv} \in \mathcal{P}_{uv}, \) and \( P_{vt} \in \mathcal{P}_{vt} \). Then a linear graph \( P_{tu} \oplus P_{uv} \oplus P_{vt} \) consists of vertices of even degree. Thus

\( P_{tu} \oplus P_{uv} \oplus P_{vt} \in C \) where \( C \) is the class of all possible circuits, \( \phi \) and edge disjoint unions of circuits in \( \mathcal{G} \). Consider \( P'_{tu} \in \mathcal{P}'_{tu} \) and \( P'_{uv} \in \mathcal{P}'_{uv} \) which have the property that

\[
P'_{tu} \oplus P'_{uv} \in P'_{vt} \in \mathcal{P}'_{vt} \quad (14-2)
\]
It is clear that there exist such $P'_{tu}$ and $P'_{uv}$. Thus

$$
\hat{\xi}_{tu} \oplus \hat{\xi}_{uv} \equiv \hat{\xi}_{vt}.
$$

(14-3)

From Eq. (10-12),

$$
\hat{\xi}_{vt} \oplus C_s \equiv \hat{\xi}_{vt}.
$$

(14-4)

Thus with Eq. (14-3), we have

$$
\hat{\xi}_{tu} \oplus \hat{\xi}_{uv} \oplus C_s \equiv \hat{\xi}_{vt}.
$$

(14-5)

Also from Eq. (10-12) for any $a,b \in \Omega(G)$,

$$
\hat{\xi}_{ab} \oplus C_s \equiv \hat{\xi}_{ab}
$$

(14-6)

or

$$
\hat{\xi}_{ab} \oplus \hat{\xi}_{ab} \equiv C_s.
$$

(14-7)

Thus by Eqs. (14-4), (14-5), and (14-7), Eq. (14-1) can be obtained.

Q.E.D.

The above theorem can be generalized as

Theorem 14-2: For any $v_1, v_2, \ldots, v_k \in \Omega(G)$ which is the vertex sequency of a path between $v_1$ and $v_k$ in $G$,

$$
\hat{\xi}_{v_1 v_2} \oplus \hat{\xi}_{v_2 v_3} \oplus \cdots \oplus \hat{\xi}_{v_{k-1} v_k} \oplus \hat{\xi}_{v_k v_1} \equiv C_s.
$$

(14-8)

Definition 14-1: The collection $\{\hat{\xi}\}$ is the collection of

ring sum of all possible combinations of $\hat{\xi}_{ij}$ for $i \neq j \in \Omega(G)$.

From Eq. (14-1), $C_s \in \{\hat{\xi}\}$, and from Eqs. (14-6) and (14-7),

we can say that $\{\hat{\xi}\}$ is a group under the ring sum operation.
Let $F_{rs}(r,s \in \Omega)$ be the switching function between vertices $r$ and $s$ of an s.c. switching network represented by a weighted non-oriented linear graph $\{w_s, G, \Omega\}$. Also let $\mathcal{L}_{rs}$ be the set representation of $F_{rs}$ (see Definition 10-3).

Definition 14-2: Unless it is specified, a switching network corresponding to a set of s.c. switching functions $F$'s between different pairs of vertices have the property that the removal of any edge will no longer satisfy all of these functions $F$'s.

From Theorem 14-1, a set of s.c. switching functions has the following property:

Lemma 14-1: Let $a$, $b$, and $c$ be in $\Omega$ of a non-oriented non-separable linear graph $\{w_s, G, \Omega\}$ corresponding to an s.c. switching network. Then

$$\left(\mathcal{L}_{ab} \oplus \mathcal{L}_{bc} \oplus \mathcal{L}_{ac}\right) \subset C_s \quad (14-9)$$

where $\mathcal{L}_{ab}$, $\mathcal{L}_{ac}$, and $\mathcal{L}_{bc}$ are set representations of s.c. switching functions $F_{ab}$, $F_{ac}$, and $F_{bc}$, respectively, and $C_s$ is the class of all possible circuits in $\{w_s, G, \Omega\}$ and the empty set.

Lemma 14-1 can be generalized as follows:

Lemma 14-2: Let $\zeta_1, \zeta_2, \ldots, \zeta_k \in \Omega$ of $\{w_s, G, \Omega\}$. Then

$$\mathcal{L}_{\zeta_1 \zeta_2} \oplus \mathcal{L}_{\zeta_2 \zeta_3} \oplus \mathcal{L}_{\zeta_3 \zeta_4} \oplus \cdots \oplus \mathcal{L}_{\zeta_k \zeta_k \zeta_1} \subset C_s \quad (14-10)$$

where $\mathcal{L}_{\zeta_1 \zeta_2}, \ldots, \mathcal{L}_{\zeta_k \zeta_k}, \text{ and } \mathcal{L}_{\zeta_1 \zeta_k}$ are set representations of s.c. switching functions $F_{\zeta_1 \zeta_2}, \ldots, F_{\zeta_k \zeta_k}, \text{ and } F_{\zeta_k \zeta_1}$ of $\{w_s, G, \Omega\}$. 
Definition 14-2: $\mathcal{L}(d)$ is a subclass of class $\mathcal{L}$ which consists of all sets that contain edge $d$. Likewise, $\mathcal{C}_s(d)$ is the subclass of $\mathcal{C}_s$ which consists of all sets that contain edge $d$.

Definition 14-3: $\mathcal{E}_{-d}$ is the subclass of $\mathcal{E}$ obtained by deleting $d$ from every set in $\mathcal{E}$ which contains $d$ and deleting empty sets produced by the deletion of $d$. $\mathcal{E}$ can be either $\mathcal{L}$ or $\mathcal{C}$.

Example 14-1: Suppose $\mathcal{L}$ consists of $\{d\}$ and $\{\text{def}\}$. Then $\mathcal{L}_{-d}$ consists of $\{\text{ef}\}$.

With these definitions, if edge $d$ is between vertices $\zeta_a$ and $\zeta_b$ in $\{w_s, G, \Omega\}$, $\mathcal{L}_{\zeta_a \zeta_b}$ can be expressed as

$$\mathcal{L}_{\zeta_a \zeta_b} = \{d, \mathcal{C}_s(d)\}_{-d}.$$  

Since

$$\mathcal{C}_s = \mathcal{L}_{rs}^2,$$

we have

$$\mathcal{C}(d) = \mathcal{L}_{rs}(d) \ominus \{\mathcal{L}_{rs} - \mathcal{L}_{rs}(d)\}.$$  

Hence, Eq. (14-7) can be written as

$$\mathcal{L}_{\zeta_a \zeta_b} = \left\{d, \{\mathcal{L}_{rs}(d)\}_{-d} \ominus \{\mathcal{L}_{rs} - \mathcal{L}_{rs}(d)\} \right\}.$$  

Definition 14-4: $\mathcal{L}_{\zeta_a \zeta_b}$ will be denoted as $\mathcal{L}_{[d]}$ when $d$ is connected between $\zeta_a$ and $\zeta_b$.

Suppose $\zeta_{ab} \in \mathcal{L}_{ab}$ consists of $\{e_1, e_2, \ldots, e_k\}$. Then by Lemma 14-2, we have

$$\mathcal{L}_{ab} = \mathcal{L}_{[e_1]} \ominus \mathcal{L}_{[e_2]} \ominus \cdots \ominus \mathcal{L}_{[e_k]}.$$
Consider $L_{ab}$ and $L_{ac}$ of a linear graph in Figure 14-1. Suppose edge $t$ is added between vertices $a$ and $b$. Then it is clear that not only $L_{ab}$ changes but also $L_{ac}$ will change. Let the resultant linear graph be $G_t$. Also set representations of s.c. switching functions between $a$ and $b$ and $a$ and $c$ to be $\mathcal{F}_{ab}$ and $\mathcal{F}_{ac}$, respectively. Then it is clear that

$$\mathcal{F}_{ab} = \{t, L_{ab}\}.$$  \hfill (14-16)

$\mathcal{F}_{ac}$ can be obtained by the following procedure: (1) Obtain $C_s$ by $L_{ab}$

\begin{enumerate}
\item Obtain $C_s$ by $L_{ab}$
\item Let $\mathcal{T}_{ac} \subseteq L_{ac}$ consist of edges $e_1, e_2, \ldots, e_k$. Form $L_{[e_j]}$ from $C_s$ by Eq. (14-15). Notice that
\end{enumerate}

$$\mathcal{F}_{ac} \supset L_{ac}^*.$$  \hfill (14-17)

Figure 14-1. Separable linear graph.

Suppose we like to synthesize a connected linear graph corresponding to an s.c. switching network which satisfies a given set of s.c. switching functions. Suppose set representations of these switching functions are $L_{a,b}^j$ for $j = 1, 2, \ldots, k$. Then we can modify $L_{a,b}^j$ by the successive applications of Eq. (14-12), the procedure discussed previously,
so that edge $t_j$ will be connected between vertices $a_j$ and $b_j$ for $j = 1, 2, \ldots, k$. Let these modified set representations be $\tilde{E}_{a_j b_j}$. Then it is clear that a linear graph $G$ containing all $t_j$ which satisfy $\tilde{E}_{a_j b_j}$ for $j = 1, 2, \ldots, k$ is non-separable. Thus

$$\left(\tilde{E}_{a_j b_j}\right)^2 \equiv C_s \text{ of } G \quad (14-18)$$

and

$$\{\tilde{E}_{a_j b_j} - \tilde{E}_{a_j b_j} - \tilde{E}_{a_j b_j} - \cdots - \tilde{E}_{a_j b_j}\} \equiv E_{a_j b_j} \quad (14-19)$$

for all $j$.

Suppose all given $\tilde{E}_{a_j b_j}$ satisfy Eqs. (14-14) and (14-15). Then we can synthesize $G$ by using any one of $\tilde{E}_{a_j b_j}$ by the method given in the previous sections, and we will have $G$ by deleting all $t_j$. The location of $t_j$ gives the two vertices $a_j$ and $b_j$ between which $F_{a_j b_j}$ will be satisfied.

15. On Multi-Contact Switching Networks and Remarks.

A set representation of an s.c. switching function is a class of all possible paths between two vertices in a linear graph corresponding to an s.c. switching network. However, for an m.c. (multi-contact) switching function, $E_{a b}$ does not contain all possible paths between vertices $a$ and $b$. Let $P$ be a path between $a$ and $b$ in $\{w_s, G, \Omega\}$. If $x$ and $\bar{x}$ are the weights of edges in $P$, then regardless of values of $x$, this path will never be closed. Thus this path is not in $F_{a b}$. Such a path is called an open-path. In other words, $E_{a b}$ of an m.c. switching network is a set of all possible non-open-paths between $a$ and $b$. Thus,
in order to synthesize m.c. switching functions, we must modify the necessary condition given by either Eqs. (10-18) and (10-19) or Eq. (10-20) as follows: The ring sum of any even number of $\eta_{k_j}$ is not in $\mathcal{L}_{ab}$ and the ring sum of any odd number of $\eta_{k_j}$'s in $\mathcal{L}_{ab}$ is one of the following three:

1. $\eta_{k_1} \Theta \eta_{k_2} \Theta \cdots \Theta \eta_{k_n} \equiv \rho_k$; 
2. $\eta_{k_1} \Theta \eta_{k_2} \Theta \cdots \Theta \eta_{k_n} \cup \kappa_k$; and
3. $\eta_{k_1} \Theta \eta_{k_2} \Theta \cdots \Theta \eta_{k_n} \equiv \eta_0$ which is an open-path.

If $\mathcal{L}_{ab}$ satisfies the above conditions, then we must insert proper open-paths in order that the resultant class of sets represents the class of all possible paths between vertices a and b. From these paths, we can obtain a cut set matrix which leads to a desired switching network.

There is a program for a digital computer to synthesize a switching function by using Theorem 11-2 in order to obtain a minimum switching network (which consists of the minimum number of switching elements) for a given switching function. Let $G$ be a switching network which satisfies a given switching function $F_{ab}$. In $G$, there may be many edges whose weights are identical; that is, $G$ is not, in general, an s.c. switching network. Suppose switching variable $x$ is the weight of $m$ edges in $G$. Then we can give $x_1, x_2, \ldots, x_m$ rather than $x$ to each of these edges to make a switching network which is equivalent to $G$ where $x_i \cap x_j = x_i \cup x_j = x_i = x$ for all $i = 1, 2, \ldots, m$. By replacing all multi-contact elements in $G$ by single contact elements in this manner,
we can obtain a switching network which consists only of a single contact element. From this network, we can obtain $\mathcal{L}_{ab}$, which is the class of all possible paths between a and b. At present there is no way of obtaining $\mathcal{L}_{ab}$ directly from $F_{ab}$ other than cut and try, which will be a good future problem.
After a multi-flow problem is introduced [11], several papers [12,13,14,15,16,41,42] which deal with communication nets are published. Here, we will discuss lossless [11,14] and lossy [29] communication nets.

16. Lossless Non-Oriented Communication Nets.

A lossless non-oriented communication net is a weighted non-oriented linear graph \( G = (V, E, c) \) with \( c \) an edge capacity (originally named as a branch capacity) which indicates the maximum amount of flow by the edge in either direction. \( Y(e) \) is the amount of flow assigned to edge \( e \) which must satisfy that \( Y(e) \leq c \).

**Definition 16-1:** Let a path from \( r \) to \( u \) be \( P_{ru} = \{e_1, e_2, \ldots, e_k\} \). A flow \( Y \) being assigned to path \( P_{ru} \) means that assigning flow \( Y \) to every edge in \( P_{ru} \) makes the total flow assigned to these edges become

\[
Y(e_j) = Y + Y_o(e_j) \quad \text{for } j = 1, 2, \ldots, k
\]

where \( Y_o(e_j) \) is the amount of flow which has been assigned to \( e_j \) previously. Notice that \( Y(e_j) \leq c_j \) must satisfy for every edge in the path. Hence

\[
Y \leq \min (c_1 - Y_o(e_1), c_2 - Y_o(e_2), \ldots, c_k - Y_o(e_k))
\]

in order that \( Y \) can be assigned to \( P_{ru} \).
Definition 16-2: A flow $\Psi_{ij}$ being assigned from vertex $i$ to vertex $j$ of $\{w_y, w_c, G, \Omega\}$ means to assign $\Psi_{kij}$ to a path $P_k$ from $i$ to $j$ in $\{w_y, w_c, G, \Omega\}$ for $k = 1, 2, \ldots$ so that $\Psi_{ij} = \sum_{(k)} \Psi_{kij}$.

Definition 16-3: Terminal capacity $t_{ij}$ from vertex $i$ to vertex $j$ is the maximum flow which can be assigned from $i$ to $j$ in $\{w_y, w_c, G, \Omega\}$.

Definition 16-4: Let $S$ be a cut set in $\{w_y, w_c, G, \Omega\}$. Then the value of cut set $S$ denoted by $V[S(c)]$ is defined as

$$V[S(c)] = \sum_{e_p \in S} c_p$$

where $c_p$ is the edge capacity of edge $e_p$.

Ford and Fulkerson (1956) [30] give the relationship between $t_{ij}$ and cut sets in $\{w_y, w_c, G, \Omega\}$ as follows: Let $S_p$ ($p = 1, 2, \ldots, k$) be all possible cut sets which separate $i$ and $j$ (Definition 1-20). Then

$$t_{ij} = \min \{V[S_1(c)], V[S_2(c)], \ldots, V[S_k(c)]\} \quad (16-4)$$

One way of specifying a communication net is to give terminal capacities between all possible pairs of vertices in the net such as using a matrix $T = [t_{ij}]$ called a terminal capacity matrix defined by

$$t_{ij} = \text{terminal capacity from } i \text{ to } j \text{ for } i \neq j$$

and

$$t_{ii} = d \text{ for all } i. \quad (16-5)$$

As far as study of a communication net is concerned, $d$ in the above definition can be anything unless we give some meaning to a terminal capacity from vertex $i$ to itself. In some cases, it may be convenient to define $d$ to be zero. Here we use the symbol $d$ to indicate that the entry is a diagonal entry.
In order to discuss the properties of a terminal capacity matrix of a communication net, we define a principal partition as follows:

Definition 16-5: The partition of a square matrix $M$ in which every diagonal entry is $d$ and every off diagonal entry is a real number as

$$ M = \begin{bmatrix}
  M_d & M_c \\
  \hline
  M_a & M_b \\
\end{bmatrix} \quad (16-6) $$

is called a "principal partition" if (1) every element in $M_c$ is identical and is the smallest number in $M$ without considering $d$ and (2) $M_a$ and $M_b$ are square submatrices with every diagonal entry being $d$. $M_a$ and $M_b$ are called the "resultant main submatrices" by a principal partition of $M$.

Because nets which will be discussed in this section are non-oriented, $t_{ij} = t_{ji}$. Thus a terminal capacity matrix $T$ of a communication net is symmetric. Furthermore, $T$ has the following interesting properties: Let $\{w_p, w_c, G, \Omega\}$ be a lossless non-oriented communication net. Then by Eq. (16-2), there exists at least one cut set $S$ which separates vertices $p$ and $q$ such that the value of $S$ is equal to terminal capacity $t_{pq}$. Let $t_{i_1j_1}$ be the smallest among all terminal capacities in the net. Also let $S_1$ be a cut set which separates vertices $i_1$ and $j_1$ whose value is equal to $t_{i_1j_1}$ and $\Omega_{i_1}$ and $\Omega_{j_1}$ be the corresponding vertex sets of $S_1$ (see Definition 1-20), where $i_1 \in \Omega_{i_1}$ and $j_1 \in \Omega_{j_1}$. Consider $t_{rs}$ where $r \in \Omega_{i_1}$ and $s \in \Omega_{j_2}$. Since the class of cut sets which separate vertices $r$ and $s$ contains $S_1$, and the value of $S_1$ is smallest among all cut sets in the net, $t_{rs} = t_{i_1j_1}$. Thus a principal partition of $T$ as in
Eq. (16-7) is possible by rearranging rows and columns such that $T_1$ consists of rows corresponding to the vertices in $\Omega_{j_1}$.

$$T = \begin{bmatrix} T_a & T_1 \\ T_1^t & T_b \end{bmatrix}.$$  \hspace{1cm} (16-7)

(Each entry $(r', s')$ in $T_1$ is equal to $t_{i_1 j_1}$ where $r' \in \Omega_{i_1}$ and $s' \in \Omega_{j_1}$.)

Let $t_{i_2 j_2}$ be the smallest entry in the resultant main submatrices $T_a$ and $T_b$. Without the loss of generality, let $t_{i_2 j_2}$ be in $T_a$. Also let $S_2$ be a cut set which separates $i_2$ and $j_2$ whose value is equal to $t_{i_2 j_2}$.

Let $\Omega_{i_2}$ and $\Omega_{j_2}$ be the corresponding vertex sets of cut set $S_2$. Then any terminal capacity $t_{uv}$, where $u \in \Omega_{i_2} \cap \Omega_{i_1}$ and $v \in \Omega_{j_2} \cap \Omega_{j_1}$, is equal to $t_{i_2 j_2}$ because the class of cut sets which separate $u$ and $v$ contains $S_2$, and the value of $S_2$ is the smallest among the values of all cut sets in the class by assumption. Hence a principal partition of $T_a$ is possible as

$$T = \begin{bmatrix} T_{aa} & T_{al} \\ T_{al}^t & T_{11} \\ T_{bl} & T_{ab} \\ T_{bl} & T_{1b} \end{bmatrix}.$$  \hspace{1cm} (16-8)

where $T_{al}$ consists of rows corresponding to the vertices in $\Omega_{i_2} \cap \Omega_{i_1}$ and the columns corresponding to the vertices in $\Omega_{j_2} \cap \Omega_{j_1}$. Thus the entries in $T_{al}$ are identical and are smallest among all entries in $T_a$. Similarly, we can pick a terminal capacity which is smallest among all entries in the resultant main submatrices $T_{aa}$, $T_{ab}$, and $T_b$ so that a
principal partition can be applied, and so on until there are no resultant main submatrices which contain more than one entry. Thus we have the following theorem.

Theorem 16-1: A symmetric matrix \( T \) of order \( v \) is a terminal capacity matrix of a lossless non-oriented communication net if and only if we can obtain principal partition of the matrix and principal partitions of all resultant main submatrices which contain more than one entry which are obtained by the preceding principal partitions.

Proof: In order to use mathematical induction to prove the theorem, we define the following.

Definition 16-6: A sequence \( \Gamma_p \) \( (p = 1, 2, \ldots, v) \) is a sequence of sets where \( \Gamma_p \) is the set of submatrices of \( T \) defined by

1. \( \Gamma_1 = \{T\} \);
2. \( \Gamma_p \) is obtained from \( \Gamma_{p-1} \) by replacing a submatrix \( T_{m} \) in \( \Gamma_{p-1} \) that contains more than one entry by two submatrices which are the resultant main submatrices of \( T_m \) obtained by a principal partition of \( T_m \). For example, if a principal partition of \( T \) is one in Eq. (16-7), then

\[
\Gamma_1 = \{T_a, T_b\}.
\]

Suppose a sequence \( \Gamma_1 \Gamma_2 \cdots \Gamma_k \) is obtained for \( k < v \). Thus we need only to prove that there exists a submatrix in \( \Gamma_k \) such that we can apply principal partition on the submatrix to obtain \( \Gamma_{k+1} \). It is clear that there always exists a matrix which contains more than one entry in \( \Gamma_k \) if \( k < v \). Let \( T_m \) be a matrix in \( \Gamma_k \) which consists of more than one entry. Also let \( \Omega_m \) be the collection of vertices corresponding
to the rows in $T_m$. Let $t_{i_kj_k}$ be the smallest in $T_m$ and $S_m$ be a cut set which separates vertices $i_k$ and $j_k$ whose value is equal to $t_{i_kj_k}$. Also $\Omega_{i_k}$ and $\Omega_{j_k}$ are to be the corresponding vertex sets of $S_m$. Then the terminal capacity $t_{r_ku_k}$ where $r_k \in \Omega_{i_k} \cap \Omega_{m}$ and $u_k \in \Omega_{i_k} \cap \Omega_{m}$ cannot be smaller than $t_{i_kj_k}$ because $t_{i_kj_k}$ is the smallest among all entries in $T_m$. Also $t_{r_ku_k}$ cannot be larger than $t_{i_kj_k}$ because $S_m$ separates $r_k$ and $u_k$. Thus $t_{r_ku_k} = t_{i_kj_k}$. This is true for any terminal capacity $t_{ab}$ where $a \in \Omega_{i_k} \cap \Omega_{m}$ and $b \in \Omega_{j_k} \cap \Omega_{m}$. Thus, there is a principal partition of $T_m$ as

$$T_m = \begin{bmatrix} T_{ma} & T_{ml} \\ T_{ml} & T_{mb} \end{bmatrix} \quad (16-9)$$

where $T_{ml}$ consists of rows corresponding to the vertices in $\Omega_{i_k} \cap \Omega_{m}$ and columns corresponding to the vertices in $\Omega_{j_k} \cap \Omega_{m}$. Thus, we can obtain $\Gamma_{k+1}$ by deleting $T_m$ from $\Gamma_k$ and adding $T_{ma}$ and $T_{mb}$. This proves the necessary part of the theorem.

In order to prove the sufficiency, we will construct a lossless non-oriented communication net whose terminal capacity matrix is equal to the given matrix. Let $T$ be a square matrix of order $v$ in which every diagonal entry is $d$. Also let a sequence

$$\Gamma_1 \Gamma_2 \cdots \Gamma_v$$

be obtained from $T$ by the process discussed previously. Hence there exists a pair of matrices in $\Gamma_p$ and one matrix in $\Gamma_{p-1}$ for $p > 1$ such that these two matrices in $\Gamma_p$ are the resultant main submatrices of the
matrix in $\Gamma_{p-1}$. Let these two matrices in $\Gamma_p$ be $T_a$ and $T_b$, and the matrix in $\Gamma_{p-1}$ be $T_m$. Let $\Omega_a$ be the set of vertices corresponding to the rows in $T_a$, $\Omega_b$ be the set of vertices corresponding to the rows in $T_b$, $\Omega_m$ be the set of vertices corresponding to the rows in $T_m$, and $t_m$ be the smallest entry in $T_m$.

Consider $\{\omega_f, \omega_c, G, \Omega\}$ where $G$ consists of $v-1$ edges $e_p$ where

1. $f(e_p) = (r_p, s_p), r_p \in \Omega_a, s_p \in \Omega_b$;
2. $c_p \in \omega_c$ which is the edge capacity of edge $e_p$ is equal to $t_m$ for $p = 2, 3, \cdots, v$; and
3. $\Omega$ is the set of vertices corresponding to the rows in $T$.

Let $\zeta$ be a vertex in $\Omega$. Then there exists a matrix in $\Gamma_r$ but not in $\Gamma_{r-1}$ which consists of one entry corresponding to $\zeta$. Let this matrix be $T_{a_r}$. Then $\Omega_a$ consists of $\zeta$. Thus edge $e_r$ is connected to $\zeta$. Hence, $\{\omega_f, \omega_c, G, \Omega\}$ is a connected linear graph. Since $G$ consists of $v-1$ edges, $\{\omega_f, \omega_c, G, \Omega\}$ has the tree-property. Hence there exists exactly one cut set $S_p$ which contains $e_p$. Notice that $S_p = \{e_p\}$. Thus the terminal capacity between $r_p$ and $s_p$ is equal to $t_m$. Since this is the smallest among all entries in $T_m$, this is also the terminal capacity between any vertices $\zeta_1$ and $\zeta_2$ as long as $\zeta_1 \in \Omega_a$ and $\zeta_2 \in \Omega_b$ where $\Omega_a$ and $\Omega_b$ are the set of vertices corresponding to the rows of $T_a$ and $T_b$, respectively, and $T_a$ and $T_b$ are the pair of resultant main submatrices of $T_m$. Thus, the terminal capacity matrix of $\{\omega_f, \omega_c, G, \Omega\}$ is $T$ which proves the sufficient part of the theorem. Q.E.D.

It is easily seen from the proof of the above theorem that there are many lossless non-oriented communication nets whose terminal
capacity matrices are identical. Thus, several papers [12,41] have been published which are dealing with the optimum synthesis of lossless non-oriented communication nets under some restrictions.

Example 16-1: The matrix

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
1 & d & 3 & 2 & 2 & 4 \\
2 & 3 & d & 2 & 2 & 3 \\
3 & 2 & 2 & d & 2 & 2 \\
4 & 2 & 2 & 2 & d & 2 \\
5 & 4 & 3 & 2 & 2 & d \\
\end{array}
\]

is a terminal capacity matrix, because the matrix can be partitioned as

\[
\begin{array}{ccccc}
1 & 5 & 2 & 4 & 3 \\
1 & d & 4 & 3 & 2 & 2 \\
5 & 4 & d & 3 & 2 & 2 \\
2 & 3 & 3 & d & 2 & 2 \\
4 & 2 & 2 & 2 & d & 2 \\
3 & 2 & 2 & 2 & d & 2 \\
\end{array}
\]

which means that the matrix satisfies the theorem 16-1.

A sequence \( \Gamma_1, \Gamma_2, \ldots, \Gamma_5 \) will be

\[
\Gamma_1 = \{T\}
\]

\[
\Gamma_2 = \left\{ \begin{array}{cc}
1 & 5 & 2 \\
1 & d & 4 & 3 \\
5 & 4 & d & 3 \\
2 & 3 & 3 & d \\
\end{array}, \begin{array}{cc}
4 & 3 \\
4 & d & 2 \\
2 & 3 & 2 & d \\
\end{array} \right\}
\]
A communication net corresponding to the given matrix can be obtained by the following procedure:

(1) Place the edge with edge capacity $t_1 = 2$ between any one of the vertices 1, 5, and 2 and any one of the vertices 3 and 4, say between the vertices 1 and 3;

(2) Place the edge with edge capacity $t_2 = 2$ between the vertices 3 and 4;

(3) Place the edge with edge capacity $t_3 = 3$ between the vertex 2 and any one of the vertices 1 and 5, say between the vertices 1 and 2;

(4) Finally, place the edge with edge capacity $t_4 = 4$ between the vertices 1 and 5. The resultant net is shown in Figure 16-1.
17. Lossless Oriented Communication Nets.

A lossless oriented communication net is a weighted oriented linear graph \( \{w_v, w_e, G, \Omega\} \) with the weight \( c_p \in w_c \) of each edge \( e_p \) representing the capacity of flow by the edge only in the direction specified by the orientation of the edge. Assigning flow \( y_{ij} \) to a path \( P_{ij} \) from vertex \( i \) to \( j \) is the same as that in Definition 16-1 except that the path must be a directed path from \( i \) to \( j \). Similarly, assigning flow \( y_{ij} \) from \( i \) to \( j \) of \( \{w_v, w_e, G, \Omega\} \) is the same as that of non-oriented communication net (Definition 16-2), and the terminal capacity from \( i \) to \( j \) is the maximum flow we can assign from vertex \( i \) to vertex \( j \) of \( \{w_v, w_e, G, \Omega\} \).

In order to discuss the properties of an oriented communication net, we define a semi-cut as follows:

**Definition 17-1:** Let \( S_{ij} \) be a cut set and \( \Omega_i \) and \( \Omega_j \) the corresponding vertex sets of \( S_{ij} \). The semi-cuts \( s_{ij} \) and \( s_{ji} \) of \( S_{ij} \) are defined as a disjoint set of edges such that

\[
s_{ij} \cup s_{ji} = S_{ij}
\]

where every edge \( e_p \) in \( s_{ij} \) has the property that the orientation of the
edge is from the vertex in $\Omega_i$ to the vertex in $\Omega_j$. Similarly, the
orientation of every edge in $s_{ji}$ is from the vertex in $\Omega_j$ to the vertex
in $\Omega_i$.

Let $S_{ pij}$ for $p = 1, 2, \ldots, k$ be all possible cut sets which
separate $i$ and $j$. Also let $s_{ pij}$ be a semi-cut of $S_{ pij}$. Then the rela-
tionship between a terminal capacity and cut sets given by Ford and
Fulkerson [30] can be stated as

$$t_{ ij} = \min \{V[s_{ i1j} (c)], V[s_{ 2 ij} (c)], \ldots, V[s_{ k1j} (c)]\}$$  \hspace{1cm} (17-1)

where $V[s_{ pij} (c)]$ is

$$V[s_{ pij} (c)] = \sum_{q}^{s_{ pij}} c_q$$  \hspace{1cm} (17-2)

where $c_q$ is the edge capacity of edge $e_q$.

From Theorem 16-1, we can see that there are at most $v-1$
different terminal capacities in a lossless non-oriented communication
net consisting of $v$ vertices. Consider a terminal capacity matrix $T$
which is realizable as a lossless non-oriented communication net. We
can realize $T$ by lossless oriented communication net by the method which
is exactly the same as that given in the proof of Theorem 16-1 except
using two edges $e_p$ and $e_p^2$ rather than one edge $e_p$ with the edge capa-
ties of $e_p$ and $e_p^2$ being equal to $t_p$ but the orientations of $e_p^1$ and $e_p^2$
being opposite. Thus it is clear that there is a lossless ori-
ented communication net consisting of $v$ vertices which has $v-1$ different
terminal capacities.
It is clear that a terminal capacity matrix of a lossless oriented communication net is not generally symmetric. However, by using the same procedure as that used to prove the necessity part in Theorem 16-1, it can be shown that we can obtain principal partitions of T and all resultant main submatrices which contain more than one entry and which are obtained by the preceding principal partitions.

The question is what the maximum number of different terminal capacities is which an oriented communication net can have. Consider a lossless oriented communication net in Figure 17-1.

Let p and q be vertices in the net in Figure 17-1 such that p > q. Consider flow \( \Psi_{pq} = t \) consists of \( \Psi_{pq}^1, \Psi_{pq}^2, \Psi_{pq}^3, \) and \( \Psi_{pq}^4 \) (that is, \( \Psi_{pq} = \Psi_{pq}^1 + \Psi_{pq}^2 + \Psi_{pq}^3 + \Psi_{pq}^4 \)) where \( \Psi_{pq}^1 \) is the flow which passes through the edge connected from p to q, \( \Psi_{pq}^2 \) is the flow which passes through vertex \( r' > p \), \( \Psi_{pq}^3 \) is the flow which passes through the edges connected from vertex p to vertices \( s' < q \), and \( \Psi_{pq}^4 \) is the flow which passes through the edges connected from vertex q to vertices \( t' \) where \( q < t' < p \). It is clear that the edge capacity of the edge
connected from vertex $p$ to vertex $q$ is $v^{v-p+2}$. Thus $\Psi_1^{pq} = v^{v-p+2}$. For $\Psi_2^{pq}$, the edge connected from vertex $p$ to vertex $p+1$ is the only edge by which $\Psi_2^{pq}$ can be assigned. Also the edge capacity of the edge connected from vertex $p+1$ to vertex $q$ is larger than that of the edge from vertex $p$ to vertex $p+1$, which gives that $\Psi_2^{pq} = v-p$. For $\Psi_3^{pq}$, we consider all possible directed paths from $p$ to $s'$ ($< q$). Since all of these paths must contain the edge connected from vertex $q-1$ to vertex $q$ whose edge capacity is $v-q-1$, and the edge capacity of the edge connected from $p$ to $q-1$ is larger than $v-q-1$, $\Psi_3^{pq} = v-q-1$. Finally, every directed path from vertex $p$ to vertex $q$ passing through vertex $t'$ but not vertices $r'$ and $s'$ contains the edge from vertex $p$ to vertex $t'$ whose edge capacity is $v^{v-p+2}$ and the edge capacity of the edge from vertex $t'$ to vertex $q$ is larger than $v^{v-p+2}$ for all $t'$, $\Psi_4^{pq} = (p-q-1)v^{v-p+2}$. Thus $\Psi^{pq}$ is equal to

$$t^{pq} = (p-q)v^{v-p+2} + 2v-p-q-1 \text{ for } p \neq v \text{ and } q \neq 1.$$  \hspace{1cm} (17-3)

When $p = v$ or $q = 1$, $\Psi_2^{pq}$ or $\Psi_3^{pq}$ will be zero, so

$$t^{pq} = \begin{cases} (p-q)v^{v-p+2} + v-q-1 & \text{for } p = v, q \neq 1 \\ (p-q)v^{v-p+2} + v-p & \text{for } p \neq v, q = 1 \\ (p-q)v^{v-p+2} & \text{for } p = v, q = 1 \end{cases}.$$  \hspace{1cm} (17-4)

From these we can see that only if $p_1 = p_2$ and $q_1 = q_2$

$$t^{p_1q_1} = t^{p_2q_2}.$$  \hspace{1cm} (17-5)
For $p < q$, we have

$$t_{pq} = t_{q+1} = v-q+1$$

(17-6)

which is less than $v$. Thus we have the terminal capacity matrix of the net in Figure 17-1 in which upper diagonal entries consist of $v-1$ different values and lower diagonal entries are all different values. Thus we can state that there is a lossless oriented communication net of $v$ vertices which has exactly $(v-1)(v+2)/2$ distinct terminal capacities.

Definition 17-2: Let $\Omega_1$ and $\Omega_2$ be a pair of disjoint vertex sets such that $\Omega_1 \cup \Omega_2 = \Omega$ for a given $\{\omega, \omega_c, G, \Omega\}$. Let $T$ be the terminal capacity matrix of $\{\omega, \omega_c, G, \Omega\}$. Then an $S$-submatrix is a submatrix of $T$ obtained by deleting rows corresponding to all the vertices in $\Omega_2$ and columns corresponding to all vertices in $\Omega_1$. Hence an $S$-submatrix of $T$ does not contain $d$ as an entry.

Let $S$ be a cut set in $\{\omega, \omega_c, G, \Omega\}$. Then by using the corresponding vertex sets of $S$, we can obtain an $S$-submatrix of $T$. Thus, there is a definite relationship between terminal capacities and $S$-submatrices of $T$.

Theorem 17-1: Every entry in $T$ except $d$, there exists at least one $S$-submatrix such that the entry is the largest entry in the $S$-submatrix.

Proof: Let $t_{pq}$ be a terminal capacity which is an entry in $T$. Also let $S_{pq}$ be a cut set which separates vertices $p$ and $q$ whose value is equal to $t_{pq}$. Let $\Omega_p$ and $\Omega_q$ be the corresponding vertex sets of $S_{pq}$ where $p \in \Omega_p$ and $q \in \Omega_q$. It is clear that the $S$-submatrix obtained from $T$ by deleting the rows corresponding to all vertices in $\Omega_q$ and the...
columns corresponding to all vertices in $\Omega_p$ contains $t_{pq}$. Suppose there exists a $t_{ru}$ in the $S$-submatrix which is larger than $t_{pq}$. In order for $t_{ru}$ to be in the $S$-submatrix, $r$ must be in $\Omega_p$ and $u$ must be in $\Omega_q$. Thus cut set $S_{pq}$ separates $r$ and $u$. This means that the semi-cut $s_{pq}$ is one of the semi-cuts in Eq. (17-1) which determine $t_{ru}$. Hence it is impossible to have $t_{ru} > t_{pq}$. Q.E.D.

Corollary 17-1: Let $t_{pq}$ be a terminal capacity and $S_{pq}$ be a cut set which separates vertices $p$ and $q$ whose value is $t_{pq}$. If there are $k$ $S$-submatrices which contain $t_{pq}$ of $T$ and $t_{pq}$ is the largest entry in each of $k$ $S$-submatrices for $k \geq 1$, then there exists at least one $S$-submatrix in these $k$ $S$-submatrices such that the vertex sets $\Omega_1$ and $\Omega_2$ of the $S$-submatrix is the corresponding vertex set of $S_{pq}$.

This corollary is obvious because we know that there exists an $S$-submatrix corresponding to $S_{pq}$.

Gomory and Hu [15] and Tang and Chien [13] give the relationship among terminal capacities in a lossless communication net as follows:

$$t_{ij} \geq \min(t_{ik}, t_{kj}) \quad (17-7)$$

for all $k \neq i,j$.

The relationship given by Eq. (17-7) is not sufficient for realizability of terminal capacity matrix of a lossless oriented communication net which can be shown as follows. Consider the following matrix:
show that $R$ is not realizable as a terminal capacity matrix. It can be seen that $R$ satisfies Theorem 17-1 and the relationship given by Eq. (17-7). There is only one $S$-submatrix $R_1$ in which 1 is the largest entry.

$$R_1 = \begin{bmatrix} a & c \\ d & 1 \\ 1 \end{bmatrix}$$

The vertex sets $\Omega_1$ and $\Omega_2$ corresponding to $R_1$ are

$$\Omega_1 = \{a, b\} \text{ and } \Omega_2 = \{c, d\}$$

which are the corresponding vertex sets of cut set $S_{ac}$ as shown in Figure 17-2. Notice that $S_{ac}$ is the cut set whose value is equal to $t_{ac}, t_{bc}, t_{ad},$ and $t_{bd}.$

There exists only one $S$-submatrix $R_2$ in which 2 is the largest entry and similarly there exists only one $S$-submatrix $R_3$ in which 3 is the largest.

$$R_2 = a[3 1 1] \quad R_3 = b[1]$$

$$R_3 = c[2]$$
These S-submatrices give cut sets $S_{ab}$ and $S_{cd}$ as shown in Figure 17-2 where $\mathcal{V}[s_{ab}(c)] = 3$ and $\mathcal{V}[s_{cd}(c)] = 2$. Thus

$$t_{ac} = 1 = c_{ad} + c_{ac} + c_{bc} + c_{bd}$$

(17-12)

and

$$t_{ab} = 3 = c_{ab} + c_{ad} + c_{ac}.$$  

(17-13)

Because the following S-submatrix is the only S-submatrix in $R$ in which $t_0$ is the largest, the cut set corresponding to $t_0$ must be $S_{dc}$ whose corresponding vertex sets $\Omega_4$ and $\Omega_3$ are $\Omega_4 = \{a, b, d\}$ and $\Omega_3 = \{c\}$ as shown in Figure 17-3.

---

**Figure 17-2. A communication net for $R$.**

**Figure 17-3. A communication net with $S_{dc}$ and $S_{db}$.**
Thus \[ \mathbf{R}_4 = \begin{bmatrix} a & 1 \\ b & 1 \\ d & t_o \end{bmatrix} \] (17-14)

Since there is only one \( S \)-submatrix \( \mathbf{R}_5 \) in which \( t_o + \varepsilon \) is the largest where

\[ \mathbf{R}_5 = \begin{bmatrix} a & b & c \\ 3 & 1 \\ d & t_o + \varepsilon & t_o \end{bmatrix} \] (17-16)

\( S_{db} \) in Figure 17-3 is the corresponding cut set of \( t_o + \varepsilon \), and

\[ t_o + \varepsilon = c_{ab} + c_{ac} + c_{db} + c_{dc} \] (17-17)

From Eqs. (17-15) and (17-17) we have

\[ \varepsilon = c_{ab} + c_{db} - c_{bc} \] (17-18)

Since every edge capacity is non-negative, Eq. (17-12) gives

\[ c_{ad} + c_{ac} \leq 1 \] (17-19)

Thus with Eq. (17-13), Eq. (17-19) gives

\[ 3 \geq c_{ab} \geq 2 \] (17-20)
Eq. (17-12) also gives

\[ c_{bc} \leq 1. \]  \hspace{1cm} (17-21)

Hence from Eqs. (17-20), (17-21), and (17-18), we have

\[ \varepsilon > c_{db} + 1. \]  \hspace{1cm} (17-22)

Since \( c_{db} \) is non-negative, \( \varepsilon \geq 1 \), which contradicts the assumption that \( 0 < \varepsilon < 1 \). Hence, \( R \) is not realizable as a terminal capacity matrix of a lossless oriented communication net. Thus the relationship among terminal capacities given by Eq. (17-7) and Theorem 17-1 are not sufficient for the realizability of terminal capacity matrix of a lossless oriented communication net.

No one has found a necessary and sufficient condition for a terminal capacity matrix of a lossless oriented communication net. However, there is a sufficient condition which allows at most \( 2(v-1) \) different terminal capacities in the net [14]. J. Resh [43] shows that if we can use a negative edge capacity, Theorem 17-1 is a necessary and sufficient one.

18. Lossy Communication Nets.

A lossy oriented communication net \( G \) is a weighted linear graph \( \{\omega_y,\omega_e,\omega_a,G,\Omega\} \) where \( \{\omega_y,\omega_e,\omega_a,G,\Omega\} \) is a lossless oriented communication net. \( \alpha_p \in \omega_a \) is an edge efficiency of edge \( e_p \in G \) with \( 0 \leq \alpha_p \leq 1 \) such that \( \Upsilon(e_p) \in \omega_y \) is a flow given to the edge \( e_p \) at the vertex \( \zeta_p \) and makes flow \( \alpha_p \Upsilon(e_p) \) at the vertex \( \zeta_p' \) where \( f(e_p) = (\zeta_p,\zeta_p') \). Hence flow \( (1-\alpha_p)\Upsilon(e_p) \) can be considered as the flow which is lost by the edge \( e_p \). For convenience we will call such a net a "lossy net."
Definition 18-1: $V[s(c)]$ and $V[s(\alpha c)]$ indicate the values of semi-cut $s$ defined as

\[
V[s(c)] = \sum_{e \in s} c_q
\]

\[
V[s(\alpha c)] = \sum_{e \in s} \alpha_q c_q
\]

Because of existence of loss of flow in edges in lossy nets, we will discuss the way of assigning flow $\Psi$ to a net as follows.

Definition 18-2: Let $P = \{e_1, e_2, \ldots, e_k\}$ be a directed path in a lossy net from vertex $i$ to vertex $j$. Assigning a flow $\Psi$ to path $P$ means that adding flow $\Psi'(e_p)$ to edges $e_p$ ($p = 1, 2, \ldots, k$) which are

\[
\Psi'(e_1) = \Psi_{ij}, \quad \Psi'(e_2) = \alpha_1 \Psi'(e_1), \ldots, \Psi'(e_k) = \alpha_{k-1} \Psi'(e_{k-1}).
\]

Under the condition that $\Psi'(e_p)$ must satisfy

\[
\Psi'(e_p) \leq c_p - \Psi_o(e_p) \quad \text{for} \quad p = 1, 2, \ldots, k
\]

where $\Psi_o(e_p)$ ($p = 1, 2, \ldots, k$) is the flow which had been assigned to $e_p$ previously. Thus the total flow assigned to edge $e_p$ becomes

\[
\Psi'(e_p) + \Psi_o(e_p)
\]

Let $P_1, P_2, \ldots, P_k$ be the all possible directed paths from vertex $i$ to vertex $j$ in net $G$. Then we can assign flows $\Psi_{ij}^{1}, \Psi_{ij}^{2}, \ldots, \Psi_{ij}^{k}$ to $P_1, P_2, \ldots, P_k$, respectively, in order to assign this total flow $\Psi_{ij}$. Also it is clear that any flow $\Psi_{ij}$ assigned to net $G$ can be broken into flows $\Psi_{ij}^{1}, \Psi_{ij}^{2}, \ldots, \Psi_{ij}^{k}$ which are assigned to $P_1, P_2, \ldots, P_k$, respectively.
Definition 18-3: Assigning flow $\Psi_{ij}$ to a lossy net $G$ means assigning flow $\Psi$, where $\sum_{p=1}^{k} \Psi_{p_{ij}} = \Psi_{ij}$, to directed paths $P_1, P_2, \ldots$, and $P_k$ which are from vertex $i$ to vertex $j$ in $G$.

Depending on the values of $\Psi_{1_{ij}}, \Psi_{2_{ij}}, \ldots, \Psi_{k_{ij}}$, the total flow $\Psi_{ij} = \sum_{p=1}^{k} \Psi_{p_{ij}}$ will be different. However, for any given $G$ which consists of finite number of edges with finite branch capacity of each edge, there will exist a largest total flow $\Psi_{ij}$ which is finite. It can be seen that Eq. (17-1) given by Ford and Fulkerson is no longer true in a lossy net. On the other hand, when all $\alpha_p = 1$ the net becomes lossless. Hence any property of lossy net which is dependent on $\alpha$'s will be closely related to that of a lossless net. We will next show that there exists a cut set in a lossy net which determines the maximum flow.

Theorem 18-1: Suppose flow $\Psi_{ij}$ from vertex $i$ to vertex $j$ has been assigned to a lossy net. If there exists no cut set $S_{ij}$ such that

$$\sum_{e_r \in S_{ij}} \Psi(e_r) = V[S_{ij}(c)]$$

(18-5)

then there exists a direct path from vertex $i$ to vertex $j$ such that an additional flow $\Psi > 0$ can be assigned.

Proof: Let $G'$ be a net obtained from a lossy net $G$ by deleting every edge $e$ whose assigned flow $\Psi(e)$ is equal to the branch capacity $c$ of the edge and every edge in $S_{ij}$. Since there exists no cut set $S_{ij}$ such that

$$\sum_{e_r \in S_{ij}} \Psi(e_r) = V[S_{ij}(c)]$$

(18-6)

then $G'$ is connected and at least one directed path $P = \{e_1, e_2, \ldots, e_k\}$ is in $G'$. Hence, we can assign flow $\Psi$ to $G$ if
where $\gamma_0(e_p)$ is the flow which has been assigned to edge $e_p$ in $G$.

Definition 18-3: Cut set $S_{ij}$ is said to be saturated or $S_{ij}$ is a saturated cut set with a flow $\gamma_{ij}$ assigned to $G$ if

$$\sum_{e_r \in S_{ij}} \gamma(e_r) = V[S_{ij}(c)].$$

(18-8)

It is clear from Theorem 18-1 that if there exists a saturated cut set $S_{ij}$ with a flow $\gamma_{ij}$ assigned to $G$, it is impossible to assign additional flow $\gamma'_{ij}$ to $G$.

Definition 18-4: A saturated cut set $S_{ij}$ is said to be a basic or $S_{ij}$ is a basic saturated cut set with a flow $\gamma_{ij}$ assigned to $G$ if

$$\sum_{e_t \in S_{ij}} \gamma(e_t) = 0.$$           \hspace{1cm} (18-9)

Let $P_1, P_2, \ldots, P_k$ be the all possible directed paths from vertex $i$ to vertex $j$ in $G$. Suppose $S_{ij}$ is a basic saturated cut set with flow $\gamma_{ij}$ assigned to $G$. Then it is clear that we can write $\gamma_{ij} = \sum_{p=1}^{k} \gamma_{pij}$ where $\gamma_{pij}$ is a flow assigned to $P_p$ for $p = 1, 2, \ldots, k$ such that

$$\gamma_{pij} = 0$$

for all $p$ which contains at least two edges in $S_{ij}$. \hspace{1cm} (18-10)

Definition 18-5: Let $\Omega_{1}$ and $\Omega_{2}$ be the corresponding vertex sets of cut set $S_{ij}$ for $p = 1, 2$, where vertex $i \in P_{1i}$. Then if $\Omega_1 \supset \Omega_2$ we say that $S_{2ij}$ follows $S_{1ij}$ or $S_{1ij}$ leads $S_{2ij}$.
Theorem 18-2: For a lossy net $G$, there exists at least one flow $\psi_{ij}$ assigned to $G$ such that there exists at least one basic saturated cut set $S_{ij}$ in $G$ with this flow.

Proof: Suppose there exists no such flow. Among all possible flows which can be assigned to $G$, let $\{\psi_{ij}\}$ be the set of all possible flows each of which is maximum flow we can assign to $G$. Then by Theorem 18-1, any flow in $\{\psi_{ij}\}$ which is assigned to $G$ will produce at least one saturated cut set. Among those flows in $\{\psi_{ij}\}$, let $\psi_{ij}$ be the flow such that assigning it to $G$ produces a minimum number $k$ of saturated cut sets as shown in Figure 18-1 where $k \geq 1$ by Theorem 18-1. Let $S_{ij}$ be the saturated cut set such that there is no saturated cut set which follows $S_{ij}$. Since $S_{ij}$ is not a basic saturated cut set by assumption, there exists a path $P_{ij}$ from $i$ to $j$ which contains more than one edge in $S_{ij}$ and $\psi_{ij} (> 0)$ of $\psi_{ij} = (q)q_{ij}$ has been assigned to $P_{ij}$ in order to assign $\psi_{ij}$ to $G$. Notice that flow $\psi_{ij}$ has already been assigned to $G$ in Figure 18-1. Let $e_{ij}$ be the edge where $f(e_{ij}) = (v_{ij}, v'_{ij})$ which is the first edge in $S_{ij}$ appearing in the edge sequence corresponding to $P_{ij}$. Let $\psi_{ij}(e_{ij})$ be the flow assigned to $e_{ij}$ when $\psi_{ij}$ is assigned to $P_{ij}$. Because there is no saturated cut set which follows $S_{ij}$, there must be at least one path $P'$ from $v'_{ij}$ to $j$ in $G$ such that we can assign a flow $\epsilon$ where $\psi_{ij}(e_{ij}) > \epsilon > 0$. Let $P''$ be the path from $v'_{ij}$ to $j$ which is part of $P_{ij}$. By assigning flow $\psi_{ij}(e_{ij}) - \epsilon$ to $P''$ rather than $\psi_{ij}$, we have the same flow $\psi_{ij}$ from $i$ to $v'_{ij}$ in $G$. However, $S_{ij}$ is no longer saturated because flows assigned to all edges except $e_{ij}$ in $P_{ij}$ which are also in $S_{ij}$ are reduced. This modification of assigning flow to $G$ does not change the flow into $i$. Thus, this flow $\psi_{ij}'$ must be in $\{\psi_{ij}\}$. The
number of saturated cut sets in $G$ by assigning $\psi_{ij}$ is less than that by assigning $\psi_{ij}$ which contradicts the assumption that $\psi_{ij}$ assigned to $G$ gives the minimum number of saturated cut sets. Thus, $k = 0$. By Theorem 18-1, there exists at least one saturated cut set if $\psi_{ij}$ is maximum. Thus, there must be a maximum flow which can be assigned to $G$ and which produces at least one basic saturated cut set. Q.E.D.

![Figure 18-1. Saturated cut sets in G.](image)

Definition 18-6: A maximum flow $\psi_{ij}$ from $i$ to $j$ which can be assigned to a net $G$ in order to receive the maximum flow at $j$ is called a source terminal capacity symbolized by $t_{ij}$. The maximum flow which will be received at vertex $j$ when $\psi_{ij}$ is assigned to $G$ is called a sink terminal capacity symbolized by $\bar{t}_{ij}$.

Let $e_1, \ldots, e_k$ be the edges connected to vertex $j$ whose orientation is to $j$. Then when $\psi_{ij}$ is assigned to $G$ in order to receive maximum flow at $j$,

$$t_{ij} = \sum_{r=1}^{k} \alpha_r \psi(e_r),$$  \hspace{1cm} (18-11)
For any vertices i, j, and k in a lossy net G, the source and sink terminal capacities between i, j, and k have the following relationship:

**Theorem 18-3:** For any vertices i, j, and k, either \( t_{ij} \geq t_{kj} \) or \( t_{ij} > t_{kj} \).

In order to prove this theorem, we define the following:

**Definition 18-7:** When a maximum flow \( \psi_{ij} \) is assigned to G, which produces a basic saturated cut set \( S_{ij} \), we say that \( S_{ij} \) is a basic saturated cut set under \( t_{ij} \).

**Proof:** Let \( S_{ij} \) be a basic saturated cut set under \( t_{ij} \) and \( \Omega_{i1} \) and \( \Omega_{i2} \) be the corresponding vertex sets of \( S_{ij} \) where \( i \in \Omega_{i1} \) and \( j \in \Omega_{i2} \). Let \( S_{ik} \) be a basic saturated cut set under \( t_{ik} \) and \( \Omega_{k1} \) and \( \Omega_{k2} \) be the corresponding vertex sets of \( S_{ik} \) where \( k \in \Omega_{k2} \) and \( i \in \Omega_{k1} \). Also let \( S_{kj} \) be a basic saturated cut set under \( t_{kj} \) and \( \Omega_{j1} \) and \( \Omega_{j2} \) be the corresponding vertex sets of \( S_{kj} \) where \( k \in \Omega_{j1} \) and \( j \in \Omega_{j2} \).

**Case 1:** Vertex \( k \in \Omega_{i1} \). There are two situations to consider here which are either \( j \in \Omega_{k2} \) or \( j \notin \Omega_{k2} \).

**Case 1-A:** When \( j \in \Omega_{k2} \). Then either \( i \in \Omega_{j1} \) or \( i \in \Omega_{j2} \) as shown in Figure 18-2(a) and (b). For \( i \in \Omega_{j1} \), \( S_{kj} \) and \( S_{ij} \) must be basic saturated cut sets under \( t_{kj} \). Thus,

\[
\xi_{ij} = t_{kj},
\]

For \( i \notin \Omega_{j2} \), \( S_{ij} \) may or may not be a basic saturated cut set under \( t_{kj} \). Thus,

\[
\xi_{ij} \geq t_{kj}.
\]
Case 1-B: When $j \in \Omega_{k2}$, we have the situations which are either $i \in \Omega_{j1}$ or $i \notin \Omega_{j1}$, as shown in Figure 18-3(a) and (b). For $i \in \Omega_{j1}$, $S_{ij}$ must be a basic saturated cut set under $\tilde{e}_{kj}$. Thus,

$$t_{ij} = \frac{e_{kj}}{e_{kj}} \quad (18-14)$$

For $i \notin \Omega_{j1}$, $S_{ij}$ need not be a basic saturated cut set under $\tilde{e}_{kj}$. Hence,

$$t_{ij} \geq \frac{e_{kj}}{e_{kj}} \quad (18-15)$$

Case 2: Vertex $k \in \Omega_{i2}$. We only need to consider the case when $\Omega_{k2} \subseteq \Omega_{i2}$. Thus, either $j \in \Omega_{k2}$ or $j \notin \Omega_{k2}$. Suppose $j \notin \Omega_{k2}$ as shown in Figure 18-4. Then $S_{ij}$ need not be a basic saturated cut set under $\tilde{e}_{ik}$. Thus,

$$t_{ij} \geq \frac{e_{ik}}{e_{ik}} \quad (18-16)$$

If $j \in \Omega_{k2}$, $S_{ij}$ must be a basic saturated cut set under $\tilde{e}_{ik}$. Thus,

$$t_{ij} = \frac{e_{ik}}{e_{ik}} \quad (18-17)$$

It is important to notice that in the case of lossless communication nets, $t_{ij} = \frac{e_{ij}}{e_{ij}}$. Thus the above theorem becomes the one in Eq. (17-7), which is known as a triangular relationship.

We next define the following matrices which are similar to the terminal (capacity) matrices for lossless communication nets.

Definition 18-8: A source terminal matrix $\bar{T}$ is defined as

$$\bar{T} = [\tilde{e}_{ij}] \quad (18-18)$$
Figure 18-2. Location of saturated cut sets (1).

Figure 18-3. Location of saturated cut sets (2).

Figure 18-4. Location of saturated cut sets (3).
and a sink terminal matrix $\mathbf{T}$ is defined as

$$\mathbf{T} = [t_{ij}]$$

(18-19)

where $t_{ii} = t_{ij} = 0$. Notice that even if lossy net is non-oriented (i.e., $\{w_y, w_c, G, \Omega\}$ is non-oriented), $\mathbf{T}$ and $\mathbf{T}$ may not be symmetric.

As we defined in the case of lossless communication net, we define an $S$-submatrix $\mathbf{M}$ of $\mathbf{T}$ by deleting either row $i$ or column $i$ of $\mathbf{T}$ for all $i$. We also define an $S$-submatrix $\mathbf{M}$ of $\mathbf{T}$ in a similar way. We call $\mathbf{M}$ and $\mathbf{M}$ a pair of $S$-submatrices of $\mathbf{T}$ and $\mathbf{T}$ if row $i$ of $\mathbf{T}$ is in $\mathbf{M}$, row $i$ of $\mathbf{T}$ is in $\mathbf{M}$, and if column $j$ of $\mathbf{T}$ is in $\mathbf{M}$, column $j$ of $\mathbf{T}$ is in $\mathbf{M}$.

With the above definitions, we can obtain a property of a source and sink terminal matrix as follows.

**Theorem 18-4:** For any entries $t_{rs}$ and $\tilde{t}_{rs}$ in $\mathbf{T}$ and $\mathbf{T}$, respectively, there exists a pair of $S$-submatrices $\mathbf{M}$ and $\mathbf{M}$ of $\mathbf{T}$ and $\mathbf{T}$ such that

- $t_{rp}$ is the largest in column $p$ for all $p$ in $\mathbf{M}$, and
- $\tilde{t}_{qs}$ is the largest in row $q$ for all $q$ in $\mathbf{M}$.

**Proof:** Let $S_{rs}$ be a basic saturated cut set under $t_{rs}$. Also let $\Omega_{r1}$ and $\Omega_{r2}$ be the corresponding vertex sets of $S_{rs}$ where $r \in \Omega_{r1}$ and $s \in \Omega_{r2}$. Let $\mathbf{M}$ and $\mathbf{M}$ be the pair of $S$-submatrices of $\mathbf{T}$ and $\mathbf{T}$ such that the rows in $\mathbf{T}$ and $\mathbf{T}$ which correspond to the vertices in $\Omega_{r1}$ and the columns in $\mathbf{T}$ and $\mathbf{T}$ which correspond to the vertices in $\Omega_{r2}$ are in $\mathbf{M}$ and $\mathbf{M}$. By Case 1 of the proof of Theorem 18-3, we have

$$t_{rs} \geq t_{ks} \quad \text{for all } k \in \Omega_{r1},$$

(18-20)
and by Case 2 of the proof of Theorem 18-3,

\[ \tilde{t}_{rs} > \tilde{t}_{rk} \quad \text{for all } k \in \Omega_{r2}. \]  

(18-21)

Thus, the theorem is true for \( p = s \) and \( q = r \).

Consider \( t_{rp} \) where \( p \in \Omega_{r1} \). Let \( S_{rp} \) be a basic saturated cut set of \( t_{rp} \). If \( S_{rs} \) need not be a basic saturated cut set under \( t_{rp} \), then \( \Omega_{p2} \) (which is one of the corresponding vertex sets \( \Omega_{pl} \) and \( \Omega_{p2} \) of \( S_{rp} \) where \( p \in \Omega_{p2} \)) is a subset of \( \Omega_{r2} \). Thus, \( t_{rp} \geq t_{kp} \) for all \( k \in \Omega_{r1} \). On the other hand, if \( S_{rs} \) is a basic saturated cut set under \( t_{rp} \), we consider \( S_{rs} \) rather than \( S_{rp} \) as a basic saturated cut set under \( t_{rp} \). Then

\[ t_{rp} \geq t_{kp} \quad \text{for all } k \in \Omega_{r1}. \]  

(18-22)

Thus, \( t_{rp} \) is the largest in column \( p \) for all \( p \) in \( M \). The proof for \( t_{qs} \geq t_{qk} \) for all \( k \in \Omega_{12} \) is almost the same as the above case.

Notice that for the lossless case, Theorem 18-4 becomes the \( S \)-submatrix condition of terminal capacity matrices.

From the results given, we see that the theory of lossless communication net becomes a special case of the theory of lossy communication nets. Here, the fundamental properties of lossy nets have been introduced. There remain many unsolved problems, one of the most important being to determine the conditions for a matrix to be \( \mathcal{X} \) or \( \mathcal{X} \) of a lossy net. No simple method has been given for evaluating \( t_{ij} \) and \( \tilde{t}_{ij} \). It is possible in some lossy communication nets that a flow from \( i \) which is less than \( \tilde{t}_{ij} \) can result in a flow which is equal to \( t_{ij} \).
at j. Hence, we can define \( \bar{t}_{ij} \) which indicates the minimum required flow at i in order to receive the flow equal to \( t_{ij} \) at j. Then it is easily seen that Theorem 18-3 will hold with \( \bar{t}_{ij} \) and \( t_{ij} \). Furthermore, if we define \( T = [\bar{t}_{ij}] \) then Theorem 18-4 will hold with \( T \) and \( T \).

Clearly, every theorem given is applicable to non-oriented lossy communication nets. However, some modifications must be made in the definitions; for example, that given for the basic saturated cut set. If we restrict the problem to the non-oriented case, we can show that basic saturated cut sets \( S_{ij} \) and \( S_{pq} \) of \( \bar{t}_{ij} \) and \( \bar{t}_{pq} \), respectively, will never cross each other, i.e., either \( \Omega_{il} \subset \Omega_{pl} \) or \( \Omega_{il} \supset \Omega_{pl} \) where \( \Omega_{il} \) and \( \Omega_{pl} \) are one of the corresponding vertex sets of \( S_{ij} \) and \( S_{pq} \), respectively. This property may help to solve some problems.
The terms and symbols used in this paper are given below.

A -- Incidence matrix.

A\_p -- A matrix obtained by deleting row p from A.

Adjacent edges -- Edges e\_1 and e\_2 in \( \{G_1, \Omega\}_f \) are adjacent if there exists \( \zeta \in \Omega \) which is one of end-points of e\_1 and e\_2.

Basic saturated cut set -- See Definition 18-4.

B -- Circuit matrix.

C -- Class of all possible circuits, edge disjoint unions of circuits, and the empty set.

C\_s -- Class of all possible circuits in a linear graph and the empty set.

C\_i\_j -- Class of all possible paths from vertex i to vertex j and all possible edge disjoint unions of path from vertex i to vertex j and circuits in a linear graph.

For Classes R and U,

\[ R \oplus U -- \text{Class } R \oplus U \equiv (\text{Min } W) \text{ where } W \text{ is a class of all distinct } r \oplus u \text{ for } r \in R \text{ and } u \in U. \text{ See Definition 10-2}. \]

\[ R^n -- R^n \equiv R^{n-1} \oplus R \text{ for } n > 1 \text{ where } R^1 \equiv R. \]

Common tree -- \( G_1 \) such that \( \{G_1, \Omega\}_f^i \) is a tree in \( \{G, \Omega\}_f^i \) and \( \{G_1, \Omega\}_f^v \) is a tree in \( \{G, \Omega\}_f^v \). The name "complete tree" is used in the original papers [16,17] which introduce this type of tree.

Corresponding vertex sets of a cut set -- See Definition 1-20.

Current edge -- Edge in G of a weighted linear graph \( \{w_y, w_1, G, \Omega\}_f^i \).

Current graph -- A weighted linear graph \( \{w_y, w_1, G, \Omega\}_f^i \).
Degree of vertex \( \zeta \) -- Number of edges each of which has vertex \( \zeta \) as one of its end-points.

Disjoint sets -- \( U_1, U_2, \cdots, U_k \) are disjoint sets if \( U_1 \cap U_2 \cap \cdots \cap U_p = \emptyset \) for all \( r \)'s where \( 1 \leq r_1, r_2, \cdots, r_p \leq k \), \( r_1 \neq r_2 \neq \cdots \neq r_p \), and \( 1 \leq p \leq k \).

Edge disjoint -- \( \{ G_1, \Omega_1 \} \) and \( \{ G_2, \Omega_2 \} \) are edge disjoint if \( G_1 \cap G_2 = \emptyset \).

Edge sequence of a circuit -- See Definition 1-13.

Edge Sequence of a path -- See Definition 1-16.

End-points of edge -- Vertices \( \zeta_i \) and \( \zeta_j \) are end-points of edge if \( f(e) = (\zeta_i, \zeta_j) \).

\( \eta_i \) -- \( \eta_i (e \in \mathcal{F}) \) is a set of all switching variables in \( \eta_i \) where \( f_i \) is the product of switching variables in \( \mathcal{F} = \bigcup f_i \). Thus, \( \eta_i \) can represent a path in a switching network.

f -- Mapping function.

Flow \( \Psi \) -- See Definitions 16-1 and 16-2.

G -- Set of edges.

\( G \) -- Representation of two weighted linear graphs \( \{ w, \omega, G, \Omega \}_{f, v} \) and \( \{ w, \omega, G, \Omega \}_{f, v} \) together.

\( G(\Omega_1 \times \Omega_2) \) of \( \{ G, \Omega \}_{f} \) -- Subgraph of \( \{ G, \Omega \}_{f} \) such that every edge \( e \) in \( G(\Omega_1 \times \Omega_2) \subset G \), \( f(e) \) is in \( \Omega_1 \times \Omega_2 \). When \( \Omega_1 \neq \Omega_2 \), \( G(\Omega_1 \times \Omega_2) = G(\Omega_1) \). See Definition 1-2.

\( G(\Omega) \) -- \( G(\Omega \times \Omega) \).

\( \{ G_1, \Omega_1 \} \cong \{ G_2, \Omega_2 \} \) -- \( G_1 \cong G_2 \) and \( \Omega_1 \cong \Omega_2 \).

\( \{ G_1, \Omega_1 \} \supset \{ G_2, \Omega_2 \} \) -- \( G_1 \supset G_2 \) and \( \Omega_1 \supset \Omega_2 \).

\( \{ G_1, \Omega_1 \} \cap \{ G_2, \Omega_2 \} \neq \emptyset \) -- \( G_1 \cap G_2 \neq \emptyset \) and \( \Omega_1 \cap \Omega_2 \neq \emptyset \).
H-submatrix -- See Definition 11-1.

Isolated vertex -- A vertex in Ω - Ω(g) of \{G,Ω\}.

\( \xi \) -- Set representation of s.c. switching function F, which is a class of sets \( \eta_i \).

Linear graph \( \{G,Ω\}_f \) -- A class of two sets G and Ω with a mapping function f which is many to one such that for every edge e in G,

\[ f(e) \in Ω \times Ω. \]

\( \{M\}_P \) -- Set of M-submatrices.

\( \{M_v\}_v \) -- Set of minimum M-submatrices.

M-submatrix -- See Definition 11-2.

Maximal connected subgraph -- See Definition 1-9.

Minimum M-submatrix -- An M-submatrix \( M(i_1,i_2,\ldots,i_k) \) which consists of rows \( i_1,i_2,\ldots,i_k \). See Definitions 11-3 and 11-4.

Min R -- A special subclass of class R. See Definition 10-1.

Non-oriented edge -- Edge e in G of \( \{G,Ω\}_f \) such that \( f(e) = (ζ_i,ζ_j) \) where \( (ζ_i,ζ_j) \) is a non-ordered pair in \( Ω \times Ω \).

Non-oriented linear graph -- linear graph with \( Ω \times Ω \) being a set of non-ordered pairs.

Ω -- Set of vertices.

Ω(G) -- Set of vertices such that every vertex \( Ω(G) \) is the end-point of at least one edge in G.

Oriented edge -- Edge e in G of \( \{G,Ω\}_f \) such that \( f(e) = (ζ_i,ζ_j) \) where \( (ζ_i,ζ_j) \) is an ordered pair in \( Ω \times Ω \). The orientation of edge e is from \( ζ_i \) to \( ζ_j \).

Oriented linear graph -- Linear graph with \( Ω \times Ω \) being a set of ordered pairs.
\( \Phi_{ij} \) -- Class of all possible paths from vertex i to vertex j in a linear graph.

Q -- Cut set matrix.

\( R^t \) -- Transpose of matrix R.

\( \Theta \) -- Ring sum.

\( \rho \) (Number of maximal connected subgraphs) -- See Definition 1-9.

Saturated cut set -- See Definition 18-3.

S.C. switching function -- a switching function of a single contact switching network.

Self loop -- Edge e such that \( f(e) = (\zeta, \zeta) \) where \( \zeta \in \Omega \).

Separated graph -- See Definition 1-8.

For sets U and W,

\[ U \cup W \] -- A set consisting of all elements in U and W.

\[ U \cap W \] -- A set consisting of all elements which are in both U and W.

\[ U - W \] -- A set consisting of all elements which are in U but not in W.

\[ U \oplus W \] -- A set consisting of all elements which are in either U or W but not both U and W. \( U \oplus W = (U- W) \cup (W-U) \).

S-submatrix -- See Definition 17-2.

\( \Gamma_p \) -- See Definition 16-6.

\( t_{ij} \) -- Amount of maximum flow which can be assigned from vertex i to vertex j of a communication net.

\( t^*_{ij} \) -- See Definition 18-6.

\( \xi_{ij} \) -- See Definition 18-6.
\( T \) -- See Definition 18-8.

\( \overline{T} \) -- See Definition 18-8.

\( T_2 \) -- 2-tree which consists of two maximal connected subgraphs \( G_1 \) and \( G_2 \) which have the property that \( p_1, p_2, \ldots, p_n \in \Omega(G_1) \) and \( q_1, q_2, \ldots, q_n \in \Omega(G_2) \).

Terminal capacity matrix \( T \) -- Matrix \([t_{ij}]\). See Eq. (16-5).

Transadmittance -- \( y_{ij,mn} \) which satisfy \( y_{ij,mn} v_{mn} = i_{ij} \).

Tree property -- See Definition 1-23.

\( V[s(w_c)] \) -- See Definition 18-1.

Vertex sequence of a path -- See Definition 1-16.

Voltage graph -- A linear graph \( \{w_y, w_v, G, \Omega\}_f \).

Weighted linear graph -- \( \{w_1, w_2, \ldots, w_p, G, \Omega\}_f \) where \( \{G, \Omega\}_f \) is a linear graph and there is one to one correspondence between \( w_i \) and \( G \) for all \( i = 1, 2, \ldots, p \).

\( W_{p,o} \) -- See Eqs. (2-13), (2-14), and (7-5).

\( W_{pq,o} \) -- See Eq. (3-8).

\( W_{pq,ro} \) -- See Eq. (3-11).

\( W_{p,o;q,r} \) -- See Eq. (7-10).

\( U_{p,o;q,r} \) -- See Eq. (8-4).

\( \Sigma U \) -- See Eqs. (3-23), (3-24), and (3-30).

\( \times \) -- Cross product.
REFERENCES


<table>
<thead>
<tr>
<th>Distribution list as of February 1, 1964</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Director</td>
</tr>
<tr>
<td>Air University Library</td>
</tr>
<tr>
<td>Maxwell Air Force Base, Alabama</td>
</tr>
<tr>
<td>Attn: CR-4803a</td>
</tr>
<tr>
<td>1 Redstone Scientific Information Center</td>
</tr>
<tr>
<td>U.S. Army Missile Command</td>
</tr>
<tr>
<td>Redstone Arsenal, Alabama</td>
</tr>
<tr>
<td>1 Electronics Research Laboratory</td>
</tr>
<tr>
<td>University of California</td>
</tr>
<tr>
<td>Berkeley, California</td>
</tr>
<tr>
<td>2 Hughes Aircraft Company</td>
</tr>
<tr>
<td>Florence and Teale</td>
</tr>
<tr>
<td>Culver City, California</td>
</tr>
<tr>
<td>Attn: N. E. Devereux Technical Document Center</td>
</tr>
<tr>
<td>3 Autonetics</td>
</tr>
<tr>
<td>(Unclassified)</td>
</tr>
<tr>
<td>9150 East Imperial Highway</td>
</tr>
<tr>
<td>Downey, California</td>
</tr>
<tr>
<td>Attn: Tech. Library, 3041-11</td>
</tr>
<tr>
<td>1 Dr. Arnold T. Nordiesick</td>
</tr>
<tr>
<td>General Motors Corporation</td>
</tr>
<tr>
<td>Defense Research Laboratories</td>
</tr>
<tr>
<td>6797 Hollister Avenue</td>
</tr>
<tr>
<td>Goleta, California</td>
</tr>
<tr>
<td>1 University of California</td>
</tr>
<tr>
<td>(Unclassified)</td>
</tr>
<tr>
<td>Lawrence Radiation Laboratory</td>
</tr>
<tr>
<td>P. O. Box 808</td>
</tr>
<tr>
<td>Livermore, California</td>
</tr>
<tr>
<td>1 Mr. Thomas L. Hartwick</td>
</tr>
<tr>
<td>Aerospace Corporation</td>
</tr>
<tr>
<td>P. O. Box 9508</td>
</tr>
<tr>
<td>Los Angeles 45, California</td>
</tr>
<tr>
<td>1 Lt. Colonel Willard Levin</td>
</tr>
<tr>
<td>Aerospace Corporation</td>
</tr>
<tr>
<td>P. O. Box 9508</td>
</tr>
<tr>
<td>Los Angeles 45, California</td>
</tr>
<tr>
<td>1 Professor Zorab Kaprelian</td>
</tr>
<tr>
<td>University of Southern California</td>
</tr>
<tr>
<td>University Park</td>
</tr>
<tr>
<td>Los Angeles 7, California</td>
</tr>
<tr>
<td>1 Sylvania Electronic Systems - West</td>
</tr>
<tr>
<td>Electronic Defense Laboratories</td>
</tr>
<tr>
<td>P. O. Box 205</td>
</tr>
<tr>
<td>Mountain View, California</td>
</tr>
<tr>
<td>Attn: Documents Center</td>
</tr>
<tr>
<td>1 Varian Associates</td>
</tr>
<tr>
<td>61 Hansen Way</td>
</tr>
<tr>
<td>Palo Alto, California</td>
</tr>
<tr>
<td>Attn: Dr. Ir. Weissman</td>
</tr>
<tr>
<td>1 Huston Denslow</td>
</tr>
<tr>
<td>(Unclassified)</td>
</tr>
<tr>
<td>Library Supervisor</td>
</tr>
<tr>
<td>Jet Propulsion Laboratory</td>
</tr>
<tr>
<td>California Institute of Technology</td>
</tr>
<tr>
<td>Pasadena, California</td>
</tr>
<tr>
<td>1 Professor Nicholas George</td>
</tr>
<tr>
<td>California Institute of Technology</td>
</tr>
<tr>
<td>Electrical Engineering Department</td>
</tr>
<tr>
<td>Pasadena, California</td>
</tr>
<tr>
<td>1 Space Technology Labs., Inc.</td>
</tr>
<tr>
<td>One Space Park</td>
</tr>
<tr>
<td>Redondo Beach, California</td>
</tr>
<tr>
<td>Attn: Acquisitions Group</td>
</tr>
<tr>
<td>STL Technical Library</td>
</tr>
<tr>
<td>2 Commanding Officer and Director</td>
</tr>
<tr>
<td>U.S. Naval Electronics Laboratory</td>
</tr>
<tr>
<td>San Diego 51, California</td>
</tr>
<tr>
<td>Attn: Code 2800, C. S. Menning</td>
</tr>
<tr>
<td>1 Commanding Officer and Director</td>
</tr>
<tr>
<td>U.S. Navy Electronics Laboratory</td>
</tr>
<tr>
<td>San Diego 51, California</td>
</tr>
<tr>
<td>Attn: Library</td>
</tr>
<tr>
<td>1 Office of Naval Research Branch</td>
</tr>
<tr>
<td>Office 1000 Geary Street, San Francisco, California</td>
</tr>
<tr>
<td>1 The RAND Corporation</td>
</tr>
<tr>
<td>1700 Main Street</td>
</tr>
<tr>
<td>Santa Monica, California</td>
</tr>
<tr>
<td>Attn: Library</td>
</tr>
<tr>
<td>1 Stanford Electronics Laboratories</td>
</tr>
<tr>
<td>(Unclassified)</td>
</tr>
<tr>
<td>Stanford University</td>
</tr>
<tr>
<td>Attn: SEL Documents Librarian</td>
</tr>
<tr>
<td>1 Dr., L. F. Carter</td>
</tr>
<tr>
<td>Chief Scientist Air Force</td>
</tr>
<tr>
<td>Room 3E-324, Pentagon</td>
</tr>
<tr>
<td>Washington 25, D.C.</td>
</tr>
<tr>
<td>1 Mr. Robert L. Fink</td>
</tr>
<tr>
<td>Associate Director for Research</td>
</tr>
<tr>
<td>Technology and Division AFSC</td>
</tr>
<tr>
<td>1 Captian Paul Johnson (USN-Res)</td>
</tr>
<tr>
<td>National Aeronautics and Space</td>
</tr>
<tr>
<td>Administration</td>
</tr>
<tr>
<td>1 Dr. Edwin M. Myers</td>
</tr>
<tr>
<td>Headquarters USAF (AFRDR)</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>1 Mr. James Ward</td>
</tr>
<tr>
<td>Office of Deputy Director (Research</td>
</tr>
<tr>
<td>and Info) Department of Defense</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>1 Dr. Alan T. Waterman</td>
</tr>
<tr>
<td>Director, National Science Foundation</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>1 Mr. G. D. Watson</td>
</tr>
<tr>
<td>Defense Research Member</td>
</tr>
<tr>
<td>Canadian Joint Staff</td>
</tr>
<tr>
<td>2450 Massachusetts Ave., N. W.</td>
</tr>
<tr>
<td>Washington 8, D. C.</td>
</tr>
<tr>
<td>1 Mr. Arthur G. Winer</td>
</tr>
<tr>
<td>Chief Scientist</td>
</tr>
<tr>
<td>Air Force Systems Command</td>
</tr>
<tr>
<td>Andrews Air Force Base</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>1 Director, Advanced Research Projects</td>
</tr>
<tr>
<td>Agency</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>1 Airforce Office of Scientific Branch</td>
</tr>
<tr>
<td>Directorate of Engineering Sciences</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>Attn: Electronics Division</td>
</tr>
<tr>
<td>1 Director of Science and Technology</td>
</tr>
<tr>
<td>Headquarters, USAF</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>Attn: AFRDR-EL/GO</td>
</tr>
<tr>
<td>1 AFRBR - SC</td>
</tr>
<tr>
<td>Headquarters, USAF</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>1 Headquarters, R &amp; T Division (Unclassified)</td>
</tr>
<tr>
<td>Rolling Air Force Base</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>Attn: RTFR</td>
</tr>
<tr>
<td>1 Headquarters, U.S. Army Material</td>
</tr>
<tr>
<td>Command Research Division, R &amp; D</td>
</tr>
<tr>
<td>Directorate</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>Attn: Physics &amp; Electronics Branch</td>
</tr>
<tr>
<td>Electronics Section</td>
</tr>
<tr>
<td>1 Commander</td>
</tr>
<tr>
<td>Diamond Ordnance Fuse Laboratories</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>Attn: Librarian, Room 211, Bldg. 92</td>
</tr>
<tr>
<td>1 Operation Evaluation Group</td>
</tr>
<tr>
<td>Chief of Naval Operations (CP-03EG)</td>
</tr>
<tr>
<td>Department of Navy</td>
</tr>
<tr>
<td>Washington, D. C. 20350</td>
</tr>
<tr>
<td>1 Chief of Naval Operations (Code OP-ONT)</td>
</tr>
<tr>
<td>Department of the Navy</td>
</tr>
<tr>
<td>Washington, D. C. 20350</td>
</tr>
<tr>
<td>1 Commanding Officer</td>
</tr>
<tr>
<td>U.S. Army Personal Research Office</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>1 Commanding Officer &amp; Director</td>
</tr>
<tr>
<td>Code 142 Library</td>
</tr>
<tr>
<td>David W. Taylor Model Basin</td>
</tr>
<tr>
<td>Washington, D. C. 20097</td>
</tr>
<tr>
<td>1 Chief, Bureau of Ships (Code 666)</td>
</tr>
<tr>
<td>Department of the Navy</td>
</tr>
<tr>
<td>Washington, D. C. 20360</td>
</tr>
<tr>
<td>1 Chief, Bureau of Ships (Code 732)</td>
</tr>
<tr>
<td>Department of the Navy</td>
</tr>
<tr>
<td>Washington, D. C. 20360</td>
</tr>
<tr>
<td>1 Chief, Bureau of Naval Weapons</td>
</tr>
<tr>
<td>Technical Library, DLI-3</td>
</tr>
<tr>
<td>Department of the Navy</td>
</tr>
<tr>
<td>Washington, D. C. 20360</td>
</tr>
<tr>
<td>1 Director, (Code 5140)</td>
</tr>
<tr>
<td>U.S. Naval Research Laboratory</td>
</tr>
<tr>
<td>Washington, D. C. 20390</td>
</tr>
<tr>
<td>1 Chief of Naval Research (Code 437)</td>
</tr>
<tr>
<td>Department of the Navy</td>
</tr>
<tr>
<td>Washington, D. C. 20360</td>
</tr>
<tr>
<td>1 Dr. H. Wallace Sinaiko (Unclassified)</td>
</tr>
<tr>
<td>Institute for Defense Analyses</td>
</tr>
<tr>
<td>Research &amp; Support Division 1666</td>
</tr>
<tr>
<td>Connecticut Ave., N. W.</td>
</tr>
<tr>
<td>Washington 9, D. C.</td>
</tr>
<tr>
<td>1 Data Processing Systems Division</td>
</tr>
<tr>
<td>National Bureau of Standards</td>
</tr>
<tr>
<td>Conn. at Van Ness</td>
</tr>
<tr>
<td>Room 219, Bldg. 10</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>Attn: A. K. Smilow</td>
</tr>
<tr>
<td>1 National Bureau of Standards</td>
</tr>
<tr>
<td>(Unclassified)</td>
</tr>
<tr>
<td>Research Information Center &amp;</td>
</tr>
<tr>
<td>Advisory Service on Information</td>
</tr>
<tr>
<td>Processing</td>
</tr>
<tr>
<td>Data Processing Systems Division</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>1 Exchange and Gift Division</td>
</tr>
<tr>
<td>(Unclassified)</td>
</tr>
<tr>
<td>The Library of Congress</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>1 NASA Headquarters</td>
</tr>
<tr>
<td>Office of Applications</td>
</tr>
<tr>
<td>400 Maryland Avenue, S. W.</td>
</tr>
<tr>
<td>Washington 25, D. C.</td>
</tr>
<tr>
<td>Attn: Mr. A. M. Greg Andrus</td>
</tr>
<tr>
<td>Code FC</td>
</tr>
<tr>
<td>1 CPDC (PGAP)</td>
</tr>
<tr>
<td>Arlington Air Force Base</td>
</tr>
<tr>
<td>Florida</td>
</tr>
<tr>
<td>1 Martin Company</td>
</tr>
<tr>
<td>F. O. Box 5837</td>
</tr>
<tr>
<td>Orlando, Florida</td>
</tr>
<tr>
<td>Attn: Engineering Library</td>
</tr>
<tr>
<td>MP-30</td>
</tr>
<tr>
<td>1 Commanding Officer</td>
</tr>
<tr>
<td>Office of Naval Research, Chicago</td>
</tr>
<tr>
<td>Branch 6th Floor, 230 North Michigan</td>
</tr>
<tr>
<td>Chicago 1, Illinois</td>
</tr>
<tr>
<td>1 Laboratories for Applied Sciences</td>
</tr>
<tr>
<td>University of Chicago</td>
</tr>
<tr>
<td>6220 South Drexel</td>
</tr>
<tr>
<td>Chicago 37, Illinois</td>
</tr>
<tr>
<td>1 Librarian</td>
</tr>
<tr>
<td>School of Electrical Engineering</td>
</tr>
<tr>
<td>Purdue University</td>
</tr>
<tr>
<td>Lafayette, Indiana</td>
</tr>
</tbody>
</table>