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    - Let \( L = (p_1, p_2, \ldots, p_n) \) be a list of real numbers in the interval \((0,1]\). The one-dimensional bin packing problem is to place the \(p_i\)'s into a minimum number of unit-capacity bins. For any algorithm \(A\), let \(A(L)\) denote the number of bins used by \(A\) in packing \(L\) and let \(OPT(L)\) denote the minimum number of bins needed to pack \(L\). It is shown that, for any on-line algorithm \(A\),
      \[
      \lim_{n \to \infty} \max \left\{ \frac{A(L)}{OPT(L)} \right\} > 1.536.
      \]
A LOWER BOUND FOR ON-LINE ONE-DIMENSIONAL
BIN PACKING ALGORITHMS

by

Donna J. Brown

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A LOWER BOUND FOR ON-LINE ONE-DIMENSIONAL BIN PACKING ALGORITHMS

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December 1979

Abstract

Let $L = (p_1, p_2, \ldots, p_n)$ be a list of real numbers in the interval $(0, 1]$. The one-dimensional bin packing problem is to place the $p_i$'s into a minimum number of unit-capacity bins. For any algorithm $A$, let $A(L)$ denote the number of bins used by $A$ in packing $L$ and let $OPT(L)$ denote the minimum number of bins needed to pack $L$. It is shown that, for any on-line algorithm $A$,

$$\lim_{n \to \infty} \left\{ \frac{\max_{OPT(L)=n} A(L)}{OPT(L)} \right\} > 1.536.$$
I. Introduction

Let \( L = (p_1, p_2, \ldots, p_n) \) be a list of real numbers in the interval \((0, 1]\). The one-dimensional bin packing problem is to place the \( p_i \)'s into a minimum number of unit-capacity bins; i.e., the sum of the numbers in each bin can be at most 1. Because this problem is known to be NP-hard [8], much work has been done in the study of heuristic algorithms with guaranteed performance bounds [12, 13, 14, 16].

In this paper we are concerned with algorithms for which the pieces (numbers) in list \( L \) are available one at a time, and each piece must be placed in some bin before the next piece is available; such an algorithm is referred to as on-line [12, 13, 16]. The performance measure used is the ratio of the number of bins used by an algorithm \( A \) in packing list \( L \), \( A(L) \), to the optimum (minimum) number of bins required to pack the list, \( \text{OPT}(L) \).

Example 1. Consider the list \( L_1 = (3/4, 1/6, 1/6, 2/3, 1/4) \). One possible packing algorithm is the well known First-Fit (FF) Algorithm [12,13,14], which places each piece in the first bin which has enough available space. As shown in Figure 1a, this algorithm leads to a packing which uses three bins. An optimal packing requires only two bins (see Figure 1b). Notice that \( FF(L_1) = \frac{3}{2} \text{OPT}(L_1) \).

We are interested, however, in the ratio \( \frac{A(L)}{\text{OPT}(L)} \) for lists \( L \) with many pieces. In particular, we wish to determine a lower bound on the performance ratio

\[
\lim_{n \to \infty} \max_{\text{OPT}(L) = n} \frac{A(L)}{\text{OPT}(L)}.
\]
a) Packing $L_1$ by the First-Fit Algorithm: $FF(L_1) = 3$.

b) An optimal packing of $L_1$: $OPT(L_1) = 2$.

Figure 1. Packings of $L_1$ from Example 1.
Example 2. For $n$ even, let the list $L_2$ consist of $n$ pieces of size $3/8$ and $n$ pieces of size $5/8$. The First-Fit Algorithm uses $\frac{3n}{2}$ bins, compared to an optimal packing of $n$ bins (see figures 2a and 2b). Thus, we know that, for the First-Fit Algorithm,

$$\text{FF}(L_2) \geq \frac{3}{2} \text{OPT}(L_2).$$

(In fact, it is known [12,13], that there is a list $L$ for which $\text{FF}(L) = \frac{17}{10} \text{OPT}(L)$.)

We shall show that there is no algorithm which can always use fewer than $1.536 \text{OPT}(L)$ number of bins. Thus, for any packing algorithm $A$,

$$\lim_{n \to \infty} \max \left\{ \frac{A(L)}{\text{OPT}(L)} \right\} > 1.536$$

This lower bound is an improvement over the bound of 1.5 proved by Yao [16].


Much work has recently been done with two-dimensional bin packing. Various algorithms [1, 2, 3, 7, 9] have been proposed, many using ideas from one-dimensional packing algorithms [12,13,14]. Some work on two-dimensional lower bounds has also been done [5,6,15]. In particular, the 1.536 lower bound presented in this paper extends immediately to two dimensions and gives a 1.536 lower bound for any on-line two-dimensional algorithm which packs pieces in order of decreasing or increasing height or increasing width [6].
a) Packing $L_2$ by the First-Fit Algorithm: $\text{FF}(L_2) = \frac{3n}{2}$.

b) An optimal packing of $L_2$: $\text{OPT}(L_2) = n$.
II. An Example

Yao [16] used a list consisting of pieces of sizes $\frac{1}{6} - 2\varepsilon$, $\frac{1}{3} + \varepsilon$, $\frac{1}{2} + \varepsilon$ in order to obtain his $\frac{3}{2}$ lower bound for any on-line bin packing algorithm. In this section we show that the result can be improved to $\frac{109}{71} > 1.535$ by considering a list with pieces sized $\frac{1}{42} - 3\varepsilon$, $\frac{1}{7} + \varepsilon$, $\frac{1}{3} + \varepsilon$, $\frac{1}{2} + \varepsilon$. In Section III the method is generalized to a list with pieces of $t$ different sizes. The work in this section is therefore only a special case of what will be shown, but it is presented here to illustrate the method and therefore make the proof of the main theorem easier to understand. (Also, $\frac{109}{71}$ is not much smaller than 1.536.)

Let $\varepsilon$ be a small positive number, $0 < \varepsilon < \frac{1}{43 \cdot 42 \cdot 3}$. For $n$ a multiple of 42, consider the list $L = L_1 L_2 L_3 L_4$, where

$L_1$ consists of $n$ pieces of size $\frac{1}{42} - 3\varepsilon$,
$L_2$ consists of $n$ pieces of size $\frac{1}{7} + \varepsilon$,
$L_3$ consists of $n$ pieces of size $\frac{1}{3} + \varepsilon$,
$L_4$ consists of $n$ pieces of size $\frac{1}{2} + \varepsilon$.

Noting that

$$\text{OPT}(L_1) = \frac{n}{42},$$
$$\text{OPT}(L_1 L_2) = \frac{n}{6},$$
$$\text{OPT}(L_1 L_2 L_3) = \frac{n}{2},$$
$$\text{OPT}(L) = n,$$

we can define the ratios

$$r_1(n) = \frac{A(L_1)}{\text{OPT}(L_1)} = \frac{42}{n} A(L_1),$$
$$r_2(n) = \frac{A(L_1 L_2)}{\text{OPT}(L_1 L_2)} = \frac{6}{n} A(L_1 L_2),$$

(2.1)
We shall prove that

$$\max\{r_1(n), r_2(n), r_3(n), r_4(n)\} \leq \frac{109}{71}.$$ 

Let $B$ denote the set of bins packed by an algorithm $A$, after the pieces in $L_1 L_2 L_3$ have been packed. Each bin $b \in B$ ($1 \leq w \leq |B|$) contains $m_{1,w}$ pieces of size $\frac{1}{42} - 3\varepsilon$, $m_{2,w}$ pieces of size $\frac{1}{7} + \varepsilon$, and $m_{3,w}$ pieces of size $\frac{1}{3} + \varepsilon$. (Note that $m_{1,w}, m_{2,w},$ and $m_{3,w}$ are nonnegative integers, $0 \leq m_{1,w} \leq 42$, $0 \leq m_{2,w} < 7$, $0 \leq m_{3,w} < 3$.) For notational convenience, we shall omit the double subscript and simply write $m_j$ when we mean $m_{j,w}$. We define the set of bins $\alpha_i (1 \leq i \leq 3)$ as follows:

$$\alpha_i = \{b \in B \mid b \text{ is at least half full, } m_1 \neq 0, \text{ and } m_j = 0 \text{ for } 1 \leq j < i\}.$$ 

In other words, a bin $b_w$ is in

$$\alpha_1 \quad \text{if} \quad \frac{1}{42}m_1 + \frac{1}{7}m_2 + \frac{1}{3}m_3 > \frac{1}{2} \quad \text{and} \quad m_1 \neq 0$$
$$\alpha_2 \quad \text{if} \quad \frac{1}{7}m_2 + \frac{1}{3}m_3 > \frac{1}{2} \quad \text{and} \quad m_2 \neq 0, \ m_1 = 0$$
$$\alpha_3 \quad \text{if} \quad \frac{1}{3}m_3 > \frac{1}{2} \quad \text{and} \quad m_3 \neq 0, \ m_1 = m_2 = 0.$$ 

Similar, we define $\beta_i (1 \leq i \leq 3)$ to be:

$$\beta_i = \{b \in B \mid b \text{ is less than half full, } m_1 \neq 0, \text{ and } m_j = 0 \text{ for } 1 \leq j < i\}.$$ 

Thus, a bin $b_w$ is in
Letting $|\alpha_4|$ represent the number of bins in $\alpha_4$, we have

\[
A(L_1) = |\alpha_1| + |\beta_1| \\
A(L_1 L_2) = |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| \\
A(L_1 L_2 L_3) = |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| + |\alpha_3| + |\beta_3|
\]

Notice that no two pieces of size $\frac{1}{2} + \varepsilon$ will fit in the same bin, nor will any of the $n$ pieces of size $\frac{1}{2} + \varepsilon$ fit in an $\alpha_4$, $\alpha_2$, or $\alpha_3$ bin, so

\[
A(L) = n + |\alpha_1| + |\alpha_2| + |\alpha_3| + n
\]

Let us assume that

\[
\max\{r_1(n), r_2(n), r_3(n), r_4(n)\} < \frac{109}{71}.
\]

Combining equations (2.1), (2.2), and (2.3), this tells us

\[
\frac{n}{42} \cdot \frac{109}{71} > |\alpha_1| + |\beta_1| \\
\frac{n}{6} \cdot \frac{109}{71} > |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| \\
\frac{n}{2} \cdot \frac{109}{71} > |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2| + |\alpha_3| + |\beta_3| \\
n \cdot \frac{109}{71} > |\alpha_1| + |\alpha_2| + |\alpha_3| + n
\]

Because there are $n$ pieces of size $\frac{1}{42} - 3\varepsilon$, $n$ of size $\frac{1}{7} + \varepsilon$, and $n$ of size $\frac{1}{3} + \varepsilon$,
From (2.6), we immediately have

\[ \begin{align*}
- \frac{4}{42} n &= - \frac{4}{42} \sum_{b \in B} m_1 \\
- \frac{1}{2} n &= - \frac{1}{2} \sum_{b \in B} m_2 \\
- n &= - \sum_{b \in B} m_3
\end{align*} \]  

(2.7)

Summing equations (2.5) and (2.7),

\[
\frac{109}{71} n \left( \frac{1}{42} + \frac{1}{6} + \frac{1}{2} + 1 \right) - n \left( \frac{4}{42} + \frac{1}{2} + 1 \right) > 4 |\alpha_1| + 3 |\beta_1| + 3 |\alpha_2| + 2 |\beta_2| + 2 |\alpha_3| + |\beta_3| + n
\]  

(2.8)

\[
- \frac{4}{42} \sum_{b \in B} m_1 - \frac{1}{2} \sum_{b \in B} m_2 - \sum_{b \in B} m_3
\]

Simplifying inequality (2.8) and rearranging terms:
\[ \sum_{b_w \in B} \left( \frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3 \right) > 4|\alpha_1| + 3|\beta_1| + 3|\alpha_2| + 2|\beta_2| + 2|\alpha_3| + |\beta_3| \]

By considering separately each of the summations on the left hand side, we show that inequality (2.9) gives a contradiction.

(a) For \( b_w \in \alpha_1 \):
\[ \frac{1}{42} m_1 + \frac{1}{7} m_2 + \frac{1}{3} m_3 \leq 1 \]
\[ \frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3 < 4 \]

(b) For \( b_w \in \beta_1 \):
\[ \frac{1}{42} m_1 + \frac{1}{7} m_2 + \frac{1}{3} m_3 < \frac{1}{2} \]
\[ \frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3 < 2 \]

(c) For \( b_w \in \alpha_2 \):
\[ \frac{1}{7} m_2 + \frac{1}{3} m_3 \leq 1 \]
\[ m_2 + 2 m_3 \leq 6 + \frac{1}{7} m_2 \]

Since the left hand side is an integer, \( m_2 + 2m_3 \leq 6 \)
\[ \frac{1}{2} m_2 + m_3 \leq 3 \]
(d) For $b_w \in \beta_2$: \[
\frac{1}{7} m_2 + \frac{1}{3} m_3 < \frac{1}{2}
\]
\[
\frac{1}{2} m_2 + m_3 < 2
\]
(e) For $b_w \in \alpha_3$: \[
\frac{1}{3} m_3 < 1
\]
\[
m_3 \leq 2
\]
(f) For $b_w \in \beta_3$: \[
\frac{1}{3} m_3 < \frac{1}{2}
\]
\[
m_3 \leq 1
\]

Combining (a) - (f),
\[
\sum_{b_w \in \alpha_1} \left( \frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3 \right) + \sum_{b_w \in \beta_1} \left( \frac{4}{42} m_1 + \frac{1}{2} m_2 + m_3 \right)
\]
\[
+ \sum_{b_w \in \alpha_2} \left( \frac{1}{2} m_2 + m_3 \right) + \sum_{b_w \in \beta_2} \left( \frac{1}{2} m_2 + m_3 \right) + \sum_{b_w \in \alpha_3} m_3 + \sum_{b_w \in \beta_3} m_3
\]
\[
< 4|\alpha_1| + 3|\beta_1| + 3|\alpha_2| + 2|\beta_2| + 2|\alpha_3| + |\beta_3|
\]

This contradicts inequality (2.9). The assumption in (2.4) must be incorrect, from which we conclude that
\[
\max \left\{ \frac{A(L_1)}{\text{OPT}(L_1)}, \frac{A(L_1 L_2)}{\text{OPT}(L_1 L_2)}, \frac{A(L_1 L_2 L_3)}{\text{OPT}(L_1 L_2 L_3)}, \frac{A(L)}{\text{OPT}(L)} \right\} \geq \frac{109}{71}.
\]
III. The Main Result

Define the sequence of integers \( \{a_n\} \), for \( n \geq 1 \), by

\[
a_1 = 2 \\
a_{n+1} = 1 + \prod_{i=1}^{n} a_i
\]

(3.1)

Thus, \( \{a_n\} = \{2, 3, 7, 43, 1807, 3263443, \ldots\} \),

and notice that

\[
\sum_{i=1}^{\infty} \frac{1}{a_i} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \ldots = 1.
\]

This sequence has been studied by Golomb [10,11] and it is conjectured that the closest approximation to 1 from below, which is a sum of \( k \) reciprocal integers, is given by

\[
\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_k} = 1 - \frac{1}{a_{k+1}},
\]

for every positive integer \( k \).

In the proof of our lower bound result, we shall make use of the following simple lemma.
Lemma. Let \( \{a_k\} \) be the sequence of integers defined above in (1). Then, for \( 1 \leq k \leq j \),

\[
\frac{j+1}{a_k} \geq \frac{k}{a_{k-1}}
\]

Proof:

We first observe that

\[
a_k \geq k + 1
\]

Then

\[
(k+1)a_k - (k+1) \geq k a_k
\]

\[
\frac{k+1}{a_k} \geq \frac{k}{a_{k-1}}
\]

and so, for \( j \geq k \),

\[
\frac{j+1}{a_k} \geq \frac{k}{a_{k-1}}.
\]

Motivated by the work in Section II, we now state and prove our main result.

Theorem. For any on-line one-dimensional packing algorithm \( A \),

\[
\lim \left\{ \max_{n \to \infty} \frac{A(L)}{OPT(L)} \right\} \geq \sum_{i=1}^{t} \frac{1}{a_i - 1} \geq 1.5363
\]

Proof:

For any positive integer \( t \geq 3 \), let \( \varepsilon \) be a small fixed number,

\[
0 < \varepsilon < \frac{1}{a_t(a_t-1)(t-1)}.
\]
We define pieces $p_1^*, \ldots, p_t^*$ to be of sizes

$$p_1 = \frac{1}{a_{t-1}} - (t-1)\epsilon$$

and

$$p_j = \frac{1}{a_{t+1-j}} + \epsilon,$$

for $2 \leq j \leq t$. Consider the list $L = L_1 L_2 \ldots L_t$, where each $L_i$ consists of $n$ pieces of size $p_i$, for $n$ some multiple of $a_t - 1$. Then, for $1 \leq k \leq t$,

$$\text{OPT}(L_1 L_2 \ldots L_k) = \frac{n}{a_{t+1-k}-1}$$

and we can define the ratios

$$r_k(n) = \frac{A(L_1 L_2 \ldots L_k)}{\text{OPT}(L_1 L_2 \ldots L_k)}$$

We shall prove that

$$\max_{1 \leq k \leq t} \{r_k(n)\} \geq R_t$$

where

$$R_t = \sum_{i=1}^{t} \frac{i}{a_i - 1}$$

Let $B$ denote the set of bins packed by an algorithm $A$, after the $(t - 1)n$ pieces in list $L_1 L_2 \ldots L_{t-1}$ have been packed. Each bin $b_w \in B$ $(1 \leq w \leq |B|)$ contains $m_{i,w}$ pieces of size $p_i$, for all $1 \leq i \leq t - 1$. For
notational convenience, we shall omit the double subscript and simply write \( m_{ij} \) when we mean \( m_{i,j} \). Note that \( 0 \leq m_{ij} < a_{t+1-j} \) for \( 1 \leq j \leq t-1 \). For \( 1 \leq k \leq t-1 \), the set \( \alpha_k \) is defined to consist of those bins \( b_w \in B \) which are at least half full and in which the smallest piece has size \( p_k \). Similarly, we define \( \beta_k \) to be the set of bins \( b_w \in B \) which are less than half full and in which the smallest piece has size \( p_k \). So \( |\alpha_k| (|\beta_k|) \) represents the number of bins in \( \alpha_k \) (\( \beta_k \)), and, for \( 1 \leq k \leq t-1 \)

\[
A(L_1 L_2 \ldots L_k) = \sum_{i=1}^{k} (|\alpha_i| + |\beta_i|). \tag{3.6}
\]

Having packed \( L_1 L_2 \ldots L_{t-1} \), we note that it will not be possible to place any of the remaining \( n \) pieces of size \( p_t \) in any \( \alpha_k \) bin. So we also have

\[
A(L_1 L_2 \ldots L_t) \geq n + \sum_{i=1}^{t-1} |\alpha_i|. \tag{3.7}
\]

Let us assume that

\[
\max_{1 \leq i \leq t} \{r_i(n)\} < R_t. \tag{3.8}
\]

Making use of equations (3.2), (3.3), (3.6), and (3.7), this assumption leads to the following inequalities, for \( 1 \leq k \leq t-1 \):

\[
\frac{n}{a_{t+1-k-1}} \cdot R_t > \sum_{i=1}^{k} (|\alpha_i| + |\beta_i|),
\]

\[
n \cdot R_t > n + \sum_{i=1}^{t-1} |\alpha_i|, \tag{3.9}
\]
Because there are \( n \) pieces of each size \( p_i \), we note that

\[
n = \sum_{w \in B} m_{t-k+1}
\]

for all \( k \) in the range \( 2 \leq k \leq t \). Thus,

\[
\frac{k}{a_{k-1}} \cdot n = -\frac{k}{a_{k-1}} \sum_{w \in B} m_{t-k+1}
\]

(3.10)

Summing equations (3.9) and (3.10) over \( k \) gives

\[
nR_t \sum_{k=1}^{t-1} \frac{1}{a_{t+1-k-1}} + nR_t - n \sum_{k=2}^{t} \frac{k}{a_{k-1}} + \sum_{k=2}^{t} \frac{k}{a_{k-1}} \sum_{w \in B} m_{t-k+1}
\]

From (3.5), we observe that

\[
R_t = \frac{1 + \sum_{k=2}^{t} \frac{k}{a_{k-1}}}{1 + \sum_{k=1}^{t-1} \frac{1}{a_{t+1-k-1}}}
\]

and so inequality (3.11) can be simplified to give

\[
\sum_{k=2}^{t} \frac{k}{a_{k-1}} \sum_{w \in B} m_{t-k+1} > \sum_{k=1}^{t-1} \sum_{i=1}^{k} (|\alpha_i| + |\beta_i|) + \sum_{i=1}^{t} |\alpha_i|
\]

(3.12)

Inequality (3.12) further simplifies to give
The remainder of this proof consists of showing that (3.13) gives a contradiction. In particular, we shall show that

\[ \sum_{k=2}^{t} \frac{k}{a_{k-1}} m_{t-k+1} > \sum_{j=1}^{t-1} ((j+1) |a_{t-j}| + j |b_{t-j}|) \]  

(3.13)

for any bin \( b_w \in \alpha_{t-j} \) \((1 \leq j \leq t - 1)\) and that

\[ \sum_{k=2}^{t} \frac{k}{a_{k-1}} m_{t-k+1} \leq j + 1 \]  

(3.14)

for any bin \( b_w \in \beta_{t-j} \) \((1 \leq j \leq t - 1)\). From this we deduce that the assumption in (3.8) is incorrect, thereby proving the assertion of (3.4). The theorem follows immediately.

We first prove assertion (3.14). For \( b_w \in \alpha_{t-j} \), then

\[ p_1 m_1 + p_2 m_2 + \ldots + p_{t-1} m_{t-1} \leq 1 \]  

(3.16)

and \( p_{t-j} m_{t-j} \) is the first nonzero term. There are two cases.

(i) Assume that \( j \leq t - 2 \). Then

\[ \sum_{i=2}^{j+1} \frac{1}{a_i} m_{t-i+1} \leq 1 \]

and

\[ \frac{1}{a_{j+1}} m_{t-j} + \sum_{i=2}^{j} \frac{1}{a_i} m_{t-i+1} \leq 1 + \frac{1}{a_{j+2}} m_{t-j} \]  

(3.17)
Recalling that \( m_j < a_{t+1-j} \), then we know

\[ m_{t-j} < a_{j+1} \quad (3.18) \]

Also, as a consequence of (3.1),

\[ a_{j+2} - 1 = a_{j+1}(a_{j+1} - 1) \quad (3.19) \]

Using (3.18) and (3.19), inequality (3.17) gives

\[ \frac{1}{a_{j+1} - 1} m_{t-j} + \sum_{i=2}^{j} \frac{1}{a_i} m_{t-i+1} < 1 + \frac{1}{a_{j+1} - 1} \quad (3.20) \]

From (3.1), we note that \( a_{j+1} - 1 \) is divisible by \( a_i \), for all \( i \leq j \). Thus, the left hand side of (3.20) is a multiple of \( \frac{1}{a_{j+1} - 1} \), and we have

\[ \frac{1}{a_{j+1} - 1} m_{t-j} + \sum_{i=2}^{j} \frac{1}{a_i} m_{t-i+1} \leq 1. \]

Thus,

\[ \frac{j+1}{a_{j+1} - 1} m_{t-j} + \sum_{i=2}^{j} \frac{j+1}{a_i} m_{t-i+1} \leq j + 1. \]

Applying the Lemma,

\[ \frac{j+1}{a_{j+1} - 1} m_{t-j} + \sum_{i=2}^{j} \frac{i}{a_i - 1} m_{t-i+1} \leq j + 1 \]

and we have proved inequality (3.14) for \( j \leq t - 2 \).
(ii) Assume that \( j = t - 1 \); i.e., \( b_w \in \alpha_1 \). Since \( p_i > \frac{1}{a_{t+1-i}} \) for \( 2 \leq i \leq t - 1 \), we conclude from (3.16) that

\[
\left[ \frac{1}{a_{t-1}} - (t-1)\varepsilon \right] m_1 + \sum_{i=2}^{t-1} \frac{1}{a_i} m_{t-i+1} \leq 1.
\]

Recalling how we chose \( \varepsilon \),

\[
\frac{1}{a_{t-1}} m_1 + \sum_{i=2}^{t-1} \frac{1}{a_i} m_{t-i+1} < 1 + \frac{m_1}{t(a_t-1)} \quad (3.21)
\]

Because \( m_1 \leq a_t - 1 \), the right hand side of (3.21) is less than \( 1 + \frac{1}{a_t} \).

As in case (i), we also note that the left hand side of (3.21) is a multiple of \( \frac{1}{a_{t-1}} \) and that \( \frac{1}{a_{t-1}} > \frac{1}{a_t} \). Thus,

\[
\frac{1}{a_{t-1}} m_1 + \sum_{i=2}^{t-1} \frac{1}{a_i} m_{t-i+1} \leq 1 \quad (3.22)
\]

Similar to case (i), we multiply both sides of (3.22) by \( t \) and apply the Lemma in order to obtain the desired result:

\[
\sum_{i=2}^{t} \frac{i}{a_{i-1}} m_{t-i+1} \leq t.
\]

We now prove assertion (3.15). For \( b_w \in \beta_{t-j} \), then

\[
p_1 m_1 + p_2 m_2 + \ldots + p_{t-1} m_{t-1} < \frac{1}{2}
\]

and \( m_{t-j} \) is the first nonzero term. There are two cases.
(i) Assume that \( j \leq t - 2 \). Then

\[
\sum_{i=2}^{j+1} \frac{1}{a_i} m_{t-i+1} < \frac{1}{2} \tag{3.23}
\]

Multiplying both sides of (3.23) by \( j + 2 \) and then applying the Lemma,

\[
\sum_{i=2}^{j+1} \frac{i}{a_i-1} m_{t-i+1} < \frac{j+2}{2} \tag{3.24}
\]

For \( j \geq 2 \), \( \frac{j+2}{2} \leq j \) and the result is proved. For \( j = 1 \), (3.24) reduces to \( m_{t-1} < \frac{3}{2} \). Since \( m_{t-1} \) is an integer, this says \( m_{t-1} \leq 1 \) and once again the desired result holds.

(ii) Assume that \( j = t - 1 \); i.e., \( b \in B_1 \). Similar to inequality (3.21), we have

\[
\frac{1}{a_{t-1}} m_1 + \sum_{i=2}^{t-1} \frac{1}{a_i} m_{t-i+1} < \frac{1}{2} + \frac{1}{a_t} \tag{3.25}
\]

Multiplying both sides of (3.25) by \( t \) and applying the Lemma,

\[
\sum_{i=2}^{t} \frac{i}{a_i-1} m_{t-i+1} < \frac{t}{2} + \frac{t}{a_t}
\]

For \( t \geq 3 \),

\[
\frac{t}{a_t} < \frac{t - 2}{2}
\]

and so

\[
\sum_{i=2}^{t} \frac{i}{a_i-1} m_{t-i+1} < t - 1
\]

and the theorem is proved. \( \blacksquare \)
References


