COMPACT REPRESENTATION OF THE SEPARATING k-SETS OF A GRAPH

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**Abstract:**

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**Subject Terms:** graph theory, data structures, connectivity
Compact Representation of the Separating $k$-sets of a Graph

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ABSTRACT

We present an $O(n)$ space representation for the separating $k$-sets of an undirected $k$-connected graph $G$ for fixed $k$, where $n$ is the cardinality of the vertex set of $G$. Namely, the total space used by the representation is $O(k^2 n)$. We also improve the upper bound on the number of separating $k$-sets of $G$ to $O(2^k n^k/k)$, which has a matching lower bound.

1. Introduction

Connectivity is an important graph property and there has been a considerable amount of work on algorithms for determining connectivity of graphs [BeX, Ev2, EvTa, Ga, GiSo, LiLoWi]. An undirected graph $G = (V,E)$ is $k$-connected if for any subset $V'$ of $k-1$ vertices of $G$ the subgraph induced by $V - V'$ is connected [Ev]. A subset $V'$ of $k$ vertices is a separating $k$-set for $G$ if the subgraph induced by $V - V'$ is not connected. For $k = 1$ the set $V'$ becomes a single vertex which is called an articulation point, and for $k = 2,3$ the set $V'$ is called a separating pair and a separating triplet, respectively. Efficient algorithms are available for finding all separating $k$-sets in $k$-connected undirected graphs for $k \leq 3$ [Ta, HoTa, MiRa, KaRa].

In [KaRa2, Ka] we addressed the question of the maximum number of separating pairs, triplets and $k$-sets in biconnected, triconnected and $k$-connected undirected graphs, respectively.

An undirected graph $G$ on $n$ vertices has a trivial upper bound of $\binom{n}{k}$ on the number of separating $k$-...
sets, \( k \geq 1 \). The graph that achieves this bound for all \( k \) is a graph on \( n \) vertices without any edges. For \( k=1 \) the maximum number of articulation points in a connected graph is \( (n-2) \) and a graph that achieves it is a path on \( n \) vertices. For \( k=2 \) the maximum number of separating pairs in an undirected biconnected graph is \( \frac{n(n-3)}{2} \) and a graph that achieves it is a cycle on \( n \) vertices [KaRa2]. Further, we observed that there is an \( O(n) \) representation for the separating pairs in any biconnected graph (although the number of such pairs could be \( \Theta(n^2) \)) [KaRa2]. For \( k=3 \) the maximum number of separating triplets in a triconnected graph is \( \frac{(n-1)(n-4)}{2} \) and we presented a graph, namely the wheel [Tu], that achieves it [KaRa2]. The number of separating \( k \)-sets in a \( k \)-connected graph is \( O(3^kn^2) \) and we show that the bound is tight up to the constant [Ka]. The lower bound on the number of separating \( k \)-sets in a \( k \)-connected undirected graph is \( \Omega(2^k \frac{n^2}{k^2}) \).

In this paper we present a linear representation of separating \( k \)-sets in \( k \)-connected undirected graphs. For \( k=2 \) representation is different from the one presented in [KaRa2]. We also give the alternative prove of the upper bound on the number of separating \( k \)-sets, which match the previous upper bounds for \( k=2 \) and \( k=3 \), and improves the upper bound for general \( k \) to \( O(2^k \frac{n^2}{k}) \). We will first present representation for \( k=2 \) and \( k=3 \) and then generalized the technique for general \( k \).

2. Graph-theoretic definitions

An undirected graph \( G=(V,E) \) consists of a vertex set \( V \) and an edge set \( E \) containing unordered pairs of distinct elements from \( V \). A path \( P \) in \( G \) is a sequence of vertices \(<v_0, \ldots, v_k>\) such that \((v_{i-1}, v_i) \in E, i=1, \ldots, k \). The path \( P \) contains the vertices \( v_0, \ldots, v_k \) and the edges \((v_0,v_1), \ldots, (v_{k-1},v_k) \) and has endpoints \( v_0, v_k \), and internal vertices \( v_1, \ldots, v_{k-1} \).

We will sometimes specify a graph \( G \) structurally without explicitly defining its vertex and edge sets. In such cases, \( V(G) \) will denote the vertex set of \( G \) and \( E(G) \) will denote the edge set of \( G \). Also, if \( V' \subseteq V \) and \( v \in V \) we will use the notation \( V' \cup v \) to represent \( V' \cup \{v\} \).

An undirected graph \( G=(V,E) \) is connected if there exists a path between every pair of vertices in \( V \). For a graph \( G \) that is not connected, a connected component of \( G \) is an induced subgraph of \( G \) which is maximally connected.
A vertex \( v \in V \) is an articulation point of a connected undirected graph \( G = (V, E) \) if the subgraph induced by \( V - \{v\} \) is not connected. \( G \) is biconnected if it contains no articulation point.

Let \( G = (V, E) \) be a biconnected undirected graph. A pair of vertices \( v_1, v_2 \in V \) is a separating pair for \( G \) if the induced subgraph on \( V - \{v_1, v_2\} \) is not connected. \( G \) is triconnected if it contains no separating pair.

A triplet \( (v_1, v_2, v_3) \) of distinct vertices in \( V \) is a separating triplet of a triconnected graph if the subgraph induced by \( V - \{v_1, v_2, v_3\} \) is not connected. \( G \) is four-connected if it contains no separating triplets.

Let \( G = (V, E) \) be an undirected graph and let \( V' \subseteq V \). A graph \( G' = (V', E') \) is a subgraph of \( G \) if \( E' \subseteq E \cap \{(v_i, v_j) | v_i, v_j \in V'\} \). The subgraph of \( G \) induced by \( V' \) is the graph \( G'' = (V', E'') \) where \( E'' = E \cap \{(v_i, v_j) | v_i, v_j \in V'\} \).

3. Representation for \( k=2 \)

Let \( G = (V, E) \) be an undirected biconnected graph with \( n \) vertices and \( m \) edges. We denote with \( g(n) \) the upper bound on the size of a compact representation of separating pairs of a graph on \( n \) vertices. Let \( \{v_1, v_2\} \) be a separating pair that divides \( G \) into nonempty \( G_1 \) and \( G_2 \). Let \( \{w_1, w_2\} \) be a "cross" separating pair with \( w_1 \in G_1 \) and \( w_2 \in G_2 \). It divides \( G_1 \) into \( G'_1 \) and \( G''_1 \), and divides \( G_2 \) into \( G'_2 \) and \( G''_2 \) (see Figure 1).

![Figure 1. Representation for \( k=2 \).](image)

Consider a maximal set of vertices \( u \) in \( G_2 \) such that \( \{w_1, u\} \) is a cross separating pair and, analogously, consider a
maximal set of vertices \( x \) in \( G_1 \) such that \( \{x, w_2\} \) is a cross separating pair. The set of \( u \)'s is the set of articulation points in \( G_2 \). Moreover, the set of \( u \)'s along with the subgraphs of \( G_2 \) between them is a path from \( v_1 \) to \( v_2 \).

Analogously, the set \( x \)'s is a set of articulation points of \( G_1 \) with additional condition that the \( x \)'s along with the subgraphs of \( G_1 \) between them is a path from \( v_1 \) to \( v_2 \). Number the vertices \( v_1, u \)'s, \( v_2, x \)'s by \( y_1, y_2 \) and so on going clockwise along the paths. We denote by \( G_i \) the subgraph of \( G \) between \( y_i \) and \( y_{i+1} \). Note that some \( G_i \) can be empty (consists of a single edge). Thus, the graph \( G \) becomes a cycle with vertices \( y \)'s and \( G_i \)'s alternating on it.

Every pair of vertices \( y \)'s give a separating pair of \( G \) unless they are adjacent and the subgraph between them is empty. Hence, we can represent all of them by the following structure:

1) the cycle: the set of vertices \( y \)'s

2) a vertex for every \( G_i \) with a flag to specify if \( G_i \) is empty. Edges between \( G_i \) and \( y_i, y_{i+1} \).

Note that when there are no cross separating pairs then we get a trivial cycle with two vertices \( v_1 \) and \( v_2 \) and two edges connecting them. Since the sets \( x \)'s and \( u \)'s are maximal all other separating pairs are inside \( G_i \cup y_i \cup y_{i+1} \).

Note that \( G_i \) can be the union of disconnected components, but each of them is connected to \( y_i \) and \( y_{i+1} \). Let the cardinality of set of vertices \( y \)'s be \( l \). Based upon the above observations we get the following recurrence relation

\[
g(n) \leq \sum_{i=1}^{l} (n_i + 2) + 4l,
\]

where \( g(n_i + 2) \) represent the upper bound for all separating pairs inside \( G_i \cup y_i \cup y_{i+1} \). The cardinality of \( G_i = n_i \), and \( \sum_{i=1}^{l} (n_i + 1) = n \). Any \( g(n) \) that satisfy the recurrence will be an upper bound on the size of representation of separating pairs of \( G \). Clearly, linear \( g(n) \) is one of them (see Appendix).

### 4. Representation for \( k=3 \)

The wheel \( W_n \) [Tu] is \( C_{n-1} \) together with a vertex \( v \) and an edge between \( v \) and every vertex on \( C_{n-1} \). It is easy to see that \( W_n \) is triconnected and has \( \frac{(n-1)(n-4)}{2} \) separating triplets.

Assume there exists a separating triplet \( \{v_1, v_2, v_3\} \) in \( G \), which separates \( G \) into nonempty \( G_1 \) and \( G_2 \) (see Figure 2).

**Lemma 1:** Only one of these three vertices has type 3 separating triplets \( \{w_1, v_i, w_2\} \) such that \( w_1 \in G_1 \) and \( w_2 \in G_2 \) [KaRa2].
Figure 2. Separating G into $G_1$ and $G_2$ by separating triplet $\{v_1, v_2, v_3\}$

Proof: Assume there is separating triplet $\{w_1, v_2, w_2\}$ of the third type in $G$, where $w_1 \in G_1$ and $w_2 \in G_2$. It separates $G_1$ into $K_1$ and $K_2$, and separates $G_2$ into $K_3$ and $K_4$. Vertices $v_1$ and $v_3$ must belong to the different components with respect to separating triplet $\{w_1, v_2, w_2\}$, otherwise either $\{w_1, v_2\}$ is a separating pair, or $\{w_2, v_2\}$ is a separating pair, or both.

Claim 1 Vertex $v_2$ has a direct edge to every nonempty subgraph $K_1, K_2, K_3, K_4$.

W.L.O.G. assume that $K_1$ is not empty and $\forall x \in K_1, (x, v_2) \in E$. Then $\{v_1, w_1\}$ is a separating pair of $G$, which separates $K_1$ from the rest of the graph.

Now, we will prove that there are no separating triplets of the third type which use $v_1$ or $v_3$. We will prove this by contradiction. W.L.O.G. assume there is a separating triplet $\{u_1, v_1, u_2\}$, where $u_1 \in G_1$ and $u_2 \in G_2$ ($u_1$ may be equal to $w_1$ and $u_2$ may be equal to $w_2$).

Case 1: $u_1 \in K_2$, if $K_2$ is not empty (see Figure 3).

By Claim 1 for $v_1$ and the existence of separating triplet $\{u_1, v_1, u_2\}$, $K_1$, $w_1$, $K_2 - u_1$ belong to the same connected component with respect to separating triplet $\{u_1, v_1, u_2\}$. If $v_2$ belongs to the same component then $\{v_1, u_1\}$ is a separating pair which separates $K_3 \cup w_2 \cup K_4 \cup v_3$ from the rest of the graph. If $v_2$ does not belong to the same component then $\{v_1, u_1\}$ is a separating pair which separates $K_1 \cup w_1 \cup K_2 - u_1$ from the rest of the graph.

Analogously, $u_2 \not\in K_4$.

Case 2: $u_1 = w_1$. 


Illustrating Case 1 in the proof of Lemma 1.

Since \( \{u_1, v_1, u_2\} \) is a separating triplet then \( v_2 \) does not have any edges to \( K_1 \) and hence, \( K_1 \) is empty by Claim 1. But then \( \{v_1, u_2\} \) is a separating pair, if \( \{u_1, v_1, u_2\} \) is a separating triplet.

Analogously, \( u_2 \neq w_2 \).

**Case 3:** \( u_1 \in K_1 \) and \( u_2 \in K_3 \).

If \( \{u_1, v_1, u_2\} \) is a separating triplet then either \( \{u_1, u_2\} \), or \( \{u_1, v_1\} \), or \( \{v_1, u_2\} \) is a separating pair.

That means that if there is a separating triplet of the third type which uses one of the \( v_i, i=1,2,3 \) then there are no separating triplets of the third type that use the other \( v_j, j=1,2,3, j\neq i \).

Let \( \{v_1, v_0, v_2\} \) be a separating triplet of a graph \( G \) on \( n \) vertices, and \( v_0 \) be the only one of the three vertices of this separating triplet which might participate in a separating triplets of the third type with respect to \( \{v_1, v_0, v_2\} \). Consider all separating triplets of the third type \( \{w_1, v_0, w_2\} \) such that \( w_1 \in G_1 \) and \( w_2 \in G_2 \), together with \( \{v_1, v_0, v_2\} \). All such separating triplets use \( v_0 \) as the "central" vertex. Rename the vertices \( w_1 \)'s, \( w_2 \)'s, \( v_1 \) and \( v_2 \) into \( \{v_1, v_2, \cdots, v_l\} \) going clockwise, such that they form the wheel with \( v_0 \) in a center, where any two nonadjacent vertices form a separating triplet with \( v_0 \). The subgraphs between \( v_i \) and \( v_{i+1} \) are denoted with \( G_i \), and some of them may be empty. Now, the graph \( G \) looks like a wheel with \( v_0 \) in a center \( v_i \), and \( G_i \) \((i=1, \cdots, l)\) on a cycle.

Every pair of vertices on the cycle of the wheel form a separating triplet with \( v_0 \) unless they are adjacent \( (v_i \) and \( v_{i+1}\)) and the subgraph \( (G_i) \) between them is empty. Hence, we can represent these separating triplets by the following structure:
1) the wheel: \( \{v_0, v_1, \ldots, v_k\} \) with edges of \( G \)

2) a vertex for every \( G_i \) with a flag to specify if \( G_i \) is empty. The edges between \( G_i \) and \( v_i, v_{i+1} \) and between \( v_0 \) and \( v_i, G_i \) with flags to specify if the edge is real.

Let us see where the rest of separating triplets of \( G \) lie.

Observation The remaining separating triplets belong to \( G_i \cup v_0 \cup v_i \cup v_{i+1} \cup \) the neighbor of \( v_i \) in \( G_{i-1} \) if such a neighbor is unique \( \cup \) the neighbor of \( v_{i+1} \) in \( G_{i+1} \) if such a neighbor is unique.

Let \( \{w_1, w_2, w_3\} \) be a separating triplet with \( w_1 \in G_1 \) and \( w_2, w_3 \in G_2 \). The separating triplet \( \{w_1, w_2, w_3\} \) separates \( G_1 \) into \( L_1 \) and \( L_2 \), and separates \( G_2 \) into \( L_3 \) and \( L_4 \) (Figure 4).

Let us see how the original separating triplet \( \{v_1, v_2, v_3\} \) is separated by the separating triplet \( \{w_1, w_2, w_3\} \).

The vertices \( \{v_1, v_2, v_3\} \) cannot belong to the same connected component of \( G \) with respect to the separating triplet \( \{w_1, w_2, w_3\} \), otherwise either \( w_1 \) would be an articulation point, or \( \{w_2, w_3\} \) would be a separating pair, or both. W.L.O.G. assume that \( v_1 \) belongs to one connected component and \( v_2, v_3 \) to the other.

Subgraph \( L_1 \) must be empty, otherwise \( \{w_1, v_1\} \) becomes a separating pair. Since the graph is triconnected, we have
1) \((w_1, v_1) \in E\),
2) \(\exists x, y \in L_3 \cup w_2 \cup w_3: (x, v_1) \in E, (y, v_1) \in E\) and
3) \(\forall z \in L_2 \cup L_4 \cup v_2 \cup v_3: (z, v_1) \notin E\).

Hence, vertex \(w_1\) is the unique neighbor of vertex \(v_1\) in \(G_1\). Moreover, if there are any separating triplets with one vertex in \(G_1\) and two in \(G_2\) which separate \(v_1\) from \(v_0\) and \(v_2\), then \(w_1\) is one of the vertices of the triplet.

A separating triplet cannot have all its three vertices in three different \(G_i\)'s otherwise two of these vertices would form a separating pair. From the proof of the Lemma 1 and the fact that the set \(\{v_1, v_2, \cdots, v_k\}\) is maximal, we know that if there is a separating triplet which involves a vertex from \(G_i\), then the other two vertices belong to \(\{v_i\} \cup \{v_{i+1}\} \cup \{v_0\} \cup G_i\) and the neighbor of \(v_i\) in \(G_{i-1}\), if such a neighbor is unique, and symmetrically a 'unique' neighbor of \(v_{i+1}\) in \(G_{i+2}\). This proves the Observation.

Let \(g(n)\) be the size of a compact representation of the separating triplets in a graph on \(n\) vertices, and let the number of vertices in \(G_i\) be \(n_i\). Then \(\sum_{i=1}^{k} (n_i + 1) + 1 = n\), and we can write the following recurrence relation

\[
g(n) = \sum_{i=1}^{l} g(n_i + 5) + (6l + 1),
\]

where \((6l + 1)\) stands for the space used to store the wheel information including multiple edges. The solution to this recurrence is clearly linear (see Appendix). This proves that there is a succinct \(O(n)\) size representation of the separating triplets.

5. Representation for general \(k\)

Let \(G=(V,E)\) be an undirected \(k\)-connected graph with \(n\) vertices and \(m\) edges. We denote with \(g(n)\) and \(f(n)\) the upper bounds on the size of representation and the number of separating \(k\)-sets for \(k\)-connected graph on \(n\) vertices. Let \(V' = \{v_1, v_2, \cdots, v_k\}\) be a separating \(k\)-set, whose removal separates \(G\) into nonempty \(G_1\) and \(G_2\) (see Figure 5). A separating \(k\)-set \(\{w_1, w_2, \cdots, w_k\}\) of \(G\) is a cross separating \(k\)-set with respect to \(V'\) if \(\exists i, j: w_i \in G_1\) and \(w_j \in G_2\). Let the cardinalities of \(G_1\) and \(G_2\) be \(l\) and \(n-l-k\), respectively. Let the upper bound on the size of the representation of the cross separating \(k\)-sets be \(h(l, n-l)\), and the maximum number of cross separating \(k\)-sets be \(r(l, n-l)\). Then any \(g(n)\) and \(f(n)\) that satisfy the recurrences
Figure 5.
Dividing $G$ into $G_1$ and $G_2$ by separating $k$-set $\{v_1, \cdots, v_k\}$

$$g(n) = \left[ g(l+k) + g(n-l) + h(l, n-l) \right],$$

$$f(n) = \left[ f(l+k) + f(n-l) + r(l, n-l) + \right],$$

are upper bounds on the size of representation and the number of separating $k$-sets in $G$. Now we will derive upper bounds for the functions $h$ and $r$ and tune up the recurrences.

Let $\{w_1, w_2, \cdots, w_k\}$ be a cross separating $k$-set with $\{w_1, \cdots, w_k\} \subset G_1$, $\{w_{r+k+1}, \cdots, w_k\} \subset G_2$ and $\{w_{r+1}, \cdots, w_{r+k}\} \subset \{v_1, \cdots, v_r\}$. The separating $k$-set $\{w_1, w_2, \cdots, w_k\}$ separates $G_1$ into $G_3$ and $G_4$, separates $G_2$ into $G_5$ and $G_6$, and divides $\{v_1, \cdots, v_r\}$ into $\{v_1, \cdots, v_r\}$, $\{v_{r+k+1}, \cdots, v_k\}$ and $v_{r+i} = w_{s+i}$, $i = 1, \cdots, l$. (see Figure 6)

Case 1 None of $G_i$, $i = 3, 4, 5, 6$ are empty. (see Figure 6)

The sets $\{w_1, w_2, \cdots, w_{r+k}, v_1, \cdots, v_r\}$, $\{w_1, w_2, \cdots, w_{r+k}, v_{r+1}, \cdots, v_k\}$, $\{v_1, \cdots, v_{r+k}, w_{r+1}, \cdots, w_k\}$ and $\{v_1, \cdots, v_k, w_{r+k}, \cdots, w_{r+k}\}$ are separating sets of $G$ that separate $G_3$, $G_4$, $G_5$ and $G_6$ respectively, so their cardinalities are greater than or equal to $k$. Then,

$$\begin{cases}
s + t + r \geq k \\
r + t + k - s - t \geq k \\
s + t + k - r - t \geq k \\
k - r + k - s - t \geq k
\end{cases}
\Rightarrow
\begin{cases}
r + s + t \geq k \\
r \geq s \\
s \geq r \\
k \geq r + s + t
\end{cases}
\Rightarrow
\begin{cases}
r = s \\
r + s + t = k
\end{cases}

From now on we replace the subscript $r$ by $s$. Let $A = \{v_1, \cdots, v_r\}$, $B = \{v_{r+s+1}, \cdots, v_k\}$, $C = \{w_1, \cdots, w_r\}$, $D = \{w_{r+s+1}, \cdots, w_k\}$, and $T = \{v_{r+1}, \cdots, v_{r+s}\} = \{w_{s+1}, \cdots, w_{r+s}\}$. For Case 1

$|A| = |B| = |C| = |D| = \frac{k-t}{2}$. 
Figure 6.
Dividing $G$ into nonempty components by separating $k$-sets
$\{v_1, \cdots, v_k\}$ and $\{w_1, \cdots, w_k\}$.

Claim 2 \( \forall \ i = s+1, \ldots, t \ \exists \ x_j \in G_j, \ j = 3,4,5,6 \colon (v_i,x_j) \in E. \)

Proof: W.L.O.G. assume \( \exists v_i; \forall x \in G_3 \colon (x,v_i) \in E. \) Then $\{v_1, \cdots, v_{s+t}, w_1, \cdots, w_s\} - \{v_i\}$ is a separating (k-1)-set.

Claim 3 For every $x \in A$ there are $y \in G_3$ and $z \in G_5$, such that $(x,y) \in E$ and $(x,z) \in E$. Analogously, for every vertex $x$ of $B$, $C$ and $D$ there are vertices $y$ and $z$ in appropriate neighboring $G_i$, $i=3,4,5,6$, which are adjacent to $x$.

Proof: W.L.O.G. assume there is $x \in A$ such that for every $y \in G_3$ $(x,y) \in E$. Then $A \cup C \cup \{x\}$ is a separating (k-1)-set.

Lemma 2 All cross separating $k$-sets containing $C \cup T$ and at least one fixed vertex of $D$ can be represented in $O((\frac{k-t}{2})^2)$ space, and their number is $O(2^{\frac{k-t}{2}})$.

Proof: Assume we have a separating $k$-set $\{w_1, \cdots, w_{s+t+a}, x_{s+t+a+1}, \cdots, x_{s+t+a+b}, y_{s+t+a+b+1}, \cdots, y_k\}$, where $x_s \in G_5$, $y_s \in G_6$, $a \geq 1$, and either $b$ or $k-s-t-a-b$ is greater or equal to 1 (the new cross separating $k$-set is different from the old one) (see Figure 7).

Let $H = \{x_{s+t+a+1}, \cdots, x_{s+t+a+b}\}$ ($x'$s) and $I = \{y_{s+t+a+b+1}, \cdots, y_k\}$ ($y'$s), and let $D$ be divided into $D' = \{w_{s+t+1}, \cdots, w_{s+t+a}\}$, $E$ which is in the same connected component as $G_3$, $A$, and part of $G_5$, and $F$ which is in the
same connected component as $G_4$, $B$ and part of $G_6$. Also let $H$ divide $G_5$ into $G_5'$ and $G_5''$, and let $I$ divide $G_6$ into $G_6'$ and $G_6''$ (see Figure 7).

Separating sets $T+D'+E+H$ and $T+D'+F+I$ separate $G_5'$ and $G_6''$, respectively. The cardinalities of these separating sets are less than $k$. Hence, $G_5'$ and $G_6''$ are empty. Moreover, since $C+T+D'+H+F$ and $C+T+D'E+I$ are separating sets and $C+T+D$ and $C+T+D'+H+I$ are separating $k$-sets, $|E|=|H|$, and $|I|=|F|$. Note that the argument still holds if either $H$ or $I$ are empty.

Next, we will show that if we replace part of $E$ and/or part of $F$ we will necessarily use only vertices of $H$ and/or $I$ for it, regardless of whether we replace part of $D'$ or not. In other words, $H$ and $I$ are unique for $E$ and $F$.

The proof is by contradiction.

Assume that there exist $I_1+H_1 \neq I+H$, such that $C+T+D'+H_1+I_1$ is a separating $k$-set. Let $H_1 \subseteq G_5$ and $I_1 \subseteq G_6$. Also, let $I_1+H_1$ divide $E$ into $E_1$ and $E_2$, and divide $F$ into $F_1$ and $F_2$ (see Figure 8).

Let $H_1$ be separated into two parts, $H_1'$ adjacent to $E$ and $E_1''$ adjacent to $F$. By the above arguments $H_1'$ is adjacent to $E_1$, $H_1''$ is adjacent to $F_2$, and $I_1$ is adjacent to $E_2+F_1$. Since all neighbors of $E$ in $G_6$ are also in $I$, and all neighbors of $F$ in $G_5$ are also in $H$, $H_1'' \subseteq H$ and $I_1$ is divided into $I_1'=I \cup I_1$ and $I_1''=I_1-I_1'$. Let $H''=H-H_1''$ and let $I'=I-I_1'$.
Figure 8.
Illustrating the uniqueness of a replacement for a part of cross separating \( k \)-set.

The separating set \( T + D' + H' + H \) separates \( E_1 \) from the rest of the graph and has cardinality is less than \( k \). Hence, \( E_1 \) is empty and we have \( I = I' \), \( E = E_2 \) and \( H_1 = H''_1 \). Analogously, the separating set \( T + D' + I_1 + H \) separates \( F_1 \) from the rest of the graph and has cardinality is less than \( k \). Hence, \( F_1 \) is empty and we have \( F = F_2 \), \( E = E_1 \), \( H = H_1 \) and \( I = I_1 \). This contradict the assumptions.

Note that the arguments still hold if either \( H \) or \( I \) are empty, or if we replace only parts of \( E \) and \( F \). If part of \( D' \) is replaced as well, then we will not replace it, so that we will look only at the replacements for \( E \) and \( F \). Also, if there exists a separating \( k \)-set that replaces \( F \) by \( H \), then there is no \( I_1 \subseteq G_6 \) that replaces any part of \( F \) for any cross separating \( k \)-set described in Lemma 2.

Thus, any replacement of any part of \( F \) for any cross separating \( k \)-set specified by Lemma 2 lies in \( H \). The set of vertices which is used for all possible replacement of any part of \( D \) for a cross separating \( k \)-sets specified by Lemma 2 will be called the fringe of \( D \), where \( H \) is the fringe of \( F \) and \( I \) is the fringe of \( E \). Note that there could be parts of \( D \) which do not have any replacements. The cardinality of the fringe of \( D \) is less than \( \frac{k-t}{2} = |D| \). Hence, the representation of all cross separating \( k \)-sets with \( C + T \) fixed along with at least one vertex from \( D \) takes \( O((\frac{k-t}{2})^2) \) space, where \( O((\frac{k-t}{2})^2) \) space is needed to specify all edges between \( D \) and its fringe. This proves the space complexity for the representation.
The number of different subsets of $D$ is $2^{|D|}$. Since for every subset $E+F$ of $D$ there is a unique replacement, (if it exists) that a separating $k$-set specified by Lemma 2, the number of separating $k$-sets with $C+T$ fixed along with at least one vertex from $D$ is upper bounded by $O \left( \frac{k-t}{2} \right)$. This proves the second part of the Lemma.

\[ \square \]

Corollary All cross separating $k$-sets containing $T+D$ and at least one vertex from $C$ can be represented in $O \left( \frac{k-t}{2} \right)^2$ space, and their number is $O \left( \frac{k-t}{2} \right)$.

Take the maximal set $X$ of disjoint $C\in G_1$ such that $C+T+D$ is a separating $k$-set. Analogously, take the maximal set $Y$ of disjoint $D\in G_2$ such that $C+T+D_i$ is a separating $k$-set. For $T$ fixed, all cross separating $k$-sets are upper bounded by $O \left( \frac{k-t}{2} |X| \frac{k-t}{2} |Y| \right) = O \left( 2^{k-t} |X| |Y| \right)$, and are represented in $O \left( \left( \frac{k-t}{2} \right)^2 (|X| + |Y|) \right)$ space. Next we will see how many different $T$'s we need to consider.

Take the smallest $T = T_1$ such that a cross separating $k$-set will have nonempty $G_i$, $i=3,4,5,6$, if it exist. If there exist a separating $k$-set with different $T = T_2$, $T_1 \neq T_2$, then it can be of four different types:

Type 1). $T_2 \cap A \neq \emptyset$ and $T_2 \cap B \neq \emptyset$,

Type 2). $\left[ T_2 \cap A = \emptyset \text{ or } T_2 \cap B = \emptyset \right]$ and $T_1 \cap T_2 \neq \emptyset$,

Type 3). $\left[ T_2 \cap A = \emptyset \text{ or } T_2 \cap B = \emptyset \right]$ and $T_1 \cap T_2 = \emptyset$,

Type 4). $T_2 \cap A = \emptyset$ and $T_2 \cap B = \emptyset$.

Let us first consider type 4 cross separating $k$-sets. Since $T_2$ must lie completely inside $T_1$ and $T_1$ has the smallest cardinality, then $T_2 = T_1$. Let the cardinality of $X$, the maximal disjoint set of $C$'s, be $l_1$, and let the cardinality of the maximal disjoint set $Y$ be $l_2$, where $l_1 + l_2 = l$. Let us number $A$, the set $X$, $B$ and the set $Y$. So $A$ becomes $A_1$, the "nearest" $D$ from $Y$ becomes $A_2$, and so on going clockwise. The cardinality of this set is $l + 2$.

From the proof of the Lemma 2 we know that all cross separating $k$-sets of type 4 consist of three parts: $T_1, C$ which is inside $G_1$ and is inside some $C$'s from set $X$ and its fringe, and $D$ which is inside $G_2$ and is inside some $D$'s from set $Y$ and its fringe. Note that $T \cup$ any two $A_i, i=1, \cdots, l+2$ are also separating $k$-sets if the parts of the graph between them are nonempty. We can also replace parts of $A_i$ by its fringe as long the above condition will be true.

Let the part of the graph $G$ between $A_i$ and $A_{i+1}, i=1, \cdots, l+2$ be $G_i, i=1, \cdots, l+2$ (i in this case taken mod $l+2$).

Let $G_i$ - the fringe of $A_i$ in $G_i$ - the fringe of $A_{i+1}$ in $G_i$ be $G_i, i=1, \cdots, l+2$. The only case when $T \cup A_i \cup A_j$ (or
parts of the fringe of $A_i$ and $A_{i+1}$) $i < j$ is not a separating $k$-set when $i = j-1$ and $G'_i = \emptyset$.

Based upon above observations the structure (structure 1) which covers all cross separating $k$-sets of type 4 will be the following:

1) $A_i$ with its fringes for all $i=1, \cdots, l+2$,

2) For every nonempty $G'_i, i=1, \cdots, l+2$ we fill all nonexistent edges of the complete graph on the neighbors of $G'_i$ as real edges. If $G'_i, i=1, \cdots, l+2$ is empty for some $i$ then we fill these edges as virtual edges. All of the edges of $G$ between $A_i$ and $G_{i+1}, i=1, \cdots, l+2$ are in the structure as real edges.

Let us see where the rest of the separating $k$-sets lie assuming there are no cross separating $k$-sets of type 1 and type 2. Note that we allow separating $k$-sets of type 3. Let us first the definition of the exceptional separating $k$-sets. The separating $k$-set is *exceptional* if it separates only part of $A_i$ and nothing else for $i=1, \cdots, l+2$.

**Lemma 3:** All separating $k$-sets which are not covered by the structure 2 and not of type 1 and 2 and not exceptions are inside $G_i \cup A_i$ and its fringes inside $G_{i-1} \cup A_{i+1}$ and its fringes inside $G_{i+1}$.

**Proof:** Since there are no type 1 and type 2 and no exceptions in separating $k$-sets, no separating $k$-set is using $T$.

There are also no cross separating $k$-set which are not covered by the structure 1. Let us see what happens if a separating $k$-set crosses some $A_i, i=1, \cdots, l+2$. (see Figure 9).

W.L.O.G. let $E \cup F \cup H$ is this separating $k$-set, which crosses $A_i$, where $E \subset G_5$, $F \subset G_6$ and $H \subset A_i$. It divides $A_i$ into $A'_i$, $A''_i$, and $H$. It also divides $G_5$ into $G'_i$ and $G''_i$, and it divides $G_6$ into $G'_6$ and $G''_6$. Both $A'_i$ and $A''_i$ are nonempty, otherwise the set $Y$ is not maximal, or there is no cross separating $k$-sets. If $G''_5$ and $G''_6$ are nonempty then $E \cup H \cup A''_i$ and $F \cup H \cup A''_6$ are separating sets with cardinalities bigger or equal to $k$. But both of them can not have cardinality bigger or equal to $k$, hence, one of $G''_5$ or $G''_6$ must be empty. W.L.O.G. let $G''_6$ be empty. Since $A_{i+1} \cup H \cup A_i$ and $A_{i+1} \cup H \cup A_i \cup F$ are separating $k$-set and separating set, respectively, $|F| \geq |A''_i 1|$. Since $E \cup H \cup A''_i$ is a separating set, since both $G''_5$ and $G''_6$ can not be empty (exception), $|A''_i 1| \geq |F|$. Hence, $A''_i | = |F|$, and $F$ is part of the fringe of $A_i$.

Let us see if a cross separating $k$-set crosses two adjacent $A_i$'s. W.L.O.G. $E \cup H \cup F \cup H \cup I$ is a separating $k$-set, which divides $A_i$ into $A'_i$, $H_1$, and $A''_i$, and divides $A_{i+1}$ into $A'_{i+1}$, $H_2$, and $A''_{i+1}$. It separates $G_{i-1}$ into $G'_{i-1}$ and $G''_{i-1}$, it separates $G_i$ into $G'_i$ and $G''_i$, it separates $G_{i+1}$ into $G'_{i+1}$ and $G''_{i+1}$. By the above argument,
G'_{i-1} and G'_{i+1} are empty, and E belongs to the fringe of A_i, and I belongs to the fringe of A_{i+1}. Note that we don't need to use the assumption that there are no exceptions. A cross separating k-set cannot cross three adjacent A_i's, since with respect to the middle A_i non of G'_{i-1} and G'_{i+1} can be empty. Hence, all other separating k-set, except exceptions, belong to G_i ∪ A_i ∪ its fringes in G_{i-1} ∪ A_{i+1} u its fringes in G_{i+1}.

Let us now consider exceptions. W.L.O.G. let there exist an exceptional separating k-set, which separates part of A_i. In other words, there is a separating k-set which separates part of A_i (A'_i), such that all of the vertices not in A_i ∪ T are neighbors of A'_i. The number of the neighbors of A'_i in G_{i-1} ∪ A_{i-1} ∪ G_i ∪ A_{i+1} is less than k. Consider the minimal set of subsets of A_i that covers all vertices of A_i which can be separated by some exceptional separating k-set. The number of subsets in this set is less than or equal to the cardinality of A_i, whence is at most \( \frac{k-t}{2} \). The number of neighbors of A_i that are used for separating these subsets is less than or equal to k vertices per subsets, so their total is at most \( \frac{k^2}{2} \). Note that \( \frac{k^2}{2} - k \) such vertices can be inside either G_{i-1} ∪ A_{i-1} or G_i ∪ A_{i+1}. Moreover, if \( v \in A_i \) participates in some subset of A_i, that can be separated by an exceptional separating k-set, then v has less than k vertices in G_{i-1} ∪ A_{i-1} ∪ G_i ∪ A_{i+1}. Hence, if we take the union of the following sets

Figure 9.
Illustrating the proof of Lemma 3.
1) $G_i \cup A_i \cup A_{i+1}$  
2) the neighbors of $A_i$ in $G_{i-1} \cup A_{i-1}$, that are used for exceptional separating $k$-sets  
3) the fringe of $A_i$  
4) the neighbors of $A_{i+1}$ in $G_{i+1} \cup A_{i+2}$, that are used for exceptional separating $k$-sets  
5) the fringe of $A_{i+1}$ for all $i$'s,  
will contain all separating $k$-sets which are not covered by the structure.

The number of exceptional separating $k$-set for $A_i$ is bounded by the number of different subsets of $A_i$. Hence, it is less than or equal to $2^{k-\frac{i}{2}}$. Thus, the number of exceptional separating $k$-sets is at most $(l+2)^{2^{k-\frac{i}{2}}}$.  

Based upon this Lemma and the above observation about exceptions, and using structure 1, we can write the following recurrence, which is valid if there are no type 1 or type 2 separating $k$-sets:  

$$s(n) = \sum_{i=1}^{l+2} s(n_i+k(k-t)+t) + (l+2)(\frac{k-t}{2})k + t ,$$

where every term inside the sum covers one of the $G_i$'s, and $(l+2)(\frac{k-t}{2})+t$ is the upper bound on the size of the structure 1. Note that $\sum_{i=1}^{l+2} (n_i + (l+2)(k-t)+t = n$. The solution to this recurrence is $O(nk + k^3)$ (see Appendix). Note that each $(n_i + k(k-t)+t)$ is less than $n$ itself.

Analogously, the recurrence on the upper bound on the number of separating $k$-sets become  

$$f(n) = \sum_{i=1}^{l+2} f(n_i+k(k-t)+t) + 2^{k-t}l \frac{1+2}{2} + 2^{k-\frac{i}{2}} (l+2).$$

The solution to this recurrence is $O(2^k n^2_k)$. Note that all cross separating $k$-set of type 3 are covered by these recurrences.

Now we will look at type 1. Let $T_2 \cap A = T_2$, $T_2 \cap B = T_2'$, and $T_1 \cap T_2 = T_2'$. With respect to a new cross separating $k$-set which uses $T_2$ some $G_i$, $i=3,4,5,6$ could be empty. Let us first look at a harder case when none of $G_i$, $i=3,4,5,6$ are empty with respect to a new cross separating $k$-set.  

A new cross separating $k$-set must cross $C$ and $D$ of the old cross separating $k$-set which uses $T_1$, otherwise the Claim 2 with respect to the new cross separating $k$-set will be violated (see Figure 10).  

Second, $T_2 = T_1$, otherwise Claim 2 will be contradicted for the old cross separating $k$-set.
Third, $C'_1 + C'_2 + H_1 + T_1 + T''_2$, $C''_1 + C''_2 + H_1 + T_1 + T''_2$, $D'_1 + D'_2 + H_2 + T_1 + T''_2$, and $D''_1 + D''_2 + H_2 + T_1 + T''_2$ are separating sets with cardinalities less than $k$, which separate $G''^4$, $G''^3$, $G''^6$, and $G''^5$, respectively. Hence, $G''^3$, $G''^4$, $G''^5$, and $G''^6$ are empty.

Fourth, $C'_1 + H_1 + C''_2 + T_2 + D'_2 + H_2 + D''_2$, $C''_2 + H_1 + C''_2 + T_2 + D'_2 + H_2 + D''_2$, and $C''_2 + H_1 + T_2 + D'_1 + H_2 + D''_2$ are separating sets. Hence, $|C'_1| \geq |C'_2|$, $|D'_1| \geq |D'_2|$, $|C''_1| \geq |C''_2|$, and $|D''_1| \geq |D''_2|$. Also, $C'_1 + H_1 + C''_2 + T_2 + T_1 + D'_1 + H_2 + D''_1$, $C''_1 + H_1 + H_2 + D'_1 + H_2 + D''_1$, and $C'_1 + H_1 + C''_1 + T_1 + T''_2 + D'_2 + H_2 + D''_1$ are separating sets. Hence,

$$
\begin{align*}
|C'_1| + |C''_2| & \geq |C'_1| \geq |C'_2| > 0 \\
|C''_1| + |T'_2| & \geq |C''_1| \geq |C''_2| > 0 \\
|D'_1| + |D''_2| & \geq |D'_1| \geq |D'_2| > 0 \\
|D''_1| + |T'_2| & \geq |D''_1| \geq |D''_2| > 0 
\end{align*}
$$

Also since we are still in a Case 1 with respect to both old and new cross separating $k$-sets, we have the following equalities

$$
\begin{align*}
|T'_2| &= |T''_2| \\
|A_2| &= |B_2| = |D'_2| + |H_2| + |D''_2| = |C'_2| + |H_1| + |C''_2|
\end{align*}
$$
Note that the set $T_2'$ has edges to the set $D_1''$, the set $T_2''$ has edges to the set $D_1'$, the set $T_2''$ has edges to the set $C_1$, and the set $T_2$ has edges to the set $C_1''$, because of the Claim 2 with respect to the new cross separating $k$-set. Hence, the maximal disjoint sets for $C$'s and $D$'s ($X$ and $Y$) will have cardinalities equal to 1.

Let us take a maximal $T_2$, and let us take the fringes of $A_2, B_2, C$ and $D$ (see Figure 11).

$C_1'$ does not have the fringe in $G_4$, otherwise part of $C_1'$ which has a fringe becomes a part of $I_1'$. If $C_1'$ has the fringe in $G_3$ then the part of $C_1'$ which has the fringe can be separated from the rest of the graph by a separating set $C_2'+T_2''+T_1$+ the fringe of $C_1'$ in $G_3$, whose cardinality is less than $k$. Hence, $C_1'$ does not have the fringe. Analogously, $C_1''$, $D_1$, and $D_1''$ do not have the fringes. Symmetrically, $T_2'$ and $T_2''$ do not have the fringes.

Let $\hat{T}_2$ be the union of vertices which are used for all possible $T_2$ which create a cross separating $k$-sets with nonempty $G_i$ $i=3,4,5,6$. Let $\hat{D}_1$ be the union of all possible $D_1$, $\hat{D}_1''$ be the union of all possible $D_1''$, $\hat{C}_1$ be the union of all possible $C_1$, $\hat{C}_1''$ be the union of all possible $C_1''$, $\hat{C}_2$ be the union of all possible $C_2$, $\hat{C}_2''$ be the union of all possible $C_2''$, $\hat{D}_2$ be the union of all possible $D_2$, and $\hat{D}_2''$ be the union of all possible $D_2''$. Let us show that all of these sets are disjoint.

Figure 11.
Illustrating the representation of separating $k$-sets of Case 1 if two or more different intersecting $T$’s exist.
(Structure 2).
Since all of them are symmetric we will prove it only for $C'$ and $C''$. Assume there are $T_3$ and $T_4$ such that $C''$ for $T_3$ is not disjoint from $C'$ for $T_4$. Then nonempty intersection of $C''$ for $T_3$ and $C'$ for $T_4$ is separated from the rest of the graph by a separating set $C'$ for $T_3 \cup T_3' \cup T_1 \cup T_4' \cup C'$ for $T_4$, whose cardinality is less than $k$. This contradiction proves the statement.

The cardinality of the union $\hat{D}''_2 \cup \hat{D}'_2 \cup I''_4 \cup I'_4$ is less than $\frac{k-t}{2}$, and analogously, the cardinality of $\hat{C}''_2 \cup \hat{C}'_2 \cup I' \cup I''_2$ is less than $\frac{k-t}{2}$. Let us call $\hat{C}'_2$, $\hat{C}''_2$, $\hat{D}'_2$, and $\hat{D}''_2$ the pseudofringe. Note that $A$ and $B$ might have fringes, but by the symmetry $T_2 - T_1$ does not have any fringes.

The structure which represent all separating $k$-sets for all possible $T$'s will the following (structure 2):

1) the original separating $k$-set with its fringes,

2) the cross separating $k$-set with minimum cardinality $T_1$ with its fringes and pseudofringes,

3) for every nonempty $G_i$ $i=3,4,5,6$ we will fill all nonexistent edges of the complete graph on the neighbors of $G_i$, if $G_i$ is empty for any $i=3,4,5,6$ we will fill these nonexistent edges of this complete graph by the virtual edges. (For $G_3$ we fill the edges between the vertices of the fringe of $A$ in $G_3$, $T_1$, $T_2'$, part of $A_2$ which does not have any fringes, $\hat{C}_i$, $I_1$, $H_1$, $I''_2$ and $\hat{C}''_2$).

From the construction of the structure it is easy to see that this structure covers all cross separating $k$-sets for all possible $T$'s, of type 1. Let us see now where the rest of the separating $k$-sets lie, if we have separating $k$-sets of type 1.

If there exists $T_2$ with at least one of the $G_i$ empty $i=3,4,5,6$, assuming it is not exception, such that there is another $T_2$ with $T_2 \cap T_1$ is nonempty along with nonempty $T_2 \cap B$ and $T_2 \cap A$, then all cross separating $k$-sets of this $T_2$ are covered by the above structure. (They belong to the fringes of $A$ and/or $B$ in $G_1$ or $G_2$ and the rest belong to the original cross separating $k$-set with its fringes or pseudofringes). So all cross separating $k$-sets are covered by this structure, assuming there are no exceptions, hence, all separating $k$-sets are either inside $G_1 \cup A \cup B \cup T_1 \cup$ the fringes of $A$ and $B$ in $G_2$, or $G_2 \cup A \cup B \cup T_1 \cup$ the fringes of $A$ and $B$ in $G_1$, or cross separating $k$-sets covered by the structure. Since the structure is symmetric, we can look at the cross separating $k$-sets where the original separating $k$-set is $C \cup D \cup T_1$. Then the pseudofringes of $C$ and $D$ become the pseudofringes of $A$ and $B$. With respect to this separation of $G$ all separating $k$-sets are either inside $G_3 \cup G_5 \cup C \cup D \cup T_1 \cup$ the fringe of $C$ in $G_4$ and the fringe of
D in G₆, or inside G₄∪C∪D∪T₁∪ the fringe of C in G₃ and the fringe of D in G₅, or separating k-sets covered by the structure. But since in both cases they are the same separating k-sets, all separating k-sets are either inside G₃∪A∪T₁∪C∪ the fringe of C in G₄∪ the fringe of A in G₅, or inside G₄∪B∪C∪T₁∪ the fringe of B in G₆, or inside G₅∪A∪D∪T₁∪ the fringe of A in G₃∪ the fringe of D in G₆, or inside G₆∪B∪D∪T₁∪ the fringe of B in G₄∪ the fringe of D in G₅, or the separating k-sets covered by the structure. To cover all exceptions we will do what we did for types 3 and 4 separating k-sets, we will add k(k−t) neighbors of A, B, C and D to each of G₃, G₄, G₅ and of G₆ which can participate in exceptional separating k-sets. Hence, the size of representation is

\[ g(n) = \sum_{i=1}^{4} g(n_i + k(k-t)+t) + 8\frac{(k-t)}{2}k + t, \]

where every term inside the sum covers one of Gᵢ i=3,4,5,6 along with its appropriate neighbors and fringes, and \( 8\frac{(k-t)}{2}k + t \) is the upper bound on the size of the structure. Note that \( \sum_{i=1}^{4} n_i + 2k - t = n \), hence the solution to the above recurrence is \( O(nk + k^3) \) (see Appendix). The number of exceptional separating k-sets is upper bounded by \( 4 \frac{k-t}{2} \). The upper bound on the number of separating k-sets become

\[ f(n) = \sum_{i=1}^{4} f(n_i + k(k-t)+t) + 4^2 \cdot 2^{k-t} + 4\cdot\frac{k-t}{2}. \]

The solution to it is \( O(2^kn + 2^kk^2) \) (see Appendix).

Let us now see what happens if we are in type 2 and no separating k-sets of type 1 exist. W.L.O.G. assume there is a separating k-set which uses \( T_2 = T_2' \cup \bar{T}_2', \) where \( T_2' \in A \) and \( \bar{T}_2' \in T_1 \), and no separating k-set of type 1 exist (see Figure 12).

If \( G_i \)'s i=3,4,5,6 are nonempty with respect to a new cross separating k-set then we become in the Case 1 with respect to a new cross separating k-set, hence \( |A_2| = |B_1| \) which is impossible. Hence, one of the \( G_i \) i=3,4,5,6 with respect to a new cross separating k-set must be empty. W.L.O.G. let the empty \( G_i \) be either \( G_3 \) or \( G_4 \) with respect to the new cross separating k-set. If \( G_4 \) is empty then \( G_5 \) with respect to the new cross separating k-set must be empty, otherwise \( T_1 \cup T_2' \cup A_2 \cup D_2 \) of the new cross separating k-set becomes a separating set with cardinality less than \( k \). Hence, if \( G_4 \) is empty then all cross separating k-set of type 2 belong to the original separating k-set with its fringes. Then all separating k-set are either inside \( G_1 \cup A \cup B \cup T_1 \cup the fringe of A in G_5 \cup the fringe of B in G_6, or G_2 \cup A \cup B \cup T_1 \cup the fringe of A in G_3 \cup the fringe of B in G_4, or they belong to the union of A \cup B \cup T_1 \cup the fringes of A and B. Note that the latter separating k-sets are covered by the structure 2. We can write the recurrences
similar to the above ones except for the sum which will be up to 2 instead of up to 4. The solution will be still of the same order. If \( G_3 \) is empty then \(|C_2| \geq |A_2|\), otherwise \( C_2 \cup T_2 \cup T_1 \cup B \) is a separating set with cardinality less than \( k \). If \( D_2 \) crosses \( D_1 \) (see Figure 12) then \( A_2 \cup T_2 \cup T_2 \cup D_2 \) is a separating set, so \(|C_2| = |A_2|\).

\( C \cup T_1 \cup D_1 \cup H \cup D' \) is a separating set, so \(|D'_{2}| \geq |D'_{1}|\). Also \( C_2 \cup T_2 \cup D_2 \cup H \cup D' \) is a separating set, so \(|D'_{1}| \geq |D'_{2}|\). Combining these two we get \(|D'_{1}| = |D'_{2}|\). Since, \( C \cup T_1 \cup T_2 \cup D_2 \cup H \cup D' \) and \( C_2 \cup T_2 \cup T_1 \cup D_1 \cup H \cup D' \) are separating sets, so \(|T_2 \cup D_2 | \geq |D_2| \geq |D_1|\). Since \( T_1 \cup D_2 \cup H \cup D' \) separates \( G'_{6} \) from the rest of the graph, and since the cardinality of this separating set is less than \( k \), \( G'_{6} \) is empty.

Hence, \( D'_{2} \) belongs to the fringe of \( D \) in \( G_6 \). \( T_2 = T_1 \) in order for the Claim 2 with respect to the old cross separating \( k \)-set to be true. And since \(|C_2| + |T_2| = |A|\) and since the cardinality of the new cross separating \( k \)-set is \( k \), \(|D'_{2}| = |D'_{1}|\). So, all cross separating \( k \)-sets of this type belong to \( G_5 \cup A \cup D \cup T_1 \cup \) the fringe of \( A \) in \( G_3 \cup \) the fringe of \( D \) in \( G_6 \), if there are no exceptional separating \( k \)-sets. Also in the maximal set of disjoint \( D \)'s (\( Y \)) all of \( D \)'s except \( D_1 \) belong to \( G_6 \). If \( G_5 \) with respect to the new cross separating \( k \)-set is nonempty, then by the above argument \( C_2 \) will belong to the fringe of \( A \). Hence, all cross separating \( k \)-sets belong to the set mentioned above, namely, \( G_4 \cup A \cup T \cup D_1 \cup \) the fringe of \( A \) in \( G_1 \cup \) the fringes of \( D_1 \) in \( G_5 \).

Let us take the maximal set of \( C \)'s and \( D \)'s (\( X \) and \( Y \)). We know that all cross separating \( k \)-sets of type 2 with nonempty \( G_5 \) belong to \( G_5 \cup A \cup D \cup T_1 \cup \) the fringe of \( A \) in \( G_3 \cup \) the fringe of \( D \) in \( G_6 \). Since we need to consider
all symmetric cases, and since we don't have any cross separating \( k \)-sets of type 1, all cross separating \( k \)-sets of the type 2 belong to \( G_3 \cup A \cup C \cup T_1 \cup \) the fringe of \( A \) in \( G_5 \cup \) the fringe of \( C \) in \( G_4 \), or \( G_4 \cup B \cup C \cup T_1 \cup \) the fringe of \( B \) in \( G_6 \cup \) the fringe of \( C \) in \( G_3 \), or \( G_5 \cup A \cup D \cup T_1 \cup \) the fringe of \( A \) in \( G_2 \cup \) the fringe of \( D \) in \( G_6 \), or \( G_6 \cup B \cup D \cup T_1 \cup \) the fringe of \( B \) in \( G_4 \cup \) the fringe of \( D \) in \( G_5 \). Note that \( C \)'s and \( D \)'s are not the same in these sets.

In case of \( G_3 \) \( C \) is "nearest" to \( A \), in case of \( G_4 \) \( C \) is "nearest" to \( B \), in case of \( G_5 \) \( D \) is "nearest" to \( A \), and in case of \( G_6 \) \( D \) is "nearest" to \( B \). Let us see where the rest of separating \( k \)-sets must lie. First, if there are no cross separating \( k \)-sets with \( G_5 \) nonempty (or same other appropriate symmetric \( G_i \) \( i=3,4,5,6 \)) then it is still possible to have a cross separating \( k \)-sets.

All cross separating \( k \)-sets consist of three parts: part one is in \( G_1 \), part two is in \( G_2 \) and part three is \( T_1 \). Part one belongs to some \( C \) from the set \( X \) or its fringe or the fringe of \( A \) in \( G_3 \) or the fringe of \( B \) in \( G_4 \). Part two belongs to some \( D \) from the set \( Y \) or its fringe or the fringe of \( A \) in \( G_5 \) or the fringe of \( B \) in \( G_6 \). That covers all cross separating \( k \)-sets which use \( T_1 \), otherwise either set \( X \) or set \( Y \) is not maximal. We don't have any cross separating \( k \)-sets of type 1. All cross separating \( k \)-sets of type 2 with nonempty appropriate \( G_i \) with respect to them belong to the part of the graph between \( A \) and the nearest \( D \) in \( G_2 \) along with \( A \) and its fringe and \( D \) and its fringe. Hence, all other separating \( k \)-sets belong to \( G_1 \cup A \cup B \cup T_1 \) with its fringes, or \( G_2 \cup A \cup B \cup T_1 \) with its fringes.

Hence, all cross separating \( k \)-sets of type 2, except exceptions are covered by the structure 2 or inside the the subgraphs associated by \( G_1, G_{i+1}, G_{l+2} \) and \( G_{l+2} \). As for the exceptions the upper bounds we got for types 3 and 4 still hold, since no part of \( T_1 \) can be separated by them (otherwise Claim 2 is contradicted). So, the recurrence which were written for the type 3 and 4 separating \( k \)-sets covers type 2 cross separating \( k \)-sets also, including exceptions. That conclude Case 1.

\[ \square \]

Case 2 For any separating \( k \)-set every cross separating \( k \)-set will have one of the \( G_i \) \( i=3,4,5,6 \) empty. Not every vertex in both \( G_1 \) and \( G_2 \) can be used for cross separating \( k \)-sets.

W.L.O.G. let \( G_3 \) will be empty (see Figure 13).

Since \( G_4 \) is nonempty by assumption, and \( G_5 \) is nonempty since there are no exception, \( C \cup T \cup B \) and \( A \cup T \cup D \) are separating sets. So their cardinalities are bigger or equal to \( k \), hence, \( |C| = |A| \) and \( |B| = |D| \). So, \( C \) is part of the fringe of \( A \) in \( G_1 \). Since this true for every \( T \), all cross separating \( k \)-sets belong to \( G_1 \cup A \cup T \cup B \cup \) the fringes of
A and B in $G_2$, or $G_2 \cup A \cup T \cup B$ the fringes of $A$ and $B$ in $G_1$, except for exceptions. So all separating $k$-sets including the exceptions are either inside $G_1 \cup A \cup B \cup T$ appropriate at most $k^2$ neighbors of $A \cup T \cup B$ in $G_2$ or inside $G_2 \cup A \cup B \cup T$ appropriate at most $k^2$ neighbors of $A \cup T \cup B$ in $G_1$ which are used in exceptional separating $k$-sets. Hence,

$$g(n) = g(n_1 + k(k-1)) + g(n_2 + k(k-1)) + 4k^2,$$

where $n_1$ and $n_2$ are the cardinalities of $G_1$ and $G_2$. We still have that $n_1 + n_2 + k = n$, and the solution to this recurrence is $O(k^2 + n)$ (see Appendix). Note that $n_i + k(k-1) < n$ for $i = 1, 2$.

For the upper bound on the number of separating $k$-sets we get the following equality

$$f(n) = f(n_1 + 2k) + f(n_2 + 2k) + 2^k,$$

where $2^k$ covers all exceptional separating $k$-sets. And its solution is clearly smaller than $O(2^k \frac{n^2}{k})$ (see Appendix).

That conclude Case 2.

Case 3 For every separating $k$-set all cross separating $k$-sets are lopsided (one of the $G_i$ $i=3,4,5,6$ will be empty).

And either $G_1$ or $G_2$ are such that every vertex of them is used for some cross separating $k$-set.

W.L.O.G. let $G_3$ be empty and the smallest $G_1$ every vertex of $G_1$ is used for some cross separating $k$-set (see Figure 13). There are two subcases: either $G_5$ or $G_6$ are empty, otherwise we will be in Case 2. Take $C$ as large as
possible.

If $G_6$ is empty then $A \cup B \cup C \cup D \cup T$ with all edges between them and filling real edges for nonempty $G_5$ and $G_4$ and virtual otherwise (analogous to the structure 1) will specify all cross separating $k$-sets. If $G_5$ is empty then $C \cup T \cup D$ separate $A$ from the rest of the graph. Hence, $C \cup T \cup D$ is an exceptional separating $k$-set. So the third structure will be the following:

1) $A, B$ and $T$ - the original separating $k$-set,

2) All the neighbors of $A \cup B \cup T$ that are used for a cross separating $k$-sets with edges between them and the original separating $k$-set.

since the remaining separating $k$-sets are inside $G_2 \cup A \cup B \cup T$, we derive the following recurrence relation:

$$g(n) = g(n-1) + k^2,$$

whose solution is $f(n) = O(k^2n)$. Analogously, we have the following recurrence relation for the upper bound on the number of separating $k$-sets

$$f(n) = f(n-1) + 2^k,$$

whose solution is $O(2^k n)$.

That conclude the proof of all cases. Our final result is that all separating $k$-sets have $O(k^2 n)$ space representation, and their number is $O(2^k n^2 / k)$.

REFERENCES


APPENDIX

\[ \sum_{i=1}^{l} (n_i + 1) = n \quad 2 \leq l \leq n \quad n_i \geq 0 \]

\[ g(n) \leq \max \left( \sum_{i=1}^{l} g(n_i + 2) + 4l \right) \]

Let \( g(n) = 4n - 16, \)

\[ g(n) \leq \max \left( \sum_{i=1}^{l} g(n_i + 2) + 4l \right) = \max \left( \sum_{i=1}^{l} (4(n_i + 2) - 16) + 4l \right) = \]

\[ \max \left( 4 \sum_{i=1}^{l} (n_i + 1) + 4l - 16l + 4l \right) = \max (4n - 8l) \leq 4n - 16 \]

\[ \sum_{i=1}^{l} (n_i + 1) + 1 = n \quad 2 \leq l \leq n-1 \quad n_i \geq 0 \]

\[ g(n) \leq \max \left( \sum_{i=1}^{l} g(n_i + 5) + 6l + 1 \right) \]

Let \( g(n) = 6n - 55, \)

\[ g(n) \leq \max \left( \sum_{i=1}^{l} g(n_i + 5) + 6l + 1 \right) = \max \left( \sum_{i=1}^{l} (6(n_i - 55) + 6l + 1) \right) = \]

\[ \max (6 \sum_{i=1}^{l} (n_i + 1) + 1) - 31l + 6l + 1) = \max (6n - 25l - 5) \leq 6n - 55 \]

\[ \sum_{i=1}^{l} (n_i + \frac{k-t}{2}) + t = n \quad 0 \leq t \leq k-2 \quad 2 \leq l \leq \frac{2n-t}{k-t} \quad n_i \geq 0 \]

\[ g(n) \leq \max \left( \sum_{i=1}^{l} g(n_i + (k-t) + \frac{1}{2}k^2 - 3kt - t) \right) \]

Let \( g(n) = 2nk - 4k^3 + 2k^2t + \frac{1}{2}k^2 - 3kt - t, \)
\[ g(n) \leq \max \left( \sum_{i=1}^{l} g(n_i + (k - t)k + t) + lk\frac{k-t}{2} + t \right) \leq \]
\[ \max \left( \sum_{i=1}^{l} 2k(n_i + k(k-t)+t) - 4k^3l + 2k^2tl + \frac{1}{2}k^2l - ktl - tl + lw\frac{k-t}{2} + t \right) = \]
\[ \max(2k\left( \sum_{i=1}^{l} n_i + \frac{k-t}{2} \right) + t) - 2kl\frac{k-t}{2} - 2kt + 2k^2l(k-t) + 2ktd - 4k^3l + 2k^2tl + \frac{1}{2}k^2l - 3ktd - tl + lw\frac{k-t}{2} + t) = \]
\[ \max(2kn + 2k^3(l-2l) + 2k^2t(-l + l) + k^2\left( \frac{1}{2}l + \frac{l}{2} - l \right) + kt(l - 2 + 2l - \frac{1}{2} - 3l) + t(-l + 1)) \leq \]
\[ 2kn - 4k^3 - 3kt + t \leq 2kn - 4k^3 + 2k^2t + \frac{1}{2}k^2 - 3kt - t \]

Hence, \( g(n) = O(nk + k^3) \).

\[ \sum_{i=1}^{l}(n_i + \frac{k-t}{2}) + t = n \quad 2 \leq l \leq \frac{k-t}{k-t} \quad 0 \leq t \leq n-2 \]

\[ f(n) = \max \left( \sum_{i=1}^{l} f(n_i + k(k-t)+t) + 2k^{-\frac{1}{2}}(l-2) + \frac{k-t}{2} \right) \]

Let

\[ f(n) = 2^{k-t}nl - 2^{k-t}k^2l + 2^{k-t}kl + \frac{1}{2}2^{k-t}kl - 3 \frac{1}{2}2^{k-t}l + 2^{k-t}kl + \frac{1}{2}2^{k-t}k - 2 \frac{1}{2}2^{k-t}k^2 - 2^{k-t}l - \frac{1}{2}2^{k-t}l - 2 \frac{k-t}{2} , \]

\[ f(n) \leq \max \left( \sum_{i=1}^{l} \left( n_i(k(k-t)+t) 2^{k-t}l - 2^{k-t}k^2l^2 + 2^{k-t}kl^2 + \frac{1}{2}2^{k-t}kl^2 - 3 \frac{1}{2}2^{k-t}l^2 + 2^{k-t}kl^2 - 2^{k-t}k^2l^2 + 2^{k-t}l^2 + \frac{1}{2}2^{k-t}kl^2 \right) \right) \]

\[ 2^{k-t}kl^2 - 2^{k-t}k^2l - \frac{1}{2}2^{k-t}l^2 - 2^{k-t}kl^2 + \frac{1}{2}2^{k-t}l^2 - \frac{1}{2}2^{k-t}l^2 - \frac{k-t}{2} = \max(2^{k-t}ln - \]

\[ \frac{1}{2}2^{k-t}kl^2 + \frac{1}{2}2^{k-t}l^2 - 2^{k-t}l + 2^{k-t}k^2l^2 - 2^{k-t}kl^2 + 2^{k-t}l^2 - 2^{k-t}k^2l^2 + 2^{k-t}kl^2 + \frac{1}{2}2^{k-t}l^2 - \]

\[ \frac{3}{2}2^{k-t}l^2 + 2^{k-t}kl + \frac{1}{2}2^{k-t}kl - 2 2^{k-t}k^2l - 2^{k-t}tl - \frac{1}{2}2^{k-t}tl - 2 \frac{2}{2} 2^{k-t}tl + \frac{1}{2}2^{k-t}l^2 - \frac{k-t}{2} ) = \]

\[ \max(2^{k-t}ln - 2 2^{k-t}k^2l + 2^{k-t}kl + \frac{1}{2}2^{k-t}kl - 2 2^{k-t}kl - 2 2^{k-t}tl - \frac{1}{2}2^{k-t}l - 2 \frac{2}{2} 2^{k-t}l + \frac{k-t}{2} ) \leq \]

\[ \max(2^{k-t}ln - 2 2^{k-t}k^2l + 2^{k-t}kl + \frac{1}{2}2^{k-t}kl - \frac{3}{2}2^{k-t}l + 2^{k-t}kl + \frac{1}{2}2^{k-t}k - 2 2^{k-t}k^2 - 2^{k-t}l - \frac{1}{2}2^{k-t}l - 2 \frac{2}{2} 2^{k-t}l + \frac{k-t}{2} ) \]

Hence, \( f(n) = O \left( \frac{n^2}{k} + 2^nk \right) \).
\[ \sum_{i=1}^{4} n_i + 2k - t = n \quad 0 \leq t \leq k - 2 \]

\[ g(n) \leq \sum_{i=1}^{4} g(n_i + k(k - t) + t) + 8k \frac{k - t}{2} + t \]

Let \( g(n) = 4nk - \frac{16}{3} k^3 + \frac{16}{3} k^2 t + \frac{4}{3} k^2 - \frac{16}{3} kt - \frac{1}{3} t, \)

\[ g(n) \leq \sum_{i=1}^{4} g(n_i + k(k - t) + t) + 4(k - t)k + t \]

\[ \sum_{i=1}^{4} (4(n_i + k(k - t) + t)k - \frac{16}{3} k^3 + \frac{16}{3} k^2 t + \frac{4}{3} k^2 - \frac{16}{3} kt - \frac{1}{3} t) + 4(k - t)k + t = \]

\[ 4k(\sum_{i=1}^{4} n_i + 2k - t) - 8k^2 + 4kt + 16k^3 - 16k^2 t + 16kt - \frac{64}{3} k^3 + \frac{64}{3} k^2 t + \frac{16}{3} k^2 - \frac{64}{3} kt - \frac{4}{3} t + 4k^2 - 4kt + t = \]

\[ 4kn + k^3(16 - \frac{64}{3}) + k^2(t(\frac{64}{3} - 16)) + k^2(\frac{16}{3} - 8 + 4) + kt(4 + 16 - \frac{64}{3} - 4) + t(1 - \frac{4}{3}) = \]

\[ 4kn - \frac{16}{3} k^3 + \frac{16}{3} k^2 t + \frac{4}{3} k^2 - \frac{16}{3} kt - \frac{1}{3} t \]

Hence, \( g(n) = O(nk + k^3). \)

\[ \sum_{i=1}^{4} (n_i + \frac{k - t}{2}) + t = n \quad 0 \leq t \leq n - 2 \]

\[ f(n) \leq \sum_{i=1}^{4} f(n_i + k(k - t) + t) + 6 \cdot 2^{k-t} + 4 \cdot 2^{\frac{k-t}{2}} \]

Let \( f(n) = 2^{k-t} n - \frac{4}{3} 2^{k-t} k^2 + \frac{4}{3} 2^{k-t} kt - \frac{5}{3} 2^{k-t} k - 2 \cdot 2^{k-t} - \frac{4}{3} \cdot 2^{\frac{k-t}{2}}, \)

\[ f(n) \leq \sum_{i=1}^{4} f(n_i + k(k - t) + t) + 6 \cdot 2^{k-t} + 4 \cdot 2^{\frac{k-t}{2}} \leq \sum_{i=1}^{4} (2^{k-t}(n_i + k(k - t) + t) - \frac{4}{3} 2^{k-t} k^2 + \frac{4}{3} 2^{k-t} kt - \frac{5}{3} 2^{k-t} k - 2 \cdot 2^{k-t} - \frac{4}{3} \cdot 2^{\frac{k-t}{2}} = \]

\[ \frac{5}{3} 2^{k-t} t + \frac{2}{3} 2^{k-t} k - 2 \cdot 2^{k-t} - \frac{4}{3} \cdot 2^{\frac{k-t}{2}} + 6 \cdot 2^{k-t} + 4 \cdot 2^{\frac{k-t}{2}} = 2^{k-t} n - 2^{k-t} k + 2 \cdot 2^{k-t} t - 2^{k-t} + \]

\[ 4 \cdot 2^{k-t} k^2 - 4 \cdot 2^{k-t} kt + 4 \cdot 2^{k-t} t - \frac{16}{3} \cdot 2^{k-t} k^2 + \frac{16}{3} \cdot 2^{k-t} kt - \frac{20}{3} \cdot 2^{k-t} t + \frac{8}{3} \cdot 2^{k-t} - \frac{16}{3} \cdot 2^{k-t} + 6 \cdot 2^{k-t} + 4 \cdot 2^{\frac{k-t}{2}} = \]

\[ 2^{k-t} n - \frac{4}{3} 2^{k-t} k^2 + \frac{4}{3} 2^{k-t} kt - \frac{5}{3} 2^{k-t} t + \frac{2}{3} 2^{k-t} k - 2 \cdot 2^{k-t} - \frac{4}{3} \cdot 2^{\frac{k-t}{2}} \]
\[ n_1 + n_2 + k = n \quad n_1, n_2 \geq 0 \]

\[ g(n) \leq g(n_1 + k(k - 1)) + g(n_2 + k(k - 1)) + 4k^2 \]

Let \( g(n) = n - 6k^2 + 3k, \)

\[ g(n) \leq n_1 + k^2 - k - 6k^2 + 3k + n_2 + k^2 - k - 6k^2 + 3k + 4k^2 = n - 6k^2 + 3k \]

\[ n_1 + n_2 + k = n \quad n_1, n_2 \geq 0 \]

\[ f(n) \leq f(n_1 + 2k) + f(n_2 + 2k) + 2^k \]

Let \( f(n) = 2^k n - 3 2^k k - 2^k, \)

\[ f(n) \leq 2^k n_1 + 2^k 2^k - 3 2^k k - 2^k + 2^k n_2 + 2^k 2^k - 3 2^k k - 2^k + 2^k = 2^k n - 3 2^k k - 2^k \]