DISTRIBUTED NETWORK SYNTHESIS 
AND APPROXIMATION 
IN THE TIME DOMAIN

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ABSTRACT

Conditions are obtained for the realization of the best approximation to a desired time domain input-output relation by means of a distributed network of a given class. Practical constraints are considered in obtaining the optimal (one-dimensional) spatial dependence for a series R-L, shunt G-C distributed network. The general results are specialized to L-C and R-C lines.

(This is a reprint.)
I. Introduction

Presumed dead by many, circuit theory is again being given an opportunity to show its mettle in the solution of distributed networks problems [1-13]. Nonlumped devices appear to offer a compactness and economy not associated with classical lumped circuits. However, much of the investigation of distributed networks which has been undertaken recently has overly forced analogies with lumped networks, consequently subverting this inherent utility to the expedient of a solution. With one exception [13], which presents a complete frequency domain analysis and practical synthesis scheme for lossless lines, recent work falls into two broad classes:

(1) The analysis of simple tapers which happen to be amenable to the classical separation methods for partial differential equations; and

(2) Analysis and synthesis of specialized circuits through analogy with classical lumped circuits.

Neither of the above two categories covers the fundamental synthesis problem for distributed networks. The general problem to which this paper is addressed is that of generating a distributed network from a given class which yields the best approximation to a desired time domain input-output relation. The general problem is presented schematically in Figure 1; the actual output signal, r(t), is to be the best approximation to a desired signal, s(t), when the input is e(t)—under the physical and practical constraints dictated by the intervening network. The problem is attacked as that of "mapping" the boundary conditions (input, output, and initial) into the interior of the network; the
element values are then easily obtained from knowledge of interior
signal distributions. Such a synthesis scheme must be contrasted with
the usual methods of few-element subsection removal synthesis.

In the present work a general procedure for a series R-L,
shunt G-C distributed network (in one spatial dimension) is evolved.
An additional feature, which is nonetheless necessary to the synthesis,
is that the approximation problem is treated concurrently with the
synthesis. The resulting network then yields the best approximation to
the desired time domain input-output relation under the constraints of
construction imposed.

II. Formulation and Solution of the Synthesis Problem

The distributed series R-L, shunt G-C network of given length
λ is assumed to be driven by voltage source \( v_s(t) \) in series with source
resistance \( R_s \); the time domain synthesis--and approximation--problem is
that of choosing the tapers \( r(x) \), \( \ell(x) \), \( g(x) \), and \( c(x) \) so as to mini­
mize the error between a desired output voltage \( v_o(t) \) and the actual
output voltage \( v(\lambda,t) \) appearing across load resistance \( R_L \). The synthesis
situation is depicted symbolically in Figure 2. It is assumed, moreover,
that bounds on the element values can be dictated a priori:

\[
\begin{align*}
R_m & \leq r(x) \leq R_M; \\
L_m & \leq \ell(x) \leq L_M; \\
g_m & \leq g(x) \leq G_M; \\
c_m & \leq c(x) \leq C_M;
\end{align*}
\]
For example, local passivity would imply that all elements be positive, whereas actual manufacturing conditions might dictate bounds even more stringent. In order to insure that the optimal line be time-invariant, one must impose the following obvious set of restrictions:

\[
\begin{align*}
\frac{\partial r(x)}{\partial t} &= 0; \\
\frac{\partial f(x)}{\partial t} &= 0; \\
\frac{\partial g(x)}{\partial t} &= 0; \\
\frac{\partial c(x)}{\partial t} &= 0.
\end{align*}
\]

The partial differential equations which describe the distributed network of Figure 2 are

\[
\begin{align*}
\frac{\partial}{\partial x} v(x,t) + L(x) \frac{\partial}{\partial t} i(x,t) + r(x) i(x,t) &= 0 \\
\frac{\partial}{\partial x} i(x,t) + c(x) \frac{\partial}{\partial t} v(x,t) + g(x) v(x,t) &= 0,
\end{align*}
\]

while the given boundary conditions are

\[
\begin{align*}
v_s(t) - v(0,t) - R_s i(0,t) &= 0, \\
v(\lambda,t) - R_L i(\lambda,t) &= 0, \\
v(x,0) &= 0,
\end{align*}
\]

and

\[
\begin{align*}
i(x,0) &= 0,
\end{align*}
\]
where the network is assumed initially relaxed. The performance goal is to match the output response to some desired time function; i.e.,
to minimize the performance index \(^2\)
\[
J = \int_{0}^{\infty} [v(\lambda,t) - kv_0(t)]^2 dt,
\]
where the gain factor \(k\) must be adjusted to insure physical realizability (i.e., the minimum and maximum allowable values of \(r(x)\) and \(g(x)\) determine the admissible steady-state error).

The solution to the minimization problem is obtained by means of the calculus of variations (see the Appendix for a complete treatment), which yields a number of partial differential equations and conditions which must be satisfied by the optimal distributed network. The results of the Appendix are repeated here for convenient reference. The optimal line must satisfy the following four partial differential equations (A.6)\(^3\):

\[
\begin{align*}
\frac{\partial}{\partial x} v(x,t) + \lambda(x) \frac{\partial}{\partial t} i(x,t) + r(x) i(x,t) &= 0; \\
\frac{\partial}{\partial x} i(x,t) + c(x) \frac{\partial}{\partial t} v(x,t) + g(x) v(x,t) &= 0; \\
\frac{\partial}{\partial x} \phi(x,t) + c(x) \frac{\partial}{\partial t} \psi(x,t) - g(x) \psi(x,t) &= 0; \\
\frac{\partial}{\partial x} \psi(x,t) + \lambda(x) \frac{\partial}{\partial t} \phi(x,t) - r(x) \phi(x,t) &= 0.
\end{align*}
\]

Moreover, the following four quantities absorb the time-invariance condition (2) and ultimately determine the parameters of the optimal line (A.7):

\[
\begin{align*}
\frac{\partial}{\partial x} v(x,t) + \lambda(x) \frac{\partial}{\partial t} i(x,t) + r(x) i(x,t) &= 0; \\
\frac{\partial}{\partial x} i(x,t) + c(x) \frac{\partial}{\partial t} v(x,t) + g(x) v(x,t) &= 0; \\
\frac{\partial}{\partial x} \phi(x,t) + c(x) \frac{\partial}{\partial t} \psi(x,t) - g(x) \psi(x,t) &= 0; \\
\frac{\partial}{\partial x} \psi(x,t) + \lambda(x) \frac{\partial}{\partial t} \phi(x,t) - r(x) \phi(x,t) &= 0.
\end{align*}
\]
\begin{align*}
\frac{\partial}{\partial t} \alpha(x,t) &= \dot{\phi}(x,t) \ i(x,t); \quad (7a) \\
\frac{\partial}{\partial t} \beta(x,t) &= \dot{\psi}(x,t) \ \frac{\partial}{\partial t} \ i(x,t); \quad (7b) \\
\frac{\partial}{\partial t} \gamma(x,t) &= \dot{\psi}(x,t) \ v(x,t); \quad (7c) \\
\frac{\partial}{\partial t} \eta(x,t) &= \dot{\psi}(x,t) \ \frac{\partial}{\partial t} \ v(x,t); \quad (7d)
\end{align*}

All of the above eight equations must evolve a solution from the following twelve boundary conditions (A.8):

\begin{align*}
v_s(t) - v(0,t) - R_s i(0,t) &= 0; \quad (8a) \\
v(\lambda,t) - R_1 i(\lambda,t) &= 0; \quad (8b) \\
v(x,0) &= 0; \quad (8c) \\
i(x,0) &= 0; \quad (8d) \\
2v(\lambda,t) - 2kv_0(t) + \dot{\phi}(\lambda,t) + \frac{1}{R_L} \ \psi(\lambda,t) &= 0; \quad (8e) \\
\dot{\phi}(0,t) - \frac{1}{R_s} \ \psi(0,t) &= 0; \quad (8f) \\
\dot{\phi}(x,\infty) &= 0; \quad (8g) \\
\psi(x,\infty) &= 0; \quad (8h) \\
a(x,0) - a(x,\infty) &= \begin{cases} 
> 0, & r(x) = R_M \\
= 0, & R_m \leq r(x) \leq R_M \\
< 0, & r(x) = R_M \\
\end{cases} \quad (8i) \\
b(x,0) - b(x,\infty) &= \begin{cases} 
> 0, & L(x) = L_M \\
= 0, & L_m \leq L(x) \leq L_M \\
< 0, & L(x) = L_M \\
\end{cases} \quad (8j)
\end{align*}
The vagueness expressed in the middle inequality conditions in the final four boundary conditions need cause no concern since for these "singular" solutions [14] to hold over a finite line length, all partial derivatives with respect to \( x \) of the left hand sides must be zero also; this additional condition yields the element values exactly.

The above results are simply specialized to series L, shunt C lines by the omission of \( r(x), g(x), a(x,t) \) and \( \gamma(x,t) \) from the above; i.e., by disregarding (7a), (7c), (8i) and (8k) and, of course, suitably modifying the remaining conditions. The specialization to series R, shunt C lines follows as simply as the foregoing by the omission of \( \ell(x), g(x), B(x,t) \) and \( \gamma(x,t) \) from the above; i.e., by disregarding (7b), (7c), (8j), and (8k) and, of course, suitably modifying the remaining conditions.

It should be emphasized that the above results are but the fundamental necessary conditions for optimality. Sufficiency conditions in general can be obtained only for specific problems and they are usually too horrendous for meaningful interpretation. However, in specific problems, one can always rely on the size of the final output error as an indication of optimality.
III. Example: Attenuated Delay

In this simple problem the following values are specified:

\[ v_s(t) = 1(t); \quad (9a) \]
\[ v_o(t) = l(t-\tau); \quad (9b) \]
\[ R_s = R_L = R_o; \quad (9c) \]
\[ \{r(x), \ell(x), g(x), c(x)\} \geq 0. \quad (9d) \]

It is immediately obvious that the condition that all elements be positive (9d) imposes a further proviso on the gain factor,

\[ k \leq \frac{1}{2}. \quad (10) \]

The reader can verify that the constant parameter line with

\[ r = -\frac{R_o}{\lambda} \ell_n \quad (2k), \quad (11a) \]
\[ \ell = \frac{\lambda R_o}{\tau}, \quad (11b) \]
\[ g = -\frac{1}{\lambda R_o} \ell_n \quad (2k), \quad (11c) \]

and

\[ c = \frac{\lambda}{\tau R_o} \quad (11d) \]

satisfies the necessary conditions (6-8).
IV. Conclusions

In this paper a new approach to the network synthesis problem has been evolved which is particularly suited to distributed networks. To the knowledge of the authors, this is the first such approach which circumvents all of the classical steps and directly relates an input-output relation to optimal element values. A number of obvious extensions of the above are currently being investigated:

(1) Extension of the technique to two spatial dimensions (planar networks) and three spatial dimensions (general networks);

(2) Extension of the technique to frequency domain input-output specifications;

(3) Extension of the technique to multi-input, multi-output situations;

(4) Extension of the technique to include more stringent element value constraints (e.g., on the rate of taper).
REFERENCES


15. Elsgolc, L. E., Calculus of Variations, Addison-Wesley, Reading, Massachusetts, pp. 47-54; 1962.

APPENDIX: Variational Solution of the Synthesis Problem

When the network equations (3) are appended by means of Lagrange multipliers $\xi(x,t)$ and $\Upsilon(x,t)$ and the time-invariance constraints (2) are appended by means of Lagrange multipliers $\alpha(x,t), \beta(x,t), \gamma(x,t),$ and $\eta(x,t)$, the minimization (5) can be rewritten in terms of the first variation

$$\delta \bar{J} \geq 0,$$

where

$$\bar{J} = \int_0^\infty \left[ v(\lambda, t) - kv_o(t) \right]^2 dt$$

$$+ \int_\Sigma \left[ \xi(x,t) \left[ \frac{\partial}{\partial x} v(x,t) + \epsilon(x) \frac{\partial}{\partial t} i(x,t) + r(x) i(x,t) \right] 
+ \Upsilon(x,t) \left[ \frac{\partial}{\partial x} i(x,t) + c(x) \frac{\partial}{\partial t} v(x,t) + g(x) v(x,t) \right] 
+ \alpha(x,t) \frac{\partial r(x)}{\partial t} + \beta(x,t) \frac{\partial \epsilon(x)}{\partial t} 
+ \gamma(x,t) \frac{\partial c(x)}{\partial t} + \eta(x,t) \frac{\partial v(x,t)}{\partial t} \right] d\sigma,$$

and the surface of integration $\Sigma$ is indicated in Figure A.1. Simple application of the calculus of variations to continuum systems [15,16] yields the following expression for $\delta \bar{J}$.
\[
\delta \overline{J} = \int_0^\infty 2[v(\sigma,t) - kv_0(t)] \delta v(\sigma,t) \, dt \\
+ \int_\Sigma [\delta \psi \frac{\partial v}{\partial x} + \delta \frac{\partial i}{\partial t} + ri] + \delta \psi \frac{\partial i}{\partial x} + c \frac{\partial v}{\partial t} + gv \\
+ \psi \left[ (\delta \frac{\partial v}{\partial x}) + (\delta \psi) \frac{\partial i}{\partial t} + \psi (\delta \frac{\partial i}{\partial t}) + (\delta r) i + r(\delta i) \right] \\
+ \psi \left[ (\delta \frac{\partial i}{\partial t}) + (\delta c) \frac{\partial v}{\partial t} + c(\delta \frac{\partial v}{\partial t}) + (\delta g) v + g(\delta v) \right] \\
+ \left( \delta \alpha \right) \frac{\partial r}{\partial t} + \alpha (\delta \frac{\partial r}{\partial t}) + \left( \delta B \right) \frac{\partial R}{\partial t} + B(\delta \frac{\partial R}{\partial t}) \\
+ \left( \delta \gamma \right) \frac{\partial g}{\partial t} + \gamma (\delta \frac{\partial g}{\partial t}) + \left( \delta \eta \right) \frac{\partial c}{\partial t} + \eta (\delta \frac{\partial c}{\partial t}) \] \, d\sigma.
\tag{A.3}
\]

The variations of partial derivatives are exchanged for partial derivatives of variations in the usual manner \([15,16]\), yielding

\[
\delta \overline{J} = \int_0^\infty 2 [v(\sigma,t) - kv_0(t)]^2 \delta v(\sigma,t) \, dt \\
+ \int_\Sigma [\delta \psi \frac{\partial v}{\partial x} + \delta \frac{\partial i}{\partial t} + ri] + \delta \psi \frac{\partial i}{\partial x} + c \frac{\partial v}{\partial t} + gv \\
+ \delta v \left[ - \frac{\partial \psi}{\partial x} - c \frac{\partial v}{\partial t} + g\psi \right] + \delta i \left[ - \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi}{\partial x} + r\psi \right] \\
+ \delta r \left[ \psi \frac{\partial i}{\partial t} - \frac{\partial \psi}{\partial t} \right] + \delta \epsilon \left[ \psi \frac{\partial i}{\partial t} - \psi \frac{\partial \psi}{\partial t} \right] \\
+ \delta g \left[ \psi \frac{\partial v}{\partial t} - \frac{\partial \psi}{\partial t} \right] + \delta c \left[ \psi \frac{\partial v}{\partial t} - \frac{\partial \psi}{\partial t} \right] \\
+ \left( \delta \alpha \right) \frac{\partial r}{\partial t} + \left( \delta B \right) \frac{\partial R}{\partial t} + \left( \delta \gamma \right) \frac{\partial g}{\partial t} + \left( \delta \eta \right) \frac{\partial c}{\partial t} \\
+ \frac{\partial}{\partial x} \left( \delta \psi v + \psi \delta i \right) + \frac{\partial}{\partial t} \left( \psi \delta i + \psi \delta v + \alpha \delta r + \psi \delta c + \psi \delta g + \eta \delta c \right) \] \, d\sigma.
\tag{A.4}
\]
The coefficients of $\delta \phi$ and $\delta \psi$ in (A.4) are zero since they are the original line equations (3) as are the coefficients of $\delta \alpha$, $\delta \beta$, $\delta \gamma$, and $\delta \eta$ since they are the original time-invariance constraints. Furthermore, the final two terms involving space and time derivatives can be converted to contour integrals by means of the two-dimensional divergence theorem:

$$\delta J = \int_0^\infty \left[ [2v(\lambda, t) - 2k_v(t) + \phi(\lambda, t)] \delta v(\lambda, t) + \psi(\lambda, t) \delta i(\lambda, t) \right] dt$$

$$- \int_0^\lambda \left[ \phi(x, \infty) \ell(x) \delta i(x, \infty) + \psi(x, \infty) c(x) \delta v(x, \infty) \right] dx$$

$$+ a(x, \infty) \delta r(x) + b(x, \infty) \delta \ell(x) + \gamma(x, \infty) \delta g(x) + \eta(x, \infty) \delta c(x) \} dx$$

$$- \int_0^\infty \left[ \phi(0, t) \delta v(0, t) + \psi(0, t) \delta i(0, t) \right] dt$$

$$+ \int_0^\lambda \left[ \phi(x, 0) \ell(x) \delta i(x, 0) + \psi(x, 0) c(x) \delta v(x, 0) \right] dx$$

$$+ a(x, 0) \delta r(x) + b(x, 0) \delta \ell(x) + \gamma(x, 0) \delta g(x) + \eta(x, 0) \delta c(x) \} dx$$

$$+ \sum_{\sigma} \left[ s v \left[ - \frac{\delta \phi}{\delta x} - c \frac{\partial \psi}{\partial t} + g \psi \right] + \delta i \left[ - \frac{\partial \psi}{\partial x} - \ell \frac{\partial \phi}{\delta t} + r \phi \right] \right]$$

$$+ \delta r \left[ \phi - \frac{\partial \alpha}{\partial t} \right] + \delta \ell \left[ \phi \frac{\partial i}{\partial t} - \frac{\partial \beta}{\partial t} \right]$$

$$+ \delta g \left[ \psi - \frac{\partial \gamma}{\partial t} \right] + \delta c \left[ \psi \frac{\partial v}{\partial t} - \frac{\partial \eta}{\partial t} \right] \sigma.$$  \hspace{1cm} (A.5)

From (A.4) and (A.5) the variations of the free quantities, $\psi(x, t)$, $\psi(x, t)$ $v(x, t)$ $i(x, t)$, $a(x, t)$, $b(x, t)$, $\gamma(x, t)$, and $\eta(x, t)$ yield eight Euler equations:
\begin{align*}
\frac{\partial}{\partial x} v(x,t) + \ell(x) \frac{\partial}{\partial t} i(x,t) + r(x) i(x,t) &= 0; \quad (A.6a) \\
\frac{\partial}{\partial x} i(x,t) + c(x) \frac{\partial}{\partial t} v(x,t) + g(x) v(x,t) &= 0; \quad (A.6b) \\
\frac{\partial}{\partial x} \hat{y}(x,t) + c(x) \frac{\partial}{\partial t} \Psi(x,t) - g(x) \Psi(x,t) &= 0; \quad (A.6c) \\
\frac{\partial}{\partial x} \Psi(x,t) + \ell(x) \frac{\partial}{\partial t} \hat{y}(x,t) - r(x) \hat{y}(x,t) &= 0; \quad (A.6d) \\
\frac{\partial}{\partial t} r(x) &= 0; \quad (A.6e) \\
\frac{\partial}{\partial t} \ell(x) &= 0; \quad (A.6f) \\
\frac{\partial}{\partial t} g(x) &= 0; \quad (A.6g) \\
\frac{\partial}{\partial t} c(x) &= 0. \quad (A.6h)
\end{align*}

Moreover, the variations of the constrained quantities, \( r(x), \ell(x), g(x), \) and \( c(x), \) yield four additional Euler equations (the arbitrariness of the multipliers \( \alpha(x,t), \beta(x,t), \gamma(x,t), \) and \( \eta(x,t) \) allows equality here even though inequality may be indicated as possible):

\begin{align*}
\frac{\partial}{\partial t} \alpha(x,t) &= \hat{y}(x,t) \ i(x,t); \\
\frac{\partial}{\partial t} \beta(x,t) &= \hat{y}(x,t) \ \frac{\partial}{\partial t} i(x,t); \\
\frac{\partial}{\partial t} \gamma(x,t) &= \Psi(x,t) \ v(x,t); \\
\frac{\partial}{\partial t} \eta(x,t) &= \Psi(x,t) \ \frac{\partial}{\partial t} v(x,t).
\end{align*}

(A.7a) \quad (A.7b) \quad (A.7c) \quad (A.7d)
From the original set of boundary conditions (4), (A.5) yields the complete set of boundary conditions:

\[ v_s(t) - v(0,t) - R_i(0,t) = 0; \]  
(A.8a)

\[ v(\lambda, t) - R_L i(\lambda, t) = 0; \]  
(A.8b)

\[ v(x,0) = 0; \]  
(A.8c)

\[ i(x,0) = 0; \]  
(A.8d)

\[ 2v(\lambda, t) - 2kv_0(t) + \phi(\lambda, t) + \frac{1}{R_L} \psi(\lambda, t) = 0; \]  
(A.8e)

\[ \psi(0,t) - R_s \phi(0,t) = 0; \]  
(A.8f)

\[ \phi(x,\infty) = 0; \]  
(A.8g)

\[ \psi(x,\infty) = 0; \]  
(A.8h)

\[ \alpha(x,0) - \alpha(x,\infty) \begin{cases} > 0, \delta(x) = R_M \\ = 0, R_m \leq \delta(x) \leq R_M \\ < 0, \delta(x) - R_M \end{cases}; \]  
(A.8i)

\[ \beta(x,0) - \beta(x,\infty) \begin{cases} > 0, \phi(x) = L_M \\ = 0, L_m \leq \phi(x) \leq L_M \\ < 0, \phi(x) = L_M \end{cases}; \]  
(A.8j)

\[ \gamma(x,0) - \gamma(x,\infty) \begin{cases} > 0, \theta(x) = G_M \\ = 0, G_m \leq \theta(x) \leq G_M \\ < 0, \theta(x) = G_M \end{cases}; \]  
(A.8k)

\[ \eta(x,0) - \eta(x,\infty) \begin{cases} > 0, \zeta(x) = C_M \\ = 0, C_m \leq \zeta(x) \leq C_M \\ < 0, \zeta(x) = C_M \end{cases}; \]  
(A.8l)
Conditions (A.8g) and (A.8h) assure no steady-state error between $v(\lambda, t)$ and $kv_0(t)$ and their attainment may involve adjustment of $k$; moreover, in exotic situations they may be replaced by the equally valid

$$\lim_{t \to \infty} \left[ [\delta(x, t) \delta v(x, t) + \Psi(x, t) c(x) \delta v(x, t)] 
+ [\alpha(x, t) \delta r(x) + \beta(x, t) \delta \lambda(x) + \gamma(x, t) \delta g(x) + \eta(x, t) \delta c(x)] \right] = 0.$$  \hspace{1cm} (A.9)

The above twenty-four conditions (A.6 - A.9) constitute a complete set of fundamental necessary conditions adequate to solve for the optimal line.
1. Although seemingly trivial, these restrictions are of fundamental importance; nowhere in the remainder of the problem can a mathematical distinction between a time-varying and a time-invariant network be made.

2. The square-error criterion is used here for convenience; the method can be extended to other performance indices merely by the introduction of the proper weighting function in the integral (5).

3. The time-invariance conditions (2) are omitted here since they are absorbed by means of the auxiliary variables introduced in Equations (7).

4. For notational convenience, from this point on in Appendix A, time and space arguments are dropped when no mininterpretation can result.
FIGURE CAPTIONS

Figure 1. Symbolic Representation of Synthesis-Approximation Problem.

Figure 2. Series R-L, Shunt G-C Distributed Network Synthesis Situation.

Figure A.1. Surface of Integration for Equation (A.2).
Distributed series R-L, shunt G-C network
Network to be synthesized.
(subject to assumed physical and practical constraints.)