THE SENSITIVITY OF GENERAL MULTIVARIABLE FEEDBACK SYSTEMS

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The new approach recently proposed by the authors for treating the parameter variation problem for a class of multivariable systems is extended to a general feedback configuration for multivariable systems. It is shown that the general linear, multivariable, feedback system can be represented by two plant transfer function matrices and three controller transfer function matrices. A formula for the sensitivity matrix of the general structure is derived. Also, sufficient conditions are obtained for assuring a feedback design that is less affected by parameter variations than another feedback design with the same overall transfer matrix.
INTRODUCTION

The potential benefits of feedback in reducing the effects of parameter variations as well as the effects of noise disturbances are well known [1]. Feedback per se does not guarantee improvement in performance. It had been shown that feedback systems could be worse than open-loop systems in reducing the effects of parameter variations [2,3]. In the case of single-variable systems, Bode's sensitivity function [1] can be incorporated in design procedures [2,3]. The sensitivity function is the key to a quantitative measure of the effects of parameter variations. Bode's sensitivity is intrinsically for single-input, single-output systems and no similar sensitivity function has been developed for multivariable systems until very recently [4,5]. The authors proposed a new approach to the parameter variation problem which is applicable to systems which may have several inputs and several outputs. A direct comparison of feedback performance with open-loop performance leads to a useful sensitivity matrix. This sensitivity matrix provides a basis for designing a multivariable feedback system whose performance is less affected by parameter variations than that for a corresponding open-loop system.

Let \( y_c(t) \) represent the n-dimensional output of the feedback system, \( y_o(t) \) the n-dimensional output of an open-loop system with the same nominal transfer characteristic, \( r(t) \), the p-dimensional input of the system, and \( y'_c(t) \) and \( y'_o(t) \) the corresponding outputs when some system parameters change. (See Figures 1 and 2). Define

\[
e_c(t) = y_c(t) - y'_c(t)
\]  

(1)
As a performance index, the integrated square of the error is chosen:

\[
\text{Performance Index} = \int_{0}^{t_1} \mathbf{g}^T(t) \mathbf{g}(t) \, dt
\]

where \( \mathbf{g}^T(t) \) is the transpose of the vector \( \mathbf{g}(t) \). For a sensible comparison, it is assumed that when there is no parameter variation, for the same input \( \mathbf{z}(t) \), \( \mathbf{y}(t) = \mathbf{y}_c(t) \). Then the feedback system is said to be better than the open-loop system if

\[
\int_{0}^{t_1} \mathbf{g}_c^T(t) \mathbf{g}_c(t) \, dt < a \int_{0}^{t_1} \mathbf{g}_o^T(t) \mathbf{g}_o(t) \, dt, \quad a < 1
\]

for any arbitrary \( t_1 \) and for any arbitrary input \( \mathbf{z}(t) \), provided only that the integrals in Equation (4) exist.

It has been shown \([4,5]\) that for a class of multivariable systems wherein all the feedback signals are derived from the output vector \( \mathbf{y}_c(t) \), the Laplace transforms of \( \mathbf{g}_o(t) \) and \( \mathbf{g}_c(t) \) are linearly related:

\[
\mathbf{E}_c(s) = \mathbf{S}(s) \mathbf{E}_o(s)
\]

where \( \mathbf{S}(s) \) is defined as the sensitivity matrix. A sufficient condition for satisfying Equation (4) is

\[
\mathbf{S}^T(-j\omega) \mathbf{S}(j\omega) - a\mathbf{I}_n \leq 0 \quad \text{(negative semi-definite for all } \omega), a < 1. \quad (6)
\]
The sensitivity matrix for the above class of systems is also related to the loop transmission matrix \( L'(s) \) with the loop opened at the output and parameter variations are included. Specifically,

\[
S(s) = [I_n - L'(s)]^{-1}
\]  

(7)

Similar relations hold for discrete-time systems. The sensitivity formulation was carried out also for systems described by state variable equations [5]. However, it was assumed for simplicity that the system structure is such that the output vector and the state vector are identical.

**GENERAL FEEDBACK SYSTEMS**

In this paper, the authors' formulation of the parameter variation problem is extended to more general systems. (See Figure 3). A linear system to be controlled which may be subject to parameter variation is denoted the plant. The plant may have several inputs. The set of inputs is denoted by a vector \( y(t) \) with a Laplace transform \( Y(s) \). The plant may have several outputs. These outputs are classified into two groups. One group, denoted by \( y(t) \) (with a Laplace transform \( Y(s) \)), is the set of primary outputs, and another group, denoted by \( z(t) \) (with a Laplace transform \( Z(s) \)) is the set of secondary outputs. A controller is to be designed to produce the plant input \( y(t) \) in such a manner that the overall system transfer characteristic relating the command input vector \( x(t) \) to the primary output \( y(t) \) is within the class of specified transfer characteristics in spite of parameter variations.
in the plant. It is assumed that the vectors \( \mathbf{y}(t), \mathbf{y}(t) \), and \( \mathbf{z}(t) \) are available as inputs to the controller.

Since the plant is linear, it may be represented by two linear subsystems \( P_1 \) and \( P_2 \) as shown in Figure 4. Likewise, if the controller is linear, \( u \) is obtained by linear operations on \( \mathbf{y}(t), \mathbf{y}(t) \), and \( \mathbf{z}(t) \). Thus, the controller may be represented by three linear subsystems \( G, H_1, \) and \( H_2 \) as shown in Figure 4. The relationships among the Laplace transforms of the various signals are

\[
\mathbf{Y}(s) = P_1(s) \mathbf{U}(s), \quad (8)
\]

\[
\mathbf{Z}(s) = P_2(s) \mathbf{U}(s), \quad (9)
\]

and

\[
\mathbf{U}(s) = G(s) \mathbf{R}(s) + H_1(s) \mathbf{Y}(s) + H_2(s) \mathbf{Z}(s) \quad (10)
\]

Let \( P'_1 \) and \( P'_2 \) denote the perturbed plant matrices. It should be observed that in the representation of Figure 4, the perturbations in \( P_1 \) and \( P_2 \) may not be independent. For example, the value of a specific amplifier gain may influence both \( P_1 \) and \( P_2 \). The open-loop realization for the same transmission characteristics is shown in Figure 5, where neither the signals \( \mathbf{y}(t) \) nor \( \mathbf{z}(t) \) are fed back. We now obtain the sensitivity matrix \( S \) for the structure in Figure 4.

Eliminating \( \mathbf{Z}(s) \) from Equations (9) and (10),

\[
\mathbf{U} = (I_m - H_2 P'_2)^{-1} [G \mathbf{R} + H_1 \mathbf{Y}] \quad (11)
\]
where $\mathbb{I}_m$ is an $m$-dimensional identity matrix, $m$ being the dimension of $U$. Eliminating $U$ from Equations (8) and (11), the following is obtained:

$$Y = [I_n - P_1(I_m - H_2 P_2)^{-1} H_1]^{-1} P_1(I_m - H_2 P_2)^{-1} G R$$  (12)

For the open-loop realization, the transfer relation is

$$Y = P_1 G R$$  (13)

With Equations (12) and (13) expressions for the error vectors $E_c$ and $E_o$ can be obtained. Since $U$ in the open-loop system does not vary with plant perturbations, and since it is equal to $U$ in the closed-loop system when there is no parameter variation,

$$E_o = Y - Y_0 = (P_1 - P_1') U = (P_1 - P_1')(I_m - H_2 P_2)^{-1}(G R + H_1 Y)$$  (14)

From Equation (12), $Y - Y_c$ can be formed:

$$E_c = Y - Y_c = Y - [I_n - P_1'(I_m - H_2 P_2')^{-1} H_1]^{-1} P_1'(I_m - H_2 P_2')^{-1} G R$$

$$= [I_n - P_1'(I_m - H_2 P_2')^{-1} H_1]^{-1} [I_n - P_1'(I_m - H_2 P_2')^{-1} H_1] Y$$

$$- P_1'(I_m - H_2 P_2')^{-1} G R$$

$$= [I_n - P_1'(I_m - H_2 P_2')^{-1} H_1]^{-1} \left\{ Y - P_1'(I_m - H_2 P_2')^{-1} (H_1 Y + G R) \right\}$$  (15)

But

$$Y = P_1 U = P_1(I_m - H_2 P_2)^{-1} (G R + H_1 Y)$$

Hence

$$E_c = [I_n - P_1'(I_m - H_2 P_2')^{-1} H_1]^{-1} [P_1 - P_1'(I_m - H_2 P_2')^{-1}(I_m - H_2 P_2')]$$

$$(I_m - H_2 P_2')^{-1} (G R + H_1 Y)$$  (17)
If \((P_1 - P_1')\) is nonsingular, we finally have

\[
E_c = ([I - P_1'(I - H_2 P_2')^{-1} H_1]^{-1} [P_1(I - H_2 P_2)^{-1} - P_1'(I - H_2 P_2')^{-1}]
\]

\[
(I - H_2 P_2) (P_1 - P_1')^{-1} E_o
\]

where \(I\) is the \(n\)-dimensional identity matrix. Hence

\[
S = ([I - P_1'(I - H_2 P_2')^{-1} H_1]^{-1} [P_1(I - H_2 P_2)^{-1} - P_1'(I - H_2 P_2')^{-1}]
\]

\[
(I - H_2 P_2) (P_1 - P_1')^{-1}
\]

For single-input, single-output systems, Equation (19) reduces to

\[
S = \frac{(P_1 - P_1') - (P_1' P_2 - P_1 P_2')}{(P_1 - P_1') (1 - P_1' H_1 - P_2' H_2)}
\]

Note that in general \((P_1 - P_1')\) is not a factor of the numerator. Hence in general \((P_1 - P_1')\) does not cancel out. If this is the case for the single variable case, then in general, \((P_1 - P_1')\) does not factor out of the numerator of Equation (19) for the multivariable case either. Thus there is no other way of deriving Equation (19) which does not involve assuming that \((P_1 - P_1')\) is nonsingular. We conclude that for the structure of Figure 4, it is not possible to relate \(E_o\) to \(E_c\) unless \((P_1 - P_1')\) is nonsingular. This is in contrast to the structure of Figure 1, where the \(P\) matrix does not even have to be square.

For the class of problems where \((P_1 - P_1')\) is nonsingular, the formula for \(S\) in Equation (19) may be substituted in Equation (6).
$G, H_1,$ and $H_2$ are to be chosen so as to satisfy the low sensitivity criterion as well as the transmission characteristic. Although it is possible to ignore the auxiliary signals $Z(t)$ and use the structure of Figure 1 and still have a feedback system which is less sensitive to parameter variation than an open-loop design, the required bandwidths of the compensators may be excessively large and introduce significant noise and saturation problems. The use of auxiliary feedback may lead to compensators with lower gain-bandwidth products. The capability of minor loop feedback in single-input, single-output systems to reduce bandwidth in the compensators is well known [3].

**COMPARISON OF FEEDBACK SYSTEMS**

The sensitivity criterion discussed above insures that a feedback realization is less affected by plant parameter variations than an open-loop realization. It is also of interest to compare feedback designs. In particular, one may wish to compare a design based on the structure of Figure 1 to a design based on the structure of Figure 4. A simple criterion is derived below.

Let $e_{c_1}(t)$ be the output error vector due to parameter variations for feedback system number 1. Similarly, let $e_{c_2}(t)$ be the corresponding error vector for feedback system number 2, and let $e_o(t)$ be the error vector for an open-loop structure. If

$$\int_0^t e_{c_1}^T(t) e_{c_1}(t) \, dt \leq a \int_0^t e_{c_2}^T(t) e_{c_2}(t) \, dt \leq b \int_0^t e_o^T(t) e_o(t) \, dt \quad (21)$$
for $a < b < 1$, and all inputs, then we say that system 1 is less affected by parameter variations than system 2, and system 2, in turn, is less affected by parameter variations than the open-loop system.

By a procedure analogous to the derivation in reference 1, we can show that

$$S_1^{-j\omega} S_1(j\omega) - a S_2^T(-j\omega) S_2(j\omega) \leq 0 \quad (22)$$

and

$$S_2^{-j\omega} S_2(j\omega) - b I \leq 0 \quad (23)$$

for $a < b < 1$ and for all $\omega$ are sufficient for insuring the satisfaction of Equation (21) for all inputs and all $t$. $S_1(j\omega)$ is the sensitivity matrix for system 1 and $S_2(j\omega)$ is the sensitivity matrix for system 2. Since the sensitivity matrix for an open-loop system is $I$, Equation (22) is a generalized sensitivity criterion which includes the previous criterion as a special case, when system 2 is an open-loop system. A positive definite weighting matrix $Q$ may be used in Equation (21) resulting in a sensitivity criterion similar to Equation (22).

**CONCLUSION**

Employing the approach used earlier by the authors, the formula for the sensitivity matrix was derived for a class of linear, time-invariant, multivariable, feedback systems. The class is more general than the one considered previously in the sense that intermediate output variables are available for feedback. The sensitivity matrix relating the open-loop and closed-loop error vectors due to plant parameter
variations can be derived only if a plant variation transfer matrix is nonsingular. Since the critical factor appears in the denominator for the single variable case, and since the factor does not cancel out for arbitrary parameter variations, it is concluded that the nonsingularity of the plant perturbation matrix is, in general, necessary for the existence of the sensitivity matrix. Although the expression for $S$ is more complicated than the one obtained previously for the class of systems where no intermediate outputs are available, the structure affords more degrees of freedom and controllers with smaller gain-bandwidths may be possible for approximately the same sensitivity properties. A formula for comparing the sensitivity performances of two feedback systems is also derived.

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Fig. 1. A Feedback Configuration

Fig. 2. An Open-loop Configuration

Fig. 3. A General Feedback System with Secondary Outputs
Fig. 4. Closed-loop Realization of a System with Secondary Outputs

Fig. 5. Open-loop Realization of a System with Secondary Outputs
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