CRITERIA FOR SYSTEM SENSITIVITY TO PARAMETER VARIATIONS

J.B. Cruz Jr. and W. R. Perkins

R-250 March, 1965
This work was supported in part by the Joint Services Electronics Programs (U. S. Army, U. S. Navy, and U. S. Air Force) under Contract No. DA 28 043 AMC 00073(E).

Portions of this work were also supported by:

Air Force Office of Scientific Research

Contract AF 49(638)-1383.

Reproduction in whole or in part is permitted for any purpose of the United States Government.

DDC Availability Notice: Qualified requesters may obtain copies of this report from DDC. DDC release to OTS is NOT authorized.
CRITERIA FOR SYSTEM SENSITIVITY TO PARAMETER VARIATIONS

J. B. Cruz, Jr. and W. R. Perkins

University of Illinois
Urbana, Illinois, U.S.A.

ABSTRACT

A characterization of the sensitivity of a class of linear multivariable control systems based on a comparison of the output deviations due to parameter variations with those of an open loop system with the same input-output characteristic was previously introduced by the authors. This direct comparison led to a meaningful definition of insensitiveness of a system relative to an open loop realization. For time-invariant systems, and for a performance index involving a time-domain integral of a quadratic form of the error, sufficiency criteria in the frequency domain were obtained for relative sensitivity for a wide class of inputs. In this paper, other frequency domain criteria which are sufficient as well as necessary are obtained. A simple time-domain criterion for linear time-discrete systems is also derived.

The sensitivity of nonlinear systems is similarly investigated. Necessary and sufficient conditions for the insensitiveness of one system relative to another are derived for a class of nonlinear systems with small parameter perturbations. Furthermore, a possible suboptimization scheme for choosing a feedback realization is suggested.
1. Introduction

A basic consideration in any control system engineering analysis or synthesis is that of the sensitivity of the system to parameter variations. One of the earliest works on sensitivity of feedback systems is Bode's [1]. He considered the effects of a parameter change on the transfer function of a linear time-invariant system with a single input and a single output. The sensitivity function he defined has been useful in the study of such classes of systems [1, 2, 3, 4, 5]. Linear time-invariant multivariable control systems have been treated only fairly recently [6, 7, 8, 9, 10, 11]. The sensitivity of multivariable linear time-invariant systems has been approached in several ways [3, 12, 13, 14, 15]. Linear time-varying system sensitivity is more involved computationally but conceptually it is not much more complicated [15, 16, 17]. In all of the above, the sensitivity is concerned with the effect of parameter variations on the transfer function matrix or operators for linear systems. Closely related to the sensitivity of transfer functions is the sensitivity of pole and zero locations [18, 19, 20, 21, 22, 23].

Another class of sensitivity problems pertains to optimal systems [24, 25, 26, 27], that is, the effect of parameter variations on some performance index to be optimized is of interest. Finally, the sensitivity of nonlinear trajectories has been studied by considering the partial derivations of the trajectories with respect to the parameters [28, 29, 30]. In this paper, the earlier results of the authors [12, 13, 15, 17] are extended and unified with new results for nonlinear systems.
2. Formulation of the Sensitivity Problem

Consider a control system with an n-vector output $\xi(t)$ and a q-vector input function $\zeta(t)$ where $\xi(t)$ and $\zeta(t)$ are real valued vector functions of time $t$, $t \in [0, t_1]$ and $t_1$ is an arbitrary positive number. Let the real p-vector $\alpha$ denote the p parameters of the system which are subject to variation. These parameters may be known to lie in various intervals. In many practical situations the parameters may be random variables or even random processes with either known or unknown probability distributions. For simplicity we assume that the parameters are independent of time. This simplifying assumption does not exclude cases where a time-varying parameter may be expressed as a known function of time and several time-invariant parameters. To emphasize the dependence of the output on the parameter vector $\alpha$, we shall also write $\xi(t)$ as $\xi(t, \alpha) \Delta y(t, \alpha, \alpha_1, \alpha_2, \ldots, \alpha_p)$. In many control applications the input-output relationship between $\zeta(t)$ and $\xi(t)$ must approximate in some sense a specified ideal input-output characteristic. We assume that when $\alpha$ takes on a nominal value $\alpha = \alpha_0$, the system realizes the desired input-output characteristic. Let the deviation in the output be denoted by

$$e(t, \Delta \alpha) \Delta y(t, \alpha_0) - y(t, \alpha_0 + \Delta \alpha)$$

for an arbitrary input $\zeta(t)$, $t \in [0, t_1]$. In general there is no unique structure which will realize the desired input-output characteristic for $\alpha = \alpha_0$. Let $e_1(t, \Delta \alpha)$ and $e_2(t, \Delta \alpha)$ denote the errors corresponding to two realizations where the same sample values for $\alpha$ and the same input $\zeta(t)$ are involved. For all allowed inputs $\zeta(t)$ we assume that
\( y_1(t, \Delta \alpha) = y_2(t, \Delta \alpha) \), for all \( t \in [0, t_1] \). From a synthesis standpoint it is desirable to compare \( e_1(t, \Delta \alpha) \) to \( e_2(t, \Delta \alpha) \). Choosing a norm appropriate for the application we shall say that system realization 1 is less sensitive to variations in \( \alpha \) than system realization 2 if there exists a real positive \( a < 1 \) such that

\[
\| e_1(t, \Delta \alpha) \| \leq a \| e_2(t, \Delta \alpha) \|
\]

(2)

for all allowable inputs and all allowable \( \Delta \alpha \)'s. Note that since the same sample \( \Delta \alpha \) is involved in the two norms, and since the inequality must be satisfied for every allowable \( \Delta \alpha \) the fact that \( \Delta \alpha \) may be random is irrelevant to our definition of relative insensitiveness of two systems. A less stringent comparison may be formulated by performing a statistical operation on \( e(t, \Delta \alpha) \). For instance, the norms in (2) may be replaced by the statistical expectations of the corresponding norms.

In many control system applications the output \( y(t) \) is an output of a subsystem called the plant, with an input function \( u(t) \) which is a real valued m-vector function of time. Usually, the \( \alpha \) vector pertains to the plant subsystem. The function \( u(t) \) in turn is the output of a subsystem called the controller. The controller inputs may include \( \tau(t), y(t), \) and \( z(t) \) where \( z(t) \) is a real valued r-vector function output of the plant other than \( y(t) \). If neither \( y \) nor \( z \) are inputs of the controller, the system is called an open loop control system. Otherwise the system is called a closed loop or feedback system. It is convenient to compare a feedback realization to an open loop realization using inequality (2). That any feedback realization is not automatically less sensitive than
an open loop realization is well known even for the linear time-invariant single-variable system [3]. Since feedback may introduce certain difficulties, e.g. instability, it is important to justify the use of a feedback realization. For instance, we may require that it be less sensitive than an open loop realization. In the following sections we shall formulate conditions for the satisfaction of inequality (2). These conditions or criteria may be incorporated in various procedures for synthesis of control systems, or they may be used to analyze and evaluate a proposed realization.

3. Frequency Domain Criteria for Linear Time-Invariant Systems

If a control system can be described by ordinary linear differential equations with constant coefficients, then we may conveniently use the Laplace transforms of all the signal variables. We shall denote the Laplace transforms of the various time functions by the corresponding capital letters. Let $T(s, \alpha)$ denote the transfer function matrix relating $R(s)$ to $Y(s, \alpha)$. Then

$$E(s, \Delta \alpha) = [T(s, \alpha_0) - T(s, \alpha_0 + \Delta \alpha)] R(s) \Delta T R.$$ (3)

When there is no danger of confusion we shall drop the function arguments for convenience. We shall continue to use the subscript $c$ for closed loop and $0$ for open loop.

Suppose that the system feedback structure is described by

$$Y_c(s, \alpha) = P(s, \alpha) U_c(s, \alpha)$$ (4)

$$U_c(s, \alpha) = G(s) \left[ T(s) - H(s) Y_c(s, \alpha) \right]$$ (5)
We wish to compare this system to an open loop system described by

\[ Y_0(s, a) = P(s, a) U_0(s) \]  

\[ U_0(s) = G_1(s) R(s) \]  

where \( Y_0(s, a_0) = Y_c(s, a_0) \) for the same \( R(s) \). In a previous paper [12] we have proved that for such systems

\[ E_c \triangleq S E_0 = [I + PGH]^{-1} E_0 \]  

where \( I \) is the identity matrix of order \( n \). If we consider only those systems for which any allowable \( R(s) \) results in a unique \( Y(s, a) \) or unique \( U(s, a) \), the inverse in (8) always exists. The plant matrix \( P \) in (8) is for \( a = a_0 + \Delta a \). When \( n = q = m = p = l \) and \( \Delta a \to 0 \), \$ reduces to a scalar and in fact reduces to the Bode formula. In our formulation, however, there is no restriction on \( n, q, m, \) or \( p \) so long as they are finite.

One convenient norm is the \( L^2[0, t_2] \) norm leading to

\[ \|e_c\|^2 = \int_0^{t_2} e_c^T(t)e_c(t)dt \leq a \int_0^{t_2} e_0^T(t)e_0(t)dt = \|e_0\|^2 \]  

with \( 0 < a < 1 \). For finite \( t_2 \), the resulting frequency domain criterion is rather involved unless we restrict \( e(t) \) to be periodic. We then consider the class of all periodic inputs of finite but otherwise arbitrary period \( t_2 \) such that

\[ \|r\|^2 = \int_0^{t_2} r^T(t)r(t)dt \]  

is bounded. We also assume that the system is stable. Otherwise e
will not approach a periodic function. Applying the finite version of Parseval's Theorem to (9) we obtain

\[ \sum_{k = -\infty}^{\infty} |\tilde{e}_k|^2 \leq a \sum_{k = -\infty}^{\infty} |\tilde{e}_0|^2 \tag{11} \]

where \( \tilde{e}_n \) is the Fourier coefficient of \( e \) and \( \tilde{e}_n^* \) is the conjugate transpose of \( \tilde{e}_n \). Since the Laplace transforms of \( e_c(t) \) and \( e_0(t) \) are related by \( S \) and since the system is stable the Fourier coefficients are simply related by

\[ \tilde{e}_k = S(k_j \omega_0, \alpha) \tilde{e}_0 \tag{12} \]

where \( \omega_0 = 2\pi/t_2 \). Hence (11) is equivalent to

\[ \sum_{k = -\infty}^{\infty} \tilde{e}_0^*(S^*S - aI) \tilde{e}_0 \leq 0 \tag{13} \]

where \( S^* \) is the conjugate transpose of \( S \). Since \( S \) is a matrix of rational function in \( s \) with real coefficients, \( S^* = S^T(-j\omega_0, \alpha) \). Since the Fourier coefficients \( \tilde{e}_k \) are arbitrary and span the unitary \( q \)-space, the dimension of \( \tilde{r}(t) \), whereas \( \tilde{e}_k \) and \( \tilde{e}_0 \) are in unitary \( n \)-space, the \( \tilde{e}_0 \)'s will span \( n \)-space only if \( n < q \) and if \( \Delta T_0 \) is of rank \( n \) for all \( \omega \) and all \( \Delta \alpha \). Also, since \( t_2 \) (and hence \( \omega_0 \)) is arbitrary (13) must be satisfied for all real \( \omega \). Consequently, for the case when \( \tilde{e}_0 \) spans \( n \)-space, a necessary and sufficient condition for the satisfaction of (13) is

\[ S^T(-j\omega, \alpha) S(j\omega, \alpha) - aI \leq 0 \tag{14} \]

for all real \( \omega \) and all \( \alpha \) in the allowed space. If \( q > n \) or if the rank
of $\Delta T_0$ is less than $n$ for some $\omega$ or some $\Delta \omega$ then (14) is only a sufficient condition.

Condition (14) is the same condition previously derived as a sufficient condition [12, 13]. We reiterate our assumption that $\tau(t)$ is restricted to the class of all periodic inputs in $L^2[0, t_2]$ where $t_2$ is arbitrary. This assumption is not clearly stated in the previous papers [12, 13, 15]. From a practical viewpoint, we may interpret the criterion (9) as follows: Suppose we are interested in the output error for an arbitrary input waveform $\tau(t)$, $t \in [0, t_2]$ and a specific $\Delta \omega$ and a specific $t_2$. We apply a sequence of pulses where each pulse is an exact replica of the specified waveform. We choose $Mt_2$ to be several time constants of the system. Then the wave shapes for $e_c(t)$ and $e_0(t)$ for computing the norm in (9) are taken to be those corresponding to the $M$th time interval.

By Parseval's Theorem the same conditions hold when $t_2$ is taken as infinity provided \{$\tau(t)$\} is the class of $L^2[0, \infty)$ functions, and provided the system is stable so that \{$e_c(t)$\} and \{$e_0(t)$\} are also in $L^2[0, \infty)$.

We may rewrite (13) as

$$
\sum_{k = -\infty}^{\infty} R_k^* \Delta T_0^*(S^* S - a \bar{I}) \Delta T_0 \bar{R}_k \leq 0.
$$

(15)

Since (15) must be satisfied for arbitrary $\omega_0$ and arbitrary $R_k$ which spans the unitary $q$-space, it is equivalent to

$$
\Delta T_0^T(-j\omega, \omega)S^T(-j\omega, \omega)S(j\omega, \omega) - a \bar{I} \Delta T_0(j\omega, \omega) \leq 0
$$

(16)

for all $\omega$ and all $\omega$ in the allowed parameter space. That is (16) is a
necessary and sufficient condition for the satisfaction of (9) for all periodic inputs \( x(t) \) in \( L^2[0, t_2] \).

Criteria involving \( S^{-1} \) are often times more convenient to use since \( S^{-1} = (I + PGH) \) and one matrix inversion is avoided. Condition (14) may be rewritten in terms of \( S^{-1} \). Similarly, (16) may be rewritten as

\[
\Delta T_c^* \{ (S^{-1}) S^{-1} - \frac{1}{a} I \} \Delta T_c > 0
\]  

The energy norm may be modified slightly by using a positive definite constant matrix \( Q \) in the criterion

\[
\int_0^{t_2} e_T(t) Q e_T(t) dt \leq a \int_0^{t_2} e_0^T(t) Q e_0(t) dt
\]

resulting in analogous frequency domain criteria \([13, 15]\).

Finally, for feedback systems where not all the plant outputs are system outputs, the sensitivity matrix formula may be too unwieldy to be practically useful \([15]\). In such cases, by a derivation analogous to the above we obtain

\[
\Delta T_c^* (-j\omega, \alpha) Q \Delta T_c (j\omega, \alpha) - a \Delta T_0^* (-j\omega, \alpha) Q \Delta T_0 (j\omega, \alpha) \leq 0
\]

for \( 0 < a < 1 \), all real \( \omega \), and all allowable \( \alpha \), as a necessary and sufficient condition for the satisfaction of (18).

In all practical cases, it is reasonable to assume that the input signals are band limited. Furthermore, the control system usually has a los-pass or band-pass frequency characteristic. This means that all the above criteria may be relaxed in practice by requiring the inequalities to hold only over the frequency band of interest rather than for all real \( \omega \).
Furthermore, if parameter perturbations are small and if $\Delta T_0$ is essentially linearly proportional to the $\Delta a_i$'s for all $\Delta a_i$'s, then by simply verifying (14) for $\alpha = \alpha_0$, (9) is satisfied for some neighborhood $0 < \| \Delta \alpha \| < \delta$. (See Section 5 for a detailed discussion of a more general case.) This is analogous to the well known scalar case requirement of $|S(j\omega)| < 1$ over the frequency band of interest, where $S$ is computed only at the nominal parameter value.

Similar criteria can be derived for systems described by linear difference equations with constant coefficients. The frequency criteria are of course in the $z$-transform domain, i.e., on the unit circle of the $z$-plane [13].

Simple second order multivariable systems were synthesized with and without the criteria as synthesis constraints. The various systems were simulated on the analog computer [12]. From a qualitative inspection of the step responses for various settings of $\alpha$, the system synthesized using the criteria as constraints seemed less affected by change in $\alpha$. Another interesting feature of the criteria is that (14), (16), (17), or (19) is a sufficient condition for the system to be optimal for some quadratic performance index [31, 13].

4. Time Domain Considerations for Linear Systems

For the time-invariant linear system we may relate the signals $x(t), y(t, \alpha), e_c(t, \Delta \alpha), e_d(t, \Delta \alpha), u_0(t), \text{ and } u_c(t, \alpha)$ by appropriate matrix convolutions. For time-varying systems, the convolutions are replaced by more general linear operators. Thus for a feedback structure
where the plant output is the system output which is fed back, instead of (8) we have

\[ \tilde{e}_c(t, \Delta \alpha) = \int_0^t \mathcal{J}(t, \tau, \alpha) e_0(t) \Delta \alpha \, dt \triangleq \hat{S} e_0(t, \Delta \alpha) \]  

(20)

where the operator \( \hat{S} \) is given by

\[ \hat{S} = [I + \hat{P} G H]^{-1} \]  

(21)

Again, we use the \( L^2[0, t_2] \) norm and the definition of relative insensitivity of (9) or (18). However, we do not have to assume that \( r(t) \) is periodic so long as \( \tilde{e}_c \) and \( e_0 \) are in \( L^2[0, t_2] \) for finite \( t_2 \).

Instead of the criterion in (14) we have [15, 17]

\[ \hat{S}^* \hat{S} - \alpha I \leq 0 \]  

(22)

a negative semi-definite operator where \( \hat{S}^* \) is the adjoint of the operator \( \hat{S} \). Similarly, the operator analog of (19) is

\[ \Delta \hat{T}^* \bar{Q} \Delta \hat{T} - \alpha \Delta \hat{T}^* \bar{Q} \Delta \hat{T} \leq 0 \]  

(23)

Condition (23) is a necessary and sufficient condition whereas (22) is sufficient but may not be necessary for the satisfaction of (9) for arbitrary inputs in \( L^2[0, t_2] \).

A simple case of the above criteria arises when the system is a single-input single-output time-discrete linear system. The input and output time sequences are related by

\[ y(k) = \sum_{j=1}^{k} t(k,j)r(j) \]  

(24)

where \( t(k,j) \) is the discrete version of the system impulse response.
The relation may be written in matrix form as

\[
\begin{bmatrix}
y(1) \\
y(2) \\
\vdots \\
y(n)
\end{bmatrix} =
\begin{bmatrix}
t(1,1) & 0 & \cdots & 0 \\
t(2,1) & t(2,2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
t(n,1) & \cdots & t(n,n)
\end{bmatrix}
\begin{bmatrix}
r(1) \\
r(2) \\
\vdots \\
r(n)
\end{bmatrix}
\]

or

\[
y = T \tilde{r}
\]

where \( T \) is a lower triangular matrix, assuming that the system is nonanticipative. Using this notation for a simple feedback system with equations analogous to (4), (5), (6), and (7), we obtain

\[
\tilde{T} = (\tilde{I} + \tilde{P}\tilde{G}\tilde{H})^{-1} \tilde{P}\tilde{G}.\]

Defining \( \tilde{e}_c \) and \( \tilde{e}_0 \) in analogous fashion we obtain

\[
\tilde{e}_c = (\tilde{I} + \tilde{P}\tilde{H}\tilde{G})^{-1} \tilde{e}_0 \triangleq \tilde{S} \tilde{e}_0
\]

where the operators are now reduced to simple matrices of real constants.

Note that assuming nonanticipative plant and controllers, \( \tilde{P}\tilde{G}\tilde{H} \) is lower triangular and if none of the main diagonal elements of \( \tilde{P}\tilde{G}\tilde{H} \) is equal to -1, \( (\tilde{I} + \tilde{P}\tilde{G}\tilde{H})^{-1} \) exists. If \( (\tilde{I} + \tilde{P}\tilde{G}\tilde{H})^{-1} \) does not exist, then the system has no unique solution for \( \tilde{y} \). As norm we choose

\[
\|\tilde{e}\|^2 = \tilde{e}^T \tilde{S} \tilde{e} = \sum_{k=1}^{n} e^2(k).
\]

Then we say that the feedback system is less sensitive than an open loop system if (2) is satisfied for all input sequences \( \tilde{r} \) of length \( n \), \( r(k) \in \mathbb{R}^2, k = 1, \ldots, n \), for all allowable \( \tilde{a} \). We then have

\[
\tilde{e}_0^T (\tilde{S}^T \tilde{S} - \tilde{a} \tilde{I}) \tilde{e}_0 \leq 0
\]

If \( \Delta \tilde{T}_0 \) is nonsingular then since \( \tilde{e}_0 = \Delta \tilde{T}_0 \tilde{r} \), \( \tilde{e}_0 \) spans the Euclidean n-space.
Hence, a necessary and sufficient condition for the satisfaction of (28) is

\[ S^T S - a I \leq 0 \]  \hspace{1cm} (29)

If \( \Delta T_0 \) is singular, (29) is only a sufficient condition. Again we emphasize that \( S \) is just a matrix of real constants. Alternatively we have

\[ (S^{-1})^T S^{-1} - \frac{1}{a} I \geq 0 . \]  \hspace{1cm} (30)

If the system is time-invariant then \( S \) is related to the inverse \( z \)-transform of the scalar sensitivity function. Although the time domain test is much more complicated than checking \( |S(z)| < 1 \) on the unit circle, the frequency domain criterion assumes a periodic input sequence for a stable system under steady state conditions whereas the time-domain criterion is not so restricted. The positive definite matrix testing may be carried out routinely on a digital computer if \( n \) is large.

If the range of \( \Delta \alpha \) is differentially small then \( S \) may be computed at the nominal value of \( \alpha \) and only one positive definite test is carried out to verify whether (2) is satisfied or not. Contrast this to the impossibility of checking (2) by direct computer simulation of the closed loop and open loop systems for all inputs and all small \( \Delta \alpha \). Even if only one \( \Delta \alpha \) is chosen, we still have the practically impossible task of applying all kinds of inputs if we insist on simulation. Compared to this then, the criterion in (29) or (30) results in a vast simplification.
5. Nonlinear Systems with Small Parameter Perturbations

Let us assume that for all allowable inputs and all $t \in [0, t^*]$, $y_c(t, \alpha)$ and $y_0(t, \alpha)$ are smooth functions of $\alpha$ so that all their first order and second order partial derivatives with respect to all the parameters $\alpha_i$, $i = 1, \ldots, p$ exist in some neighborhood of $\alpha_0$. We also assume that when $\Delta \alpha = 0$,

$$y_c(t, \alpha_0) = y_0(t, \alpha_0)$$

for every input. Then for $\alpha$ in the neighborhood of $\alpha_0$, we may write

$$\varepsilon(t, \Delta \alpha) = y(t, \alpha_0 + \Delta \alpha) - y(t, \alpha_0) =$$

$$\sum_{i=1}^{p} \frac{\partial y}{\partial \alpha_i}(t, \alpha_0) \Delta \alpha_i$$

$$+ \frac{1}{2} \sum_{r=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 y}{\partial \alpha_r \partial \alpha_j}(t, \alpha_0 + k \Delta \alpha) \Delta \alpha_r \Delta \alpha_j$$

where $K$ is a diagonal matrix with elements $0 < K_{ii} < 1$. We also assume that

$$\lim_{\|\Delta \alpha\| \to 0} \frac{\|\frac{1}{2} \sum_{r=1}^{p} \sum_{j=1}^{p} \frac{\partial^2 y}{\partial \alpha_r \partial \alpha_j}(t, \alpha_0 + K \Delta \alpha) \Delta \alpha_r \Delta \alpha_j\|}{\|\frac{\partial y}{\partial \alpha_i}(t, \alpha_0) \Delta \alpha_i\|} \to 0$$

for all allowable inputs. That is, for any real $\epsilon > 0$ there exists a $\delta > 0$ such that for all $\Delta \alpha$ satisfying $0 < \|\Delta \alpha\| < \delta$
for all allowable inputs. Thus, applying the triangle inequality to (32) we have

\[ ||e(t, \Delta a)|| \leq \sum_{i=1}^{P} \frac{\partial y}{\partial \alpha_i} (t, a_0) \Delta \alpha_i || (1 + \epsilon) \quad (35) \]

Similarly

\[ ||e(t, \Delta a)|| \geq \sum_{i=1}^{P} \frac{\partial y}{\partial \alpha_i} (t, a_0) \Delta \alpha_i || (1 - \epsilon) \quad (36) \]

If the above assumption in (33) holds, then there exists a neighborhood of \( \Delta a \), \( 0 < ||\Delta a|| < \delta \) such that for all \( \Delta a \) in the neighborhood and for all allowable inputs condition (2) is equivalent to the condition that there exists a real number \( b \), \( 0 < b < 1 \) such that

\[ \sum_{i=1}^{P} \frac{\partial y_c}{\partial \alpha_i} (t, a_0) \Delta \alpha_i || \leq b \sum_{r=1}^{P} \frac{\partial y_0}{\partial \alpha_i} (t, a_0) \Delta \alpha_i || \quad (37) \]

To show that (37) implies (2) we choose

\[ \epsilon = \frac{a - b}{a + b} \quad (38) \]

where \( a \) is any real number such that \( b < a < 1 \). Clearly \( 0 < \epsilon < 1 \). Then we may rewrite (37) as

\[ \sum_{i=1}^{P} \frac{\partial y_c}{\partial \alpha_i} \Delta \alpha_i || (1 + \epsilon) \leq \left[ \frac{b(1+\epsilon)}{(1-\epsilon)} \right] \sum_{r=1}^{P} \frac{\partial y_0}{\partial \alpha_i} \Delta \alpha_i || (1 - \epsilon) \quad (39) \]

Using (35) and (36) we have

\[ ||e_c|| \leq \left[ \frac{b(1+\epsilon)}{(1-\epsilon)} \right] ||e_0|| \quad (40) \]
From the choice of $\varepsilon$

$$\frac{b(1+\varepsilon)}{(1-\varepsilon)} = a < 1$$  \hfill (41)

and hence we have (2) for some neighborhood $0 < ||\Delta a|| < \delta$ corresponding to the choice of $\varepsilon$. To show that (2) implies (37) we choose $\varepsilon$ in the range $0 < \varepsilon < 1$. Using (35) and (36), (2) implies

$$2 \sum_{r=1}^{p} \frac{\partial y_{c}}{\partial a_{i}} \Delta a_{i} (1 - \varepsilon) \leq a \sum_{r=1}^{p} \frac{\partial y_{0}}{\partial a_{i}} \Delta a_{i} (1 + \varepsilon),$$ \hfill (42)

$$\sum_{r=1}^{p} \frac{\partial y_{c}}{\partial a_{i}} \Delta a_{i} \leq \left[ \frac{a(1+\varepsilon)}{1-\varepsilon} \right] \sum_{i=1}^{p} \frac{\partial y_{0}}{\partial a_{i}} \Delta a_{i}.$$ \hfill (43)

For a choice of

$$\varepsilon = \frac{b-a}{b+a}$$ \hfill (44)

where $b$ is any real number satisfying $a < b < 1$, then clearly $0 < \varepsilon < 1$ and (43) reduces to (37) as required.

We note that since the $\Delta a_{i}$'s are independent parameter variations which are arbitrary except for the restriction inside the $\delta$ neighborhood, then (37) is equivalent to

$$\sum_{i=1}^{p} K_{i} \frac{\partial y_{c}}{\partial a_{i}} \leq b \sum_{i=1}^{p} K_{i} \frac{\partial y_{0}}{\partial a_{i}} \text{ for } -1 \leq K_{i} \leq 1$$ \hfill (45)

for all $i$, $i = 1, \ldots, p$. Thus (45) is equivalent to (2) provided (33) holds. Note also that since $\delta$ is a monotonically increasing function of $\varepsilon$, it is desirable to make $\varepsilon$ as large as possible. From (38) we see that if (45) is satisfied with as small a $b$ as possible, then for $a$ close to 1,
€ can be made as close to 1 as possible resulting in the satisfaction of (2) for a δ as large as possible.

Let us consider a nonlinear control system, possible time-varying also, described in state vector form:

\[
\frac{\partial}{\partial t} x(t, \alpha) = f(x, u, \alpha, t) \tag{46}
\]

\[
y(t, \alpha) = h(x, r, \alpha, t) \tag{47}
\]

\[
u(t, \alpha) = g(x, y, r, t) \tag{48}
\]

\[
x(0, \alpha) = x_0 \tag{49}
\]

for \( t \in [0, t_1] \), where \( x(t, \alpha) \) is the state vector, \( y(t, \alpha) \) is the system output, \( r(t) \) is the system input, \( u(t, \alpha) \) is the plant input, \( g \) is the control law, \( x(0, \alpha) \) is the initial state, and \( \alpha \) is the parameter vector subject to variation, with a nominal value \( \alpha_0 \). Let the dimensions of \( x, y, u, r, \) and \( \alpha \) be \( r, n, m, q, \) and \( p \) respectively. For an open loop system, the control law \( g \) does not depend on \( x \) nor \( y \). We now wish to compare different systems with exactly the same plant equations as in (46), (47), and (49) but with different control laws. However, for \( \alpha = \alpha_0 \), the corresponding plant inputs \( u_1(t, \alpha_0) \) and \( u_2(t, \alpha_0) \) are equal for all allowable \( r(t) \) and all \( t \in [0, t_1] \). We may wish to compare \( \|e_1\| \) with \( \|e_2\| \) for all inputs and a particular initial state or for all allowed initial states. The specific application will of course dictate the choice of criteria. Taking the partial derivative of (45), (46), and (47) with respect to \( \alpha_1 \) and simplifying, we have
\[
\frac{\partial}{\partial t} \frac{\partial x(t, \alpha)}{\partial \alpha_i} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} \frac{\partial h}{\partial x} - \frac{\partial x(t, \alpha)}{\partial \alpha_i} \\
+ \frac{\partial f}{\partial u} \frac{\partial g}{\partial y} \frac{\partial h}{\partial \alpha_i} + \frac{\partial f}{\partial \alpha_i} \tag{50}
\]

and

\[
\frac{\partial y}{\partial \alpha_i} (t, \alpha) = \frac{\partial h}{\partial x} \frac{\partial x}{\partial \alpha_i} + \frac{\partial h}{\partial \alpha_i} \tag{51}
\]

We of course assume that the indicated partial derivatives in the various Jacobian matrices exist. Since \(x(0, \alpha)\) is assumed to be a constant vector in (49) we have

\[
\frac{\partial x}{\partial \alpha_i} (0, \alpha_0) = 0 \tag{52}
\]

For any \(\alpha(t)\) and \(\alpha_0\), \(x(t, \alpha_0)\) and \(y(t, \alpha_0)\) are obtained from (46), (47), (48), and (49). The various Jacobians in (50) and (51) are then known functions of time and thus (50), (51), and (52) constitute a set of linear time-varying differential equations for \(\frac{\partial x}{\partial \alpha_i}\) and \(\frac{\partial y}{\partial \alpha_i}\) which are the sensitivity functions in Tomovic [28, 29, 30].

All these calculations may be automatically carried out on an analog computer to compare two systems. Unfortunately it is practically impossible to verify (45) for an infinite number of inputs. A less ambitious criterion might be to compare two systems only for a small number of important inputs and initial conditions, and thus modify criterion (2) accordingly. The analog simulation then includes a concurrent computation of the chosen norms for the problem.
As a possible suboptimization scheme suppose that the control law contains some control parameters \( q_1, q_2, \ldots, q_s \), represented by a vector \( q \) such that for \( q \in Q \) and \( \alpha = \alpha_0 \)

\[
u(t, \alpha) + g(x, y, r, t, q) = g(x, y, r, t, q_0)
\]  

(53)

This means that for all \( q \in Q \), the control laws are all equivalent whenever the parameter vector \( \alpha = \alpha_0 \). Suppose that instead of (2) we have the criterion

\[\sum_{k=1}^{m} \| e_{\alpha}(t, \Delta \alpha, q) \|_k \leq a \sum_{k=1}^{m} \| e_{0}(t, \Delta \alpha) \|_k \]  

(54)

where \( ||e||_k \) is the norm corresponding to a chosen input \( r_k(t) \) and initial condition \( x_k(0) \), and the summations are over the \( M \) selected pair of inputs and initial conditions, and \( 0 < a < 1 \). In addition to requiring (59) we may choose \( q \) such that (54) is satisfied with as small a number \( a \) as possible. If desired, weighting coefficients may be used in (54). If (33) holds for the selected \( M \) pair of inputs and initial conditions, then this minimization is equivalent to the minimization of

\[ J = \sum_{k=1}^{M} \left\| \frac{\partial y}{\partial \alpha_i}(t, \alpha_0) \right\|_k \]  

(55)

with respect to \( q, q \in Q \). This then yields the least sensitive system within a class of control law realizations.
6. Conclusion

The sensitivity of a feedback control system may be studied by comparing the susceptibility of the system of parameter variations to that of an open loop system with the same nominal input-output characteristics. By comparing appropriate norms of the output errors due to parameter variations a meaningful definition of relative insensitivity was introduced. In the case of linear time-invariant multivariable systems where the system outputs are also the plant outputs used as feedback signals, the Laplace transforms of the errors are simply related by a sensitivity matrix. If the norm used is the average energy of the steady state error due to periodic inputs of arbitrary periods, then sufficient as well as necessary and sufficient frequency domain conditions guarantee the relative insensitivity of the feedback system for all periodic inputs. The same frequency domain conditions apply for arbitrary inputs which cause errors with bounded total energy. Analogous conditions apply in the time domain. For the linear case, and using an energy norm, the criteria involve testing certain operators for positive semi-definiteness. The operators reduce to ordinary matrices for time-discrete systems. For the nonlinear case with small parameter perturbations and with a small number of possible inputs, a small number of analog computer runs can verify whether there is a neighborhood of $\alpha_0$ such that a system is less sensitive compared to another, for all (infinitely many) small perturbations in the small neighborhood of $\alpha_0$. 
Acknowledgment

The research reported in this paper was supported in part by the Joint Services Electronics Program by the Department of the Army, Department of the Navy (Office of Naval Research) and Department of the Air Force (Office of Scientific Research), and by the Advanced Research Projects Agency under Department of the Army Contract DA-28-043-AMC-00073(E) and Department of the Air Force (Office of Scientific Branch) Contract AF 49(638)-1383.

References


### Distribution list as of March 1, 1965 (Cont’d.)

<table>
<thead>
<tr>
<th>Order</th>
<th>Institution, Address</th>
<th>Contact</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Lincoln Laboratory, Massachusetts Institute of Technology</td>
<td>F. O. Box 71</td>
<td>*Dr. Robert Kingston</td>
</tr>
<tr>
<td>1</td>
<td>APOC (MAFF)</td>
<td>Bolling Air Force Base, Florida</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Mr. Alan Bessom</td>
<td>Boeing Air Development Center, Griffiss Air Force Base</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Director</td>
<td>Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts, 02139</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Polytechnic Institute of Brooklyn</td>
<td>33 Johnson Street, Brooklyn, New York 11201</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Director</td>
<td>Radiation Laboratory, Columbia University, 539 West 128th Street, New York, New York 10027</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Director</td>
<td>Graduate Science Laboratory, University of Illinois, Urbana, Illinois 61801</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Director</td>
<td>Stanford Electronics Laboratories, Stanford University, Stanford, California</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Professor A. A. Beugel</td>
<td>Director, Laboratories for Electronics and Related Sciences Research, University of Texas, Austin, Texas 78712</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Professor J. E. Applegar</td>
<td>Department of Electrical Engineering, University of Texas, Austin, Texas 78712</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Director of Engineering Applied Physics</td>
<td>710 Pierce Hall, Harvard University, Cambridge, Massachusetts 02138</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>NASA Headquarters</td>
<td>Office of Applications, 400 Maryland Avenue, N.W. Washington, D.C. GECO PC No. A. M. Greg Andres</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>National Bureau of Standards</td>
<td>Research Information Center and Advisory Serv., Mail Processing, Data Processing Systems Division, Washington 25, D.C.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Dr. Wallace Sillhoto</td>
<td>Institute for Defense Analyses, Research &amp; Eng. Support Div., 1466 Connecticut Avenue, N.W., Washington 9, D.C.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Data Processing Systems Division</td>
<td>National Bureau of Standards, Comm. at Von Nees, Room 239, Bldg. 10, Washington 25, D.C. Attn: A. E. Behlow</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Exchange and Gift Division</td>
<td>The Library of Congress, Washington 25, D.C.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Dr. Alan T. Waterman</td>
<td>Director, National Science Foundation, Washington 25, D.C.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>R. E. Courtes</td>
<td>Oak Ridge National Laboratory, P. O. Box 6, Oak Ridge, Tennessee</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>H. S. Atkinson</td>
<td>Energy Commission, Office of Technical Information Extension, P. O. Box 62, Oak Ridge, Tennessee</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Mr. C. D. Watson</td>
<td>Defense Research Board, Canadair Joint Staff, 4500 Massachusetts Avenue, N.W., Washington 8, D.C.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Martin Company</td>
<td>P. O. Box 5837, Orlando, Florida</td>
<td>Attn: Engineering Library NW-30</td>
</tr>
<tr>
<td>1</td>
<td>Laboratories for Applied Sciences</td>
<td>University of Chicago, 5200 South Decatur, Chicago, Illinois 60637</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Order</th>
<th>Institution, Address</th>
<th>Contact</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Librarian</td>
<td>School of Electrical Engineering, Purdue University, Lafayette, Indiana</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Donald L. Ely</td>
<td>Dept. of Electrical Engineering, University of Iowa, Iowa City, Iowa</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Instrumentation Laboratory</td>
<td>Massachusetts Institute of Technology, 68 Albany Street, Cambridge 39, Massachusetts</td>
<td>Attn: Library V2-129</td>
</tr>
<tr>
<td>1</td>
<td>Sylvania Electric Products, Inc.</td>
<td>Electronics System Division, Waltman Labs., 345 First Avenue, P.O. Box 34, Massachusetts</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Hughes Aircraft Company</td>
<td>California, California</td>
<td>Attn: E. C. Rosenburg, Supervisor Company Technical Document Center</td>
</tr>
<tr>
<td>3</td>
<td>Automobiles</td>
<td>9550 East Deerfield Highway, Downey, California</td>
<td>Attn: Tech. Library, 304-11</td>
</tr>
<tr>
<td>1</td>
<td>Dr. Arnold T. Norddahl</td>
<td>General Motors Corporation, Defense Research Laboratories, 6747 Bullis Street, Glendale, California</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>University of California</td>
<td>Lawrence Radiation Laboratory, P. O. Box 228, Livermore, California</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Mr. Thomas J. Burtwell</td>
<td>Aerospace Corporation, P. O. Box 9543, Los Angeles 45, California</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Lt. Col. William Levin</td>
<td>Aerospace Corporation, P. O. Box 9543, Los Angeles 45, California</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Sylvania Electronics Systems-West</td>
<td>Electronic Science Laboratories, P. O. Box 305, Mountain View, California</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Varian Associates</td>
<td>611 Hansen Way, Palo Alto, California 94303</td>
<td>Attn: Tech. Library</td>
</tr>
<tr>
<td>1</td>
<td>Rocket Data</td>
<td>Library Supervisor, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Professor Nicholas George</td>
<td>California Institute of Technology, California Engineering Department, Pasadena, California</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Space Technology Labs., Inc.</td>
<td>One Space Park, Redondo Beach, California</td>
<td>Attn: Acquisitions Group STL Technical Library</td>
</tr>
<tr>
<td>1</td>
<td>The Rand Corporation</td>
<td>1700 Main Street, Santa Monica, California</td>
<td>Attn: Library</td>
</tr>
<tr>
<td>1</td>
<td>IBM</td>
<td>1100 North Broadway, Palisades Park, California 94303</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Navy Department</td>
<td>Office of Naval Research, Attn: Library 4361-11</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Dr. L. Torres Quevedo</td>
<td>Office of Naval Research, Attn: Library WI-109</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>National Physical Laboratory</td>
<td>London, England</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Dr. J. J. C. Macrae</td>
<td>Behavioral Science Division, Advanced Research Projects Agency, The Pentagon (Room 2072), Washington, D.C. 20301</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Dr. Glenn A. Young</td>
<td>Ortho, Personnel and Training Branch, Office of Naval Research, Navy Department, Washington, D.C. 20360</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Instituto de Fisica Aplicada</td>
<td>&quot;L. Torres Quevedo&quot; High Vacuum Laboratory, Madrid, Spain</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Stanford Research Institute</td>
<td>Attn: G-607 External Reports (for J. Goldberg) Menlo Park, California 94025</td>
<td></td>
</tr>
</tbody>
</table>
CRITERIA FOR SYSTEM SENSITIVITY TO PARAMETER VARIATIONS

A characterization of the sensitivity of a class of linear multi-variable control systems based on a comparison of the output deviations due to parameter variations with those of an open loop system with the same input-output characteristic was previously introduced by the authors. This direct comparison led to a meaningful definition of insensitiveness of a system relative to an open loop realization. For time-invariant systems, and for a performance index involving a time-domain integral of a quadratic form of the error, sufficiency criteria in the frequency domain were obtained for relative sensitivity for a wide class of inputs. In this paper, other frequency domain criteria which are sufficient as well as necessary are obtained. A simple time-domain criterion for linear time-discrete systems is also derived.

The sensitivity of nonlinear systems is similarly investigated. Necessary and sufficient conditions for the insensitiveness of one system relative to another are derived for a class of nonlinear systems with small parameter perturbations. Furthermore, a possible suboptimization scheme for choosing a feedback realization is suggested.
Control systems
sensitivity
parameter variations
feedback