ELEMENTARY COMPLETE TREE TRANSFORMATION

Wataru Mayeda

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by

Wataru Mayeda

Introduction

It is known that a passive electrical network without mutual couplings can be analyzed by knowing all possible trees of a linear graph corresponding to the network. Since there exists a reasonably simple method of generating all possible trees of a linear graph without duplications, analysis of such a network by a computer becomes indubitably simple.

When a pair of linear graphs is used, an active network can be analyzed by knowing all possible complete trees each of which is a tree of both linear graphs. At present there is no simple method of generating all possible complete trees without duplications. Hence, in order to obtain all possible complete trees by a computer, one of the best available methods at present is to generate all possible trees of each linear graph to obtain two collections of trees, then intersecting the two collections. It is not difficult to design an active network such that there are more than a thousand of trees in each of a pair of linear graphs corresponding to the net, but there are less than one hundred complete trees. Hence to obtain a simple method of generating all possible complete trees is undoubtedly important for analysis of active networks by a computer.

Obtaining one tree from another tree which is used to generate all possible trees in a linear graph is accomplished by so-called "elementary tree transformation." In this paper, this transformation is generalized so that a complete tree can be obtained from another complete tree by the generalized
transformation namely an "elementary complete tree transformation."

Furthermore, any complete tree can be obtained from any other complete tree by successive elementary complete tree transformations. The study of the properties of such transformation will hopefully lead to find a method of generating all possible complete trees without duplications in the future.

**Edge-permutations**

In this paper, a pair of linear graphs $G_1$ and $G_2$ is assumed to have the properties that

\[
\Omega_1 = \Omega_2 \quad (1)
\]

and

\[
\xi_1 = \xi_2 \quad (2)
\]

where $\Omega_p$ and $\xi_p$ are the sets of all vertices and the set of all edges in $G_p$ respectively for $p=1,2$. It is easily seen that these properties must be satisfied when the pair of linear graphs represents an active electrical network. Notice that the location of an edge $e$ in $G_1$ and that of edge $e$ in $G_2$ may not be the same.

A complete tree $t$ of a pair of linear graphs $G_1$ and $G_2$ is a set of edges in $\xi_1$ which is a tree of $G_1$ as well as a tree of $G_2$. In a pair of linear graphs $G_1$ and $G_2$ given in Fig. 1, a set of edges $\{cd\}$ is a complete tree. However, a set of edges $\{bc\}$ is not a complete tree.

For an edge $e$ in a complete tree $t$, the symbol $S^p_e(t)$ is defined to be the fundamental cut set with respect to $t$ which contains edge $e$ in $G_p$ for $p = 1,2$. For a complete tree $t = \{cd\}$ in Fig. 1, $S^1_c(t) = \{bc\}$ and $S^2_c(t) = \{abc\}$. 
Definition 1: Let $t_1$ and $t_2$ be complete trees of a pair of linear graphs $G_1$ and $G_2$. Also let

$$t_1 - t_2 = \{a_1 \ldots a_m\}^*$$

and

$$t_2 - t_1 = \{e_1 \ldots e_m\}.$$

Then edge-permutation $\mathcal{E}(t_1, t_2)$ of $t_1$ and $t_2$ is

$$\mathcal{E}(t_1, t_2) = \begin{pmatrix}
    e_1' & e_2' & \ldots & e_m' \\
    e_1'' & e_2'' & \ldots & e_m'' \\
\end{pmatrix}$$

where each column represents edge $a_p$ $(p = 1, 2, \ldots, m)$ and entries in column $a_p$ are edges $e_p'$ and $e_p''$ which satisfy that

1. $e_p' \in S^1_{a_p}(t_1)$
2. $e_p'' \in S^2_{a_p}(t_1)$

and

3. $\{e_1' e_2' \ldots e_m'\} = \{e_1'' e_2'' \ldots e_m''\} = \{e_1 e_2 \ldots e_m\}$

* $A - B$ is a set of edges in $A$ but not in $B.$
Consider a pair of linear graphs $G_1$ and $G_2$ in Fig. 2. Let

$$t_0 = \{a_1a_2a_3\}$$
$$t_1 = \{e_1e_2a_3\}$$

then $t_0 - t_1 = \{a_1a_2\}$, $t_1 - t_0 = \{e_1e_2\}$

and the edge permutation $\epsilon(t_0t_1)$ is

$$\epsilon(t_0t_1) = \begin{pmatrix} e_1 \\ e_2 \\ e_2 \\ e_1 \end{pmatrix}$$

Fig. 2. Complete tree $t_0 = \{a_1a_2a_3\}$

Since $t_1$ and $t_2$ are trees of $G_1$, it is easily seen that we can assign each edge $e_p$ to only one of fundamental cut sets $S_{a_p}^1(t_1)$ ($p = 1,2,\ldots,m$) such that every fundamental cut set has exactly one of edges $e_1,e_2,\ldots,$ and $e_m$ and each edge $e_p$ ($1 \leq p \leq m$) is assigned to exactly one of the fundamental cut sets. This is also true for $G_2$. Thus we can always obtain an edge permutation $\epsilon(t_1t_2)$ of any two complete trees $t_1$ and $t_2$ of a pair of linear graphs $G_1$ and $G_2$. 
Consider a pair of linear graphs $G_1$ and $G_2$ in Fig. 2. Let

$t_0 = \{a_1a_2a_3\} \text{ and } t_1 = \{e_1e_2a_3\}$, then $t_0 - t_1 = \{a_1a_2\}$, $t_1 - t_0 = \{e_1e_2\}$

and the edge permutation $\varepsilon(t_0t_1)$ is

$$\varepsilon(t_0t_1) = \begin{pmatrix} e_1 & e_2 \\ e_2 & e_1 \end{pmatrix}$$

Since $t_1$ and $t_2$ are trees of $G_1$, it is easily seen that we can assign each edge $e_p$ to only one of fundamental cut sets $S^{1}_{a_p}(t_1)$ ($p = 1, 2, \ldots, m$) such that every fundamental cut set has exactly one of edges $e_1, e_2, \ldots, e_m$ and each edge $e_p$ ($1 \leq p \leq m$) is assigned to exactly one of the fundamental cut sets. This is also true for $G_2$. Thus we can always obtain an edge permutation $\varepsilon(t_1t_2)$ of any two complete trees $t_1$ and $t_2$ of a pair of linear graphs $G_1$ and $G_2$. 

![Fig. 2. Complete tree $t_0 = \{a_1a_2a_3\}$](image)
Elementary Common Tree Transformation

An elementary tree transformation is defined to be

\[ t_2 = f_1 \oplus \{ ea \} \]

where \( a \in t_1 \), \( t_2 \cdot t_1 = e \) and \( t_1 \) and \( t_2 \) are trees of a linear graph. However, if \( t_1 \) is a complete tree of a pair of linear graphs, there may exist no edge \( e \) such that \( t_1 \oplus \{ ea \} \) is a complete tree. Hence, an elementary complete tree transformation is defined in such a way to include transformation \( t_1 \oplus \{ ea \} \) if \( t_1 \oplus \{ ea \} \) is a complete tree. The formal definition is given as

**Definition 2:** Let \( t_1 \) and \( t_2 \) be complete trees of a pair of linear graphs \( G_1 \) and \( G_2 \). Also let

\[ t_1 - t_2 = \{ a_1 a_2 \ldots a_m \} \]

and

\[ t_2 - t_1 = \{ e_1 e_2 \ldots e_m \} \]

Then

\[ t_1 \oplus \{ a_1 a_2 \ldots a_m e_1 e_2 \ldots e_m \} = t_2 \]

is an elementary complete tree transformation if there exists no complete tree \( t' \) of \( G_1 \) and \( G_2 \) such that

\[ t_1 \oplus \{ a_1 a_2 \ldots a_p e_1 e_2 \ldots e_p \} = t' \]

and \( \{ a_1 a_2 \ldots a_p e_1 e_2 \ldots e_p \} \) is a proper subset of \( \{ a_1 a_2 \ldots a_m e_1 e_2 \ldots e_m \} \).

With this definition, we have an important property of elementary complete tree transformations given by the following theorem.

**Theorem 1:** Let \( t_0 \) and \( t_m \) be complete trees of a pair of linear graphs \( G_1 \) and \( G_2 \). Also let

\[ t_0 - t_m = \{ a_1 \ldots a_m \} \]
and
\[ t_m - t_0 = \{e_1 \ldots e_m\} . \]  

Then
\[ t_m = t_0 \oplus \{a_1 \ldots a_m e_1 \ldots e_m\} \]

is an elementary complete tree transformation if and only if the following two conditions are satisfied:

**Condition 1:** An edge-permutation \( \varepsilon(t_0, t_m) \) is not factorizable.

**Condition 2:**
\[ e_p \in S^1_p(t_0) \cup \bigcup_{j=1}^{m} S^1_p(t_0) \]
\[ e'_p \in S^2_p(t_0) \cup \bigcup_{j=1}^{m} S^2_p(t_0) \]

for all \( p = 1, 2, \ldots, m \) where \( \{e'_1 \ldots e'_m\} = \{e_1 \ldots e_m\} \).

Before proving Theorem 1, we will study about elementary tree transformations of a linear graph.

**Definition 3:** Let \( t_0, t_1, \) and \( t_2 \) be trees of a linear graph \( G \). Then the transformation
\[ t_2 = t_1 \oplus \{ae\} \]

where \( \{a\} = t_1 - t_2 \) and \( \{e\} = t_2 - t_1 \), is a M-tree transformation under tree \( t_0 \) if
\[ e \in S_a(t_1) \wedge S_a(t_0) . \]

**Definition 4:** A sequence of trees \( t_1, t_2, \ldots, t_m \) is a fundamental tree sequence from \( t_1 \) to \( t_m \) under \( t_0 \) in \( G \) if \( t_p \) is obtained from \( t_{p-1} \) by a M-tree transformation under \( t_0 \) for \( p = 1, 2, \ldots, m-1 \).
Consider two trees \( t_0 \) and \( t_m \) in \( G \). Suppose

\[
t_0 - t_m = \{a_1 \ldots a_m\}
\]

and

\[
t_m - t_0 = \{e_1 \ldots e_m\}
\]

Then it is known \(^2\) that for any given sequence of \( a \)'s, such as \( a_1 a_2 \ldots a_m \), there exists a fundamental tree sequence from \( t_0 \) to \( t_m \) under \( t_0 \). Also we know that if sequence \( a_1 a_2 \ldots a_m \) is a M-sequence, there exists exactly one fundamental tree sequence from \( t_0 \) to \( t_m \) under \( t_0 \). In the other words, if we give a sequence \( a_1 a_2 \ldots a_m \). Then there exists a fundamental tree sequence \( t_0 t_1 \ldots t_m \) under \( t_0 \) such that

\[
t_p = t_{p-1} \oplus \{a_p e_p\}
\]

where

\[
e_p \in S_{a_p} (t_{p-1}) \cap S_{a_p} (t_0)
\]

for \( p = 1, 2, \ldots, m \). This means that for a given sequence \( a_1 a_2 \ldots a_m \), there exists a sequence \( e_1 e_2 \ldots e_m \) such that the sequence of pairs \( \{a_1 e_1\} \{a_2 e_2\} \ldots \{a_m e_m\} \) gives a fundamental tree sequence \( t_0 t_1 \ldots t_m \) under \( t_0 \) by

\[
t_p = t_0 + \{a_1 e_1\} \oplus \ldots \oplus \{a_p e_p\}
\]

for \( p = 1, 2, \ldots, m \).

Suppose we give a sequence \( e'_1 e'_2 \ldots e'_m \) of edges \( e_1 e_2 \ldots e_m \) where \( \{e'_1 e'_2 \ldots e'_m\} = \{e_1 e_2 \ldots e_m\} \). The question is whether there exists a sequence \( a'_1 a'_2 \ldots a'_m \) where \( \{a'_1 a'_2 \ldots a'_m\} = \{a_1 a_2 \ldots a_m\} \) such that the sequence of pairs \( \{a'_1 e'_1\} \{a'_2 e'_2\} \ldots \{a'_m e'_m\} \) gives a fundamental tree sequence which satisfy Eq. (17). In order to answer the above question, we will form a fundamental tree sequence \( t_0 t_1 \ldots t_m \) backward as follows: Suppose a sequence \( e'_1 e'_2 \ldots e'_m \) is given. Let \( G^0 \) be a linear
graph obtained from $G$ by shorting all edges in $t_0 \cap t_m$ and opening all edges in $G - \{t_0 \cup t_m\}$. Also let $t_0^0$ and $t_m^0$ be obtained from $t_0$ and $t_m$ by shorting all edges in $t_0 \cap t_m$ respectively. Notice that $t_0^0$ and $t_m^0$ are trees in $G^0$.

For example, from $G$, $t_0$ and $t_m$ given in Fig. 3a, b, and c, we can obtain $G^0$, $t_0^0$, and $t_m^0$ given in Fig. 3d, e, and f respectively.

Let $S_{e_m}(t_m^0) = \{e_{a_1}a_2...a_k\}$ in $G^0$ where $k' \leq m$. Then at least one of $S_{e_j}^i(t_0^0)$ in $G^0$ ($i = 1, 2, ..., k$) contains $e_m$ because $S_{e_m}(t_m^0)$ can be expressed as

$$S_{e_m}(t_m^0) = S_{a_1}^1(t_0^0) \oplus S_{a_2}^1(t_0^0) \oplus ... \oplus S_{a_k}^1(t_0^0) \quad (18)$$

Suppose $e_m \in S_{a_m}(t_0^0)$ where $a_m \in \{a_1...a_k\}$. Then

$$t_{m-1}^0 = t_m^0 \oplus \{e_{a_m}\} \quad (19)$$

is a M-tree transformation under $t_0^0$ in $G^0$. Thus there exist a tree $t_{m-1}^0$ in $G$ such that

$$t_m = t_{m-1}^0 \oplus \{e_{a_m}\} \quad (20)$$

is a M-tree transformation under $t_0^0$ in $G$.

Shorting $a_m$ and opening $e_m$ in $G^0$, we obtain a linear graph $G'$. Let $t_{m-1}'$ and $t_0'$ be obtained from $t_{m-1}^0$ and $t_0^0$ by shorting $a_m$. Then $t_0'$ and $t_{m-1}'$ are trees in $G'$. Let $S_{e_{a_m-1}'}(t_{m-1}') = \{e_{a_{m-1}''}a''_m\}$ in $G'$. Then there exists at least one cut set in $S_{a_{m-1}''}(t_0')$, ..., and $S_{a_m''}(t_0')$ in $G'$ which contains edge $e_{m-1}$. Let $e_{m-1} \in S_{a_{m-1}'}(t_0')$ where $a_{m-1} \in \{a_1''...a_m''\}$. Then

$$t_{m-2}' = t_{m-1}' \oplus \{e_{m-1}a_{m-1}'\} \quad (21)$$

is a M-tree transformation under $t_0'$ in $G'$. Hence, there exists $t_{m-2}$ in $G$ such that

$$t_{m-1} = t_{m-2} \oplus \{e_{m-1}a_{m-1}'\} \quad (22)$$
Fig 3. Example of $G, t_0, t_n, G^0, t_0^0$ and $t_n^0$. 

(a) $G$  
(b) $t_0$  
(c) $t_n$  
(d) $G^0$  
(e) $t_0^0$  
(f) $t_n^0$
is a M-tree transformation under \( t_0 \) in \( G \).

The above process called a "backward M-process" can be continued for any sequence of \( e_1, e_2, \ldots, e_m \). Thus for any sequence \( e_1 e_2 \ldots e_m \), there exists a sequence \( a_1 a_2 \ldots a_m \) such that the sequence of pairs \( \{e_1 a_1\}, \{e_2 a_2\}, \ldots, \{e_m a_m\} \) gives a fundamental tree sequence \( t_0 t_1 \ldots t_m \) under \( t_0 \) given by Eq. (17).

Now we are ready to discuss the properties of elementary complete tree transformations. For convenience, the symbol \( W \) indicates a two rows matrix

\[
W = \begin{bmatrix}
a_1' e_1' \ldots a_m' e_m' \\
e_1' e_2' \ldots e_m'
\end{bmatrix}
\]  

(23)

where \( \{a_1' a_2' \ldots a_m'\} = t_0 - t_m \), \( \{e_1' e_2' \ldots e_m'\} = t_m - t_0 \) and \( t_0 \) and \( t_m \) are complete trees of a pair of linear graphs \( G_1 \) and \( G_2 \).

Definition 5: A matrix \( W \) is a 1 FTS (Fundamental Tree Sequence) matrix of \( t_0 \) and \( t_m \) if

1. \( t_{p-1} \oplus \{e'_p a'_p\} = t_p \) \( \text{for } p = 1, 2, \ldots, m-1 \)  

   \[
   t_{m-1} \oplus \{e'_1 a'_1\} = t_m
   \]

   (25)

which are M-tree transformations under \( t_0 \) in \( G_1 \) and

2. \( t_{p-1} \oplus \{e'_p a'_p\} = t_p \) \( \text{for } p = 1, 2, \ldots, m \)  

   \[
   t_m \oplus \{e'_1 a'_1\} = t_m
   \]

   (26)

which is a M-tree transformation under \( t_0 \) in \( G_2 \).

Definition 6: A matrix \( W \) is a k FTS matrix if \( W \) can be partitioned as

\[
W = \begin{bmatrix} W_1 & W_2 & \ldots & W_k \end{bmatrix}
\]  

(27)

such that each \( W_i \) \( (i = 1, 2, \ldots, k) \) is a 1 FTS matrix.

Consider two complete trees \( t_0 \) and \( t_m \) of a pair of linear graphs \( G_1 \) and \( G_2 \). Let

\[
t_0 - t_m = \{a_1 \ldots a_m\}
\]  

(28)
and

\[ t_m - t_0 = \{e_1 \ldots e_m\} \]  \hspace{1cm} (29)

Then from any edge in \( \{e_1 \ldots e_m\} \), we can form a k FTS matrix by the following process:

**Step 1:** For edge \( e \), we can find an edge, say edge \( a_m \), in \( \{a_1 \ldots a_m\} \) by a backward M-process such that there exists \( t_{m-1} \) in \( G_1 \) and

\[ t_{m-1} \oplus \{ea_m\} = t_m \]  \hspace{1cm} (30)

is a M-tree transformation under \( t_0 \) in \( G_1 \). We put an edge \( a_m \) in (1,m) position of \( W \).

**Step 2:** With \( a_m \), we can find an edge in \( \{e_1 \ldots e_m\} \), say \( e_m \), such that there exists \( t_{m-1}' \) in \( G_2 \) and

\[ t_{m-1}' \oplus \{a_m e_m\} = t_m \]  \hspace{1cm} (31)

is a M-tree transformation under \( t_0 \) in \( G_2 \). We put \( e_m \) in (2,m) position of \( W \). If \( e_m = e \), we pick a new edge in \( \{e_1 \ldots e_m\} \) which has not been used before and back to Step 1. If there is no new edge, the process will terminate, otherwise, we proceed to Step 3.

**Step 3:** With edge \( e_p \) which has been obtained by the previous step, we can obtain an edge in \( \{a_1 \ldots a_m\} \) which has not been used in the 1st row of \( W \), say \( a_{p-1}' \), by a backward M-process so that we can find \( t_{p-2}' \) which satisfies

\[ t_{p-2}' \oplus \{ea_{p-1}\} = t_{p-1} \]  \hspace{1cm} (32)

is a M-tree transformation under \( t_0 \) in \( G_1 \). We put \( a_{p-1} \) in (1,p-1) position of \( W \). With \( a_{p-1} \), we can obtain an edge, say \( e_{p-1} \), in \( \{e_1 \ldots e_m\} \) which has not been used in the 2nd row of \( W \), such that

\[ t_{p-2}' \oplus \{e_{p-1}a_{p-1}\} = t_{p-1}' \]  \hspace{1cm} (33)
is a M-tree transformation under $t_0$ in $G_2$. We put $e_{p-1}$ in $(2,p-1)$ position of $W$.

If $e_{p-1} = e$, we pick a new edge in $\{e_1 \ldots e_m\}$ which has not been used in the 2nd row of $W$ and go back to Step 1. If every edge in $\{e_1 \ldots e_m\}$ is used in the 2nd row of $W$, the process is terminated. Otherwise, we repeat Step 3.

When the above process is terminated, a matrix $W$ is formed. It is clear by the above process, the resultant matrix is a $k$ FTS matrix.

Consider a pair of linear graphs $G_1$ and $G_2$ as shown in Fig. 4.

Let complete trees $t_0$ and $t_m$ be $t_0 = \{a_1,a_2,a_3\}$ and $t_m = \{e_1,e_2,e_3\}$. Suppose edge $e_1$ is chosen to start the process in order to obtain a matrix $W$. By Step 1, we can choose any one of $a_1$, $a_2$, and $a_3$. Suppose we choose $a_3$. Then by Step 2, we can use either $e_2$ or $e_3$. Suppose we choose $e_3$. At this point, we obtain a part of $W$ as

$$W = \begin{bmatrix}
1 & 2 & 3 \\
1 & & a_3 \\
2 & & e_3
\end{bmatrix}$$

(34)
By Step 3 with \( e_3 \), we obtain \( a_2 \) and \( e_2 \). Then by \( e_2 \), we obtain \( a_1 \). Finally by \( a_1 \) we obtain \( e_1 \), all of which are by Step 3. The resultant matrix \( W \) is

\[
W = \begin{bmatrix}
1 & 2 & 3 \\
1 & a_1 & a_2 & a_3 \\
2 & e_1 & e_2 & e_3
\end{bmatrix}
\]  

(35)

Instead of choosing \( a_3 \), if we choose \( a_1 \) at the first part of the above process and choose \( a_1 \) in Step 2, we have the situation to choose a new edge from \( \{e_2, e_3\} \). By choosing \( e_2 \) to start the process again, we have the following matrices depending on the choice we made at the second step.

\[
\begin{align*}
1 & 2 & 3 \\
1 & a_2 & a_3 & a_1 \\
2 & e_3 & e_2 & e_1
\end{align*}
\]

\[
\begin{align*}
1 & 2 & 3 \\
1 & a_2 & a_3 & a_1 \\
2 & e_2 & e_3 & e_1
\end{align*}
\]

Now we are ready to prove Theorem 1.

**Proof:** Case 1 - Suppose Condition 2 does not satisfy: Let edge \( e_j \) is in \( S^1(t_0) \) for \( j = 1, 2, \ldots, r \) but not in any other fundamental cut sets \( S^1_{a_j}(t_0) \) for \( r < q \leq m \) in \( G_1 \). By starting with edge \( e_j \), we can form a \( k \) FTS matrix by the previously described process. Suppose \( k > 1 \). Let

\[
W = [W_1, W_2, \ldots, W_k]
\]  

(36)

and

\[
W_1 = \begin{bmatrix}
a_1' & \ldots & a_k' \\
e_1' & \ldots & e_k'
\end{bmatrix}
\]  

(37)

is a 1 FTS matrix. This means that in \( G_1 \), there exists a fundamental tree sequence \( t_0t_1\ldots t_u \) under \( t_0 \) such that
\[ t_p = t_{p-1} \oplus \{ e^{'}_{p+1} a^{'}_p \} \quad \text{for} \ p = 1, 2, \ldots, u-1 \quad (38) \]

and

\[ t_u = t_{u-1} \oplus \{ e^{'}_u a^{'}_u \} \quad (39) \]

Similarly, there exists a fundamental tree sequence \( t_0' = t_0', t_1', \ldots, t_{u-1}', t_u' \) under \( t_0 \) in \( G_2 \) such that

\[ t_p' = t_{p-1}' \oplus \{ e^{'}_p a^{'}_p \} \quad \text{for} \ p = 1, 2, \ldots, u \quad (40) \]

Furthermore it is clear from Eq. (17) that \( t_u = t_u' \). Thus there exists a complete tree \( t_u \) such that

\[ t_0 \oplus \{ e_1' \ldots e_u' a_1' \ldots a_u' \} = t \quad (41) \]

and \( \{ e_1' \ldots e_u' a_1' \ldots a_u' \} \) is a proper subset of \( \{ e_1 \ldots e_u a_1 \ldots a_u \} \). Hence the theorem is true for \( k > 1 \).

Suppose \( k = 1 \). Let fundamental tree sequence of \( G_1 \) and \( G_2 \) corresponding to 1 FTS matrix \( W \)

\[ W = \begin{bmatrix}
  a_1' & a_2' & \ldots & a_m' \\
  e_1' & e_2' & \ldots & e_m'
\end{bmatrix} \quad (42) \]

be

\[ t_0', t_1', \ldots, t_m \]

and

\[ t_0' = t_0', t_1', \ldots, t_m' \]

respectively, where \( e_1 = e_1' \).

For fundamental cut set \( S_{a_j}^1 (t_0) \), let \( a_j = a_j' \) for \( j = 1, 2, \ldots, r \). Notice that \( e_1 \in S_{a_j}^1 (t_0) \) by assumption. Furthermore, \( e_1' \) is the edge among \( a_1', a_2', \ldots, a_r' \) appearing first in the sequence \( a_1'a_2' \ldots a_r' \) which is the first row of \( W \).
Consider tree \( t_{u_1} \) in \( G_1 \) which is
\[
t_{u_1} = t_{u_1-1} \oplus \{e'_{u_1+1} a'_{u_1} \}
\] (43)

Since \( e_1 = e'_1 \) is not in any fundamental cut set \( S_{a'}^1(t_0) \) for \( p = 1,2,...,u_1-1 \), it is clear that
\[
S_{a'}^1(t_{u_1-1}) = S_{a'}^1(t_0)
\] (44)

Thus edge \( e_1 \) satisfies that
\[
e_1 \in S_{a'}^1(t_0) \cap S_{a'}^1(t_{u_1-1})
\] (45)

which means that we can use \( e_1 \) rather than \( e'_{u_1+1} \) to obtain \( t_{u_1} \) from \( t_{u_1-1} \). By doing so, we have the fundamental tree sequence \( t_0 t_1 ... t_{u_1-1} t'' \) under \( t_0 \) in \( G_1 \) where
\[
t''_{u_1} = t_{u_1-1} \oplus \{e'a'_1 \}
\] (46)

Compare with the subsequence \( t_0 t'_1 ... t'_u \) of fundamental tree sequence \( t'_0 = t'_0 t'_1 ... t'_m \) under \( t_0 \) in \( G_2 \), we can see that
\[
t''_{u_1} = t'_{u_1}
\] (47)

Thus \( t'_{u_1} \) is a complete tree where
\[
t'_{u_1} = t_0 \oplus \{e'_{u_1} e'_{u_1+1} a_{u_1} a'_{u_1} \}
\] (48)

and \( \{e'_{u_1} e'_{u_1+1} a_{u_1} a'_{u_1} \} \) is a proper subset of \( \{e'_{u_1} e'_{u_1+1} e'_{m+1} a_1 a'_{u_1} a'_{m+1} \} \). Thus the theorem is true for \( k = 1 \).

Case 2 - Suppose Condition 2 is satisfied but Condition 1 does not satisfy:
Suppose the edge permutation $\xi(t_0 t_m)$ can be factorized as
\[
\xi(t_0 t_m) = \begin{pmatrix}
e_1 e_2 \ldots e_u \\
e'_1 e'_2 \ldots e'_u \\
e_{u+1} \ldots e_m \\
e'_{u+1} \ldots e'_m
\end{pmatrix}
\] (49)
where $\begin{pmatrix} e_1 e_2 \ldots e_u \\ e'_1 e'_2 \ldots e'_u \end{pmatrix}$ is not factorizable. Without the loss of generality, suppose the columns of $\xi(t_0 t_m)$ represent edges $a_1, a_2, \ldots, a_m$. In $G_1$, we can form a fundamental tree sequence under $t_0$ as $t_0 t_1 t_2 \ldots t_u$ where
\[
t_p = t_{p-1} \oplus \{ e_a \} \\
p = 1, 2, \ldots, u.
\] (50)
These trees satisfy the definition of M-tree transformation under $t_0$ because $e_p$ is not in any of the fundamental cut sets $S^1_{a_j}(t_0)$ for $j = 1, 2, \ldots, m$ except $j = p$. Thus
\[
S^1_{a_j}(t_0) = S^1_{a_j}(t_p)
\] (51)
for $j = p+1, \ldots, u$.

Similarly in $G_2$, a fundamental tree sequence $t'_0 = t_0, t'_1, t'_2, \ldots, t'_u$ where
\[
t'_p = t'_{p-1} \oplus \{ e'_a \} \\
p = 1, 2, \ldots, u
\]
can be formed. Since $\{ e_1 \ldots e_u \} = \{ e'_1 \ldots e'_u \}$,
\[
t' = t_u.
\] (52)
Hence $t_u$ is a complete tree which can be expressed as
\[
t_u = t_0 \oplus \{ e_1 e_2 \ldots e_u a_1 a_2 \ldots a_u \}
\] (53)
where $\{ e_1 e_2 \ldots e_u a_1 a_2 \ldots a_u \}$ is a proper subset of $\{ e_1 e_2 \ldots e_m a_1 a_2 \ldots a_u \ldots a_m \}$.
Thus the theorem is true for this case. By Cases 1 and 2, we prove the half
of the theorem. The remaining to be proven is that if

\[ t_m = t_0 \oplus \{ e_1 \ldots e_m a_1 \ldots a_m \} \tag{54} \]

is not an elementary complete tree transformation, then either Condition 1 or Condition 2 will be violated.

Suppose \( t_0 \oplus \{ e_1 \ldots e_m a_1 \ldots a_m \} \) is not an elementary complete tree transformation. Then there exists a complete tree \( t_u \) such that

\[ t_0 \oplus \{ e'_1 \ldots e'_u a'_1 \ldots a'_u \} = t_u \tag{55} \]

where \( \{ e'_1 \ldots e'_u a'_1 \ldots a'_u \} \) is a proper subset of \( \{ e_1 \ldots e_m a_1 \ldots a_m \} \). This means that there exists an edge-permutation which is factorizable if Condition 2 is satisfied. Thus either Condition 1 or Condition 2 will be violated. Q.E.D.

Consider a pair of linear graphs \( G_1 \) and \( G_2 \) in Fig. 5.

![Fig. 5. A pair of linear graphs \( G_1 \) and \( G_2 \).](image)

Let \( t_0 = \{ a_1 a_2 \} \) and \( t_2 = \{ e_1 e_3 \} \). Then

\[ t_0 \oplus \{ e_1 e_2 a_1 a_2 \} = t_2 \]

is not an elementary complete tree transformation because in \( G_1 \), \( e_1 \) and \( e_3 \) are in \( S_{a_1} t_0 \) which violate Condition 2 in Theorem 1.
However, for $t_3 = \{e_1e_2\}$,

$$t_0 \oplus \{e_1e_2, a_1a_2\} = t_3$$

is an elementary complete tree transformation. Theorem 1 gives the following Lemma.

**Lemma 1:** Any complete tree can be obtained from a complete tree by successive elementary complete tree transformations.

**Remarks**

It can be seen that if two linear graphs $G_1$ and $G_2$ are identical, an elementary complete tree transformation becomes a known elementary tree transformation. An important future problem is to generate all possible complete trees without duplications by the use of elementary complete tree transformations.

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It is known that a passive electrical network without mutual couplings can be analyzed by knowing all possible trees of a linear graph corresponding to the network. Since there exists a reasonably simple method of generating all possible trees of a linear graph without duplications, analysis of such a network by a computer becomes indubitably simple.

When a pair of linear graphs is used, an active network can be analyzed by knowing all possible complete trees each of which is a tree of both linear graphs. At present there is no simple method of generating all possible complete trees without duplications. Hence, in order to obtain all possible complete trees by a computer, one of the best available methods at present is to generate all possible trees of each linear graph to obtain two collections of trees, then intersecting the two collections. It is not difficult to design an active network such that there are more than a thousand of trees in each of a pair of linear graphs corresponding to the net, but there are less than one hundred complete trees. Hence to obtain a simple method of generating all possible complete trees is undoubtedly important for analysis of active networks by a computer.

(continued on separate sheet)
linear graphs
active network analysis
tree generation
elementary tree transformation
analysis by computer
ABSTRACT (continued)

Obtaining one tree from another tree which is used to generate all possible trees in a linear graph is accomplished by so-called "elementary tree transformation." In this paper, this transformation is generalized so that a complete tree can be obtained from another complete tree by the generalized transformation, namely an "elementary complete tree transformation."

Furthermore, any complete tree can be obtained from any other complete tree by successive elementary complete tree transformations. The study of the properties of such transformation will hopefully lead to find a method of generating all possible complete trees without duplications in the future.