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AND COMPLETE TREES
Wataru Mayeda

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GENERATION OF TREES AND COMPLETE TREES

Wataru Mayeda

Abstract

A method of generating all complete trees of a pair of linear graphs which can represent active networks is given. This method consists of two parts, one of which is to obtain one complete tree and the other is to generate all possible complete trees of distance one from that already determined.

All complete trees of distance one from a given tree \( t_0 = \{a_1, a_2, \ldots, a_n\} \) can easily be obtained by

\[
\bigcup_{p=1}^{n} T^P(t_0) = \bigcup_{p=1}^{n} \{t| t = t_0 \oplus \{a_p e\}, e \in S_a(t_o, G_i) \cap S_a(t_o, G_v)\}
\]

where \( S_a(t_o, G_i) \) is a fundamental cut set containing edge \( a_p \) with respect to tree \( t_0 \) in \( G_i \), \( S_a(t_o, G_v) \) is a fundamental cut set containing edge \( a_p \) with respect to \( t_0 \) in \( G_v \), and \( G_i \) and \( G_v \) are the pair of linear graphs representing an active network. When we have cut set matrices \( A_i \) and \( A_v \) corresponding to \( G_i \) and \( G_v \) respectively, we can find a tree \( t_o \) by changing these matrices to the fundamental form as \([A_i U]_{11}\) and \([A_v U]_{11}\) in which the edges corresponding to the unit matrix \( U \) form a complete tree.

These two processes are easily carried out by the use of computers. There will be no duplications when complete trees are generated by this method. Furthermore, complete trees are generated by sets of complete trees...
classified by edges in initial complete tree $t_0$. Thus it will be easy to factorize according to the weights of these edges.
Introduction

Whether or not the use of computers for analysis of electrical networks is useful and an important part of modern electronic industry depends on the computer time necessary to analyze such networks. One way of saving computer time is to find an effective method of generating all possible trees in the case of passive networks, and to generate all possible complete trees in the case of active networks and networks with transformers.

There are several methods available at present. However, none of them is satisfactory for generating all complete trees by computers. Obviously, if a method can generate all complete trees effectively, the same method can generate all trees effectively. On the other hand, even though a method can generate all trees effectively, to obtain all complete trees by simply modifying the method is not generally very effective.

It has been known for a long time that to generate all trees which are distance one from a given tree is easy. Also it is easily seen that generating all complete trees of distance one from a given complete tree is likewise simple. Hence if a method can generate all possible complete trees by generating complete trees of distance one from some set of complete trees, the method will be a desirable one. Here such a method is introduced, and a computer program using this method is in preparation. This method is simple in technique and no difficult theories are involved. Also finding the sign of each complete tree can be found without any additional computations.
Generation of Trees

Let $A = [A_{ij}]$ be the fundamental cut set matrix of a connected linear graph $G$ consisting of $v$ vertices. Also let $t_o$ be the tree corresponding to the unit matrix in $[A_{11}]$. Consider another tree $t$ in $G$. Let

$$t - t_o = \{e_1, e_2, \ldots, e_n\}$$

and

$$t_o - t = \{a_1, a_2, \ldots, a_n\}.$$

Then

$$t_o \cap t = \{a_{n+1}, \ldots, a_{v-1}\}$$

be the edges in both $t_o$ and $t$. Notice that $t - t_o$ is a set of edges in $t$ but not in $t_o$. For convenience, let $P$ and $Q$ be sets of edges such that $P \cap Q = \emptyset$. We define the symbol $G(P;Q)$ to represent the linear graph obtained from $G$ by shorting all edges in $P$ and opening all edges in $Q$. With this definition, $G(t_o \cap t; t_o - t)$ is a linear graph obtained from $G$ by shorting all edges in $t_o \cap t$ and opening all edges in $t_o - t$.

**Theorem 1:** Let $t_o$ and $t$ be trees in $G$. Then $t - t_o$ is a tree in $G(t_o \cap t; t_o - t)$.

**Proof:** Since $t$ is a tree of $G$, shorting any edge $a$ in $t$ makes $t - \{a\}$ a tree of $G(\{a\}; \emptyset)$. Also opening any edge $e \notin t$ does not destroy $t$ as a tree of $G(\emptyset; \{e\})$. Q.E.D.

We define a distance between two trees $t_o$ and $t$ as the number of edges in $t_o - t$. The symbol $T_{t_o}^{a_1a_2\ldots a_n}$ is the set of all possible trees $t$ such that

$$t_o - t = \{a_1, a_2, \ldots, a_n\}.$$
Thus any tree $t$ in $T_{a_{1}a_{2}\ldots a_{n}}[t_{o}]$ is distance $n$ from $t_{o}$.

**Theorem 2:** Let $t \in T_{a_{1}a_{2}\ldots a_{n}}[t_{o}]$. Then $t - t_{o}$ is a tree in $G(t_{o} \cap t; t_{o} - t)$. Furthermore, any tree $t'$ in $G(t_{o} \cap t; t_{o} - t)$ with all edges in $t_{o} \cap t$ is a tree in $T_{a_{1}a_{2}\ldots a_{n}}[t_{o}]$.

**Proof:** The first part of this theorem follows directly from Theorem 1. The proof of the second part is as follows: If $t' \cup (t_{o} \cap t)$ is not a tree of $G$, then it must contain at least one circuit. Since $t'$ is a tree in $G(t_{o} \cap t; t_{o} - t)$ and $t_{o} \cap t$ is part of $t_{o}$, both subgraphs themselves do not contain any circuit. Thus only the circuits which possibly exist in $t' \cup (t_{o} \cap t)$ will be those consisting of some of edges in $t_{o} \cap t$ and some of edges in $t'$. However, if this is the case, when all edges in $t_{o} \cap t$ are shorted, those edges in $t'$ must form circuits including self-loops which contradicts the assumption that $t'$ is a tree in $G(t_{o} \cap t; t_{o} - t)$. Q.E.D.

It is known that to obtain $T^{a_{i}}[t_{o}]$ where $a_{i} \in t_{o}$ is easily done by the scheme

$$T^{a_{i}}[t_{o}] = \{t|t = t_{o} \oplus \{ae\}, \ e \in S_{a}(t_{o})\}.$$  

That is, we can obtain $T^{a_{i}}[t_{o}]$ directly from the fundamental cut set matrix

$$A = [A_{11}U]$$

by taking edge $e$ whose corresponding row in $A_{11}$ is non-zero at the column corresponding to the cut set $S_{a}(t_{o})$. The following example will illustrate this point.
Example 1: Let fundamental cut set matrix $A$ of the linear graph shown in Figure 1 be

$$
A = \begin{bmatrix}
    e_1 & e_2 & e_3 & e_4 & a_1 & a_2 & a_3 & a_4 \\
    1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
    0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
    1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 
\end{bmatrix}
$$

![Figure 1. A linear graph $G$](image)

where $t_0 = \{a_1 a_2 a_3 a_4\}$. Then $T^1_{a_1}[t_0]$ can be obtained from $t_0$ by replacing edge $a_1$ by edge $e_1$ because only column $e_1$ which is non-zero at the first row corresponding to $S_{a_1}(t_0)$. $T^2_{a_1}[t_0]$ can be obtained from $t_0$ by changing edge $a_2$ by edges $e_2$, $e_3$ and $e_4$ one at the time because columns $e_2$, $e_3$ and $e_4$ have non-zero at the second row. Thus

$$
T^2_{a_1}[t_0] = \{\{e_2 a_1 a_3 a_4\}, \{e_3 a_1 a_3 a_4\}, \{e_4 a_1 a_3 a_4\}\}.
$$

Similarly

$$
T^3_{a_2}[t_0] = \{\{e_2 a_1 a_2 a_4\}, \{e_4 a_1 a_2 a_4\}\}
$$

and

$$
T^4_{a_3}[t_0] = \{\{e_1 a_1 a_2 a_3\}, \{e_3 a_1 a_2 a_3\}, \{e_4 a_1 a_2 a_3\}\}.
$$
Thus to obtain all possible trees which are distance one from \( t_0 \) is a very simple task. The problem is to obtain trees which are not distance one from \( t_0 \).

From Theorems 1 and 2, the following process will give \( \mathcal{T}_{t_0}^{a_1a_2} \) from \( G(t_0 - \{a_1a_2\};\{a_1a_2\}) \).

**Step 1:** Take one tree of \( G(t_0 - \{a_1a_2\};\{a_1a_2\}) \). Let this tree be \( t_0(a_1a_2) = \{e_1e_2\} \).

**Step 2:** Since \( \mathcal{T}_{t_0}^{a_1a_2} \) can be obtained by knowing \( \mathcal{T}_{t_0}^{e_1}(a_1a_2) \), \( \mathcal{T}_{t_0}^{e_2}(a_1a_2) \) and \( \mathcal{T}_{t_0}^{e_1e_2}(a_1a_2) \) which are the set of all trees in \( G(t_0 - \{a_1a_2\};\{a_1a_2\}) \), we first obtain \( \mathcal{T}_{t_0}^{e_1}(a_1a_2) \) and \( \mathcal{T}_{t_0}^{e_2}(a_1a_2) \) which consists of trees of distance one from \( t_0(a_1a_2) \). Notice that to obtain these trees, we only need to know the fundamental cut sets with respect to tree \( t_0(a_1a_2) \). To obtain \( \mathcal{T}_{t_0}^{e_1e_2}(a_1a_2) \), we go back to Step 1 using (i) tree \( t'(a_1a_2) \) which consists of edges in \( t_0 - \{a_1a_2\} \) and edges \( e_1 \) and \( e_2 \), (ii) edges \( e_1 \) and \( e_2 \) rather than \( a_1 \) and \( a_2 \), and (iii) linear graph \( G(t_0 - \{a_1a_2\};\{a_1a_2\}) \) rather than \( G \).

When we use a cut set matrix rather than a linear graph to obtain all trees, with the definition given below, the above process becomes simply as follows:

**Definition 1:** A matrix \( Q \) is said to be a fundamental form if \( Q \) is expressed as \( [Q_{11}U] \) where \( U \) is a unit matrix. Notice that the rank of \( Q \) must be the same as the number of rows in the matrix.

Suppose a cut set matrix \( A \) is given. Then first we change it to a fundamental form of \( [A_{11}U] \) by elementary row operations. Notice that all edges in tree \( t_0 \) correspond to the columns of \( U \) in \( [A_{11}U] \).
Step 1: Let \( A(a_1 a_2) \) be the submatrix of \( A_{11} \) obtained by deleting all rows which do not have non-zero at the columns corresponding to \( a_1 \) and \( a_2 \) in \([A_{11} U]\), and delete all columns of \( U \). It can be seen that \( A(a_1 a_2) \) is a cut set matrix of the linear graph \( G(t_0 - \{a_1 a_2\};\{a_1 a_2\}) \).

Step 2: By elementary row operations, change \( A(a_1 a_2) \) into a fundamental form as \( [A(a_1 a_2)_{11} U] \) which is a fundamental cut set matrix of \( G(t_0 - \{a_1 a_2\};\{a_1 a_2\}) \), and the columns corresponding to \( U \) represents edges \( e_1 \) and \( e_2 \) which form tree \( t_0(a_1 a_2) \) in this modified linear graph. If we can not obtain a fundamental form of \( A(a_1 a_2) \) by the elementary row operations, then we know that there is no tree in the linear graph \( G(t_0 - \{a_1 a_2\};\{a_1 a_2\}) \).

Each tree in \( T^{e_1}[t_0(a_1 a_2)] \) can be obtained by replacing edge \( e_1 \) by edge \( e_p \) whose corresponding column of \( A(a_1 a_2)_{11} \) has a non-zero at the first row which represents \( S_{e_1}[t_0(a_1 a_2)] \). Similarly, we replace edge \( e_2 \) by edge \( e_q \) whose corresponding column of \( A(a_1 a_2)_{11} \) has a non-zero at the second row to obtain \( T^{e_2}[t_0(a_1 a_2)] \). To obtain trees in \( T^{e_1 e_2}[t_0(a_1 a_2)] \), we go back to Step 1 by (i) using \( A(a_1 a_2) \) rather than \( A \), and (ii) considering edges \( e_1 \) and \( e_2 \) rather than \( a_1 \) and \( a_2 \). Notice that there is no rows to remove in order to obtain a new cut set matrix. Actually, \( A(a_1 a_2)_{11} \) is a new cut set matrix to be considered next. The next example will illustrate the above process.

Example 2: For a given linear graph shown in Figure 2, a cut set matrix with the corresponding tree \( t_0 = \{a_1 a_2 a_3 a_4\} \) is
To obtain $T_{a_1a_2}[t_0]$, we only need to consider linear graph $G(t_0 - \{a_1, a_2\}; \{a_1, a_2\})$. A cut set matrix $A(a_1a_2)$ corresponding to this linear graph is the first two rows of $A$ without columns $a_1, a_2, a_3$ and $a_4$; that is

$$A(a_1a_2) = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

By rearranging rows, we have

$$A(a_1a_2) = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$
Thus tree $t_o(a_1 a_2)$ of $G(t_o - \{a_1 a_2\};\{a_1 a_2\})$ will be $\{e_3 e_4\}$. Since each row in the above matrix represents a fundamental cut set with respect to this tree $t_o(a_1 a_2)$,

$$T^{e_4}[t_o(a_1 a_2)] = \{e_1 e_3\}$$

which is obtained according to Step 2 in the above process. That is, since only column $e_1$ has a non-zero at the first row which represents cut set $S_{e_1}^{t_o(a_1 a_2)}$, only edge $e_1$ can replace edge $e_4$ in order to obtain $T^{e_4}[t_o(a_1 a_2)]$. Similarly, both edges $e_1$ and $e_2$ can replace edge $e_3$ because columns $e_1$ and $e_2$ have non-zero at the second row,

$$T^{e_3}[t_o(a_1 a_2)] = \{\{e_1 e_4\},\{e_2 e_4\}\}.$$

To obtain $T^{e_3 e_4}[t_o(a_1 a_2)]$, we use the cut set matrix $A(a_1 a_2 e_3 e_4)$ obtained from $A(a_1 a_2)_{11} U$ by deleting all columns belonging to $U$, i.e.,

$$A(a_1 a_2 e_3 e_4) = \begin{bmatrix} e_1 & e_2 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

This matrix can easily be changed into a fundamental form which is just a unit matrix. Notice that this matrix is a cut set matrix of a linear graph $G(t_o - \{a_1 a_2\};\{a_1 a_2\} U \{e_3 e_4\})$. Also there is only one tree in this graph which is $t_o(a_1 a_2 e_3 e_4) = \{e_1 e_2\}$. Hence all possible trees of linear graph $G(t_o - \{a_1 a_2\};\{a_1 a_2\})$ are $\{e_3 e_4\}$, $\{e_1 e_3\}$, $\{e_1 e_4\}$, $\{e_2 e_4\}$, and $\{e_1 e_2\}$. Thus we can obtain all trees in $T^{a_1 a_2}[t_o]$ by replacing edges $a_1$ and $a_2$ by edges in each one of the above trees as

$$T^{a_1 a_2}[t_o] = \{\{e_3 e_4 a_3 a_4\}, \{e_1 e_3 a_3 a_4\}, \{e_1 e_4 a_3 a_4\}, \{e_2 e_4 a_3 a_4\}, \{e_1 e_2 a_3 a_4\}\}.$$
In general, we can obtain $T^{a_1a_2...a_p}$ of linear graph $G$ by the following process, where $t_o = \{a_1, a_2, ..., a_n\}$ and $1 < p \leq n$.

**Step 1:** Form a new linear graph $G(t_o - \{a_1, a_2, ..., a_p\}; \{a_1, a_2, ..., a_p\})$ from $G$. Then pick a tree in this graph. Let this be $t' = \{e_1, e_2, ..., e_p\}$. If it is impossible to pick a tree, there is no tree in $T^{a_1a_2...a_n}$.

**Step 2:** The all possible trees in $G(t_o - \{a_1, a_2, ..., a_p\}; \{a_1, a_2, ..., a_p\})$ are those in

\[ T^{e_1}[t_o'], T^{e_2}[t_o'], ..., T^{e_p}[t_o'] \]

\[ T^{e_1e_2}[t_o'], T^{e_1e_3}[t_o'], ..., T^{e_1e_p}[t_o'], T^{e_2e_3}[t_o'], ..., T^{e_p-1e_p}[t_o'] \]

\[ \ldots \ldots \]

\[ T^{e_1e_2...e_p}[t_o'], \ldots, \text{where } 1 \leq j_1 < j_2 < \ldots < j_q < p \]

\[ \ldots \ldots \]

\[ T^{e_1e_2...e_p}[t_o'] \].

Obtain each tree in $T^{e_1}[t_o'], T^{e_2}[t_o'], \ldots, T^{e_p}[t_o']$ which is a tree of distance one from $t_o$ in $G(t_o - \{a_1, a_2, ..., a_p\}; \{a_1, a_2, ..., a_p\})$. Obtaining trees in $T^{e_1e_2...e_p}[t_o']$ is the same as obtaining trees in $T^{a_1a_2...a_n}[t_o]$. Thus we go back to Step 1 by considering $t_o'$ rather than $t_o$ as a given tree and $G(t_o - \{a_1, a_2, ..., a_p\}; \{a_1, a_2, ..., a_p\})$ rather than $G$ as a given linear graph.

With cut set matrix $A$ rather than linear graph $G$ to obtain all trees, the above process becomes as follows: Let $A$ be a given cut set matrix of a given linear graph $G$. First we change $A$ into a fundamental form as $[A_{11}U]$ by elementary operations. Let $t_o = \{a_1, ..., a_n\}$ be a tree corresponding to the unit matrix $U$ in $[A_{11}U]$. The following steps gives all trees in $T^{a_1...a_p}[t_o]$ where $1 < p \leq n$. 
**Step 1:** Form a cut set matrix $A(a_1...a_p)$ obtained from $A = [A_{11}U]$ by deleting all rows corresponding to fundamental cut sets $S_{ar}(t_o)$ for all edges $a_r$ in $t_o - \{a_1...a_p\}$, (notice that the row corresponding to $S_{ar}(t_o)$ has 1 at the column representing $a_r$) and deleting all columns of the unit matrix $U$ in $[A_{11}U]$. Change $A(a_1...a_p)$ into a fundamental form as $[A(a_1...a_p)_{11}U]$. Suppose the tree $t'_o$ consists of edges $e_1,e_2,...,e_p$ corresponding to columns of $U$ in $[A(a_1...a_p)_{11}U]$.

**Step 2:** Obtain $T_{er}(t'_o)$ for $r = 1,2,...,p$ by replacing edge $e_r$ by edge $e'$ whose corresponding column has a non-zero at the row representing fundamental cut set $S_{er}(t'_o)$. Notice that row representing $S_{er}(t'_o)$ has 1 at the column corresponding to edge $e_r$.

In order to obtain $T_{ej_1}^e_{j_u}(t'_o)$, we go back to Step 1 considering $A(a_1...a_p)$ rather than $A$ as a given cut set matrix, $t'_o$ rather than $t_o$ as a tree and edges $e_{j_1},...,e_{j_u}$ rather than $a_1,...,a_p$ as edges to be replaced.

The following example will illustrate the above process.

**Example 3:** Suppose a linear graph $G$ shown in Figure 3 is given. A fundamental cut set matrix $A$ with respect to tree $t_o = \{a_1a_2a_3a_4\}$ is

$$A = \begin{bmatrix}
e_1 & e_2 & e_3 & e_4 & a_1 & a_2 & a_3 & a_4 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
2 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
3 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
4 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}$$
Then trees in $T^{1\text{a}}_{t_0}$ are those obtained from $t_0$ by replacing edge $a_1$ by edges which has 1 at row 1 representing $S^{a_1}_{a_1}(t_0)$. Those edges are $e_1$ and $e_2$. Thus

$$T^{1\text{a}}_{t_0} = \{e_1a_2a_3a_4, e_2a_2a_3a_4\}.$$ 

Similarly

$$T^{2\text{a}}_{t_0} = \{\{a_1e_1a_3a_4, \{a_1e_2a_3a_4, \{a_3e_4a_3a_4\}\}}\}$$

$$T^{3\text{a}}_{t_0} = \{\{a_1a_2e_1a_4, \{a_1a_2e_3a_4\}\}}\}$$

and

$$T^{4\text{a}}_{t_0} = \{\{a_1a_2a_3e_2, \{a_1a_2a_3e_3, \{a_1a_2a_3e_4\}\}\}}\}$$

To obtain $T^{1a_2}_{a_1}[t_0]$, we consider a new cut set matrix $A(a_1a_2)$ from $A$ by deleting row 3 and row 4 and columns belonging to the unit matrix; that is,

$$A(a_1a_2) = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 \end{bmatrix}$$
by elementary row operations and rearranging columns, we can change the matrix into a fundamental form as

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1
\end{bmatrix}
\]

Thus

\[
t_o(a_1a_2) = \{e_2e_4\},
\]

\[
T^4[t_o(a_1a_2)] = \emptyset
\]

and

\[
T^2[t_o(a_1a_2)] = \{e_1e_4\}
\]

which gives

\[
T^{a_1a_2}[t_o] = \{\{e_2e_4a_3a_4\}, \{e_1e_4a_3a_4\}\}.
\]

To obtain \(T^{e_1e_2}[t_o(a_1a_2)]\), we consider a new cut set matrix \(A(a_1a_2e_1e_2)\) obtained from \([A(a_1a_2)_{11U}]\) by deleting all columns in \(U\) as

\[
\begin{bmatrix}
e_1 & e_3 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

Notice that in this case, there is no row to be deleted because edges \(e_1\) and \(e_2\) form \(t_o(a_1a_2)\). This matrix, obviously, can not be changed into a fundamental form. Thus

\[
T^{e_1e_2}[t_o(a_1a_2)] = \emptyset.
\]

To obtain \(T^{a_1a_3}[t_o]\), we consider a matrix \(A(a_1a_2)\) obtained from \([A_{11U}]\) by deleting rows 2 and 4 (because these rows correspond to \(S_{a_2}(t_o)\) and \(S_{a_4}(t_o)\) where \(a_2\) and \(a_4\) are in \(t_o - \{a_1a_3\}\)), and deleting all columns.
belonging to \( U \) in \([A_{11}U]\) corresponding to edges \( a_1, a_2, a_3, \) and \( a_4 \).

\[
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 \\
1 & 1 & 0 & 0 \\
3 & 1 & 0 & 1 & 0
\end{bmatrix}
\]

\( A(a_1a_3) = \)

This matrix can be changed into a fundamental form as

\[
\begin{bmatrix}
e_1 & e_4 & e_2 & e_3 \\
1 & 0 & 1 & 0 \\
3 & 0 & 0 & 1
\end{bmatrix}
\]

\([A(a_1a_3)_{11}U]\) =

Then

\[
t_o(a_1a_3) = \{e_2e_3\},
\]

\[
T^{e_2}[t_o(a_1a_3)] = \{e_1e_3\}
\]

and

\[
T^{e_3}[t_o(a_1a_3)] = \{e_1e_2\} .
\]

Furthermore from \( A(a_1a_3e_2e_3) \)

\[
\begin{bmatrix}
e_1 & e_4 \\
1 & 1 & 0 \\
3 & 1 & 0
\end{bmatrix}
\]

which is obtained from \([A(a_1a_3)_{11}U]\) by deleting columns belonging to \( U \),

we can see that

\[
T^{e_2e_3}[t_o(a_1a_3e_2e_3)] = \emptyset .
\]

Thus

\[
T^{a_1a_3}[t_o] = \{\{e_2a_2e_3a_4\}, \{e_1a_2e_3a_4\}, \{e_1a_2e_2a_4\}\} .
\]
For $a_1a_4[t_0]$, we consider $A(a_1a_4)$ obtained from $[A_{11}U]$ by deleting row 2 and 3 and columns belonging to $U$ as

$$A(a_1a_4) = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ 1 & 1 & 1 & 0 & 0 \\ 4 & 0 & 1 & 1 & 1 \end{bmatrix}$$

which can be written as

$$[A(a_1a_4)_{11U}] = \begin{bmatrix} e_2 & e_3 & e_1 & e_4 \\ 1 & 1 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Thus

$$t_0(a_1a_4) = \{e_1e_4\},$$

$$T^1[t_0(a_1a_4) = \{e_2e_4\}$$

$$T^4[t_0(a_1a_4) = \{\{e_2e_1\},\{e_3e_1\}\}.$$

In order to obtain $a_1a_4[t_0]$, we must obtain $T^1e_4[T^0(a_1a_4)]$ which can easily be found by considering the following matrix $A(a_1a_4e_1e_4)$

$$A(a_1a_4e_1e_4) = \begin{bmatrix} e_2 & e_3 \\ 1 & 0 \\ 4 & 1 \end{bmatrix}$$

which can be changed into a fundamental form as

$$[A(a_1a_4e_1e_4)_{11U}] = \begin{bmatrix} e_2 & e_3 \\ 1 & 0 \\ 4 & 1 \end{bmatrix}$$
Hence \( t_0(a_1a_4e_1e_4) = \{e_2e_3\} \) and \( T^{e_1e_4}[t_0(a_1a_4)] = \{e_2e_3\} \). Thus

\[
T^{a_1a_4}[t_0] = \{\{e_1a_2a_3e_4\}, \{e_2a_2a_3e_4\}, \{e_1a_2a_3e_2\}, \{e_1a_2a_3e_3\}, \{e_2a_2a_3e_3\}\}
\]

For \( T^{a_2a_3}[t_0] \), we change a matrix \( A(a_2a_3) \)

\[
A(a_2a_3) = \begin{bmatrix}
2 & 1 & 1 & 0 & 1 \\
3 & 1 & 0 & 1 & 0
\end{bmatrix}
\]

as

\[
[A(a_2a_3)_{11}] = \begin{bmatrix}
2 & 1 & 1 & 1 & 0 \\
3 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

Hence

\[
t_0(a_2a_3) = \{e_3e_4\},
\]

\[
T^{e_4}[t_0(a_2a_3)] = \{\{e_1e_3\}, \{e_2e_3\}\}
\]

\[
T^{e_3}[t_0(a_2a_3)] = \{e_1e_4\}.
\]

Furthermore, \( T^{e_3e_4}[t_0(a_2a_3)] \) can be obtained by considering matrix \( A(a_2a_3e_3e_4) \)

\[
A(a_2a_3e_3e_4) = \begin{bmatrix}
2 & 1 & 1 \\
3 & 1 & 0
\end{bmatrix}
\]

which can be changed into a fundamental form as

\[
[A(a_2a_3e_3e_4)_{11}] = \begin{bmatrix}
3 & 1 & 0 \\
2 & 0 & 1
\end{bmatrix}
\]
Hence $T^{e_3 e_4} [t_0 (a_2 a_3)] = \{e_1 e_2\}$. Thus

$$\begin{align*}
T^{a_2 a_3} [t_0] &= \{\{a_1 e_3 e_4 a_4\}, \{a_1 e_1 e_3 a_4\}, \{a_1 e_2 e_3 a_4\}, \{a_1 e_1 e_4 a_4\}, \\
&\{a_1 e_1 e_2 a_4\}\}.
\end{align*}$$

For $T^{a_2 a_4} [t_0]$, we consider $A(a_2 a_4)$ as

$$A(a_2 a_4) = 2 \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}
\quad 4 \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}$$

which can be rewritten as

$$[A(a_2 a_4)]_{11} = 2 \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}
\quad 4 \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}$$

Hence $t_0 (a_2 a_4) = \{e_1 e_3\}$,

$$T^{e_1}[t_0 (a_2 a_4)] = \{\{e_2 e_3\}, \{e_4 e_3\}\}$$

and

$$T^{e_3}[t_0 (a_2 a_4)] = \{\{e_2 e_1\}, \{e_4 e_1\}\}.$$

Since $A(a_2 a_4 e_1 e_3)$ for $T^{e_1 e_3} [t_0 (a_2 a_4)]$

$$A(a_2 a_4 e_1 e_3) = 2 \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

cannot be changed into a fundamental form, there is no tree in $T^{e_1 e_3} [t_0 (a_2 a_4)]$.

Hence
For $T^{a_3a_4}[t_o]$, matrix $A(a_3a_4)$ which is

$$
A(a_3a_4) = \begin{bmatrix}
e_1 & e_2 & e_3 & e_4 \\
3 & 1 & 0 & 1 \\
4 & 0 & 1 & 1
\end{bmatrix}
$$

will be changed into a fundamental form as

$$
[A(a_3a_4)_{11}U] = \begin{bmatrix}
e_3 & e_4 & e_1 & e_2 \\
3 & 1 & 0 & 1 \\
4 & 1 & 1 & 0
\end{bmatrix}
$$

Thus $t_0(a_3a_4) = \{e_1e_2\}$ and we have

$T^{e_1}[t_0(a_3a_4)] = \{e_3e_2\}$

$T^{e_2}[t_0(a_3a_4)] = \{\{e_3e_1\},\{e_4e_1\}\}$

and for $T^{e_1e_2}[t_0(a_3a_4)]$, we consider

$$
A(a_3a_4e_1e_2) = \begin{bmatrix}
e_3 & e_4 \\
3 & 1 \\
4 & 1
\end{bmatrix}
$$

which can be changed to

$$
[A(a_3a_4e_1e_2)_{11}U] = \begin{bmatrix}
e_3 & e_4 \\
3 & 1 \\
4 & 0
\end{bmatrix}
$$

Hence

$T^{e_1e_2}[t_0(a_3a_4)] = t_0(a_3a_4e_1e_2) = \{e_3e_4\}$. 

$T^{a_2a_4}[t_o] = \{\{a_1e_1a_3e_3\},\{a_1e_2a_3e_3\},\{a_1e_3a_3e_4\},\{a_1e_1a_3e_2\},\{a_1e_1a_3e_4\}\}$. 
Thus
\[ T^{3a_4}[t_0] = \{a_1a_2e_1e_2, a_1a_2e_2e_3, a_1a_2e_1e_3, a_1a_2e_1e_4, a_1a_2e_3e_4 \}. \]

Now to obtain \( T^{1a_2a_3}[t_0] \), we consider matrix \( A(a_1a_2a_3) \) obtained from \( [A_{11}U] \) by deleting row 4 and columns belonging to \( U \) as
\[
A(a_1a_2a_3) = \begin{bmatrix}
e_1 & e_2 & e_3 & e_4 \\
1 & 1 & 1 & 0 & 0 \\
2 & 1 & 1 & 0 & 1 \\
3 & 1 & 0 & 1 & 0
\end{bmatrix}
\]

This can be changed into a fundamental form as
\[
[A(a_1a_2a_3)_{11}U] = \begin{bmatrix}
e_1 & e_2 & e_4 & e_3 \\
1 & 1 & 1 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 \\
3 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

Thus \( t_0(a_1a_2a_3) = \{e_2, e_3, e_4\} \) and we have
\[
T^{2e_2}[t_0(a_1a_2a_3)] = \{e_1e_3e_4\}
\]
\[
T^{e_1}[t_0(a_1a_2a_3)] = \emptyset
\]
\[
T^{e_3}[t_0(a_1a_2a_3)] = \{e_1e_2e_4\}
\]

For \( T^{e_2e_4}[t_0(a_1a_2a_3)] \), we consider matrix
\[
A(a_1a_2a_3e_2e_4) = \begin{bmatrix}
e_1 \\
1 \\
2 & 0
\end{bmatrix}
\]
obtained from \([A(a_1 a_2 a_3)]_{11} U\) by deleting row 3 and all columns belonging
to U. It is clear that this matrix cannot be a fundamental form. Thus
\[T^{e_2 e_4}[t_0(a_1 a_2 a_3)] = \emptyset.\] Similarly,
\[T^{e_2 e_3}[t_0(a_1 a_2 a_3)] = T^{e_3 e_4}[t_0(a_1 a_2 a_3)] = T^{e_4 e_3}[t_0(a_1 a_2 a_3)] = \emptyset.\]
Hence
\[T^{a_1 a_2 a_3}[t_0] = \{e_2 e_3 e_4 a_4, e_3 e_4 a_4, e_1 e_2 e_4 a_4\}.

For \(T^{a_1 a_2 a_4}[t_0]\), we consider
\[
\begin{array}{cccc}
e_1 & e_2 & e_3 & e_4 \\
1 & 1 & 1 & 0 & 0 \\
2 & 1 & 1 & 0 & 1 \\
4 & 0 & 1 & 1 & 1 \\
\end{array}
\]
\[A(a_1 a_2 a_4) =
\]
By changing this matrix into a fundamental form as
\[
\begin{array}{cccc}
e_1 & e_2 & e_4 & e_3 \\
1 & 1 & 1 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 \\
4 & 1 & 0 & 0 & 1 \\
\end{array}
\]
\[[A(a_1 a_2 a_4)]_{11} U\] we have \(t_0(a_1 a_2 a_4) = \{e_2 e_3 e_4\},\)
\[T^{e_2}[t_0(a_1 a_2 a_4)] = \{e_1 e_3 e_4\}.
\]
\[T^{e_4}[t_0(a_1 a_2 a_4)] = \emptyset\]
and
\[T^{e_3}[t_0(a_1 a_2 a_4)] = \{e_1 e_2 e_4\}.
\]
Because $A(a_1 a_2 a_4)_{11}$ consists of one column,

$$e^2 e^4_{[t_0(a_1 a_2 a_4)]} = e^2 e^3_{[t_0(a_1 a_2 a_4)]} = e^4 e^3_{[t_0(a_1 a_2 a_4)]}$$

$$= T e^2 e^4 e^3_{[t_0(a_1 a_2 a_4)]} = \emptyset .$$

Hence

$$T a^2 a^4_{[t_0]} = \{e^2 e^3 e^4, e_1^3 e_3 e_4, e_1 e_2 e_4\} .$$

For $T a^3 a^4_{[t_0]}$, we consider

$$A(a_1 a_3 a_4) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 1 & 1 \end{bmatrix}$$

which can be changed into a fundamental form as

$$[A(a_1 a_3 a_4)_{11}] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence

$$t_0(a_1 a_3 a_4) = \{e_1 e_3 e_4\}$$

$$T e^2_{[t_0(a_1 a_3 a_4)]} = \{e_1 e_3 e_4\}$$

$$T e^3_{[t_0(a_1 a_3 a_4)]} = \{e_1 e_2 e_4\}$$

and

$$T e^4_{[t_0(a_1 a_3 a_4)]} = \emptyset .$$

Furthermore

$$T e^2 e^3_{[t_0(a_1 a_3 a_4)]} = T e^2 e^4_{[t_0(a_1 a_3 a_4)]} = T e^3 e^4_{[t_0(a_1 a_3 a_4)]}$$

$$= T e^2 e^3 e^4_{[t_0(a_1 a_3 a_4)]} = \emptyset .$$
Hence
\[ T_{a_1a_2a_3a_4}^{t_o} = \{e_2a_2e_3e_4, e_1a_2e_3e_4, e_1a_2e_2e_4\}. \]

To obtain \( T_{a_2a_3a_4}^{t_o} \), we use matrix

\[
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 \\
2 & 1 & 1 & 0 & 1 \\
3 & 1 & 0 & 1 & 0 \\
4 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
\]

which can be changed into

\[
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 \\
2 & 1 & 1 & 0 & 0 \\
3 & 1 & 0 & 1 & 0 \\
4 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

which contains a row of zeros. Thus, there is no trees in \( T_{a_2a_3a_4}^{t_o} \).

Finally, to obtain \( T_{a_1a_2a_3a_4}^{t_o} \), we consider

\[
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 \\
1 & 1 & 1 & 0 & 0 \\
2 & 1 & 1 & 0 & 1 \\
3 & 1 & 0 & 1 & 0 \\
4 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
\]

This can be changed to

\[
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
which contains a row of zeros. Thus there is no tree in $T^{a_1a_2a_3a_4}[t_o]$. The trees obtained by the above process are all trees in the given linear graph $G$.

**Generation of Complete Trees**

Let $A_i$ and $A_v$ be cut set matrices of the current graph $G_i$ and voltage graph $G_v$ of a given network $G$. Also let the $i$th column of both $A_i$ and $A_v$ represent edge $e_i$ of $G$.

With matrices $A_i$ and $A_v$, the following operations are called elementary operations,

1. Adding and subtracting one row from another
2. Multiplying a row by (-1)
3. Interchanging rows
4. Interchanging columns.

Associated with operations (2) and (3), we define a number $N$ called a "M-number" which indicates the number of times we use these two operations in order to make a given matrix to a desired form. The M-number $N_i$ and $N_r$ which are respectively necessary in order to change given pair of cut set matrices $A_i$ and $A_r$ into a fundamental form as $[A_{i11}, U]$ and $[A_{v11}, U]$ are called fundamental M-numbers with respect to tree $t_o$ where $t_o$ corresponds to the unit matrix of $[A_{i11}, U]$ and $[A_{v11}, U]$. Notice that $t_o$ is a complete tree in $G$. The sign of $t_o$ which is important for topological analysis of such networks is given by $(-1)^{N_i + N_v}$.

**Example 4:** Let $A_i$ and $A_v$ of a given network $G$ be
We can change these into a fundamental form as

\[
\begin{align*}
A_i &= \begin{bmatrix}
e_1 & e_2 & e_3 & e_4 & e_5 \\
1 & 1 & 0 & 1 & 0 \\
2 & 0 & 1 & -1 & 1 \\
3 & 0 & -1 & 0 & 0 \\
\end{bmatrix} \\
A_v &= \begin{bmatrix}
e_1 & e_2 & e_3 & e_4 & e_5 \\
1 & 1 & -1 & -1 & 0 \\
2 & 0 & 0 & 1 & 0 \\
3 & -1 & 0 & 0 & -1 \\
\end{bmatrix}
\end{align*}
\]

with fundamental M-numbers \(N_i = 1\) and \(N_v = 4\). Thus the sign of \(t_\circ = \{e_3, e_4, e_5\}\) is \((-1)^{N_i+N_v} = -1\).

For convenience, the sign of a complete tree \(t_\circ\) is given by the superscript as \(t_\circ^\pm\). For example, since the sign of \(\{e_3, e_4, e_5\}\) is \(-1\), we express it as \(\{e_3, e_4, e_5\}^-.\)

It is clear that all complete trees of distance one from \(t_\circ\) are in the set

\[
\bigcup_{a \in t_\circ} T^a[t_\circ] = \bigcup_{a \in t_\circ} \{t | t = t_\circ \oplus \{ae\}, e \in S_{a}^{(i)}(t_\circ) \cap S_{a}^{(v)}(t_\circ)\}
\]

where \(S_{a}^{(i)}(t_\circ)\) is a fundamental cut set containing edge \(a\) with respect to \(t_\circ\) in current graph \(G_i\) and \(S_{a}^{(v)}(t_\circ)\) is a fundamental cut set containing \(a\) with respect to \(t_\circ\) in voltage graph \(G_v\) for all \(a \in t_\circ\). In order to obtain these complete trees directly from matrices \(A_i\) and \(A_v\), we define the following operation.

**Definition:** The operation \(\Theta\) of two matrices \(P = [p_{ij}]\) and \(Q = [q_{ij}]\) of order \(m\) and \(n\) is the product of corresponding entries of \(P\) and \(Q\), i.e.,

\[
P \Theta Q = [p_{ij}q_{ij}]
\]
For example, if $P$ and $Q$ are

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

Then $P \odot Q$ is

$$P \odot Q = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

With this definition, all complete trees of distance one from $t_o$ can be obtained by the following steps when $[A_{i_{11}} U]$ and $[A_{v_{11}} U]$, with $t_o$ corresponding to the unit matrix $U$, are given:

**Step 1:** Form $[A_{i_{11}} U] \odot [A_{v_{11}} U]$. Notice that each row $j$ of $[A_{i_{11}} U] \odot [A_{v_{11}} U]$ can represent $S_{a_j}^{(i)}(t_o) \cap S_{a_j}^{(v)}(t_o)$ where row $j$ of $[A_{i_{11}} U]$ and $[A_{v_{11}} U]$ represent respectively $S_{a_j}^{(i)}(t_o)$ and $S_{a_j}^{(v)}(t_o)$.

**Step 2:** Form all complete trees in $T_{a_j}[t_o]$, where $a_j \in t_o$, by replacing $a_j$ by edge $e$ whose corresponding column in $[A_{i_{11}} U] \odot [A_{v_{11}} U]$ has a non-zero at row $j$ which represents $S_{a_j}^{(i)}(t_o) \cap S_{a_j}^{(v)}(t_o)$. If this non-zero is +1, the sign of the newly obtained complete tree is the same as that of $t_o$. Otherwise, the sign of the new complete tree is opposite from that of $t_o$. The following example will illustrate this point.

**Example 5:** All complete trees of distance one from $t_o$ in the linear graph in Example 4 can be obtained by first forming $[A_{i_{11}} U] \odot [A_{v_{11}} U]$ as

\[
\begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} A_{i_{11}} U \odot A_{v_{11}} U = 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}
\]
Notice that \( t_o \) in Example 4 is \( \{e_3e_4e_5\}^- \). (The sign of \( t_o \) is -1.) \( T^{e_3}[t_o] \) is obtained from \( t_o \) by replacing edge \( e_3 \) by edge \( e_1 \) because only column \( e_1 \) has non-zero at the first row representing \( S_{e_3}^{(i)}(t_o) \cap S_{e_3}^{(v)}(t_o) \). Thus

\[
T^{e_3}[t_o] = \{e_1e_4e_5\}^+
\]

The sign of \( \{e_1e_4e_5\} \) is the opposite of the sign of \( t_o \) because of the non-zero entry being -1. Similarly,

\[
T^{e_4}[t_o] = \{e_1e_3e_5\}^-
\]

and

\[
T^{e_5}[t_o] = \{e_2e_3e_4\}^-
\]

All complete trees in \( T^{a_1a_2\ldots a_p}(t_o) \) can be obtained by the following steps where \( t_o = \{a_1a_2\ldots a_n\} \) and \( 1 < p \leq n \).

**Step 1:** From \( [A_{v,11}] \) and \( [A_{v,11}] \), form matrices \( A_v(a_1\ldots a_p) \) and \( A_v(a_1\ldots a_p) \) respectively by deleting all rows representing \( S_{a_r}^{(i)}(t_o) \) and \( S_{a_r}^{(v)}(t_o) \) for all \( r = p+1, p+2, \ldots, n \) and deleting all columns belonging to the unit matrix \( U \). Notice that the row representing \( S_{a_r}^{(i)}(t_o) \) in \( [A_{v,11}] \) has 1 at the column corresponding to edge \( a_r \). Similarly, the row representing \( S_{a_r}^{(v)}(t_o) \) in \( [A_{v,11}] \) has 1 at the column corresponding to edge \( a_r \).

**Step 2:** Change \( A_v(a_1\ldots a_p) \) and \( A_v(a_1\ldots a_p) \) into fundamental forms as \( [A_{v,11}(a_1\ldots a_p)11U] \) and \( [A_{v,11}(a_1\ldots a_p)11U] \). Notice that jth columns in both of these matrices must represent edge \( e_j \) for all columns. Suppose columns of the unit matrix \( U \) in these matrices represent edges \( e_1, e_2, \ldots, e_p \), that is, \( t_o' = \{e_1\ldots e_p\} \). The sign of \( t_o' \), by definition, is the same as that of \( t_o' \cup (t_o-\{a_1\ldots a_p\}) \) which is a complete tree in a given network \( G \).
Let the fundamental M-numbers be \( N_1(a_1...a_p) \) and \( N_v(a_1...a_p) \) which are necessary to change \( A_1(a_1...a_p) \) and \( N_v(a_1...a_p) \) in fundamental forms. Then if \((-1)^{N_1(a_1...a_p)+N_v(a_1...a_p)} \) is -1, the sign of \( t^i_o \) is opposite from that of \( t_o \). Otherwise, the sign of \( t^i_o \) is the same as that of \( t_o \). Since all complete trees in the resultant current and voltage graphs whose fundamental cut set matrices are \([A_1(a_1...a_p)_{11} U]\) and \([A_v(a_1...a_p)_{11} U]\) respectively are in

\[
T^1_{[t^i_o]}, T^2_{[t^i_o]}, ..., T^p_{[t^i_o]}
\]

\[
T^e_1 e^2_{[t^i_o]}, ..., T^e_1 e^j_{2_{[t^i_o]}}, ..., T^e_p e^p_{2_{[t^i_o]}}
\]

\[
T^e_j e^j_{1_{[t^i_o]}}, ..., T^e_j e^j_{r_{[t^i_o]}}, ..., T^e_j e^r_{p_{[t^i_o]}}
\]

\[
T^e_j e^j_{1_{[t^i_o]}}, ..., T^e_j e^j_{r_{[t^i_o]}}, ..., T^e_j e^r_{p_{[t^i_o]}}
\]

\[
T^e_j e^j_{1_{[t^i_o]}}, ..., T^e_j e^j_{r_{[t^i_o]}}, ..., T^e_j e^r_{p_{[t^i_o]}}
\]

The complete trees in \( T^u_{[t^i_o]} \) for \( u = 1, 2, ..., p \) are of distance one from \( t^i_o \). Thus these can be obtained by the process discussed previously. The process of obtaining complete trees in \( T^e_j e^j_{1_{[t^i_o]}}, ..., T^e_j e^j_{r_{[t^i_o]}} \) is exactly the same as that of complete trees in \( T^a_1...a_p_{[t^i_o]} \). Thus we go to Step 1 by using (1) \( t^i_o \) as \( t^i_o \), (2) \( e_1, e_2, ..., e_r \), and (3) \([A_1(a_1...a_p)_{11} U]\) and \([A_v(a_1...a_p)_{11} U]\) as \([A_1 U]\) and \([A_v U]\).

**Example 6:** The following steps will give all complete trees of a linear graph corresponding to \( A_1 \) and \( A_r \) where...
We can make $A_i$ and $A_v$ in a fundamental form as

$$
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 & e_5 \\
1 & 1 & 0 & 1 & 0 \\
\end{bmatrix}
\quad \quad
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 & e_5 \\
1 & -1 & -1 & 0 & 0 \\
\end{bmatrix}

\quad \quad
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 & e_5 \\
3 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\quad \quad
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 & e_5 \\
2 & -1 & 0 & 0 & -1 \\
\end{bmatrix}
$$

with $N_i = 1$ and $N_v = 4$. Thus the sign of $t_o = \{e_3 e_4 e_5\}$ is $(-1)^{N_i+N_v}$ which is -1, or $t_o = \{e_3 e_4 e_5\}$.

To obtain all complete trees of distance one from $t_o$, we operate

$$
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 & e_5 \\
1 & -1 & 0 & 1 & 0 \\
\end{bmatrix}
\quad \quad
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 & e_5 \\
2 & 1 & 0 & 0 & 1 \\
\end{bmatrix}

\quad \quad
\begin{bmatrix}
e_1 & e_2 & e_3 & e_4 & e_5 \\
3 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
$$

Then

$$
T^3[t_o] = \{e_1 e_4 e_5\}
$$

$$
T^4[t_o] = \{e_1 e_3 e_5\}
$$

and

$$
T^5[t_o] = \{e_2 e_3 e_4\}
$$

To obtain $T^3 e_4[t_o]$, we consider $A_i(e_3 e_4)$ and $A_v(e_3 e_4)$ where

$$
A_i(e_3 e_4) =
\begin{bmatrix}
e_1 & e_2 \\
1 & 1 \\
\end{bmatrix}
\quad \quad
A_v(e_3 e_4) =
\begin{bmatrix}
e_1 & e_2 \\
1 & -1 \\
\end{bmatrix}
$$
which can be changed into a fundamental form as

\[
[A_1(e_3 e_4)]_{11} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \quad \quad [A_v(e_3 e_4)]_{11} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}
\]

with \( N_1(e_3 e_4) = 0 \) and \( N_v(e_3 e_4) = 1 \). Thus \( t_0(e_3 e_4) = \{e_1 e_2\} \) with the sign

\[
N_1(e_3 e_4) + N_v(e_3 e_4)
\]

which is \((-1)\) times the sign of \( t_0 \), which is \(-1\). Thus \( t_0(e_3 e_4) \) is \(+1\), or \( t_0(e_3 e_4) = \{e_1 e_2\} \). Hence \( T^{e_3 e_4}[t_0] = \{e_1 e_2 e_5\} \).

To obtain \( T^{e_3 e_5}[t_0] \), we form

\[
A_1(e_3 e_5) = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \quad \quad A_v(e_3 e_5) = \begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix}
\]

Since \( A_v(e_3 e_5) \) is singular, it is impossible to change it into a fundamental form. Thus \( T^{e_3 e_5}[t_0] = \emptyset \).

To obtain \( T^{e_4 e_5}[t_0] \), we consider

\[
A_1(e_4 e_5) = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \quad \quad A_v(e_4 e_5) = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}
\]

which can be changed into a fundamental form by elementary operations as

\[
[A_1(e_4 e_5)]_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \quad [A_v(e_4 e_5)]_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
with fundamental M-numbers being $N_1(e_4 e_5) = 0$ and $N_v(e_4 e_5) = 0$. Thus $t_0(e_4 e_5) = \{e_1 e_2\}^-$ (the sign of $t_0(e_4 e_5)$ is the same as that of $t_0$ which is -1). Hence

$$T_4^5[t_0] = \{e_1 e_2 e_3\}^-.$$

The matrix $A_1(e_4 e_5)$ for $T_3^4 e_5 \{t_0\}$ consists of two columns and three rows which is impossible to change into a fundamental form. Thus $T_3^4 e_5 \{t_0\} = \emptyset$.

An important operation in this technique is to change matrices into a fundamental form. There are several ways of doing this. One way is as follows:

1. Take any column in $A_1$, say $e$, which (i) has non-zero in $A_v$ and (ii) has a non-zero at the first row of $A_1$. Move this column in both $A_1$ and $A_v$ to the first column of the place for $U$ in a fundamental form. If this non-zero is -1, multiply this row by -1.

2. Let Kth row of $A_v$ be a non-zero at column $e$. Move this Kth row to the first row. Remove all other non-zeros in $A_1$ and $A_v$ in column $e$ by elementary operations.

3. Now we consider submatrices $A_1$ and $A_v$ obtained from $A_1$ and $A_v$ by deleting the first row and column $e$, and go back to (1) except when all other non-zeros are to be removed in (2), we consider $A_1$ and $A_v$ rather than $A_1$ and $A_v$. 
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### ABSTRACT

A method of generating all complete trees of a pair of linear graphs which can represent active networks is given. This method consists of two parts, one of which is to obtain one complete tree and the other is to generate all possible complete trees of distance one from that already determined.

All complete trees of distance one from a given tree \( t_o = \{a_1, a_2, \ldots, a_n\} \) can easily be obtained by

\[
U \mathcal{T}_P(t_o) = U \{t | t = t_o \oplus \{a_p\}, e \in S_{a_p}(t_o, G_i) \cap S_{a_p}(t_o, G_v)\}
\]

Where \( S_{a_p}(t_o, G_i) \) is a fundamental cut set containing edge \( a_p \) with respect to tree \( t_o \) in \( G_i \), \( S_{a_p}(t_o, G_v) \) is a fundamental cut set containing edge \( a_p \) with respect to \( t_o \) in \( G_v \), and \( G_i \) and \( G_v \) are the pair of linear graphs representing an active network. When we have cut set matrices \( A_i \) and \( A_v \) corresponding to \( G_i \) and \( G_v \), respectively, we can find a tree \( t_o \) by changing these matrices to the fundamental form as \([A_i, U]\) and \([A_v, U]\) in which the
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edges corresponding to the unit matrix U form a complete tree. These two processes are easily carried out by the use of computers. There will be no duplications when complete trees are generated by this method. Furthermore, complete trees are generated by sets of complete trees classified by edges in initial complete tree $t_0$. Thus it will be easy to factorize according to the weights of these edges.