Result Verification Algorithms for Optimization Problems

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In this article we discuss the design of result verification algorithms for optimization problems. In particular, we design time-optimal result verification algorithms which verify the solution of all-pairs shortest paths, maximum flow, and matching problems. We prove that polynomial-time verification algorithms for \( NP \)-complete problems do not exist, unless \( P = NP \). Result verification problems for most of the \( \text{NPC} \)-hard problems are not believed to be in \( NP \). We also consider verification algorithms for approximation algorithms for \( NP \)-complete and \( \text{NPC} \)-hard problems.
Result Verification Algorithms for Optimization Problems

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Abstract

In this article we discuss the design of result verification algorithms for optimization problems. In particular, we design time-optimal result verification algorithms which verify the solution of all-pairs shortest paths, maximum flow, and matching problems. We prove that polynomial-time verification algorithms for NP-complete problems do not exist, unless P \neq NP. Result verification problems for most of the NP-hard problems are not believed to be in NP. We also consider verification algorithms for approximation algorithms for NP-complete and NP-hard problems.

1 Introduction

Consider a program A written to evaluate a function f. In this article we address the issue of being able to verify that the program A works correctly for all inputs, i.e., A(x) = f(x) for all inputs x. Most of the approaches to this problem fall under the following three broad categories: formal verification, program testing/correcting, and result verification.

Formal verification of an algorithm or program requires a formal, rigorous proof of the correctness of the program, using the syntactic and semantic rules of the language in which the program is written. Formal verification has had limited success because even small programs are hard to verify formally. Because the technique of formal verification checks the correctness of the code as written on paper, formal verification doesn't prove the correctness of the executable code of the algorithm verified. The executable code may still be faulty because of errors in the compilation process, hardware faults, etc.

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Blum et al. [1] introduced the theory of self-testing/correcting, where a self-tester for a program A estimates the probability that \( A(x) \neq f(x) \) for a random input \( x \) and a self-corrector for A computes \( f(x) \) correctly, if the error probability of A is sufficiently low. Both the self-tester and the self-corrector can make calls to A, and they are considered efficient if their total time is linear in the running time of A. This approach has two drawbacks. First, self-testers and self-correctors are both program specific. Second, they are allowed to make multiple calls to the original program. Hence, in most cases the total running time of the tester or corrector is more than the time of the original program itself.

Result verification, as the name suggests, involves checking the correctness of a proposed solution to an instance of the problem. A result verification algorithm for a problem \( \Pi \) can be used to test the output of any program solving the problem \( \Pi \). Moreover, a result verification algorithm does not call a program solving the original problem.

In this manuscript, we design time-optimal result verification algorithms for some optimization problems. We discuss the existence of polynomial-time verification algorithms for \( NP \)-complete or \( NP \)-hard problems, and for approximation algorithms for \( NP \)-hard problems. The question of existence of polynomial-time verification algorithms becomes more interesting when we allow an algorithm to leave behind a trail of data, polynomial in length of the input, called certification trail [12], [13], in addition to its normal output. A result verification algorithm might use this additional information cleverly to verify the result in much less time.

The rest of the article is organized as follows. In the next section we formally define some terms and design time-optimal result verification algorithms which verify the solution of all-pairs shortest paths, maximum flow, and matching problems. In the third section we discuss the existence of polynomial-time verification algorithms for \( NP \)-complete and \( NP \)-hard problems, and for approximation algorithms for \( NP \)-hard optimization problems. In the fourth section, we discuss result verification with the help of certification trails. Finally, we conclude in section 5.

## 2 Definitions

In this section we define some terms and present some examples.

**Definition 2.1** A problem \( \Pi \) is formalized as a binary relation, i.e., a set of ordered pairs. The domain of \( \Pi \) is the set of instances, and the range is the set of solutions. Thus, \( \Pi \subseteq I \times S \), where \( I \) is the set of instances and \( S \) is the set of solutions for the problem. We say an algorithm \( A \) solves a problem \( \Pi \) if for all \( x \) in \( I \), when \( x \) is input to \( A \), a \( y \) in \( S \) is output such that \((x, y) \in \Pi \). In some cases, when each instance has a unique solution, it is possible to formalize the problem as a function.

**Definition 2.2** Given a problem \( \Pi \subseteq I \times S \), the verification problem for the problem \( \Pi \) is denoted by \( V(\Pi) \). The verification problem \( V(\Pi) \) is a subset of \((I \times S) \times \{0, 1\}\) such that
Consider an algorithm A for a problem II and a verification algorithm B solving the corresponding verification problem V(II). By definition, if B takes as an input the ordered pair \((x, y)\), where \(y\) is the output generated by A on the input \(x\), then B outputs 1 if and only if A produced a correct output on the input \(x\).

We illustrate the above definitions using the maximum clique size problem as an example. For the maximum clique size problem, II, the set of instances is the set of graphs, and the set of solutions is the set of natural numbers: \((G, k) \in \Pi\) if \(k\) is the size of the maximum clique in \(G\). For the verification problem, if \(k\) is the size of the largest clique in \(G\), then \( ((G, k), 1) \in V(\Pi) \); otherwise, \( ((G, k), 0) \in V(\Pi) \) when \(k\) is not the size of the largest clique in \(G\).

### 2.1 Verifiers for Network Flow Problems

In this section we discuss verification algorithms for maximum flow and minimum-cost flow problems in networks, and for bipartite and general matching problems.

#### 2.1.1 Verifier for the Maximum Flow Problem

Consider a directed graph \(G\) with two distinguished vertices \(s\), the source, and \(t\), the sink. Let the vertex set and edge set of \(G\) be \(V(G)\) and \(E(G)\) respectively. Let \(m = |E(G)|\) and let \(c(e)\) be the nonnegative maximum capacity of an edge \(e\) in \(E(G)\). Let the flow in arc \((x, y)\) be denoted by \(f(x, y)\). Also, let \(\mathbb{R}\) be the set of real numbers. Then an \(s-t\) flow of value \(v\) is defined by the following constraints:

\[
\sum_{(u, w) \in E(G)} f(u, w) = \sum_{(w, u) \in E(G)} f(w, u), \quad \text{for each node } u \neq s, t
\]

\[
\sum_{(s, w) \in E(G)} f(s, w) = \sum_{(w, s) \in E(G)} f(w, s) - v
\]

\[
\sum_{(t, w) \in E(G)} f(t, w) = \sum_{(w, t) \in E(G)} f(w, t) + v
\]

\[
f \leq c
\]

\[
f \geq 0
\]

where \(f, c \in \mathbb{R}^m\) are the flow and the capacity vectors, respectively. A feasible flow is a function \(f\) on vertex pairs which satisfies the above constraints. The value of a flow \(f\) is \(v\), the flow out of the source \(s\). The maximum flow problem is that of finding a feasible flow...
$f$ of maximum value $v$. This problem has a rich and elegant theory and many applications both in operations research and in combinatorics [5], [14].

We need to define a few terms. A residual capacity for a flow $f$ is the function on vertex pairs given by $\text{res}(v, w) = c(v, w) - f(v, w)$. The residual graph $R_f$ for a flow $f$ is the graph with vertex set $V(G)$, source $s$, sink $t$, and an edge $(v, w)$ of capacity $\text{res}(v, w)$ for every pair $v, w$ such that $\text{res}(v, w) > 0$. An augmenting path for $f$ is a path $p$ from $s$ to $t$ in $R_f$.

We state the following lemma without proof [14].

**Lemma 2.1** A flow $f$ is a maximum flow if and only if there is no augmenting path for $f$.

Let MF be the maximum flow problem. For the corresponding verification problem $V(\text{MF})$, the set of instances is $\Gamma \times F$, where $\Gamma$ is the set of graphs with edge capacities, and $F$ is the set of flows on graphs. An algorithm for $V(\text{MF})$ takes a graph $G$ with a capacity vector $c$ and a flow $f$ as an input. It outputs 1 if and only if $f$ is a flow with maximum value in $G$. According to Theorem 2.1, the algorithm outputs 1 if and only if there is no augmenting path for $f$. The existence of an augmenting path for $f$ in $G$ can be easily checked in $O(m)$ time, because there exists an augmenting path in $G$ if and only if there exists an $(s, t)$ path in the residual graph $R_f$, which can be constructed in $O(m)$ time.

**Theorem 2.1** The maximum flow result verification problem can be solved by an $O(m)$ time verification algorithm.

### 2.1.2 Verifier for the Minimum-Cost Flow Problem

Let $G$ be a network such that each edge $(v, w)$ has a cost per unit of flow, $\text{cost}(v, w)$, in addition to a capacity $c(v, w)$. Let $v_0$ be a positive flow value given. The min-cost problem is to find a feasible $s$-$t$ flow of value $v_0$ that has minimum cost, where the cost of a flow $f$ is $\text{cost}(f) = \sum_{f(v, w) > 0} \text{cost}(v, w)f(v, w)$.

We define the residual graph $R_f$ for a flow $f$ exactly as we did in the section 2.2.1, with the extension that $\text{cost}(v, w)$ is the same on $R_f$ as on $G$.

We state the following lemma [8] without proof.

**Lemma 2.2** An $s$-$t$ flow $f$ in a network is an optimal min-cost flow if and only if there are no negative cost cycles in the residual graph $R_f$.

From the above theorem, verifying the optimality of a given flow $f$ on a given network $G$ amounts to checking for a negative cost cycle in the residual graph $R_f$. Negative cycles in a weighted graph can be detected in $O(n^3)$ time, where $n$ is the number of vertices in the graph [2].

**Theorem 2.2** The minimum-cost flow verification problem can be solved by an $O(n^3)$ time verification algorithm.
2.2 Verifiers for Matching

A matching \( M \) of a graph \( G = (V, E) \) is a subset of edges with the property that no two edges of \( M \) share the same node. The unweighted matching problem is to find a maximum size matching \( M \) of an unweighted graph \( G \). Edges in \( M \) are called matched edges; the other edges are free. Nodes that are not incident upon any matched edge are called exposed. A path \( p = [u_1, u_2, u_3, \ldots, u_k] \) is called alternating if the edges \([u_1, u_2], [u_3, u_4], \ldots, [u_{2j-1}, u_{2j}]\) are free, whereas \([u_2, u_3], [u_4, u_5], \ldots, [u_{2j}, u_{2j+1}]\) are matched. The alternating path \( p \) is called augmenting if both \( u_1 \) and \( u_k \) are exposed vertices.

**Lemma 2.3** A matching \( M \) in an unweighted graph \( G \) is maximum if and only if there is no augmenting path in \( G \) with respect to \( M \).

See [8] for the proof.

One is tempted to devise the analog of the max-flow algorithm for matching: Start with any matching, and repeatedly discover augmenting paths. Indeed, all known algorithms for matching are based on exactly this idea.

Bipartite matching problems can be viewed as special cases of network flow problems [5], [8].

For the non-bipartite unweighted matching problems, the verification of a solution involves checking for an augmenting path in the graph with respect to the matching presented. Tarjan [14] describes an algorithm to detect an augmenting path in \( O(m) \) time.

For the non-bipartite weighted matching problem, the following lemma is of great import [14]. We state the theorem without proof. Here, the weight of an augmenting path is the sum of the weights of the edges on the augmenting path.

**Lemma 2.4** Let \( M \) be a matching of maximum weight among matchings of size \(|M|\), let \( p \) be an augmenting path for \( M \) of maximum weight, and let \( M' \) be the matching formed by augmenting \( M \) using \( p \). Then \( M' \) is of maximum weight among matchings of size \(|M| + 1\).

Lemma 2.4 implies that the augmenting path method will compute maximum weight matchings of all possible sizes if we always augment using a maximum weight augmenting path. All known algorithms use exactly this idea to solve the general weighted matching problem. Essentially, all algorithms work in stages, where each stage searches for a maximum weight augmenting path. For verification purposes, one stage of such an algorithm solving the general weighted matching problem suffices to check for correctness of the matching presented. The best known algorithm for solving the weighted nonbipartite matching problem is by Galil, Micali, and Gabow [6] which runs in \( O(mn \log n) \) time. Each result verification stage of the algorithm runs in \( O(m \log n) \) time.

**Theorem 2.3** The unweighted general matching problem has a result verification algorithm which runs in \( O(m) \) time. The weighted general matching problem has a result verification algorithm which runs in \( O(m \log n) \) time.
2.3 Verifier for the All-Pairs Shortest Paths

Let $G$ be a directed graph with $n$ vertices and $m$ edges, with nonnegative weights on edges. The all-pairs shortest paths problem is to find a shortest path, i.e., a path with minimum weight, between each pair of vertices in $G$. Here, the weight of a path is defined as the sum of the weights of its edges. The length of a path is the number of edges in the path. The most widely known algorithms for the all-pairs shortest paths problem are those of Dijkstra [3] and Floyd [4]. Dijkstra's algorithm has a running time of $\Theta(mn + n^2 \log n)$ when implemented with a heap. Floyd's algorithm runs in $\Theta(n^3)$ time. Bellman and Ford [2] also developed an algorithm which runs in $\Theta(n^3)$ time but also handles graphs with negative weights.

Many algorithms for the shortest paths problem use edge weights only to compute and compare the weights of paths. We therefore define a version [7] of the decision tree model that captures this behavior.

**Definition 2.3** A path-comparison-based algorithm $A$ solving the all-pairs shortest paths problem accepts as input a graph $G$ and a weight function on its edges. The algorithm $A$ can perform all standard operations. However, the only way it can access the edge weights is to compute or compare the weights of paths in the graph.

A verification algorithm for the all-pairs shortest paths problem takes as an input a directed graph $G$ on $n$ vertices, a weight function, and $n(n-1)$ paths corresponding to the pairs of vertices in $G$; the algorithm outputs 1 if and only if for each pair of vertices, the path presented in the input is actually a shortest path between that pair of vertices in $G$. Karger et al. [7] show that any path-comparison-based algorithm for verification of all-pairs shortest paths for directed graphs requires $\Omega(mn)$ path-weight comparisons. We present a path-comparison-based verification algorithm for the all-pairs shortest paths problem which runs in $O(mn)$ time. This is an improvement over the program checker of Rubinfeld [11], which runs in $O(n^3)$ time.

There are graphs with $\Theta(n^2)$ pairs of vertices whose connecting paths have lengths $\Theta(n)$ each. Hence, we cannot verify all-pairs shortest paths in $O(mn)$ time if the verification algorithm is given as an input all those shortest paths explicitly, because the size of the input itself could be $\Omega(n^3)$. Thus the verification algorithm takes as an input only a data structure from which shortest connecting paths can be constructed in time proportional to their lengths. This data structure is the predecessor matrix $C$, where for each vertex pair $i \neq j$, the entry $C[i, j]$ is a vertex $k$ such that the edge $(k, j)$ lies on the shortest path from $i$ to $j$. In addition to $C$, the verification algorithm also takes the distance matrix $D$ as an input, where each entry $D[i, j]$ is the weight of the shortest path from $i$ to $j$. By definition, $D[i, i] = 0$.

We explain the algorithm briefly before presenting the pseudo code. The algorithm takes a weighted directed graph, and an arbitrary solution to the all-pairs shortest paths problem, i.e., matrices $C$ and $D$, as an input. The algorithm outputs 0 if the given solution is incorrect; otherwise, it outputs 1. The weight of the edge $(i, j)$ is denoted by $w(i, j)$. The first part of the algorithm determines for each entry $C[i, j]$ of the matrix $C$ whether $D[i, j] = D[i, C[i, j]] + w(C[i, j], j)$. If not, then an error is detected and 0 is output.
The second part of the algorithm checks for each source $i$ and each edge $(u, v) \in E(G)$ whether using an edge $(u, v)$ could improve the given shortest path from the source $i$ to $v$. If $D[i, u] + w(u, v) < D[i, v]$, then the path from $i$ to $u$ plus the edge $(u, v)$ yields a shorter path from $i$ to $v$, and hence, an error is detected. Following the algorithm, we prove that the algorithm outputs 1 if and only if the matrices $C$ and $D$ presented to it are correct.

Let $\mathbb{R}^+$ denote the set of nonnegative real numbers and let $w(i, j)$ denote the value of the weight function $w : E(G) \to \mathbb{R}^+$ at $(i, j)$.

**Algorithm 1: Verification algorithm for all-pairs shortest paths**

**Input:** Directed graph $G$, a weight function $w : E(G) \to \mathbb{R}^+$, the predecessor matrix $C$, and the distance matrix $D$.

1. for $i = 1$ to $n$  /* for each source */
2. for $j = 1$ to $n$  /* for each destination */
3. if ($j = i$ and $D[i, j] \neq 0$)
4. Return(0)
5. elseif ($j \neq i$)
6. $k := C[i, j]$
7. if ($D[i, j] \neq D[i, k] + w(k, j)$)

    /* From the definition of the predecessor matrix, the edge $(k, j)$ */
    /* lies on the given shortest path from $i$ to $j$. Hence, if $C$ and $D$ */
    /* are correct, then $D[i, j]$ should equal $D[i, k] + w(k, j)$. */

8. Return(0)
9. endif
10. endif
11. endfor
12. for each edge $(u, v) \in E(G)$

    /* Check whether the edge $(u, v)$ could be used to improve the */
    /* given shortest path from $i$ to $v$ */
13. if ($D[i, u] + w(u, v) < D[i, v]$)
14. Return(0)
15. endif
16. endfor
17. endfor
18. Return(1)
Proof of Correctness:

Because the algorithm outputs 0 only if there is an obvious error in the input matrices C or D, Algorithm 1 outputs 1 if the input matrices C and D are correct. Conversely, we prove that if Algorithm 1 outputs 1 then the input matrices C and D are correct.

The verification algorithm verifies single-source shortest paths for source i in the ith iteration of the outermost for loop. Define, for each i, the function \( p_i : [1..n] \rightarrow [1..n] \) as \( p_i(j) = C[i,j] \). Interpretation of \( p_i(j) \) as the parent of \( j \) gives a tree \( T_i \) for each i.

We claim that if for some \( i, j \), the entry \( D[i,j] \) is not the distance of the vertex \( j \) from the root \( i \) in \( T_i \), then lines 3–10 of Algorithm 1 detect an error, and it returns 0. If some entry in the ith row of the matrix \( D \) is incorrect, then there exists a vertex \( j \) such that \( D[i,j] \) is not the distance of \( j \) from \( i \) in \( T_i \). If there is more than one such vertex in \( T_i \), then consider the one whose path from \( i \) in \( T_i \) has the fewest edges. Let that vertex be \( j \), and let \( k = C[i,j] \). The path from \( i \) to \( k \) has fewer edges than the path from \( i \) to \( j \) in \( T_i \), and hence, by the selection of \( j \), the entry \( D[i,k] \) equals the distance of \( k \) from \( i \) in \( T_i \). Thus, the distance from \( i \) to \( j \) in \( T_i \) should be \( D[i,k] + w(k,j) \). Because the entry \( D[i,j] \) is incorrect, \( D[i,j] \neq D[i,k] + w(k,j) \) and hence, Line 8 returns 0.

Finally, lines 12–16 check whether the tree \( T_i \) correctly represents the shortest path tree for the source \( i \).

We prove that each row \( i \) of \( D \) is correct by showing that for every source \( i \), the distance of a vertex \( j \) from \( i \) in \( T_i \) is the correct shortest distance of \( j \) from \( i \), for all \( j \). Note that by preceding discussion, \( D[i,j] \) is the correct distance of the vertex \( j \) from the root \( i \) in \( T_i \) if the algorithm outputs 1. Let \( T_i' \) be the correct shortest path tree for the source \( i \). We denote the distance of a vertex \( j \) from the root \( i \) in \( T_i' \) by \( D'[i,j] \). We show that the distance of a vertex \( j \) in \( V(G) \) from the root is the same in \( T_i \) and \( T_i' \), i.e., \( D[i,j] = D'[i,j] \) for every \( j \). We prove this by induction on the level of the vertex \( j \) in \( T_i' \), where level of a vertex in a tree is defined as the number of edges on the path from the root of the tree to that vertex. By convention, the level of the root is 0.

**Basis.** Level(\( j \)) = 0. Here, \( j = i \) and the claim is obviously true.

**Induction Step.** Level(\( j \)) = \( l \). Suppose the predecessor of \( j \) in \( T_i' \) is \( k \). Because the level of the vertex \( k \) is \( l - 1 \), the induction hypothesis implies \( D'[i,j] = D'[i,k] + w(k,j) = D[i,k] + w(k,j) \). Because of the check made in line 13 for the edge \((k,j)\) and the algorithm outputs 1, \( D[i,j] \leq D[i,k] + w(k,j) \). Also, because \( D'[i,j] \) is the correct shortest distance, \( D'[i,j] \leq D[i,j] \). Therefore, \( D'[i,j] = D[i,j] \).

Therefore, all the entries of the row \( i \) of \( D \) are correct. This implies that \( T_i \) is a correct shortest tree path for the source \( i \). Hence, the matrix \( D \) and the predecessor matrix \( C \), which defines the trees, are correct.

Thus, if Algorithm 1 outputs 1, then the input matrices \( D \) and \( C \) are correct. □

It is easy to see that Algorithm 1 is a path-based-comparison algorithm that runs in \( O(mn) \) time for a connected graph \( G \). The algorithm can easily be parallelized to run on \( O(mn) \) processors in \( O(1) \) time on an EREW PRAM machine.
Theorem 2.4 Algorithm 1 is a $\Theta(mn)$ time path-based-comparison verification algorithm for the all-pairs shortest paths problem.

3 Verifiers for Intractable Problems

In this section we discuss the existence of polynomial-time verification algorithms for NP-complete and NP-hard problems.

3.1 Verification Problems for NP-complete Problems

Consider a decision problem $\Pi: I \rightarrow \{0, 1\}$. The verification problem $V(\Pi)$ for the problem $\Pi$ is the relation $V(\Pi) \subseteq (I \times \{0, 1\}) \times \{0, 1\}$ such that $((x, y), 1) \in V(\Pi)$ if and only if $(x, y) \in \Pi$. An algorithm $A$ for a decision problem $\Pi$ can be interpreted as a Turing machine that accepts the language $\{x|(x, 1) \in \Pi\}$. The *language accepted* by an algorithm $A$ for a decision problem $\Pi$ is denoted by $L(\Pi)$.

Because there is a one-to-one correspondence between decision problems and languages, we define a problem $\Pi$ to be NP-complete if $L(\Pi)$ is AP-complete. We now show that the verification problem for every AP-complete problem is AP-complete. We show this by proving that for every decision problem $\Pi$, $L(\Pi)$ is Karp-reducible to the language $L(V(\Pi))$.

Reducing $L(\Pi)$ to $L(V(\Pi))$

Define $f: I \rightarrow J$, where $J$ is $I \times \{0, 1\}$, as $f(x) = (x, 1)$. It's easy to see that $f(x) \in L(V(\Pi))$ if and only if $x \in L(\Pi)$. Because $f$ is computable in polynomial time, $L(\Pi)$ is Karp-reducible to $L(V(\Pi))$.

Also, $L(V(\Pi)) \in NP$, for any $L(\Pi)$ in NP.

Thus, we have proved the following result.

**Theorem 3.1** The verification problem $V(\Pi)$ for any NP-complete problem $\Pi$ is NP-complete.

3.2 Verification for NP-hard Problems

Let $\mathbb{N}$ be the set of natural numbers. Consider an NP-hard problem $H \subseteq I \times \mathbb{N}$ satisfying the following property: If $(x, y) \in H$, then $y \leq f(x)$, where the integer value of $f(x)$ is polynomial in size of $x$ and the function $f$ is computable in polynomial time. The value $f(x)$ is essentially the upper bound on the integer value of a solution of $x$. Let $B$ be a verification algorithm solving the problem $V(H)$. We show that a polynomial number of
calls to a verification algorithm \( B \) is sufficient to solve \( H \). Consequently, if \( P \neq NP \), then a polynomial time algorithm for \( V(H) \) does not exist.

**An algorithm for solving \( H \) using \( B \)**

**Input:** \( x \)

Compute \( f(x) \)

for \( k = 1 \) to \( f(x) \)

if \( (B \) outputs 1 on the input \((x, k)\))

Return \((k)\)

endif

endfor.

Because the value of \( f(x) \) is polynomial in the length of the input \( x \), the \( NP \)-hard problem \( H \) is Cook-reducible to the verification problem \( V(H) \). For many graph optimization problems — minimum vertex cover size, maximum clique size, minimum clique cover size, chromatic number, maximum cycle length size, etc. — the value of the solution \( y \) for a given instance graph \( x \) is bounded above by the number of vertices in the graph. Hence, each of these \( NP \)-hard graph optimization problems is Cook-reducible to its corresponding verification problem.

We state the above result in the form of the following theorem.

**Theorem 3.2** Consider an \( NP \)-hard problem \( H \subseteq I \times \mathbb{R} \), such that if \((x, y) \in H\) then \( y \leq f(x) \), where \( f \) is computable in polynomial time. Also, suppose that the value of \( f(x) \) is polynomial in size of \( x \). A polynomial time verification algorithm for \( V(H) \) does not exist, unless \( P = NP \).

Let \( \mathbb{R} \) be the set of real numbers. Consider an \( NP \)-hard optimization problem \( H \subseteq I \times \mathbb{R} \). For each \( x \in I \), \( x \) has a certain set of real candidate solutions. We illustrate the term "real candidate solutions" through the following examples. For the maximum clique size problem, \( y \) is a candidate solution of a graph \( G \) if \( G \) has a clique of size \( y \). For the traveling salesman problem, \( y \) is a candidate solution of an instance \( G \) if there exists a tour of cost \( y \) in \( G \). In contrast, a feasible solution of an instance \( x \) is a combinatorial structure or vector which is a valid solution to the problem. For example, a feasible solution of the minimum vertex coloring problem is an actual valid coloring of the vertices of the input graph. Similarly, a feasible solution of the maximum clique size problem is the set of vertices which induce a clique in the input graph. Intuitively, a candidate solution is the cost of a feasible solution, and the optimization problem is to find the optimum candidate solution.

For some optimization problems, it is possible to compute the optimum feasible solution of an instance in polynomial time using an oracle which computes the optimum cost of any given instance. Such optimization problems are known as self-reducible problems. For example, it is an easy to design a polynomial time bounded graph coloring algorithm \( A \), which colors a given graph with minimum number of colors (the chromatic number of the graph), using an algorithm which returns the chromatic number of any given graph. Similarly, finding
the maximum clique in a graph is also a self-reducible problem because it is easy to find a maximum clique in a graph, using an algorithm which returns the maximum clique size of any given graph. Note that for these optimization problems, it may not possible to compute the optimum solution of an instance $I$ in polynomial time, if we are given only the optimum cost of the instance $I$. We should be able to query the optimum costs of the “subinstances” of $I$ also in constant time.

$H$ is an $NP$-hard maximization problem if for each $(x, y) \in H$, $y$ is the largest candidate solution of $x$. Also we define the corresponding decision problem, $N_H$, for a maximization problem $H$ as: $((x, y), 1) \in N_H$ if $(x, z) \in H$ for some $z \geq y$. Suppose $H$ is an $NP$-hard maximization problem and the corresponding decision problem $N_H$ is $NP$-complete (as is the case with many $NP$-hard optimization problems).

**Observations**

- $N_H \subseteq (I \times \mathbb{R}) \times \{0, 1\}$. $((x, y), 1) \in N_H$ if and only if $x$ has a candidate solution $z$ such that $z \geq y$.

- $V(H) \subseteq (I \times \mathbb{R}) \times \{0, 1\}$. $((x, y), 1) \in V(H)$ if and only if $y$ is the maximum candidate solution of $x$.

Let $\Pi$ be the maximum clique size problem, which is an $NP$-hard maximization problem. If $(G, k) \in \Pi$, then the value of $k$ is bounded by the number of vertices in $G$. Hence, for the problem $\Pi$, $f(G) = |V(G)|$, which is polynomial in the size of $G$. For a graph $G$, $y$ is a candidate solution of $G$ if and only if $G$ has a clique of size $y$. An algorithm solving $\Pi$ outputs the maximum candidate solution of $x$. The corresponding decision problem for $\Pi$ is known as the Clique-Size decision problem where $(G, k) \in \text{Clique-Size}$ if the graph $G$ has a clique of size greater than or equal to $k$. Moreover, $(G, k) \in V(\Pi)$, the verification problem for $\Pi$, if and only if $k$ is the size of the largest clique in the graph $G$.

**Definition 3.1** A language $L$ is in the class $DP$ if and only if there are two languages $L_1 \in NP$ and $L_2 \in \text{coNP}$ such that $L = L_1 \cap L_2$ [10], [9]. Obviously, $NP \subseteq DP$ and all $DP$-complete languages are $NP$-hard.

$V(H)$ is the “exact cost” version of the $NP$-complete optimization problem $N_H$. The “exact cost” versions of many of the known $NP$-complete optimization problems (independent set, knapsack, max-cut, max-sat, maximum clique size, etc.) have been shown to be $DP$-complete [10], [9]. Also, if $V(H)$ is $DP$-complete, then $V(H) \notin NP$ unless $DP = NP$.

**Theorem 3.3** Consider an $NP$-hard problem $H$ such that the “exact-cost” version of its corresponding decision problem $N_H$ is $DP$-complete. The verification problem $V(H)$ for $H$ is $DP$-complete. Consequently, $V(H) \notin NP$ unless $DP = NP$.  


3.3 Approximate Verifiers

Consider an \( NP \)-hard maximization problem, \( H \subseteq I \times \mathbb{R} \), where, as defined in the previous section, \( (x, z) \in H \) if \( z \) is the largest candidate solution of the instance \( x \). An algorithm for \( H \) is an \( (a, c) \) approximation algorithm if on input \( x \) in \( I \), it outputs \( y \) in \( \mathbb{R} \) such that \( y \) is a candidate solution of \( x \), and if \( (x, z) \in H \), then \( z \leq a \cdot y + c \), where \( a, c \in \mathbb{R} \). Similarly, an \( (a, c) \) approximation verifier for the problem \( H \) is defined as the problem \( AH \subseteq (I \times \mathbb{R}) \times \{0, 1\} \), such that \( ((x, y), 1) \in AH \) if \( y \leq z \leq a \cdot y + c \), where \( z \) is such that \( (x, z) \in H \). Note that \( y \) need not be a candidate solution of \( x \).

An \( (a, c) \) approximation algorithm for the maximum clique size problem would take a graph \( G \) as an input and would output \( y \) such that \( G \) has a clique of size \( y \), and if \( k \) is the size of the largest clique of \( G \), then \( k \leq a \cdot y + c \).

We show that if there exists a polynomial-time \( (a, c) \) approximation algorithm for an \( NP \)-hard maximization problem \( H \), then \( L(N_H) \), the language accepted by an algorithm for the \( NP \)-complete decision problem \( N_H \) corresponding to \( H \) (as defined in 3.2), is Cook-reducible to \( L(AH) \), the language accepted by the \( (a, c) \) approximation verifier for the problem \( H \). Let \( AA \) be a polynomial-time \( (a, c) \) approximation algorithm for the problem \( H \).

Observations

- \( L(N_H) \subseteq I \times \mathbb{R} \). \((x, y) \in L(N_H) \) if and only if \( x \) has a candidate solution \( z \) such that \( z \geq y \).
- \( L(AH) \subseteq I \times \mathbb{R} \). \((x, y) \in L(AH) \) if and only if \( y \leq z \leq a \cdot y + c \) where \( z \) is the maximum candidate solution of \( x \).

Cook-reduction of \( L(N_H) \) to \( L(AH) \)

**Input:** \((x, y)\)

1. Call \( AA \) with the input \( x \). Let the output be \( m \).
2. /* Let \( z \) be the maximum candidate solution of \( x \). Then \( m \leq z \leq a \cdot m + c */
3. if \((y \leq m)\) Return(1) /* Because \( y \leq m \leq z */
4. if \((y > m)\)
5. if \((y > a \cdot m + c)\) Return(0) /* Because \( z \leq a \cdot m + c < y */
6. else
7. Use the oracle for language \( L(AH) \) to determine whether \((x, y) \in L(AH)\).
8. if \(((x, y) \in L(AH))\) Return(1) /* \((x, y) \in L(AH) \Rightarrow y \leq z \leq a \cdot y + c \)*/
9. else Return(0)
10. /* \( y \geq m \Rightarrow a \cdot y + c \geq a \cdot m + c \). Because \( z \) is not between \( y \) and \( a \cdot y + c */
11. /* but is less than \( a \cdot m + c \) (line 2.), \( m \leq z \leq y */
12. endif
13. endif
14. endif

The above reduction returns 1 if and only if \( z \), the maximum candidate solution of the instance \( z \), is greater than or equal to \( y \). A similar reduction can be shown for \( NP \)-hard minimization problems.

Thus, we have proved the following result.

**Theorem 3.4** Let \( H \) be an \( NP \)-hard optimization problem whose corresponding decision problem, \( \text{NP}_H \), is \( NP \)-complete. Also, let \( H \) have a polynomial-time \((a, c)\) approximation algorithm. Then there is no polynomial-time algorithm for an \((a, c)\) approximate verifier of \( H \), unless \( P = NP \).

It should be noted here that given a polynomial time \((a, c)\) approximation algorithm \( A \), there always exists a polynomial time verification algorithm which verifies the correctness of \( A \), i.e., given an ordered pair \( (x, y) \), it checks if \( y \) is the output to be produced by fault-free \( A \) when run on the input \( x \). The fault-free algorithm \( A \) itself would suffice as such a verification algorithm. Such verification algorithms are algorithm specific, in contrast to the algorithms solving the \((a, c)\) approximation verifier problem. For example, for bin-packing, an \( NP \)-hard minimization problem, there exists a polynomial time approximation algorithm which implements the "first-fit" heuristic and is guaranteed to return a solution of value no more than \( \frac{17}{10} \cdot \text{OPT} + 1 \), where \( \text{OPT} \) is the optimal solution. Now, given an input \( x \) and an arbitrary candidate solution \( y \), it is easy to verify whether \( y \) could be output by the "first-fit" heuristic algorithm on input \( x \), but it is not easy to verify whether \( y \leq \frac{17}{10} \cdot \text{OPT} + 1 \).

## 4 Verification using Certification Trails

Sullivan and Masson [12, 13] introduced a conceptually novel and powerful technique to achieve fault tolerance in software systems. We consider using this technique in the context of verifiers, as defined in this article.

Let us consider a problem \( \Pi \subseteq I \times S \) and an algorithm \( A \) solving \( \Pi \). The algorithm \( A \) takes \( x \in I \) as an input, and outputs \( y \in S \) such that \((x, y) \in \Pi \). The verification problem \( V(\Pi) \) is a relation on \((I \times S) \times \{0, 1\} \). A verification algorithm verifying the correctness of \( A \) takes the ordered pair \((x, y)\) as an input, where \( y \) is the output produced by \( A \) on the input \( x \), and outputs 1 if and only if \( A \) produced a correct output on the input \( x \), i.e., \((x, y) \in \Pi \). To make the task easier for the verification algorithm, we modify the algorithm \( A \) to \( A_c \) so that \( A_c \) now produces a trail of data, polynomial in the length of the input, which we call certification trail, in addition to its normal output. The verification algorithm would try to use this additional information to verify the result more quickly.
Hence, we define a certification-trail verification algorithm \( V_Ac \) for the modified algorithm \( A_c \) as one which takes \((x, y, c)\) as an input, where \( y \) and \( c \) are the output and certification trail respectively produced by the algorithm \( A_c \) on the input \( x \), and outputs \( 1 \) if \( y \) and \( c \) are the correct output and certification trail generated by \( A_c \); otherwise, \( V_Ac \) outputs \( 0 \). Note that \( V_Ac \) is expected to output \( 0 \) even when \( A_c \) outputs both \( y \) and \( c \) incorrectly on the input \( x \). Hence, this definition differs slightly from the one proposed in [12], which allows the verification algorithm to behave erratically when both \( y \) and \( c \) are incorrect.

We illustrate the concept of a certification-trail verification algorithm with the help of an example. Let us consider the single-pair shortest path problem \( SP \). Let \( A_{SP} \) be an algorithm solving \( SP \), and \( V_A{SP} \) be a verification algorithm for \( SP \). The input to \( A_{SP} \) is a weighted graph \( G \) and a pair of vertices \((s, t)\). The output of the algorithm is the shortest path from \( s \) to \( t \) in \( G \). The verification algorithm \( V_{A_{SP}} \) takes a weighted graph \( G \) and a path between a pair of vertices \((s, t)\) in \( G \) as an input, and \( V_{A_{SP}} \) checks whether the path given is actually the shortest path from \( s \) to \( t \). Now, if \( A_{SP} \) also calculates shortest paths from \( s \) to every other vertex in \( G \) (most of the known algorithms solving \( SP \) work this way), then \( A_{SP} \) could provide the verification algorithm \( V_{A_{SP}} \) with some additional information. In this case, we require \( A_{SP} \) to output for each other vertex, \( u \in V(G) \), its predecessor in the shortest path from \( s \) to \( u \) and the weight of the shortest path from \( s \) to \( u \). This additional information is output as a certification trail. The verification algorithm \( V_{A_{SP}} \), hence, takes a weighted graph \( G \), the shortest path between \( s \) and \( t \), and the certification trail consisting of predecessor and shortest distance for each other vertex, as the input. As shown in section 2.2, this information can be used to detect an error in \( O(m) \) time, where \( m \) is the number of edges in \( G \).

Next, we claim that if the certification-trail verification algorithm \( V_Ac \) runs in polynomial time, then the language accepted by the original verification problem \( L(V(II)) \in NP \). Consequently, for an \( NP \)-hard problem \( H \) whose corresponding decision problem is \( NP \)-complete, the verification problem for \( H \) does not have even a certification-trail verification algorithm which runs in polynomial time, because \( V(H) \) for most of the well known “exact cost” version optimization problems is not known (or believed) to be in \( NP \) (refer to Theorem 3.3).

To show that \( L(V(II)) \in NP \), we use the algorithm \( V_Ac \) to develop a polynomial time non-deterministic algorithm \( ND_1 \) that accepts \( L(V(II)) \). On input \((x, y)\), \( ND_1 \) guesses a string \( c \), polynomial in the length of \((x, y)\), and calls \( V_Ac \) with the input \((x, y, c)\). If \( V_Ac \) outputs \( 0 \), then \( ND_1 \) loops forever; otherwise, \( ND_1 \) outputs \( 1 \) when \( V_Ac \) outputs \( 1 \). To see that \( ND_1 \) accepts \( L(V(II)) \), we observe that \( ND_1 \) accepts an input \((x, y)\) if and only if there exists a string \( c \) such that \( V_Ac \) outputs \( 1 \) on \((x, y, c)\) if and only if \((x, y) \in V(II) \). Hence, \( L(V(II)) \in NP \).

Thus, we have proved the following theorem.

**Theorem 4.1** If there exists a polynomial time certification-trail verification algorithm \( V_Ac \) for an algorithm solving the problem \( II \), then language accepted by the verification problem \( L(V(II)) \in NP \).
5 Conclusions

We have shown that verification problems for \(NP\)-complete problems are \(NP\)-complete. Verification problems for many well known \(NP\)-hard maximization problems, whose corresponding decision problems are \(NP\)-complete, have been proved to be \(DP\)-complete and \(NP\)-hard. Furthermore, \((a, c)\) approximate verifiers of \(NP\)-hard languages, which have polynomial time \((a, c)\) approximation algorithms, cannot have polynomial time algorithms, unless \(P = NP\).

We have also discussed the existence of polynomial time verification algorithms when they are presented with some additional information (a certification trail). We proved that if a certification trail verification algorithm \(VA_c\) exists for an algorithm \(A\) solving the problem \(\Pi\), then \(VA_c \in NP\).

The verification algorithms discussed in the section 2.1 belong to the class of primal-dual algorithms [8], which is a general framework for solving linear programming. A primal-dual algorithm works in stages, where each stage checks for the optimality of the current solution. Hence, each stage of a primal-dual algorithm is essentially a result verification stage. This leaves us with the following open question. Is there a broader characterization of algorithms consisting of stages whose each stage can be used as a result verification stage?

References


