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ABSTRACT

Based on the picture of field-particle scattering, an attempt is made to derive kinetic equations for plasmas. The system is assumed to have overall neutrality and to be under no external forces. A linear approximation which corresponds to linearizing the Vlasov equation is used to truncate the hierarchy of equations. The whole problem is reduced to exactly solving a familiar singular integral equation. A coupled set of temporal equations for the field energy and the homogeneous Fourier component of the one-particle distribution function is obtained under arbitrary initial correlations. The particle part of the equations consists of two terms: the one is the well-known Balescu-Lenard collision integral and the other is what one obtains in the quasi-linear theory of plasmas based on the Vlasov equation. The resonant scattering part of the equations agrees with the Pines-Schrieffer equations. Both the stable and unstable cases are considered, and the motions of ions is also included. The theory is valid regardless whether the system is homogeneous or non-homogeneous.
1. INTRODUCTION

The majority of work done on the subject material is, more or less, an extension of the theories developed for the case of inert gases. They are based on the picture of particle-particle scattering with the Hamiltonian

\[ H_0 = \sum_{i=1}^{n} \frac{p_i^2}{2m} + \sum_{i=1}^{n} V(r_{ij}) \]  

(1)

where \( V(r_{ij}) \) is particle-particle interaction potential at the inter-particle distance \( r_{ij} \).

Pioneering works based on the picture of field-particle scattering are those of Klimontovich\(^2\) and Pines and Schrieffer.\(^1\) For the case of electron gas, these authors start with the Hamiltonian:

\[ H = \sum_{i=1}^{n} \left[ \frac{p_i^2}{2m} + \frac{\mathbf{A}(x_i)}{c} \right]^2 + \int d^3x \frac{|E(x)|^2}{8\pi} \]  

(2)

with the subsidiary condition

\[ \text{div} \, E(x) = -4\pi e \sum_{i=1}^{n} \delta(x - x_i) \]  

(3)

where \( E(x) \) is the electric field intensity and \( A(x) \) its vector potential so that

\[ E(x) = -\frac{e}{c} \frac{\partial}{\partial t} A(x). \]

The purpose of the present work is to mathematically systematize this approach. Theories\(^3\) based on the particle-particle scattering picture
have emerged using Dupree's method to reach essential agreement with the works of Pines-Schieffer.¹

We first treat, for simplicity, an electron gas with uniform positive background. The extension to include also ion motion is then straightforward and is given in the latter part of this work. In Section 2 the Hamiltonian (2) is expressed in quantum mechanical operators and a hierarchy of operator equations is derived. This hierarchy is truncated, and in Section 3 we obtain a closed set of equations for ensemble-averaged quantities. The classical limit is taken. Then the whole problem is reduced to solving a familiar singular integral equation, which is done in the Appendix. We summarize the solution in Section 4 and obtain a coupled set of equations for the field energy and the one-particle distribution function. In Section 5 we extend the theory to include motion of ions.
2. HIERARCHY OF OPERATOR EQUATIONS

If we expand \( A(x) \) and \( E(x) \) in Fourier series:

\[
A(x) = \sum_{\tilde{k}} A_{\tilde{k}} e^{ik \cdot x}
\]

\[
E(x) = \sum_{\tilde{k}} E_{\tilde{k}} e^{ik \cdot x}
\]  

(4)

and define canonically conjugate operators \( \hat{a}_k \) and \( \hat{p}_k \) such that

\[
\tilde{A}_k = \sqrt{4\pi c^2} \hat{a}_k \hat{Q}_k
\]

\[
\tilde{E}_k = \sqrt{4\pi} \hat{a}_k \hat{P}_{-k}
\]  

(5)

where

\[
\hat{a}_k = \tilde{a}_k = \text{unit vector},
\]

then the Hamiltonian (2) reads:

\[
H = \sum_{i=1}^{n} \frac{p_{i}^2}{2m} - \frac{1}{2} \sum_k (\hat{p}_k \cdot \hat{p}_{-k} + \omega_p^2 \hat{Q}_k \cdot \hat{Q}_{-k}) + \sum_k \sum_{i=1}^{n} \frac{\sqrt{4\pi} e \hat{a}_k \cdot \hat{P}_i}{m} \left( \frac{\hbar K}{2} \right) \hat{Q}_k e^{ik \cdot x_i}
\]

\[
- \sum_k \sum_{k' \neq k} \frac{2\pi e^2}{m} (\hat{a}_k \cdot \hat{a}_{k'}) \hat{Q}_k \cdot \hat{Q}_{-k'} e^{i(k - k') \cdot x_i}
\]

where \( \omega_p^2 = \frac{4\pi ne^2}{m} \) = plasma frequency squared. The set of Hamiltonian equations together with the subsidiary condition (3) is Newtonian and Maxwellian equations:
\[ \frac{dx_i}{dt} = u_i = \frac{p_i + eA(x_i)}{m} \]
\[ \frac{du_i}{dt} = -eE(x_i) \]

\[ E(x) = -e \frac{\partial}{\partial t} A(x) \]
\[ \frac{\partial}{\partial t} \sim E(x) + 4\pi \sim j(x) = 0 \]
\[ \text{div} \sim E(x) = -4\pi e \sum_{i=1}^{n} \delta(x - x_i) \]

where
\[ j(x) = -e \sum_{i=1}^{n} u_i \delta(x - x_i) \]

Replacing
\[ p_i \rightarrow \hbar p_i \]
and using second quantized operators for the electrons:

\[ \sum_{i=1}^{n} \frac{ik \cdot x_i}{\hbar} \]

we have
\[ H = \sum_{\sim} \frac{\hbar^2}{2m} c_{p}^+ c_{p} + \frac{1}{2} \sum_{k} \omega_{p}^2 Q_{k} Q_{-k} \]
\[ + \sum_{k \in \mathcal{E}} \sum_{\sim} m_{p, k} Q_{k} c_{p}^+ c_{p-k} \]
\[ - \sum_{k \neq k'} \sum_{\sim} \delta_{k, k'} Q_{k} Q_{-k'} c_{p}^+ c_{p-k+k'} \]

(6)
where

\[
\begin{align*}
\epsilon_p &= \frac{\hbar^2 p^2}{2m} \\
m(p, k) &= \left(\frac{4\pi e^2}{k^2}\right)^{1/2} v^p_{p-k}, \quad (\hbar v^p_{p-k} = \epsilon_p - \epsilon_{p-k}) \\
\mathcal{L}(k, k') &= \frac{2\pi e^2}{m} (\hat{a}_k \cdot \hat{a}_{k'}) = \mathcal{L}(k', k).
\end{align*}
\]

Communtation relations are:

\[
\begin{align*}
[Q_k, P_{k'}] &= i\hbar \delta_{k, k'}, \text{ etc.} \\
\{c_\mathcal{P}, c_\mathcal{P}^+\} &= \delta_\mathcal{P}, \mathcal{P}'', \text{ etc.}
\end{align*}
\]

with the field operators commuting with the particle operators.

The subsidiary condition, i.e., the divergence theorem (3), is interpreted as in quantum electrodynamics: let \(S_k\) be an operator defined as

\[
S_k = \sum_{\mathcal{P}} c_\mathcal{P}^+ c_\mathcal{P} + k
\]

and \(|\Psi_n\rangle\) an eigenfunction satisfying

\[
H|\Psi_n\rangle = E_n |\Psi_n\rangle,
\]

then the subsidiary condition reads

\[
S_k |\Psi_n\rangle = 0.
\]
One can easily show that

\[ i\hbar \frac{\partial S_k}{\partial t} = [S_k, H] = 0 , \]

hence

\[ S_k = \text{constant in time.} \]

So far we have simply followed Bohm and Pines' except for the field operators \( Q_k \) and \( P_k \) which they replaced also by second quantized operators \( a_k \) and \( a_k^+ \).

We note the following: let \( \Theta \) be any operator, then

\[ \langle \psi_n | \Theta S_k | \psi_n \rangle = 0 , \]

hence, if we use this incomplete set \( | \psi_n \rangle \) for ensemble average,

\[ \langle \Theta S_k \rangle = 0 , \]

i.e.

\[ \langle \Theta P_{-k} \rangle = i \left( \frac{4\pi e^2}{\hbar k^2} \right)^{1/2} \sum_{p \in \mathcal{L}} \langle \Theta c_p^+ c_{p+k} \rangle . \tag{9} \]

We will use this relation later.

Since for any operator \( \Theta \)

\[ i\hbar \frac{\partial}{\partial t} \Theta = [\Theta, H] , \]

we have

\[ \frac{\partial}{\partial t} Q_k = -P_{-k} , \tag{10} \]
\[ i \hbar \frac{\partial}{\partial t} c_p^+ c_{p+k} = \hbar \nu_{p+k} c_p^+ c_{p+k} \]
\[ + \sum_{k', k''} \mathcal{Q}_k \left[ m(p+k', k'') c_p^+ c_{p+k-k'} - m(p+k', k''') c_p^+ c_{p+k'} \right] \]
\[ - \sum_{k', k''} \mathcal{Q}_k Q_{k'} Q_{k''} \left( c_p^+ c_{p+k-k'} + k'' - c_{p+k''-k'} c_{p+k} \right) , \]

(11)

where, when \( k = 0 \),

\[ i \hbar \frac{\partial}{\partial t} c_p^+ c_p = \sum_{k} \mathcal{Q}_k \left[ m(p, k) c_p^+ c_{p-k} - m(p+k, k) c_p^+ c_p \right] \]
\[ + \sum_{k', k''} \mathcal{Q}_k Q_{k'} Q_{k''} \left( c_p^+ c_{p+k-k'} + k'' - c_{p+k-k'} c_{p+k} \right) , \]

(12)

\[ i \hbar \frac{\partial}{\partial t} \mathcal{Q}_k c_p^+ c_{p-q} = i \hbar \mathcal{Q}_k \left( \frac{\partial}{\partial t} \right) c_p^+ c_{p-q} + i \hbar \mathcal{Q}_k \frac{\partial}{\partial t} c_p^+ c_{p-q} \]
\[ + \sum_{k'} \mathcal{Q}_k Q_k \left[ m(p-q, k') c_p^+ c_{p-q-k'} - m(p+k', k') c_p^+ c_{p-k'} \right] \]
\[ - \sum_{k', k''} \mathcal{Q}_k Q_{k'} Q_{k''} \left( c_p^+ c_{p-q-k'} + k'' - c_{p+k-k'} c_{p-q} \right) , \]

(13)

and so on.

To truncate the hierarchy we apply linear approximation to Eq. (11), i.e.,

\[ i \hbar \frac{\partial}{\partial t} c_p^+ c_{p+k} = \hbar \nu_{p+k} c_p^+ c_{p+k} \]
\[ + m(p+k, k) Q_k \left[ \langle c_p^+ c_p \rangle - \langle c_{p+k}^+ c_{p+k} \rangle \right] . \]

(14)
This is tantamount to starting with the approximate Hamiltonian

\[
H = \sum_{p,\tilde{p}} \epsilon_{\tilde{p}} c_{p,\tilde{p}}^{+} c_{p,\tilde{p}} - \frac{1}{2} \sum_{k} (p_{k} p_{-k} + \omega_{p}^{2} q_{k} q_{-k})
\]

\[
+ \sum_{k} \sum_{\tilde{p}} m(p, k) q_{k} c_{p,\tilde{p}}^{+} c_{p,\tilde{p}} .
\]

(6')

which is without the last term of (6), and then neglecting the mode-mode coupling. We note that Eq. (14) is the linearized Vlasov equation. The neglect of the last term of the Hamiltonian (6) is what Bohm and Pines called random-phase approximation.
3. FORMULATION OF THE PROBLEM

We take ensemble average of the operator equations. From Equation (12), we have, neglecting the last term,

\[
i \hbar \frac{\partial}{\partial t} \langle c_p^+ c_p \rangle = -i \sum_k \left( \frac{\partial}{\partial p} \right) \eta(p, k) \text{Im} \langle Q_k c_p^+ c_{p-k} \rangle
\]

(15)

where, since \( \langle c_p^+ c_p \rangle \) is a real quantity, only the imaginary part of \( \langle Q_k c_p^+ c_{p-k} \rangle \) contributes, and the differential operator \( \left( \frac{\partial}{\partial p} \right) \) is to be understood as

\[
\left( \frac{\partial}{\partial p} \right) \langle Q_k c_p^+ c_{p-k} \rangle = \langle Q_k c_p^+ c_{p+k} \rangle - \langle Q_k c_p^+ c_{p-k} \rangle.
\]

This notation is justified in the classical limits \( \hbar \to 0 \).

For the quantity \( \langle Q_k c_p^+ c_{p-k} \rangle \), we get, to be consistent with the linear approximation we use, from (10) and (14),

\[
i \hbar \frac{\partial}{\partial t} \langle Q_k c_p^+ c_{p-k} \rangle = -i \hbar \langle P_{-k} c_p^+ c_{p-k} \rangle - \hbar v_p \langle Q_k c_p^+ c_{p-k} \rangle
\]

\[
- m(p \cdot k) \langle Q_{-k} \rangle \left( \frac{\partial}{\partial p} \right) \langle c_p^+ c_p \rangle. \quad (16)
\]

Now, using Equation (9),

\[
\langle P_{-k} c_p^+ c_{p-k} \rangle = i \left( \frac{\Delta T e^-}{k^2} \right)^{1/2} \sum_{p'} \langle c_{p'}^+, c_{p'+k}^+ c_p c_{p-k} \rangle
\]

\[
= i \left( \frac{\Delta T e^-}{k^2} \right)^{1/2} \langle c_{p-k}^+ c_{p-k} \rangle \left[ 1 - \langle c_p^+ c_p \rangle \right]
\]

\[
+ i \left( \frac{\Delta T e^-}{k^2} \right)^{1/2} \sum'_{p' \neq L} \langle c_{p'}^+, c_{p'+k}^+ c_{p-k} \rangle,
\]

(17)
where \( \Sigma' \) means summation over \( p' \) with the term \( p' = p - k \) excluded. We get the equation for \( \langle c^+_p c^+_{p+k} c^+_p c^-_{p-k} \rangle \) from (14):

\[
\begin{align*}
&i \hbar \frac{\partial}{\partial t} \langle c^+_p c^+_{p+k} c^+_p c^-_{p-k} \rangle = \hbar (\nu^{p'k} - \nu^p) \langle c^+_p c^+_{p+k} c^+_p c^-_{p-k} \rangle \\
&- m(p' + k, k) \left( k \cdot \frac{\partial}{\partial p} \right) \langle c^+_p, c^-_p \rangle \langle Q_k c^+_p c^-_{p-k} \rangle \\
&- m(p, k) \left( k \cdot \frac{\partial}{\partial p} \right) \langle c^+_p c^-_p \rangle \langle Q_{-k} c^+_p c^-_{p+k} \rangle.
\end{align*}
\]

The equation for \( \langle Q_k Q_{-k} \rangle \) is, using (9),

\[
\frac{\partial}{\partial t} \langle Q_k Q_{-k} \rangle = - \langle P_{-k} Q_{-k} \rangle - \langle Q_k P_k \rangle
\]

\[
= \frac{4\pi e^2}{k^2} \sum_{p} \frac{1}{2} \text{Im} \langle Q_k c^+_p c^-_{p-k} \rangle, \tag{19}
\]

since, from (4), (5) and the reality condition of \( \sim A(x) \),

\[
Q_{-k} = - Q^+_k
\]

and so

\[
\sum_{p} \langle c^+_p c^+_{p+k} Q_{-k} \rangle = - \sum_{p} \langle Q_k c^+_p c^-_{p-k} \rangle^* \tag{20}
\]

where \( ^* \) means complex conjugate.

The equations (15), (16), (17), (18), and (19) constitute a closed set of equations. This closure was possible through consistent use of the linear approximation expressed in (14). We note that this approximation is in effect to keep the binary correlations like
\[ \langle Q_k c^+_p c^-_{p-k} \rangle \text{ and } \langle c^+_p c^+_{p-k} c^+_p c^-_{p-k} \rangle \]

and neglect the ternary and higher correlations like

\[ \langle Q_k Q_k' c^+_p c^-_{p-k+k'} \rangle \text{ and } \langle Q_k Q_k'' c^+_p c^-_{p-k - k''} \rangle . \]

Truncating the hierarchy in this way one can indeed obtain the closed set of equations the same as above.

Since, from (5),

\[ \frac{|E_k|^2}{4\pi} = - \langle P_k P^-_{-k} \rangle , \]

and this is physically an important quantity, we need an expression for \( \langle P_k P^-_{-k} \rangle \). We have from (16), using (9),

\[
\left( \frac{4\pi e^2}{k^2} \right)^{1/2} \sum_{P} \frac{\partial}{\partial t} \text{ Im} \langle Q_k c^+_p c^-_{p-k} \rangle \\
= \langle P_k P^-_{-k} \rangle - \omega_p^2 \langle Q_k Q^-_{-k} \rangle \\
+ \left( \frac{4\pi e^2}{k^2} \right)^{1/2} \sum_{P} \nu_{p-k} \text{ Re} \langle Q_k c^+_p c^-_{p-k} \rangle \\
= \frac{1}{2} \frac{\partial^2}{\partial t^2} \langle Q_k Q^-_{-k} \rangle ,
\]

the last equality coming from (19). This means

\[
\langle P_k P^-_{-k} \rangle = \omega_p^2 \langle Q_k Q^-_{-k} \rangle - \left( \frac{4\pi e^2}{k^2} \right)^{1/2} \sum_{P} \nu_{p-k} \text{ Re} \langle Q_k c^+_p c^-_{p-k} \rangle \tag{21}
\]
provided

\[ \left| \frac{\partial^2}{\partial t^2} \langle Q_k, Q_{-k} \rangle \right| \ll \omega_p^2 \langle Q_k, Q_{-k} \rangle \]

or

\[ \frac{\gamma_k^2}{\omega_p^2} \ll 1 \]

where \( \gamma_k \) is decay (or growth) rate of \( \langle Q_k, Q_{-k} \rangle \).

We perform Laplace transformation of the equations (16), (17), and (18), assuming \( \langle c^+_p c^-_p \rangle \) as constant in time. Let

\[ \langle \rangle_w = \int_0^\infty dt \langle \rangle e^{i \omega t} \]

\[ \equiv \text{Laplace transform of } \langle \rangle \]

and \( \langle \rangle_0 \equiv \text{value of } \langle \rangle \text{ at } t=0 \).

From (18),

\[ \langle c^+_p, c^+_p, c^+_p, c^+_p, c^-_p, c^-_p, c^-_p, c^-_p \rangle_w = - \frac{i \langle c^+_p, c^+_p, c^+_p, c^+_p, c^-_p, c^-_p, c^-_p, c^-_p \rangle_0}{\nu_{p}^{p'+k} - \nu_{p-k}^{p} - k\omega} \]

\[ + \langle Q_k c^+_p c^-_p c^-_p c^-_p \rangle_w \left( \frac{4\pi e^2}{k^2} \right)^{1/2} \frac{\nu_{p}^{p'+k} - k\omega}{\hbar} \]

\[ + \left( \frac{4\pi e^2}{k^2} \right)^{1/2} \frac{\nu_{p-k}^{p} - k\omega}{\hbar} \left( k \cdot \frac{\partial}{\partial \nu_{p}^{p'}} \right) \langle c^+_p c^-_p \rangle_w \]

\[ + \left( \frac{4\pi e^2}{k^2} \right)^{1/2} \frac{\nu_{p-k}^{p} - k\omega}{\hbar} \left( k \cdot \frac{\partial}{\partial \nu_{p-k}^{p}} \right) \langle Q_{-k} c^+_p c^-_p c^-_p c^-_p \rangle_w \]
hence

$$\sum_{\mathcal{L}'} \langle c_{p'}^+ c_{p'+k}^+ c_{p}^+ c_{p-k}^- \rangle = -i \sum_{\mathcal{L}'} \frac{\langle c_{p'}^+ c_{p'+k}^+ c_{p}^+ c_{p-k}^- \rangle}{\nu_{p'}^+ + k - \nu_{p}^+ - k \omega}$$

$$+ \langle \nu_{p-k}^+ + k \omega \rangle \langle Q_k c_{p}^+ c_{p-k}^- \rangle \frac{1}{\hbar} (\frac{4\pi e^2}{k^2})^{1/2} \frac{\partial}{\partial p} \frac{\langle c_{p}^+ c_{p}^- \rangle}{\nu_{p-k}^+ + k - \nu_{p}^+ - k \omega}$$

$$+ \frac{1}{\hbar} (\frac{4\pi e^2}{k^2})^{1/2} \nu_{p-k}^+ \langle Q_k c_{p}^+ c_{p-k}^- \rangle \sum_{\mathcal{L}'} \langle c_{p}^+ c_{p}^- \rangle \frac{\langle Q_{-k} c_{p'-k}^+ c_{p'}^- \rangle}{\nu_{p-k}^+ + k - \nu_{p}^+ - k \omega}.$$

From this and (17), we have

$$\langle p_{-k} c_{p}^+ c_{p-k}^- \rangle = -\left(\frac{4\pi e^2}{4}\right)^{1/2} \frac{1}{\hbar} \langle c_{p}^+ c_{p-k}^- \rangle \left[1 - \langle c_{p}^+ c_{p}^- \rangle\right]$$

$$+ \left(\frac{4\pi e^2}{k^2}\right)^{1/2} \sum_{\mathcal{L}'} \langle c_{p}^+ c_{p'+k}^+ c_{p}^+ c_{p-k}^- \rangle \frac{\langle Q_{-k} c_{p'-k}^+ c_{p'}^- \rangle}{\nu_{p-k}^+ + k - \nu_{p}^+ - k \omega}$$

$$+ i \nu_{p-k}^+ \langle Q_k c_{p}^+ c_{p-k}^- \rangle \frac{4\pi e^2}{\hbar k^2} \sum_{\mathcal{L}'} \nu_{p-k}^+ + k - \nu_{p}^+ + k \omega$$

$$+ i \nu_{p-k}^+ \frac{4\pi e^2}{\hbar k^2} (\nu \cdot \frac{\partial}{\partial p}) \langle c_{p}^+ c_{p}^- \rangle \sum_{\mathcal{L}'} \langle Q_{-k} c_{p'-k}^+ c_{p'}^- \rangle \frac{\langle Q_{-k} c_{p'-k}^+ c_{p'}^- \rangle}{\nu_{p-k}^+ + k - \nu_{p}^+ + k \omega}.$$
Thus from (16),

\[
\langle \nu_{p-k}^+ + k\omega \rangle \varepsilon^+ (k, \nu_{p-k}^+ + k\omega) \langle Q_k c_p^+ c_{p-k}^- \rangle_w
\]

\[
= i \left[ \langle Q_k c_p^+ c_{p-k}^- \rangle_0 \right] \sum_{\nu_{p-k}'} \frac{\langle c_p^+ c_{p-k}^+ c_{p-k}^- \rangle_0}{\nu_{p-k}' - \nu_{p-k} - k\omega}
\]

\[
+ i \left( \frac{4\pi e^2}{k^2} \right)^{1/2} \frac{1}{\omega} \langle c_{p-k}^+ c_{p-k}^- \rangle \left[ 1 - \langle c_p^+ c_p^- \rangle \right]
\]

\[
- \frac{1}{\hbar} \left( \frac{4\pi e^2}{k^2} \right)^{1/2} \langle Q_k Q_{-k} \rangle \nu_{p-k}^+ \left( k \cdot \frac{\partial}{\partial \rho} \right) \langle c_{p-k}^+ c_{p-k}^- \rangle
\]

\[
+ \frac{4\pi e^2}{\hbar k^2} \nu_{p-k}^+ \left( k \cdot \frac{\partial}{\partial \rho} \right) \langle c_{p-k}^+ c_{p-k}^- \rangle \sum_\nu \frac{\langle Q_{-k} c_{p-k}^+ c_{p-k}^- \rangle_w}{\nu_{p-k}' - \nu_{p-k} - k\omega}
\]

(22)

where

\[
\varepsilon^+ (k, \omega) = 1 - \frac{4\pi e^2}{\hbar k^2} \sum_\nu \frac{\langle k \cdot \frac{\partial}{\partial \rho} \rangle \langle c_p^+ c_p^- \rangle}{\nu_{b+k}^+ - \omega}
\]

(23)

is the dielectric constant defined in the upper-half plane of \( \omega \) and analytically continued to the lower-half plane. Equation (22) is an integral equation we are to solve in the classical limits \( \hbar \to 0 \).

Let \( \nu = \frac{\hbar p}{m} \), then

\[
\nu_{p-k}^+ \xrightarrow{\nu_{p-k}^+ \to k \cdot \nu} \frac{1}{\hbar} k \cdot \frac{\partial}{\partial \rho} \xrightarrow{1 / \hbar k \cdot \frac{\partial}{\partial \rho} \to \frac{1}{m} \frac{\partial}{\partial \nu}}
\]
We denote

\[
\begin{align*}
\langle c^+_p c^-_p \rangle &= n f_o(\nu) \\
- \langle Q_k Q_{-k} \rangle &= V_k(t) \\
\left(\frac{4\pi e^2}{k^2}\right)^{1/2} \langle Q_k c^+_p c^-_{p-k} \rangle &= n g(k, \nu, t).
\end{align*}
\]

Since \( Q_{-k} = - Q_k \),

\[
\left(\frac{4\pi e^2}{k^2}\right)^{1/2} \langle Q_{-k} c^+_p c^-_{p-k} \rangle = - n g^*(k, \nu, t).
\]

For a complex function \( \varphi(t) \) of time \( t \) we define

\[
\varphi^*(\omega) = \int_0^\infty dt \varphi^*(t) e^{i k \omega t}
\]

= Laplace transform of complex conjugate of \( \varphi(t) \).

Then

\[
\varphi^*(\omega) = \varphi^*(-\omega^*).
\]

Equation (22) in the classical limits reads:

\[
k \left( \frac{k \cdot \nu}{k} + \omega \right) \xi^+ \left( k, \frac{k \cdot \nu}{k} + \omega \right) g(k, \nu, \omega)
\]

\[
= i a_o(k, \nu, \omega)
\]

\[
+ i \left(\frac{4\pi e^2}{k^2}\right) \frac{1}{\omega} f_o(\nu) + \frac{4\pi e^2}{k^2} V_k(\omega) \left( \frac{k \cdot \frac{\partial}{\partial \nu}}{k \cdot \frac{\partial}{\partial \nu}} \right) f_o(\nu) \frac{k \cdot \nu}{m}
\]

\[
- \left(\frac{\omega^2}{k^2} \frac{k \cdot \nu}{k \cdot \nu} \right) f_o(\nu) \int d^3 v' \frac{g^*(k, \nu', \omega)}{k \cdot \nu' - k \cdot \nu - k \omega}.
\]

(26)
where
\[ n_{a_o}(k, \omega, \nu) = \left(\frac{4\pi e^2}{k^2}\right)^{1/2} \left\langle Q_k c^+ p c^+ p - \frac{4\pi e^2}{k^2} \Sigma \mathcal{E} \right\rangle \left(\frac{c^+ p c^+ p + k c^+ c c^+ c}{p^+ + k - \nu p - k - \omega}\right) \]

and
\[ \epsilon^+(k, \omega) = 1 - \frac{\omega_p}{k^2} \int d^3v \frac{(k \cdot \frac{\partial}{\partial \nu}) f_o(\nu)}{k \cdot \nu - k \omega} . \]

Let \( k = (k, o, o) \) and \( \nu = (u, v_2, v_3) \), so that \( k \cdot \nu = ku \). We define
\[ H(k, u, t) = \int_{-\infty}^{+\infty} dv_2 \int_{-\infty}^{+\infty} dv_3 g(k, \nu, t) . \]

From (26), integrating over \( v_2 \) and \( v_3 \), we have
\[ k(u + \omega) \epsilon^+(k, u + \omega) H(k, u, \omega) \]
\[ = i \bar{a}_o(k, u, \omega) \]
\[ + i \frac{4\pi e^2}{k^2} \frac{1}{\omega} \bar{f}_o(u) + \frac{4\pi e^2}{m} V_k(\omega) u \bar{f}_o(u) \]
\[ - \frac{\omega_p}{k} u \bar{f}_o(u) \int_{-\infty}^{+\infty} du' \frac{H^{(*)}(k, u', u - \omega)}{u' - u - \omega} , \]

where a barred function \( \bar{\varphi}(u) \) is defined
\[ \bar{\varphi}(u) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv_2 dv_3 \varphi(\nu) . \]
Indeed,

\[ H(k, u, t) = g(k, v, t) \quad (30) \]

and

\[ \xi^+(k, \omega) = 1 - \frac{2}{k^2} \int_{-\infty}^{+\infty} du \frac{\tilde{f}'(u)}{u - \omega} . \quad (31) \]

Since

\[ \int d^3 \tilde{v} \frac{g^{(*)}(k, v', \omega)}{\sim v' - k \cdot v - k \omega} = \frac{1}{k} \int_{-\infty}^{+\infty} du' \frac{H^{(*)}(k, u', \omega)}{u' - u - \omega}, \]

we can rewrite (26):

\[ g(k, v, \omega) = \frac{i \alpha_o(k, v, \omega) - i \left( k \cdot \frac{\partial}{\partial v} \right) f_o(\omega)}{k f_o'(u) \xi^+(k, u + \omega)} k(u + \omega) \xi^+(k, u + \omega) \]

\[ + \frac{4\pi e^2}{k} \frac{f_o(\omega) - \frac{\tilde{f}_o(u)}{k f_o'(u)} \left( k \cdot \frac{\partial}{\partial \omega} \right) f_o(\omega)}{\omega (u + \omega) \xi^+(k, u + \omega)} \]

\[ + \frac{\left( k \cdot \frac{\partial}{\partial \omega} \right) f_o(\omega)}{k f_o'(u)} H(k, u, \omega) . \quad (32) \]

Thus the whole problem is reduced to solving the integral Equation (29).

Let \( k(u + \omega) H(k, u, \omega) = \Xi(k, u, \omega) \), \quad (33)

then (29) reads:
\[ \varepsilon^+(k, u + \omega) \mathcal{K}(k, u, \omega) \]

\[ = b_k(u, \omega) - \frac{\omega^2}{k^2} \overline{\tilde{f}'_0(u)} \int_{-\infty}^{+\infty} \frac{\mathcal{K}'(k, u', \omega)}{u' - u - \omega} \]  \hspace{1cm} (34)

with \[ b_k(u, \omega) \equiv i \overline{a}_0(k, u, \omega) + i \frac{4\pi e^2}{k} \frac{1}{\omega} \overline{\chi}(u) \]

\[ + \frac{4\pi e^2}{m} \) V_k(\omega) n \overline{\tilde{f}'_0(u)} + \frac{\omega^2}{k^2} \phi_0(u) \int_{-\infty}^{+\infty} H(k, u', \omega). \]  \hspace{1cm} (35)

Technique of solving (34) is well-known. 5,6,7 We try an elegant method as outlined in Ref. 5 in the Appendix.
4. KINETIC EQUATIONS

The solution of (34) is given by (A21) of the Appendix:

\[ \delta C(k,u,\omega) = \frac{b_k(u,\omega)}{\varepsilon^+(k, u+\omega)} + \varepsilon_2(k,u) \int_{L} du' \frac{q_k(u',\omega) - p_k(u',\omega)}{(u'-u-i\delta) \varepsilon^-(k,u')} \varepsilon^+(k,u'+\omega) \]

(36)

where the line in integration L is as shown in Fig. A of the Appendix and \( q_k(z,\omega) \) and \( p_k(z,\omega) \) are defined by (A16) or equivalently by (A22):

\[
\begin{align*}
p_k^-(u,\omega) &= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} du' \frac{b_k(u',\omega)}{u' - (u-i\delta)} \quad \left\{ \begin{array}{l}
n_k(u,\omega) \quad \left[ \begin{array}{c}
b_k^+(u',\omega) \quad \left[ \begin{array}{c}
q_k(u,\omega) \quad \left[ \begin{array}{c}
p_k^+(u,\omega) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} du' \frac{b_k(u',-\omega^*)}{u' - (u+\omega+i\delta)} \quad \left[ \begin{array}{c}
-2\gamma_k t \end{array} \right] 
\end{array} \right] 
\end{array} \right] 
\end{array} \right]
\end{align*}
\]

(37)

The solution is explicitly calculated also in the Appendix, and we have:

\[
H(k,u,t) = -\frac{k}{n} \varepsilon_2(k,u) V_k(t) \\
+ \frac{4\pi e^2}{k^3} \left[ \begin{array}{c}
\frac{f_0(u)}{u} \varepsilon^+(k,u) \\int_{L} du' \frac{\varepsilon_2(k,u)}{(u'-u-i\delta)} \left| \varepsilon(k,u') \right|^2 
\end{array} \right] \\
- \frac{\varepsilon_2(k,u)}{V_k^+(u-V_k^-)} \left[ \begin{array}{c}
1 - e^{-i k (u-v_k^-) t} \end{array} \right] C_k \ e^{-2\gamma_k t} \\
- \frac{\varepsilon_2(k,u)}{V_k^+(u-V_k^-)} \left[ \begin{array}{c}
1 - e^{-i k (u-v_k^-) t} \end{array} \right] C_k \ e^{-2\gamma_k t} 
\]

(38)
with

$$C_k = \frac{1}{2nk} \left[ \varepsilon_0 - \frac{\varepsilon_0(v_k)}{2\gamma_k} \frac{16\pi^2 n^2}{k^2} \sigma_k^2(v_k) \right],$$

(39)

where $\varepsilon_0$ will soon be identified as the initial value of the field energy, $\tilde{\nu}_k^\pm$ and $\tilde{\mu}_k^\pm$ are the complex roots of the dielectric constants:

$$\varepsilon^\pm(k, \tilde{\nu}_k) = 0 = \varepsilon^\pm(k, \tilde{\mu}_k), \text{ i.e.,}$$

$$\tilde{\nu}_k^\pm = \frac{1}{k} (\nu_k \pm i\gamma_k)$$

$$\tilde{\mu}_k^\pm = \frac{1}{k} (-\nu_k \pm i\gamma_k),$$

(40)

$\nu_k$ and $\mu_k$ are the real parts of $\tilde{\nu}_k^\pm$ and $\tilde{\mu}_k^\pm$ respectively, and

$$\sigma_k(u) = \frac{\partial}{\partial u} \varepsilon_1(k,u).$$

(41)

We note:

$$\gamma_k = \frac{\pi k \varepsilon_2(k,v_k)}{\sigma_k(v_k)}$$

(42)

and, as is pointed out in (A31),

$$\frac{ik(u-\tilde{\nu}_k)}{e} \begin{cases} 2\pi i \delta(u-v_k) \quad \text{(stable)} \\ 0 \quad \text{(unstable)} \end{cases} \begin{cases} \frac{2\pi i \delta(u-v_k)}{e} \quad \text{(stable)} \\ 0 \quad \text{(unstable)} \end{cases}$$

(43)

Referring to Fig. A of the line of integration $L$, we have
\[
\int_{u'}^{+\infty} \frac{\bar{f}_o(u')}{(u'-u-i\delta)|\varepsilon(k,u')|^2} = \int_{-\infty}^{u'} \frac{\bar{f}_o(u')}{(u'-u-i\delta)|\varepsilon(k,u')|^2}
\]

\[
+ \left\{ 2\pi i \left[ \frac{\bar{f}_o(v_k)}{\sigma_k(v_k)(u-\nu_k^+)} \varepsilon^-(k,\nu_k^+) - \frac{\bar{f}_o(v_k)}{\sigma_k(v_k)(u-\nu_k^-)} \varepsilon^+(k,\nu_k^-) \right] \right\} \text{for unstable case only}
\]

where the second term comes in when the plasma is unstable, e.g., when \(\gamma_k < 0\) while \(\gamma_{-k} > 0\). Since

\[
\varepsilon^-(k,\nu_k^+) = -\frac{2i\nu_k}{k} \sigma_k(v_k) \quad \text{and} \quad \varepsilon^+(k,\nu_k^-) = \frac{2i\nu_k}{k} \sigma_k(v_k),
\]

\[
\int_{u'} \frac{\bar{f}_o(u')}{(u'-u-i\delta)|\varepsilon(k,u)|^2}
\]

\[
= i\pi \frac{\bar{f}_o(u)}{|\varepsilon(k,u)|^2} + P \int_{-\infty}^{+\infty} \frac{1}{u'-u} \frac{\bar{f}_o(u')}{|\varepsilon(k,u')|^2}
\]

\[
- \left\{ \frac{2\pi k}{\sigma_k(v_k)\gamma_k} \bar{f}_o(v_k) \frac{P}{u-v_k} \right\} \text{for unstable case only}
\]

(44)

where \(P\) indicates the principal value. Let, e.g.,

\[
\frac{1}{u - (\nu_k^- - i\delta)} \equiv \begin{cases} \frac{1}{u-\nu_k^-} - 2\pi i \delta(u-\nu_k^-) & \text{(stable)} \\ \frac{1}{u-\nu_k^-} & \text{(unstable)} \end{cases}
\]

(45)
then $\xi^-(k,\tilde{\nu}_k^-) = 0$ reads

$$
1 + \int_{-\infty}^{+\infty} \frac{\xi_2(k,u)}{u - (\tilde{\nu}_k^- - i\delta)} = 0
$$

(46)

regardless whether the case is stable or unstable; similarly, $\xi^+(k,\tilde{\nu}_k^+)=0$ is written

$$
1 + \int_{-\infty}^{+\infty} \frac{C_2(k,u)}{u - (\tilde{\nu}_k^+ + i\delta)} = 0 .
$$

(47)

Thus, if we take imaginary part of (38), noting the formula (43) and

$$
\text{Im} \frac{1}{\xi^+(k,u)} = - \frac{\pi \xi_2(k,u)}{|\xi(k,u)|^2},
$$

we have

$$
\text{Im} H(k,u,t) = - C_k e^{-2\gamma_k t} \text{Im} \frac{\xi_2(k,u)}{\tilde{\nu}_k^+[u - (\tilde{\nu}_k^- - i\delta)]}
$$

$$
- C_{-k} e^{-2\gamma_{-k} t} \text{Im} \frac{\xi_2(k,u)}{\tilde{\nu}_k^+[u - (\tilde{\nu}_k^- - i\delta)]} .
$$

(48)

Since, from (19) and (24),

$$
\frac{\partial}{\partial t} \omega_p^2 V_k(t) = - 2n \omega_p^2 \int_{-\infty}^{+\infty} \text{Im} H(k,u,t) ,
$$

(49)
we obtain, referring to (46),

\[
\frac{\partial}{\partial t} \omega_p V_k(t) = -2nk C_k \frac{\omega_p}{\omega_k} \gamma_k e^{-2\gamma_k t} - 2nk C_{-k} \frac{\omega_p}{\omega_{-k}} \gamma_{-k} e^{-2\gamma_{-k} t},
\]

or

\[
\frac{\partial}{\partial t} \sum_k \omega_p^2 V_k(t) = -4n \sum_k \gamma_k \frac{\omega_p}{\omega_k} k C_k e^{-2\gamma_k t}.
\]

Since, from (21),

\[
\left| \frac{E_k}{4\pi k} \right|^2 = \omega_p^2 V_k(t) + nk \int_{-\infty}^{+\infty} du \ u \Re H(k,u,t),
\]

we need the expression for the second term. Referring to (44), we have from (38)

\[
\int_{-\infty}^{+\infty} du \ u \Re H(k,u,t) = -\frac{\omega_p}{nk} V_k(t)
\]

\[
+ \frac{4\pi e^2}{k^3} \int_{-\infty}^{+\infty} du \left[ \frac{1}{\varepsilon_+^-(k,u)} + \xi_2(k,u) \int_{-\infty}^{+\infty} du' \frac{\rho}{u'-u} \left| \xi(k,u') \right|^2 \right]
\]

\[
- \frac{4\pi e^2}{k^3} \left\{ \frac{2\pi k}{\sigma_k^2(v_k)\gamma_k} \int_{-\infty}^{+\infty} du \frac{\xi_2(k,u)}{u-v_k} \right. \text{for unstable case only}
\]

\[
- C_k e^{-2\gamma_k t} \Re \int_{-\infty}^{+\infty} du \frac{\xi_2(k,u)}{\varepsilon_k^+ + (u-v_k)^2}
\]
The second term is calculated as follows:

\[
\int_{-\infty}^{+\infty} \left[ \bar{f}_0(u) \Re \frac{1}{\epsilon^+(k,u)} + \epsilon_2(k,u) \int_{-\infty}^{+\infty} \frac{P}{u'-u} \frac{\bar{f}_0(u')}{|\epsilon(k,u')|^2} \right] du
\]

\[
= \int_{-\infty}^{+\infty} \frac{\bar{f}_0(u)}{|\epsilon(k,u)|^2} \frac{\epsilon_1(k,u)}{|\epsilon(k,u)|^2} - \int_{-\infty}^{+\infty} \frac{\bar{f}_0(u)}{|\epsilon(k,u)|^2} \frac{\epsilon_2(k,u')}{|\epsilon(k,u')|^2} \int_{-\infty}^{+\infty} \frac{P}{u'-u} \epsilon_2(k,u') du'
\]

\[
= \int_{-\infty}^{+\infty} \frac{\bar{f}_0(u)}{|\epsilon(k,u)|^2} \left[ \epsilon_1(k,u) - 1 \right]
\]

We get an approximate expression for \( \frac{1}{|\epsilon(k,u)|^2} \) in the following way:

\[
\frac{1}{|\epsilon(k,u)|^2} \approx - \frac{1}{\pi \epsilon_2(k,u)} \Im \frac{1}{\epsilon^+(k,u)}
\]

\[
\approx - \frac{1}{\pi \epsilon_2(k,u)} \Im \left[ \frac{1}{\sigma_k(v_k)} \frac{1}{(u-v_k^+)} + \frac{1}{\sigma_k(u_k)} \frac{1}{(u-u_k^+)} \right]
\]

\[
\approx \pm \frac{\delta(u-v_k)}{\sigma_k(v_k) \epsilon_2(k,v_k)} \pm \frac{\delta(u-u_k)}{\sigma_k(u_k) \epsilon_2(k,u_k)},
\]

or, referring to (42),
\[ \frac{1}{|\xi(k,u)|^2} \simeq \pm \pi k \frac{\delta(u-v_k)}{\sigma_k^2(v_k) \gamma_k} \pm \pi k \frac{\delta(u-u_k)}{\sigma_k^2(u_k) \gamma_{-k}}, \tag{53} \]

where + for stable case (both \( \gamma_k, \gamma_{-k} > 0 \)) and - for unstable case (either \( \gamma_k \) or \( \gamma_{-k} < 0 \)). Then, regardless whether the system is stable or unstable,

\[ \int_{-\infty}^{+\infty} du \text{ Re } H(k,u,t) = -\frac{\omega^2}{nk} \nu_k(t) \]

\[ + \frac{4\pi^2 e^2}{k^2} \left[ \frac{f_o(v_k)}{\sigma_k^2(v_k) \gamma_k} + \frac{f_o(u_k)}{\sigma_k^2(u_k) \gamma_{-k}} \right] \]

\[ + C_k e^{-2\gamma_k t} + C_{-k} e^{-2\gamma_{-k} t}. \tag{54} \]

Thus (52) reads:

\[ \frac{|E_k|^2}{4\pi} = \frac{1}{2} \left[ E_k(t) + E_{-k}(t) \right] \]

with

\[ E_k(t) = 2nk C_k e^{-2\gamma_k t} + \frac{16\pi^2 ne^2}{k} \frac{f_o(v_k)}{\gamma^2_k}, \tag{55} \]

\( \varepsilon_k(t) \) is the forward progressive wave energy and \( \varepsilon_{-k}(t) \) is the backward. Referring to (39), we have

\[ \varepsilon_k(t) = \varepsilon_k^0 e^{-2\gamma_k t} + \frac{16\pi^2 ne^2}{k} \frac{f_o(v_k)}{\gamma^2_k} \frac{1 - e^{-2\gamma_k t}}{2\gamma_k}, \tag{56} \]
whence we see that

$$\varepsilon_k^0 = \text{initial value of } \varepsilon_k(t) \quad (57)$$

The physical meaning of the second term on the right-hand side of (56) becomes clear in the following way. If we take the time derivative of $\varepsilon_k(t)$:

$$\frac{d}{dt} \varepsilon_k(t) = -2\gamma_k \varepsilon_k(t) + \frac{16\pi^2 ne^2}{\sigma_k^2(v_k)} \int d^3v v f_0(v) \delta(k \cdot v - \omega_k). \quad (58)$$

This means that

$$\frac{16\pi^2 ne^2}{\sigma_k^2(v_k)} \int d^3v v f_0(v) \delta(k \cdot v - \omega_k) = \frac{16\pi^2 ne^2}{k \sigma_k^2(v_k)} f_0(v_k)$$

is the rate of spontaneous excitation of the field energy by those electrons traveling with the phase velocity $v_k$. Then

$$\frac{16\pi^2 ne^2}{k \sigma_k^2(v_k)} f_0(v_k) \frac{1 - e^{-2\gamma_k t}}{2\gamma_k} = \int_0^t d\tau \frac{16\pi^2 ne^2}{k \sigma_k^2(v_k)} f_0(v_k) e^{-2\gamma_k(t-\tau)}$$

is the total of that part of the field energy at time $t$ which has been spontaneously excited by the electrons from the initial time $t = 0$ up to $t$ while decaying exponentially.

We now turn to the particle part. From (32), disregarding the terms varying rapidly in time, we get

$$g(k, \nu, t) = \frac{k}{2\pi} \int \omega \ g(k, \nu, \omega) e^{-ik\omega t}$$
\[
= \frac{4\pi e^2}{k^3} \sum_{u \in^+ (k, u)} \left[ f_0 (v) - \frac{f_0 (u)}{k f_o' (u)} \left( k \cdot \frac{\partial}{\partial v} \right) f_0 (v) \right] 
\]

\[
- \frac{4\pi e^2}{k^3} \left[ f_0 (v) - \frac{f_0 (u)}{k f_o' (u)} \left( k \cdot \frac{\partial}{\partial v} \right) f_0 (v) \right] \times 
\]

\[
x \left[ \frac{ik (u-v_k^+)}{\sigma_k (v_k) v_k^+ (u-v_k^+)} + \frac{ik (u-u_k^+)}{\sigma_k (u_k) u_k^+ (u-u_k^+)} \right] 
\]

\[+ \frac{\left( k \cdot \frac{\partial}{\partial v} \right) f_0 (v)}{k f_o' (u)} H(k, u, t) \, . \] (59)

Equation (15) reads in classical limits:

\[
\frac{\partial}{\partial t} f_0 (v) = - \sum_k \frac{1}{\varepsilon^+ (k, u)} \left( k \cdot \frac{\partial}{\partial v} \right) \text{Im} \left[ k \cdot v g (k, v, t) \right] . \] (60)

\[
\text{Im} \frac{1}{\varepsilon^+ (k, u)} \overset{2}{=} \text{Im} \left[ \frac{1}{\sigma_k (v_k) u_v^+ (u-v_k^+)} + \frac{1}{\sigma_k (u_k) u_k^+ (u-u_k^+)} \right] 
\]

\[
r \overset{2}{=} \frac{2\pi}{\sigma_k (v_k)} \delta (u-v_k) + \frac{2\pi}{\sigma_k (u_k)} \delta (u-u_k) 
\]

with - for stable plasma and + for unstable plasma, and, as noted in (A31), e.g.,

\[
\frac{ik (u-v_k^+)}{u-v_k^+} \overset{\gamma_k t \geq 1}{\longrightarrow} \begin{cases} 
0 & \text{(stable)} \\
2\pi i \delta (u-v_k) & \text{(unstable)} 
\end{cases}
\]
hence we have,

\[ \text{Im} \left( (k \cdot v) g(k, v, t) \right) \]

\[
= \frac{4\pi^2 e^{2}}{k^3} \left[ \frac{\delta(u-v_k)}{\sigma_k(v_k)} \frac{f'_o(v_k)}{f'_o(u_k)} + \frac{\delta(u-u_k)}{\sigma_k(u_k)} \frac{f'_o(u_k)}{f'_o(v_k)} \right] \times \\
\times \left[ k \bar{f}'_o(u) f'_o(v) - \bar{f}'_o(u) (k \cdot \frac{\partial}{\partial v}) f'_o(v) \right] \\
+ \frac{(k \cdot \frac{\partial}{\partial v}) f'_o(v)}{k \bar{f}'_o(u)} \sim \sim \text{Im} H(k,u,t). 
\]

From (42),
\[
\bar{f}'_o(v_k) = -\frac{k \sigma_k(v_k)}{\Pi \omega_p^2},
\]
and, similarly,
\[
\bar{f}'_o(u_k) = -\frac{k \sigma_k(u_k)}{\Pi \omega_p^2}.
\]

Referring to (48), we have

\[ \text{Im} \left( (k \cdot v) g(k, v, t) \right) \]

\[
= \frac{4\pi^2 e^{2}}{k^4} \omega_p^2 \left[ \frac{\delta(u-v_k)}{\sigma_k(v_k)} \gamma_k + \frac{\delta(u-u_k)}{\sigma_k(u_k)} \gamma_k \right] \times \\
\times \left[ k \bar{f}'_o(u) f'_o(v) - \bar{f}'_o(u) (k \cdot \frac{\partial}{\partial v}) f'_o(v) \right] \\
+ \frac{\omega_p^2}{k^3} (k \cdot \sim) (k \cdot \frac{\partial}{\partial v}) f'_o(v) \text{Im} \left\{ \frac{-2\gamma_k t}{\bar{v}_k + [u - (\bar{v}_k - i\delta)]} \right\}
\]
\[ W_k(t) \equiv 2nk C_k e^{-2Y_k t} \]

Let

\[
W_k(t) = 2nk C_k e^{-2Y_k t} = \xi_k e^{-2Y_k t} - \frac{16\pi^2 n^2 e^2}{k \sigma_k^2(v_k)} \frac{\bar{f}_k(v_k)}{2Y_k} e^{-2Y_k t},
\]

so that \( \xi_k(t) = W_k(t) + \frac{16\pi^2 n^2 e^2}{k \sigma_k^2(v_k)} \frac{\bar{f}_k(v_k)}{2Y_k} \).

Then we have, for both the stable and unstable cases,

\[
\frac{\partial}{\partial t} f_o(\xi) = \sum_k \frac{8\pi^3 e^2 \omega_p^2}{mk \gamma_k \sigma_k^2(v_k)} \left( k \cdot \frac{\partial}{\partial \xi} \right) \delta(k \cdot \xi - \omega_k) \times
\]

\[
x \int d^3v' \delta(k \cdot \xi' - k \cdot \xi) \left[ \left( k \cdot \frac{\partial}{\partial \xi} \right) - \left( k \cdot \frac{\partial}{\partial \xi'} \right) \right] f_o(\xi) f_o(\xi')
\]

\[
- \Sigma \frac{\omega_p^2}{kmn^2} \left( k \cdot \frac{\partial}{\partial \xi} \right) \text{Im} \frac{(k \cdot \xi) W_k(t)}{w_k - i\gamma_k} \left[ k \cdot \xi - (w_k + i\gamma_k - i\delta) \xi \right] \left( k \cdot \frac{\partial}{\partial \xi} \right) f_o(\xi)
\]

with \( \frac{\partial}{\partial t} W_k(t) = -2Y_k W_k(t) \).

The first term on the right-hand side of (64) is identified, by using the approximation (53), with the well-known Balescu-Leonard collision integral in the stable case. The second is essentially what one gets in the quasi-linear theory of plasmas based on Vlasov's equation.
We will have a clear physical picture of the relaxation process (64) by considering the resonant scattering part. By grouping the terms on the right-hand side of (64) differently and defining a new field energy:

\[ \hat{E}_k(t) = \frac{u_k}{2k} \sigma_k(v_k) \varepsilon_k(t), \]  

we get

\[ \frac{\partial f_o(v)}{\partial t} \bigg|_{\text{res.}} = \sum_k \frac{8\pi^2 e^2}{mk \sigma_k(v_k)} \left( k \cdot \frac{\partial}{\partial v} \right) \left( k \cdot \frac{\partial}{\partial v} f_o(v) \right) \left[ \hat{E}_k(t) \right] - \frac{\delta(k \cdot v - w_k)}{m} \left( k \cdot \frac{\partial}{\partial v} f_o(v) \right) + \frac{\delta(k \cdot v - w_k)}{m} \left( k \cdot \frac{\partial}{\partial v} f_o(v) \right) . \]  

From (58) and (65),

\[ \frac{\partial}{\partial t} \hat{E}_k(t) = -2\gamma_k \hat{E}_k(t) + \frac{8\pi^2 ne^2}{k \sigma_k(v_k)} \left( k \cdot \frac{\partial}{\partial v} f_o(v) \right) \delta(k \cdot v - w_k). \]  

These two coupled equations (66) and (67) are essentially what Pines and Schrieffer obtained. The equilibrium value of \( \hat{E}_k(t) \) is, from (67),

\[ \hat{E}_k(t) \xrightarrow{t \to \infty} \kappa T \]  

where \( \kappa \) is Boltzmann constant and \( T \) the temperature. This is the energy of an oscillator, i.e., plasmon. The first terms on the right-hand sides of (66) and (67) represent absorption and induced emission of plasmons by electrons, and the second terms spontaneous emission.
Energy-momentum conservation for each process is expressed by
\[ \delta(k \cdot v - \omega_k). \] One can show from (66) and (67)
\[
\frac{d}{dt} \left[ n \left\langle \frac{mv^2}{2} \right\rangle_{\text{res.}} + \sum_{k} \hat{\varepsilon}_k(t) \right] = 0. \quad (69)
\]

Besides these resonant scattering processes we have in (64) non-resonant scattering because the frequency spectrum of the field appears with width of order \( \gamma_k \) instead of in the form of \( \delta \)-function. We examine total energy conservation. The total energy in our approximation is, from (6') and (21),
\[
\langle H \rangle = n \left\langle \frac{mv^2}{2} \right\rangle - \sum_{k} \langle P_k P_{-k} \rangle
\]
\[ - \frac{1}{2} \sum_{k} \left[ \langle P_k P_{-k} \rangle - \frac{\omega_p^2}{\hbar} \langle Q_k Q_{-k} \rangle \right]. \]

Since
\[ - \sum_{k} \langle P_k P_{-k} \rangle = \sum_{k} \frac{|E_k|^2}{4\pi} = \sum_{k} \varepsilon_k(t) \]
and
\[ - \langle Q_k Q_{-k} \rangle = V_k(t), \]
\[
\langle H \rangle = n \left\langle \frac{mv^2}{2} \right\rangle + \sum_{k} \hat{\varepsilon}_k(t)
\]
\[ + \frac{1}{2} \sum_{k} \left[ \varepsilon_k(t) - \frac{\omega_p^2}{\hbar} V_k(t) \right]. \quad (70)\]
From (64),

$$\frac{\partial}{\partial t} \left( \langle \frac{mv^2}{2} \rangle \right) = \sum_k w_k^2 \Im \frac{W_k(t)}{w_k - i\gamma_k} \int_{-\infty}^{+\infty} \frac{u^2}{u - (\gamma_k - i\delta)} f_o''(u) \du + \infty \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du$$

$$= \int_{-\infty}^{+\infty} \frac{u^2}{u - (\gamma_k - i\delta)} f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_o''(u) \du \right) f_2(u) = \frac{\partial}{\partial t} W_k(t) = \frac{\partial}{\partial t} V_k(t)$$

and, from (62) and (51),

$$- \Sigma_k 2\gamma_k \frac{w_k^2}{w_k} W_k(t) = \frac{\partial}{\partial t} \Sigma_k \frac{w_k^2}{w_k} V_k(t)$$
Thus we have

\[
\frac{\partial}{\partial t} \left\{ n \left\langle \frac{mv^2}{2} \right\rangle + \sum_k \varepsilon_k(t) + \frac{1}{2} \sum_k \left[ \varepsilon_k(t) - \omega_p^2 v_k(t) \right] \right\} = \frac{\partial}{\partial t} \langle H \rangle = 0 .
\] (71)

Since \( \omega_k \sim \omega_p \), and hence

\[
\frac{\partial}{\partial t} \sum_k \left[ \varepsilon_k(t) - \omega_p^2 v_k(t) \right] \sim 0 ,
\]

(71) reads

\[
\frac{\partial}{\partial t} \left\{ n \left\langle \frac{mv^2}{2} \right\rangle + \frac{1}{2} \sum_k \frac{|E_k|^2}{4\pi} \right\} = 0 .
\] (71a)

The kinetic temperature \( T \) is defined by

\[
\left\langle \frac{mv^2}{2} \right\rangle = 3\kappa T .
\]

(71a) is written

\[
\frac{\partial}{\partial t} \left[ 3n \kappa T + \frac{1}{2} \sum_k \frac{|E_k|^2}{4\pi} \right] = 0 .
\] (71b)

This is the equation based on the Hamiltonian (2), while the corresponding equation based on the Hamiltonian (1) is

\[
\frac{\partial}{\partial t} \left[ 3n \kappa T_0 + \frac{1}{2} \sum_k \frac{|E_k|^2}{8\pi} \right] = 0 .
\] (72)
We note the difference in the definition of the kinetic temperatures $T$ and $T_0$. We know that $\frac{|E_k|^2}{8\pi}$ is the potential energy part of the organized oscillatory motion, i.e., plasma oscillation. A portion of the kinetic energy of the electrons accounts for the kinetic energy part of this organized motion, so that the total energy of this oscillator is $\pi T$ where $T$ is the temperature of the heat bath which is provided by the remaining portion of the kinetic energy of the electrons, hence is different from $T_0$ and obviously $T < T_0$. 
5. MOTION OF IONS

The basic Hamiltonian is, corresponding to (2),

\[ H = \sum_{i=1}^{n} \frac{1}{2m} \left[ p_i^2 + \frac{e}{c} A(x_i) \right]^2 + \sum_{q=1}^{N} \frac{1}{2M} \left( p_q^2 - \frac{2e}{c} A(x_q) \right)^2 \]

\[ + \int \frac{d^3x}{8\pi} \frac{|E(x)|^2}{|E(x)|^2} \]  

(73)

Capital letters are used for ions. For overall neutrality

\[ NZ = n. \]  

(74)

If expressed in quantum mechanical operators, as in (6),

\[ H = \sum \mathcal{E}_p c_p^+ c_p + \sum \mathcal{E}_p c_p^+ c_p \]

\[ - \frac{1}{2} \sum_{k} (P_k p_{-k} + \omega_p^2 Q_k Q_{-k}) \]

\[ + \sum \sum_{k} m(p,k) Q_k c_p^+ c_{p-k} + \sum \sum_{k} M(p,k) Q_k c_p^+ c_{p-k} \]

\[ - \sum \sum_{k\neq k'} L(k,k') Q_k Q_{-k} c_p^+ c_{p-k+k'} \]

\[ - \sum \sum_{k\neq k'} L(k,k') Q_k Q_{-k} c_p^+ c_{p-k+k'} \]  

(75)
\[ \epsilon_p = \frac{\hbar^2 p^2}{2m}, \quad \epsilon_p = \frac{\hbar^2 p^2}{2M} \]

\[ m(p,k) = \left( \frac{4\pi e^2}{\hbar^2} \right)^{\frac{1}{2}} \nu_{p-k}, \quad (\hbar \nu_{p-k} = \epsilon_p - \epsilon_{p-k}) \]

\[ M(P,k) = \left( \frac{4\pi Z e^2}{\hbar^2} \right)^{\frac{1}{2}} N_{p-k}, \quad (\hbar N_{p-k} = \epsilon_p - \epsilon_{p-k}) \]

\[ \ell(k,k') = \frac{2\pi e}{m} (\hat{a}_k \cdot \hat{a}_{k'}) \]

\[ L(k,k') = \frac{2\pi Z e^2}{M} (\hat{a}_k \cdot \hat{a}_{k'}) \]

\[ \omega_p = \omega_0 + \omega_p^2; \quad \omega_p^2 = \frac{4\pi NZ^2 e^2}{M}, \quad \omega_0 = \frac{4\pi e^2}{m}. \]

Corresponding to (7) and (8), the subsidiary condition reads:

\[ S_k \equiv P_{-k} - i \left( \frac{4\pi e^2}{2} \right)^{\frac{1}{2}} \sum_{(p)} \left[ c_p^+ c_{p+k} - Z c_p^+ c_{p+k}^+ \right] \]

\[ S_k |\psi_n> = 0. \]

In classical limits \( \hbar \to 0 \),

\[ \nu_{p-k}, N_{p-k} \to k \cdot v \]

\[ \frac{1}{\hbar} \frac{\partial}{\partial \hat{p}} \to \frac{1}{m} \frac{\partial}{\partial \hat{v}}, \quad \frac{1}{M} \frac{\partial}{\partial \hat{v}}. \]

We denote, as in (24),
where
\[
\epsilon_p = \frac{\sqrt{2} \hbar^2}{2m}, \quad \epsilon_p = \frac{\hbar^2 p^2}{2M}
\]

\[
m(p, k) = \left( \frac{4\pi e^2}{\hbar^2} \right)^{1/2} \nu_{p-k}, \quad (\hbar \nu_{p-k} = \epsilon_p - \epsilon_{p-k})
\]

\[
M(p, k) = \left( \frac{4\pi e^2}{\hbar^2} \right)^{1/2} N_{p-k}, \quad (\hbar N_{p-k} = \epsilon_p - \epsilon_{p-k})
\]

\[
\mathcal{L}(k, k') = \frac{2\pi e^2}{m} (\hat{a}_k \cdot \hat{a}_{k'})
\]

\[
L(k, k') = \frac{2\pi e^2}{M} (\hat{a}_k \cdot \hat{a}_{k'})
\]

\[
\omega_p^2 = \omega_n^2 + \omega_n^2; \quad \omega_n^2 = \frac{4\pi e^2}{M}, \quad \omega_n^2 = \frac{4\pi e^2}{m}.
\]

Corresponding to (7) and (8), the subsidiary condition reads:

\[
S_k \equiv \nu_{p-k} - i \left( \frac{4\pi e^2}{\hbar^2} \right)^{1/2} \sum_{p, p+k} \left[ c_p^+ c_{p+k} - Z c_p^+ c_{p+k} \right]
\]

\[
S_k |\nu_n \rangle = 0.
\]

In classical limits \( \hbar \to 0 \),

\[
\nu_{p-k}, N_{p-k} \to k \cdot \nu
\]

\[
\frac{1}{\hbar} \frac{\partial}{\partial \nu} \to \frac{1}{m} \frac{\partial}{\partial \nu}, \quad \frac{1}{\hbar} \frac{\partial}{\partial \nu} \to \frac{1}{M} \frac{\partial}{\partial \nu}.
\]

We denote, as in (24),
\[ \langle c_p^+ c_p \rangle = n f_o(v) \]
\[ \langle c_p^+ c_p \rangle = N f_o(v) \]
\[ - \omega_p^2 \langle Q_k Q_{-k} \rangle = V_k(t) \]
\[ \left( \frac{4\pi e^2}{k^2} \right)^{1/2} \langle Q_k c_p^+ c_{p-k} \rangle = n g(k, v, t) \]
\[ \left( \frac{4\pi Z^2 e^2}{k^2} \right)^{1/2} \langle Q_k c_p^+ c_{p-k} \rangle = N G(k, v, t). \]

Dielectric constant, e.g., \( \varepsilon^+(k, \omega) \) is now, corresponding to (31),

\[ \varepsilon^+(k, \omega) = 1 - \frac{\omega_p^2}{k^2} \int_{-\infty}^{+\infty} \frac{\varphi_o'(u)}{u - \omega} \]

with

\[ \omega_p^2 \varphi_o(v) \equiv \omega_-^2 f_o(v) + \omega_+^2 F_o(v) \]

and

\[ \pi \varepsilon_2(k, u) \equiv \text{Im} \varepsilon^+(k, u) = - \pi \frac{\omega_p^2}{k^2} \varphi_o'(u). \]

If we define \( H(k, u, t) \), as in (28), now

\[ H(k, u, t) \equiv \bar{G}(k, u, t) - \frac{N}{n} \tilde{G}(k, u, t), \]

then we have the following set of equations:
\[
\frac{\partial}{\partial t} F_o(\nu) = - \sum_{k} \frac{1}{M} \left( k \cdot \frac{\partial}{\partial \nu} \right) (k \cdot \nu) \text{Im} G(k, \nu, t), \quad (82)
\]

\[
\frac{\partial}{\partial t} F_o(\nu) = \sum_{k} \frac{1}{M} \left( k \cdot \frac{\partial}{\partial \nu} \right) (k \cdot \nu) \text{Im} G(k, \nu, t), \quad (83)
\]

and, corresponding to (32)

\[
g(k, \nu, \omega) = \frac{i a_0(k, \nu, \omega) - i \frac{\omega^2 (k \cdot \frac{\partial}{\partial \nu} \phi_o(\nu)}{\omega_p^2 k \varphi'_o(u)} B_0(k, u, \omega)}{k(u+\omega) \epsilon^+(k, u+\omega)}
\]

\[
= \frac{i \frac{4\pi e^2}{k^4} F_o(\nu) - \omega^2 \frac{k \cdot \frac{\partial}{\partial \nu} \phi_o(\nu)}{\omega_p^2 k \varphi'_o(u)} \left[ \frac{\bar{F}_o(u)}{\epsilon} + Z \frac{\bar{F}_o(u)}{\epsilon} \right]}{\omega(u+\omega) \epsilon^+(k, u+\omega)}
\]

\[
+ \frac{\omega^2 (k \cdot \frac{\partial}{\partial \nu} \phi_o(\nu)}{\omega_p^2 k \varphi'_o(u)} H(k, u, \omega), \quad (84)
\]

\[
G(k, \nu, \omega) = \frac{i A_o(k, \nu, \omega) + i \frac{N \omega^2 (k \cdot \frac{\partial}{\partial \nu} \phi_o(\nu)}{\omega_p^2 k \varphi'_o(u)} B_0(k, u, \omega)}{k(u+\omega) \epsilon^+(k, u+\omega)}
\]

\[
= \frac{m \omega^2 (k \cdot \frac{\partial}{\partial \nu} \phi_o(\nu)}{M \omega_p^2 k \varphi'_o(u)} \left[ \frac{\bar{F}_o(u)}{\epsilon} + Z \frac{\bar{F}_o(u)}{\epsilon} \right]}{\omega(u+\omega) \epsilon^+(k, u+\omega)}
\]

\[
+ i \frac{4\pi e^2 \omega^2}{k^4} \left[ \frac{F_o(\nu) - \omega^2 \frac{k \cdot \frac{\partial}{\partial \nu} \phi_o(\nu)}{\omega_p^2 k \varphi'_o(u)} \left[ \frac{\bar{F}_o(u)}{\epsilon} + Z \frac{\bar{F}_o(u)}{\epsilon} \right]}{\omega(u+\omega) \epsilon^+(k, u+\omega)} \right]
\]
\[
\frac{n \omega_p^2 (k \cdot \frac{\partial}{\partial \nu}) F_0 (\nu)}{N k \omega_p^2 \varphi_o' (u)} H(k, u, \omega) \quad (85)
\]

while for \( H(k, u, \omega) \), similar to \( 29 \),

\[
k(u+\omega) \epsilon^+(k, u+\omega) H(k, u, \omega) = i B_o (k, u, \omega) + i \frac{4 \pi e^2}{k^3 \omega} \left[ F_0 (u) + Z F_0 (u) \right] + \frac{\omega_p^2}{n} V_k (u) u \varphi_o' (u) \]

\[
- \frac{\omega_p^2}{k} u \varphi_o' (u) \int_{-\infty}^{+\infty} H^*(k, u', \omega) \frac{\varphi_o' (u)}{u' - u - \omega} \quad (86)
\]

where

\[
B_o (k, u, \omega) = \tilde{A}_o (k, u, \omega) - \frac{N}{n} \tilde{A}_o (k, u, \omega)
\]

and \( a_o (k, \nu, \omega) \) and \( A_o (k, \nu, \omega) \) are determined by the initial conditions of the correlations.

The equation \( \epsilon^+(k, \omega) = 0 \) has now four complex roots. The first two of them are, as given by \( 40 \), associated with plasma waves, and the other two, as given below, with acoustic waves:

\[
\epsilon^+(k, \widetilde{V}_k) = 0 = \epsilon^+(k, \widetilde{U}_k)
\]

where

\[
\begin{aligned}
\widetilde{V}_k & = \frac{1}{k} \left( \Omega_k + i \Gamma_k \right) \\
\widetilde{U}_k & = \frac{1}{k} \left( - \Omega_k + i \Gamma_k \right)
\end{aligned}
\]

\[
\epsilon^-(k, \widetilde{V}_k) = 0 = \epsilon^-(k, \widetilde{U}_k)
\]
with
\[
\Omega_k^2 = \left\{ \begin{array}{c}
\frac{\omega_+^2}{\epsilon_e(k,\omega)} \\
\Gamma_k = \frac{\pi k \epsilon_2(k,\nu_k)}{\sigma_k(\nu_k)}
\end{array} \right\}
\]

where
\[
\epsilon_e(k,\omega) = 1 - \frac{\omega^2}{k^2} \left[ \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f_0'(u)}{u} \right]
\]

= electron shielding factor

and
\[
\sigma_k(\nu_k) = \frac{\partial}{\partial u} \epsilon_1(k,u) \bigg|_{u=\nu_k} \approx 2k \frac{\omega_+^2}{\Omega_k^3}.
\]

Hence, the expression corresponding to (A37) is, for real \( u \),
\[
\frac{1}{\epsilon^-(k,u)} \approx 1 + \frac{1}{\sigma_k(\nu_k)(u - \nu_k^-)} + \frac{1}{\sigma_k(\nu_k)(u - \nu_k^-)}
+ \frac{1}{\sigma_k(\nu_k)(u - \nu_k^-)} + \frac{1}{\sigma_k(\nu_k)(u - \nu_k^-)}.
\]

Inverse Laplace transform of the solution of the integral equation (86) gives, corresponding to (38),
\[
H(k, u, t) = -\frac{k}{n} \epsilon_2(k, u) \nu_k(t)
+ \frac{4\pi e^2}{k^3} \left[ \frac{\phi(u)}{u \epsilon^+(k,u)} + \frac{\epsilon_2(k,u)}{u} \int_{L}^{u'} \frac{\phi(u')}{(u' - u - i\delta)|\epsilon(k,u')|^2} du' \right]
\]
\[ \begin{align*}
\xi_2(k,u) & \left[ \frac{\text{i}k(u - \tilde{w}_k)}{1 - e^{\tilde{w}_k (u - \tilde{w}_k)}} \right] C_k^p e^{-2\gamma_k t} \\
\xi_2(k,u) & \left[ \frac{\text{i}k(u - \tilde{u}_k)}{1 - e^{\tilde{u}_k (u - \tilde{u}_k)}} \right] C_k^p e^{-2\gamma_k t} \\
\xi_2(k,u) & \left[ \frac{\text{i}k(u - \tilde{w}_k)}{1 - e^{\tilde{w}_k (u - \tilde{w}_k)}} \right] C_k^a e^{-2\Gamma_k t} \\
\xi_2(k,u) & \left[ \frac{\text{i}k(u - \tilde{u}_k)}{1 - e^{\tilde{u}_k (u - \tilde{u}_k)}} \right] C_k^a e^{-2\Gamma_k t} \tag{92}
\end{align*} \]

where \[ \phi(u) = \overline{f_0(u)} + \sum F_o(u), \tag{93} \]

\[ \begin{align*}
C_k^p & = \frac{1}{2nk} \left[ \varepsilon_k^p(\omega) - \frac{\phi_k(v_k)}{2\gamma_k} \frac{16\pi^2 n e^2}{k \sigma^2_k(v_k)} \right] \\
C_k^a & = \frac{1}{2nk} \left[ \varepsilon_k^a(\omega) - \frac{\phi_k(v_k)}{2\Gamma_k} \frac{16\pi^2 n e^2}{k \sigma^2_k(v_k)} \right] \tag{94}
\end{align*} \]

We can later identify \( \varepsilon_k^p(\omega) \) and \( \varepsilon_k^a(\omega) \) with the initial values of the plasma wave energy and acoustic wave energy respectively. If we take the imaginary part of (92), we have, corresponding to (48),
\[
\text{Im } H(k, u, t) = -C_k e^{-2\gamma_k t} \frac{\varepsilon_2(k,u)}{\nu_k + [u - (\nu_k^* - i\delta)]} + \frac{\gamma_k}{2} C_k e^{-2\gamma_k t} + \frac{\Gamma_k}{\Omega_k^2} C_{-k} e^{-2\Gamma_{-k} t}.
\]

Thus,

\[
\int_{-\infty}^{\infty} \text{Im } H(k,u,t) du = \frac{\gamma_k}{2} \frac{p}{\omega_k} e^{-2\gamma_k t} + \frac{\Gamma_k}{2 \Omega_k^2} \frac{p}{C_{-k} e^{-2\Gamma_{-k} t}}.
\]

because, e.g.,

\[
0 = \varepsilon^-(k, \nu_k^*) = 1 + \int_{-\infty}^{\infty} \frac{\varepsilon_2(k,u)}{u - (\nu_k^* - i\delta)} du.
\]

Exactly like (49) for electron plasma case, we have

\[
\frac{\partial}{\partial t} \omega_p^2 V_k(t) = -2n \omega_p^2 \int_{-\infty}^{\infty} \text{Im } H(k,u,t) du.
\]
therefore

\[ \frac{\partial}{\partial t} \sum_k \omega_p^2 V_k(t) = - \sum_k 2\gamma_k \left( \frac{\omega_p}{\omega_k^2} \right) 2\eta_k c_k e^{-2\gamma_k t} \]

\[ - \sum_k 2\Gamma_k \left( \frac{\omega_p}{\Omega_k^2} \right) 2\eta_k c_k e^{-2\Gamma_k t} . \quad (98) \]

We have the same expression as (52):

\[ \frac{|E_{\pi k}|^2}{4\pi} = \omega_p^2 V_k(t) + \eta_k \int_{-\infty}^{+\infty} du \Re H(k,u,t) . \quad (99) \]

In the same manner as we obtained (54) we get

\[ \int_{-\infty}^{+\infty} du \Re H(k,u,t) = - \frac{\omega_p^2}{\eta_k} V_k(t) \]

\[ + \frac{4ne^2}{\omega_k^2} \left[ \frac{\phi_o(v_k)}{\sigma_k^2(v_k)} \gamma_k + \frac{\phi_o(u_k)}{\sigma_k^2(u_k)} \gamma_k - \frac{\phi_o(v_k)}{\sigma_k^2(v_k)} r_k + \frac{\phi_o(U_k)}{\sigma_k^2(U_k)} r_k \right] \]

\[ + C_k e^{-2\gamma_k t} + C_{-k} e^{-2\Gamma_k t} + C_k e^{-2\gamma_k t} + C_{-k} e^{-2\Gamma_k t} . \quad (100) \]

Let

\[ \varepsilon_p^c(t) = 2\eta_k c_k e^{-2\gamma_k t} + \frac{16\pi^2 n e^2}{\omega_k} \frac{\phi_o(v_k)}{k \sigma_k^2(v_k)} \]

\[ \left\{ \begin{array}{l}
\varepsilon_k(t) = 2\eta_k c_k e^{-2\Gamma_k t} + \frac{16\pi^2 n e^2}{\omega_k} \frac{\phi_o(U_k)}{2\Gamma_k} \\
\end{array} \right\} . \quad (101) \]
then

$$\frac{1}{4\pi} \left\lvert E_{\mathbf{k}} \right\rvert^2 = \frac{1}{2} \left[ \varepsilon^p_{\mathbf{k}}(t) + \varepsilon^{-p}_{\mathbf{k}}(t) \right] + \frac{1}{2} \left[ \varepsilon^a_{\mathbf{k}}(t) + \varepsilon^{-a}_{\mathbf{k}}(t) \right].$$

(102)

$\varepsilon^p_{\mathbf{k}}(t)$ is plasma wave field energy and $\varepsilon^a_{\mathbf{k}}(t)$ acoustic wave field energy.

From (94),

$$\begin{align*}
\varepsilon^p_{\mathbf{k}}(t) &= \varepsilon^p_{\mathbf{k}}(0) e^{-2\gamma_k t} + \frac{16\pi^2 n e^2}{k \sigma^2_k(v_k)} \phi_o(v_k) \frac{1 - e^{-2\gamma_k t}}{2}\beta_k(t) \\
\varepsilon^a_{\mathbf{k}}(t) &= \varepsilon^a_{\mathbf{k}}(0) e^{-2\Gamma_k t} + \frac{16\pi^2 n e^2}{k \sigma^2_k(v_k)} \phi_o(v_k) \frac{1 - e^{-2\Gamma_k t}}{2}\beta_k(t).
\end{align*}$$

(103)

If we take the time derivative of (103),

$$\begin{align*}
\frac{\partial}{\partial t} \varepsilon^p_{\mathbf{k}}(t) &= -2\gamma_k \varepsilon^p_{\mathbf{k}}(t) + \frac{16\pi^2 n e^2}{\sigma^2_k(v_k)} \int d^3 v \left[ f_o(v) + Z F_o(v) \right] \delta(k' v - w_k) \\
\frac{\partial}{\partial t} \varepsilon^a_{\mathbf{k}}(t) &= -2\Gamma_k \varepsilon^a_{\mathbf{k}}(t) + \frac{16\pi^2 n e^2}{\sigma^2_k(v_k)} \int d^3 v \left[ f_o(v) + Z F_o(v) \right] \delta(k' v - \Omega_k).
\end{align*}$$

(104)

We have the same physical interpretation of the second terms on the right-hand side of (103) as we did for the electron plasma case, i.e.,

they are that part of the field energy which has been spontaneously excited by electrons and ions from time $t=0$ to $t$ while decaying exponentially.
From (84), taking the inverse Laplace transform, we have

\[
g(k, v, t) = -\frac{4\pi e^2}{k^3} \int \frac{u^+}{u^+(k, r)} \left\{ f_o(v) - \frac{\omega^2}{\omega_p^2} \frac{f_o(u) + Z f_o(u)}{\varphi_o'(u)} \left( k \cdot \frac{\partial}{\partial v} \right) f_o(v) \right\} \]

\[
+ \frac{4\pi e^2}{k^3} \left\{ f_o(v) - \frac{\omega^2}{\omega_p^2} \frac{f_o(u) + Z f_o(u)}{\varphi_o'(u)} \left( k \cdot \frac{\partial}{\partial v} \right) f_o(v) \right\} \times
\]

\[
\times \left[ \frac{ik(u-v_k^+ + u-u_k^+)}{\sigma_k(v_k) \gamma_k^+ (u-v_k^+)} + \frac{ik(u-u_k^+)}{\sigma_k(u_k) \gamma_k^+ (u-u_k^+)} \right]
\]

\[
+ \frac{ik(u-v_k^+ + u-u_k^+)}{\sigma_k(u_k) \gamma_k^+ (u-v_k^+)} + \frac{ik(u-u_k^+)}{\sigma_k(u_k) \gamma_k^+ (u-u_k^+)} \right]
\]

\[
+ \frac{\omega^2}{\omega_p^2} \left( k \cdot \frac{\partial}{\partial v} \right) f_o(v) \times \frac{H(k, u, t).}{\varphi_o'(u)} \tag{105}
\]

Hence

\[
\text{Im} \left[ (k \cdot v) g(k, v, t) \right]
\]

\[
= \frac{4\pi^3 e^2 \omega^2}{k^4} \left[ \frac{\delta(u-v_k)}{\sigma_k(v_k)} \gamma_k + \frac{\delta(u-u_k)}{\sigma_k(u_k)} \gamma_k + \frac{\delta(u-v_k)}{\sigma_k(v_k)} \Gamma_k + \frac{\delta(u-u_k)}{\sigma_k(u_k)} \Gamma_k \right] \times
\]

\[
x \left\{ k f_o(v) \left[ \frac{\delta'(u) + \frac{M u}{M f_o'(u)} - [f_o(u) + Z f_o(u)] \left( k \cdot \frac{\partial}{\partial v} \right) f_o(v) \right] \right\}
\]
\[
\frac{\omega^2}{k^3} \left\{ \begin{array}{c}
p C_k e^{\frac{-2\gamma_k t}{k^2}} \\
\frac{\text{Im}}{k^2} \frac{1}{\bar{u}_k^+ [u - (\bar{u}_k^- - i\delta)]} \\
\frac{1}{k} \frac{1}{a C_k e^{\frac{-2\Gamma_k t}{k^2}}} \\
\frac{1}{\bar{u}_k^+ [u - (\bar{u}_k^- - i\delta)]} \end{array} \right\} \frac{\langle k \cdot \tilde{v} \rangle}{\langle \tilde{v} \rangle} \left( k \cdot \frac{\partial}{\partial \tilde{v}} \right) f_o(\tilde{v}).
\]

Let
\[
W_k(t) \equiv \frac{p}{2nk C_k e^{\frac{-2\gamma_k t}{k^2}}}
\]
and
\[
W_k(t) \equiv \frac{a}{2nk C_k e^{\frac{-2\Gamma_k t}{k^2}}},
\]
so that
\[
\begin{array}{c}
\varepsilon_k(t) = W_k(t) + \frac{16\pi^2}{k^2} \frac{n e^2}{\sigma_k^2(\tilde{v}_k)} \frac{\bar{F}_o(\tilde{v}_k) + Z \bar{F}_o(\tilde{v}_k)}{2\gamma_k} \\
\varepsilon_k(t) = W_k(t) + \frac{16\pi^2}{k^2} \frac{n e^2}{\sigma_k^2(\tilde{v}_k)} \frac{\bar{F}_o(\tilde{v}_k) + Z \bar{F}_o(\tilde{v}_k)}{2\Gamma_k}
\end{array}
\]

Then we have, corresponding to (64),
\[
\frac{\partial}{\partial t} f_o(\tilde{v}) = \sum_k \frac{8\pi^3}{mk^2} e^2 \omega^2 \left( \frac{1}{\bar{v}} \frac{\partial}{\partial \bar{v}} \right) \left[ \delta(\bar{v} \cdot \bar{v}' - w_k) \right] \frac{\delta(\bar{v} \cdot \bar{v}' - \Omega_k)}{\sigma_k^2(\tilde{v}_k)} + \frac{\delta(\bar{v} \cdot \bar{v}' - \Omega_k)}{\Gamma_k \sigma_k^2(\tilde{v}_k)} \times
\]
\[
\times \int d^3 \tilde{v}' \delta(\bar{v} \cdot \bar{v}' - k \cdot \bar{v}) \left\{ [f_o(\tilde{v}') + Z F_o(\tilde{v}')] \left( k \cdot \frac{\partial}{\partial \tilde{v}} \right) f_o(\tilde{v}) - f_o(\tilde{v}) \left( k \cdot \frac{\partial}{\partial \tilde{v}'} \right) [f_o(\tilde{v}') + \frac{mZ}{M} F_o(\tilde{v}')] \right\}
\]
Similarly, we have

\[
\frac{\partial}{\partial t} F_0(\nu) = \sum_k \frac{8\pi^2 \alpha^2}{M_k^2} \Bigg( k \cdot \frac{\partial}{\partial \nu} \Bigg) \left( \frac{\delta(k \cdot \nu - \nu_k)}{\gamma_k \sigma_k^2(\nu_k)} + \frac{\delta(k \cdot \nu - \Omega_k)}{\Gamma_k \sigma_k^2(\nu_k)} \right) \times \\
\int d^3v \left\{ F_0(\nu') + \frac{1}{Z} \left[ F_0(\nu) + \frac{M}{mZ} F_0(\nu) \right] \right\} \left( k \cdot \frac{\partial}{\partial \nu} \right) F_0(\nu) \\
- F_0(\nu) \left( k \cdot \frac{\partial}{\partial \nu} \right) \left[ F_0(\nu) + \frac{M}{mZ} F_0(\nu) \right] \\
- \sum_k \frac{\omega_+^2}{Mnk^2} \left( k \cdot \frac{\partial}{\partial \nu} \right) \text{Im} \left[ \frac{(k \cdot \nu) W_k(t)}{(\omega_k - i\Gamma_k - (\omega_k + i\gamma_k - i\delta))} \right] \\
+ \frac{(k \cdot \nu) a}{(\Omega_k - i\Gamma_k - (\Omega_k + i\gamma_k - i\delta))} \left( k \cdot \frac{\partial}{\partial \nu} \right) F_0(\nu) \right) .
\]
In order to have a clearer physical picture of the relaxation processes (109) and (110), we consider, as we did for the electron plasma case, the resonant scattering parts of these equations.

We define

\[ \hat{\varepsilon}_k^p(t) = \frac{\omega_k}{2k} \sigma_k(v_k) \hat{\varepsilon}_k^p(t) \]

\[ \hat{\varepsilon}_k^a(t) = \frac{\Omega_k}{2k} \sigma_k(v_k) \hat{\varepsilon}_k^a(t) \]

(11)

From (109), we have

\[
\left[ \frac{\partial f_0(v)}{\partial t} \right]_{\text{res.}} = \sum_k \frac{8\pi^2 e^2}{mk} \sigma_k(v_k) \left( \frac{\partial}{\partial v} \right) \left[ \frac{\hat{\varepsilon}_k^p(t)}{m} \delta(k \cdot v - \omega_k) \left( \frac{\partial}{\partial v} \right) f_0(v) \right]
\]

\[
+ \omega_k f_0(v) \delta(k \cdot v - \omega_k)
\]

\[
+ \sum_k \frac{8\pi^2 e^2}{mk} \sigma_k(v_k) \left( \frac{\partial}{\partial v} \right) \left[ \frac{\hat{\varepsilon}_k^a(t)}{m} \delta(k \cdot v - \Omega_k) \left( \frac{\partial}{\partial v} \right) f_0(v) \right]
\]

\[
+ \Omega_k f_0(v) \delta(k \cdot v - \Omega_k)
\]

(112)

From (110), we have

\[
\left[ \frac{\partial f_0(v)}{\partial t} \right]_{\text{res.}} = \sum_k \frac{8\pi^2 e^2}{Mk} \sigma_k(v_k) \left( \frac{\partial}{\partial v} \right) \left[ \frac{\hat{\varepsilon}_k^p(t)}{M} \delta(k \cdot v - \omega_k) \left( \frac{\partial}{\partial v} \right) f_0(v) \right]
\]

\[
+ \omega_k f_0(v) \delta(k \cdot v - \omega_k)
\]
\[
+ \sum_k \frac{8\pi^2 Z^2 e^2}{M k \Omega_k \sigma_k(v_k)} \left( k \cdot \frac{\partial}{\partial \omega} \right) \left[ \hat{\mathcal{E}}_k(t) \frac{\delta(k \cdot \omega - \omega_k)}{M} \left( \sim \omega \sim \omega_k \right) \right] F_0(\omega) \\
+ \Omega_k F_0(\omega) \delta(k \cdot \omega - \Omega_k) \right].
\]

From (104) and (111),

\[
\frac{\partial}{\partial t} \hat{\mathcal{E}}_k^P(t) = -2 \gamma_k \hat{\mathcal{E}}_k^P(t) + \frac{8\pi^2 \nu e^2}{k \sigma_k(v_k)} \left[ \hat{\mathcal{F}}_k(t) + \sum_{\omega} F_0(\omega) \right] \delta(k \cdot \omega - \omega_k) + \sum_{\omega} F_0(\omega) \delta(k \cdot \omega - \Omega_k),
\]

\[
\frac{\partial}{\partial t} \hat{\mathcal{E}}_k^a(t) = -2 \Gamma_k \hat{\mathcal{E}}_k^a(t) + \frac{8\pi^2 \nu e^2}{k \sigma_k(v_k)} \Omega_k \int d^3 v \left[ \hat{\mathcal{F}}_0(\omega) + \sum_{\omega} F_0(\omega) \right] \delta(k \cdot \omega - \Omega_k). \]

The equilibrium values are:

\[
\begin{align*}
\hat{\mathcal{E}}_k^P(t) & \to \kappa T \\
\hat{\mathcal{E}}_k^a(t) & \to \kappa T
\end{align*}
\]

This means \( \hat{\mathcal{E}}_k^P(t) \) and \( \hat{\mathcal{E}}_k^a(t) \) are energies of oscillators, i.e., plasmons and phonons respectively. Therefore, the equation (112), e.g., says that the electron distribution relaxes through absorption and both induced and spontaneous emission of plasmons and phonons. One can readily show that

\[
\frac{\partial}{\partial t} \left[ n \left\{ \frac{mv^2}{2} \right\}_{\text{res.}} + N \left\{ \frac{Mv^2}{2} \right\}_{\text{res.}} + \sum_k \hat{\mathcal{E}}_k^P(t) + \sum_k \hat{\mathcal{E}}_k^a(t) \right] = 0. \]

If the temperature is so low that the ions are essentially frozen and
the number of electrons traveling with the phase-velocity of plasma waves are negligibly small, then we have electron-phonon interaction only. Our results in this case agree, again, with those of Pines and Schrieffer.¹

The average total energy in our approximation is

\[
\langle H \rangle = n\left(\frac{mv^2}{2}\right) + N\left(\frac{Mv^2}{2}\right) + \sum_k \left[ \varepsilon_k^a(t) + \varepsilon_k^p(t) \right]
\]

\[
+ \frac{1}{2} \sum_k \left[ \varepsilon_k^a(t) + \varepsilon_k^p(t) - \omega_p^2 v_k(t) \right].
\]

(117)

From (109) and (110), we get

\[
\frac{\partial}{\partial t} \left[ n\left(\frac{mv^2}{2}\right) + N\left(\frac{Mv^2}{2}\right) \right]
\]

\[
= \sum_k w_k^p(t) \left[ - \frac{\omega_p^2}{\omega_k^2} \gamma_k + 3\gamma_k \right]
\]

\[
+ \sum_k w_k^a(t) \left[ - \frac{\omega_p^2}{\Omega_k^2} \Gamma_k + 3\Gamma_k \right].
\]

With reference to (98) and (107) this equation shows energy conservation:

\[
\frac{\partial}{\partial t} \langle H \rangle = 0.
\]

(118)
6. DISCUSSIONS

We have attempted to develop a kinetic theory of plasmas, abiding by the first principle. It is based on the picture of field-particle scattering, instead of particle-particle scattering. It is developed within the limit of the linear approximation. Its resonant part, we have shown, agrees with Pines-Schrieffer equations.\(^\text{13}\)

Another noticeable aspect of the present theory is that it is valid regardless whether the system is homogeneous or non-homogeneous; we have nowhere to make an assumption of its homogeneity.

The essential feature of the linear approximation is expressed in the operator equation (14) which is the linearized Vlasov equation. This approximation is equivalent to limiting the consideration only up to the binary correlations and neglecting the higher ones. The present theory cannot explain, therefore, the stabilization process of the marginal instability. To do this, one has to go beyond this approximation, i.e., to consider the higher correlations.
REFERENCES AND FOOTNOTES


9. In case of unstable plasma, this portion of the contour is deformed in the same manner as in Fig. A so that both poles $\tilde{\nu}_k$ and $\tilde{\eta}_k$ lie above the line.

10. As an unstable plasma we are considering here the case when only $\gamma_k < 0$ while $\gamma_{-k} > 0$. If both of them are negative we have simply to add another term:

$$\frac{4\pi e^2}{k^3} \frac{2\pi k}{\sigma_k^2(u_k) \gamma_k} \int_{-\infty}^{+\infty} \int_{u-u_k} f_0(u_k) \frac{P}{u-u_k} \xi_2(k,u).$$

11. The present form is preferred because it remains the same whether the case is stable or unstable. This collision integral represents the particle-particle scattering through intermediation of the field.

13. In the scheme of Pines and Schrieffer\(^1\) plasmons and phonons are well-defined elementary excitations, i.e., \(\gamma_k\) and \(\Gamma_k\rightarrow 0\). Therefore, if one applies Born's approximation to it as they did, then one gets only resonant scattering.
SOLUTION OF THE INTEGRAL EQUATION (34)

Equation (34) is:

\[ \varepsilon^+(k, u+\omega) \tilde{\mathcal{C}}(k,u,\omega) = b_k(u,\omega) + \varepsilon_2(k,u) \int_{-\infty}^{+\infty} du' \frac{\tilde{\mathcal{C}}^*(k,u',\omega)}{u' - u - \omega} \]

where

\[ \varepsilon_2(k,u) \equiv - \frac{\omega p}{k^2} f_o'(u) \]

and

\[ \tilde{\mathcal{C}}^*(k,u,\omega) = \tilde{\mathcal{C}}^*(k,u, - \omega^*) . \]

We note that

\[ \varepsilon^+(k,\omega) = 0 \]

has 2 roots

\[ \tilde{\nu}_k^+ = \frac{1}{k}(\omega_k - i\gamma_k) \]

\[ \tilde{\nu}_k^- = \frac{1}{k}(-\omega_k - i\gamma_k) \]

where

\[ \omega_k, \omega_k \approx \omega_p \]

and

\[ \gamma_k, \gamma_k > 0 \quad \text{for stable plasma.} \]

For unstable plasma either \( \gamma_k \) or \( \gamma_k \) is negative. Similarly

\[ \varepsilon^-(k,\omega) = 0 \]
has 2 roots:

\[ \tilde{\nu}_k^+ = \frac{1}{k} (\omega_k + i\gamma_k) = (\tilde{\nu}_k^-)^* \]

\[ \tilde{\nu}_k^- = \frac{1}{k} (-\omega_k + i\gamma_k) = (\tilde{\nu}_k^+)^* \]  

(A2)

We solve (34) for stable plasma, in which case we can put \( \omega = \text{real} \) and analytically continue the solution thereof to the entire \( \omega \)-plane. For unstable plasma this process is not allowed; the solution for this case can, however, be obtained through analytical continuation of that for stable case with respect to \( \gamma_k \).

Now, putting \( \omega = \text{real} \), we have a coupled set of equations:

\[ \xi^+(k, u + \omega) \mathcal{K}(k, u, \omega) \]

\[ = b_k(u, \omega) + \xi_2(k, u) \int_{-\infty}^{+\infty} du' \frac{\mathcal{K}_*(k, u', -\omega)}{u' - (u + \omega) + i\delta} \]  

(A3)

and

\[ \xi^-(k, u - \omega) \mathcal{K}_*(k, u, -\omega) \]

\[ = b_k^*(u, -\omega) + \xi_2(k, u) \int_{-\infty}^{+\infty} du' \frac{\mathcal{K}(k, u', \omega)}{u' - (u - \omega) + i\delta} \]  

(A4)

where \( \delta = \text{small real positive number} \). Since

\[ \frac{1}{u' - u + i\delta} = \frac{p}{u' - u} + i\pi \delta (u - u') \]
and
\[ \varepsilon^+(k,u) = \varepsilon_1(k,u) + i \pi \varepsilon_2(k,u) \]

where
\[ \varepsilon_1(k,u) = \text{Re} \, \varepsilon^+(k,u) \]
\[ = 1 - \frac{\omega^2}{k^2} P \int_{-\infty}^{+\infty} \frac{f_o'(u')}{{u}' - u} \]
\[ \pi \varepsilon_2(k,u) = \text{Im} \, \varepsilon^+(k,u) = - \pi \frac{\omega^2}{k^2} f_o'(u) \]

with \( u = \text{real number} \), (A3) can be rewritten:
\[ \left[ \varepsilon_1(k, u+w) + i \pi \varepsilon_2(k, u+w) \right] \mathcal{K}(k, u, w) \]
\[ = b_k(u, w) + i \pi \varepsilon_2(k, u) \mathcal{K}^*(k, u+w, -w) \]
\[ + \varepsilon_2(k, u) P \int_{-\infty}^{+\infty} \frac{\mathcal{K}^*(k, u'+w, -w)}{u' - u}, \quad (A6) \]

while we have from (A4), putting \( u \to u+w \),
\[ \left[ \varepsilon_1(k, u) - i \pi \varepsilon_2(k, u) \right] \mathcal{K}^*(k, u+w, -w) \]
\[ = b_k^*(u+w, -w) - i \pi \varepsilon_2(k, u+w) \mathcal{K}(k, u, w) \]
\[ + \varepsilon_2(k, u+w) P \int_{-\infty}^{+\infty} \frac{\mathcal{K}(k, u', w)}{u' - u}. \quad (A7) \]
If we subtract (A7) from (A6),

\[
\varepsilon_1(k, u+\omega) \tilde{\mathcal{C}}(k, u, \omega) - \varepsilon_1(k, u) \tilde{\mathcal{C}}^*(k, u+\omega, -\omega) = c_k(u, \omega) + \epsilon_2(k, u) \mathcal{P} \int_{-\infty}^{+\infty} \tilde{\mathcal{C}}^*(k, u', \omega) \frac{u'}{u' - u}
\]

\[
- \epsilon_2(k, u+\omega) \mathcal{P} \int_{-\infty}^{+\infty} \tilde{\mathcal{C}}(k, u', \omega) \frac{u'}{u' - u}
\]

where

\[
c_k(u, \omega) = b_k(u, \omega) - b_k^*(u+\omega, -\omega).
\]

We define:

\[
\tilde{\gamma}(k, z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\tilde{\mathcal{C}}(k, u', \omega)}{u' - z} du'
\]

\[
\gamma(k, z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\tilde{\mathcal{C}}^*(k, u'+\omega, -\omega)}{u' - z} du'
\]

and

\[
\alpha_k(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{c_k(u', \omega)}{u' - z} du'
\]

\[
\eta_k(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{w_2^2}{k^2} \frac{f'_{o}(u')}{u' - z} du'
\]

\[
\zeta_k(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{w_2^2}{k^2} \frac{f'_{o}(u'+\omega)}{u' - z} du'
\]

(A10)
Then

\[ 3\zeta(k, u, \omega) = \phi^+(k, u) - \phi^-(k, u) \quad \{ \text{(A12)} \} \]

and

\[ 3\zeta^*(k, u+w, -\omega) = \psi^+(k, u) - \psi^-(k, u) \]

\[ \epsilon_2(k, u) = \eta^+_k - \eta^-_k \quad \{ \text{(A13)} \} \]

\[ \epsilon_2(k, u+w) = \zeta^+_k - \zeta^-_k \]

\[ \epsilon_1(k, u) = 1 + i\pi(\eta^+_k + \eta^-_k) \]

\[ \epsilon_1(k, u+w) = 1 + i\pi(\zeta^+_k + \zeta^-_k) \]

where, e.g., \( \phi^+(k, z) = \phi(k, z) \) for \( z \) in the upper-half \( z \)-plane and is analytically continued to the lower-half plane across real axis, and similarly, \( \phi^-(k, z) = \phi(z) \) for \( z \) in the lower-half \( z \)-plane and is analytically continued to the upper-half plane. Equation (A8), then, reads

\[ (1+2\pi i \zeta^+_k)\phi^+ - (1+2\pi i \eta^+_k)\psi^+ - \alpha^+_k \]

\[ = (1+2\pi i \zeta^-_k)\phi^- - (1+2\pi i \eta^-_k)\psi^- - \alpha^-_k \quad \{ \text{(A14)} \} \]

We assume the numerator functions of the integrands of the definition integrals (A10) and (A11) all satisfy the Hölder conditions. Then the defined functions are all sectionally holomorphic. (A14), therefore, shows that the function

\[ [1+2\pi i \zeta^+_k(z)] \phi(k, z) - [1+2\pi i \eta^+_k(z)] \psi(k, z) - \alpha^+_k(z) \]
is analytic in the whole $z$-plane, hence must be zero because it vanishes as $|z| \to \infty$, i.e.,

$$\left[1+2\pi i \xi_k(z)\right] \hat{\phi}(k,z) - \left[1+2\pi i \eta_k(z)\right] \psi(k,z) = \alpha_k(z). \quad (A15)$$

It is clear from (A14) that for the assumption of sectional holomorphy of $\phi(k,z)$ and $\psi(k,z)$ to be valid it is necessary that

$$1+2\pi i \zeta_k^+(z) \quad \text{and} \quad 1+2\pi i \eta_k^+(z)$$

should not vanish in the upper-half $z$-plane and, equivalently,

$$1+2\pi i \zeta_k^-(z) \quad \text{and} \quad 1+2\pi i \eta_k^-(z)$$

should not vanish in the lower-half $z$-plane. This is to say that

$$\xi^+(k,z) = 0$$

should not have roots in the upper-half $z$-plane, and this is actually the case for stable plasma.

For the two unknown functions $\phi(k,z)$ and $\psi(k,z)$ we have one relationship (A15). We need another. Let

$$p_k(z,\omega) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} du' \frac{b_k(u',\omega)}{u' - z}$$

and

$$q_k(z,\omega) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} du' \frac{b_k^*(u'+\omega, -\omega)}{u' - z}.$$
Since
\[ \xi^+(k, u+\omega) = 1 + 2\pi i \zeta_k^+(u) \]
and
\[ \xi^-(k, u) = 1 + 2\pi i \eta_k^-(u) , \]
we get from (A7)
\[ (1 + 2\pi i \eta_k^-)(\psi^+ - \psi^-) \]
\[ = (q_k^+ - q_k^-) + 2\pi i (\zeta_k^+ - \zeta_k^-)\phi^- \]
\[ = (1 + 2\pi i \eta_k^-)(\psi^+ + \alpha_k^- - (1 + 2\pi i \zeta_k^-)\phi^-) , \tag{A17} \]
the last equality coming from (A15). Since
\[ \alpha_k(z) = p_k(z, \omega) - q_k(z, \omega) , \]
(A17) leads to
\[ (1 + 2\pi i \eta_k^-)(\psi^+ - (1 + 2\pi i \zeta_k^+)\phi^- = q_k^+ - p_k^- , \]
or
\[ \frac{\psi^+}{1 + 2\pi i \zeta_k^+} - \frac{\phi^-}{1 + 2\pi i \eta_k^-} = \frac{q_k^+ - p_k^-}{(1 + 2\pi i \eta_k^-)(1 + 2\pi i \zeta_k^+)} , \]
i.e., since from (A15)
\[ \frac{\phi^-}{1 + 2\pi i \eta_k^-} = \frac{\alpha_k^-}{(1 + 2\pi i \eta_k^-)(1 + 2\pi i \zeta_k^-)} + \frac{\psi^-}{1 + 2\pi i \zeta_k^-} , \]
\[
\frac{\psi^+}{1+2\pi i \zeta_k^+} - \frac{\psi^-}{1+2\pi i \zeta_k^-} = \frac{1}{1+2\pi i \eta_k^-} \left[ \frac{q_k^+ - p_k^-}{1+2\pi i \zeta_k^+} + \frac{p_k^- - q_k^+}{1+2\pi i \zeta_k^-} \right]. \quad (A18)
\]

Because of stability of plasma,

\[
\frac{\psi^+(k,z)}{1+2\pi i \zeta_k^+(z)}
\]

is analytic in the upper-half \(z\)-plane, and

\[
\frac{\psi^-(k,z)}{1+2\pi i \zeta_k^-(z)}
\]

in the lower-half. Therefore, the solution satisfying the boundary condition (A18) is:

\[
\frac{\psi(z)}{\varepsilon(k,\omega + z)} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} du' \frac{1}{u' - z} \frac{1}{\varepsilon^-(k,u')} \left[ \frac{q_k^+(u') - p_k^-(u')}{\varepsilon^+(k, u' + \omega)} + \frac{p_k^- - q_k^+(u')}{\varepsilon^-(k, u' + \omega)} \right].
\]

(A19)

Now, from (A6) with the help of (A19), we have

\[
3\zeta(k,u,\omega) = \delta^+(k,u) - \delta^-(k,u)
\]

\[
= \frac{b_k(u,\omega)}{1+2\pi i \zeta_k^+} + \varepsilon_2(k,u) \frac{\psi^+(k,u)}{1+2\pi i \zeta_k^+}
\]

\[
= \frac{b_k(u,\omega)}{\varepsilon^+(k,u+\omega)}
\]
\[ \mathcal{K}(k,u) = \frac{b_k(u)}{\varepsilon^+(k,u+\omega)} + \varepsilon_2(k,u) \int_{-\infty}^{+\infty} \frac{du'}{u' - (u+i\delta)} \frac{q^+_k(u',\omega) - p^-_k(u',\omega)}{\varepsilon^-(k,u') \varepsilon^+(k,u'+\omega)} \]

The last integral vanishes, because the integrand is analytic in the lower-half plane of \( u' \) and the contour can be closed in the lower-half plane. Thus we have

\[ \mathcal{K}(k,u,\omega) = \frac{b_k(u)}{\varepsilon^+(k,u+\omega)} \]

This is the solution for the case of stable plasma.

The solution for unstable plasma is obtained, following Balescu, by analytically continuing (A20) with respect to \( \gamma_k \), thus deforming the line of integration such that the pole arising from \( \varepsilon^+(k,u'+\omega) = 0 \) remain below it and that from \( \varepsilon^-(k,u') = 0 \) above it (Fig. A).
We designate the line of integration as $L$, which is to be understood as shown in Fig. A. We have, then,

$$\mathcal{F}(k,u,\omega) = \frac{b_k^{+}(u,\omega)}{\epsilon^{+}(k, u+\omega)} + \epsilon_2(k,u) \int_{L} du' \left( \frac{q_k^{+}(u',\omega) - p_k^{-}(u',\omega)}{(u'-u-i\delta) \epsilon^{-}(k,u')} \right) \epsilon^{+}(k, u'+\omega),$$

(A21)

where, from (A16), since $\omega$ is now complex,
By definition (33)

\[ H(k,u,\omega) = \frac{\mathcal{C}(k,u,\omega)}{k(u+\omega)} \]

and

\[ H(k,u,t) = \frac{k}{2\pi} \int_{c} d\omega \, H(k,u,\omega) \, e^{-ik\omega t} \]

\[ = \frac{1}{2\pi} \int_{c} d\omega \, \frac{\mathcal{C}(k,u,\omega)}{u+\omega} \, e^{-ik\omega t} \]  

(A23)

where the line of integration \( c \) lies, in the upper-half \( \omega \)-plane, parallel to the real axis and above all the singularities of \( H(k,u,\omega) \).

We calculate \( H(k,u,t) \) explicitly. Let

\[ \int_{-\infty}^{+\infty} du \, \text{Re} \, H(k,u,t) \equiv R_k(t) \]

\[ \int_{-\infty}^{+\infty} du \, \text{Im} \, H(k,u,t) \equiv I_k(t). \]  

(A24)
Then, referring to (35) for the definition of \( b_k(u, \omega) \), we have

\[
b_k(u, \omega) = i \frac{\alpha_0(k, u, \omega)}{\omega} + i \frac{4\pi e^2}{k^3} \frac{1}{\omega} f_o(u) - \frac{k}{\omega} \epsilon_2(k, u) \left[ ku V_k(\omega) + n R_k(\omega) - \text{in } I_k(\omega) \right]
\]

and

\[
b_k^*(u, - \omega^*) = - i \frac{\alpha_0^*(k, u, - \omega^*)}{\omega} + i \frac{4\pi e^2}{k^3} \frac{1}{\omega} f_o(u) - \frac{k}{\omega} \epsilon_2(k, u) \left[ ku V_k(\omega) + n R_k(\omega) + \text{in } I_k(\omega) \right].
\]

In the last expression we used the relation

\[
V_k(-\omega^*) = V_k^*(\omega), \text{ etc.},
\]

which comes from reality of \( V_k(t) \), \( R_k(t) \) and \( I_k(t) \). We have from (A22) and (A25):

\[
p_k^-(u, \omega) = \frac{1}{2\pi} A_o^-(k, u, \omega) + \frac{1}{2\pi} \frac{4\pi e^2}{k^3} \frac{1}{\omega} \phi_o^-(u)
\]

\[
+ \frac{1}{2\pi} \frac{k}{n} \left[ 1 - \epsilon^-(k, u) \right] \left[ ku V_k(\omega) + n R_k(\omega) - \text{in } I_k(\omega) \right]
\]

\[
a_k^+(u, \omega) = \frac{1}{2\pi} A_o^+(k, u, \omega) + \frac{1}{2\pi} \frac{4\pi e^2}{k^3} \frac{1}{\omega} \phi_o^+(u+\omega)
\]

\[
+ \frac{1}{2\pi} \frac{k}{n} \left[ 1 - \epsilon^+(k, u+\omega) \right] \left[ k(u+\omega) V_k(\omega) + n R_k(\omega) + \text{in } I_k(\omega) \right]
\]

(A26)
where

\[
A_o(k,z,w) = \int_{-\infty}^{+\infty} \frac{a_o(k,u,w)}{u-z} \, du
\]

and

\[
\phi_o(z) = \int_{-\infty}^{+\infty} \frac{f_o(u)}{u-z} \, du.
\]

In (A26) use is made of the relation

\[
a_o^*(k,\nu,\omega) = -a_o(-k,\nu,\omega)
\]

which comes from the definition (27) of \(a_o(k,\nu,\omega)\). The form of the expressions (A21) and (A26) suggests to split \(H(k,u,t)\) in four terms:

\[
H(k,u,t) = h_1(k,u,t) + h_2(k,u,t) + h_3(k,u,t) + h_4(k,u,t)
\]  

with

\[
h_1(k,u,t) = \frac{1}{2\pi} \int_c^\infty dw \frac{b_k(u,w)}{(u+w) \varepsilon^+(k,u+w)} e^{-ikwt}
\]

\[
h_2(k,u,t) = \frac{1}{(2\pi)^2} \int_c^\infty dw \frac{\varepsilon_2(k,u)}{u+w} e^{-ikwt} \int_{L'} du' \frac{A_o^+(-k,u'+w,\omega) - A_o^-(k,u',\omega)}{(u'-u-i\delta) \varepsilon^-(k,u') \varepsilon^+(k,u'+w)}
\]

\[
h_3(k,u,t) = \frac{1}{(2\pi)^2} \int_c^\infty dw \frac{\varepsilon_2(k,u)}{w(u+w)} e^{-ikwt} \int_{L'} du' \frac{\phi_o^+(u'+w) - \phi_o^-(u')}{(u'-u-i\delta) \varepsilon^-(k,u') \varepsilon^+(k,u'+w)}
\]
and

\[ h_4(k,u,t) = \frac{1}{2\pi} \int \frac{d\omega}{c} \epsilon_2(k,u) \left\{ e^{-ik\omega t} \left[ \frac{1}{2\pi i} \frac{1}{n} \int \frac{du}{u^1-u-i\delta} \right] \times \right. \]

\[ \times \left[ \frac{i V_k^0}{\epsilon^+(k,u') \epsilon^-(k,u'+\omega)} \right] \right. \]

\[ - \frac{k(u'+\omega) V_k(\omega) + n R_k(\omega) + \text{in} I_k(\omega)}{\epsilon^-(k,u')} \]

\[ + \frac{ku' V_k(\omega) + n R_k(\omega) - \text{in} I_k(\omega)}{\epsilon^+(k,u'+\omega)} \bigg] \],

(A29d)

where, in (A29d),

\[ V_k^0 \equiv \text{initial value of } V_k(t) \]

(A30)

and

\[ kw V_k(\omega) + 2\text{in} I_k(\omega) = i V_k^0. \]

This equation comes from (19) with the definitions (24) and (37).

In calculating these four terms, we note the following:

1) \( g(k,\nu, t) \), hence also \( H(k,u,t) \), cannot have a pole at

\[ u = \frac{k \cdot v}{k} = 0. \]

Since

\[ n g(k,\nu, t) \equiv \left( \frac{4\pi \epsilon_0^2}{k^2} \right)^{\frac{1}{2}} \langle Q_k c_p^+ c_{p-k} \rangle \]

and

\[ \epsilon_p - \epsilon_{p-k} \xrightarrow[h\to 0]{} k \cdot \nu = ku, \]
a pole at \( u = 0 \) means a resonant field-electron scattering without change in energy of the electron, which cannot be a real scattering process. This is the reason why we disregard those terms like

\[
\frac{ikut}{e u} \quad \text{and} \quad \frac{1}{u - \frac{2i\gamma_k}{k}}.
\]

2) We may also disregard those terms varying rapidly in time, such as

\[
\frac{ikut}{e} \quad \text{and} \quad e^{2i\omega t}.
\]

We should, however, retain the terms like, e.g.,

\[
\frac{ik(u - \tilde{v}_k^+)}{e (u - \tilde{v}_k^+)}
\]

because

\[
\frac{ik(u - \tilde{v}_k^+)}{e (u - \tilde{v}_k^+)} \rightarrow \begin{cases} 
0 & (\text{stable}) \\
2\pi i \delta(u - v_k) & (\text{unstable})
\end{cases}
\]

(A31)

where

\[
v_k = \frac{\omega_k}{k}.
\]

3) As is noted in (A1) and (A2),
\[ \varepsilon^+(k, \tilde{v}_k^+) = 0 = \varepsilon^+(k, \tilde{u}_k^+) . \] (A32)

We have, therefore,

\[ \varepsilon^+(k, z) = \begin{cases} 
\sigma_k^{+}(\tilde{v}_k^+) (z - \tilde{v}_k^+) & \text{near } z = \tilde{v}_k^+ \\
\sigma_k^{+}(\tilde{u}_k^+) (z - \tilde{u}_k^+) & \text{near } z = \tilde{u}_k^+ 
\end{cases} \] (A33)

with

\[ \sigma_k^{+}(z) = \frac{\partial}{\partial z} \varepsilon^+(k, z). \] (A34)

If we calculate, e.g., \( \sigma_k^{+}(\tilde{v}_k^+) \),

\[ \sigma_k^{+}(\tilde{v}_k^+) = \int_{-\infty}^{+\infty} du \frac{\varepsilon_2(k, u)}{(u - \tilde{v}_k^+)^2} \]

\[ = \frac{w_p}{k^2} \int_{-\infty}^{+\infty} du \frac{\bar{f}_0'(u)}{u_k^2} \left( 1 + 2 \frac{u}{u_k} + 3 \frac{u^2}{u_k^2} + 4 \frac{u^3}{u_k^3} \right) \]

\[ = \frac{w_p}{k^2} \frac{2}{u_k^2} \left( 1 + \frac{6 \langle u^2 \rangle}{u_k^2} \right), \]

where it is assumed

\[ \int_{-\infty}^{+\infty} du \ u \bar{f}_0(u) \equiv \langle u \rangle = 0. \] (A35)
Since
\[ k^2 u_k^2 = \omega_k^2 \simeq \frac{\omega_p^2}{p} + 3 k^2 \langle u^2 \rangle, \]
we get
\[ \sigma_k^{(+)}(\tilde{v}_k^+) \simeq 2 k \frac{\omega_k}{\omega_p^2}. \quad (A36) \]

From (A33), since \( \text{Im} \frac{1}{\varepsilon(k,u)} \) with \( u = \text{real} \) is appreciable only in the neighborhood of \( u = \frac{\omega_k}{k} \) and \( -\frac{\omega_k}{k} \), we have, e.g.,
\[ \frac{\text{Im} \frac{1}{\varepsilon(k,u)}}{\sigma_k^{(-)}(\tilde{v}_k^-)(u - \tilde{v}_k^-)} \]
\[ + \frac{\text{Im} \frac{1}{\varepsilon(k,u)}}{\sigma_k^{(-)}(\tilde{u}_k^-)(u - \tilde{u}_k^-)}. \]

If we apply Cauchy's theorem:
\[ \oint_{\Gamma} \frac{1}{u' - (u+i\delta)} \left[ \frac{1}{\varepsilon(k,u')} - 1 \right] = 0 \]

where the contour \( \Gamma \) runs along the real axis and round the lower-half \( u' \)-plane, we have, for real \( u \),
\[ \frac{1}{\varepsilon(k,u)} \simeq 1 + \frac{1}{\sigma_k^{(-)}(\tilde{v}_k^-)(u - \tilde{v}_k^-)} + \frac{1}{\sigma_k^{(-)}(\tilde{u}_k^-)(u - \tilde{u}_k^-)}. \quad (A37) \]

Now we proceed to calculate \( h_1(k,u,t) \) first. From (A25),
\[ h_1(k,u,t) = \frac{1}{2\pi} \int dw \frac{b_k(u,\omega)}{c (u+\omega) \in^+ (k, u+\omega)} e^{-i\omega t} \]

\[ = \frac{1}{2\pi} \int dw \frac{e^{-i\omega t}}{c (u+\omega) \in^+ (k, u+\omega)} \left\{ i \frac{4\pi e^2}{k^3} \frac{1}{\omega} \frac{1}{\bar{f}_o(u)} \right\} \]

\[ + i \bar{a}_o(k, u, \omega) \]

\[ - \frac{k}{n} \varepsilon_2(k,u) \left[ ku \nu_k(w) + n R_k(w) - i n I_k(w) \right] \]

\[ = \frac{4\pi e^2}{k^3} \frac{1}{\bar{f}_o(u)} \left[ \frac{1}{u \in^+ (k,u)} - e \frac{-i\omega t}{\sigma_k(\nu_k^+) \nu_k^+ (u - \nu_k^+)} \right. \]

\[ - e \frac{-i\omega t}{\sigma_k(\nu_k^+) \nu_k^+ (u - \nu_k^+)} \]

\[ - \frac{k}{2\pi n} \varepsilon_2(k,u) \int dw \frac{e^{-i\omega t}}{c (u+\omega) \in^+ (k, u+\omega)} \left[ ku \nu_k(w) + n R_k(w) - i n I_k(w) \right] \]

(A38)

where the terms involving \( \bar{a}_o(k,u,\omega) \) are disregarded because they vary rapidly in time.

For \( h_2(k,u,t) \), we perform \( u' \)-integration first. Since \( A_o^+(-k, u'+\omega, w) \) is holomorphic in the upper-half \( u' \)-plane and

\[ A_o^+(-k, u'+\omega, w) \xrightarrow{|u'| \to \infty} \sim \frac{1}{u'} , \]
we can close the contour of the following integral in the upper-half plane, i.e.,

\[
\int_{\mathcal{L}} du' \frac{A^+_o(-k, u' + \omega, \omega)}{(u' - u - i\delta) \in^-(k, u') \in^+(k, u' + \omega)}
\]

\[
= 2\pi i \frac{A^+_o(-k, u + \omega, \omega)}{\in^-(k, u) \in^+(k, u + \omega)}
+ 2\pi i \left[ \frac{A^+_o(-k, \tilde{v}_k^- + \omega, \omega)}{\sigma_k^-(\tilde{v}_k^-) \in^+(k, \tilde{v}_k^- + \omega)} + \frac{A^+_o(-k, \tilde{\nu}_k^- + \omega, \omega)}{\sigma_k^-(\tilde{\nu}_k^-) \in^+(k, \tilde{\nu}_k^- + \omega)} \right].
\]

If we perform \(\omega\)-integration, we have

\[
\frac{1}{(2\pi)^2} \int_{\mathcal{C}} \frac{e^{-ik\omega t}}{u + \omega} \left( \int_{\mathcal{L}} du' \frac{A^+_o(-k, u' + \omega, \omega)}{(u' - u - i\delta) \in^-(k, u') \in^+(k, u' + \omega)} \right)

= e^{ik(u-\tilde{v}_k^+)} \frac{A^+_o(-k, \tilde{v}_k^+, \tilde{v}_k^- - u)}{\sigma_k^+(\tilde{v}_k^+) \tilde{v}_k^- \in^-(k, u)}

+ e^{ik(u-\tilde{\nu}_k^+)} \frac{A^+_o(-k, \tilde{\nu}_k^+, \tilde{\nu}_k^- - u)}{\sigma_k^+(\tilde{\nu}_k^+) \tilde{\nu}_k^- \in^-(k, u)}

- \frac{2\gamma_k t}{e^{\sigma_k^+(\tilde{v}_k^-) \tilde{v}_k^- \sigma_k^-(\tilde{v}_k^-) (u - \tilde{v}_k^-)}} A^+_o(-k, \tilde{v}_k^+, - \frac{2i\gamma_k}{k})

- \frac{2\gamma_{-k} t}{e^{\sigma_k^+(\tilde{\nu}_k^-) \tilde{\nu}_k^- \sigma_k^-(\tilde{\nu}_k^-) (u - \tilde{\nu}_k^-)}} A^+_o(-k, \tilde{\nu}_k^+, - \frac{2i\gamma_{-k}}{k})
\]
We make, e.g. the following approximation:

\[ \frac{\text{ik}(u-v_+^k)t}{\varepsilon^-(k,u)} \approx \frac{\text{ik}(u-v_+^k)t}{\sigma_k^-(v_+^k)(u-v_+^k)}. \]  

(A39)

This is because the numerator \( e^{ik(u-v_+^k)t} \) requires \( |u-v_+^k| \) to be small in order not to be varying rapidly in time. We then, have

\[
\frac{1}{(2\pi)^2} \int_{c} \frac{-ik\omega t}{u+w} \text{d}u - \frac{1}{(2\pi)^2} \int_{L} \text{d}u' \frac{A^+(k, u'+\omega, \omega)}{(u'-u-i\delta) \varepsilon^-(k,u') \varepsilon^+(k, u'+\omega)}.
\]

Similarly, with the contour of \( u' \)-integration now closed in the lower-half plane,

\[
\frac{1}{(2\pi)^2} \int_{c} \frac{-ik\omega t}{u+w} \text{d}u - \frac{1}{(2\pi)^2} \int_{L} \text{d}u' \frac{A^-(k, u', \omega)}{(u'-u-i\delta) \varepsilon^-(k,u') \varepsilon^+(k, u'+\omega)}.
\]
Since, from (A27),
\[ A^+(k, \tilde{\nu}_k^+, -\frac{2i\gamma}{k}) = -\left[ A^-(k, \tilde{\nu}_k^-, -\frac{2i\gamma}{k}) \right]^* \]
we have
\[ h_2(k,u,t) = \frac{2 \varepsilon_2(k,u) e^{-2\gamma t} \text{Re} A^-_o(k, \tilde{u}_k^-, -\frac{2i\gamma}{k})}{\sigma_k(\tilde{\nu}_k^+)(\tilde{u}_k^+ - \tilde{\nu}_k^-)(u-\tilde{\nu}_k^-)} \left[ 1 - e^{ik(u-\tilde{\nu}_k^-)t} \right]. \]
\[ + \frac{2 \varepsilon_2(k,u) e^{-2\gamma t} \text{Re} A^-_o(k, \tilde{\nu}_k^-, -\frac{2i\gamma}{k})}{\sigma_k(\tilde{\nu}_k^+)(\tilde{\nu}_k^- - \tilde{u}_k^+)(u-\tilde{\nu}_k^-)} \left[ 1 - e^{ik(u-\tilde{\nu}_k^-)t} \right]. \]
\[ (A40) \]

For \( h_3(k,u,t) \) (A29c) we proceed in a similar way as for \( h_2(k,u,t) \) above.

\[ \int_{L} du' \frac{\phi^+_o(u'+\omega) - \phi^-_o(u')}{(u'-u-i\delta) \in^-_o(k,u') \in^+_o(k,u'+\omega)} \]
\[ = \frac{2\pi i \phi^+_o(u+\omega)}{\in^-_o(k,u) \in^+_o(k,u+\omega)} \]
\[ - \frac{2\pi i \phi^+_o(\tilde{\nu}_k^- + \omega)}{\sigma_k(\tilde{\nu}_k^-)(u-\tilde{\nu}_k^-) \in^+_o(k, \tilde{\nu}_k^- + \omega)} - \frac{2\pi i \phi^-_o(\tilde{\nu}_k^+ - \omega)}{\sigma_k(\tilde{\nu}_k^+)(u-\tilde{\nu}_k^+ + \omega) \in^-_o(k, \tilde{\nu}_k^+ - \omega)} \]
\[ - \frac{2\pi i \phi^-_o(\tilde{u}_k^+ - \omega)}{\sigma_k(\tilde{\nu}_k^+)(u-\tilde{\nu}_k^+ + \omega) \in^-_o(k, \tilde{u}_k^+ - \omega)} - \frac{2\pi i \phi^-_o(\tilde{u}_k^+ + \omega)}{\sigma_k(\tilde{\nu}_k^+)(u-\tilde{\nu}_k^+ + \omega) \in^-_o(k, \tilde{u}_k^+ - \omega)}. \]
Therefore,

\[ h_3(k,u,t) = \frac{1}{2\pi i} \frac{4\pi}{k^3} \int \frac{\phi_o^+(u') - \phi_o^-(u')}{u' - u - i\delta} \frac{du'}{L} \]

\[ \left\{ \begin{array}{c}
\phi_o^+(\tilde{\nu}_k^+) e^{ik(u-\tilde{\nu}_k^+)t} \\
\sigma_k^+(\tilde{\nu}_k^+)\tilde{\nu}_k^+(\tilde{\nu}_k^+ - u) \in^-(k,u)
\end{array} \right\}
\]

\[ + \left\{ \begin{array}{c}
\phi_o^-(u^-) e^{ik(u-\tilde{\nu}_k^-)t} \\
\sigma_k^-(\tilde{\nu}_k^-)\tilde{\nu}_k^-(-\tilde{\nu}_k^- - \frac{2i\gamma_k}{k})(u-\tilde{\nu}_k^-)
\end{array} \right\}
\]

where the first term comes from \( \omega = 0 \) pole. We use, e.g., the following approximation, similar to (A39):

\[ \frac{ik(u-\tilde{\nu}_k^+) e}{(u-\tilde{\nu}_k^+) \in^- (k,u)} \approx \frac{ik(u-\tilde{\nu}_k^+) t}{(u-\tilde{\nu}_k^+) \sigma_k^- (\tilde{\nu}_k^-) (u-\tilde{\nu}_k^-)} \]

\[ = \frac{ik(u-\tilde{\nu}_k^+) t}{\sigma_k^- (\tilde{\nu}_k^-) (\frac{2i\gamma_k}{k})(u-\tilde{\nu}_k^+)} \]
We note
\[ \phi_o^+(u) - \phi_o^-(u) = 2\pi i \bar{f}_o(u) \]
and
\[ \phi_o^+(\nu_k^+) - \phi_o^-(\nu_k^-) \simeq \phi_o^+(\nu_k) - \phi_o^-(\nu_k) = 2\pi i \bar{f}_o(\nu_k). \]

Thus we have

\[
\begin{align*}
    h_3(k,u,t) &= \frac{4\pi e^2}{k^3} \frac{\xi_2(k,u)}{u} \int_{u'} L \left( \frac{2\pi i \bar{f}_o(u')}{(u'-u-i\delta)|\xi(k,u')|^2} \right) \\
    &\quad - \frac{4\pi e^2}{k^3} \xi_2(k,u) \left[ \frac{2\pi i \bar{f}_o(\nu_k^-) e^{\frac{ik(u-\nu_k^+)}{t}}}{\sigma_k^+(\nu_k^-) \sim_k^+ \sigma_k^-(\nu_k^-)} \left( \frac{2i\gamma_k}{k} \right)(u-\nu_k^+) \right. \\
    &\quad \left. + \frac{2\pi i \bar{f}_o(u_k^-) e^{\frac{ik(u-u_k^+)}{t}}}{\sigma_k^+(u_k^-) \sim_k^+ \sigma_k^-(u_k^-)} \left( \frac{2i\gamma_k}{k} \right)(u-u_k^+) \right] \\
    &\quad - \frac{4\pi e^2}{k^3} \xi_2(k,u) \left\{ \frac{2\pi i \bar{f}_o(\nu_k^-) e^{-2\gamma_k t}}{\sigma_k^+(\nu_k^-) \sim_k^+ \sigma_k^-(\nu_k^-)} \left[ 1 - e^{\frac{-ik(u-\nu_k^-) t}{2\gamma_k}} \right] \right. \\
    &\quad \left. + \frac{2\pi i \bar{f}_o(u_k^-) e^{-2\gamma_k t}}{\sigma_k^+(u_k^-) \sim_k^+ \sigma_k^-(u_k^-)} \left[ 1 - e^{\frac{-ik(u-u_k^-) t}{2\gamma_k}} \right] \right\}.
\end{align*}
\]
In performing $u'$-integration of $h_4(k,u,t)$ of (A29d), we substitute the approximate expression (A37) for $\xi^-(k,u)$ in the integrand. We see, then, that the new integrand vanishes like $1/u'^2$ when $|u'| \to \infty$. The contour, therefore, can be closed in the upper-half $u'$-plane. Let

$$\xi_o^-(k,u) \equiv \text{right-hand side of (A37)}. \quad (A43)$$

Then, we have

$$h_4(k,u,t) = \frac{k}{2\pi i} \int_{c} \frac{\xi_2(k,u)}{u+\omega} \ e^{-ik\omega t} \left[ \frac{i V_k^O}{\xi_o^-(k,u) \xi^+(k,u+\omega)} \right]$$

$$= -\frac{k}{2\pi i} \xi_2(k,u) \int_{c} d\omega \ e^{-ik\omega t} \left[ kV_k(\omega) + \frac{nR_k(\omega) + \text{in}I_k(\omega)}{u+\omega} \right]$$
\[ + \frac{k}{2\pi n} \varepsilon_2(k,u) \int_{c}^{\infty} dw \frac{e^{-i\omega t}}{u+w} \frac{ku V_k(\omega) + nR_k(\omega) - \text{in} \: I_k(\omega)}{\varepsilon^+(k,u+w)} \]

\[ - \frac{k}{n} \varepsilon_2(k,u) V_k^o \left\{ \frac{-2\gamma_k t}{e} \frac{\varepsilon_k^+(\varepsilon_k^+ \varepsilon_k^- \varepsilon_k^- (u-\tilde{\nu}_k^-)}{\sigma_k^+(\varepsilon_k^+ \varepsilon_k^- \varepsilon_k^- (u-\tilde{\nu}_k^-) - \sigma_k^+(\tilde{\nu}_k^+ \tilde{\nu}_k^+ \varepsilon_o^-(k,u))} \right\} + \frac{-2\gamma_k t}{e} \frac{\varepsilon_k^+(\varepsilon_k^+ \varepsilon_k^- \varepsilon_k^- (u-\tilde{\nu}_k^-)}{\sigma_k^+(\varepsilon_k^+ \varepsilon_k^- \varepsilon_k^- (u-\tilde{\nu}_k^-) - \sigma_k^+(\tilde{\nu}_k^+ \tilde{\nu}_k^+ \varepsilon_o^-(k,u))} \right\} . \]

Since \( R_k(t) \) and \( I_k(t) \) as defined by (A24) vary slowly in time, probably like \( e^{-2\gamma_k t} \), the term

\[ \int_{c}^{\infty} dw \frac{n R_k(\omega) + \text{in} \: I_k(\omega)}{u+w} e^{-i\omega t} \]

will turn out to have a pole on the imaginary axis near \( u=0 \) of \( u \)-plane, like \( 1 \left( u - \frac{2i\gamma_k}{k} \right) \), hence can be disregarded. We can also approximate, e.g.,

\[ \frac{\varepsilon_k^+(\varepsilon_k^+ \varepsilon_k^- \varepsilon_k^- (u-\tilde{\nu}_k^-)}{\sigma_k^+(\varepsilon_k^+ \varepsilon_k^- \varepsilon_k^- (u-\tilde{\nu}_k^-) - \sigma_k^+(\tilde{\nu}_k^+ \tilde{\nu}_k^+ \varepsilon_o^-(k,u))} \cong \frac{\varepsilon_k^+(\varepsilon_k^+ \varepsilon_k^- \varepsilon_k^- (u-\tilde{\nu}_k^-)}{\sigma_k^+(\varepsilon_k^+ \varepsilon_k^- \varepsilon_k^- (u-\tilde{\nu}_k^-) - \sigma_k^+(\tilde{\nu}_k^+ \tilde{\nu}_k^+ \varepsilon_o^-(k,u))} . \]

We have, then

\[ h_4(k,u,t) = - \frac{k}{n} \varepsilon_2(k,u) V_k(t) \]

\[ + \frac{k}{2\pi n} \int_{c}^{\infty} dw \frac{\varepsilon_2(k,u)}{u+w} e^{-i\omega t} \frac{ku V_k(\omega) + nR_k(\omega) - \text{in} \: I_k(\omega)}{\varepsilon^+(k,u+w)} \]
In summing up (A38), (A40), (A42) and (A44), we note the definition of $\gamma_k$:

$$\gamma_k \equiv \frac{\pi k \varepsilon_2(k,v_k)}{\sigma_k(v_k)} \quad (A45)$$

where

$$\sigma^+_k(v_k^+) \simeq \sigma^-_k(v_k^-) \simeq \sigma_k(v_k) \equiv \text{Re} \int_{-\infty}^{+\infty} \frac{\varepsilon_2(k,u)}{(u-v_k^+)^2} \quad (A46)$$

Then we have

$$H(k,u,t) = -\frac{k}{n} \varepsilon_2(k,u) v_k(t)$$

$$+ \frac{4\pi e^2}{k^3} \left[ \frac{\bar{f}_0(u)}{u} + \frac{\varepsilon_2(k,u)}{u} \int_{u'}^L \frac{\bar{f}_0(u')}{(u'-u-i\delta)\varepsilon(k,u')^2} \right]$$

$$\varepsilon_2(k,u) \left[ \frac{1 - e^{i(k-u_k^-)t}}{(u-v_k^-)^+} \right] c_k e^{-2\gamma_k t}$$

$$\varepsilon_2(k,u) \left[ \frac{1 - e^{i(k-u_k^-)t}}{(u-u_k^-)^+} \right] c_k e^{-2\gamma_k t} \quad (A47)$$
where

\[ C_k = \frac{k/n}{\sigma_k(v_k)} \left[ v_k^0 - \frac{2n}{k} \text{Re} A_o ^\prime \left( k, \tilde{\nu}_k, - \frac{2i\gamma_k}{k} \right) - \frac{f_o(v_k)}{2\gamma_k} \frac{8n^2 ne^2}{k^3} \right]. \]

In (A47) use is made of the relations:

\[ f_o(u_k) = f_o(v_k). \]

and

\[ \text{Re} A_o ^\prime \left( k, \tilde{u}_k, - \frac{2i\gamma_k}{k} \right) = \text{Re} A_o ^\prime \left( -k, \tilde{\nu}_k', - \frac{2i\gamma_k}{k} \right). \]

Let

\[ \varepsilon_o = \frac{2k^2}{\sigma_k^2(v_k)} \left[ v_k^0 - \frac{2n}{k} A_o ^\prime \left( k, \tilde{\nu}_k, - \frac{2i\gamma_k}{k} \right) \right], \]

(A48)

then

\[ C_k = \frac{1}{2nk} \left[ \varepsilon_o \frac{f_o(v_k)}{2\gamma_k} \frac{16n^2 ne^2}{k \sigma_k^2(v_k)} \right]. \]

(A49)

\( \varepsilon_o \) is identified with the initial value of the field energy.
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Integral Equation
Pines-Schrieffer Equations