COMMUNICATION NETWORKS
WITH SPECIFIED SURVIVABILITY

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ABSTRACT

This paper considers the problem of the survivability of communication networks from the point of view of graph theory. Stations are represented by nodes of a graph and communication links between stations are represented by edges of a graph. The strength of a graph is defined with respect to nodes only, edges only, and both nodes and edges. Two methods of constructing graphs with specified strength are derived. Planar graphs are studied and a bound on their strength is found. Methods of finding the strength of an arbitrary graph are given.
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1. INTRODUCTION

A communication network is composed of stations and communication links of some sort which join the stations together. If the network is to fulfill its function the stations must be able to communicate with each other. One of the many things which might prevent this is the physical destruction of sections of the network. This destruction would have to take one of three forms: destruction of stations; destruction of links joining the stations; or a combination of both. The problem considered in this paper is the design of a communication network with a certain number of stations so that it has the following property. If a specified number of stations, links, or stations and links are destroyed, the remaining network is still operational.

Since graph theoretic concepts can be easily applied to communication networks, that approach will be used here. The network will be represented by a linear graph which is a collection of line segments whose intersections are called nodes. A line segment between two nodes is called an edge. The nodes of the graph will represent stations and the edges of the graph will represent links. The destruction of stations will be equivalent to the removal of nodes from the graph, and the destruction of links will be equivalent to the removal of edges.

A solution to the problem for station destruction is arrived at using these concepts. This solution is shown to apply when link destruction
is considered and also when the destruction of both stations and links is allowed.

The reader is referred to Seshu and Reed, [3], for definitions of any standard terms used in graph theory which appear in this paper.
2. STRENGTH OF GRAPHS WITH RESPECT TO NODES

The removal of nodes from a graph will be considered first. The problem can be restated as follows. Given a number of nodes with which to form a nonseparable graph, how should the graph be formed so that a specific number of nodes must be removed before the remaining graph is separated into two or more subgraphs not connected to each other. We denote by $G_k$, a subgraph of $k$ nodes.

**Definition 1**

A nonempty connected subgraph $G_k$ of a graph $G$ will be called $n$-node-attached if the removal of at least $n$ nodes is required to separate $G_k$ from the rest of the graph.

With regard to node removal a vector $S_N = [s_1, s_2, \ldots, s_m]$ can now be formed. $s_1$ is the minimum node attachment of the $G_1$ subgraphs of $G$, $s_2$ is the minimum node attachment of the $G_2$ subgraphs of $G$, and so on through $s_m$ which is the minimum node-attachment of the $G_m$ subgraphs of $G$ where $G_m$ is the largest subgraph which can be separated and still leave some nodes remaining in the rest of $G$.

**Definition 2**

A graph $G$ is $n$-node-strong if every element of its $S_N$ vector is $\geq n$.

With this background the problem can be attacked. If only one node is to be removed the solution is to connect the given nodes in a circuit. It
can be seen that the $S_N$ vector of a circuit is composed entirely of 2's, and a circuit is therefore 2-node-strong. The problem is more difficult when two or more nodes are to be removed.

Let A and B be two n-node-strong graphs. A new graph G can be formed by establishing edges between A and B such that each node of A is connected to some node of B and each node of B is connected to some node of A. A graph formed by the preceding process will be called an $\alpha$-graph.

**Theorem 1**

An $\alpha$-graph G formed from two n-node-strong graphs A and B is $(n+1)$-node-strong.

**Proof**

Since each node of A is connected to a node of B, we see that a subgraph $G_k$ of G which is also a subgraph of A is at least $(n+1)$-node-attached. The same could be said if $G_k$ were a subgraph of B.

If $G_k$ were partly in A and partly in B both A and B would have to be separated in order to separate $G_k$ from the rest of G. Therefore $G_k$ would be $2n$-node-attached. Thus the theorem.

Graphs of any strength desired can be constructed using the results of Theorem 1. A sequence of $\alpha$-graphs can be formed by starting with two 2-node-strong circuits which can be joined together to form a 3-node-strong $\alpha$-graph as shown in Figure 1. Then two of these can be joined to form a 4-node-strong $\alpha$-graph, etc. However, the most impor-
Figure 1. Construction of 3-node-strong graph from two 2-node-strong graphs
tant result of the theorem is the fact that given any graph known to be n-node-
strong, an (n+1)-node-strong graph can be easily constructed.

Consider a special class of α-graphs which are constructed from
two n-node-strong graphs D and E with identical numbers of nodes. Estab­
lish a one to one correspondence between the nodes of D and E, then join
the corresponding nodes together by edges. The resulting graph will be called
a symmetrical α-graph.

Theorem 2

Let an n-node-strong, n ≥ 3, symmetrical α-graph G be formed
from two (n-1)-node-strong graphs D and E each having the same number
of nodes. Then G is (n+1)-node-strong if each subgraph formed by the sep­
aration of G is restricted to having at least two nodes.

Proof

Let $G_k, k > 1$, be a subgraph of G. If $G_k$ is a subgraph of D, it
is $(n-1+2) = (n+1)$-node-attached because each node of D shares an incident
edge with a different node in E. The same could be said if $G_k$ were a sub-
graph of E. If $G_k$ is partly in D and partly in E, it is 2(n-1)-node-attached
because both D and E would have to be separated in order to separate $G_k$
from the rest of G. Since $(n+1) ≤ 2(n-1)$ for $n ≥ 3$, the theorem follows.

Let C be an n-node-strong graph and let there be a node D not in
C. A new graph G can be formed by establishing edges from every node of
C to D. A graph formed in this way will be called a β-graph.
Theorem 3

A $\beta$-graph $G$ formed from an $n$-node-strong graph $C$ is $(n+1)$-node strong.

Proof

A subgraph $G_k$ of $G$ which is also a subgraph of $C$ will be $(n+1)$-node-strong, because every node of $C$ is connected to node $D$. If $G_k$ includes node $D$, $G_k$ cannot be separated from the rest of $G$ because such subgraphs are connected to every node in the rest of $G$. And of course node $D$ by itself is at least $(n+1)$-node-strong because it is connected to every node in $C$. The theorem follows.

A sequence of $\beta$-graphs could be formed with a four node circuit, the smallest 2-node-strong graph, as the starting point. This is shown in Figure 2. The addition of each new node would increase both the total number of nodes and the strength of the graph by one. If there were $N$ nodes in one of these graphs, the graph would be $(N-2)$-node-strong. By definition this is the strongest an $N$ node graph can be because there must be at least two nodes remaining to form two subgraphs after the graph has been separated by removing nodes. The nodes in the original four node circuit are most vulnerable because they are not connected to every other node by a single edge. Each new node is, however, and consequently the new nodes cannot be separated.

This sequence of $\beta$-graphs, then, would yield the strongest graphs
Figure 2. Sequence of β-graphs

2-Node-Strong

3-Node-Strong

4-Node-Strong
with a given number of nodes, but also the most expensive to realize as communication networks—so expensive in fact as to be highly impractical. A network that would prove to be economically feasible would probably be no more than 4 or 5-node-strong. These 4 or 5-node-strong networks could be constructed using circuits as a starting point and applying Theorem 1, as explained before. Or they could be constructed by the application of Theorem 3 to an existing 3 or 4-node-strong network. Of course the networks could be made as large as required by the particular application. We now switch from general theory to more detailed observations.

Now the details of how a subgraph is connected to the rest of a graph will be examined.

**Definition 3**

The nodes which must be removed to separate a subgraph $G_k$ from the rest of a graph $G$ will be called the connectors of $G_k$.

**Definition 4**

The order of a connector $A$ of $G_k$ will be the number of edges incident at $A$ which are also incident at $G_k$.

There are certain conditions which, if met, will maximize the attachment of small subgraphs of a planar graph of degree D. A $G_2$ subgraph has the possibility of having $2(D-1)$ connectors as shown in Figure 3. These $2(D-1)$ connectors will be realized if the edges incident at the subgraph do not share any connectors. If connectors are shared, some of the
Figure 3. $G_2$ subgraph

regions into which a planar graph divides the plane will have three sides. These regions of the plane are called faces. Therefore the maximum attachment of $G_2$ subgraphs of a planar graph is achieved if there are no 3-sided faces in a graph.

A $G_3$ subgraph has the possibility of having $3D-4$ connectors, as in Figure 4. These $3D-4$ connectors will be realized if the edges incident on the subgraph do not share any connectors.

Figure 4. $G_3$ subgraph
If A and C share a connector a 4-sided face results and if A and B or B and C share a connector a 3-sided face results. So to maximize the attachment of $G_3$ subgraphs, a graph must have no 3 or 4-sided faces.

A class of planar graphs can be defined, some of which have the properties described in the preceding two paragraphs.

**Definition 5**

An S-uniform graph is a planar graph whose faces all have the same number of sides, S.

Now let $F$ be the number of faces in a planar graph, $N$ be the number of nodes, $E$ be the number of edges, and $D$ be the degree. Then Euler's formula, [1],

$$N - E + F = 2$$

is one equation relating these variables. Also, we have

$$DN = 2E$$

and

$$\sum_{s=1}^{\infty} (F_s S) = 2E$$

where $F_s$ is the number of faces with S sides. The summation actually goes from 3 to $\infty$ because there are no faces of one or two sides in the graphs considered in this paper. Since we are considering S-uniform graphs,

$$FS = 2E$$

So

$$FS = DN$$
\[ F = \frac{DN}{S} \]  
\[ E = \frac{DN}{2} \]  

Substituting (6) and (7) into (1) gives

\[ N - \frac{DN}{2} + \frac{DN}{S} = 2 \]  
\[ 2NS - DNS + 2DN = 4S \]  

Solving for \( N \),

\[ N = \frac{4S}{2D + S(2-D)} \]  

If \( D=2 \) there is but one possible graph, the circuit. Therefore set \( D=3 \). (9) becomes

\[ N = \frac{4S}{6-S} \]

Immediately we see that \( S \) must be less than 6. \( S \) is set equal to 3, 4, and 5 and \( N \), \( E \), and \( F \) are calculated for each value. The results are shown as graphs 1, 2, and 3 in Table I. Now set \( D=4 \) in (9).

\[ N = \frac{2S}{4-S} \]

Therefore \( S \) must be less than 4, so \( S=3 \). The result is shown as graph 4 in Table I. If \( D=5 \)

\[ N = \frac{4S}{10-3S} \]

\( S=3 \) is the only possibility again and the result is shown as graph 5 in Table I. Since \( N \) must be > 0, the denominator of (9) gives

\[ 2D + S(2-D) > 0 \]
The maximum $D$ will occur when $S$ is as small as possible, $^*$ that is, $S=3$.

So

$$2D + 3(2-D) > 0$$

$$6-D > 0$$

and we see that $D$ must be less than 6 and our table is complete. There are not many $S$-uniform graphs. Graphs 2 and 3 maximize the attachment of $G_2$ subgraphs and Graph 3 maximizes the attachment of $G_3$ subgraphs. All three have $D=3$.

<table>
<thead>
<tr>
<th>Graph</th>
<th>$D$</th>
<th>$S$</th>
<th>$N$</th>
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<td>3</td>
<td>12</td>
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Table 1. Data for $S$-uniform graphs

The construction of these graphs is shown in Figure 5.

The conclusion was reached that the incidence of the nodes of an $S$-uniform graph can not exceed five. This is generally true for all planar graphs.

$^*$This is verified by showing that $D < \frac{2S}{S-2}$ and that $\frac{2(S+1)}{(S+1)-2} < \frac{2S}{S-2}$.
Figure 5. S-uniform graphs
Theorem 4

The minimum incidence of a planar graph G must be less than six.

Proof

Suppose that the minimum incidence of G is greater than five, and that G has E edges and N nodes. Since each edge has two ends and at least six edges are incident at each node, we have

\[ 2E \geq 6N \] (10)

Multiply equation (1) by six and substitute equation (10) into the result

\[ N - E + F = 2 \]
\[ 6N - 6E + 6F = 12 \]
\[ 2E - 6E + 6F \geq 12 \]
\[ -4E + 6F \geq 12 \]
\[ -2E + 3F \geq 6 \] (11)

Now substitute equation (3) into (11).

\[ - \sum_{s=3}^{8} (F_{s} S) + 3F \geq 6 \]

but

\[ F = \sum_{s=3}^{8} F_{s} \] (12)

therefore
and

$$\sum_{s=3}^{\infty} (F_s S) + 3 \sum_{s=3}^{\infty} F_s \geq 6$$

which is impossible since \( S \geq 3 \). Therefore the incidence of \( G \) can be no larger than five and the theorem is proved.

An immediate consequence of Theorem 4 is the following:

**Theorem 5**

A planar graph can be no more than 5-node-strong.

**Proof**

By Theorem 4 the minimum incidence of a planar graph can be no greater than five. Theorem 7 then follows from Definition 2 and the fact that the first element of an \( S_N \) vector is equal to the minimum incidence of a graph.

An important result is the following.

**Theorem 6**

A planar graph must have at least one face with less than six sides.

**Proof**

Suppose no faces of a planar graph \( G \) have less than six sides. The nodes of the dual graph of \( G \) would then have a minimum incidence
greater than or equal to six. This violates Theorem 4 so Theorem 6 is proved.

The important result of Theorem 6 is that $G_k, k > 3$, subgraphs of a planar graph can never realize their full potential for attachment. If the $G_4$ subgraphs of a graph are to have maximum attachment all faces of the graph must have more than five sides so that no connectors are shared. This is impossible by the last theorem. The same reasoning applies to large subgraphs.

Some knowledge is obviously required of the attachment of large subgraphs. Larger subgraphs can be formed from smaller ones by the following process. $G_{k+1}$ can be formed from $G_k$ by forming the union of $G_k$, one of the connectors, $A$, of $G_k$, and the edges incident on both $G_k$ and $A$. $G_{k+1}$ may have more, less, or the same number of connectors as $G_k$ had.

Let $G$ be a graph.

Let $G_k$ be a subgraph of $G$.

Let $A$ be one of several connectors of $G_k$.

Let $G_{k+1}$ be formed as explained above where the connector used is $A$.

Let $A_1, A_2, \ldots, A_x$ be nodes which share at least one edge with $A$. $A_1, A_2, \ldots, A_x$ may be in $G_k$, may be connectors of $G_k$, or may
be neither. Suppose there are N nodes in the last category. Then the net
increase in the connectors of $G_{k+1}$ over $G_k$ is $N-1$. The "-1" term appears
because $A$ is not a connector of $G_{k+1}$. So if $G_k$ is $P$-node-attached, $G_{k+1}$
would be $(P+N-1)$-node-attached.

When $G_k$ contains a relatively small number of the nodes of $G$, the
formation of $G_{k+1}$ has a good chance of being accompanied by an increase in
the attachment because some nodes attached to $A$ will probably not be in $G_k$
or be connectors of $G_k$. However, when $G_k$ contains a large number of the
nodes of $G$, the formation of $G_{k+1}$ will probably cause a decrease in the at­
tachment.

As an example of this process the cases occurring for a graph of
degree three are shown in the following list. Out of a total of nine possibil-
ities, one increases and five lower the number of connectors in forming a
larger subgraph.
Case 1

\[ G_{k+1} \text{ will have one more connector than } G_k \text{ has.} \]

Case 2

\[ G_{k+1} \text{ will have the same number of connectors as } G_k \text{ has.} \]

Case 3

\[ G_{k+1} \text{ will have the same number of connectors as } G_k \text{ has.} \]
Case 4

$G_{k+1}$ will have the same number of connectors as $G_k$ has.

Case 5

$G_{k+1}$ will have one less connector than $G_k$ has.

Case 6

$G_{k+1}$ will have one less connector than $G_k$ has.
Case 7

Case 8

Case 9

$G_{k+1}$ will have one less connector than $G_k$ has.

$G_{k+1}$ will have one less connector than $G_k$ has.

$G_{k+1}$ will have one less connector than $G_k$ has.
There is a simple matrix method for determining the node attachment of each subgraph of a graph $G$.

**Definition 6**

$C = [c_{ij}]$ is an $N \times N$ symmetrical matrix for a graph $G$ with $N$ nodes, where

- $c_{ij} = 1$ if node $i$ is directly connected to node $j$.
- $c_{ij} = 0$ if node $i$ is not directly connected to node $j$. A node is considered as connected to itself.

Suppose it is necessary that the node attachment of some subgraph $G_k$ be found. $C$ can be rearranged so that the first $k$ columns and rows correspond to the $k$ nodes in $G_k$. $C$ can now be partitioned in the following manner:

$$
\begin{bmatrix}
C_k \times k & C_k \times (n-k) \\
C_{(n-k)} \times k & C_{(n-k)} \times (n-k)
\end{bmatrix}
$$

**Theorem 7**

The node attachment of subgraph $G_k$ of a graph is equal to the number of columns of $C_k \times (n-k)$ with 1's appearing.

**Proof**

If a "1" appears in a column of $C_k \times (n-k)$, it indicates that some node of $G_k$ is directly connected to a node of $G$ not in $G_k$; therefore the number of columns which have a "1" would be the number of connectors of
Thus the theorem.

In principle this theorem could be used to find the minimum attachment of each size subgraph in $G$. With this information the $S_N$ vector for $G$ could be constructed and the node strength of $G$ determined. However, since each subgraph of $G$ must be tested and there may be many such subgraphs, the method may be lengthy.
3. STRENGTH OF GRAPHS WITH RESPECT TO EDGES

The second method of graph separation, removal of edges, will be considered now. The problem is: given a number of nodes with which to form a graph, how should the graph be drawn so that a specified number of edges must be removed before the remaining graph is separated? Definitions will be very similar to those given before for the node removal problem.

**Definition 7**

A nonempty, connected subgraph $G_k$ of a graph $G$ will be said to be $e$-edge-attached if the removal of at least $e$ edges is required to separate $G_k$ from the rest of $G$.

This is analogous to the definition of $n$-node-attached in Definition 1. A vector like $S_N^*$ can be formed. Call this new vector $S_E^*$ and make its elements the minimum edge attachment of the $G_k$'s, $k = 1, 2, \ldots, m$, as in $S_N^*$.

**Definition 8**

A graph $G$ will be $e$-edge-strong if every element of its $S_E^*$ vector is $\geq e$.

The separation of a graph by edge removal necessitates the removal of certain sets of edges from the graph. These sets of edges are disjoint unions of cut-sets. The edge strength of a graph is limited by the number of edges in the minimum cut-set and if a graph $G$ is to be
e-edge-strong, one basic requirement is that the minimum incidence of the
nodes of G be at least e. So the number of edges in the minimum cut-sets
of G should be equal to the minimum incidence of the nodes of G, if G is to
be efficient.

Larger subgraphs of a graph G can be formed from smaller sub-
graphs by the same process as was explained in the discussion of node
strength. The same notation is used as before with the addition of φ as the
order of connection of A to $G_k'$, and I as the incidence of A. Suppose $G_k$ is
e-edge-attached. Then $G_{k+1}$ would be $[e + (I-φ) - φ]$-edge-attached. The
"-φ" term is to subtract those edges connecting A to $G_k'$, and the "I-φ"
term adds edges connecting A to nodes not in $G_k$. The attachment can be
rewritten as $(e + I - 2φ)$-edge-attached. Given a graph of specified degree,
a study can be made to determine what effect increasing the size of sub-
graphs has on edge attachment. This has already been done in the case of
node attachment for $D = 3$.

Just as there was a matrix method for finding the node attachment
of a subgraph of a graph, there is also a matrix method for finding the edge
attachment of a subgraph.

**Definition 9**

$A = [a_{ij}]$ is a matrix of $n$ rows and $e$ columns for a graph of $n$
nodes and $e$ edges, where

$a_{ij} = 1$ if the edge $j$ is incident at the node $i$.

$a_{ij} = 0$ if the edge $j$ is not incident at the node $i$. 
This matrix is the incidence matrix of graph theory.

Suppose the edge attachment of a subgraph $G_k$ of a graph $G$ is to be found.

The matrix $A$ can be arranged so that the first $k$ rows correspond to the $k$ nodes of $G_k$. Then $A$ can be partitioned in the following manner.

\[
\begin{bmatrix}
A_{k \times e} \\
A_{(n-k) \times e}
\end{bmatrix}
\]

**Theorem 8**

The edge attachment of a subgraph $G_k$ of a graph $G$ is equal to the number of columns of $A_{k \times e}$ with one and only one "1" appearing.

**Proof**

If a column of $A_{k \times e}$ has one "1" appearing, then the edge corresponding to that column must have one end incident at a node in $G_k$ and the other end incident at a node not in $G_k$. Therefore that edge would have to be removed if $G_k$ were to be separated from the rest of $G$. Thus the theorem.

As in the case of Theorem 9, the application of this theorem to finding the edge strength of a graph could prove to be lengthy.

There should be some relationships between node attachment and edge attachment. Suppose node $A$ is a connector of subgraph $G_k$ of a graph $G$. Then $A$ is directly connected to $G_k$ by at least one edge. Now if $G$ were to be separated into two subgraphs, one of which was $G_k$, by node removal,
A would have to be removed. If G were to be separated by edge removal instead, we see that at least one edge would have to be removed to have the same effect as the removal of A. Therefore we have proved the following theorem:

**Theorem 9**

If a subgraph $G_k$ of graph G is $n$-node-attached, it is at least $n$-edge-attached.

Let $\phi$ be of the order of connection of A to $G_k$. Then it is easy to see that $\phi$ edges would have to be removed to have the same effect on $G_k$ as the removal of A. If the orders of connection of the connectors of $G_k$ are summed, the result would be the edge attachment of $G_k$.

From Theorem 9 it is evident that all of the theorems concerning the node strength of a graph are also applicable to the edge strength of a graph if the theorems are considered to give lower bounds on edge strength.
Lastly, we come to the third method of graph separation, removal of both nodes and edges. The statement of the problem is almost exactly as it was before. Given a number of nodes with which to form a graph, how should the graph be drawn so that a specified number of nodes and edges must be removed before the remaining graph is separated? Nodes and edges of a graph will be called components and $n/e$ will be a specified ratio of the number of removed nodes to the number of removed edges when a graph is separated. It may not always be possible to make $n/e$ exactly the specified value.

**Definition 10**

A nonempty, connected subgraph $G_k$ of a graph $G$ will be said to be $t_{n/e}$-component-attached if the removal of $t$ components in, as nearly as possible, the ratio $n/e$ is required to separate $G_k$ from the rest of $G$.

A vector $S_{C(n/e)}$ can be formed whose elements are the minimum component attachment, components in the ratio of $n/e$, of the $G_k$'s, $k = 1, 2, \ldots, m$ as in $S_N$ and $S_E$.

**Definition 11**

A graph $G$ will be $t_{n/e}$-component-strong if every element of its $S_{C(n/e)}$ vector is $\geq t$.

If $G_k$ is to be separated by node removal from the rest of a graph $G$, all of the $n$ connectors of $G_k$ must be removed. If $G_k$ is to be
separated from the rest of G by edge removal, all of the e edges incident at both \( G_k \) and the connectors of \( G_k \) must be removed. Now when removal of both nodes and edges is used to accomplish the separation, some connections of \( G_k \) to G are eliminated by removing the appropriate connector, other connections are eliminated by removing the edges incident at both a connector and \( G_k \). Suppose c components are removed in this last case. It is evident that for any \( G_k \) of G, \( n \leq c \leq e \). This results in the following theorem:

**Theorem 10**

If a subgraph \( G_k \) of a graph G is a \( n \)-node-attached, it will be at least \( \frac{n}{e} \)-component-attached where \( n/e \) is any ratio.

Networks which give protection against separation by node removal give at least as good a protection against separation by general component removal.

Let us take the viewpoint of one who is in the business of separating communication networks. If we want to separate a network by destroying a minimum number of things, Theorems 9 and 10 tell us that we should destroy stations. However, if we consider the relative cost of destroying a station or link, a different situation might arise. The cost of destroying a station may be different from the cost of destroying a link. A station may contain significant redundancy whereas a link has no redundancy in the sense that it is just a single cable or radio tower; also a station may be better
protected by defensive systems than a link. Therefore it would seem likely to cost more to destroy a station than a link. Suppose that $C_N$ is the cost of destroying a station and $C_E$ is the cost of destroying a link and that these costs are constant throughout the network. $C_N$ and $C_E$ can be related by the formula:

$$C_N = a C_E$$

where "a" is constant for a given network. Turning now to the graphic representation, let $G_k$ have a connector $A$ of order $\varphi$. If $G$ is to be separated so that one of the resulting subgraphs is $G_k$ we can see that $A$ or the edges incident at both $A$ and $G_k$ would have to be removed. If $\varphi$ was less than "a" it would be cheaper to destroy the links corresponding to the edges incident at both $A$ and $G_k$. On the other hand, if $\varphi$ was greater than "a" it would be cheaper to destroy the station corresponding to node $A$. This comparison could be performed for every connector of $G_k$ to determine in each case which it would be cheaper to destroy, stations or links.
5. CONCLUSION

Let us return to communication networks and see if any useful results have been obtained. If a network whose separation requires the destruction of \( n \) stations is known, Theorem 1 gives a method of constructing a network whose separation requires the destruction of \( n+1 \) stations. Theorem 2 points out that some of the networks formed by the method of Theorem 1 which require the destruction of \( n \) stations to separate, require the destruction of \( n+1 \) stations to separate if the separation of single stations is disregarded. Theorem 3 gives another method of constructing a network whose separation requires the destruction of \( n+1 \) stations, if a network whose separation requires the destruction of \( n \) stations is known. A special case of the application of Theorem 3 leads to the strongest networks with a given number of nodes. Networks of any desired strength can be formed by repeated application of either Theorem 1 or Theorem 3.

Planar networks were studied and Theorem 5 states that they can always be separated by the destruction of five stations. A special class of planar networks was defined which maximizes the difficulty of separation of small groups of stations. However, these networks do nothing special for large groups of stations.

A matrix method, Theorem 7, was derived for finding the number of stations it would be necessary to destroy in order to separate some group of stations from the remaining stations of a network. This method
could be applied repeatedly to find the minimum number of these stations for the entire network, and thus give some indication of the overall strength of the network.

Theorem 9 states that if \( n \) stations have to be destroyed to separate a network into two or more parts, at least \( n \) links or lines must be destroyed to accomplish the same thing. Thus the networks used to give protection against separation by station destruction give at least as good a protection against separation by link destruction.

A matrix method, Theorem 8, was derived for finding the number of links that it would be necessary to destroy in order to separate some group of stations from the remaining stations of a network. As before, this method could be applied repeatedly to give an indication of the overall strength of the network.

Theorem 10 gives assurance that a network which gives some protection against separation by station destruction gives at least as good a protection against separation if both stations and links may be destroyed.
LIST OF REFERENCES


This paper considers the problem of the survivability of communication networks from the point of view of graph theory. Stations are represented by nodes of a graph and communication links between stations are represented by edges of a graph. The strength of a graph is defined with respect to nodes only, edges only, and both nodes and edges. Two methods of constructing graphs with specified strength are derived. Planar graphs are studied and a bound on their strength is found. Methods of finding the strength of an arbitrary graph are given.
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