NUMERICAL STUDIES OF THE NON-LINEAR BOLTZMANN EQUATION
PART II: STUDIES OF NEW TECHNIQUES OF ERROR REDUCTION

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PART II: STUDIES OF NEW TECHNIQUES OF ERROR REDUCTION

B. L. Hicks, S. M. Yen and B. J. Reilly
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1. Introduction

Since 1963 a number of techniques have been developed for improving the accuracy and stability of Nordsieck's Monte Carlo method of evaluating the Boltzmann collision integral. This report summarizes all of the techniques, including the older ones that have already been described in our reports and papers and the newer ones which we have been using during the past year but which have not previously been discussed in detail. Some of the techniques are numerical corrections of the Monte Carlo method and some are new and more effective implementations of it.

In describing the various techniques and their effects on the calculations we have developed a more or less abbreviated shorthand (defined in Sect. 2) for designating them. Using the shorthand we may summarize the applicability of these corrections as follows. In all iterative Boltzmann problems the corrections LS, MB, and IS should be used. In relaxation problems, like that of the pseudoshock, the use of MB, IS would considerably improve the accuracy and interpretability of results which already showed evidence of high accuracy when only LS was used. (Increased accuracy is, of course, equivalent to greater computing speed for the same accuracy.)

Since 1965 most of our calculations have dealt with steady state, two-point boundary value problems, like that of the shock wave and heat transfer between plates in rarefied gases. We have therefore been concerned with the accuracy and stability of the iterative schemes we have been using for solving these problems. The corrections LS, MB and IS have been very useful here and also the use of variable intervals.

One of the corrections tried (SB) was not successful; the reasons are illuminating and are covered briefly in the report. The change from RAND to RANDY was made to correct a coding error (not a defect in Nordsieck's method).
2. Techniques of Error Reduction

2.1 Old Techniques (1963-66)

2.11 Variable J

The number of equal intervals (J-1) in the range of the independent variable (for steady state-Boltzmann problems) obviously will affect the magnitude of the quadrature errors made in replacing the Boltzmann differential equation by a Boltzmann difference equation. In the past five years we have therefore frequently compared "solutions" of the Boltzmann difference equation calculated for different values of J. As our methods have become successively more refined, we have come to the conclusion that studies of the effect of J, made with our present program for the CDC 1604, will be fruitful for Krook problems but not for the corresponding Boltzmann problems. This conclusion is based upon the fact that the various systematic and random errors are reasonably well balanced for the solution, in a reasonable amount of time, of typical Boltzmann problems on the 1604, and that the more accurate calculations needed for investigation of the J-dependent quadrature errors and their propagation will be too time-consuming on this computer.

Such studies of solutions of the Krook equation can be easily made on the 1604, however, and will give information about J-dependent errors for the solution of the Boltzmann equation because the solution algorithms are identical except for different means of calculation of the collision integral.
2.12 n as Independent Variable

We have used transformations of the independent variable in our shock wave calculations since 1963. In particular, transformation of the space variable $x$ to the variable $n$ ($n = \text{particle density}$) defined by

$$\frac{dn}{dx} = \int d\tilde{v} (a-bf) v_x,$$

where the Boltzmann collision integral, $(a-bf)$, is calculated by the Monte Carlo method, has been found to be especially useful.$^2$

2.13 Least Squares (LS)

We have used a least squares correction of $(a-bf)$ since 1964 to insure that the conservation equations peculiar to each gas dynamic problem are satisfied. The least squares corrections generally improve the long-term stability of iterative solutions of the Boltzmann equation for steady state problems, like the shock wave and heat transfer problems, and improve the accuracy of calculation of relaxation phenomena, like the pseudo-shock. When systematic errors are large, as for very strong shock waves, then we suspect that the least squares corrections may help to promote long-term instability.

In the relaxation problem$^3$ number density and energy are conserved, and in the shock problem the fluxes of mass, momentum and energy are conserved. In the two problems there are then, two and three side conditions respectively in the least squares treatment. Each side condition
is of the form

\[ \sum_{s=0}^{225} g_s p_s \phi_s = 0 \]

where \( p_s' \) is the corrected value of \( (a-bf)_s \); \( g_s \) is the weighting factor in numerical quadrature over velocity space; \( \phi_s \) is a product of powers of velocity components; and the sum is over the 226 velocity bins. Thus, moments of \( (a-bf)_s \) are put equal to zero in making the least squares correction, a procedure which has certain advantages compared to the procedure used earlier of forcing moments of \( f \) to be constant.

With these side conditions we minimize the mean square fractional correction to \( (a-bf) \). If we denote the uncorrected value of \( (a-bf) \) by \( p_s \), then the minimization condition reads

\[ \delta \sum_{s=0}^{225} (p_s' - p_s)^2 / (p_s')^2 = 0. \]

In this equation the weight factor is \( (p_s')^{-2} \), which seems to lead to better results than the weight factor 1, in the sense that a second type of correction (to be described next) is smaller for the weighting by \( (p_s')^{-2} \).

The least squares corrections, \( (p_s' - p_s) \), generated by the foregoing equations, are distributed equally between \( a \) and \( bf \)

\[ a_s' - a_s = -(bf)_s' + (bf)_s = \frac{1}{2}(p_s' - p_s) \]

because there is no a priori reason to distribute these unequally.

Occasionally one of the corrected values, \( a_s' \) or \( (bf)_s' \), may be negative. As such a value has no physical meaning, we make a second correction, keeping
\((a-bf)_s\) constant:

\[ a''_s = 0, \quad (bf)_s'' = (bf)'_s - a'_s \quad \text{if } a' < 0 \]

and

\[ a''_s = a'_s - (bf)'_s, \quad (bf)_s'' = 0 \quad \text{if } (bf)' < 0. \]

(The two cases are disjoint.)

For monitoring the corrections, we print in the "moment" output the values of \(\text{SIG(COR(A-BF))}\) and of \(\text{CORR(NUA)}\), for each station, where the first quantity is equal to the rms value of the LS correction of \(a(\sigma_j \delta a)\); and the second quantity is a measure of the change in the value of \(v_a = \int a \, dv\) because of the second correction of \(a\). (Note: \(\text{CORR(NUA)}\) is scaled and not weighted.) The effect upon \(\sigma_j \delta a\) of various corrections other than LS will be discussed in later sections. The value of \(\text{CORR(NUA)}\) is usually equal to zero and is never a large part of \(v_a\) in the shock wave calculations.

2.14 Fixed Sets of Collision Samples (FS)

Three types of Monte Carlo sampling of the molecular collisions were distinguished.\(^1\) The second type (fixed sets of collision samples) was used from April 1966 to December 1967 and made possible the first reliable solutions of the Boltzmann equation for strong shock waves. The third type will be discussed in Sect. 2.24.
2.2 New Techniques and Program Changes

2.21 The Maxwell-Boltzmann Correction (MB):

Various studies⁴ (see also Sect. 3.1) have been made of the two kinds of errors in the Monte Carlo evaluation of the Boltzmann collision integral. **Systematic** errors are caused by the use of a finite number of bins in velocity space; and by a finite size for each bin in velocity space, in \( \vec{\mathbf{k}} \) space and in the \( x \) or \( n \) dimension. In particular, the different estimates \((a, a', A, A', bf, (bf)'), BF, (BF)')\) of \( a \) and of \( bf \) that are combined in our Monte Carlo program contain different contributions from the so-called "look-up" (interpolation) and "planting" (inverse interpolation) errors. **Random** errors are associated with the finite size of each sample of collisions used. Both kinds of errors can be reduced, for a gas near equilibrium, by the MB (Maxwell-Boltzmann) technique first used successfully in December 1967.

We customarily use the MB corrections for all calculations, that is, for gases both far from and near to equilibrium, for the sake of consistency and uniformity of treatment. Near equilibrium the corrections produce a significant increase of accuracy of the results; far from equilibrium, the corrections are small and do no harm. With the MB corrections it is now possible to make calculations for gases much closer to equilibrium than was hitherto possible, whether the gas is in a very weak shock or is near the boundary of a strong shock.

The MB technique amounts to correcting the calculated values of \( a \) and of \( bf \) for a non-equilibrium gas in such a way that \( a-bf \) would equal zero for a gas in equilibrium that has the same values of density, temperature and gas velocity as the non-equilibrium gas. The velocity distribution function of this equilibrium gas is denoted by \( f_{MB} \). The program thus carries out the following sequence of steps:
1. Calculate $f_{MB}$.

2. For one collision, calculate $a$ and $bf$ as usual for the non-equilibrium gas and also calculate $\Delta = \frac{1}{2}(a-bf)_{MB} = \frac{1}{2}k_0(FF'-ff')_{MB} \neq 0^*$ for the equilibrium gas from $f_{MB}$.

3. Calculate corrected values of $a$ and $bf$:

   \[ a' - a = -(bf)' + bf = -\Delta. \]

4. Repeat these corrections to get $a'$, $A$, ... $(BF)'$.

5. Repeat steps 2, 3 and 4 for all collisions in the sample.

Then, as should be the case if there were no error in Monte Carlo evaluation of the collision integral,

\[
(a-bf)'_{MB} = 2 \sum_{\text{sample}} \Delta \equiv 0
\]

for $f = f_{MB}$.

When the 1S Monte Carlo technique is used (see Sect. 2.24), the MB correction adds only 14% to the computing time for $J = 9$.

2.22 RAND AND RANDY

In the period September 1963 to January 1968 the Monte Carlo program used bits 13-66 (where 95 is the left-most bit in the 96 bit double-precision product produced by the modified Juncosa generator) where bits 13-21 were used for randomizing nine signs. Calculations made in this period are designated by the code RAND. Since January 1968 bits 30-66 (22-74 in SB) have been used (bits 41-49 for randomizing signs) as they should be distributed more nearly randomly. (See Sect. 3.4.) Three calculations based on this second choice of bit positions are designated by the code RANDY.

*The uncorrected increment $\delta(a-bf) = 0$ only for $f = f^{\text{SHRF}}$, the value of $f$ generated by relaxation to a pseudo-equilibrium defined by our Monte Carlo algorithm.*
"Bender" runs (Sect. 3.1) showed larger systematic error with RAND than with RANDY, but no significant differences between results of using the two choices of bit positions have appeared in any shock or heat transfer runs.

2.23 The Sub-bin Correction (SB):

The SB correction was not successful, but the reasons for its lack of success illustrate basic features of the Monte Carlo evaluation of the collision integral.

The SB correction amounted to increasing the number of bins in the two-dimensional velocity space \((v_x, v_\perp)\) by subdividing each bin into 256 sub-bins. The qualitative arguments were that (1) interpolation errors (both look-up and planting) would be then distributed nearly at random over each sub-bin both for the incoming molecules (that is, in the calculation of \(ff'\) and planting of \(\delta(bf)\) etc.) and for the outgoing molecules (that is, for \(FF'\) and the planting of \(\delta a\), etc.), and (2) that then the systematic errors would be similar for the four estimates of \(a\) and of \(bf\) instead of different.

This symmetry will indeed, be achieved for very large Monte Carlo collision samples, but for samples small enough to be generated on the CDC 1604, the sampling over the sub-bins is very sparse, and it is therefore not surprising that no beneficial effects of the SB correction have been observed. A disadvantage of the SB correction is that it increases the random errors somewhat.

2.24 One sample (IS):

Use of the fixed sample (FS) algorithm implies that a fixed set of
difference equations are being solved, but since the collision samples at the various stations are statistically independent, the algorithm is in effect describing the interaction of (J-2) slightly different non-equilibrium gases. In spite of this characteristic of the FS calculations, it was possible, as already mentioned, to solve the Boltzmann equation for several problems.

In December 1967, the IS algorithm was introduced in which the same collision sample was used for each station so that the same gas was therefore situated at each station. This technique reduced the fluctuations (that had been observed with the FS algorithm) for a given velocity bin from one station to the next and separated even more cleanly than before the variations with n of the functions f, a and bf from their variations over velocity space. In the case of the important function n' = dn/dx the n' vs horizontal curves are "nested", that is, the curves for different IS samples are similar but displaced along the n' axis as though a different x scale corresponds to each sample.

Another virtue of the IS algorithm is that it permits an increase of computing time by a factor of as much as three. This increase is achieved by calculating for each randomly chosen collision, the contributions to a and to bf for each of the (J-2) internal stations before going on to the similar calculations for the next randomly chosen collision.

2.25 Variable Intervals in n

In the shock and heat transfer problems it is advantageous to get more detailed information near the boundaries about the variation of f, a and bf and the moments with n or x than elsewhere. The greater detail is obtained by reducing the step size in n or x near the boundaries by an
appropriate specification of the variable step size that is input to the computer before a run. Two sets of variable intervals have been used: a set of 15 intervals proportional to $1/8$, $1/8$, $1/4$, $1/2$, $1$, $1$, $1$, $1$, $1$, $1$, $1/2$, $1/4$, $1/8$, $1/8$ and a set of 13 intervals proportional to $1/2$, $1/2$, $1/2$, $1/2$, $1/2$, $1/2$, $1/2$, $1/2$, $1/2$, $1/2$, $1/2$. With the greater detail thus afforded, it has been possible to find, for the first time, various distinct forms of non-equilibrium behavior of the Knudsen layer in heat transfer in rarefied gases and to extend calculations of shock structure far up- and down-stream. For very strong shock waves (Mach numbers greater than 4) the first choice of variable intervals cannot be used because of reinforcement of the various quadrature errors near the upstream side of the shock.

2.26 New Techniques for "Bender" Runs

The basic techniques for error studies have been described. Recent improvements include the following:

a) computer calculations (rather than hand calculations) of the mean ($\mu$), root mean square (rms), and standard deviation ($\sigma$) of various error-related functions over two-dimensional velocity space;

b) introduction of a parameter "DELV", usually set equal to 3, that controls, in calculations of the various statistics, how many velocity bins are excluded because they contribute large errors to the statistics but contain only a small fraction of the molecules;

c) use four different values of $N$, in order to get greater precision in separating the random and systematic errors;

d) estimates of the standard deviation of the standard deviation.
2.27 Numerical Studies of Errors Due to Discrepancy in Integration Interval $\delta n$:

The interval $\delta n$ used in n-wise integration for the shock wave is

$$\delta n = \frac{(n_2 - n_1)}{(J-1)}$$

where $n_1$ and $n_2$ are the exact values of the number density calculated from the Rankine-Hugoniot relations. In $J$ integration steps the change of $n$ was accordingly equal to $(n_2 - n_1)$. At the end stations, however, the values of $n$ are $n'_1$ and $n'_2$, calculated by numerical integration as are the values of $n$ and all moments at interior stations. These values do not in general agree with the values $n_1$ and $n_2$.

We have studied the errors introduced by this discrepancy in the program by comparing the solutions of the Krook equation and the Boltzmann equation with the corresponding solutions obtained by using the $(\delta n)_{quad}$:

$$(\delta n)_{quad} = \frac{(m_2 - m_1)}{(J-1)}.$$

2.28 Numerical Estimates of Errors in $df/dn$

In integrating the Boltzmann equation for steady-state problems, like the shock wave and steady heat transfer in a rarefied gas, we use centered differences in calculating the derivative $df/dn$. The fractional error in this estimate can be approximated for $j = 3/2, 5/2, \ldots, (J-5/2)$ (midpoints of intervals) by the expression

$$\Delta^3 f/(24\Delta f).$$
We used this expression in a new program which evaluates this fractional error in \( \frac{df}{dn} \) for each of the 226 velocity bins at all of the midpoints specified above. We used values of \( f \) calculated with the 1S algorithm in order that they be smooth in their variation with respect to \( n \), and thereby avoided unnecessary noise in the estimates of the error of \( \frac{df}{dn} \).
3. Tests and Results

3.1 "Bender" Runs*

Certain conventions will be used in the discussions in this section. The term "three cases" will refer to three Bender runs which are identical except that the programs used are with either the RAND, or the RANDY, or the RANDY+SB technique. The term "six cases" will refer to these three types of runs made both with and without the LS correction.

Statements of statistical uncertainty in the quantities discussed will uniformly be expressed as 90% confidence limits. Thus \( \mu = (2.0 \pm 0.1) \) signifies that the 90% confidence limits of \( \mu \) are (1.9, 2.1).

The variance \( \sigma^2 \) of an error is decomposed into the contributions \( \sigma_r^2 \) and \( \sigma_s^2 \) of the random and the systematic errors:\(^6\)

\[
\sigma^2 = \sigma_r^2 + \sigma_s^2.
\]

We assumed that the random error in determining \( \sigma_r^2 \) was equal to the random error in determining \( \sigma^2 \).

The symbol \( \mu \) will be used for the mean value.

All of the results discussed (except those in Sect. 3.11) refer to the hot side of a shock of Mach number 2.5. Our Bender Boltzmann program calculates for this condition the analytical values of the (equilibrium) Maxwell-Boltzmann velocity distribution function \( f_{\text{eq}} \) and then calculates by the Monte Carlo method whatever values of \( a, \beta \) and \( a-\beta \) are

---

*In the CSL Rarefied Gas Dynamics Group the name "Bender" is applied to runs first programmed by L. S. Bender which calculate various values of \( a, \beta \), and \( a-\beta \) for a shock wave for \( J=3 \). Only the values for the equilibrium gases at the end stations ("cold", \( j=0 \); "hot", \( j=2 \)) of Bender runs will be used here.
desired, with appropriate output of isolines, tables and the three statistics mentioned above.

3.11 MB

The MB correction is usually so effective that it obscures the errors inherent in the Monte Carlo method. We therefore discuss the general effect of the MB correction before turning to the effects of the other corrections.

In Table 1 we see the very notable effects of the MB correction upon the values of \((a-b_f)/bf\). This quantity should, of course, equal zero if there were no errors in the Monte Carlo calculation of the two parts of the collision integral.

<table>
<thead>
<tr>
<th>(M_1)</th>
<th>cold side</th>
<th>hot side</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>no MB</td>
<td>7.2 x 10^{-2}</td>
</tr>
<tr>
<td></td>
<td>MB</td>
<td>0.02 x 10^{-2}</td>
</tr>
<tr>
<td>10</td>
<td>no MB</td>
<td>18 x 10^{-2}</td>
</tr>
<tr>
<td></td>
<td>MB</td>
<td>29 x 10^{-2}</td>
</tr>
</tbody>
</table>

The MB correction is thus unsuccessful only for the cold side of a very strong shock \((M_1=10)\) where both the random and systematic errors become large, for the subdivision of velocity space that we use, namely, for 226 velocity bins in the \((v_x,v_y)\) plane. The Monte Carlo method can, of course, equally well be applied to a finer subdivision of velocity space, with consequent reduction of both types of error, when a larger and faster computer
is used for solutions of the nonlinear Boltzmann equation.

The effect of the MB correction can be summarized in another way. For all six cases for \( M_1 = 2.5 \) and for \( N_7 \) between 17 and 11, we have the inequality

\[
2 \times 10^{-4} < \sigma[(a-bf)/bf] < 7 \times 10^{-4}.
\]

Further evidence of the success of the MB correction is given in Sect. 3.4.

3.12 LS and SB; RAND and RANDY

In general, the effects of the LS correction are similar to those which were found earlier. One statistic that has been studied is \( \mu[(a-bf)/bf] \), which measures the departure, owing to systematic errors in the Monte Carlo technique, of \( a/bf \) from its correct value of unity, for a gas in equilibrium. We note that, for \( N_7 = 17 \), the statistical fluctuations of \( \mu[(a-bf)/bf] \) are small but not negligible (20-30%). For smaller values of \( N_7 \) the fluctuations are so large that the observed deviations of \( \mu \) from zero are no longer significant.

For RAND runs with no LS correction the old data gave \( \mu[(a-bf)/bf] = 1.2 \times 10^{-2} \). The new value is \( (1.46 \pm 0.10) \times 10^{-2} \). With the LS correction the three values of \( \mu[(a-bf)/bf] \) for RAND, RANDY, and RANDY+SB runs are \( (5.5, 3.1, 4.2 \text{ each } \pm 0.10) \times 10^{-3} \). The LS correction apparently is more effective in reducing the mean error than in reducing the standard deviation of the systematic error.

As in reference 6, the value of \( \mu[(a-bf)/bf] \) could be calculated accurately by a linear combination of the values of \( \mu[a/a_n] \) and of \( \mu[bf/a_n] \), the accuracy now being extended to three places. Correlation of the errors
of a and bf reduces the value of $\mu[(a-bf)/bf]$ as compared with the values of $\mu[a/a_n]$ and of $\mu[bf/a_n]$, by factors of about four with LS corrections, but not appreciably without these corrections.

The effect of the LS correction can also be judged from the values of the ratio $\sigma$(no LS)/$\sigma$(LS) where $\sigma$ in each case refers to $\sigma[(a-bf)/bf]$. In reference 4 it was stated that the approximate value of this ratio was 2. For the three recent runs the values obtained were 2.7, 3.9, and 3.4. The LS correction does not change appreciably the random part $\sigma_r$ of the standard deviation of $(a-bf)/bf$.

The SB correction seems to increase the value of the mean error.

The quantity $\sigma_s$ defined earlier measures the departure of $(a-bf)/bf$ from its mean value owing to systematic errors in the Monte Carlo evaluation of the Boltzmann collision integral. The value of $\sigma_s$ reported in reference 4 was 0.018. The values determined recently lie in the range 0.016-0.026 ($\pm$ 0.002) for the six cases and thus verify the earlier value.

The quantity $\sigma_r$, also defined earlier, measured the departure of $(a-bf)/bf$ from its mean value owing to random errors in the Monte Carlo evaluation of the collision integral. The earlier value was 0.052 and the recent (RAND) value is 0.054, each for $N_r=13$. The good agreement is partly fortuitous because the earlier determination is much less accurate, having been based upon the use of just two different sample sizes instead of the four used recently. The recent results also check that $\sigma_r$ is proportional to $2^{-N_r/2}$ within the observed statistical fluctuations of these quantities.

For the six different cases, and $N=13$, the value of $\sigma_r$ was found to lie in the range 0.036-0.056 ($\pm$ 0.002).

The old value of $\sigma_r[a/a_n]$ was 0.05, the new value is 0.09, for all
six cases. From values of $\sigma_{r}[a/a_n]$ and of $\sigma_{r}[bf/a_n]$ it is possible\(^6\) to compute what the value of $\sigma_{r}[(a-bf)/bf]$ would have been had there been no correlation of the random errors of $a$ and of $bf$. The ratio of the values of $\sigma_{r}$ thus calculated and the values of $\sigma_{r}$ observed measure the extent of this correlation. This ratio has the three values 5.0, 4.6 and 6.0 for the calculations of reference 6, SB+LS and LS, respectively.

### 3.2 Errors in $df/dn$

Calculations of the fractional error in $df/dn$ were made for three shock wave runs with $M_1 = 2.5, 4.0, \text{ and } 10$ and $j=9$. The fractional error in $df/dn$ is generally less than 0.01 for all three values of $M_1$ and also throughout velocity space where one might expect to find larger errors, namely:

- a) near extrema of $f$ (where $df/dn \approx 0$)
- b) for $j=3/2, v_x \leq -11$; and for $j=13/2, v_x \geq 0$ to 13 (varies with $M_1$).

In case b) the larger errors may be due to errors in $f$ near the cold and hot side connected with the non-vanishing there of $\delta_f^c$ and $\delta_f^h$.

For $M_1 = 2.5$ and 4.0, the fractional error in $df/dn$ is generally smaller than 0.001, when $j=9/2, 11/2, (all v_x)$; and when $j=13/2 (v_x < 0)$. For $M_1 = 2.5$ the fractional error in $df/dn$ is also less than 0.001 at $j=7/2$.

These errors in the numerical evaluation of $df/dn$ from the difference ratio are generally small enough so that we may neglect them, compared to random errors of the Monte Carlo method for sample sizes that can be reached on the CDC 1604. For $M_1 \leq 1.2$, a study of these errors is probably not necessary until we need to examine integration and differentiation errors more critically, as may be required, for example, in the
Monte Carlo study of the Navier-Stokes approximation. Error runs on $\frac{df}{dn}$ for Krook shocks, with $M_1 = 2.5, 4.0,$ and $10$ and $J=9$, were made, and the numerical differentiation errors that occur are similar to those for the corresponding Boltzmann shocks.

3.3 Errors in Shock Wave Calculations Due to Discrepancy in Integration Interval $\delta n$:

We have studied this error in shock wave calculations for the Krook shock for $M_1 = 1.2, 2.5, 6,$ and $10$ and for the Boltzmann shock for $M_1 = 8$. It was found that, for the Krook shock, the percentage difference between the moments of the distribution function is small (less than 1% for $M_1 = 1.2, 2.5,$ and $6$ and less than 3% for $M_1 = 10$) but this error in $dn/dx$ and $d\mathcal{v}/dx$ is large, for $M_1 = 6$ and $10$, especially near the hot side. Errors found for the Boltzmann shock for $M_1 = 8$ are comparable to those of the Krook shock results for large Mach numbers; however, the errors in $dn/dx$ and $d\mathcal{v}/dx$ are less than the 90% confidence limit of the mean of the four samples used.

3.4 Miscellaneous Errors in $f$ and $(a-bf)$ (shock wave)

The errors $\delta_I^c$ and $\delta_I^h$ at the end stations are reduced by factors of five to ten, for $M_1 = 1.2$ as $J$ is changed from $9$ (equal intervals) to $15$ (variable intervals, Sect. 2.25). For $M_1 = 2.5$ there is no significant effect on these errors of using the variable intervals. Our studies did not extend to the effect of variable intervals on other measures of errors in $f$.

Other conclusions expressed in terms of the various kinds of $\delta_I^f$ are:

a) For $M_1 < 10$ $\delta_I^f^a$, and $\delta_I^f$ are roughly equal to one another and
vary in a regular way with $M_1$.

b) $\delta_{I_c}^f < \delta_{I_c}^f$ (for J=13 and 15) except for $M_1 = 2.5$.

c) $\delta_{I_c}^f < \delta_{I_c}^f$ generally by a factor between two and ten.

SIGCOR(A-BF) is a measure of the average change in (a-bf) made in least square corrections at a station. (It is scaled proportionally to $a_d$ rather than to $a_{AN}$; therefore, values of this quantity for different Mach numbers cannot be compared directly.) We found that the MB correction reduces the LS correction, as measured by SIGCOR(A-BF), very substantially - by a factor of six in the most unfavorable case (cold side for $M_1 = 10$) and usually by a factor of $10^2$ to $10^5$. The MB correction is thus successful in spite of the fact that the MB correction uses values of density, temperature, and gas velocity calculated from numerical integrations over velocity space rather than more accurate values.

3.5 Tests of the Random Number Generator

Many indirect tests of the modified Juncosa random number generator have been made. Using a program written by Rick Blomme we have now also made direct statistical tests.

The tests used were chi-square tests on frequency distribution and serial correlation of five successive runs of 50 blocks of 512 numbers generated by the modified Juncosa algorithm. The numbers tested were chosen as follows:

a) generate the $k$-th double precision product in the JNCBLH sequence, where $k = k_0 + j \times KK$;

b) extract three bits (bit positions JJ to JJ+2 measured from the left) to get the $j$-th number tested;
c) repeat as needed for the statistical tests.

Runs were made for KK=1 and KK=8. It was found that the three bit numbers were statistically dependent for JJ ≤ 9 and JJ ≥ 78 (for KK=1) and for JJ ≤ 9 and JJ ≥ 75 (for KK=8). In other words, nine bits at the left and 18 to 21 bits at the right of each double precision product were "polluted", and this pollution is larger for KK=8 than for KK=1. In other studies of the random number generator we showed that the (i+2) bits on the right of each of the double precision products spaced by $2^4$ are identically equal to 1!
REFERENCES


3. The Pseudo-Shock: A Non-Linear Problem of Translational Relaxation, B. L. Hicks, CSL Report 236.


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