AN APPROXIMATION OF FORCE-TORQUE EQUATION IN ROLLING MILLS

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by

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This work was supported in part by the Joint Services Electronics Program (U.S. Army, U.S. Navy, and U.S. Air Force) under Contract DAAB 07-67-C-0199; also in part by the U.S. Air Force under Grant AFOSR 68-1579A.

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ABSTRACT

After briefly discussing several rolling theories, this report gives a derivation of strip-thickness, roll force, and torque equations. These nonlinear equations are then simplified. All the basic functions are expanded in the three term-power series with respect to the "mill variables." A computer program is developed which evaluates the coefficients of the series. The program consists of two main subroutines and allows all the mill coefficients to be obtained in less than 20 seconds on a CDC-1604 computer. The above scheme remains general for any type of metal being rolled. The method of obtaining coefficients which utilizes implicit differentiations made this program considerably faster, simpler and more accurate than any published computational schemes. A numerical example is included.
1. INTRODUCTION

The theory of rolling of metals began with the pioneer work of Siebel and Karman 1924-25 [1,2]. Since then many authors have attempted to obtain a satisfactory method for calculating the force and torque to be applied to the rolls in order to reduce the thickness of the material. In all of the attempts, there are certain assumptions and some approximations are used. Therefore the "exactness" of a theory refers to the extent of the assumptions made in the development of the theory.

The method developed by Orowan [3] is the first which presented a numerical and graphical way of computing roll force and torque. Further approximations of Orowan's theory by Bland, Ford and Ellis [4,5] made it more convenient for numerical computations. The difference between [3] and [4,5] lies in the three extra assumptions introduced in [4,5] as will be discussed in Section 2 of this report.

In the sequel roll force, torque, and strip thickness equations are expanded in a Taylor series of the mill operating parameters. A computational scheme by a digital computer is introduced which gives an appreciable improvement over Bland and Ford's algorithm.

Appendix 1 presents a list of notations and the nomenclature whose familiarity is very useful to the reader throughout this report.

2. DERIVATION OF FORCE AND TORQUE EQUATIONS

This section reviews the assumptions made by Orowan and Bland-Ford, derive the force and torque equations, and compares the two theories.
2.1. Basic Assumptions

The assumptions of Orowan are listed below [3].

a. The arc of contact between rolls and the material is always "circular."

b. The coefficient of rolling friction is constant.

c. There is no appreciable "elastic compression" of the strip.

d. There is no "spread," i.e. the entire rolling process is considered as a plane strain problem.

e. The "Huber-Mises Condition of Plasticity" holds, i.e. the yield stress is independent of hydrostatic stress. The principle stress, normal to the planes of strain, \( p_1 \) and \( p_2 \) is related to yield stress as: 
   \[
   p_1 - p_2 = k = 1.15 \kappa \]

f. Homogeneous compression: the strip is consists of perpendicular plane sections to the direction of rolling which remains normal. The vertical and horizontal stresses s and q are constant over three sections, i.e. \( s - q = k \).

The theory of Bland-Ford makes the following extra three assumptions [3]:

g. The normal roll pressure, \( p \) is approximately equal to the principle vertical stress, \( s \).

h. The following inequality holds

\[
\left( \frac{P}{k} - 1 \right) \frac{d}{d\phi}(hk) \ll \ll \frac{hk}{d\phi} \frac{P}{k}
\]

Bland and Ford have shown that this assumption holds with an error of 1 - 5%. 
i. $\sin \phi = 0$, $\cos \phi = 1$ except when the coefficient of $(1 - \cos \phi)$ is large and $\phi$ is in radius.

### 2.2. Horizontal Force and Normal Pressure

A typical 5-stand rolling mill is shown in Figure 1.

![Figure 1. A typical 4-high five-stand cold rolling mill.](image)

Figure 1. A typical 4-high five-stand cold rolling mill.

The thickness of the strip is reduced after passage through each stand. There are five roll gaps one of which is shown in Figure 2. The strip is divided into a number of vertical segments. As the strip passes through the rolls these vertical segments are compressed and wrapped together, resulting in a longitudinal expansion in the direction of rolling. By virtue of assumption (d), no lateral expansion is possible. Due to this longitudinal expansion the vertical segments squeeze each other out of the roll gap which is opposed by the forces due to the friction between the rolls and the strip. At a point defined by an arc $\phi$ and a
Figure 2. The normal pressure distribution in a roll gap.
vertical plane AB, a horizontal pressure is produced which can just overcome the friction along the surface AC and BD. Note that in the neighborhood of the "entry plane" these segments are squeezed backward while around the "exit plane" they are squeezed forward. Somewhere in the middle of the arc of contact there exists a "neutral plane" such that to its right the strip segments are squeezed to the right and to its left they are squeezed to the left. Away from the neutral plane the normal pressure $p$ decreases because the areas AC and BD decrease and the friction forces acting on volume ACDB reduce. When there is no entry or exit tension the normal roll pressure vanishes on the entry and exit planes as shown in Fig. 2. The normal pressure reaches its maximum at neutral plane arc of contact, $\phi^n$. The $p$ vs. $\phi$ curve shown is called "friction hill" by Orowan.

Appendix 2 presents a detailed derivation of the "exact" rolling theory as developed by Orowan, Bland and Ford. The resulting equations for roll force, torque and neutral plane thickness are given below,

$$F = F(h^i, h^o, t^i, t^o, \mu) = R'k \left[ \int_0^{\phi^n} (1 + \frac{R'}{h^o} \phi) (1 - \frac{t^o}{k}) e^{\mu H(h^o, \phi)} d\phi + \right.$$

$$\left. \int_{\phi^n}^{\phi^i} \frac{(h^o + R' \phi)^2}{h^i} (1 - \frac{t^i}{k^i}) e^{\mu (H^i(h^o, \phi^i) - H(h^o, \phi))} \right]$$

where

$$H(h^o, \phi) = 2\sqrt{\frac{R'}{h^o}} \tan^{-1} \sqrt{\frac{R'}{h^o}} \phi$$

(2.1)
\[
\tau = \tau(h, h^0, t, t^0, \mu) = RR'k \int_{0}^{\phi} (1 + \frac{R'}{h^0} \phi^2) (1 - \frac{\kappa}{k^0}) \phi e^{\mu H(h^0, \phi)} d\phi
\]

\[
+ \int_{0}^{\phi} (\frac{h^0 + R' \phi^2}{h^1}) (1 - \frac{\kappa}{k^1}) \phi e^{\mu (H(h^0, \phi) - h(h^0, \phi))} d\phi
\]

\[
+ R \frac{h^1 - h^0}{2} .
\]

(2.3)

Similarly:

\[
h^n = h^n(h, h^0, t, t^0, \mu) = h^0 + \frac{h^0}{4} \tan \frac{2}{\sqrt{\frac{R}{h}}} \tan^{-1} \frac{1}{\sqrt{\frac{h^1}{h}}} \phi^2
\]

\[
= \frac{1}{2\mu} \ln \left( \frac{h^1 (1 - \frac{t^0}{k^0})}{h^0 (1 - \frac{t^1}{k^1})} \right) ^2 .
\]

(2.4)

Note that all quantities are nonlinear functions of five mill parameter \( h, h^0, t, t^0, \mu \) and arc of contact \( \phi \).

3. APPROXIMATION OF ROLLING EQUATIONS

In this section the rolling equations are first expanded in a Taylor's series whose coefficients are evaluated next.

3.1. Taylor Series Expansion

The roll force, torque, and neutral plane thickness all depend on the following mill variables:
\[ F = f(h^i, h^o, t^i, t^o, \mu) \]
\[ \tau = \tau(h^i, h^o, t^i, t^o, \mu) \]
\[ h^n = h^n(h^i, h^o, t^i, t^o, \mu) \] (3.1)

for simplicity let

\[ q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix}, \quad \frac{\partial F}{\partial q} = \begin{bmatrix} h^i \\ h^o \\ t^i \\ t^o \\ \mu \end{bmatrix} \] (3.2)

Then writing a Taylor series expansion and truncating it after the second term,

\[ F(q) \approx F(q^*) + \sum_{i=1}^{5} \frac{\partial F}{\partial q_i} \Delta q_i + \frac{1}{2!} \sum_{i=1}^{5} \sum_{j=1}^{5} \frac{\partial^2 F}{\partial q_i \partial q_j} \Delta q_i \Delta q_j \] (3.3)

Similarly for \( \tau \) and \( h^n \),

\[ \tau(q) \approx \tau(q^*) + \sum_{i=1}^{5} \frac{\partial \tau}{\partial q_i} \Delta q_i + \frac{1}{2!} \sum_{i=1}^{5} \sum_{j=1}^{5} \frac{\partial^2 \tau}{\partial q_i \partial q_j} \Delta q_i \Delta q_j \] (3.4)

\[ h^n(q) \approx h^n(q^*) + \sum_{i=1}^{5} \frac{\partial h^n}{\partial q_i} \Delta q_i + \frac{1}{2!} \sum_{i=1}^{5} \sum_{j=1}^{5} \frac{\partial^2 h^n}{\partial q_i \partial q_j} \Delta q_i \Delta q_j \] (3.5)

Where \( q^* \) is the vector of the "operating" values, of mill parameter which are some known specified quantities.

It is common practice [6,7] to represent the above equations in terms of relative increments such as,
\[ \hat{F} \triangleq \frac{\Delta F}{F(q^*)} = \frac{F(q) - F(q^*)}{F(q^*)} \]

\[ \hat{\tau} \triangleq \frac{\Delta \tau}{\tau(q^*)} = \frac{\tau(q) - \tau(q^*)}{\tau(q^*)} \], etc.

For the sake of simplicity we drop the caps from all variables, thus it will be understood that we mean \( \hat{F} \) by \( F \), etc. The equations 3.3, 3.4 and 3.5 become

\[ F = \sum_{i=1}^{5} k_i q_i + \sum_{i=1}^{5} \sum_{j=1}^{5} k_{ij} q_i q_j \]  \hspace{1cm} (3.6)

\[ \tau = \sum_{i=1}^{5} \ell_i q_i + \sum_{i=1}^{5} \sum_{j=1}^{5} \ell_{ij} q_i q_j \]  \hspace{1cm} (3.7)

\[ h^n = \sum_{i=1}^{5} m_i q_i + \sum_{i=1}^{5} \sum_{j=1}^{5} m_{ij} q_i q_j \]  \hspace{1cm} (3.8)

where,

\[ k_i = \left[ \frac{q_i^*}{F^*} \right] \frac{\partial F}{\partial q_i} \] \hspace{1cm} (3.6')

\[ k_{ij} = \frac{1}{2} \left[ \frac{q_i^* q_j^*}{F^*} \right] \frac{\partial^2 F}{\partial q_i \partial q_j} \]

\[ \ell_i = \left[ \frac{q_i^*}{\tau^*} \right] \frac{\partial \tau}{\partial q_i} \] \hspace{1cm} (3.7')

\[ \ell_{ij} = \frac{1}{2} \left[ \frac{q_i^* q_j^*}{\tau^*} \right] \frac{\partial^2 \tau}{\partial q_i \partial q_j} \]

\[ m_i = \left[ \frac{q_i^*}{h^n} \right] \frac{\partial h^n}{\partial q_i} \] \hspace{1cm} (3.8')

\[ m_{ij} = \frac{1}{2} \left[ \frac{q_i^* q_j^*}{h^n} \right] \frac{\partial^2 h^n}{\partial q_i \partial q_j} \]

In vector form 3.6, 3.7, and 3.8 may be written as:

\[ \hat{F} = \langle k, q \rangle + \langle q, Kq \rangle \]  \hspace{1cm} (3.9)

\[ \hat{\tau} = \langle \ell, q \rangle + \langle q, Lq \rangle \]  \hspace{1cm} (3.10)

\[ \hat{h}^n = \langle m, q \rangle + \langle q, Mq \rangle \]  \hspace{1cm} (3.11)
where \( \langle \rangle \) presents scalar product of two vectors, \( k, \lambda, \) and \( m \) are 5-dimensional vectors and \( K, L, \) and \( M \) are \( 5 \times 5 \) matrices. Note that since the function \( F, \tau, \) and \( h^n \) are smooth and continuous and the order of partial differentiation is irrelevant, hence matrices \( K, L, \) and \( M \) are all symmetric.

### 3.2. Evaluation of Series Coefficients

In this section the details of a method of evaluating the coefficients \( k, K, \lambda, L, m, \) and \( M \) will be given. The method is explained for one coefficient of each equation and the details and the remainder may be found in Appendix 3. Rewrite equations 2.1, 2.3, and 2.4 of section 2 as follows,

\[
F(q) = K' \int_0^{\phi^0} \gamma(q_2, \phi) \delta(q_2) e^{\alpha(q_2, q_5, \phi)} d\phi + \\
K' \int_0^{\phi^i} \eta(q_1, q_2, \phi) \xi(q_3) e^{\beta(q_2, q_5, \phi)} d\phi 
\]

\[
\tau(q) = K'' \int_0^{\phi^0} \gamma \phi \delta e^{\alpha} d\phi + K'' \int_0^{\phi^i} \eta \phi \xi e^{\beta} d\phi + RR' \nu(q) 
\]

\[
h^n(q) = q_2 + \lambda(q_2) \tau(q),
\]

where \( K' = R'k \) and \( K'' = RR'k \). \( \alpha, \beta, \gamma, \delta, \eta, \xi, \lambda, \) and \( \tau \) are defined in Appendix 3.

From (3.12) the expression for the coefficients \( k_1 \) and \( k_{11} \) are

\[
k_1 = K' \frac{q_1^*}{E_k} \int_0^{\phi} \frac{\partial \eta}{\partial q_1^*} \xi e^{\beta} d\phi 
\]

(3.15a)
Similarly from (3.13) and (3.14),

\[ k_{11} = \frac{1}{2} \frac{q_1^*}{F^*} F_{11} = \frac{1}{2} K' \frac{q_1^*}{F^*} \int \phi_n^1 \frac{\partial^2 \eta}{\partial q_1^*^2} \xi e^\theta d\phi. \] (3.15b)

\[ \ell_1 = \frac{q_1^*}{\tau^*} \left[ R + K'' \int \phi_n^1 \frac{\partial \eta}{\partial q_1^*} \xi e^\theta d\phi \right] \] (3.16a)

\[ \ell_{11} = \frac{1}{2} K'' \frac{q_1^*}{\tau^*} \int \phi_n^1 \frac{\partial^2 \eta}{\partial q_1^*^2} \xi e^\theta d\phi \] (3.16b)

\[ m_1 = \frac{q_1^*}{h^*} \frac{\partial h^*}{\partial q_1^*} = \frac{q_1^*}{h^*} \frac{\partial \eta}{\partial q_1^*} \lambda \] (3.17a)

\[ m_{11} = \frac{1}{2} \frac{q_1^*}{h^*} \frac{\partial^2 h^*}{\partial q_1^*^2} = \frac{1}{2} \frac{q_1^*}{h^*} \lambda \frac{\partial^2 \eta}{\partial q_1^*^2}. \] (3.17b)

There are \(3 \times 5 = 15\) coefficients of the form \(k_1, \ell_1, m_1\), and \(3 \times \frac{5(5+1)}{2} = 45\) the forms \(k_{11}, \ell_{11}, m_{11}\).

4. COMPUTATIONAL SCHEME

It is observed (see Appendix 3) that the integrands involved in evaluating force coefficients differ only by a term \(\phi\) from those of torque coefficients, thus the same subroutine may be used to compute \(k, \ell, K,\) and \(L\).

For example, if
\[ k_1 = \frac{q_1^*}{F^*} K' \int_0^{\phi_n} f_i(q_1, q_2, q_3, q_5, \phi) d\phi \]  
(4.1)

then,

\[ l_1 = \frac{q_1^*}{\tau^*} K'' \int_0^{\phi_n} \phi f_i(q_1, q_2, q_3, q_5, \phi) d\phi \]  
(4.2)

where,

\[ f_i = \eta_1 \xi e^\beta, \text{ etc.} \]

Similarly for \( k_2 \) and \( l_2 \):

\[ k_2 = \frac{q_2^*}{F^*} K' \int_0^{\phi_n} f_{2n}(q_2, q_4, q_5, \phi) d\phi + \int_0^{\phi_n} f_{2i}(q_1, q_2, q_3, q_5, \phi) d\phi. \]

Then

\[ l_2 = \frac{q_2^*}{\tau^*} K'' \int_0^{\phi_n} \phi f_{2n} d\phi + \int_0^{\phi_n} \phi f_{2i} d\phi \]

where \( f_{2n} \) and \( f_{2i} \) are those functions defined in equation (14) of Appendix 3.

There are two sets of parameters needed for computation of the coefficients. Tables 1 and 2 present these. Note that the quantities seen in Table 1 have to be given for each stand which is present in the mill.

There are two main subroutines and eight smaller subroutines. In addition a reliable integration routine is needed to compute all the sixty coefficients. An integrating routine utilizing Simpson's rule has proven to be effective [4]. Figures 4 and 5 show a flow graph for the main program and the two main subroutines, ROLCOEF and THICKNS.
### Table 1. Mill Operating Values

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Notation</th>
<th>Symbol Used in Report</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entry thickness</td>
<td>$h^*i$</td>
<td>$q_1^*$</td>
</tr>
<tr>
<td>Exit thickness</td>
<td>$h^*i$</td>
<td>$q_2^*$</td>
</tr>
<tr>
<td>Entry tension</td>
<td>$t^*i$</td>
<td>$q_3^*$</td>
</tr>
<tr>
<td>Exit tension</td>
<td>$t^{*o}$</td>
<td>$q_4^*$</td>
</tr>
<tr>
<td>Coefficient of friction</td>
<td>$\mu^*$</td>
<td>$q_5^*$</td>
</tr>
<tr>
<td>Roll force</td>
<td>$F^*$</td>
<td>$F^*$</td>
</tr>
<tr>
<td>Roll Torque</td>
<td>$\tau^*$</td>
<td>$\tau^*$</td>
</tr>
<tr>
<td>Neutral plane thickness</td>
<td>$h^*n$</td>
<td>$h^*n$</td>
</tr>
<tr>
<td>Yield stress</td>
<td>$k$</td>
<td>$k$</td>
</tr>
</tbody>
</table>

### Table 2. Mill Data

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol Used in this Report</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial thickness</td>
<td>$h_1$</td>
</tr>
<tr>
<td>Strip width</td>
<td>$w$</td>
</tr>
<tr>
<td>Rolls undeformed radius</td>
<td>$R$</td>
</tr>
<tr>
<td>Rolls elastic coefficient</td>
<td>$c$</td>
</tr>
</tbody>
</table>
Figure 4. Main program for computing coefficients.
Figure 5a. Flow graph for subroutine ROLCOEF (continued).
Figure 5a (continued)
Figure 5a. Flow graph for subroutine ROLCOEF.
Figure 5b. Flow graph for subroutine THICKNS.
5. A NUMERICAL EXAMPLE

To illustrate the computational scheme presented in the previous section, the operating parameters of a mill introduced in [4] are used to evaluate the series coefficients of force, torque, and thickness equations. Table 3 shows the data for this example.

Computations were performed using a CDC 1604 computer and a fast and accurate integration routine* using Simpson's rule was adapted. The resulting parameters, vectors, and matrices are shown below,

Table 3. Numerical Example [4]

1. Mill Operating Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entry thickness</td>
<td>( h^* = 0.0700 ) inch</td>
</tr>
<tr>
<td>Exit thickness</td>
<td>( h^0* = 0.0490 ) inch</td>
</tr>
<tr>
<td>Entry tension</td>
<td>( t^* = 3.3000 ) tons/in²</td>
</tr>
<tr>
<td>Exit tension</td>
<td>( t^0* = 5.2000 ) tons/in²</td>
</tr>
<tr>
<td>Coefficient of friction</td>
<td>( \mu^* = 0.0860 )</td>
</tr>
<tr>
<td>Roll force</td>
<td>( F^* = 26.7000 ) tons</td>
</tr>
<tr>
<td>Roll torque</td>
<td>( T^* = 3.7200 ) tons-in</td>
</tr>
<tr>
<td>Neutral plane thickness</td>
<td>( h_n^* = 0.0580 ) inch</td>
</tr>
<tr>
<td>Mean yield stress</td>
<td>( k = 27.9000 ) tons/in²</td>
</tr>
<tr>
<td>Entry yield stress</td>
<td>( k_1 = 24.7000 ) tons/in²</td>
</tr>
<tr>
<td>Exit yield stress</td>
<td>( k^0 = 28.8000 ) tons/in²</td>
</tr>
</tbody>
</table>

2. Mill Data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial thickness</td>
<td>( h_1 = 0.1000 ) in</td>
</tr>
<tr>
<td>Mean width</td>
<td>( w = 3.0000 ) in</td>
</tr>
<tr>
<td>Rolls undeformed radius</td>
<td>( R = 5.0000 ) in</td>
</tr>
<tr>
<td>Rolls elastic coefficient</td>
<td>( c = 0.000167 ) in²/tons</td>
</tr>
</tbody>
</table>

Results of Numerical Example

Deformed radius of rolls, \( R' = 5.707 \text{ in} \)
Percent reduction of entry, \( r_i = 30\% \)
Percent reduction of exit, \( r_o = 51\% \)
Arc of contact at entry, \( \phi_i = 0.06066 \text{ radian} \)
Arc of contact at neutral point, \( \phi_n = 0.01940 \text{ radian} \)

\[
\begin{bmatrix}
-0.2498 \\
0.1152 \\
-0.0049 \\
0.1118
\end{bmatrix}
\quad \begin{bmatrix}
0.2498 & -0.0705 & 0.0058 & 0 & -0.0437 \\
-0.0371 & -0.0033 & 0.0005 & -0.0283 \\
0 & 0 & -0.0020 \\
0 & 0 & -0.0005 & 0.0236
\end{bmatrix}
\]

\( k = -0.0116 \quad K = 0.0116 \) \hspace{1cm} (5.1)

\[
\begin{bmatrix}
4.0833 \\
-4.8994 \\
1.3263 \\
-0.9416 \\
0.1116
\end{bmatrix}
\quad \begin{bmatrix}
0.3470 & -0.1031 & 0.6793 & 0 & -0.0500 \\
0.0219 & -0.0048 & -0.4696 & -0.0205 \\
0 & 0 & -0.0023 \\
0 & 0 & -0.0002 & 0.0226
\end{bmatrix}
\]

\( \lambda = 0.0116 \quad \Lambda = 0.0116 \) \hspace{1cm} (5.2)

\[
\begin{bmatrix}
-0.1309 \\
0.9890 \\
-0.0061 \\
0.0055 \\
0.0001
\end{bmatrix}
\quad \begin{bmatrix}
0.12710 & -0.00226 & 0.00288 & -0.00260 & 0.06450 \\
2.17595 & -0.0062 & 0.00560 & -0.06520 \\
0 & -0.0001 & -0.00012 & 0.00302 \\
0 & 0.00023 & -0.00274 \\
0 & 0 & -0.00184
\end{bmatrix}
\]

\( m = 0.0061 \quad M = 0.0001 \) \hspace{1cm} (5.3)
Note that most of the second order coefficients are small in magnitude, indicating that a first-order approximation which is considered by many authors [4,6,7] is perhaps satisfactory for this mill. Thus considering the truncation errors involved many elements of K, L, and M are effectively zero. This will indicate that the overall Rolling Mill System with highly nonlinear force, torque, thickness equations can be modeled by a linear first order expansion plus a small (in magnitude) second order corrections. The computation time for this example was of the order of less than 20 seconds on a CDC-1604 computer.

6. CONCLUSIONS

The roll force, torque, and neutral plane thickness equations, highly nonlinear in the mill variables \( h^i, h^o, t^i, t^o, \) and M, are expanded up to second-order. The computational scheme introduced sums to be fast and efficient and is easily applicable to any type of metal being rolled. The force and torque equations coefficients were jointly obtained by one subroutine. One basic assumption made in the computation was that the metal's yield stress is slowly varying with the arc of contact, \( \phi \). This, however, need not be of too much concern if the yield-stress percent reduction curves are available for the mill under consideration. The results of the numerical example (section 5) reveals that a linear expansion of the force-torque equations can be improved by a second-order corrections.

ACKNOWLEDGMENTS

The authors wish to thank the Coordinated Science Laboratory for their support and the use of its facilities. We are also thankful to
Mrs. Rose Lane for her excellent typing of the manuscript.

This work was supported in part by the U.S. Air Force under Grant AFOSR 68-1579A, in part by the Joint Services Electronics Program (U.S. Army, U.S. Navy, and U.S. Air Force) under Contract No. DAAB-07-67-C-0199 with the Coordinated Science Laboratory, University of Illinois, Urbana, Illinois.

REFERENCES


Appendix 1. Notation and Nomenclature

The following superscripts are used throughout the report:

\( i \) = "entry" plane
\( o \) = "exit" plane
\( n \) = neutral plane
\( e \) = experimental value of a quantity
\( 1 \) = initial value
\( I \) = sometimes to exit and entry segments of the strip.

The following quantities are used throughout this report:

\( \phi \) = arc of contact, radians
\( R \) = radius of undeformed roll (effective roll radius)
\( R' \) = radius of deformer roll (arc of contact)
\( h \) = thickness of the strip
\( P \) = normal roll pressure
\( k \) = yield stress of the strip
\( t \) = tension per unit area of the strip
\( \tau \) = roll torque per unit width
\( F \) = roll force per unit width
\( w \) = width of the strip
\( r \) = percent reduction in the strip thickness
\( \mu \) = coefficient of friction
\( q, s \) = principle stresses on horizontal and vertical planes
\( f \) = horizontal force on an element of unit width
\( c = \text{elastic constant of the rolls} \)

\[
\frac{8(1-\alpha^2)}{\pi Y}
\]

where \( \alpha = \text{Poisson's ratio} \) and \( Y = \text{Young's modulus} \)

\( k^* = \text{yield stress in tension} \)

Special Notation:

Given a function of several variables, \( F(q_1, q_2, q_3, \ldots, q_n) \) then by \( F_{13} \) it is meant:

\[
F_{13} = \frac{\partial^2 F}{\partial q_1 \partial q_3}, \text{ etc.}
\]

Note that this notation is only applicable for \( F, \tau, h^n, \alpha, \beta, \gamma, \delta, \eta, \xi, \lambda, \) and \( \pi \) as they appear.
Appendix 2. Derivation of Exact Roll Force, Torque, and Neutral Plane Thickness

The horizontal force per unit width \( f(x) \) is a function of \( x \) measured from the plane of exit as shown in Figure 2. Consider a segment whose horizontal displacement is \( dx \), the force exerted on this segment is

\[
f(x) - f(x+dx) = -\frac{df}{dx} \, dx.
\]

This force is cancelled with the horizontal component of the friction forces on both top and bottom segments. The resulting horizontal force due to normal pressure is (see Figures 2 and 3)

\[
2P \sin\phi \cdot \frac{dx}{\cos\phi} = 2P \tan\phi \, dx.
\]

The horizontal frictional force acting on the top and bottom segments is

\[
\pm 2\mu P \frac{dx}{\cos\phi} = \pm 2\mu P \, dx.
\]

The balancing equation is

\[
-\frac{df}{dx} \, dx + 2P \tan\phi \, dx \pm 2\mu P \, dx = 0
\]

or

\[
\frac{df}{dx} = 2P (\tan\phi \pm \mu) \tag{1}
\]

Introducing \( \phi \) as the independent variable through the following relations:

\[
x = R'\sin\phi, \quad dx = R'\cos\phi \, d\phi \tag{2}
\]
(1) can be rewritten as:
\[
\frac{df}{d\phi} = 2R'p(s\phi \pm \mu \cos \phi). \tag{3}
\]

This equation relates the horizontal force per unit width on a plane located at an angle $\phi$ with the normal roll pressure exerted on the strip (see Figure 3).

A second relation between $f$ and $p$ is found by considering assumption (g), since $q = f/h$:

Thus:
\[
p = q + k = \frac{f}{h} + k. \tag{4}
\]

Now eliminating $f$ from (3) and (4),
\[
\frac{d}{d\phi} \left[ h(p-k) \right] = 2PR'(s\phi \pm \mu \cos \phi)
\]
or
\[
hk \frac{d}{d\phi} \left( \frac{p}{k} \right) + \left( \frac{p}{k} - 1 \right) \frac{d}{d\phi} (hk) = 2PR'(s\phi \pm \mu \cos \phi)
\]
using assumption (h):
\[
hk \frac{d}{d\phi} \left( \frac{p}{k} \right) = 2PR'(s\phi \pm \mu \cos \phi)
\]
or
\[
\frac{d}{d\phi} \left( \frac{p}{k} \right) = \frac{2R'}{h} (s\phi \pm \mu \cos \phi) \tag{5}
\]

from assumption (i):
Figure 3: Mill variables associated with each roll gap.
\[ h = h^0 + 2R'(1 - \cos \phi) = h^0 + 2R'(1 - 1 + \frac{\phi^2}{2!} - \frac{\phi^4}{4!} + \ldots) = h^0 + R'\phi^2 \] and
\[ \sin \phi \pm \mu \cos \phi \approx \phi \pm \mu \]

So integrating (5):
\[ \int \frac{d}{d\phi} \left( \frac{P}{k} \right) d\phi = \int \frac{2R'(\phi \pm \mu)}{h^0 + R'\phi^2} d\phi \]
or
\[ \ln \left( \frac{P}{k} \right) = \int \frac{d}{d\phi} \left( \frac{h}{R'} \right) \pm 2\mu \sqrt{\frac{R'}{h^0}} \tan^{-1} \sqrt{\frac{R'}{h^0}} \phi + \text{const.} \]
or
\[ P = A \cdot k \left( \frac{h}{R'} \right)^{\pm \mu H} \tag{6} \]

where; \( A \) is a constant and \( H \) is defined as:
\[ H(h^0, \phi) = 2\sqrt{\frac{R'}{h^0}} \tan^{-1}\sqrt{\frac{R'}{h^0}} \phi \] . \tag{7}

To evaluate \( A \), note that at the exit:
\[ q = -t^0 \text{ and } H(h^0, 0) = 0, \text{ from (4)}: \]
\[ P^0 = k^0 - t^0 \text{ and from (6)}: \]
Thus Equation (6) will become:

\[ A = \frac{R_i}{h_o} (1 - \frac{t_o}{k}) \]

Thus Equation (6) will become:

\[ p^o = \frac{kh}{h_o} (1 - \frac{t_o}{k}) \ e^{\mu H} \]  \hspace{1cm} (8)

and

\[ p^i = \frac{kh}{h_i} (1 - \frac{t_i}{k}) \ e^{\mu(H^i - H)} \]  \hspace{1cm} (9)

where

\[ h^i = H(\phi^i) = 2 \sqrt{\frac{R_i}{h_o}} \tan^{-1}\sqrt{\frac{R_i}{h_o}} \phi^i. \]

Neutral Plane Angle and Thickness

It is a common practice to evaluate the neutral plane angle and thickness \([6,7]\) because they will be instrumental in the computation as it became clear in Sections 3 and 4. From the previous section it was noted that,

\[ h = h^o + R' \phi^2. \]  \hspace{1cm} (10)

Furthermore at neutral plane,

\[ p^o_n = p^i_n \]  \hspace{1cm} (11)

hence from (11) it follows that
Thus from (7):

$$\phi^n = \sqrt{\frac{h^o}{R'}} \tan \left( \frac{\sqrt{\frac{h^o}{R'}} \cdot \frac{H^n}{2}}{h^o} \right)$$  \hspace{1cm} (13)

end $H^n$ is as defined in (12). Using (10), (13), and expression for $H^i$,

$$h^n = h^n(h^i, h^o, t^i, t^o, \mu) = h^o \left\{ 1 + \frac{1}{4} \tan^2 \left( \frac{R'^i}{h^o} \cdot \tan^{-1} \left( \frac{R'^i}{h^o} \phi^i \right) \right) \right. $$

$$ - \frac{1}{2\mu} \ln \left( \frac{h^i(1 - \frac{t^o}{k^o})}{h^o(1 - \frac{t^i}{k^i})} \right)^2 \} \right. $$  \hspace{1cm} (14)

**Roll Force and Torque**

With assumption g and Figure 3, the roll force per unit width of the strip is the integral of principle stress on the vertical plane, $s \approx P$ along the contacting arc,

$$F = \int_0^{\phi^i} s \cdot R' d\phi \approx R' \int_0^{\phi^i} P d\phi$$

$$F = R' \int_0^{\phi^i} P^o d\phi + \int_0^{\phi^i} P^i d\phi \cdot \phi^n$$  \hspace{1cm} (15)

Substituting for $P^o$ and $P^i$ from (8) and (9),
The torque per unit width is the integral of the functional force moments about the rolling axis taken along the arc of contact. Note that the normal components of these functional forces do not have an appreciable moment about the rolling axis (see Figure 3). The lever arm of moments is the distance between the rolling axis and the center of the rolls, i.e. the radius of the undeformed roll, R. Thus,

\[
\tau = \mu_R R' \left[ \int_{\phi_n}^{\phi_i} P_i d\phi - \int_{\phi_0}^{\phi_i} P_0 d\phi \right]
\]

A careful inspection of (17) will show that the accuracy of the calculations in torque will not be great since one deals with the difference of two very small quantities and hence a small error in either one will result in a large error in torque, thus to get around this difficulty integrate(3) with respect to \( \phi \) and noting that \( f^i = -h^i t^i \) and \( f^0 = -h^0 t^0 \),

\[
f^0 - f^i = h^0 t^0 - h^i t^i = 2R' \left[ \int_{\phi_0}^{\phi_i} P\sin\phi d\phi - \mu \left( \int_{\phi_n}^{\phi_i} P_i\cos\phi d\phi - \int_{\phi_0}^{\phi_i} P_0\cos\phi d\phi \right) \right]
\]
Finally using assumption \(i\), rearranging and substituting in (17)

\[
h^o t^o - h^i t^i = 2R' \left[ \int_0^{\phi_i} \phi^i d\phi - \mu \left( \int_0^{\phi^i} \phi^i d\phi - \int_0^{\phi^o} \phi^o d\phi \right) \right]
\]

and

\[
\tau = \mu RR' \left[ - \frac{h^o t^o - h^i t^i}{2\mu R'} + \frac{1}{\mu} \int_0^{\phi_i} \phi^i d\phi \right]
\]

Using (8) and (9),

\[
\tau = \tau(h^i, h^o, t^i, t^o, \mu) = RR' \left[ \int_0^{\phi^i} \frac{kh}{k^i} \left( 1 - \frac{\phi^0}{\phi^i} \right) \phi e^{\mu H} d\phi + \int_0^{\phi^i} \frac{kh}{k^i} \left( 1 - \frac{\phi^i}{\phi^0} \right)
\]

\[
+ \frac{h^i t^i - h^o t^o}{2R'} \right]
\]

Thus equations (14), (16), and (19) are the three desired, thickness, force and torque. They are all functions of five mill parameters \(h^i, h^o, t^i, t^o,\) and \(\mu\).
Appendix 3. Partial Derivatives of Force, Torque, and Thickness

The essential calculations is that of finding the elements of the following vectors and matrices:

1. Force \( k = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{bmatrix} \quad K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{22} & k_{23} & k_{24} & k_{25} \\ k_{33} & k_{34} & k_{35} \\ k_{44} & k_{45} \\ k_{55} \end{bmatrix} \) \( (1) \)

2. Torque \( l = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \end{bmatrix} \quad L = \begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} \\ l_{22} & l_{23} & l_{24} & l_{25} \\ l_{33} & l_{34} & l_{35} \\ l_{44} & l_{45} \\ l_{55} \end{bmatrix} \) \( (2) \)

3. Thickness \( m = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix} \quad M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{22} & m_{23} & m_{24} & m_{25} \\ m_{33} & m_{34} & m_{35} \\ m_{44} & m_{45} \\ m_{55} \end{bmatrix} \) \( (3) \)

Thus all together there are 60 coefficients to be computed, but as it will be seen 8 of them are zero.
The function introduced in equations (3.12), (3.13), and (3.14) are defined as follows:

\[ \gamma: \quad \gamma(q_2, \phi) = 1 + R' \frac{\phi^2}{q_2} \]  \hspace{1cm} (4)

\[ \delta: \quad \delta(q_4) = 1 - \frac{q_4}{k^o} \]  \hspace{1cm} (5)

\[ \eta: \quad \eta(q_1, q_2, \phi) = \frac{q_2 + R'\varphi^2}{q_1} \]  \hspace{1cm} (6)

\[ \xi: \quad \xi(q_3) = 1 - \frac{q_3}{k^I} \]  \hspace{1cm} (7)

\[ \alpha: \quad \alpha(q_2, q_5, \phi) = 2q_5 \sqrt{R'/q_2} \tan^{-1} \sqrt{R'/q_2} \phi \]  \hspace{1cm} (8)

\[ \beta: \quad \beta(q_2, q_5, \phi) = 2q_5 \sqrt{R'/q_2} \left[ \tan^{-1} \sqrt{R'/q_2} \phi^i - \tan^{-1} \sqrt{R'/q_2} \phi \right] \]

\[ = \alpha(q_2, q_5, \phi^i) - \alpha(q_2, q_5, \phi) \]  \hspace{1cm} (9)

\[ \nu: \quad \nu(q_1, q_2, q_3, q_4) = \frac{R}{2} (q_1 - q_3 - q_2q_4) \]  \hspace{1cm} (10)

\[ \lambda: \quad \lambda(q_2) = \frac{1}{4q_2} \tan^2 \sqrt{R'/q_2} \]  \hspace{1cm} (11)

\[ \pi: \quad \pi(q) = \left[ \sqrt{R'/q_2} \tan^{-1} \sqrt{R'/q_2} \phi^i - \frac{1}{2q_5} \ln \left( \frac{q_1 - q_4/k^o}{q_2 - 1 - q_3/k^I} \right) \right]^2 \]

\[ = \frac{1}{4q_5^2} \left[ \alpha(q_2, q_5, \phi^i) - \ln \frac{q_1 \delta(q_4)}{q_2 \xi(q_3)} \right]^2 \]  \hspace{1cm} (12)
with these preliminary definitions one may proceed to evaluate all the coefficients as follows

1. Force Coefficients:

Vector $k$:

$$k_i = \left. \frac{q^*_i}{F^*_i} \frac{\partial F(q)}{\partial q_i} \right|_{q^*_i} = \frac{q^*_i}{F^*_i} F_i, \quad i = 1,2,\ldots,5$$

$$F_1 = K' \int^{\phi_1}_{\phi_n} \eta_1 \xi e^{\theta} \, d\phi$$  \hspace{1cm} (13)

$$F_2 = K' \int^{\phi_n}_{\phi_1} (\gamma_2 \delta + \gamma \delta \alpha_2) e^{\theta} \, d\phi + K' \int^{\phi_1}_{\phi_n} (\eta_2 \xi + \eta \xi \beta_2) e^{\theta} \, d\phi$$ \hspace{1cm} (14)

$$F_3 = K' \int^{\phi_n}_{\phi_1} \eta_3 \xi e^{\theta} \, d\phi$$  \hspace{1cm} (15)

$$F_4 = K' \int^{\phi_n}_{\phi_1} \gamma \delta_4 e^{\theta} \, d\phi$$  \hspace{1cm} (16)

$$F_5 = K' \int^{\phi_n}_{\phi_1} \gamma \delta_5 e^{\theta} \, d\phi + K' \int^{\phi_1}_{\phi_n} \eta \xi \beta_5 e^{\theta} \, d\phi$$ \hspace{1cm} (17)

Matrix $K$:

$$k_{ij} = \frac{1}{2} \left. \frac{q^*_i}{F^*_i} \frac{q^*_j}{F^*_j} \right|_{q^*_i} F_{ij}, \quad i = 1,\ldots,5 \quad j = 1,\ldots,5$$

$$F_{11} = K' \int^{\phi_1}_{\phi_n} \eta_1 \xi e^{\theta} \, d\phi$$ \hspace{1cm} (18)
\[ F_{12} = K' \int_{\phi_n}^{\phi_i} (\eta_{12} + \eta_{12}) e^\theta d\phi \]  
(19)

\[ F_{13} = K' \int_{\phi_n}^{\phi_i} \eta_{13} e^\theta d\phi \]  
(20)

\[ F_{14} = K' \int_{\phi_n}^{\phi_i} (\eta_1^e e^\theta) d\phi = 0 \]  
(21)

\[ F_{15} = K' \int_{\phi_n}^{\phi_i} \eta_{14} e^\theta d\phi \]  
(22)

\[ F_{22} = K' \int_{\phi_n}^{\phi_i} [\gamma_{22} e^\theta + 2 \gamma_{22} \delta^2 + \gamma^2 \alpha_{22} + \gamma^2 \alpha_{22}^2] e^\theta d\phi \]  
(23)

\[ F_{23} = K' \int_{\phi_n}^{\phi_i} \xi_2 (\eta_2 + \eta_{22}) e^\theta d\phi \]  
(24)

\[ F_{24} = K' \int_{\phi_n}^{\phi_i} (\gamma_2 + \gamma \alpha_{22}) \delta^4 e^\theta d\phi \]  
(25)

\[ F_{25} = K' \int_{\phi_n}^{\phi_i} [\alpha_{25} + (\gamma_2 + \gamma \alpha_{22}) \delta e^\alpha d\phi + \]  
\[ K' \int_{\phi_n}^{\phi_i} [\beta_{25} \gamma + (\eta_2 + \eta_{22}) \beta_5] e^\theta d\phi \]  
(26)

\[ F_{33} = 0 \]  
(27)
\[ F_{34} = 0 \]  \hspace{1cm} (28)

\[ F_{35} = K' \int_{\phi^n} \eta \xi_3 \beta_5 \ e^\beta d\phi \]  \hspace{1cm} (29)

\[ F_{44} = 0 \]  \hspace{1cm} (30)

\[ F_{45} = K' \int_{0}^{\phi^n} \gamma \xi_4 \alpha_5 \ e^\alpha d\phi \]  \hspace{1cm} (31)

\[ F_{55} = K' \int_{0}^{\phi^n} \gamma \xi \xi_5 \beta_5 \ e^\beta d\phi + K' \int_{\phi^n} \eta \xi^2 \beta_5 \ e^\beta d\phi \]  \hspace{1cm} (32)

2. Torque Coefficients

The inspection of equations (3.12) and (3.13) of Section 3 indicates that the integrals involved in the torque equation differ from those of the force by only a term \( \phi \) in the integrand hence the integral part of all partials of \( \tau \) with respect to \( q \) will remain the same as \( F \) except to multiply a term \( \phi \) in the integrands. Furthermore since \( \nu \) is a linear function of \( q \) then all the second partials of \( \nu \) with respect to \( q \) are zero except for \( \frac{\partial^2 \nu}{\partial q_2 \partial q_4} \) and \( \frac{\partial^2 \nu}{\partial q_1 \partial q_3} \) in evaluating the elements of matrix \( L \). For example:

\[ \ell_4 = \frac{q_4^*}{\tau^*} \tau_4 = \frac{q_4^*}{\tau^*} \nu_4 + \int_{0}^{\phi^n} f(q_2, q_5, \phi) \phi d\phi \]

where \( f(q_2, q_4, \phi) = \gamma \delta_4 e^{\alpha} \) as seen in the expression for \( k_4 \) previously. Thus the same routines can be used to compute both \( k, K \) and \( \ell, L \) elements.
3. Thickness Coefficients

Vector $m$:

$$m_i = \frac{q_i^* h_i^n}{h_i^n} \quad i = 1, \ldots, 5$$

$$h_1^n = \lambda n_1$$  \hspace{1cm} (33)

$$h_2^n = 1 + \lambda_2 n_1 + \lambda n_2$$  \hspace{1cm} (34)

$$h_3^n = \lambda n_3$$  \hspace{1cm} (35)

$$h_4^n = \lambda n_4$$  \hspace{1cm} (36)

$$h_5^n = \lambda n_5$$  \hspace{1cm} (37)

Matrix $M$:

$$m_{ij} = \frac{1}{2} \frac{q_i^* q_j^*}{h_i^n h_j^n} \quad i = 1, \ldots, 5 \quad j = 1, \ldots, 5$$

$$h_{11}^n = \lambda n_{11} \quad i = 1, 2, \ldots, 5$$

$$h_{22}^n = \lambda_2 n_{22} + 2\lambda_2 n_2 + \lambda n_{22}$$  \hspace{1cm} (38)

$$h_{2i}^n = \lambda_2 n_{2i} + \lambda n_{2i} \quad i = 3, 4, 5$$  \hspace{1cm} (39)

$$h_{3i}^n = \lambda n_{3i} \quad i = 3, 4, 5$$  \hspace{1cm} (40)
4. Evaluation of Partial Derivatives

The following partials are needed for computations:

1. \( \gamma : \quad \gamma_2 = - \frac{R' \phi^2}{q_2^2} \)  
   \( \gamma_{22} = \frac{2R' \phi^2}{q_2^3} \)  

2. \( \delta : \quad \delta_4 = - \frac{1}{k_0} \)  

3. \( \eta : \quad \eta_1 = - \frac{q_2 + R' \phi^2}{q_1^2} \)  
   \( \eta_{11} = 2 \frac{q_2 + R' \phi^2}{q_1^3} \)  
   \( \eta_2 = \frac{1}{q_1} \)  
   \( \eta_{12} = - \frac{1}{q_1^2} \)  

4. \( \xi : \quad \xi_3 = - \frac{1}{k_1} \)
5. $\alpha$:  

\[
\begin{align*}
\alpha_2 &= -q_5 \left[ \frac{\sqrt{R'}/q_2^5}{2} \tan^{-1} \sqrt{R'}/q_2 \phi + \frac{R'\phi}{q_2 + R'q_2\phi^2} \right] \quad (51) \\
\alpha_{22} &= q_5 \left[ \frac{3}{2} \frac{\sqrt{R'}/q_2^5}{2} \tan^{-1} \sqrt{R'}/q_2 \phi + \frac{R'\phi}{2q_2^2(q_2 + R'\phi^2)} \right. \\
&\left. + \frac{R'(q_2 + R'\phi^2)}{(q_2 + R'q_2\phi^2)^2} \right] \quad (52) \\
\alpha_5 &= 2\frac{\sqrt{R'}/q_2^5}{2} \tan^{-1} \sqrt{R'}/q_2 \phi \quad (53) \\
\alpha_{25} &= -\left[ \frac{\sqrt{R'}/q_2^5}{2} \tan^{-1} \sqrt{R'}/q_2 \phi - \frac{R'\phi}{q_2(q_2 + R'\phi^2)} \right] \quad (54)
\end{align*}
\]

6. $\beta$:  

\[
\begin{align*}
\beta_2 &= \alpha_2^i - \alpha_2 \text{, where } \alpha_2^i = \alpha(q_2, q_5, \phi_i) \quad (55) \\
\beta_{22} &= \alpha_{22}^i - \alpha_{22} \quad (56) \\
\beta_5 &= \alpha_5^i - \alpha_5 \quad (57) \\
\beta_{25} &= \alpha_{25}^i - \alpha_{25} \quad (58)
\end{align*}
\]

7. $\nu$:  

\[
\begin{align*}
\nu_1 &= \frac{R}{2} q_3 \quad (59) \\
\nu_2 &= -\frac{R}{2} q_4 \quad (60)
\end{align*}
\]
\( \nu_3 = \frac{R}{2} q_1 \)  
(61)  

\( \nu_4 = -\frac{R}{2} q_2 \)  
(62)  

\( \nu_{13} = \frac{R}{2} \)  
(63)  

\( \nu_{24} = -\frac{R}{2} \)  
(64)  

8. \( \lambda \): \quad \text{let} \quad \rho = \sqrt{q_2/R'}  

\( \lambda_2 = \frac{1}{4} \tan^2 \rho + \frac{1}{4} \rho \cdot \tan(\rho) \sec^2(\rho) \)  
(65)  

\( \lambda_{22} = \frac{3}{8R'} \rho \tan(\rho) \sec^2(\rho) + \frac{1}{8R'} \sec^2(\rho) (\sec^2(\rho) + 2 \tan^2(\rho)) \)  
(66)  

9. \( \pi \): \quad \text{Let} \quad b = \alpha^i - \ln \frac{q_1 \delta}{q_2 \sigma}  
(67)  

\( \pi_{11} = -\frac{1}{2q_1 q_5} \cdot b \)  
(68)  

\( \pi_{11} = \frac{1}{2q_1 q_5} \cdot (b + 1) \)  
(69)  

\( \pi_2 = \frac{1}{2q_5} \left( \alpha^i \cdot \frac{1}{q_2} + \frac{1}{q_2} \right) \cdot b \)  
(70)  

\( \pi_{22} = \frac{1}{2q_5} \left[ \left( \alpha^i_{22} \cdot \frac{1}{q_2} \right) \cdot b + \left( \alpha^i_{2} + \frac{1}{q_2} \right) \right] \)  
(71)
\[
\begin{align*}
\Pi_3 &= - \frac{1}{2k} \frac{q_5^2 \xi^2}{q_5^2} \cdot b \\
\Pi_{33} &= \frac{1}{2q_5^2k^2} \xi^2 (1 - b) \\
\Pi_4 &= \frac{1}{2k^2q_5^2} \cdot b \\
\Pi_{44} &= \frac{1}{2q_5^2k^2} \xi^2 (1 + b) \\
\Pi_5 &= \frac{b \ln(\cdot)}{2q_5^3} = \frac{(\alpha^i - b) b}{2q_5^3}, \quad \Pi = [f(q_2, \phi^i) - \frac{1}{2q_5} \ln(\cdot)]^2 \\
\Pi_{55} &= \frac{\ln(\cdot)}{2q_5^3} \left( \alpha^i_5 - \frac{3b}{q_5} \right) = \frac{(\alpha^i - b) \left( \alpha^i_5 - \frac{3b}{q_5} \right)}{2q_5^3} \\
\Pi_{12} &= - \frac{1}{2q_1q_5^2} \frac{\alpha^i_2}{q_2} + \frac{1}{q_2} \\
\Pi_{13} &= + \frac{1}{2k} \frac{q_1^2q_5^2 \xi^2}{q_5^2} \\
\Pi_{14} &= - \frac{1}{2k} \frac{q_1^2q_5^2 \xi^2}{q_5^2} \\
\Pi_{15} &= \frac{1}{q_1^2q_5^2} \left( \frac{b}{q_5} - \frac{1}{2} \alpha^i_5 \right)
\end{align*}
\]
\[ \pi_{23} = \frac{-1}{2k \cdot q_3} \left( \alpha_5^i + \frac{1}{q_2} \right) \] (82)

\[ \pi_{24} = \frac{1}{2k \cdot q_2} \left( \alpha_2^i + \frac{1}{q_2} \right) \] (83)

\[ \pi_{25} = \frac{1}{2q_3} \left[ \left( \alpha_2^i + \frac{1}{q_2} \right) \alpha_5^i + \alpha_2^i \cdot \alpha_2^i - \ln \frac{q_1 \delta}{q_2 \xi} \right] \] (84)

\[ \pi_{34} = \frac{-1}{2k \cdot q_3} \] (85)

\[ \pi_{35} = \frac{-1}{2k \cdot q_5} \left[ \alpha_5^i - \frac{2}{q_3} \left( \alpha^i - \ln \frac{q_1 \delta}{q_2 \xi} \right) \right] \] (86)

\[ \pi_{45} = \frac{1}{2k \cdot q_5} \left[ \alpha_5^i - \frac{2}{q_3} \left( \alpha^i - \ln \frac{q_1 \delta}{q_2 \xi} \right) \right] \] (87)
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After briefly discussing several rolling theories, this report gives a derivation of strip-thickness, roll force, and torque equations. These nonlinear equations are then simplified. All the basic functions are expanded in the three term-power series with respect to the "mill variables." A computer program is developed which evaluates the coefficients of the series. The program consists of two main subroutines and allows all the mill coefficients to be obtained in less than 20 seconds on a CDC-1604 computer. The above scheme remains general for any type of metal being rolled. The method of obtaining coefficients which utilizes implicit differentiations made this program considerably faster, simpler and more accurate than any published computational schemes. A numerical example is included.
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