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IN A COLD BEAM–PLASMA SYSTEM

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Abstract

The coupling of 3 fixed phase waves in a cold beam-plasma system is discussed for all cases that result from locating the waves on the positive and negative energy branch. Explicit expressions are given for the spatial variation of the wave amplitudes including the influence of wave number mismatch and spatial growth rates. The coupling constants are computed for the decay and the explosive mode for representative beam and plasma parameters.
I. INTRODUCTION

The interaction of waves with waves can be considered as one of the basic processes of nonlinear plasma dynamics and consequently plays a dominant role for the understanding of many nonlinear plasma phenomena. The simplest situation obtains when only three waves participate in the interaction and it is this case that has received most attention in the recent literature.\(^{1-18}\)

For an effective interaction between three waves to occur it is necessary that the selection rules
\[ \omega_1 + \omega_2 = \omega_0 \]
and
\[ k_1 + k_2 = k_0 \]
are satisfied. Whether these conditions can be met depends on the dispersion properties of the waves in question.

One system which is particularly well adapted for the study of three wave coupling phenomena is a beam-plasma system in which a cold electron beam interacts with a cold plasma in which collisions of electrons with ions or neutral atoms are important. The mode structure of this system allows for an effective three wave coupling over a wide range of frequency and wave number. Furthermore, since the system has a positive and a negative energy branch, the waves can distribute on these two branches in such a way that the two nonlinear instabilities, namely the decay- and the explosive instability, can occur.
This report is organized as follows: In section II we review the wave-wave coupling equations for monochromatic waves as derived from perturbation theory, in section III we discuss the solution for the different cases under the assumption that there is no spatial linear growth. In section IV we include the linear spatial growth (or damping) for the special case that all growth (damping) rates are equal. In section V we present numerical calculations of the coupling constants for physically relevant situations.
II. GENERAL CONSIDERATIONS

The interaction of a monoenergetic electron beam with a cold plasma in the presence of collisions is described by the dispersion relation

\[ \epsilon(k, \omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\nu_c)} - \frac{\omega_b^2}{(\omega - k\nu_o)^2} = 0 \]

The solutions \( \omega(k) \) and \( k(\omega) \) will be referred to as the temporal and spatial case respectively. The latter one is of particular interest in experimental situations and will be treated in detail. Fig. 1 shows a plot of the real parts of the frequency and wavenumber for the temporal and spatial case for selected parameters that are typical of experimental situations of interest to us. One can see by inspection that for the unstable waves the selection rules for 3-wave coupling can be approximately satisfied with a small mismatch in either \( \omega \) or \( k \), so that we have in general for the temporal case

\[ k_1 + k_2 = k_o \]
\[ \omega_1 + \omega_2 = \omega_o + \Delta \omega \]

for the spatial case

\[ \omega_1 + \omega_2 = \omega_o \]
\[ k_1 + k_2 = k_o + \Delta k \]

We shall see later that this mismatch is typically of the order of 1%.

The derivation of kinetic equations for the waves can either start from the hydrodynamic equations or from the Vlasov equation. In
the case under consideration either method may be followed. The hydro-
dynamic equations allow to include collisions in a straightforward way.
Collisions, however, turn out to be relatively unimportant for the wave
coupling dynamics and can be neglected in the resulting coupling constants.
The coupling constants obtained from the hydrodynamic equations are then
identical to those obtained from the Vlasov equation. It should be
noted, however, that collisions must be retained in computing the mode
structure. Starting from the Vlasov equation and expanding the distribu­
tion function in powers of the wave amplitude to second order we obtain
a kinetic equation for the wave potential in the form\(^2,12\)

\[
2 \varepsilon(k, \omega) \varphi(k, \omega) + \sum_{k' + k'' = k} \varepsilon^{(2)}(k, \omega; k', \omega'; k'', \omega'') \varphi(k', \omega') \varphi(k'', \omega'') = 0
\]

where \(\varepsilon(k, \omega)\) and \(\varepsilon^{(2)}(k, \omega; k', \omega', k'', \omega'')\) are linear and the second order
dielectric functions. \(\varepsilon^{(2)}(k, \omega; k', \omega'; k'', \omega'')\) is given by

\[
\varepsilon^{(2)}(k, \omega; k', \omega'; k'', \omega'') = \sum_{j} \frac{4\pi e_i^3}{2k^2m_j^2} \int \frac{1}{w-kv} \left[ \begin{array}{cccc} k\cdot\hat{\delta} & 1 & k''\cdot\hat{\delta} & k'\cdot\hat{\delta} \\ \omega'' & \omega'' & \omega'' & \omega'' \\ \omega' & \omega' & \omega' & \omega' \\ \omega' & \omega' & \omega' & \omega' \end{array} \right] f_{oj}
\]

where the summation runs over the different plasma constituents. We con­
sider the interaction of 3 colinear longitudinal waves of fixed phase
(this corresponds to the experimental situation where the waves are launched
externally.). Introducing the electric field via \(\imath k E(k, \omega) = \varphi(k, \omega)\) and
assuming a slow variation of the amplitude in time we obtain
The modes \( \omega_i, k_i \) are solutions of the dispersion relation

\[ \epsilon(k_i, \omega_i) = 0 \]

Furthermore we have used the definitions

\[ M(k, \omega; k', \omega'; k'', \omega'') = \frac{\omega^2}{\left| k \right| |k'| |k''|} \epsilon(2)(k, \omega; k', \omega'; k'', \omega'') ; \]

\[ S_k = \text{sgn} \frac{\partial \epsilon}{\partial \omega_k} ; \Delta \omega = \omega_k - \omega_{k_2} - \omega_o \]

The coupling constants satisfy the symmetry relations

\[ M(k, \omega; k', \omega'; k'', \omega'') \]
\[ = M(k', \omega'; k'', \omega''; k, \omega) \]
\[ = M(k', \omega'; -k'', -\omega''; k, \omega) \]
\[ = M(k'', \omega''; -k', -\omega'; k, \omega) \]

with \( k = k' + k'' \).

\( \gamma_{k_i} \) is the linear temporal growth rate of the \( i \)th mode. It is convenient to introduce the notation
(7) \[ C_{k_i}(t) = \frac{1}{8\pi} \frac{\partial \varepsilon}{\partial \omega_{k_i}} \left| E_{k_i}(t) \right|^{\frac{1}{2}}, \quad (i = 0, 1 \text{ and } 2), \]

\[ V_{k,k',k''} = \frac{1}{8\pi} \frac{\partial \varepsilon}{\partial \omega_k} \frac{1}{\partial \omega_{k'}} \frac{1}{\partial \omega_{k''}} \left| \frac{1}{8\pi} \frac{\partial \varepsilon}{\partial \omega_{k'}} \frac{1}{\partial \omega_{k''}} \right|^{\frac{1}{2}} = \frac{k^2}{8\pi} \frac{\epsilon(2)}{\partial \omega_k} \frac{\partial \varepsilon}{\partial \omega_{k'}} \frac{\partial \varepsilon}{\partial \omega_{k''}} \left| \frac{1}{8\pi} \frac{\partial \varepsilon}{\partial \omega_{k'}} \frac{1}{\partial \omega_{k''}} \right|^{\frac{1}{2}} . \]

\[ |C_{k_i}|^2 \] is the effective occupation number of the wave \( \omega_i, k_i \).

Eq. 4 then take the form

(8) \[ \frac{\partial}{\partial t} C_{k_0} = - S_0 V C_{k_1} C_{k_2} e^{-i\Delta\omega t}, \]

\[ \frac{\partial}{\partial t} C_{k_1} = S_1 V C_{k_2} C_{k_0} e^{i\Delta\omega t}, \]

\[ \frac{\partial}{\partial t} C_{k_2} = S_2 V C_{k_0} C_{k_1} e^{i\Delta\omega t}, \]

with \( S_0 = \text{Sgn}(\omega S_{k_0} \omega_{k_1} \omega_{k_2}) \),

(9) \[ S_1 = \text{Sgn}(\omega S_{k_1} \omega_{k_2} \omega_{k_0}) , \]

\[ S_2 = \text{Sgn}(\omega S_{k_2} \omega_{k_0} \omega_{k_1}) . \]

Here we have used the fact that \( C_{-k} = C_k^* \)

For the case where \( \gamma_{k_0} = \gamma_{k_1} = \gamma_{k_2} = 0 \) and \( \Delta\omega = 0 \), we obtain

from Eq. (8):

(10) \[ \frac{\partial^2}{\partial t^2} C_{k_0} = -\left| V \right|^2 \left[ S_0 S_2 |C_{k_1}|^2 + S_0 S_1 |C_{k_2}|^2 \right] C_{k_0}, \]

(11) \[ \frac{\partial^2}{\partial t^2} C_{k_1} = \left| V \right|^2 \left[ S_1 S_2 |C_{k_0}|^2 - S_0 S_1 |C_{k_2}|^2 \right] C_{k_1}. \]
$$\frac{\partial^2}{\partial t^2} c_{k_2} = |v|^2 [s_1 s_2 |c_k|^2 - s_o s_2 |c_{k_1}|^2] c_{k_2}.$$ 

These equations show that the amplitude of the three waves will either grow or decrease monotonically with time, if

(11) \( S_{o1} < 0 \) and \( S_{o2} < 0 \),

that is, "the explosive type instability" can occur, as shown by Coppi et al.\(^{17}\).

On the other hand, if we consider the case where

(12) \( S_{o1} > 0 \) and \( S_{o2} > 0 \)

we obtain "the decay instability" as first described by Oraevskii and Sagdeev\(^{5}\).

For the spatial case we obtain in an analogous fashion

(13) \( \left( \frac{\partial}{\partial x} - \kappa_o \right) E_{o} (x) = - \frac{M(k_o, \omega_o; k_1, \omega_1; k_2, \omega_2)}{\text{sgn} \{\omega_o \omega_1 \omega_2 S_k \} \frac{2\pi}{\partial k_o}} \frac{E_1 E_2 e^{i\Delta k x}}{\omega_1 \omega_2} \)

(14) \( \left( \frac{\partial}{\partial x} - \kappa_1 \right) E_{1} (x) = - \frac{M(k_1, \omega_1; -k_2, -\omega_2; k_o, \omega_o)}{\text{sgn} \{\omega_1 (-\omega_2) S_k \} \frac{2\pi}{\partial k_1}} \frac{E_1 e^{-i\Delta k x}}{-\omega_2 \omega_o} \)

(13) \( \left( \frac{\partial}{\partial x} - \kappa_2 \right) E_{2} (x) = - \frac{M(k_2, \omega_2; k_o, \omega_o; -k_1, -\omega_1)}{\text{sgn} \{\omega_2 (-\omega_1) S_k \} \frac{2\pi}{\partial k_2}} \frac{E_2 e^{-i\Delta k x}}{\omega_o - \omega_1} \)

\( \kappa_1 \) is the linear spacial growth rates for the mode \( (\omega_1 k_1) \) and \( \Delta k = k_1 + k_2 - k_o \). Again it is convenient to introduce the notation

\( D_{\omega_i} (x) = \left| \frac{1}{8\pi} \frac{\partial \delta}{\partial k_i} \right|^\frac{1}{2} E_{\omega_i} (x) \) \( (i = 0, 1, 2) \)

(14) \( W_{\omega_o, \omega_1, \omega_2} = \frac{1}{8\pi} \frac{M(k_o, \omega_o; k_1, \omega_1; k_2, \omega_2)}{\left| \frac{1}{8\pi} \frac{\partial \delta}{\partial k_o}, \frac{1}{8\pi} \frac{\partial \delta}{\partial k_1}, \frac{1}{8\pi} \frac{\partial \delta}{\partial k_2} \right|^\frac{1}{2}} \)
The coupling constants $W$ obey the symmetry relations

$$W_{\omega, \omega_1, \omega_2} = W_{\omega_1, \omega_2, \omega} = W_{\omega_2, \omega, \omega_1} = W$$

Equation 13 then go over into

$$\frac{\partial}{\partial x} \kappa_0 D_{\omega_0} = - S_{0, \omega_1} W D_{\omega_2} e^{i\Delta kx}$$

(15)$$\frac{\partial}{\partial x} \kappa_1 D_{\omega_1} = S_{1, \omega_2} W^* D_{\omega_0} e^{-i\Delta kx}$$

$$\frac{\partial}{\partial x} \kappa_2 D_{\omega_2} = S_{2, \omega_0} W D^*_1 e^{-i\Delta kx}$$

We note that equ. (15) have the same structure as those for the temporal case and the same remarks about decay and explosive instability therefore apply.

As mentioned before there are two modes associated with the spatial case of the beam plasma instability, a positive energy branch with $\omega_1 S_1 > 0$ and a negative energy branch with $\omega_1 S_1 < 0$. Based on the relations (11) and (12) we summarize the possible interactions leading to a decay- and an explosive instability in Table 1 and Fig. 2. The mode $k_{\omega, \omega_0}$ is referred to as the pump wave, there is, however, no distinction between the modes $(k_1, \omega_1$) and $(k_2, \omega_2)$.

We note the following points:

i) The results are unchanged when the frequency of each of the three modes changes sign.

ii) In the case of decay instability

a) If, all the three modes are on the same branch, the pump mode decays into lower frequency modes, $|\omega| = |\omega_1| + |\omega_2|$. 

(Fig. 1 (a) and (b)). Here $|\omega_1|$ is the absolute value of $\omega_1$.

(b) If the three modes are distributed on the positive energy branch as well as the negative one, then the pump mode can decay into a higher frequency mode, $|\omega_1| + |\omega_2| = |\omega_3|$ (Fig. 1 (c) and (d)).

In the case of explosive instability

(a) It is necessary that the three modes are located on the negative energy branch as well as the positive one.

(b) If the pump mode is on one branch and the other two modes are on the other branch, then $|\omega_0| = |\omega_1| + |\omega_2|$ (Fig. 1 (e) and (f)).

(c) If the pump and another modes (say $\omega_2$) are on one branch and the other mode is on the other branch, then $|\omega_0| + |\omega_2| = |\omega_1|$. (Fig. 1 (g) and (h)).

The general treatment in the presence of a negative energy wave was made by Hasegawa et al.\textsuperscript{13,14} and they suggested that the anti-stokes' mode could be excited in this situation. As described above.
### Table 1

<table>
<thead>
<tr>
<th>Sign of wave energy</th>
<th>Sign of frequency</th>
<th>Explosive or decay instability</th>
<th>Figure 2</th>
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The coupled amplitude equations (8) can be written in terms of their real and imaginary parts as follows:

Expressing the complex amplitude as

\[
D_i(x) = a_i(x) e^{i \omega_i(x)} \quad (i = 0, 1 \text{ and } 2),
\]

we obtain from Equ. (15)

\[
\begin{align*}
\frac{da_0}{dx} - \kappa_0 a_0 &= - S_0 W a_1 a_2 \sin \theta, \\
\frac{da_1}{dx} - \kappa_1 a_1 &= S_1 W a_2 a_0 \sin \theta, \\
\frac{da_2}{dx} - \kappa_2 a_2 &= S_2 W a_0 a_1 \sin \theta,
\end{align*}
\]
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The coupled amplitude equations (8) can be written in terms of their real and imaginary parts as follows:

Expressing the complex amplitude as

$$D_i(x) = a_i(x)e^{i\phi_i(x)} \quad (i = 0, 1 \text{ and } 2),$$

we obtain from Equ. (15)

$$\frac{da_0}{dx} - \kappa_0 a_0 = -S_0 W a_1 a_2 \sin \theta,$$

$$\frac{da_1}{dx} - \kappa_1 a_1 = S_1 W a_2 a_0 \sin \theta,$$

$$\frac{da_2}{dx} - \kappa_2 a_2 = S_2 W a_0 a_1 \sin \theta,$$
\[
\frac{d\theta}{dx} = \Delta k + \text{ctg } \theta \left\{ \frac{d}{dx} \ln(a_o a_1 a_2) - (\kappa_o + \kappa_1 + \kappa_2) \right\} ,
\]

where
\[
\theta(x) = \Delta k x + \varphi_1(x) + \varphi_2(x) - \omega_o(x) + \frac{\pi}{2} .
\]

Using Eqn. (17) we obtain
\[
\frac{d}{dx} (a_o^2 + S_o S_1 a_1^2) - 2 (\kappa_o a_o^2 + S_o S_1 \kappa_1 a_1^2) = 0 ,
\]

\[
\frac{d}{dx} (a_o^2 + S_o S_2 a_2^2) - 2 (\kappa_o a_o^2 + S_o S_2 \kappa_2 a_2^2) = 0 ,
\]

\[
\frac{d}{dx} (a_1^2 - S_1 S_2 a_2^2) - 2 (\kappa_1 a_1^2 - S_1 S_2 \kappa_2 a_2^2) = 0 ,
\]

which go over into the vectorial Manley-Rowe relations taken in the direction of propagation when the linear growth rates vanish.

The last of Eq. (17) can be rewritten as
\[
\left\{ \frac{d}{dx} - (\kappa_o + \kappa_1 + \kappa_2) \right\} (a_o a_1 a_2 \cos \theta) = \frac{\Delta k}{2S_o \omega} \left( \frac{d}{dx} - 2\kappa_o \right) a_o^2 .
\]

Furthermore, the conservation of power flow is expressed by
\[
\frac{d}{dx} (S_o \omega_o a_o^2 + S_1 \omega_1 a_1^2 + S_2 \omega_2 a_2^2) - 2 (S_o \kappa_o \omega_o a_o^2 + S_1 \kappa_1 \omega_1 a_1^2 + S_2 \kappa_2 \omega_2 a_2^2) = 0
\]

which is derived from Eq. (17) with \( \omega_o = \omega_1 + \omega_2 \).

In the following we discuss the detailed character of the decay and explosive type instabilities, respectively.
III. SPATIAL DEVELOPMENT OF NONLINEAR INSTABILITIES

i) Decay Instabilities

This instability occurs when the conditions \( S_0 S_1 > 0 \) and \( S_0 S_2 > 0 \), are satisfied as described above. Here we assume that \( S_0 = S_1 = S_2 = -1 \). (The results are the same as those for \( S_0 = S_1 = S_2 = +1 \) where the initial value of \( \Theta(x) \) is chosen appropriately.) Let us first consider the case where the linear growth can be neglected \( (\kappa_0 = \kappa_1 = \kappa_2 = 0) \).

The coupled equations (17) reduce to

\[
\begin{align*}
\frac{d a_0}{d x} &= W a_1 a_2 \sin \Theta, \\
\frac{d a_1}{d x} &= - W a_0 a_2 \sin \Theta, \\
\frac{d a_2}{d x} &= - W a_0 a_1 \sin \Theta, \\
\frac{d \Theta}{d x} &= \Delta k + \cot \Theta \frac{d}{d x} \ln (a_0 a_1 a_2).
\end{align*}
\]

(22)

From the Manley-Rowe relations, Eq. (19), and Eq. (20), we can define the following three invariants:

\[
\begin{align*}
a_0^2 + a_1^2 &= P, \\
a_0^2 + a_2^2 &= Q, \\
a_0 a_1 a_2 \cos \Theta + \frac{\Delta k}{2W} a_0^2 &= G.
\end{align*}
\]

(23)

The solution of the coupled equation can be obtained, following a method discussed by Armstrong et al.\(^{14}\) Substituting Eq. (23) into the first of Eq. (22), we obtain

\[
\frac{d}{d x} a_0^2 = \pm 2W \left[ a_0^2 (P - a_0^2)(Q - a_0^2) - (G - \frac{\Delta k}{2W} a_0^2)^2 \right]^{1/2},
\]

(24)

where the \( \pm \) sign is determined by the sign of \( \sin \Theta \) for \( x = 0 \). It corresponds
to the amplifying and to the decaying solution, respectively. The intensity of the wave \( a_0 \) decreases when \( -\frac{\pi}{2} < e < 0 \), while it increases when \( 0 < e < \frac{\pi}{2} \). The general solution for the intensity of the three waves is

\[
a_0^2(x) = b_1^2 + (b_2^2 - b_1^2) \text{ sn}^2 \left[ \frac{\pi}{W(b_3^2 - b_1^2)^{\frac{1}{2}}(x-x_0)}, \gamma \right],
\]

(25) \[a_1^2(x) = a_1^2(0) + a_0^2(0) - b_1^2(b_2^2 - b_1^2) \text{ sn}^2 \left[ \frac{\pi}{W(b_3^2 - b_1^2)^{\frac{1}{2}}(x-x_0)}, \gamma \right],\]

\[a_2^2(x) = a_2^2(0) + a_0^2(0) - b_1^2(b_2^2 - b_1^2) \text{ sn}^2 \left[ \frac{\pi}{W(b_3^2 - b_1^2)^{\frac{1}{2}}(x-x_0)}, \gamma \right].\]

The \( b_i^2 \) are the roots of the radicand

(26) \[Y(a_0^2) = a_0^2(P-a_o^2)(Q-a_0^2) - \frac{\Delta k}{2W} a_0^2 = (a_0^2 - b_3^2)(a_0^2 - b_2^2)(a_0^2 - b_1^2) = 0\]

and it is assumed that

\[b_3^2 \geq b_2^2 \geq b_1^2\]

\[\gamma^2 \text{ and } x_0 \text{ are defined as}\]

(27) \[\gamma^2 = \frac{b_2^2 - b_1^2}{b_3^2 - b_1^2} \leq 1\]

(28) \[\pm x_0 = \frac{1}{W(b_3^2 - b_1^2)^{\frac{1}{2}}} \text{ sn}^{-1} \left[ \frac{a_0^2(0) - b_1^2}{b_2^2 - b_1^2} \right], \gamma\]

(a) Perfect matching, \( \Delta k = 0 \)

Case (1a)

The simplest case of physical interest occurs when \( a_0^2(0) \gg a_1^2(0) > 0 \) and \( a_2^2(0) = 0 \). This corresponds to the case where the energy of a strong pump wave \( \omega_0 \) is converted to the waves \( \omega_1 \) and \( \omega_2 \). Three roots of Eq. (26) are
\[ b_3^2 = a_o^2(0) + a_1^2(0) > b_2^2 = a_o^2(0) > b_1^2 = 0. \]

The amplitudes of the three waves are

\[ a_o^2(x) = a_o^2(0) \text{ sn}^2 [W[a_o^2(0) + a_1^2(0)]^{1/2}(x-x_o), \gamma], \]

\[ a_1^2(x) = a_1^2(0) + a_o^2(0) \text{ cn}^2 [W[a_o^2(0) + a_1^2(0)]^{1/2}(x-x_o), \gamma], \]

\[ a_2^2(x) = a_o^2(0) \text{ cn}^2 [W[a_o^2(0) + a_1^2(0)]^{1/2}(x-x_o), \gamma], \]

where

\[ 1 - \gamma^2 = \frac{a_1^2(0)}{a_o^2(0) + a_1^2(0)} \ll 1. \]

The initial condition \( \text{ sn}^2 [W[a_o^2(0) + a_1^2(0)]^{1/2}x_o, \gamma] = 1 \) leads to

\[ x_o = \frac{K(\gamma)}{W[a_o^2(0) + a_1^2(0)]^{1/2}} \approx \frac{1}{W[a_o^2(0)]} \left\{ \ln \frac{4 a_o^2(0)}{a_1^2(0)} \right\}, \]

where \( K(\gamma) \) is the elliptic integral of the first kind. This value of \( x_o \) is the decay length which means that within this length the intensity of initial wave \( a_o^2(0) \) decreases to zero; the period of the variation of energy is \( 2x_o \).

**Case (1b)**

Next consider the case where the initial intensity of the pump wave \( a_o^2(0) \) is smaller than that of \( a_1^2(0) \), that is, \( a_1^2(0) \gg a_o^2(0) > 0 \) and \( a_2^2(0) = 0 \). Then the three roots of Eq. (26) are

\[ b_3^2 = a_o^2(0) + a_1^2(0) \gg b_2^2 = a_o^2(0) > b_1^2 = 0. \]
and therefore

\[ a_0^2(x) = a_0^2(0) \sin^2 \left[ \frac{\pi}{2} \left( a_1(0) \left[ 1 + \frac{a_0^2(0)}{a_1^2(0)} \right]^{\frac{1}{2}} (x-x_0) \right) \right] , \]

where

\[ \gamma^2 = \frac{a_0^2(0)}{a_0^2(0) + a_1^2(0)} \approx \frac{a_0^2(0)}{2 a_1^2(0)} \approx 0 . \]

For small \( \gamma^2 \) we may approximate the elliptic function \( \text{sn}(x,y) \) by \( \text{sn}[x,0] = \sin x \).

The intensity of three waves is then given approximately by

\[ a_0^2(x) = a_0^2(0) \sin^2 \left[ \frac{\pi}{2} a_1(0) \left[ 1 + \frac{a_0^2(0)}{a_1^2(0)} \right]^{\frac{1}{2}} (x-x_0) \right] , \]

\[ a_1^2(x) = a_1^2(0) + a_0^2(0) \cos^2 \left[ \frac{\pi}{2} a_1(0) \left[ 1 + \frac{a_0^2(0)}{a_1^2(0)} \right]^{\frac{1}{2}} (x-x_0) \right] , \]

\[ a_2^2(x) = a_2^2(0) \cos^2 \left[ \frac{\pi}{2} a_1(0) \left[ 1 + \frac{a_0^2(0)}{a_1^2(0)} \right]^{\frac{1}{2}} (x-x_0) \right] . \]

The decay length, \( x_0 \), is obtained as

\[ x_0 \approx \frac{\pi/2}{\text{sn} a_0^2(0) + a_1^2(0)} \approx \frac{1}{a_1^2(0)} \frac{\pi}{2} \left[ 1 - \frac{a_0^2(0)}{2 a_1^2(0)} \right] . \]

It is noted that if we consider the case where \( a_1^2(0) \gg a_2^2(0) > 0 \) and \( a_0^2(0) = 0 \), the intensity of three waves is given by Eq. (31) with the exception that \( (x-x_0) \) is replaced by \( x \). Case 1a and 1b are sketched schematically in Fig. 3.

Case (1c)

In the case where \( a_1^2(0) = a_2^2(0) \gg 0 \) and \( a_0^2(0) = 0 \), the three roots
of Eq. (26) are

\[ b_3^2 = b_2^2 = a_1^2(0) = a_2^2(0) \gg 0, \quad b_1^2 = 0 \]

and \( \gamma = 1 \).

The elliptic function can then be expressed in terms of elementary functions
and we obtain

\[
\begin{align*}
a_o^2(x) &= a_1^2(0) \text{sn}^2[W a_1(0)x, 1] = a_1^2(0) \tanh^2[W a_1(0)x], \\
a_1^2(x) &= a_2^2(x) = a_1^2(0) \text{sech}^2[W a_1(0)x].
\end{align*}
\]

In this case, the interaction length becomes infinite.

(b) Imperfect Matching \((\Delta k \neq 0)\).

In this section we examine the effects of finite mismatch on the
variation of the wave intensity and compute the magnitude of the mismatch which
can be tolerated in order to observe the decay instability.

**Case (2a)**

For the initial condition \( a_o^2(0) \gg a_2^2(0) > 0 \) and \( a_2^2(0) = 0 \), Eq. (26)
becomes

\[
Y(a_o^2) = \left\{ a_o^2 - a_o^2(0) \right\} \left\{ a_o^4 - \left[ a_o^2(0) + a_1^2(0) + (\Delta k/2W)^2 \right] a_o^2 + a_1^2(0) \left( \frac{\Delta k}{2W} \right)^2 \right\} = 0
\]

This equation has three different real roots, that are ordered according to

\[ b_3^2 > b_2^2 = a_o^2(0) > b_1^2 > 0. \]

Since the intensity of pump wave \( a_o^2(x) \) oscillates between \( a_o^2(0) \) and \( b_1^2 \),
and approaches \( a_o^2(0) \) with the increasing mismatch \((\Delta k/2W)\). The amplitude of
oscillation therefore becomes smaller and the decay length increases as the
mismatch is increased.

For example, in the case where \( a_o^2(0) \gg (\Delta k/2W) \gg a_1^2(0) > 0 \)
we have

\[ b_3^2 \approx a_o^2(0) + a_1^2(0) + \frac{1}{a_o^2(0)} \left( \frac{\Delta k}{2W} \right)^4. \]
\[ b_1^2 = \left( \frac{\Delta k}{2\hbar} \right)^2 \left[ 1 - \frac{1}{2} \frac{\Delta k}{a_0^2(0)} \right] \] .

The amplitudes are then given by

\[ a_o^2(x) = \left( \frac{\Delta k}{2\hbar} \right)^2 \left[ 1 - \frac{1}{2} \frac{\Delta k}{a_0^2(0)} \right] + [a_o^2(0) - \left( \frac{\Delta k}{2\hbar} \right)^2] \left[ 1 - \frac{1}{2} \frac{\Delta k}{a_0^2(0)} \right] \]

\[ \times \text{sn}^2 \left[ \frac{\Delta k}{2\hbar} \left[ a_o^2(0) + a_1^2(0) - \left( \frac{\Delta k}{2\hbar} \right)^2 \right] \left[ 1 - \frac{2}{a_0^2(0)} \left( \frac{\Delta k}{2\hbar} \right)^2 \right] \left( x-x_0 \right), \gamma \right] , \]

\[ a_1(x) = a_1^2(0) + a_0^2(0) - a_0^2(x) , \]

\[ a_2(x) = a_0^2(0) - a_0^2(x) . \]

We note that \( a_o^2(x) \) oscillates approximately between \( a_o^2(0) \) and \( (\Delta k/2\hbar)^2 \).

With

\[ 1 - \gamma = \frac{a_1^2(0) + \frac{1}{2} \frac{\Delta k}{a_0^2(0)} \left( \frac{\Delta k}{2\hbar} \right)^4}{a_o^2(0) + a_1^2(0) - \left( \frac{\Delta k}{2\hbar} \right)^2} \approx 0 \]

The decay length is given by

\[ x_0 = \frac{1}{2\hbar \left[ a_o^2(0) + a_1^2(0) - \left( \frac{\Delta k}{2\hbar} \right)^2 \left[ 1 - \frac{2}{a_0^2(0)} \left( \frac{\Delta k}{2\hbar} \right)^2 \right] \right] \text{ln} \frac{4 \left[ a_o^2(0) + a_1^2(0) - \left( \frac{\Delta k}{2\hbar} \right)^2 \left[ 1 - \frac{2}{a_0^2(0)} \left( \frac{\Delta k}{2\hbar} \right)^2 \right] \right]}{a_1^2(0) + \frac{1}{2} \frac{\Delta k}{a_0^2(0)} \left( \frac{\Delta k}{2\hbar} \right)^4} . \]
The ratio of this value of $x_0$ to that in the case of $\Delta k = 0$ given by Eq. (30) is

\[
\frac{x_0(\Delta k)}{x_0(\Delta k=0)} \approx \left\{1 + \frac{1}{2} \frac{1}{a_0(0)} \left(\frac{\Delta k}{2N}\right)^2\right\} \ln \frac{4\{a_0^2(0) - \left(\frac{\Delta k}{2N}\right)^2\}}{a_1^2(0) + \frac{1}{4} \left(\frac{\Delta k}{2N}\right)^4}.
\]

If the mismatch is so large that

\[
\left(\frac{\Delta k}{2N}\right)^2 = a_0^2(0) - a_1^2(0), \text{ i.e., } \left(\frac{\Delta k}{2N}\right)^2 = 1 - \frac{a_1^2(0)}{a_0^2(0)},
\]

the roots are

\[
b_3 = a_0^2(0) + a_0(0) a_1(0),
\]

\[
b_1 = a_0^2(0) - a_0(0) a_1(0),
\]

and the intensity of three waves is given by

\[
a_0^2(x) = a_0^2(0) - a_0(0) a_1(0) \begin{array}{c}
\text{cn}^2\left[W a_0(0)\sqrt{\frac{2}{a_0(0)} a_1(0)}(x-x_0), 2^{-\frac{1}{2}}\right],
\end{array}
\]

\[
a_1^2(x) = a_1^2(0) + a_0(0) a_1(0) \begin{array}{c}
\text{cn}^2\left[W a_0(0)\sqrt{\frac{2}{a_0(0)} a_1(0)}(x-x_0), 2^{-\frac{1}{2}}\right],
\end{array}
\]
\[ a_2^2(x) = a_o(0)a_1(0) \text{cn}^2 \left[ W a_o(0) \sqrt{\frac{2 a_1(0)}{a_o(0)}} (x-x_0), 2^{\frac{1}{2}} \right] . \]

The amplitude of oscillation becomes so narrow that \( a_o^2(x) \) oscillates between \( a_o(0) \) and \( a_o(0)a_1(0) \). In addition, the interaction length becomes long and is given by

\[ \frac{x}{x_o} = \frac{K(2^{\frac{1}{2}})}{W a_o(0)} \sqrt{\frac{a_o(0)}{2 a_1(0)}}. \]

If the mismatch increases further we may put

\[ \left( \frac{\Delta k}{2W} \right)^2 = a_o^2(0) - a_1^2(0) + 2\Delta^2, (\Delta^2 > 0), \]

to obtain

\[ b_1^2 = a_o^2(0) + \Delta^2 - \sqrt{a_o^2(0)a_1^2(0) + \Delta^4} \]

\[ a_o^2(0) - a_o(0)a_1(0) + \Delta^2 \left[ 1 - \frac{1}{2} a_1^2(0) a_0^2(0) \right] \text{ for } \Delta^2 \ll a_o(0)a_1(0), \]

\[ a_o^2(0) - \frac{a_o^2(0)a_1^2(0)}{2 \Delta^2} \text{ for } \Delta^2 \gg a_o(0)a_1(0) \]

The wave interaction becomes negligibly small as \( \Delta^2 \) is increased. Case (2a) is shown schematically in Fig. 4.

**Case (2b)**

In the case where \( a_1^2(0) \gg a_o^2(0) > 0 \) and \( a_2^2(0) = 0 \) and the mismatch obeys the inequality

\[ a_1^2(0) \gg (\Delta k/2W)^2 \gg a_o^2(0), \]
we have
\[ b_3^2 \approx a_1^2(0) + a_0^2 + \frac{(\Delta k)^2}{2W} \left\{ 1 - \frac{a_2^2(0)}{a_1^2(0)} \right\}, \]

\[ b_2^2 = a_0^2(0), \quad b_1^2 = \frac{a_2^2(0)}{a_1^2(0)} \frac{(\Delta k)^2}{2W}, \]

\[ \gamma^2 \approx \frac{\frac{a_2^2(0)}{a_1^2(0)}}{1 + \frac{a_0^2(0)}{a_1^2(0)} + \frac{1}{a_1^2(0)} \frac{(\Delta k)^2}{2W}} \approx 0 \]

and the intensity of three waves is approximately given by

\[ a_o^2(x) = \frac{a_o^2(0)}{a_1^2(0)} \frac{(\Delta k)^2}{2W} + a_0^2(0) \left\{ 1 - \frac{1}{a_1^2(0)} \frac{(\Delta k)^2}{2W} \right\} \]

\[ \times \sin^2 \left\{ W a_1(0) \left[ 1 + \frac{a_0^2(0)}{a_2^2(0)} + \frac{1}{a_1^2(0)} \frac{(\Delta k)^2}{2W} \right]^2 \right\} (x-x_0) \]

\[ a_1^2(x) = a_1^2(0) + a_2^2(x) \]

\[ a_2^2(x) = a_0^2(0) \left\{ 1 - \frac{1}{a_1^2(0)} \frac{(\Delta k)^2}{2W} \right\} \cos^2 \left\{ W a_1(0) \left[ 1 + \frac{a_0^2(0)}{a_2^2(0)} + \frac{1}{a_1^2(0)} \frac{(\Delta k)^2}{2W} \right]^2 \right\} (x-x_0) \]

The decay length \( x_0 \) is given by

\[ x_0 \approx \frac{1}{W a_1(0)} \frac{\pi}{2} \left\{ 1 - \frac{a_0^2(0)}{2 a_1^2(0)} - \frac{1}{2 a_1^2(0)} \frac{(\Delta k)^2}{2W} \right\}. \]
which is somewhat smaller than that in the case of perfect matching, Eq. (32).

**Case (2c)**

In the case where \( a_1(0) = a_2(0) \gg 0 \) and \( a_0(0) = 0 \) and the mismatch is so small that

\[
(\Delta k/2W)^2 \ll a_1(0),
\]

the roots are

\[
b_3^2 \approx a_1^2(0) + a_1(0) (\Delta k/2W) + \frac{1}{2} (\Delta k/2W)^2,
\]

\[
b_2^2 = a_1^2(0) - a_1(0) (\Delta k/2W) + \frac{1}{2} (\Delta k/2W)^2,
\]

\[
b_1^2 = 0
\]

and the intensity of three waves is given by

\[
a_0^2(x) = \{a_1^2(0) - a_1(0) (\Delta k/2W)\} \sin^2[\omega \{a_1^2(0) + a_1(0) \Delta k/2W\}] x, \gamma
\]

(42)

\[
a_1^2(x) = a_2^2(x) = a_1^2(0) - a_0^2(x),
\]

where

\[
1 - \gamma^2 = \frac{2 a_1(0) \Delta k}{a_1^2(0) + a_1(0) (\Delta k/2W) + \frac{1}{2} (\Delta k/2W)^2} \approx \frac{1}{a_1(0)} \frac{\Delta k}{2W} \ll 1.
\]

The decay length \( x_o \) becomes

\[
x_o = \frac{K(\gamma)}{W \{a_1^2(0) + a_1(0) \Delta k/2W\}} \approx \frac{1}{2} \ln \frac{a_1(0)}{\Delta k/2W}
\]

(43)

It is noticeable that when \( \Delta k \neq 0 \), \( a_0^2(x) \) oscillates between zero and

\[
a_1^2(0) - a_1(0) (\Delta k/2W)
\]

with the decay length \( x_o \) given by Eq. (43), though
\[ a^2_0(x) \text{ approaches } a^2_1(0) \text{ at } x \to \infty \text{ when } \Delta k = 0. \]

\textbf{ii) Explosive Instabilities}

The interaction becomes explosive when \( S_0 S_1 < 0 \) and \( S_0 S_2 < 0 \) as given by Eq. (10). We assume that \( S_0 = -1 \), and \( S_1 = S_2 = +1 \). (The results are the same as those for \( S_0 = +1 \) and \( S_1 = S_2 = -1 \) where the initial value of \( \theta(x) \) is chosen appropriately). Again we consider the case where the linear growth can be neglected \( (\kappa_o = \kappa_1 = \kappa_2 = 0) \).

The coupled equations (17) reduce to

\[
\frac{da}{dx} = W a_1 a_2 \sin \theta ,
\]

\( (44) \)

\[
\frac{da_1}{dx} = W a_2 a_0 \sin \theta ,
\]

\[
\frac{da_2}{dx} = W a_0 a_1 \sin \theta ,
\]

\[
\frac{d\theta}{dx} = \Delta k + \cot \theta \frac{d}{dx} \ln (a_0 a_1 a_2) .
\]

From the Manley - Rowe relations, Eq. (19) and Eq. (20), we can define the following three invariants:

\[ a^2_0 - a^2_1 = P , \]

\( (45) \)

\[ a^2_0 - a^2_2 = Q , \]

\[
a_o a_1 a_2 \cos \theta + \frac{\Delta k}{2W} a^2_o = G .
\]

The solution of the coupled equation can be obtained following a method discussed by Coppi et al. Substituting Eq. (45) into the first of Eq. (44),
we obtain

\[ \frac{d}{dx} a_0^2 = \pm 2W \left[ a_0^2 (a_0^2 - p)(a_0^2 - q) - \left( G - \frac{\Delta k}{2W} a_0^2 \right)^2 \right]^{1/2}, \]

where the \( \pm \) sign is determined by the sign of \( \sin \theta_0 \) for \( x = 0 \) and corresponds to the amplifying and to the decaying solution respectively. The intensity of the wave \( (a_0) \) increases when \( 0 < \theta_0 < \pi/2 \), while it decreases when \( -\pi/2 < \theta_0 < 0 \).

The solution is given by

\[ W(x) = \pm \frac{1}{2} \int_0^{a_0^2(0)} \left[ (a_0^2 - b_1^2)(a_0^2 - b_2^2)(a_0^2 - b_3^2) \right]^{1/2} \frac{a_0^2(x)}{a_0^2(0)} \frac{da_0}{da_0^2}, \]

where \( b_1^2, b_2^2, b_3^2 \) are roots of the cubic

\[ Y(a_0) = a_0^2 (a_0^2 - p)(a_0^2 - q) - \left( G - \frac{\Delta k}{2W} a_0^2 \right)^2, \]

and it is assumed that \( b_3^2 \geq b_2^2 \geq b_1^2 \geq 0 \).

We assume that the initial intensity of three waves are \( a_0^2(0) \gg a_1^2(0) > 0 \) and \( a_2^2(0) = 0 \). Then eq. (48) becomes

\[ Y(a_0^2) = \left[ a_0^2 - a_0^2(0) \right] \left[ a_0^4 - \left( a_0^2(0) - a_1^2(0) + \left( \frac{\Delta k}{2W} \right)^2 a_0^2 + \frac{\Delta k^2}{2W^2} \right) \right] = 0. \]

One root of Eq. (49) is \( a_0^2(0) \). The explosive instability can occur as long as this is the largest root, as described in the following (cases (1) \( \sim \) (4)).

However, as the mismatch \( (\Delta k/2W) \) is increased, the wave-wave coupling becomes weak and the explosion distance increases as expected. It will be shown that if the mismatch is so large that \( \Delta k/(2W a_0(0)) \geq 1 \) and \( b_1^2 = a_0^2(0) \), then the explosive instability does not occur, but that the three waves show oscillatory
behavior similar to the decay instability, (cases (5) and (6)). On the other hand, a quenching effect will be seen when the initial intensity of the three waves is given by

\[ a_0^2(0) > a_1^2(0) > a_2^2(0) > 0 \text{ (case (7)).} \]

a) Perfect matching \((\Delta k = 0)\)

We assume that Eq. (48) has three real roots with different values

\[ b_3^2 > b_2^2 > b_1^2 > 0 . \]

Then, the general solution of Eq. (47) (taking the + sign) is given by

\[
(50) \quad a_0^2(x) = b_1^2 + \frac{(b_2^2 - b_1^2)}{\text{sn}^2 \left[ - W \left( b_3^2 - b_1^2 \right)^{\frac{1}{2}} (x-x_0), \gamma \right]}. 
\]

From Eq. (45), the intensity for the other waves is given by

\[
(51) \quad a_1^2(x) = a_1^2(0) - a_0^2(0) + a_0^2(x),
\]

\[
(52) \quad a_2^2(x) = a_2^2(0) - a_0^2(0) + a_0^2(x),
\]

with

\[
(52) \quad \gamma = \frac{b_2^2 - b_1^2}{b_3^2 - b_1^2} \leq 1
\]

and

\[
(53) \quad x_0 = \frac{1}{W \left( b_3^2 - b_1^2 \right)^{\frac{1}{2}}} \text{sn}^{-1} \left[ \frac{b_3^2 - b_1^2}{a_0^2(0) - b_1^2} \right], \gamma > 0.
\]

It is clear from Eq. (50) that \(a_0^2(x) \to \infty\) as \(x = x_0\) and that the explosion
length is given by \( x_{\text{exp}} = x_0 \).

**Case (1)**

For the initial condition, \( \Delta k = 0, a_1^2(0) \geq a_1^2(0) > 0 \), the three roots of Eq. (49) are given by

\[
\begin{align*}
\text{b}_3 &= a_0^2(0) > \text{b}_2 = a_0^2(0) - a_1^2 \geq 0
\end{align*}
\]

The intensity of the three waves is

\[
a_0^2(x) = \frac{a_0^2(0)}{\text{sn}^2[-W a_0(0)(x-x_0), \gamma]} ,
\]

\[
a_1^2(x) = a_1^2(0) + a_0^2(0) \text{cn}^2[-W a_0(0)(x-x_0), \gamma] ,
\]

\[
a_2^2(x) = a_2^2(0) \text{cn}^2[-W a_0(0)(x-x_0), \gamma] ,
\]

where

\[
1 - \gamma^2 = \frac{a_1^2(0)}{a_0^2(0)} \approx 0.
\]

The explosion length is given by

\[
x_{\text{exp}} = x_0 = \frac{K(\gamma)}{W a_0(0)} \approx \frac{1}{W a_0(0)} \left( \frac{1}{2} \ln \frac{16a_0^2(0)}{a_1^2(0)} \right).
\]

Thus the intensity of all the three waves increases simultaneously and reaches infinity at \( x = x_{\text{exp}} \). This case is demonstrated in Fig. (5a).

**b) Imperfect Matching (\( \Delta k \neq 0 \)).**

Here we examine effects of a mismatch on the variation of wave intensity and ask for the magnitude of \( \Delta k \) which is permissible to observe the explosive
instability.

**Case (2)**

If the mismatch is so small that \( a_0^2(0) \gg (\Delta k/2W)^2 \gg a_1^2(0) > 0 \) and \( a_2^2(0) = 0 \), the three roots of Eq. (49) are given by

\[
\begin{align*}
b_3^2 &= a_0^2(0), \\
b_2^2 &= a_0^2(0) - a_1^2(0) + \frac{1}{a_0^2(0)} (\Delta k/2W)^2, \\
b_1^2 &= (\Delta k/2W)^2 \left[ 1 - \frac{1}{a_0^2(0)} (\Delta k/2W)^2 \right].
\end{align*}
\]

Substituting these values into Eq. (50), we obtain

\[
a_0^2(x) = (\Delta k/2W)^2 \left[ 1 - \frac{1}{a_0^2(0)} (\Delta k/2W)^2 \right]
\]

\[
+ \frac{a_0^2(0) - (\Delta k/2W)^2 \left[ 1 - \frac{1}{a_0^2(0)} (\Delta k/2W)^2 \right]}{\text{sn}^2 [-W a_0(0)] \left[ 1 - \frac{1}{a_0^2(0)} (\Delta k/2W)^2 \right] \left( 1 - \frac{1}{a_0^2(0)} (\Delta k/2W)^2 \right)^{1/2} (x-x_0), \gamma]}
\]

where

\[
1 - \gamma^2 = \frac{a_1^2(0) - \frac{1}{a_0^2(0)} (\Delta k/2W)^4}{a_0^2(0) - (\Delta k/2W)^2 \left[ 1 - \frac{1}{a_0^2(0)} (\Delta k/2W)^2 \right]} \approx 0.
\]

The explosion length is given approximately by
Comparing Eq. (57) with Eq. (55), the explosion length becomes larger as the mismatch $\Delta k$ is increased.

**Case (3)**

If the mismatch $\Delta k$ is increased so that

$$\left| \Delta k \right| = a_0(0) - a_1(0) ,$$

then the three roots of Eq. (49) are given by

$$b_3^2 = \frac{a_0(0)}{a_1(0)} > b_2^2 = b_1^2 = a_0(0)\left[ a_0(0) - a_1(0) \right] .$$

For this case, where the two lower roots coincide, the general solution of Eq. (47) is given by

$$a_0(x) = b_3^2 + (b_3^2 - b_2^2) \tan^2 \left[ (b_3^2 - b_2^2) W(x - x_0) \right] .$$
where
\[ x_0 = - \frac{1}{W(b_3^2 - b^2)^{\frac{1}{2}}} \tan^{-1} \left( \frac{a_1(0) - b_2}{b_3 - b^2} \right) \]  

From Eq. (60), the explosion distance \( x_{\text{exp}} \) becomes
\[ x_{\text{exp}} = \frac{\pi}{W(b_3^2 - b^2)^{\frac{1}{2}}} + x_0. \]

Substituting the values for \( b_3 \) and \( b \) given by Eq. (59) into Eq. (60), we obtain
\[ a_0^2(x) = a_0^2(0) + a_0(0)a_1(0)\tan^2 \left[ a_0(0)a_1(0) \right] \exp \left[ a_0(0) - a_1(0) \right] \]

\[ a_1^2(x) = a_1^2(0) + a_0(0)a_1(0)\tan^2 \left[ a_0(0)a_1(0) \right] \exp \left[ a_0(0) - a_1(0) \right] \]

\[ a_2^2(x) = a_0(0)a_1(0)\tan^2 \left[ a_0(0)a_1(0) \right] \exp \left[ a_0(0) - a_1(0) \right] \]

and
\[ x_{\text{exp}} = \frac{\pi}{W} \left( \frac{a_0(0)}{a_1(0)} \right)^{\frac{1}{2}}. \]

Comparing this explosion length with that for \( \Delta k = 0 \) given by Eq. (55),
\[ \frac{x_{\text{exp}} (\Delta k = 0)}{x_{\text{exp}} (\Delta k = 0)} = \frac{a_0(0)^{\frac{1}{2}}}{a_1(0)} \ln \left\{ \frac{16 a_0^2(0)/a_1^2(0)}{1} \right\} \]

which means that the explosion length becomes much larger.

**Case (4)**

When the mismatch \( \Delta k \) is increased more the two lower roots of Eq. (49) \( b_2^2 \) and \( b_1^2 \), become conjugate complex. Following the procedure outlined in reference 12, we write
\[ Y(a^2_0) = (a^2_0 - b^2_0)(a^4_0 - 2b_0 a^2_0 + c) = 0 \]

where \( b^2_0, b, c \) are real and \( c > b^2 \).

The solution of Eq. (47) is obtained as

\[ a^2_0(x) = \frac{(H + b^2_3) - (H - b^2_3) \text{cn}[2H^k_3 W(x-x^0_0), \gamma]}{1 + \text{cn}[2H^k_3 W(x-x^0_0), \gamma]} , \]

where

\[ H = (b^4_3 - 2b^2_3 + c)^{1/2} , \]

and

\[ \gamma = \left( \frac{H + b - b^2_3}{2H} \right)^{1/2} \]

and

\[ x^0 = -\frac{1}{2H_3 W} \text{cn}^{-1} \left[ \frac{H + b^2_3 - a^2(0)}{2H^k_3 W} - \frac{H - b^2_3 + a^2(0)}{2H} \right] , \gamma \] .

From Eq. (67) we find the explosion length as

\[ x_{exp} = x_0 + \frac{K(\gamma)}{H^k_3 W} . \]

The condition, where Eq. (49) has conjugate complex roots, is given by

\[ \left| \frac{\Delta k}{W a_0(0)} \right| > 1 - \frac{a^2(0)}{2a^2_0(0)} + \left( \frac{\Delta k}{2 W a_0(0)} \right)^2 \]

On the other hand for the instability to be explosive, it is necessary that \( b^2_3 = a^2(0) > \text{Re} b^2_1 = \text{Re} b^2_1 \), which requires

\[ 1 + \frac{a^2_1(0)}{a^2_0(0)} > \left( \frac{\Delta k}{2 W a_0(0)} \right)^2 \]
Under these conditions we have

\[ H = a_0(0)a_1(0) , \]

\[
\gamma = \frac{(\Delta k)^2 - (a_0(0) - a_1(0))^2}{4a_0(0)a_1(0)}
\]

and

\[ x = 0 . \]

Eq. (67) then becomes

\[
a^2(x) = a_0(0) + a_0(0)a_1(0) + \left\{ \frac{2^2}{2^2} - a_0(0)a_1(0) \right\} \text{cn} \left[ 2\{a_0(0)a_1(0)\}^{1/2}, x, \gamma \right] \]

and the explosion length is given by

\[
x_{\text{exp}} = \frac{K(\gamma)}{W} a_0(0) \left\{ \frac{a_0(0)}{a_1(0)} \right\}^{1/2} .
\]

**iii) Non-explosive instability**

When the mismatch \( \Delta k \) is increased beyond that in the case (4), the explosive instability does not occur, but there appears an instability similar to the decay instability. In this case the amplitudes of all three waves increases or decrease simultaneously.

We assume that Eq. (49) has three real roots which are given by

\[
b_3^2 \approx b_2^2 > b_1^2 = a_0^2(0) .
\]

Using the same method as that for discussing the decay instability we obtain

\[
a^2(x) = b_1^2 + (b_2^2 - b_1^2) \text{sn}^2 \left[ W (b_3^2 - b_1^2)^{1/2} x, \gamma \right] .
\]
By virtue of the Manley - Rowe relations Eq. (45), the other two waves are given by

\begin{align*}
2a_1(x) &= a_1^2(0) - a_2^2(0) + a_0^2(x), \\
2a_2(x) &= a_2^2(0) - a_0^2(0) + a_0^2(x),
\end{align*}

where

\[ \gamma = \frac{b_2^2 - b_1^2}{b_2^2 - b_3^2} \quad \text{and} \quad x_0 = 0. \]

**Case (5)**

If the mismatch becomes

\begin{align*}
\left| \frac{\Delta k}{2W} \right| &= a_0(0) + a_1(0),
\end{align*}

then

\[ b_3^2 = b_2^2 = a_0(0)[a_0(0) + a_1(0)] > b_1^2 = a_0^2(0). \]

Since \( \gamma = 1 \), Eq. (74) reduces to

\begin{align*}
a_0^2(x) &= a_0^2(0) + a_0^2(0)a_1(0)\tanh^2[W[a_0(0)a_1(0)]^\frac{1}{2}x],
\end{align*}

Thus the intensity of the all three waves increases asymptotically and reach the values \( a_0(0)a_1(0) \) plus initial values, respectively, for \( x \to \infty \).

**Case (6)**

If the mismatch \( \Delta k \) increases further and is given by

\begin{align*}
\left( \frac{\Delta k}{2W} \right)^2 &= a_0^2(0) + a_1^2(0) + 2\Delta^2,
\end{align*}

where

\[ \Delta^2 > a_0(0)a_1(0). \]
then
\[ b_3^2 > b_2^2 > b_1^2 = a_0^2(0) \]
and
\[ b_3^2, b_2^2 = (a_0^2(0) + \Delta^2) \pm (\Delta^4 - a_0^2(0)a_1^2(0))^{1/2}. \]

The intensity is given by
\[
{2_a^2}(x) = {2_a^2}(0) + [\Delta^2 - \{\Delta^4 - a_0^2(0)a_1^2(0)\}]^{1/2} \text{sn}^2[\sqrt{W} \{\Delta^2 + (\Delta^4 - a_0^2(0)a_1^2(0))^{1/2}\} x, \gamma],
\]

where
\[
W = \frac{\Delta^2 - \{\Delta^4 - a_0^2(0)a_1^2(0)\}^{1/2}}{\Delta^2 + \{\Delta^4 - a_0^2(0)a_1^2(0)\}^{1/2}}.
\]

Thus, the intensities of the three waves oscillate in phase, and its amplitude is \([\Delta^2 - \{\Delta^4 - a_0^2(0)a_1^2(0)\}]^{1/2}\). (Fig. 5b).

iv) Quenching

In the above discussion (Case (1) ~ (6)), we assumed that the initial phase is \(0 < \theta_0 < \pi/2\); the intensity of the three waves therefore always increases. As mentioned before, if the initial phase is \(-\pi/2 < \theta_0 < 0\), then the intensity of all the three waves decreases at first, that is, there appears a quenching effect.

The initial intensity of the three wave is assumed to obey the inequality
\[ a_0^2(0) > a_1^2(0) > a_2^2(0) > 0 \]
and that \(\Delta k = 0\). Then the three roots of Eq. (48) become
\[ a_0^2(0) > b_3^2 > b_2^2 > b_1^2 > 0. \]
Taking the -sign in Eq. (47), the intensity of the wave is given by

\[
a^2_o(x) = b_1^2 + \frac{b_3^2 - b_1^2}{\text{sn}^2[\text{sn}^{-1}(b_3^2 - b_1^2)^{1/2}(x + x_0), \nu]}
\]

\(a_1^2(x), a_2^2(x)\) are given by Eq. (51) and \(x_0\) by Eq. (53).

It follows from Eq. (81) that \(a^2_o(x)\) at first decreases and that it reaches the minimum value \(b^2_3\) at a distance

\[
x_1 = \frac{K(\gamma)}{W(b_3^2 - b_1^2)^{1/2}} - x_0
\]

As \(x\) increases further, \(a^2_o(x)\) increases and the explosive increase in amplitude takes place at a distance

\[
x_2 = \frac{2K(\gamma)}{W(b_3^2 - b_1^2)^{1/2}} - x_0
\]

By contrast, in the case of the explosive instability, \(a^2_o(x)\) is given by Eq. (50) and therefore \(a^2_o(x)\) increases monotonically until an explosive increase in amplitude takes place at \(x_0\). In Fig. 6 the variation of \(a^2_o(x)\), \(a_1(x)\), \(a_2(x)\) as a function of \(x\) are shown schematically.
IV. NONLINEAR INSTABILITIES FOR LINEARLY GROWING (DAMPED) WAVES

Under realistic conditions, e.g., in the beam plasma system nonlinear instabilities occur between plasma waves which are linearly growing (or damped). If their linear growth (damping) rates are large compared with those due to the nonlinear effects, the results described in Section III are changed. For example, the decay (explosion) length becomes shorter for the linearly growing waves and longer for the linearly damped waves. Unfortunately, however, it is difficult to solve the coupled equations in general in these situations. Some insight into the modifications that are introduced by finite linear growth (damping) rates can be gained by discussing the case where all $\kappa_i$ are equal = $\kappa_0$. Then, introducing the quantities $\alpha_i(x)$ by

$$a_i(x) = a_i(x)e^{\kappa_0 x} \quad (i = 0, 1, \text{and } 2)$$

the coupled equations (17) become

$$\frac{d\alpha_0}{dx} = - S_0 \alpha_0 \alpha_2 e^{\kappa_0 x} \sin \theta$$

$$\frac{d\alpha_1}{dx} = S_1 \alpha_0 \alpha_2 e^{\kappa_0 x} \sin \theta$$

$$\frac{d\alpha_2}{dx} = S_2 \alpha_0 \alpha_0 \alpha_1 e^{\kappa_0 x} \sin \theta$$

$$\frac{d\theta}{dx} = \Delta \kappa + \cot \theta \frac{d}{dx} \ln \left( \alpha_0 \alpha_1 \alpha_2 \right)$$

For the corresponding Manley-Rowe relations, we have the two invariants;
\[ \frac{2}{o_1^2} + S_0 S_1 a_1^2 = P \]
\[ \frac{2}{o_2^2} + S_0 S_2 a_2^2 = Q \]

The last of Eq. (84) becomes

\[ (86) \frac{d}{dx} (\alpha_0 \alpha_1 \alpha_2 \cos \theta) - \frac{\Delta k}{2S_0^2} e^{-\kappa x \frac{d}{dx} \alpha_0^2} = 0 \]

For the perfect matching, $\Delta k = 0$, we have the other invariant, i.e.,

\[ (87) \alpha_0 \alpha_1 \alpha_2 \cos \theta = G \]

Using the above three variants, we can discuss the detailed character of the decay and explosive instabilities in analogy to the treatment in Section III.

i) Decay instabilities

(a) Linearly growing waves ($\kappa > 0$)

Here assume that $S_0 = S_1 = S_2 = -1$ as in III. Then, from Eq. (84), (85) and (87) we obtain the following equation

\[ (88) \frac{d}{dx} \alpha_0^2 = \pm 2Ne^{\kappa x} \left\{ \alpha_0^2 (P-a^2_0)(Q-a^2_0) - G^2 \right\} \]

Eq. (88) can be solved in the similar way as Eq. (84) in III and the general solution for the intensity of the three waves is

*Eq. (86) contains the term, $e^{-\kappa x \Delta k}$, which may be interpreted as a linearly growing wave having the effect of reducing and a linearly damped wave to increasing the mismatch.*
\[
\alpha_0^2(x) = b_2^2 + (b_2^2 - b_1^2) \text{sn}^2[\pm W(b_2^2 - b_1^2)^{\frac{1}{2}} \frac{1}{\kappa}(e^x - e^{-x})], \gamma
\]

(89)
\[
\alpha_2^2(x) = \alpha_1^2(0) + \alpha_0^2(0) - b_1^2 - (b_2^2 - b_1^2) \text{sn}^2[\pm W(b_2^2 - b_1^2)^{\frac{1}{2}} \frac{1}{\kappa}(e^x - e^{-x})], \gamma
\]
\[
\alpha_2^2(x) = \alpha_2^2(0) + \alpha_0^2(0) - b_1^2 - (b_2^2 - b_1^2) \text{sn}^2[\pm W(b_2^2 - b_1^2)^{\frac{1}{2}} \frac{1}{\kappa}(e^x - e^{-x})], \gamma
\]

where

(90)
\[
\gamma = \frac{b_2^2 - b_1^2}{b_3^2 - b_1^2} \leq 1
\]

and

(91)
\[
\mp (e^{\kappa x} - 1) = \text{sn}^{-1}\left[\frac{\alpha_2^2(0) - b_1^2}{b_2^2 - b_1^2}\right], \gamma
\]

Case (1a)

We consider the case of the boundary condition

\[
\alpha_0^2(0) = a_0^2(0) \gg \alpha_1^2(0) = a_1^2(0) > 0 \quad \text{and} \quad \alpha_2^2(0) = a_2^2(0) = 0
\]

which is the same as the case (1a) in III. The solution of Eq. (89) becomes

(92)
\[
a_0^2(x) = a_0^2(0) e^{2\kappa x} \text{sn}^2[\mp W(a_0^2(0) + a_1^2(0))^{\frac{1}{2}} \frac{1}{\kappa}(e^x - e^{-x})], \gamma
\]
\[
a_1^2(x) = a_1^2(0) e^{2\kappa x} + a_0^2(0) e^{2\kappa x} \text{cn}^2[\mp W(a_0^2(0) + a_1^2(0))^{\frac{1}{2}} \frac{1}{\kappa}(e^x - e^{-x})], \gamma
\]
\[
a_2^2(x) = a_2^2(0) e^{2\kappa x} \text{cn}^2[\mp W(a_0^2(0) + a_1^2(0))^{\frac{1}{2}} \frac{1}{\kappa}(e^x - e^{-x})], \gamma
\]

where

(93)
\[
\gamma = \frac{a_0^2(0)}{a_0^2(0) + a_1^2(0)} \approx 1
\]
and
\[ \frac{1}{\kappa (e^{\kappa x_0})} = \frac{K(\gamma)}{W[a_0^2(0) + a_1^2(0)]^{1/2}} \approx \frac{1}{W a_0(0)} \ln \left\{ \frac{4a_0(0)}{a_1(0)} \right\}. \]

Case (1b)

In the case where
\[ \alpha_1^2(0) = a_1^2(0) \gg \alpha_0^2(0) > 0 \quad \text{and} \quad \alpha_2^2(0) = a_2^2(0) = 0, \]
we have
\[ a_0^2(x) = a_0^2(0) e^{2\kappa x} \co \frac{2}{W[a_0^2(0) + a_1^2(0)]^{1/2}} \ln \left[ \frac{4a_0(0)}{a_1(0)} \right], \gamma \]
\[ a_1^2(x) = a_0^2(0) e^{2\kappa x} \co \frac{2}{W[a_0^2(0) + a_1^2(0)]^{1/2}} \ln \left[ \frac{4a_0(0)}{a_1(0)} \right], \gamma \]
\[ a_2^2(x) = a_0^2(0) e^{2\kappa x} \co \frac{2}{W[a_0^2(0) + a_1^2(0)]^{1/2}} \ln \left[ \frac{4a_0(0)}{a_1(0)} \right], \gamma \]

For small \( \gamma \), the intensity of the three waves is then given approximately by
\[ a_0^2(x) = a_0^2(0) e^{2\kappa x} \co \frac{2}{W[a_0^2(0) + a_1^2(0)]^{1/2}} \ln \left[ \frac{4a_0(0)}{a_1(0)} \right], \gamma \]
\[ a_1^2(x) = a_0^2(0) e^{2\kappa x} \co \frac{2}{W[a_0^2(0) + a_1^2(0)]^{1/2}} \ln \left[ \frac{4a_0(0)}{a_1(0)} \right], \gamma \]
\[ a_2^2(x) = a_0^2(0) e^{2\kappa x} \co \frac{2}{W[a_0^2(0) + a_1^2(0)]^{1/2}} \ln \left[ \frac{4a_0(0)}{a_1(0)} \right], \gamma \]
where
\[ \frac{1}{\kappa (e^{\kappa x_0})} = \frac{\pi/2}{W[a_0^2(0) + a_1^2(0)]^{1/2}} \approx \frac{\pi/2}{W a_0(0)} \left\{ 1 - \frac{1}{2} \frac{a_0^2(0)}{a_1^2(0)} \right\}. \]

It is interesting to compare the above results with those where \( \kappa = 0 \). In the latter case the decay length \( x_0 \) is given by
\[ x_0 = \frac{K(\gamma)}{W\{a_0^2(0) + a_1^2(0)\}^{\frac{1}{2}}} = \frac{C}{\kappa_{NL}} \]

\( x_0 \) is of the order of \( \kappa_{NL}^{-1} \).

On the other hand, in the case where \( \kappa \neq 0 \), the decay length is given by

\[ e^{\kappa x_0} = 1 + \kappa x_0 = 1 + C \frac{\kappa}{\kappa_{NL}} \]

or

\[ x_0^* = \frac{1}{\kappa} \ln (1 + \kappa x_0) \approx \frac{1}{\kappa} \ln \kappa x_0 \quad (\kappa > \kappa_{NL}) \]

It follows from Eq. (100) that if the linear growth rate is much larger than the nonlinear growth rate, i.e., \( \kappa >> \kappa_{NL} \), the decay length \( x_0^* \) becomes much shorter than \( x_0 \) and is of the order of \( \kappa_{NL}^{-1} \) and nearly independent of \( \kappa_{NL} \). Thus, in the case of \( \kappa \neq 0 \), the decay instability can occur for rather weak initial amplitudes \( a_o(0) \), and \( a_1(0) \). In Fig. 8 the decay length \( x_0^* \) is plotted as a function of \( \kappa_{NL} \) with \( \kappa \) as a parameter.

In the case of \( \kappa = 0 \) the maximum amplitudes of \( a_o \), \( a_1 \), and \( a_2 \) are fixed at the respective initial values. In the case of \( \kappa \neq 0 \) on the contrary, their maximum values are growing exponentially and the minimum values of \( a_o(0) \) and \( a_1(0) \) become zero.

---

*Here it is convenient to take

\[ \kappa_{NL} = W a_o(0) \] and \[ C = \frac{K(\gamma)}{a_1(0) \{2^{\frac{1}{2}} \} \left[ 1 + \left( \frac{1}{a_o(0)} \right) \right]} \]

for the case where \( a_o^2(0) > a_1^2(0) \). The constant \( C \) is plotted in Fig. 7 as a function of \( a_1(0)/a_o(0) \).
(b) Linearly damped wave ($\kappa = |\kappa| < 0$)

For this case, the intensity of three waves is the same as that given in (a) with $\kappa$ replaced by $-|\kappa|$. However, the condition $\kappa_{NL} > |\kappa|$ must be satisfied in this case. This is a natural limit because otherwise the waves are damped out before the nonlinear coupling occurs.

Using Eq. (91), the condition for a decay instability to occur and the definition of the decay length, is given by

$$
(101) \quad (1 - e^{-|\kappa|x^*_o}) = \frac{|\kappa|}{W(b_3^2 - b_1^2)^{1/2}} \text{sn}^{-1}\left[\frac{\sigma_0^2(0) - b_1^2}{b_2^2 - b_1^2}\right], \gamma < 1
$$

For the same boundary condition as the case (1a and b), the above condition gives

$$
(102) \quad 1 - e^{-|\kappa|x^*_o} = |\kappa|x^*_o < 1
$$
or

$$
|\kappa| < \frac{\kappa_{NL}}{C}
$$

The decay length is given by

$$
(103) \quad x^*_o = -\frac{\kappa n(1 - |\kappa|)}{|\kappa|} > 0
$$
and $x^*_o$ becomes long compared with $\kappa_{NL}^{-1}$.

As mentioned in II, in the beam-plasma system there are four kinds of interaction resulting in the decay instability. However, the modes on the positive energy branch are linearly damped waves. This imposes rather severe conditions in the occurrence of the decay instability for these modes compared with the case where all three modes are on the negative energy branch.
ii) Explosive instabilities

(a) Linearly growing waves \((\kappa > 0)\)

Here we assume that \(S_0 = 1, S_1 = S_2 + 1\) as in III. Then from Eq. (84) (85) and (87) we obtain the following equation

\[
\frac{d}{dx} \sigma_0^2 = \pm 2 \omega e^{\kappa x} \left\{ \sigma_0^2 (\sigma_0^2 - p) (\sigma_0^2 - q) - \mathcal{G} \right\}^{1/2}
\]

The general solution for the intensity of the three waves is

\[
\sigma_0(x) = b_1^2 + \frac{b_3^2 - b_1^2}{\text{sn}^{-1} \left[ \omega \left( \frac{b_3^2 - b_1^2}{\text{sn}^{-1} \left[ \sigma_0^2 (\sigma_0^2 - p) (\sigma_0^2 - q) - \mathcal{G} \right]^{1/2}} \right] , \gamma \right]}
\]

\[
\sigma_1(x) = \sigma_1(0) - \sigma_0(0) + \sigma_0(x)
\]

\[
\sigma_2(x) = \sigma_2(0) - \sigma_0(0) + \sigma_0(x)
\]

with

\[
\gamma^2 = \frac{b_3^2 - b_1^2}{b_3^2 - b_1^2}
\]

and

\[
\pm (e^{\kappa x} - 1) = \frac{\sigma_0^2 (\sigma_0^2 - p)}{\text{sn}^{-1} \left[ \frac{b_3^2 - b_1^2}{\sigma_0^2 (\sigma_0^2 - p) - \mathcal{G}} \right]^{1/2}} \text{sn}^{-1} \left[ \frac{b_3^2 - b_1^2}{\sigma_0^2 (\sigma_0^2 - p) - \mathcal{G}} \right]^{1/2}, \gamma \right]
\]

The wave becomes explosive at \(x_{\text{exp}}^* = x_0^*\).

If we consider the case where the boundary condition is

\[
\sigma_0^2(0) = a_0^2(0) \gg \sigma_1^2(0) = a_1^2(0) > 0 \text{ and } \sigma_2^2(0) = a_2^2(0) = 0,
\]

which is the same as the case (1) in III, the intensity of the three waves is
\[
\frac{2}{a_0(x)} = \frac{a_0^2(0) e^{2\kappa x}}{\text{sn}^2[W a_0(0) \frac{1}{\kappa} (e^{\kappa x} - e^{-\kappa x})], y]}
\]

\[
a_1(x) = a_1^2(0) e^{2\kappa x} + a_0^2(0) e^{2\kappa x} \text{cn}^2[W a_0(0) \frac{1}{\kappa} (e^{\kappa x} - e^{-\kappa x}), y]
\]

(108)

\[
a_2(x) = a_0^2(0) e^{2\kappa x} \text{cn}^2[W a_0(0) \frac{1}{\kappa} (e^{\kappa x} - e^{-\kappa x}), y]
\]

with

\[
\gamma^2 = 1 - \frac{a_1^2(0)}{a_0^2(0)} \approx 1
\]

(109)

and

\[
\frac{1}{K(\kappa^*-e_0)} = \frac{K(\kappa)}{W a_0(0)} \approx \frac{1}{W a_0(0)} \ln \frac{4 a_0(0)}{a_1(0)}
\]

The explosion distance is therefore given by

\[
\frac{\kappa^*}{x^*_{\text{exp}}} = \frac{\kappa}{x^*_{\text{exp}}} = \frac{1}{\kappa} \ln (1 + \kappa^*-e_{\text{exp}})
\]

(111)

and is of the order of \(\kappa^{-1}\) when \(\kappa \gg \kappa_{\text{NL}}\) analogous to that in the decay instability with \(\kappa \neq 0\).

(b) Linearly damped wave \((\kappa = -|\kappa| < 0)\)

The solution is the same as in (a) if we replace \(\kappa\) by \(-|\kappa|\). The condition for the explosive instability to occur and the definition of the explosion distance is given by

\[
(1 - e^{-|\kappa| x^*_0}) = \frac{|\kappa|}{W(b_3^2 - b_1^2)^{\frac{1}{2}}} \frac{1}{\text{sn}^{-1}[\frac{b_3^2 - b_1^2}{a_0(0) - b_1^2}]}, y < 1
\]

(112)

For the same boundary condition as in (a), the explosion distance is
\[ x_{\text{exp}}^* = x_0^* = -\log \left( 1 - \frac{|\kappa|x_{\text{exp}}}{|\kappa|} \right) > 0 \]

where it was assumed that

\[ |\kappa| < \frac{1}{x_{\text{exp}}} \frac{\kappa_{\text{NL}}}{K(\gamma)} \]

\( x_{\text{exp}}^* \) becomes longer than that in the case where \( |\kappa| = 0 \).

For the explosive instability to occur in a beam-plasma system, at least one mode must lie on the positive energy branch and is therefore linearly damped. This situation seems to be unfavorable for the occurrence of the explosive instability.

### iii) Quenching of growing waves (\( \kappa > 0 \))

If we consider the boundary condition

\[ \alpha_0^2(0) = a_0^2(0) \geq \alpha_1^2(0) = a_1^2(0) \geq \alpha_2^2(0) = a_2^2(0) > 0 \]

and \(-\pi/2 < \alpha_0 < 0\), there appears a quenching of the three waves.

Taking the lower sign of Eq. (105), we obtain

\[ a_0^2(x) = b_1^2 e^{2\kappa x} + \frac{(b_3^2 - b_1^2) e^{2\kappa x}}{\text{sn}^2 \left[ W(b_3^2 - b_1^2)^{1/2} \frac{1}{x} \right] (e^{-e^{-x_0^*}}), \gamma} \]

where \( x_0^* \) is determined by

\[ 1 - e^{-x_0^*} = \frac{x}{W(b_3^2 - b_1^2)^{1/2}} \text{sn}^{-1} \left[ \frac{b_3^2 - b_1^2}{a_0^2(0) - b_1^2} \right], \gamma \]

and \( x_0^* < 0 \).
Equation (114) shows that $\alpha^2_o(\gamma)$ decrease at first and reaches the minimum value $\alpha^2_o(x_1) = b_3^2$ at $x = x_1$ and then increases and becomes infinite at $x = x_2$, where $x_1$ and $x_2$ are defined through the relations

$$\frac{1}{\nu} \left( e^{x_1} - e^{x_0} \right) = \frac{K(\gamma)}{W(b_3^2 - b_1^2)^{1/2}}$$  \hspace{1cm} (116)$$

and

$$\frac{1}{\nu} \left( e^{x_2} - e^{x_0} \right) = \frac{2K(\gamma)}{W(b_3^2 - b_1^2)^{1/2}}$$  \hspace{1cm} (117)$$
V. COMPUTATION OF THE COUPLING COEFFICIENT

The strength of the interaction is determined by the quantity $W(w_o, w_1, w_2)$ defined by Eq. II. (14). From a practical point of view it is convenient to express the coupling coefficient in terms of the nonlinear growth rate

\[(118) \quad \kappa_{NL} = W a_0(0)\]

that was already introduced in the previous section. This quantity has an immediate physical significance. It is the rate at which the waves 1 and 2 would grow if $a_0$ would stay constant.

For the case of a monoenergetic electron beam in a cold plasma the distribution function is

\[f_o(v) = n_p \delta(v) + \alpha n_B \delta(v - v_o)\]

where $\alpha = n_B/n_p \ll 1$ is the ratio of beam density to plasma density. $\kappa_{NL}$ then becomes

\[(119) \quad \kappa_{NL} = \frac{e}{m} = \frac{E_o(0)}{v_o^2} \Gamma_{NL}\]

where

\[(120) \quad \Gamma_{NL} = \frac{1}{4} (\Omega_1 - K_1)^{2/3} (\Omega_2 - K_2)^{2/3} \left\{ \frac{1}{\alpha \Omega_1 \Omega_2 \Omega_o} \left( \frac{K_1}{\Omega_1} + \frac{K_2}{\Omega_2} + \frac{K_o}{\Omega_o} \right) \right\} + \frac{1}{(\Omega_1 - K_1)(\Omega_2 - K_2)(\Omega_o - K_o)} \left[ \frac{1}{\Omega_1 - K_1} + \frac{K_2}{\Omega_2 - K_2} + \frac{K_o}{\Omega_o - K_o} \right]\]

where

\[\Omega = \frac{w}{w_p}, \quad K = \frac{kv_o}{w_p}\]
If we compute $\Gamma_{NL}$ from the moment equations including collisions in the equations of motion for the plasma electrons we find

$$
\Gamma_{NL} = \frac{1}{4} (\Omega_1 - K_1)^{3/2} (\Omega_2 - K_2)^{3/2} \left[ \frac{1}{\alpha \Omega_0 (\Omega_1 + i\nu)(\Omega_2 + i\nu)} \left( \frac{K_0}{\Omega_0 + i\nu} + \frac{K_1}{\Omega_1} + \frac{K_2}{\Omega_2} \right) \right]$

(121)

$$
+ \frac{1}{(\Omega_1 - K_1)(\Omega_2 - K_2)(\Omega_0 - K_0)} \left[ \frac{K_1}{\Omega_1 - K_1} + \frac{K_2}{\Omega_2 - K_2} + \frac{K_0}{\Omega_0 - K_0} \right]
$$

The dominant contribution comes from the second term in the curly bracket. The effect of collisions is therefore small and we are justified in neglecting them in the sense that we replace Eq. (121) by Eq. (120). It is, however, essential to retain the collisions in the expressions for $K (K = K(\Omega, \nu))$ because they influence the mode structure in a profound way.

In Figures 9 and 12 we present representative examples of $\Gamma_{NL}$ and the mismatch $\Delta \nu_0 / \omega_p$ for the case of the decay and the explosive instability corresponding to Case b) and g) in Figure 2.

One of the noteworthy features of these plots is the fact that the nonlinear coupling is a minimum when two of the primary modes are in the vicinity of the most unstable mode. Experimentally this corresponds to the case when the primary waves have their largest amplitude due to linear growth.
REFERENCES

Figure 1. Mode structure of a beam-plasma system for the temporal (la) and spatial (lb) case. In the upper diagram the solid curve corresponds to the unstable branch.
Figure 2. Possible ways to couple 3 modes on different energy branches leading to the decay - or explosive instability.
Figure 3. Spatial variation of wave amplitudes for the decay instability.
Figure 4. Effect of finite mismatch on the spatial variation of wave amplitudes for the decay instability.
Figure 5. Spatial variation of wave amplitudes for the explosive instability (Case 1) and the non-explosive instability (Case 2).
Figure 6. Quenching effect due to finite amplitude in $a_2(0)$ for the explosive instability.
Figure 7. The constant C: as a function of $a_1(0)/a_0(0)$
Figure 8. Influence of linear growth rate on the decay length.
Figure 9. Nonlinear growth rate for the decay instability when all 3 waves are located on the same branch of the dispersion relation (Figure (2b)).
Figure 10. Wave number mismatch for the cases of Figure 8. 
\( \Delta k = k_1 + k_2 - k_0 \)
Figure 11. Nonlinear growth rate for explosive coupling to the sum-frequency (Figure 2g).
Figure 12. Wave number mismatch for the cases of Figure 11. \[ \Delta k = k_2 - k_1 - k_0 \]
THREE WAVE COUPLING IN A COLD BEAM-PLASMA SYSTEM

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The coupling of 3 fixed phase waves in a cold beam-plasma system is discussed for all cases that result from locating the waves on the positive and negative energy branch. Explicit expressions are given for the spatial variation of the wave amplitudes including the influence of wave number mismatch and spatial growth rates. The coupling constants are computed for the decay and the explosive mode for representative beam and plasma parameters.
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