H-Infinity Control in Infinite-Dimensional Spaces and Related Partial Differential Equations

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1308 West Main Street, Urbana, IL 61801
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<th>1a. REPORT SECURITY CLASSIFICATION</th>
<th>Unclassified</th>
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<tr>
<td>2a. SECURITY CLASSIFICATION AUTHORITY</td>
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<td>2b. DECLASSIFICATION/DOWNGRADING SCHEDULE</td>
<td></td>
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<tr>
<td>4. PERFORMING ORGANIZATION REPORT NUMBER(S)</td>
<td>UILU-ENG-97-2224 DC-181</td>
</tr>
<tr>
<td>5a. NAME OF PERFORMING ORGANIZATION</td>
<td>Coordinated Science Laboratory University of Illinois</td>
</tr>
<tr>
<td>6b. OFFICE SYMBOL (if applicable)</td>
<td>N/A</td>
</tr>
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<td>7a. NAME OF MONITORING ORGANIZATION</td>
<td>National Science Foundation</td>
</tr>
<tr>
<td>6c. ADDRESS (City, State, and ZIP Code)</td>
<td>1308 West Main Street Urbana, IL 61801</td>
</tr>
<tr>
<td>7b. ADDRESS (City, State, and ZIP Code)</td>
<td>Washington, DC 20050</td>
</tr>
<tr>
<td>8a. NAME OF FUNDING/SPONSORING ORGANIZATION</td>
<td>National Science Foundation</td>
</tr>
<tr>
<td>8c. ADDRESS (City, State, and ZIP Code)</td>
<td>Washington, DC 20050</td>
</tr>
<tr>
<td>11. TITLE (Include Security Classification)</td>
<td>H-Infinity Control in Infinite-Dimensional Spaces and Related Partial Differential Equations</td>
</tr>
<tr>
<td>12. PERSONAL AUTHOR(S)</td>
<td>XIAO, Mingqing</td>
</tr>
<tr>
<td>13a. TYPE OF REPORT</td>
<td>Technical</td>
</tr>
<tr>
<td>13b. TIME COVERED</td>
<td>FROM TO</td>
</tr>
<tr>
<td>14. DATE OF REPORT (Year, Month, Day)</td>
<td>1997 July</td>
</tr>
<tr>
<td>15. PAGE COUNT</td>
<td>326</td>
</tr>
<tr>
<td>16. SUPPLEMENTARY NOTATION</td>
<td></td>
</tr>
<tr>
<td>17. COSATI CODES</td>
<td>FIELD</td>
</tr>
<tr>
<td>18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)</td>
<td>Infinite-dimensional systems; H-infinity-optimal control; Differential Games; Evolution equations; Viscosity solutions Parabolic systems; Robust boundary control.</td>
</tr>
<tr>
<td>19. ABSTRACT (Continue on reverse if necessary and identify by block number)</td>
<td>See attachment.</td>
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<tr>
<td>20. DISTRIBUTION/AVAILABILITY OF ABSTRACT</td>
<td>UNCLASSIFIED/UNLIMITED</td>
</tr>
<tr>
<td>21. ABSTRACT SECURITY CLASSIFICATION</td>
<td>Unclassified</td>
</tr>
<tr>
<td>22a. NAME OF RESPONSIBLE INDIVIDUAL</td>
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<td>22b. TELEPHONE (Include Area Code)</td>
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<td>22c. OFFICE SYMBOL</td>
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H-INFINITY CONTROL IN INFINITE-DIMENSIONAL SPACES
AND RELATED PARTIAL DIFFERENTIAL EQUATIONS

BY

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 1997

Urbana, Illinois
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
THE GRADUATE COLLEGE

JULY 1997
(date)

WE HEREBY RECOMMEND THAT THE THESIS BY
MINGQING XIAO

ENTITLED
H-INFINITY CONTROL IN INFINITE-DIMENSIONAL SPACES
AND RELATED PARTIAL DIFFERENTIAL EQUATIONS

BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF

DOCTOR OF PHILOSOPHY

Director of Thesis Research

Head of Department

Committee on Final Examination†

Chairperson

† Required for doctor's degree but not for master's.
Abstract

This thesis furthers the $H^\infty$-optimal control theory for infinite-dimensional systems and develops new results on related partial differential equations.

The thesis consists of nine chapters, and begins with a general introduction to $H^\infty$-optimal control problems for infinite-dimensional systems, and to the corresponding theory of (viscosity or operator) solution for Hamilton-Jacobi-Isaacs equations (HJI). The second chapter studies the $H^\infty$ Control of a class of infinite-dimensional linear systems with nonlinear outputs. By using a differential game-theoretic approach, a set of necessary and sufficient conditions are obtained for the existence of a state-feedback controller under which a given $H^\infty$ bound (on disturbance attenuation) is achieved. Characterization of such a controller is given, and the result is applied to a disturbance attenuation problem with control constraints.

The third chapter demonstrates the existence of finite-dimensional compensators in connection with the $H^\infty$-optimal control of infinite-dimensional linear time-invariant systems. The approach adopted uses a Galerkin-type approximation, where there is no requirement for the system operator to have a complete set of eigenvectors. It is shown that if there exists an infinite-dimensional compensator satisfying the $H^\infty$ robustness property, then a finite-dimensional compensator exists and achieves the same level of robustness. The order of the finite-dimensional compensator depends on the requirement of robustness which is measured by the difference between the desired attenuation level and the critical value of the system, which is inherent to a given system. A procedure for constructing finite-dimensional compensators based on approximate solution of the optimal regulator and optimal observer via Galerkin-type approximations is provided.
A relationship between two generalized Riccati equations and two critical parameter values, which are related to the attenuation of $H^\infty$-optimal control problems associated with a class of time-variant linear evolution systems on infinite-dimensional spaces, is established in chapter 4. Explicit formulas for calculating these two critical parameters as well as the closed-form solutions of these two generalized Riccati equations are given. The connection between these two parameters and a two-person differential game is also investigated. The approximation of the infinite-dimensional Riccati equations is also considered.

The fifth chapter deals with the $H^\infty$-optimal control problem in infinite dimensional linear systems when only sampled values of the state are available. For finite-horizon time-varying systems the optimum controller is characterized in terms of the solution of a particular generalized Riccati evolution equation, with the optimum performance determined by a family of generalized Riccati evolution equations. For the infinite-horizon time-invariant problem the optimum controller is characterized in terms of the solution of a particular generalized Riccati evolution equation, and the performance is determined in terms of the conditions of a single generalized Riccati equation, defined on the longest sampling interval. It is shown that when the sampling interval is chosen appropriately the sampled feedback controller adopted can achieve the desired attenuation level for which usually regular feedback controller is needed. Moreover an upper bound on the sampling interval is estimated to guarantee desired performance. Two algorithms are presented for designing sampled-data controllers, one for the finite-horizon case, and the other one for the infinite-horizon case.

The sixth chapter is considered to be a further study of the topics of the fifth chapter, when only sampled values of the state are available, with control acting on the boundary.
Under appropriate assumptions the existence and uniqueness of an optimal disturbance attenuating controller are proven. Properties of the solutions of the associated generalized Riccati equations are also studied. The problem is formulated in a differential game framework by associating a zero-sum differential game with the original disturbance attenuation control problem. The minimizing player's minimax strategy in this game corresponds to the optimal controller in the disturbance attenuation problem, which is linear and is characterized in terms of the solution of a particular generalized Riccati evolution equation. The optimum achievable performance is determined by the condition of existence of a solution to another family of generalized Riccati evolution equations. The formulation allows for the control to be time-varying between two consecutive sampling times, and in this respect this chapter presents optimum choices for these waveforms as functions of sampled values of the state. A relationship is obtained between the length of the longest sampling interval and the performance of the derived controllers, and it is shown that as the former goes to zero the latter converges to the optimum performance obtained under continuous feedback information.

In the seventh chapter, two Hamilton-Jacobi-Bellman (HJB) equations corresponding to nonlinear control of piecewise deterministic systems are considered in which the control could be unbounded. Since these two HJB equations are coupled partial differential equations, their Hamiltonian structures are different from the standard one. Existence and uniqueness results have been obtained in the sense of viscosity solutions. Explicit control structures are also provided.

The eighth chapter studies the solvability of HJI equations that arise in finite and infinite-horizon nonlinear $H^\infty$ control problems where the system is affine in the control
and the disturbance, while the cost function is not necessarily continuous in the state and the control. In each case the existence of viscosity supersolutions is proven, under the assumption that the value function is finite. A result on global asymptotic stability of closed-loop system under the $H^\infty$ controller and the worst-case disturbance is obtained.

The thesis ends with chapter 9, which provides a summary of contents and contributions of the thesis, and discusses possible extensions of the results obtained, as well as some future challenges.
Dedication

To my parents
ACKNOWLEDGEMENTS

I am deeply grateful to my advisor, Professor Tamer Başar, who not only took me into the Decision and Control Laboratory but also brought me into the very frontier of research. His dedication to academic excellence and enthusiasm to work have been exemplary to me. I have learned much from him, and his patience and knowledge have been an inspiration to me. Without his support, this thesis would not have been possible.

I would also like to express my sincere gratitude to Professor Robert W. Carroll, who has served as my academic advisor since I have been with the University of Illinois, for supervision in the early stages of my doctoral research. It was through his encouragement, help and care that I was able to start doing research in Control Theory before I met Professor Başar. During those years, even though his research area is different from Control Theory, he still did his best to help me in my research. I am in debt of gratitude for all he has done for me during these years.

I also want to thank Professor Jerry Uhl, who brought me into the Calculus & Mathematica Development Team two years ago. I benefited a lot from this new, creative style of teaching, so did my research from the Mathematica.

This thesis benefits a lot from the courses taught by Professor Robert W. Carroll, Professor Robert Jerrard, Professor Nick Firoozey, and Professor Bassam Bamieh, as well as from many discussions I had with them. I also would like to thank Professor Juraj Medanic for being a member of my final exam committee with his busy schedule, and for providing valuable comments. Finally, I would like to express my deep appreciation for the help of Lori Dick, graduate secretary of the Department of Mathematics, and Becky Lonberger and Francie Bridges, secretaries of the Coordinated Science Laboratory.
University of Illinois.

This work was supported in part by U.S. Department of Energy under Grant DOE-DEFG-02-94ER13939, in part by the National Science Foundation under Grant ECS-93-12807 and in part by a University of Illinois Fellowship.

Finally I would like to thank my wife for her patience and understanding during these years. I also would like to thank my parents for their constant encouragement, support, and love.
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1. Introduction

Infinite-dimensional systems are generally governed by partial differential equations (PDEs) or functional differential equations (FDEs), and hence also cover distributed parameter systems. PDE and FDE representations are natural for many practical systems such as flexible beams, delay systems, parabolic systems, airfoil dynamics, combustion systems, economic systems, robot control, etc. Therefore, one of the reasons behind using distributed-parameter models is that such infinite-dimensional models may represent the dynamics of physical systems more accurately than their finite-dimensional approximates. Yet another reason is that, in some cases, infinite-dimensional models with only a few parameters can provide a better characterization of physical phenomena than finite-dimensional models with a large number of parameters. Thus, the economical representation of the system is another important reason why distributed models are used in practice. Control of systems governed by PDEs or FDEs is a fascinating and challenging topic, from a mathematical as well as an applications point of view.

One of the major concentrated activities of the past decade in control theory has been the development of the so called “$H^\infty$-optimal control theory”, which involves control of systems under uncertainties. The goal is to design a controller such that the performance of the controlled system is “optimal” in certain sense regardless of unknown disturbances. In a typical control system there are two kinds of uncertainties: (i) disturbances or noises, and (ii) modeling errors. The purpose of control is to achieve certain performance specifications in the closed loop system despite these uncertainties. We shall introduce the problem in Section 1.1 below in more technical details. The mathematical symbol “$H^\infty$” stands for the Hardy space of all complex-valued functions of a complex variable, which are
analytic and bounded in the open right-half complex plane. Adoption of this terminology stems from the historical fact that the fundamental $H^\infty$-optimal problem was originally formulated in the frequency domain (by Zames [Zam]). Later, it was discovered that time-domain methods based on state-space representations lead to simpler and more complete derivations for a large class of systems in finite or infinite-dimensional spaces. One of the most important features of the $H^\infty$-optimal control theory is that one can incorporate both disturbances and unmodeled dynamics in the plant into a single design criterion, which makes it fundamentally different from the classical linear quadratic Gaussian (LQG) control theory. Moreover, there are many control engineering problems such as sensitivity minimization, model matching, tracking, robust stabilization, etc, which can be put in the framework of $H^\infty$-optimal control theory.

The study of $H^\infty$-optimal control problem for distributed parameter systems started at the beginning of 90's, and the theory has not yet been fully developed. One of the major reasons is that control problems of distributed parameter systems present some challenging issues that are not present in the finite-dimensional case. For instance, it is no longer easy to establish necessary and sufficient conditions to describe the optimal level of disturbance attenuation, $\gamma_c^{*}$, an example in point being the boundary control problem for distributed-parameter systems. Another major difficulty is that for infinite-dimensional systems, the corresponding Hamilton-Jacobi-Isaacs (HJI) equations are evolution equations, and the existence of solutions to these HJI evolutions as well as verification of other properties are still not clear, in particular, for nonlinear systems.

Pritchard and Townley were one of the first to study the singular $H^\infty$-control problem of distributed parameter systems for the so-called Pritchard-Salamon class, in connec-
tion with stability radius optimization ([PT]). Keulen, Peters and Curtain generalized the
finite-dimensional result of Zhou and Khargonekar [ZK] to linear time-invariant infinite-
dimensional systems ([KPC]). Bensoussan and Bernhard extended the finite-dimensional
result of Baṣar and Bernhard [BB] to infinite-dimensional systems when there are dis-
turbances in output measurements ([BeBe]). McMillan and Triggiani developed the $H^\infty$
-optimal control theory for boundary control problems described by some particular type of
partial differential equations ([MT 1-3]). Barbu mainly studied the $H^\infty$-optimal boundary
control for the hyperbolic case ([B1]), while Keulen focused on Pritchard-Salamon systems,
whose inputs and outputs may be unbounded ([Ke]).

1.1. Motivation of the $H^\infty$-Problem

First, we sketch the so-called standard $H^\infty$-optimal control problem, following the de-
velopment in the book of Baṣar and Bernhard ([BB]). Let $\Sigma_p$ be some (infinite-dimensional)
plant with two inputs and two outputs, which can be expressed in the following compact
form:

\begin{equation}
\begin{aligned}
z &= G_{11}(w) + G_{12}(u) \\
y &= G_{21}(w) + G_{22}(u) \\
u &= \mu(y)
\end{aligned}
\end{equation}

where $z$ represents the to-be-controlled output, $y$ the measured output, $w$ the disturbances,
and $u$ the dynamic controller, and they belong to appropriate Hilbert spaces, denoted by
$\mathcal{H}_z$, $\mathcal{H}_y$, $\mathcal{H}_w$, and $\mathcal{H}_u$, respectively. $G_{ij}, i, j = 1, 2$ are appropriate causal linear operators$^1$.

For simplicity, we also assume that $\mu \in \mathcal{M}$ is a causal linear operator; here $\mathcal{M}$ is the

$^1$ i.e., if $\phi_1(t) = \phi_2(t)$ for almost all $t \in I$ we have $(G_{ij} \phi_1)(t) = (G_{ij} \phi_2)(t)$ for almost all $t \in I$, where $I$
is any allowable time interval, and $\phi_1, \phi_2$ belong to appropriate spaces.
controller space, which is assumed to be compatible with the information available to the controller.

Considering now the closed-loop system of (1.1.1) describing the relationship between \( w \) and \( z \) only, we have the operator \( T : \mathcal{H}_w \to \mathcal{H}_z \) given by

\[
(1.1.2) \quad T = G_{11} + G_{12} \mu ([I - G_{22} \mu]^{-1} G_{21})
\]

provided the inverse involved exists. The \( H^\infty \)-optimal controller design problem is to find a control \( \mu^* \) such that the operator norm \( \| T \|_{\mathcal{L}(\mathcal{H}_w, \mathcal{H}_z)} \) is minimized. Therefore the \( H^\infty \)-optimal control problem for above system is to

\[
\text{minimize} \quad \| T \|_{\mathcal{L}(\mathcal{H}_w, \mathcal{H}_z)}
\]

over all admissible controls in \( \mathcal{M} \). If this problem has a solution, we define

\[
(1.1.3) \quad \gamma^*_c = \inf_{\mathcal{M}} \| T \|_{\mathcal{L}(\mathcal{H}_w, \mathcal{H}_z)}
\]

and \( \gamma^*_c \) is called the minimax disturbance attenuation level under the compatible output measurement. The word minimax is used here because

\[
(1.1.4) \quad \inf_{\mathcal{M}} \| T \|_{\mathcal{L}(\mathcal{H}_w, \mathcal{H}_z)} = \inf_{\mathcal{M}} \sup_{\| w \| = 1} \| Tw \|_{\mathcal{H}_z}
\]

and thus this turns out to be a minimax optimization problem.

A direct calculation of the minimax disturbance attenuation level \( \gamma^*_c \) is technically complicated in general. Hence, a reasonable formulation of the original basic problem may be the following: Given a positive number \( \gamma > \gamma^*_c \), find, if it exists, a controller that will ensure satisfaction of the inequality

\[
(1.1.5) \quad \| T \|_{\mathcal{L}(\mathcal{H}_w, \mathcal{H}_z)} < \gamma
\]
which is sometimes called the relaxed $H^\infty$-optimal control problem or suboptimal $H^\infty$-control problem. The relaxed $H^\infty$-optimal control problem is clearly simpler than the original $H^\infty$-optimal control problem, and solving it would be adequate for most application problems in practice.

1.2. The $H^\infty$-control problem in a semigroup framework

That the theory of semigroups known as Hille-Yosida's theory is capable of far-reaching applications to the initial and mixed problems of parabolic and hyperbolic equations with coefficients independent of the time variable was recognized in 50's. It was Kato who first produced significant results concerning the semigroup-theoretic treatment of equations with coefficients that depend on the time variable ([K2]). Since then research in this field has become very active and an extensive literature has appeared (see Krein [Kr] and others listed in the bibliography). At the same time, Lions [L2], and Carroll [C] have used representation theorems of linear functionals by quadratic forms in the study of equations of evolution, which is different from the semigroup approach but is also a very effective way of dealing with linear time-varying partial differential equations.

Most of the infinite-dimensional linear systems involving state-feedback control and disturbances can formally be described by equations of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + D(t)w(t), \quad x(0) = x_0$$

$$z(t) = C(t)x(t) + D_{11}(t)u(t) + D_{12}(t)w(t)$$

where the state $x(\cdot)$, the control $u(\cdot)$, the disturbance $w(\cdot)$, and the output $z(\cdot)$ are all functions of time with values in certain Hilbert spaces. Furthermore for each fixed $t$, $A(t)$ is the infinitesimal generator of a $C_0$-semigroup, $B$ and $D$ are the input operators, $C$ is the
output operator, $D_{11}$ and $D_{12}$ are the feedthrough operators. \((1.2.1)\) is usually represented by the so-called mild solution in conjunction with \((1.1.1)\) as:

\[
(1.2.2) \quad x(t) = U(t, 0)x_0 + \int_0^t U(t, \eta)(B(\eta)u(\eta) + D(\eta)w(\eta))d\eta \\
\quad \quad z(t) = C(t)x(t) + D_{11}(t)u(t) + D_{12}(t)w(t)
\]

where $U(\cdot, \cdot)$ is the evolution operator associated with $A(\cdot)$. In some cases such as boundary control, operators $B$ and $C$ may be unbounded, and the representation \((1.2.2)\) must be interpreted in the sense of a "bigger" space ([Ber], [B2], [MT 1-3]).

Let us denote the state space by $X$, the control space by $U$, the disturbance space by $W$, and the output space by $Z$. Suppose that $F \in L(X, U)$ is such that the evolution operator $T_F(\cdot, \cdot)$ associated with $A + BF$ is exponentially stable. Then the influence of $w(\cdot)$ on $z(\cdot)$ is measured by the induced operator norm of $T$, where $T$ is the bounded linear operator from $L_2(0, \infty; W)$ to $L_2(0, \infty; Z)$ mapping $w(\cdot)$ to $z(\cdot)$ in \((1.2.2)\) with $x_0 = 0$ and $u(\cdot) = Fx(\cdot)$:

\[
(1.2.3) \quad (Tw)(t) = (C + D_{11}F) \int_0^t T_F(t, s)D_{11}w(s)ds + D_{12}w(t).
\]

Since $T_F(\cdot, \cdot)$ is exponentially stable, it follows that $T \in L(L_2(0, \infty; W), L_2(0, \infty; Z))$. The suboptimal $H^\infty$ problem is to seek an exponentially stabilizing $F \in L(X, U)$ such that $\|T\| < \gamma$ for some prespecified $\gamma > 0$, where $\|T\|$ denotes the induced operator norm mentioned above. The optimal $H^\infty$ problem is to find the smallest possible $\gamma$ for which $\|T\| < \gamma$ holds.

1.3. A Relationship Between $H^\infty$-Optimal Control and LQ Zero-Sum Dynamic Games

The fact that the $H^\infty$-optimal control problem is also a differential game was expounded
by Başar and Bernhard in [BB], the techniques of which are now widely used in dealing with $H^\infty$-optimal control problems in finite or infinite dimensional spaces. Recall that the $H^\infty$-optimal control problem is to find a controller $\mu \in \mathcal{M}$ such that

$$ \gamma_c^* = \inf_{\mathcal{M}} \| T \|_{\mathcal{L}(\mathcal{H}_w, \mathcal{H}_z)} = \inf_{\mathcal{M}} \sup_{\| w \| = 1} \| T w \|_{\mathcal{H}_z} $$

has a solution. This requirement is equivalent to the following (i)& (ii):

(i)

$$ \| T_\mu (w) \|_{\mathcal{H}_z}^2 \leq \gamma_c^* \| w \|_{\mathcal{H}_w}^2, \quad \forall w \in \mathcal{H}_w $$

(ii) there is no other $\mu \in \mathcal{M}$, say $\hat{\mu}$, and a corresponding $\hat{\gamma} < \gamma_c^*$, such that

$$ \| T_{\hat{\mu}} (w) \|_{\mathcal{H}_z} \leq \hat{\gamma}^2 \| w \|_{\mathcal{H}_w}^2, \quad \forall w \in \mathcal{H}_w $$

Now, introducing the parametrized (in $\gamma \geq 0$) family of cost functions:

$$ J_\gamma (\mu, w) := \| T_\mu (w) \|_{\mathcal{H}_z}^2 - \gamma^2 \| w \|_{\mathcal{H}_w}^2 $$

(i)&(ii) above become equivalent to the problem of finding both the "smallest" value of $\gamma \geq 0$ under which the upper value of the associated game with objective function $J_\gamma (\mu, w)$ is bounded above by zero, and the corresponding controller that achieves this upper value. This is called the soft-constrained game associated with the disturbance attenuation problem, because in this game there is no hard bound on the disturbance $w$.

It turns out that for a large class of linear-quadratic systems, the zero-sum differential game has equal upper and lower values (whenever they are both finite), which makes the theory on saddle-point solutions of linear-quadratic differential games directly applicable to such classes of worst-case design problems.
1.4. Hamilton-Jacobi-Isaacs Equations

A large of class of optimal control problems are approached by the method of dynamic programming. The fundamental equation of dynamic programming is a nonlinear evolution equation for the value function, which is called the Hamilton-Jacobi-Bellman (HJB) partial differential equation. It has its roots in the classical works on calculus of variations by Hamilton, Jacobi and Carathéodory, and was applied extensively to optimal control by Bellman.

The connection between dynamic programming and two-person, zero-sum differential games was observed by Isaacs in the early 1950's (see [I]). In his pioneering book on two-persons zero-sum differential games, Isaacs used extensively the dynamic programming principle, which he called "tenet of transition", and the associated first order PDE which now carries his name. He showed that if the values of various differential games are regular enough, then they solve certain first order PDEs with "max-min" or "min-max" type nonlinearity (the Isaacs equations). His ingenuous ideas and their developments are still very useful, but they seem hard to use for systems with high state-space dimension, and do not provide a general theory for the solution of the Isaacs' equations.

If the system is deterministic, HJI equation is of first order nonlinear partial differential equation, and is well known that such equations do not have in general global classical solutions. At the same time the value functions of optimal control problems and games are not differentiable in general, and they are not even continuous for some problems (see [Sor], [XB4], [FQ]). Only in some special cases value functions are smooth and dynamic programming works, typically for linear systems and quadratic costs. Many later papers on the subject have worked around this difficulty: see especially Fleming [Fle 1-2], Friedman
[F 3-4], Elliott-Kalton [EK 1-3], Krassovski-Subbotin [KrSu], Subbotin [Su], etc., and the references therein.

1.5. Viscosity Solution

The theory of viscosity solutions, first introduced by M.G. Crandall and P.L. Lions [CrLi2], provides a convenient framework in which to study HJB equations. The theory was reformulated and streamlined by Crandall, Evans and Lions [CEL]. Lions proved that the value functions of optimal control problems are viscosity solutions of the corresponding HJB equations whenever they are merely continuous [L], and the same was done for some zero-sum differential games by Barron, Evans, Jensen, and Souganidis under various definitions of upper and lower value ([BEJ], [ES], [Sou]). In addition, the viscosity solution framework is well suited to proving continuous dependence of solutions on problem data. The theory has grown enormously in the last ten years and many different and interesting applications have been found.

$H^\infty$-optimal control theory provides a worst-case design methodology to tackle the problem in the presence of unknown disturbances. The theory of differential games and the study of the associated HJI equations appear to be basic tools of the theory. In order to construct robust optimal controllers, the solvability of HJI equations plays a key role. In the case of nonlinear systems, even in a special form as nonlinear in the state and affine in the controls (see [BB], [Sor]), the HJI equations generally do not have classical solutions because the values of differential games are not smooth enough to satisfy the HJI equations in the classical sense.

That the values of differential games in certain classes are viscosity solutions of the
corresponding HJI equations have been proven by [ES] and [BS]. Although the differential game that arises in the $H^\infty$-problem is somewhat similar to that considered in [ES], there are, however, four significant differences in the consideration of $H^\infty$-control problem: (1) the control for the maximizing player must be allowed to be unbounded in order to correspond to ($L^2$) disturbances, (2) the dynamics are usually Lipschitz but not necessarily bounded, and cost functions may grow quadratically, (3) stability is a major issue in the $H^\infty$ problem setting, (4) the $H^\infty$-problem emphasizes measurement feedback. Such differences prevent the traditional game theory from direct application to $H^\infty$-problems, and necessitates a separate study of the differential game arising in $H^\infty$-problems.

1.6. Contribution and Organization of the Thesis

The main contribution of this thesis can be summarized as: (i) the development of the $H^\infty$-optimal control theory for infinite-dimensional systems (Chapter 2-Chapter 6); (ii) the development of the viscosity solution theory for some classes of nonlinear partial differential equations which are related to optimal control problems as well as $H^\infty$-optimal control problems (Chapter 7-Chapter 8). Although the contents in different chapters are closely related to each other, each chapter has been written to be self-contained so that to understand one chapter it is not necessary to go over the earlier ones. The connections between chapters will be emphasized in the introduction of each chapter. The following provides some detail on the contents and contributions of the thesis.

In the Chapter 2, we consider the control of a class of infinite-dimensional uncertain systems with nonlinear regulated outputs. The result obtained constitutes a nontrivial generalization of the standard one (Curtain 1993 [KPC], Barbu 1994 [Ba1]). The basic
approach is to incorporate the nonlinear output into an unconstrained optimization framework by choosing discontinuous weighting functions, and study a particular differential game associated with the disturbance attenuation problem. We develop a general theory for existence and characterization of disturbance attenuating time-varying state-feedback controllers for $H^\infty$ problems in infinite dimensions. Such a generalization has allowed us to accommodate in our analysis a large class of practical cases such as control with constraints; state with constraints; state with different weights in different regions; and so on.

The existence of finite-dimensional controllers for the $H^\infty$-optimal control of infinite-dimensional systems is completely solved in Chapter 3. This was an open problem posed by Curtain in 1990 [C1]. The approach adopted there uses Galerkin-type approximation, where there is no requirement for the system to have a complete set of eigenvectors. This constitutes the first work that addresses the derivation of finite dimensional compensators in infinite-dimensional $H^\infty$-optimal control. The case without disturbances is a special case under the problem setting in this chapter (Ito 1990, [It]). The results obtained can be applied to a large class of practical models such as the structural acoustic model, the delay system model, and the hereditary differential system model.

Chapter 4 studies two generalized Riccati evolution equations, which are related to the attenuation of $H^\infty$-optimal control problems of time-varying linear evolution systems in infinite-dimensional spaces. Two intrinsically defined sharp values of parameters (called critical values) are identified. We assert that when parameters are below these critical values, the corresponding generalized evolution Riccati equation does not admit a solution, while when parameters are larger than the critical values, their corresponding generalized
Riccati evolution equation admits a solution. Moreover two explicit closed-form solutions corresponding to these two Riccati equations in terms of known operators are obtained. Such a finding makes it possible not only to directly calculate the minimax disturbance attenuation bounds, but also to study systems with hybrid controls of $H^\infty$-optimal control problems for infinite-dimensional systems.

Based on the results obtained in Chapter 4, Chapter 5 presents the solution to the infinite-dimensional $H^\infty$-optimal control problem when only sampled values of states are available for linear time-varying systems in finite-horizon and linear time-invariant systems in infinite-horizon. It is shown that when sampling intervals are chosen appropriately, the designed sampled-data feedback controller achieves the desired attenuation level for which usually a regular continuous feedback controller is required. The upper bound of the sampling intervals is estimated to guarantee the performance. This is the only one of two works (another one is in the next chapter) so far which deals with sampled-data $H^\infty$ problem in infinite-dimensional spaces.

Chapter 6 extends the results in Chapter 5 further to the boundary control problem for general linear time-varying parabolic systems by using a blend of semigroup operator theory and differential game theory. To the best of our knowledge, this chapter is the first one to address and resolve the question of optimum (time-varying) intersample behavior in the context of $H^\infty$ optimization for boundary-value problems.

Chapter 7 considers two coupled nonlinear partial differential equations, which arise in the optimal control of nonlinear piecewise deterministic systems. The major challenge here is that these two coupled nonlinear partial differential equations are different from the standard one. We extend the standard concept for uncoupled viscosity solution to coupled
partial differential equations, obtain uniqueness by generalizing the standard comparison theorem as well as the existence by generalizing the standard dynamic programming argument.

Chapter 8 discusses the solvability of a class of Hamilton-Jacobi-Isaacs equations arising in nonlinear $H^\infty$ control problems. It has been shown that such HJI equations admit a super-solution, which is a storage function associated with a class of nonlinear systems and is only lower semicontinuous. Such a super-solution can be used to construct an $H^\infty$-optimal controller which, in the standard case, is required to be continuous.

Chapter 9 provides a summary of the thesis, and discusses some future challenges in $H^\infty$-optimal control problems for infinite-dimensional systems.
2. $H^\infty$-Optimal Control of a Class of Infinite-Dimensional Linear Systems with Nonlinear Outputs

2.1. Introduction

In a standard infinite-dimensional linear $H^\infty$-control problem, we have the input-output relationship

\begin{align*}
(2.1.1) \quad \dot{x}(t) &= Ax(t) + Bu(t) + Dw(t) \quad t \in (t_0, t_f] \\
         x(t_0) &= x_0 \\
(2.1.2) \quad z(t) &= C_1 x(t) + D_{12} u(t)
\end{align*}

where $A$ is the infinitesimal generator of a $C_0-$ semigroup $e^{At}$ on a real separable Hilbert space $X$ (state space), and $B \in \mathcal{L}(U, X)$, $D \in \mathcal{L}(W, X)$ are linear operators defined on real separable Hilbert spaces $U$ (control space) and $W$ (disturbance space) respectively. Let $Z$ be the controlled output space (another real Hilbert space) and $C_1 \in \mathcal{L}(X, Z)$, $D_{12} \in \mathcal{L}(U, Z)$. Let us use the notation $(\cdot, \cdot)_X$ to denote the inner product on $X$, and $\| \cdot \|_X$ to denote the norm of an element out of $X$, and adopt the natural counterparts of this convention for the other Hilbert spaces $U$, $W$, and $Z$. Under the standard hypothesis, $D^*_{12}[C_1, D_{12}] = [0, I]$, where $^*$ denotes the adjoint, we have

\begin{equation}
(2.1.3) \quad (z, z)_Z = (C_1 x, C_1 x)_Z + (u, u)_U
\end{equation}

Introduce the quadratic cost:

\begin{align*}
(2.1.4) \quad L_{x_0}(u_{[t_0, t_f]}, w_{[t_0, t_f]}) &= \int_{t_0}^{t_f} (z(\tau), z(\tau))_Z d\tau + (Q_f x(t_f), x(t_f))_X \\
&= \int_{t_0}^{t_f} ((C_1 x(\tau), C_1 x(\tau))_Z + (u(\tau), u(\tau))_U) d\tau + (Q_f x(t_f), x(t_f))_X
\end{align*}
where \( Q_f \in \mathcal{L}(X, X) \) with \( Q_f = Q_f^* \geq 0 \). For a fixed (admissible feedback) control \( u(t) = \mu(x(t)), L_{x_0} \) induces a natural mapping of \( L^2([t_0, t_f]; W) \) into nonnegative reals which we denote by \( \mathcal{F}_\mu \). The minimax \((H^\infty \text{ - optimal})\) controller design problem with state feedback is one of finding a state-feedback controller \( \hat{\mu} \) that minimizes the index

\[
(2.1.5) \quad \sup_{w \in L^2([t_0, t_f]; W)} \rho(\mu, w)
\]

where

\[
(2.1.6) \quad \rho(\mu, w) := \left\{ \frac{\mathcal{F}_\mu^{1/2}}{\|w\|_2} \right\}, \quad \text{and} \quad \|w\|_2 := \left\{ \int_{t_0}^{t_f} \|w(t)\|^2_W dt \right\}^{1/2}
\]

If this minimum value (whenever it exists) is denoted by \( \gamma^* \), then a related problem is the following: Given a \( \gamma > \gamma^* \), find a state-feedback controller, \( \hat{\mu}_\gamma \), such that

\[
(2.1.7) \quad \inf_{\mu \in \mathcal{M}_{CL}} \sup_w \rho(\mu, w) < \gamma
\]

Implicit in these definitions is the assumption that the initial state of the system is zero (i.e. \( x_0 = 0 \)), so that \( \mathcal{F}_\mu \) would be associated with a linear gain from \( w \) to \( z \), if \( \mu \) is chosen to be linear. Another point to mention is that, as shown in [BB]\(^2\) the study of such \( H^\infty \)-optimal control problems is equivalent to the study of a differential game with dynamics (2.1.1) and kernel function

\[
(2.1.8) \quad J_\gamma(t_0, x_0; u, w) = \int_{t_0}^{t_f} \left( \|C_1 x(t)\|^2_Z + \|u(t)\|^2_U - \gamma^2 \|w(t)\|^2_W \right) dt + (Q_f x(t_f), x(t_f))_X
\]

which is to be minimized with respect to the controller \( u(\cdot) = \mu(\cdot, x(\cdot)) \), and maximized with respect to \( w \). This equivalence is as far as the upper value of this differential game

\^2\)Even though this reference covers only the finite-dimensional case, the same arguments used there apply to the infinite-dimensional case as well, to prove this equivalence under some appropriate assumptions.
goes, though one can show that under some appropriate conditions, for all \( \gamma > \gamma^* \) this differential game has a saddle point, while for \( \gamma < \gamma^* \), its upper value is infinite. Such a result is true regardless of whether \( x_0 = 0 \) or not; see Chapter 4 of this thesis and [MT1] for the infinite-dimensional case, and [BB] for the finite-dimensional case.

The fundamental \( H^\infty \)-control problem for finite-dimensional linear systems was originally formulated by Zames [Zam], and first studied in the frequency domain [Fr]. Later, it was discovered that time-domain methods based on state-space representations lead to simpler and more complete derivations for a large class of systems ([BB], [DGKF], [KPR], [Pe], [Sc], [St], [Tad]).

The fact that the \( H^\infty \)-optimal control problem is also a differential game was expounded by Başar and Bernhard in [BB], the techniques of which are now widely used in dealing with \( H^\infty \)-optimal control problems in finite or infinite dimensional spaces. The differential game whose performance index is given by (2.1.8) (or more generally by \( J_\gamma(t_0,0;\mu,w) = J_\mu - \gamma^2\|w\|^2 \)) is called the soft-constrained game associated with the disturbance attenuation problem, because in this game there is no hard bound on the disturbance \( w \). It turns out that for a large class of systems, the zero-sum dynamic game has equal upper and lower values (whenever they are both finite), which makes the theory on saddle-point solutions of linear-quadratic dynamic games directly applicable to such classes of worst-case design problems.

In the infinite-dimensional case, the state-space approach was first used to treat the singular \( H^\infty \)-control problem for the Pritchard-Salamon class, in connection with stability radius optimization([PT']). The results obtained in this framework were then generalized to a larger class of linear systems ([Ic], [KPC]). For the boundary control problems we refer
the reader to ([B2], [MT1], [MT2], [MT3]). As far as we know, in the infinite-dimensional case, almost all systems considered heretofore were linear and the cost functions (or kernel functions in terms of the terminology of game theory) were quadratic. The main advantage of dealing with quadratic cost functions is that they lead to linear optimal controllers: 
\[ \mu(x) = Fx, \]
under which it is relatively easy to prove the stability of the feedback system. In spite of such nice features, a quadratic cost function also has its limitations. It cannot handle, for example, constraints on control, or the state, which can, however, be incorporated into an unconstrained optimization framework by choosing discontinuous weighting functions.

Motivated by these considerations, we consider in this chapter, a class of (infinite-dimensional) systems with nonlinear regulated outputs, defined as follows:

\[ z(t) = f(x(t), u(t)) \quad (x(\cdot), u(\cdot)) \in L^2([t_0, t_f]; X) \times L^2([t_0, t_f]; U) \]

where \( f : X \times U \to Z \) is nonlinear with respect to \( x \) or \( u \). This nonlinearity in the output translates into a cost function of the form:

\[ L_{x_0}(u, w) = \int_{t_0}^{t_f} \{ g(x(t)) + h(u(t))\} dt + \phi_0(x(t_f)) \]

where \( g, h \) and \( \phi_0 \) satisfy certain regularity conditions listed in the next section. The cost function (2.1.10) covers a large class of practical problems, such as problems with control constraints, state constraints, as well as problems where the state is weighted differently in different regions. Of course, the quadratic cost function (2.1.4) is a special case of (2.1.10). The basic approach to the problem here is game-theoretic, which associates a particular differential game with the disturbance attenuation problem. Here we only consider the finite horizon case, with the extension to the infinite-horizon problem to
be dealt with in another forthcoming paper. Under a rather weak assumption on the
infinitesimal generator $A$ (which is that $A$ generates a $C_0$ semigroup on $X$), we develop a
general theory for existence and characterization of disturbance attenuating time-varying
state-feedback controllers for $H^\infty$ problems in infinite dimensions. The study leads to
original results not only in the infinite-dimensional case, but also for finite-dimensional
systems with nonlinear outputs. Three examples included in the chapter serve to illustrate
the theory. The chapter ends with two Appendices, containing some auxiliary results used
in the main body of this chapter.

2.2. General Formulation of the Problem and the Main Result

2.2.1. Problem Formulation

Consider a class of uncertain dynamical systems governed by the set of evolution equa-
tions with nonlinear output:

\begin{equation}
\dot{x}(t) = Ax(t) + Bu(t) + Dw(t), \quad t_0 < t \leq t_f
\end{equation}

\begin{equation}
x(t_0) = x_0
\end{equation}

\begin{equation}
z(t) = f(x(t), u(t))
\end{equation}

where $x(t) \in X$ is the state, $u(t) \in U$ the control, $w(t) \in W$ the disturbance, and $z(t) \in Z$
the controlled output, with $X, U, W, Z$ being real separable Hilbert spaces. $A$ is the
infinitesimal generator of a $C_0$-semigroup $e^{At}$ on $X$; $B \in \mathcal{L}(U, X)$, $D \in \mathcal{L}(W, X)$ are linear
operators defined on $U$ and $W$, respectively; and $f : X \times U \to Z$ is such that

\begin{equation}
\|f(x, u)\|_Z^2 = g(x) + h(u) \quad (x, u) \in X \times U
\end{equation}
where $g$ and $h$ satisfy certain regularity conditions, which will be made precise shortly. Here and throughout in the sequel we shall use the asterisk symbol to denote dual operators and $(\cdot, \cdot)$ to denote inner products on appropriate Hilbert spaces. Denote the cost function as

\[(2.2.1.3) \quad L_{x_0}(u, w) = \int_{t_0}^{t_f} \{g(x(t)) + h(u(t))\} dt + \phi_0(x(t_f))\]

which we wish to minimize under worst choices for the unknown disturbance. For every $\gamma \in \mathbb{R}^+$, let $J_\gamma : [t_0, t_f] \times X \times L^2([t_0, t_f]; U) \times L^2([t_0, t_f], W) \to \mathbb{R}$ be given by

\[(2.2.1.4) \quad J_\gamma(t, x, u, w) = \int_t^{t_f} \{g(x(s)) + h(u(s)) - \gamma^2\|w(s)\|^2\} ds + \phi_0(x(t_f))\]

where $t \to x(t)$ is the solution of (2.2.1.1) (in "mild" sense).

We now make the notion of an admissible control policy (or law) precise.

**Definition 2.2.1.1.** An admissible feedback control policy for system (2.2.1.1) is a (nonlinear, multivalued) mapping $F : [t_0, t_f] \times X \to U$ such that for every $x_0 \in X$ and $w \in L^2([t_0, t_f]; W)$ the Cauchy problem

$$\dot{x}(t) \in Ax + BFx + Dw \quad t \in (t_0, t_f], \quad x(t_0) = x_0$$

has at least one mild solution.

**Remark 2.2.1.2.** Note that in accordance with the definition above, an admissible controller has access to instantaneous perfect measurements of the state of the system (2.2.1.1).

**Remark 2.2.1.3.** The reason for allowing $F$ to be multivalued is because various terms in the cost function $L$, such as $g$, $h$, $\phi_0$, are not necessarily differentiable, thus making
the value function also nondifferentiable. Because of this, the optimal control is generally expressed in terms of subdifferentials, which generally contain more than one element.

We shall denote by $\mathcal{M}_{CL}$ the class of admissible feedback control policies for system (2.2.1.1), as per Definition 2.2.1.1. Also, by a possible abuse of notation, we will write $L_{x_0}(\mu, w)$ for (2.2.1.3) when $u$ is generated by the control policy $\mu$.

**Definition 2.2.1.4.** System (2.2.1.1) is $\gamma$-robust if there exists a $\mu \in \mathcal{M}_{CL}$ and a nonnegative function $U : X \to \mathbb{R}$ such that for some $\epsilon \in (0, \gamma)$

\begin{equation}
L_{x_0}(\mu, w) \leq \epsilon^2 \int_{t_0}^{t'} \|w(t)\|_{W^2}^2 dt + U(x_0), \quad \forall (x_0, w) \in X \times L^2([t_0, t_f]; W)
\end{equation}

**Remark 2.2.1.5.** In the above definition, the initial condition $x_0$ plays a role since the system (2.1.1) becomes nonlinear when a nonlinear feedback control is applied. In the quadratic cost function case, a possible choice for $U$ is $U(x_0) = \alpha \|x_0\|^2$, where $\alpha > 0$ is a constant positive parameter. Note that in this case $U(0) = 0$, and hence the above definition subsumes the case of standard linear $H^\infty$-optimal control problem (see e.g. [BB], [BeBe], [Ic], [KPC]).

The differential game associated with this problem is defined by kernel (2.2.1.4), whose upper value is the quantity of interest:

\begin{equation}
\inf_{\mu \in \mathcal{M}_{CL}} \sup_{w \in L^2([t_0, t_f]; W)} J_\gamma(t_0, x_0; \mu, w)
\end{equation}

It will in fact be shown that if this upper value is finite for some $\gamma = \gamma^0$, then the game has a value for all $\gamma > \gamma^0$, which is attained by some state-feedback (saddle-point) policies $(u(t) = \hat{\mu}(t, x(t)), w(t) = \hat{\nu}(t, x(t)))$, where the feedback policy $\nu(\cdot, \cdot)$ for the disturbance is defined in a way analogous to that of $\mu$. This result will be established under the following
assumptions on \(g, h\) and \(\phi_0\).

**Assumptions**

(A1) \(g\): \(X \rightarrow \mathbb{R}\) and \(\phi_0\): \(X \rightarrow \mathbb{R}\) are proper nonnegative lower semicontinuous convex functions.

(A2) \(h\): \(U \rightarrow \mathbb{R}\) is a lower semicontinuous nonnegative convex function, non-identically \(+\infty\).

(A3) \[
\int_{t_0}^{t_f} h(u_1(t))dt = \int_{t_0}^{t_f} h(u_2(t))dt \implies \int_{t_0}^{t_f} \|u_1(t)\|_{I_U}^2 dt = \int_{t_0}^{t_f} \|u_2(t)\|_{I_U}^2 dt
\]

and there exits a proper convex, nondecreasing function \(\omega: \mathbb{R}^+ \rightarrow \mathbb{R}^+\) with the properties \(\omega(0) = 0, \lim_{r \rightarrow +\infty} \omega(r^2)/r = +\infty\) such that \(h(u) \geq \omega(\|u\|_{I_U}^2)\).

(A4) \(g\) is locally Lipschitz, and there exist constants \(C_1\) and \(C_2\) such that

\[
\sup\{\|\xi\| : \xi \in \partial g(x)\} \leq C_1 \|x\| + C_2
\]

(A5) The function \(u \rightarrow h(u)\) is continuous and \(u \rightarrow \partial h(u)\) is continuous and bounded on every bounded subset of \(U\) and \(u \rightarrow h^*(u)\) is Fréchet differentiable, where \(h^*\) is the conjugate function of \(h\), i.e. \(h^*(u) = \sup_{v \in U} \{(u, v) - h(v)\}\).

(A6) \(\phi_0: X \rightarrow \mathbb{R}\) is convex, locally Lipschitz.

**Remark 2.2.1.6.** Assumptions (A1)-(A6) are automatically satisfied when the output is the standard output defined in (2.1.1), (2.1.2).

Let us denote by \(\Phi\) a class of functions \(\{\varphi\}\) satisfying the following properties:

1. \(\varphi: [t_0, t_f] \times X \rightarrow \mathbb{R}\);

2. For every \(s \in [t_0, t_f]\), \(\varphi(s, \cdot)\) is convex and continuous on \(X\);

3. For every \(x \in W^{1,2}([t_0, t_f]; X)\), \(t \rightarrow \varphi(t, x(t))\) is absolutely continuous and the following
rule of differentiation holds
\[
\frac{d}{dt} \varphi(t,x(t)) = \varphi_t(t,x(t)) + (\eta(t,x(t)), \dot{x}(t)) \quad \text{a.e.} \quad t \in [t_0, t_f]
\]
where \( \eta \in \partial \varphi \), where \( \partial \varphi \) denotes the subdifferential of \( \varphi \); i.e.
\[
\partial \varphi(x_0) = \{ y \in X : \varphi(x_0) \leq \varphi(h) + (y, x_0 - h) \} \quad \forall h \in X
\]
and \( W^{1,2}([t_0, t_f]; X) \) is the typical Sobolev space (see e.g. [Ad]).

2.2.2. Statement of the Main Result

Under conditions (A1)-(A3) and (A5), let the following Hamilton-Jacobi-Isaacs equation admit a solution \( \varphi_\gamma \in \Phi \) for \( \eta \in \partial \varphi_\gamma \) for a given \( \gamma > 0 \):

(2.2.2.1)
\[
\frac{\partial \varphi_\gamma(s, x)}{\partial s}(s, x) + (Ax, \eta(x)) - h^*(-B^*\eta(x)) + g(x) + \frac{\gamma^{-2}}{4} \|D^*\eta(x)\|_W^2 = 0
\]

\( \text{a.e. } s \in (t_0, t_f) \)

\( \varphi_\gamma(t_f, h) = \phi_0(h) \quad \forall h \in X \)

Furthermore, let the following properties hold:

(i) \( x \to \eta(x) \) is continuous;

(ii) \( D(\eta) \supseteq D(A) \) and \( \|\eta(x)\|_X \leq c_1\|x\|_X + c_2 \) for all \( x \in D(A) \), for some constants \( c_1 > 0, c_2 > 0 \).

Then, the zero-sum differential game (2.2.1.4) admits a feedback saddle-point solution, which is given by

(2.2.2.2)
\[
\hat{\mu}(t; x(t)) = \partial h^*(-B^*\eta(x(t))) \quad \hat{\mu}(\cdot; x(\cdot)) \in C([t_0, t_f]; U)
\]

(2.2.2.3)
\[
\hat{\nu}(t; x(t)) = \frac{\gamma^{-2}}{2} D^*\eta(x(t)) \quad \hat{\nu}(\cdot; x(\cdot)) \in C([t_0, t_f]; W)
\]
and \( \varphi_\gamma(t, x) \) is the value of the game. Moreover (2.2.1.5) holds with \( \epsilon = \gamma \).

Conversely, under conditions (A1)-(A6), let the \( \gamma \)-robustness property hold and let

\[
\varphi_\gamma(t, x) := \inf_{\mu \in \mathcal{M}_{CL}} \sup_{w \in L^2([t_0, t_f]; W)} J_\gamma(t, x, \mu, w)
\]

Then, \( \varphi_\gamma(t, x) \) is finite for any \( x_0 \in X \), and \( \varphi_\gamma \in \Phi \) and satisfies equation (2.2.2.1) almost everywhere with respect to \( s \in [t_0, t_f] \). Moreover, if (i) and (ii) above are satisfied, then (2.2.2.2) is an admissible optimal feedback control policy.

In the next two sections, we will present several theorems, which will capture separately the necessity and sufficiency parts of the main result above.

2.3. Proof of the Main Result: Necessity

We first consider the following sup inf problem, which constitutes the lower value of the game:

\[
\varphi_\gamma(t_0, x_0) := \sup_{w \in L^2([t_0, t_f]; W)} \inf_{u \in L^2([t_0, t_f]; U)} J_\gamma(t_0, x_0; u, w)
\]  

(2.3.1)

Theorem 2.3.1. Let assumptions A1-A3 hold. Suppose that for a given \( \gamma > 0 \), the \( \gamma \)-robustness property holds for system (2.2.1.1). Then:

(i) (2.3.1) has at least one solution \( (\hat{u}, \hat{w}) \);

(ii) There exists a function \( p \in C([t_0, t_f]; X) \) such that

\[
\dot{\hat{x}}(t) = A\hat{x}(t) + B\hat{u}(t) + D\hat{w}(t)
\]  

(2.3.2)

\[
\dot{\hat{p}}(t) \in -A^*p(t) + \partial g(\hat{x}(t))
\]  

(2.3.3)

\[
B^*p(t) \in \partial h(\hat{u}(t))
\]  

(2.3.4)
(2.3.5) \[ p(t_f) + \partial \phi_0(\hat{x}(t_f)) \geq 0 \]

(2.3.6) \[ \hat{x}(t_0) = x_0 \]

(2.3.7) \[ D^* p(t) = -2\gamma^2 \hat{w}(t) \]

where solutions in (2.3.2), (2.3.3) are both defined in the “mild” sense.

(iii) \((\hat{u}, \hat{w})\) is an open-loop saddle-point solution of the differential game \(J_\gamma(t_0, x_0; u, w)\) on the space \(L^2([t_0, t_f]; U) \times L^2([t_0, t_f]; W)\).

Proof. (i) Under the hypotheses of the theorem, there exist a \(\mu \in \mathcal{M}_{CL}\) and \(\varepsilon \in (0, \gamma)\) such that

\[ \int_{t_0}^{t_f} \left( g(x(t)) + h(\mu(t, x(t))) \right) dt + \phi_0(x(t_f)) \leq \varepsilon^2 \int_{t_0}^{t_f} \|w(t)\|_W^2 dt + U(x_0) \]

for all \((x_0, w) \in X \times L^2([t_0, t_f]; W)\) and for some nonnegative function \(U : X \to \mathbb{R}\). This yields that \(\sup_w J_\gamma(t_0, x_0; \mu, w)\) is finite for a given \(x_0 \in X\). Since \(J_\gamma(t_0, x_0; \cdot, w)\) is convex, lower semicontinuous, coercive and \(\not\equiv \infty\), it has a unique minimum, say \(u^w(t)\). Let \(\Psi(w) = -\inf_{L^2([t_0, t_f]; U)} J_\gamma(t_0, x_0; u, w)\). Then there exists \(\Gamma : L^2([t_0, t_f]; W) \to L^2([t_0, t_f]; U)\) with \(\Gamma w := u^w\) such that

\[ \Psi(w) = -J_\gamma(t_0, x_0; u^w, w) = \gamma^2 \|w\|_2^2 - L_{x_0}(\Gamma w, w) \geq (\gamma^2 - \varepsilon^2)\|w\|_2^2 - U(x_0) \]

Let \(\{w_k\}_{k=1}^\infty \subset L^2([t_0, t_f]; W)\) be a minimizing sequence,

\[ \Psi(w_k) \to \inf_{w \in L^2([t_0, t_f]; W)} \Psi(w) \]

Note that \(\{w_k\}_{k=1}^\infty\) is bounded in \(L^2([t_0, t_f]; W)\). Hence \(\exists \hat{w} \in L^2([t_0, t_f]; W)\) such that

\[ w_k \to \hat{w} \text{ weakly in } L^2([t_0, t_f]; W) \]

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Clearly we have

\[ (2.3.8) \quad L_{x_0}(\Gamma w_k, w_k) \leq \int_{t_0}^{t_f} \left( g(x_k(t)) + h(\Gamma \hat{w}(t)) \right) dt + \phi_0(x_k(t_f)) \]

where \( x_k(t) \) satisfies

\[ \dot{x}_k(t) = Ax_k(t) + B\Gamma \hat{w}(t) + Dw_k(t) \]
\[ x_k(t_0) = x_0 \]

It is not difficult to see that

\[ x_k \to \hat{x} \text{ in } C([t_0, t_f]; X) \]

by the Arzela-Ascoli Theorem (see Appendix II), and that \((\hat{x}, \Gamma \hat{w}, \hat{w})\) satisfies \((2.2.1.1)\).

Assumption \((A3)\) and \((2.3.8)\) further imply that \(\Gamma w_k\) is bounded, and thus there exists a \(\bar{u} \in L^2([t_0, t_f]; U)\) such that

\[ \Gamma w_k \to \bar{u} \text{ weakly on } L^2([t_0, t_f]; U) \]

Let us introduce:

\[ \dot{x}_k(t) = Ax_k(t) + B\Gamma w_k(t) + Dw_k(t) \]
\[ x_k(t_0) = x_0 \]

Again by the Arzela-Ascoli Theorem, there exists a subsequence of \( x_{k} \) (to be denoted by \( x_{k} \) again) that converges to \( \hat{x} \in C([t_0, t_f]; X) \). Now making use of \((2.3.8)\), we have

\[ L_{x_0}(\bar{u}, \hat{w}) \leq \int_{t_0}^{t_f} \left( g(x(t)) + h(\Gamma \hat{w}(t)) \right) dt + \phi_0(x(t)) \]

Therefore

\[ \Gamma \hat{w} = \bar{u}, \quad x = \hat{x} \]
By assumption (A3), we have

$$\int_{t_0}^{t_f} \| \Gamma \hat{w} \|^2_Y dt = \int_{t_0}^{t_f} \| u \|^2_Y dt$$

and hence

$$\Gamma w_k \rightarrow \Gamma w \text{ strongly in } L^2([t_0, t_f]; U)$$

Thus $w \rightarrow -\Psi(w)$ is weakly lower semicontinuous; therefore,

$$\sup \{-\Psi(w), w \in L^2([t_0, t_f]; W)\}$$

has at least one solution. Clearly $(\Gamma \hat{w}, \hat{w})$ is a solution of (2.3.1).

(ii) Let $(\hat{u}, \hat{w}) = (\Gamma \hat{w}, \hat{w})$ be a solution of (2.3.1). Note that

$$\inf_u J_\gamma(t_0, x_0; u, \hat{w}) = J_\gamma(t_0, x_0; \hat{u}, \hat{w})$$

By Theorem 1.1 of chapter 4 of [BP1], there exists a function $p \in C([t_0, t_f]; X)$ such that (2.3.2)–(2.3.6) are true. Now we claim (2.3.7). Let $w \in L^2([t_0, t_f], W)$ be arbitrary but fixed, and $x \in C([t_0, t_f]; X)$ be such that

(2.3.9) \[ \hat{x}(t) = A x(t) + B \hat{u}(t) + D w(t), \quad x(t_0) = x_0 \]

Let $q(t) \in \partial g(\hat{x}(t))$. From the definitions of $\partial g, \partial h$ and $\partial \phi_0$, and (2.3.5), we have

(2.3.10) \[ g(\hat{x}(t)) \leq g(x(t)) + (\hat{x}(t) - x(t), q(t)) \]

(2.3.11) \[ \phi_f(\hat{x}(t_f)) \leq \phi_f(x(t_f)) + (\hat{x}(t_f) - x(t_f), -p(t_f)) \]

Hence

$$J_\gamma(t_0, x_0; \hat{u}, \hat{w}) \leq J_\gamma(t_0, x_0; \hat{u}, w) + \int_{t_0}^{t_f} (\hat{x}(t) - x(t), q(t)) dt$$

$$+ (\hat{x}(t_f) - x(t_f), -p(t_f)) + \gamma^2 (\| w \|^2_{L^2([t_0, t_f], W)} - \| \hat{w} \|^2_{L^2([t_0, t_f], W)})$$

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By using (2.2.1.1) and (2.3.9), we have

\[ \int_{t_0}^{t_f} (\hat{x}(t) - x(t), q(t))dt = \int_{t_0}^{t_f} \left( \int_{t_0}^{t} e^{A(t-s)}D(\hat{w}(s) - w(s))ds, q(t) \right)dt \]

Applying Fubini’s theorem to interchange the order of integration and using (2.3.3) yields:

\[
\int_{t_0}^{t_f} (\hat{x}(t) - x(t), q(t))dt = \int_{t_0}^{t_f} (D(\hat{w}(s) - w(s)), \int_{s}^{t} e^{A^*(t-s)}q(t)dt)ds \\
= \int_{t_0}^{t_f} (D(\hat{w}(s) - w(s)), e^{A^*(t_f-s)}p(t_f) - p(s))ds \\
= \int_{t_0}^{t_f} (\hat{w}(s) - w(s), D^*e^{A^*(t_f-s)}p(t_f) - D^*p(s))w ds
\]

Using (2.2.1.1) and (2.3.9) again, we arrive at

\[(\hat{x}(t_f) - x(t_f), -p(t_f)) = \int_{t_0}^{t_f} (e^{A(t_f-s)}D(\hat{w}(s) - w(s)), -p(t_f))ds,\]

and hence

\[
\int_{t_0}^{t_f} (\hat{x}(t) - x(t), q(t))dt + (\hat{x}(t_f) - x(t_f), -p(t_f)) \\
+ \gamma^2(||w||^2_{L^2([t_0,t_f], W)} - ||\hat{w}||^2_{L^2([t_0,t_f], W)}) \\
= \int_{t_0}^{t_f} (\hat{w}(s) - w(s), -D^*p(s))w + \gamma^2(||w||^2_{L^2([t_0,t_f], W)} - ||\hat{w}||^2_{L^2([t_0,t_f], W)}) \\
= \gamma^2 \{ \int_{t_0}^{t_f} ||w(s) + \frac{\gamma^2}{2}D^*p(s)||_W^2 ds - \int_{t_0}^{t_f} ||\hat{w}(s) + \frac{\gamma^2}{2}D^*p(s)||_W^2 ds \}
\]

Therefore

\[ J_\gamma(t_0, x_0; \hat{u}, \hat{w}) \leq J_\gamma(t_0, x_0; \hat{u}, w) \\
+ \gamma^2 \{ \int_{t_0}^{t_f} ||w(s) + \frac{\gamma^2}{2}D^*p(s)||_W^2 ds - \int_{t_0}^{t_f} ||\hat{w}(s) + \frac{\gamma^2}{2}D^*p(s)||_W^2 ds \} \]
Suppose $\hat{w}(s) \neq -\frac{1}{2}D^*p(s)$, and choose $w(s) = -\frac{1}{2}D^*p(s)$. Then we have

$$J_\gamma(t_0, x_0; \hat{u}, \hat{w}) < J_\gamma(t_0, x_0; \hat{u}, w)$$

which contradicts our initial hypothesis that $(\hat{u}, \hat{w})$ is a solution of (2.3.1).

(iii) To prove that $(\hat{u}, \hat{w})$ is also a saddle point of $J_\gamma(t_0, x_0; u, w)$, we need to show that

$$J_\gamma(t_0, x_0; u, w) \leq J_\gamma(t_0, x_0; \hat{u}, \hat{w}) \leq J_\gamma(t_0, x_0; u, \hat{w})$$

In (i) we have already proven that the right-hand side inequality is true. Now we show that the above left-hand side inequality also holds. Fix $(y, v) \in C([t_0, t_f]; X) \times L^2([t_0, t_f], U)$ such that

$$\dot{y}(t) = Ay(t) + Bv(t) + Dw(t), \quad y(t_0) = \bar{x}$$

where $\bar{x} \in X$. Let $q(t) \in \partial g(\hat{x}(t))$; then (2.3.3) and (2.3.4) yield

(2.3.12)

$$\int_{t_0}^{t_f} (h(\hat{u}(t)) + g(\hat{x}(t))) dt$$

$$\leq \int_{t_0}^{t_f} (\hat{u}(t) - u(t), B^*p(t))_U dt + \int_{t_0}^{t_f} (\hat{x}(t) - y(t), q(t))_X dt$$

We also note:

$$\int_{t_0}^{t_f} (\hat{x}(t) - y(t), q(t))_X dt = \int_{t_0}^{t_f} (\hat{w}(t) - w(t), D^*e^{A^*(t_f-t)}p(t_f) - D^*p(t))_w dt$$

$$+ \int_{t_0}^{t_f} (\hat{u}(t) - u(t), B^*e^{A^*(t_f-t)}p(t_f) - B^*p(t))_U ds + \int_{t_0}^{t_f} (e^{A(t-t_0)}(x_0 - \bar{x}), q(t))_X dt$$

Since

$$\int_{t_0}^{t_f} (e^{A(t-t_0)}(x_0 - \bar{x}), q(t))_X dt = (x_0 - \bar{x}, \int_{t_0}^{t_f} p(t_f) - p(t_0))_X,$$
and (2.3.5) provides

\[(\ddot{x}(t_f) - y(t_f), -p(t_f)) = (x_0 - \ddot{x}, e^{A(t_f - t_0)}p(t_f)) + \int_{t_0}^{t_f} (e^{A(t_f - t)}D(\ddot{w}(t) - w(t), -p(t_f))) ds + \int_{t_0}^{t_f} (e^{A(t_f - t)}B(\ddot{u}(t) - v(t)), -p(t_f)) dt,\]

(2.3.12) becomes

\[(2.3.13) \quad J_\gamma(t_0, x_0, \ddot{u}, \ddot{w}) - J_\gamma(t_0, \ddot{x}, v, w) \leq (x_0 - \ddot{x}, -p(t_0)) + \gamma^2 \int_{t_0}^{t_f} \|w(t) + \frac{\gamma^{-2}}{2}D^*p(t)\|^2_W dt\]

Letting \(\ddot{x} = x_0, w = -\frac{\gamma^{-2}}{2}D^*p(t)\), the conclusion follows. \(\Box\)

**Corollary 2.3.2.** Under the hypotheses of Theorem 2.3.1, we have

\[\{-p(t_0)\} = \partial \varphi_\gamma(t_0, x_0)\]

**Proof.** From (2.3.13) of Theorem 2.3.1, we have

\[(2.3.14) \quad \varphi_\gamma(t_0, x_0) - \inf_v J_\gamma(t_0, \ddot{x}, v, w) \leq (x_0 - \ddot{x}, -p(t_0)) + \gamma^2 \int_{t_0}^{t_f} \|w(t) + \frac{\gamma^{-2}}{2}D^*p(t)\|^2_W dt\]

Let \(w(t) = \ddot{w}(t)\); then we get

\[\varphi_\gamma(t_0, x_0) - \inf_v J_\gamma(t_0, \ddot{x}, v, \ddot{w}) \leq (x_0 - \ddot{x}, -p(t_0))\]

Since

\[\inf_v J_\gamma(t_0, \ddot{x}, v, \ddot{w}) \leq \sup_w \inf_v J_\gamma(t_0, \ddot{x}, v, w) = \varphi_\gamma(t_0, \ddot{x}),\]

we have

\[(2.3.15) \quad \varphi_\gamma(t_0, x_0) - \varphi_\gamma(t_0, \ddot{x}) \leq (x_0 - \ddot{x}, -p(t_0))\]
i.e. $-\frac{1}{\lambda} p(t_0) \in \partial \varphi(t_0, x_0)$. To complete the proof, let us define $Px_0 = \{-p(t_0)\}$ and next show that $P$ is maximal monotone. Let $y_0 \in X$ be arbitrary but fixed. We claim that the following equation

$$\lambda x_1 + Px_1 = y_0$$

has at least one solution for some $\lambda$. Let us introduce the sup inf problem

$$\sup_{w \in L^2([t_0, t_f]; W)} \inf_{(u, x_0) \in L^2([t_0, t_f]; U) \times X} \left\{ \int_{t_0}^{t_f} \left( g(x(t)) + h(u(t)) - \gamma^2 \|w(t)\|_{L^2_W}^2 \right) dt \right.$$  

$$+ \phi_0(x(t_f)) - \frac{\lambda^2}{2} \|x_0 - \frac{1}{\lambda^2} y_0\|^2 \right\}.$$  

where $x(t), x_0$ satisfy (2.2.1.1) and $\lambda \in \mathbb{R}^+$. We claim that when $\lambda$ is sufficiently large, the above sup inf admits a finite solution.

By Theorem 1.1 of chapter 4 of [BP1], for every $w \in L^2([t_0, t_f]; W)$ a unique solution for the minimization problem is given by

\begin{align}
(2.3.16) & \quad B^* p^w(t) \in \partial h(u^w(t)) \\
(2.3.17) & \quad \dot{p}(t) \in -A^* p(t) + \partial g(x^w(t)) \\
(2.3.18) & \quad p^w(t_f) + \partial \phi_0(x^w(t_f)) \ni 0 \\
(2.3.19) & \quad p^w(t_0) = \lambda^2 x_0 - y_0
\end{align}

Because of the $\gamma$-robustness assumption, when $\lambda$ is sufficiently large, we have

$$\int_{t_0}^{t_f} \left( g(x^w(t)) + h(u^w(t)) - \gamma^2 \|w(t)\|_{L^2_W}^2 \right) dt + \phi_0(x^w(t_f)) - \frac{\lambda^2}{2} \|x_0 - \frac{1}{\lambda^2} y_0\|^2 \chi$$

$$\leq - (\gamma^2 - \epsilon^2) \int_{t_0}^{t_f} \|w(t)\|_{L^2_W}^2 dt - \frac{\lambda^2}{2} \|x_0 - \frac{1}{\lambda^2} y_0\|^2 \chi + U(x_0)$$

$$\leq - (\gamma^2 - \epsilon^2) \int_{t_0}^{t_f} \|w(t)\|_{L^2_W}^2 dt$$

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Thus, the above sup inf problem has a solution \((x_0, \bar{u}, \bar{w})\). For such \(x_0\), let \((\hat{u}, \hat{w})\) be a solution of the lower value game (2.3.1). By letting \(w = \hat{w}\) in (2.3.16)-(2.3.18), we have
\[
\hat{w}(t) = -\frac{x}{2} D^*p\hat{w}(t),
\]
and hence
\[
p\hat{w}(t_0) \in Px_0
\]
Note that (2.3.15) yields \(Px_0 = \lambda^2 x_0 - y_0\), and therefore completes the proof. \(\square\)

**Remark 2.3.3.** More generally, it is easy to see that \(-p(t) = \partial\varphi_\gamma(t, x(t))\) is true for \(t_0 \leq t \leq t_f\).

**Remark 2.3.4.** Since \(B^*p(t) \in \partial h(\hat{u}(t))\) implies \(\hat{u}(t) \in \partial h^*(B^*p(t))\), using previous remark, we have the following relationship between the output and the optimal controller:
\[
\hat{u}(t) \in \partial h^*(B^*(-\partial\varphi_\gamma(t, \hat{x}(t))))
\]
As mentioned before, the control we are mainly interested in is the state feedback control. Later we shall see that the \(H^\infty\)-optimal feedback control for system (2.2.1.1) with cost function (2.2.1.3) is in fact
\[
\hat{u}(t, x(t)) \in \partial h^*(B^*(-\partial\varphi_\gamma(t, x(t))))
\]

The following lemma establishes some important properties of the lower value function \(\varphi_\gamma(t, x)\), defined by (2.3.1) with \((t_0, x_0)\) replaced by an arbitrary \((t, x)\).

**Lemma 2.3.5.** Suppose that for a given \(\gamma > 0\), the \(\gamma\)-robustness property holds. Then:

(i) For every \(x \in D(A)\), \(s \to \varphi_\gamma(s, x)\) is Lipschitz on \([t_0, t_f]\);

(ii) For every \(s \in [t_0, t_f]\), \(x \to \varphi_\gamma(s, x)\) is Lipschitz on every bounded subset of \(X\);

(iii) \(t \to \varphi_\gamma(t, x(t))\) is absolutely continuous and for \(x \in W^{1,2}([t_0, t_f]; X)\) we have
\[
\frac{d}{dt} \varphi_\gamma(t, x(t)) = \varphi_\gamma(t, x(t)) + \left(\eta(t, x(t)), \dot{x}(t)\right) \quad \text{a.e. } t \in (t_0, t_f)
\]

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where \( \eta(t, x(t)) \in \partial \varphi_\gamma(t, x(t)) \).

**Proof.** (i) Recall that

\[
\varphi_\gamma(t_0, x_0) = \sup_w \inf_u J_\gamma(t_0, x_0; u, w),
\]

and note that the \( \gamma \)-robustness property guarantees \( \varphi_\gamma(t_0, x_0) \) to be finite for any \( x_0 \in X \).

As in the proof of Theorem 2.3.1, for each \( t \geq 0 \), and fixed \( x \in X \) we introduce the mapping

\[ \Gamma^t_x : W \to U \]

defined by

\[
\varphi_\gamma(t, x) = \sup_w J_\gamma(t, x; \Gamma_x^t w, w) = J_\gamma(t, x; \Gamma_x^t \hat{w}_{[t,t_f]}, \hat{w}_{[t,t_f]})
\]

assuming at the last step the existence of a maximizing \( w \), without any loss of generality.

Let \( u_{[t_1,t_f]} \in L^2([t_0,t_f]; U) \) be the open-loop control defined by

\[ u_{[t_1,t_f]} = \Gamma_{x_0}^{t_1} \hat{w}_{[t_1,t_f]} \]

Note that by our assumption on \( h \), \( \exists u_0 \in U \) such that \( h(u_0) < +\infty \).

Define

\[
\bar{u} = \begin{cases} 
  u_0, & t_0 \leq t < t_1 \\
  \Gamma_{x_0}^{t_1} \hat{w}_{[t_1,t_f]}, & t_1 \leq t \leq t_f
\end{cases}
\]

Then we have

\[
\varphi_\gamma(t_0, x_0) - \varphi_\gamma(t_1, x_0) \leq J_\gamma(t_0, x_0; \bar{u}, \hat{w}_{[t_0,t_f]}) - J_\gamma(t_1, x_0; \Gamma_{x_0}^{t_1} w_{[t_1,t_f]}, \hat{w}_{[t_0,t_f]})
\]

\[
= \int_{t_0}^{t_1} \left( g(x_{\bar{u}}(s)) + h(\bar{u}) - \gamma^2 \| \hat{w}_{[t_0,t_f]} \| (s) \right) ds + \int_{t_1}^{t_f} \left( g(x_{\Gamma_x^{t_1} w_{[t_1,t_f]}}(s)) - g(x_{\Gamma_{x_0}^{t_1} \hat{w}_{[t_1,t_f]}}(s)) \right) ds
\]

\[
+ \phi_0(x_{\bar{u}}(t_f)) - \phi_0(x_{\Gamma_{x_0}^{t_1} \hat{w}_{[t_1,t_f]}}(t_f))
\]

Note the following inequalities:

\[
\| x_{\bar{u}}(t) - x_{u_{[t_0,t_f]}}(t) \| \leq \| e^{A(t-t_0)} x_0 - e^{A(t-t_1)} x_1 \|
\]

\[
+ \int_{t_0}^{t_1} \| e^{A(t-s)} (Bu_0 + D\hat{w}_{[t_0,t_f]}) \| ds
\]

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and
\[ \|e^{A(t-t_0)}x_0 - e^{A(t-t_1)}x_0\| \leq C\|Ax_0\||t_1 - t_0| \]

Since \( g, \phi_0 \) are locally Lipschitz, the result follows.

(ii) Let \( x_0, x_1 \in X \). Without any loss of generality, assume that \( \varphi_\gamma(t_0, x_0) \geq \varphi_\gamma(t_0, x_1) \).

Introduce again the mapping \( \Gamma_{x_1}^{t_0} : W \rightarrow U \) such that
\[ \varphi_\gamma(t_0, x_1) = \sup_w J_\gamma(t_0, x_1; \Gamma_{x_1}^{t_0} w, w) = J_\gamma(t_0, x_1; \Gamma_{x_1}^{t_0} \hat{w}_0, \hat{w}_0) \]

Now note the inequality:
\[ \varphi_\gamma(t_0, x_0) - \varphi_\gamma(t_0, x_1) \leq J_\gamma(t_0, x_0; \Gamma_{x_1}^{t_0} \hat{w}_0, \hat{w}_0) - J_\gamma(t_0, x_1; \Gamma_{x_1}^{t_0} \hat{w}_0, \hat{w}_0) \]
\[ = \int_{t_0}^{t_1} (g(x^{x_0}(s)) - g(x^{x_1}(s)) ds + \phi_0(x^{x_1}(t_f)) - \phi_0(x^{x_0}(t_f)) \]

Clearly
\[ \|x^{x_1}(t) - x^{x_0}(t)\| \leq \|e^{A(t-t_0)}(x_0 - x_1)\| \leq C\|x_0 - x_1\| \]

Again since \( g, \phi_0 \) are locally Lipschitz, the desired result readily follows.

(iii) Let \( x_s \) be the mild solution of
\[ \dot{x}_s(t) = Ax_s(t) + Bu_s(t) + Dw_s(t) \quad s \leq t \leq t_f \]
\[ x_s(s) = x \]

where \((u_s, w_s) \in L^2(s, t_f; U) \times L^2(s, t_f; W)\) and \( x \in X \). Introduce
\[ \varphi_\gamma(s, x) := \sup_{w \in L^2([s, t_f]; W)} \inf_{u \in L^2([s, t_f]; U)} J_\gamma(s, x; u, w) \]

Let \((\hat{u}_s, \hat{w}_s)\) be a solution for \( \varphi_\gamma(s, x) \) such that
\[ \varphi_\gamma(s, x) = \int_s^{t_f} (g(\hat{x}_s) + h(\hat{u}_s) - \gamma^2 \|\hat{w}_s\|_W) ds + \phi_0(\hat{x}_s(t_f)) \]
Note that \( t \to g(\tilde{x}_s(t)) + h(\tilde{u}_s(t)) - \gamma^2 \| \tilde{w}_s(t) \| \) is continuous by assumption (A5). Let \( s \in [t_0, t_f] \) be such that the function \( t \to \varphi_\gamma(t, x_s(t)) \) is differentiable at \( t = s \). If \( x \in W^{1,2}([t_0, t_f]; X) \), then

\[
\frac{d}{dt} \varphi_\gamma(t, x_s(t))|_{t=s} = \lim_{t \to s} (\varphi_\gamma(t, x_s(t)) - \varphi_\gamma(s, x_s(s)))(t - s)^{-1} \\
+ \lim_{t \to s} (\varphi_\gamma(t, x_s(t)) - \varphi_\gamma(t, x_s(s)))(t - s)^{-1}
\]

and there exists \( \eta \in \varphi_\gamma(x) \) such that

\[
\lim_{t \to s} (\varphi_\gamma(t, x_s(t)) - \varphi_\gamma(t, x_s(s)))(t - s)^{-1} = \langle \eta, \dot{x}_s(t) \rangle
\]

by virtue of (ii) and the mean value property (c.f. [BP1] or [Au]). \( \square \)

**Remark 2.3.6.** The preceding lemma infers that system (2.2.1.1) with cost function defined in (2.2.1.3) being \( \gamma \)-robust implies \( \varphi_\gamma \in \Phi \) (see definition of \( \Phi \) in section 2).

**Remark 2.3.7.** Lemma 2.3.5 (i) implies that for \( x \in D(A) \), \( \varphi_\gamma(\cdot, x) \) is absolutely continuous on \([t_0, t_f]\), and thus is almost everywhere differentiable on \([t_0, t_f]\).

**Theorem 2.3.8.** Suppose that for a given \( \gamma > 0 \), the \( \gamma \)-robustness property holds for system (2.2.1.1) with cost (2.2.1.3). Then, there exists \( \eta \in \partial \varphi_\gamma \) such that \( \forall x \in D(A) \) the following equation holds

\[
\frac{\partial \varphi_\gamma}{\partial s}(s, x) + (Ax, \eta(s, x)) - h^*(-B^* \eta(s, x)) + g(x) + \frac{\gamma^{-2}}{4} \| D^* \eta(s, x) \|_W^2 = 0 \\
(2.3.20)
\]

a.e. \( s \in (t_0, t_f) \)

\[
\varphi_\gamma(t_f, h) = \phi_0(h) \quad \forall h \in X
\]
\textit{Proof.} For each fixed \( x \in D(A) \) we may assume the existence of a mapping \( \Gamma_x : W \to U \) such that

\[
\varphi_\gamma(s, x) = \sup_{L^2([s,t_f]; U)} \inf_{L^2([s,t_f]; U)} J_\gamma(s, x; u_s, w_s) = \sup_{L^2([s,t_f]; W)} J_\gamma(s, x; \Gamma_x w_s, w_s)
\]

\( = J_\gamma(s, x; \Gamma_x \hat{w}_s, \hat{w}_s) \)

Let us consider the following system, where we henceforth drop the subscript on \( \Gamma \):

\begin{equation}
(2.3.21) \quad \dot{x}_s(t) = Ax_s(t) + B \Gamma \hat{w}_s(t) + D \hat{w}_s(t) \quad s \leq t \leq t_f
\end{equation}

\( x_s(s) = x \)

Since the pair \((\Gamma \hat{w}_s(t), \hat{w}_s(t))\) is continuous on \([t_0, t_f]\) by Theorem 2.3.1, and since (2.3.21) implies

\[
x_s(t) = e^{A(t-s)}x + \int_s^t e^{A(t-\tau)}(B \Gamma \hat{w}_s(\tau) + D \hat{w}_s(\tau))d\tau,
\]

and \( x \in D(A) \), we have

\begin{equation}
(2.3.22) \quad \lim_{t \to s} \frac{x_s(t) - x}{t - s} = Ax_s(s) + B \Gamma \hat{w}(s) + D \hat{w}_s(s)
\end{equation}

Note that

\begin{equation}
(2.3.23) \quad \frac{d\varphi_\gamma}{dt}(t, x_s(t)) + g(x_s(t) + h(\Gamma \hat{w}_s(t)) - \gamma^2 \| \hat{w}_s(t) \|^2) = 0, \quad a.e. \ t \in [t_0, t_f]
\end{equation}

and

\begin{equation}
(2.3.24) \quad \frac{d\varphi_\gamma}{dt}|_{t=s} = \varphi_\gamma(s, x) + (\eta(s, x), \dot{x}(s)) \quad a.e. \ t \in [t_0, t_f]
\end{equation}

where \( \eta(s, x) \in \partial \varphi_\gamma(s, x) \). Combining (2.3.22), (2.3.23), and (2.3.24) yields

\begin{align}
\varphi_\gamma(t, x) + (\eta(s, x)A x + B \Gamma \hat{w}_s(s) + D \hat{w}_s(s)) \\
+ g(x) + h(\Gamma \hat{w}_s(s)) - \gamma^2 \| \hat{w}_s(s) \|^2_W = 0
\end{align}

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Since
\[ h(\Gamma \hat{\omega}_s) + h^*(-B^*\eta) = (-B\Gamma \hat{\omega}_s, \eta), \]
by virtue of Corollary 2.3.2 we may take \( p(s) = -\eta(s, x) \in \varphi_\gamma(s, x) \), and under this choice (2.3.25) becomes
\[ \varphi_\gamma(s, x) + (\eta(s, x), Ax) - h^*(-B^*\eta(s, x)) + \gamma^{-2} \| D^*\eta(s, x) \|^2 W = 0 \]
The proof of the theorem is thus complete. \( \square \)

2.4. Proof of the Main Result: Sufficiency

The following theorem provides a sufficient condition for existence of a solution of the system (2.2.1.1) when a nonlinear feedback control policy is used.

**Theorem 2.4.1.** Let \( \varphi \) be a proper lower semicontinuous convex function: \( X \rightarrow \mathbb{R} \), \( D(A) \cap D(\partial \varphi) \neq \emptyset \), and \( \partial \varphi : X \rightarrow X \) satisfy

\[ \sup\{\|\eta\| : \eta \in \partial \varphi(x)\} \leq C_1\|x\|_X + C_2 \]

where \( C_1, C_2 \) are constants. Then the Cauchy problem

\[ \dot{x}(t) \in Ax(t) + B\partial \varphi(x(t)) + Dw(t), \quad x(t_0) = x_0 \]

has at least one mild solution over \([t_0, t_f]\) for every \( w \in L^2([t_0, t_f]; W)\).

**Proof.** By the assumptions on \( \varphi, \partial \varphi \) is a maximal monotone operator. Define \( \varphi_\lambda : X \rightarrow \mathbb{R} \) as follows

\[ \varphi_\lambda(y) = \|y - J_\lambda y\|^2 / 2\lambda + \varphi(J_\lambda y) \quad \lambda > 0 \]
where \( J_\lambda = (I + \lambda \partial \varphi)^{-1} \) is a nonexpansive single-valued operator from \( X \) to \( D(\partial \varphi) \) where \( D(\partial \varphi) \) denotes the valid domain for \( \partial \varphi \). \( \varphi_\lambda \) is Fréchet differentiable and \( \nabla \varphi_\lambda = (\partial \varphi)_\lambda = \lambda^{-1}(I - J_\lambda) \) is Lipschitzian. Consider the approximation equation

\[
(2.4.4) \quad \dot{x}_\lambda(t) = Ax_\lambda(t) + B\nabla \varphi_\lambda(x_\lambda(t)) + Dw(t) \quad x_\lambda(t_0) = x_0
\]

which is equivalent to the following integral equation

\[
x_\lambda(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}(B\nabla \varphi_\lambda(x_\lambda(s)) + Dw(s))ds =: (Qx_\lambda)(t)
\]

Set \( X_1 = C([t_0, t_f]; X) \). It follows from the nonexpansive property of \( J_\lambda \) by induction that

\[
\|Q^n(x_\lambda^1) - Q^n(x_\lambda^2)\|_{X_1} \leq C\|B\|^n(t_f - t_0)^n \lambda^n n! \|x_\lambda^1 - x_\lambda^2\|_{X_1}, n = 1, 2, \ldots
\]

and thus \( Q \) has a unique fixed point \( x_\lambda \) in \( X_1 \), which is the mild solution of (2.4.4) for each \( \lambda > 0 \). By the assumption made on \( \partial \varphi \), we have

\[
\|\nabla \varphi_\lambda(x_\lambda)\| \leq C_3\|x_\lambda\| + C_4,
\]

and in view of Gronwall's lemma (see Appendix II) it is readily seen that \( \{x_\lambda\} \) is uniformly bounded on \([t_0, t_f]\). Arzela-Ascoli theorem infers that there is a subsequence \( \{x_{\lambda_n}\} \) of \( \{x_\lambda\} \) that is uniformly convergent to a continuous function \( x = x(t) \) on \([t_0, t_f]\) as \( \lambda_n \to 0 \). Since \( \nabla \varphi_\lambda(x_\lambda) \in \partial \varphi((I+\lambda \partial \varphi)^{-1}x_\lambda) \), \( \nabla \varphi_\lambda(x_\lambda) \) is also uniformly bounded, and furthermore there exists a subsequence \( \nabla \varphi_{\lambda_n}(x_{\lambda_n}) \) and \( y \in X \) such that \( \nabla \varphi_{\lambda_n}(x_{\lambda_n}) \to y \) weakly on \( X \) and \( y \in \partial \varphi(x) \). Note that \( x_\lambda \) is equicontinuous on \([t_0, t_f]\), which implies that there exists a subsequence of \( \nabla \varphi_{\lambda_n}(x_{\lambda_n}) \) which strongly converges to \( y \in \partial \varphi(x) \). Thus \( x = x(t) \) is a mild solution of (2.4.2), which completes the proof. \( \square \)
We now return to the cost function $J_\gamma$, and consider the following problem: Given a fixed but arbitrary $w \in L^2([t_0, t_f]; W)$, minimize

$$(2.4.5) \quad J_\gamma(t_0, x_0; u, w) = \int_{t_0}^{t_f} \{g(x) + h(u) - \gamma^2 \|w\|^2_{W} \} ds + \phi_0(x(t_f))$$

over all $u \in L^2([t_0, t_f]; U)$. Define

$$(2.4.6) \quad \psi^w(x_0) := \inf_u J_\gamma(t_0, x_0; u, w)$$

**Corollary 2.4.2.** Suppose that

(i) $\sup\{|\xi| : \xi \in \partial g(x)\} \leq C_1 \|x\| + C_2$;

(ii) $u \rightarrow h^*(u)$ is Fréchet differentiable on $U$.

Then there exists an optimal feedback control $\mu^w(t, x(t))$ for (2.4.6), given by

$$\mu^w(t, x(t)) \in \partial h^*(B^*(-\partial \psi^w(x(t))))$$

**Proof.** Since $J_\gamma(t_0, x_0; u, w)$ is convex, lower semicontinuous, coercive and $\neq \infty$, it has at least one minimum, say $\mu^w(t)$. In this case we know that there exists a function $p^w(t) \in C([t_0, t_f]; X)$ such that

$$(2.4.7) \quad \dot{p}^w(t) \in -A^*p^w(t) + \partial g(x(t))$$

$$(2.4.8) \quad Bp^w(t) \in \partial h(u^w(t))$$

$$(2.4.9) \quad p^w(t_f) + \phi_0(x(t_f)) \geq 0$$

(2.4.7) implies that

$$p^w(t) = e^{-A^*(t-t_f)}p(t_f) - \int_t^{t_f} e^{-A^*(t-t')} q(x(t)) dt$$

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where \( q(x(t)) \in \partial g(x(t)) \). By the assumption on \( \partial g \), we have

\[
\|p^w(t)\| \leq C_1 \|x\| + C_2
\]

Since \( \partial \psi^u(x(t)) = \{-p^w(t)\} \) (c.f. [BP1]),

\[
(2.4.10) \quad \sup\{\|\zeta\| : \quad \zeta \in \partial \psi^u(x)\} \leq C_3 \|x\| + C_4
\]

From (2.4.8) we have

\[
u^w(t) = \partial h^*(Bp^w(t))\]

since \( h \) is lower semicontinuous. Hence \( \exists \eta \in \partial \psi^u(x(t)) \) such that

\[
\mu^w(t) = \partial h^*(-B\eta(x(t)))
\]

From (2.4.10) and Theorem 2.4.1, we know that the feedback control

\[
\mu^w(t, x(t)) = \partial h^*(-B\eta(x(t)))
\]

is admissible in the sense of Definition 2.2.1.1. Note that when this feedback control is used, \( p^w \) still satisfies (2.4.7)- (2.4.9), and hence, it is also an optimal feedback control. This completes the proof of the Corollary. \( \Box \)

We are now in a position to study the inf sup problem, i.e. the upper value of the game. Recall that \( \mathcal{M}_{CL} \) is defined as the class of (nonlinear) feedback controllers \( \mu : X \to U \), and in terms of this notation, and for \( s \in (t_0, t_f) \) and \( x \in X \), let

\[
(2.4.11) \quad \varphi_\gamma(s, x) = \inf_{\mu \in \mathcal{M}_{CL}} \sup_{w \in L^2([t_0, t_f]; W)} J_\gamma(s, x; \mu, w)
\]
Theorem 2.4.3. Suppose that \( u \to h^*(u) \) is Fréchet differentiable and for \( x \in D(A) \) the following equation admits a solution \( \varphi_\gamma \in \Phi \) for \( \eta \in \partial \varphi_\gamma \):

\[
\frac{\partial \varphi_\gamma}{\partial s}(s, x) + (Ax, \eta(x)) - h^*(-B^*\eta(x)) + g(x) + \frac{\gamma^{-2}}{4} \|D^*\eta(x)\|^2_W = 0
\]

a.e. \( s \in (t_0, t_f) \)

\[\varphi_\gamma(t_f, h) = \phi_0(h) \quad \forall h \in X\]

Moreover assume that

(i) \( x \to \eta(x) \) is continuous;

(ii) \( D(\eta) \supseteq D(A) \) and \( \|\eta(x)\|_X \leq c_1\|x\|_X + c_2 \) for all \( x \in D(A) \).

Then the differential game (2.4.11) admits a feedback saddle-point solution, which is given by

\[
\dot{\mu}(t; x(t)) = \partial h^*(-B\eta(x(t)))
\]

\[
\dot{\nu}(t; x(t)) = \frac{\gamma^{-2}}{2} D^*\eta(x(t))
\]

and \( \varphi_\gamma(s, x) \) is the optimal value of (2.4.11). In particular

\[
\varphi_\gamma(t_0, x_0) = \inf_{\mu \in M_{CL}} \sup_{w \in L^2([t_0, t_f]; W)} J_\gamma(t_0, x_0; \mu, w)
\]

Proof. Since system (2.2.1.1) (when initial condition is \( x(s) = x \)) has just a mild solution, \( x(t) \) is not necessarily differentiable; thus we consider its Yosida approximation (c.f. [Paz]):

\[
\dot{x}_n(t) = A_n x_n(t) + B u_n(t) + D w_n(t)
\]

\( x_n(s) = x \)

where \( A_n = n^2 R(n; A) - nI \) is the Yosida approximation of \( A \) and \( R(n; A) = (nI - A)^{-1} \), \( w_n \in C^1([t_0, t_f]; W) : w_n(\cdot) \to w(\cdot) \) and \( u_n \in C^1([t_0, t_f]; U) : u_n(\cdot) \to u(\cdot) \). Since
\( A_n \in L(X) \), (2.4.13) has a unique solution \( x_n \in W^{1,2}([t_0, t_f]; X) \cap L^2([t_0, t_f]; D(A)) \) for each sufficiently large positive integer \( n \) and \( x_n(t) \to x(t) \) uniformly on \( X \) and \( x(t) \) is the mild solution of (2.2.1.1). In (2.4.12), let \( x = x_n(t) \), \( s = t \); then we have

\[
\varphi_{\gamma}(t, x_n(t)) + (A_n x_n(t), \eta(x_n(t))) - h^*(-B^* \eta(x_n(t))) + g(x_n(t)) + \frac{\gamma^{-2}}{4} \|D^* \eta(x_n(t))\|^2_W = 0
\]

In view of \( \varphi_{\gamma} \in \Phi \) and (2.4.12),

(2.4.14)

\[
\frac{d\varphi}{dt}(t, x_n(t)) - (Bu_n + Dw_n, \eta(x_n(t))) - h^*(-B^* \eta(x_n(t))) + g(x_n(t)) + \frac{\gamma^{-2}}{4} \|D^* \eta(x_n(t))\|^2_W = 0
\]

Let \( \bar{u}_n = \bar{u}(t; x_n(t)) = \partial h^*(-B^* \eta(x_n(t))) \), and according to Theorem 2.4.1 \( \bar{u}(t; x_n(t)) \) is an admissible feedback control policy for system (2.4.13). Thus we have

\[
h(\bar{u}_n) + h^*(-B^* \eta(x_n(t))) = (-Bu_n, \eta(x_n(t))),
\]

and hence (2.4.14) becomes

\[
\frac{d\varphi}{dt}(t, x_n(t)) + (B\bar{u}_n - Bu_n, \eta(x_n(t))) + h(\bar{u}) - h(u)
\]

\[
(g(x_n(t)) + h(u(t)) - \gamma^2 \\|w(t)\|^2_W) + \gamma^2 \\|w(t) - \frac{\gamma^{-2}}{2} D^* \eta(x(t))\|^2_W = 0
\]

Integrating the above equation from \( t = s \) to \( t = t_f \), and letting \( n \to \infty \), yields

(2.4.15)

\[
\int_s^{t_f} \{(B\bar{u}(t; x(t)) - Bu(t), \eta(x(t))) + h(\bar{u}(t; x(t))) - h(u(t))\} dt
\]

\[
+ \gamma^2 \int_s^{t_f} \|w(t) - \frac{\gamma^{-2}}{2} D^* \eta(x(t))\|^2_W dt + J_\gamma(s, x; u, w) = \varphi_{\gamma}(t_0, x_0)
\]

Since \( \bar{u} \in \partial h^*(-B^* \eta(x)) \), it follows that

\[
(B\bar{u}(t; x(t)) - Bu(t), \eta(x(t))) + h(\bar{u}(t; x(t))) - h(u(t)) \leq 0
\]

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Minimizing $J_\gamma(s, x; \cdot, w)$ over $u$, we have
\[
\int_s^t \{(B\bar{u}(t; x(t)) - Bu(t), \eta(x(t))) + h(\bar{u}(t; x(t))) - h(u(t))\} dt = 0
\]
Maximizing $J_\gamma(s, x; u, \cdot)$ over $w$, we have
\[
w(t) = \frac{\gamma^{-2}}{2} D^*\eta(x(t))
\]
Let
\[
\mu(t; x(t)) = \partial h^*(-B^*\eta(x(t)))
\]
\[
\nu(t; x(t)) = \frac{\gamma^{-2}}{2} D^*\eta(x(t))
\]
Then, for any $(u, w) \in L^2([s, t_f]; U) \times L^2([s, t_f]; W)$, we have
\[
J_\gamma(s, x; u, w) \leq J_\gamma(s, x; \mu, \nu) \leq J_\gamma(s, x; u, \nu)
\]
and therefore it is readily seen that $(\mu, \nu)$ is a saddle-point solution and
\[
\varphi_\gamma(s, x) = \inf_{\mu \in \mathcal{M}_{CL}} \sup_{w} J_\gamma(s, x; \mu, w) = \sup_{w} \inf_{\mu \in \mathcal{M}_{CL}} J_\gamma(s, x; \mu, w)
\]
In particular, letting $s = t_0$ and $x = x_0$, we have
\[
\varphi_\gamma(t_0, x_0) = \inf_{\mu \in \mathcal{M}_{CL}} \sup_{w} J_\gamma(t_0, x_0; \mu, w) = \sup_{w} \inf_{\mu \in \mathcal{M}_{CL}} J_\gamma(t_0, x_0; \mu, w)
\]
\[\square\]

**Corollary 2.4.4.** Under the hypotheses of Theorem 2.4.3, for any $(x_0, w)$ belonging to the space $X \times L^2([t_0, t_f]; W)$ the following holds:
\[
\int_{t_0}^{t_f} \{g(x(t)) + h(\mu(x(t)))\} dt + \phi_0(x(t_f)) \leq \gamma^2 \int_{t_0}^{t_f} \|w(t)\|_W^2 dt + \varphi_\gamma(t_0, x_0)
\]
where $\mu(x(t)) = \partial h^*(-B\eta(x(t)))$.

**Proof.** This is a direct consequence of (2.4.14). $\square$
Corollary 2.4.5. Under the hypotheses of Theorem 2.4.3, with \( \varphi_{\infty}(t_0, x_0) = 0 \), we have

\[
\inf_{\mu \in \mathcal{M}_{CL}} \sup_w \rho(\mu, w) \leq \gamma^2
\]

where \( \rho \) was defined by (2.1.6).

Proof. Let

\[
\omega(u(t)) := \int_{t_0}^{t_f} \{(B\bar{u}(t; x(t)) - Bu(t), \eta(x(t))) + h(\bar{u}(t; x(t))) - h(u(t))\} dt,
\]

where

\[
\bar{u}(t; x(t)) = \partial h^*(-B^*\eta(x(t))).
\]

According to (2.4.14), we have

\[
\sup_w \rho(u, w) = \rho(u, \bar{w}) = \gamma^2 - \gamma^2 \frac{\omega(u(t))}{\int_{t_0}^{t_f} \|w(t)\|_W^2 dt}
\]

Since \( \omega(u) \geq 0 \) for any \( u \in L^2([t_0, t_f]; U) \), we have

\[
\inf_{\mu \in \mathcal{M}_{CL}} \sup_w \rho(\mu, w) \leq \gamma^2 \quad \Box
\]

2.5. Application to a Class of Problems with Control Constraints

In this section, we consider the \( H^\infty \)-optimal control for system (2.2.1.1) under control constraints. Let \( U_0 \) be a closed convex subset of \( U \), and the system be described by

(2.5.1) \[
\dot{x}(t) = Ax(t) + Bu(t) + Dw(t), \quad x(t_0) = x_0
\]

\[
z(t) = Cx(t) + R^{\frac{1}{2}}u(t)
\]

where \( C \in L(X, X), \ R \in L(U, U) \) with \( R^*[C, R] = [0, I] \).
The $\gamma$-robustness property with the control constraint $U_0$ amounts to the existence of

$\mu^c \in \mathcal{M}_{CL} \subset \mathcal{M}_{CL}$ and a nonnegative function $V : X \to \mathbb{R}$ such that

$$\int_{t_0}^{t_f} \{ ||C(x(t))||_Z^2 + ||\mu^c(t,x(t))||_U^2 \} dt + \phi_0(x(t_f)) \leq \epsilon^2 \int_{t_0}^{t_f} ||w(t)||_W^2 dt + V(x_0)$$

where $0 < \epsilon < \gamma$, and $\mathcal{M}_{CL}$ represents a subclass of admissible feedback controls in $\mathcal{M}_{CL}$ with $\mu^c(t;x(t)) = Fc\ x(t) \in U_0$ for all $w \in L^2([t_0,t_f],W)$ and $x_0 \in X$, \quad $Fc : [t_0,t_f] \times X \to U_0$.

Let $I_{U_0}$ be the indicator function of $U_0$, i.e.

$$I_{U_0}(u) = \begin{cases} 0 & \text{if } u \in U_0 \\ +\infty & \text{if } u \notin U_0 \end{cases}$$

Define $P_{U_0} : U \to U_0$ by $P_{U_0}(u) = u_0$ where $u_0 \in U_0$ is such that

$$\inf_{v \in U_0} \|v - u\|_U = \|u_0 - u\|_U$$

From the Best Approximation Theorem (see e.g. Theorem 2.3 of [Au]), we know that $P_{U_0}$ is well-defined. If $U_0$ is a subspace of $U$, $P_{U_0}$ is just the projection operator of $U$ on $U_0$.

Let $h(u) = ||u||_U^2 + I_{U_0}(u)$. Now we can restate the $\gamma$-robustness property with control constraints as follows: There exists a control $\mu \in \mathcal{M}_{CL}$ (not necessarily assuming that $\mu$ takes values in $U_0$) and a nonnegative function $V : X \to \mathbb{R}$ such that

$$\int_{t_0}^{t_f} \{ ||C(x(t))||_Z^2 + h(u(t)) \} dt + \phi_0(x(t_f)) \leq \epsilon^2 \int_{t_0}^{t_f} ||w(t)||_W^2 dt + V(x_0)$$

holds for any $(x_0,w) \in X \times L^2([t_0,t_f];W)$ where $0 < \epsilon < \gamma$.

If (2.5.3) is satisfied we know that in fact $u \in U_0$. Note that

$$h^*(u) = \frac{1}{4}(\|u\|^2_U - \|u - P_{U_0}(u)\|^2_U)$$

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and hence that \( \partial h^*(u) \) consists of a single element \( P_{U_0} \) (hence it is Gâteaux differentiable); in fact \( \partial h^* \) is Fréchet differentiable (see Appendix I). This now brings us to the following theorem.

**Theorem 2.5.1.** Suppose that the following equation admits a solution \( \varphi_\gamma \in \Phi \) for \( \eta(s, x) \in \partial \varphi_\gamma(s, x); \)

\[
\frac{\partial \varphi_\gamma}{\partial s}(s, x) + (Ax, \eta(s, x)) - h^*(-B^*\eta(s, x)) + (Cx, Cx) + \frac{\gamma^{-2}}{4} \|D^*\eta(s, x)\|_W^2 = 0
\]

a.e. s \( \in (t_0, t_f) \)

\[\varphi_\gamma(t_f, h) = \phi_0(h), \quad \forall h \in X\]

and also that

(i) \( x \to \eta(\cdot, x) \) is continuous;

(ii) \( D(\eta) \supseteq D(A) \) and \( \|\eta(t, x)\|_X \leq c_1 \|x\|_X + c_2 \) for all \( x \in D(A) \).

Then we have

\[
\int_{t_0}^{t_f} \{ \|C(x(t))\|_Z^2 + h(\mu(t)) \} dt + \phi_0(x(t_f)) \leq \gamma^2 \int_{t_0}^{t_f} \|w(t)\|_W^2 dt + V(x_0)
\]

where \( \mu(t, x(t)) = P_{U_0}(-B^*\eta(t, x(t))) \), which is an admissible feedback control for system (2.5.1).

Conversely, if the \( \gamma \)-robustness property with control constraints (i.e. (2.5.3) holds, then \( \varphi_\gamma(s, x) \) defined as below satisfies (2.5.5):

\[
\varphi_\gamma(t_0, x_0) = \inf_{\mu \in \mathcal{M}_{CL}} \sup_{w} J_\gamma(t_0, x_0; \mu, w)
\]

where

\[
J_\gamma(t_0, x_0; \mu, w) = \int_{t_0}^{t_f} \{ \|C(x(t))\|_Z^2 + h(\mu(t; x(t))) - \gamma^2 \|w(t)\|_W^2 \} dt + \phi_0(x(t_f))
\]

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When the control constraint set \( U_0 \) is a subspace of the control space \( U \), we can obtain a more explicit expression. Note that in this case, \( P_{U_0} \) is a linear operator from \( U \) to \( U_0 \). Thus we can easily verify:

(i) \( D(\varphi_\gamma)(t, \cdot) = X \);

(ii) \( x \to \partial \varphi_\gamma(t, x) \) is linear;

provided that the \( \gamma \)-robustness property with control constraints holds and \( \varphi_\gamma \) is given by (2.5.7).

For convenience, let \( \phi_f \equiv 0 \). Since \( \varphi_\gamma \) is maximal monotone on \( X \), and \( \partial \varphi_\gamma \) is self-adjoint on \( X \) for every \( t \in [t_0, t_f] \), we have

\[
\varphi_\gamma(t, x) = (Z_\gamma(t)x, x) \quad \forall x \in X, \quad \partial \varphi_\gamma(t) = 2Z_\gamma(t)
\]

where \( P_{U_0} \) is the projection operator of \( U \) on \( U_0 \). In view of this, (2.5.5) now leads to:

(2.5.8)

\[
\dot{Z}_\gamma(t) + A^*Z_\gamma(t) + Z_\gamma(t)A - Z_\gamma(t)(2BP_{U_0}B^* - B^*P_{U_0}B^* - \frac{1}{\gamma^2}DD^*)Z_\gamma(t) + C^*C = 0
\]

\[
Z_\gamma(t_f) = 0
\]

Let

\[
K_{x_0}(u, w) := \int_{t_0}^{t_f} \{\|C(x(t))\|_Z^2 + h(u(t))\} dt
\]

and

\[
\rho(u, w) := K_0(u, w)/\int_{t_0}^{t_f} \|w(t)\|_W^2 dt
\]

**Corollary 2.5.2.** Given \( \gamma > 0 \),

\[
\sup_{w} \inf_{u} \rho(u, w) < \gamma^2
\]
if and only if (2.5.8) admits a unique mild solution. In this case the optimal feedback control is given by

$$\mu^*(t, x(t)) = -Pu_0B^*Z_\gamma(t)x(t), \quad 0 \leq t \leq t_f$$

and the worst disturbance is (given in state feedback form)

$$\nu^*(t, x(t)) = \gamma^{-2}D^*Z_\gamma(t)x(t), \quad 0 \leq t \leq t_f$$

Moreover

$$\sup_{w} \inf_{u} \rho(u, w) < \gamma^2 \text{ implies } \inf_{\mathcal{M}} \sup_{w} \rho(u, w) = \sup_{w} \inf_{u} \rho(u, w) < \gamma^2$$

Proof. If (2.5.8) admits a mild solution, and for $x \in X$ we let

$$\varphi_\gamma(t, x) = (Z_\gamma(t)x, x),$$

then $\varphi_\gamma$ verifies (2.5.5) and conditions (i), (ii) of Theorem 2.5.1; hence we have

$$\int_{t_0}^{t_f} \{\|C(x(t))\|^2_Z + h(\mu(x(t)))\} dt \leq \gamma^2 \int_{t_0}^{t_f} \|w(t)\|^2_w dt \quad \forall w \in L^2([t_0, t_f]; W)$$

where $\mu(t, x(t)) = Pu_0(-B^*Z_\gamma(t)x(t))$ and $x$ can be viewed as the solution of

$$\dot{x} = \left(A - (BPu_0B^*Z_\gamma - \frac{1}{\gamma^2}DD^*Z_\gamma)\right)x + D(w - \frac{1}{\gamma^2}D^*Z_\gamma x), x(t_0) = 0$$

Hence immediately

$$\int_{t_0}^{t_f} \|x(t)\|^2_X dt \leq c_0 \int_{t_0}^{t_f} \|w - \frac{1}{\gamma^2}D^*Z_\gamma x\|^2_W dt$$

Combining (2.5.9) and (2.5.10) we have

$$\sup_{w} \inf_{u} \rho(u, w) < \gamma^2$$
Conversely, if (2.5.11) is true for a given \(\gamma\), according to definition of \(\rho\) we know that there exists a \(\mu \in \mathcal{M}_{CL}\) such that

\[
\int_{t_0}^{t_f} (\|C_x(t)\|_V^2 + h(\mu(x(t))))dt \leq (\gamma^2 - \delta^2) \int_{t_0}^{t_f} \|w(t)\|_W^2 dt
\]

where \(\delta : 0 < \delta < \gamma\). Thus we can define

\[
\varphi_\gamma(t, x) = \sup_{u, w} \inf_{u} \{K_0(u, w) - \gamma^2 \|w\|_W^2\}.
\]

Following the discussion preceding Corollary 2.5.2, we know that \(\partial \varphi_\gamma\) is the solution of the generalized Riccati equation (2.5.8). Next we claim

\[
\sup_{u, w} \inf \rho(u, w) = \inf_{\mathcal{M}_{CL}} \sup \rho(\mu, w)
\]

Since \(\inf_{\mathcal{M}_{CL}} \sup_{u, w} \rho(u, w) < \gamma^2\), (2.5.8) admits a unique mild solution. Again let \(\varphi_\gamma(t, x) = (Z_\gamma(t) x, x)\). By (2.4.14), for \((u, w) \in L^2([t_0, t_f]; U_0) \times L^2([t_0, t_f]; W)\), we have

\[
\int_{t_0}^{t_f} \{(-BPu_0 B^* Z_\gamma(t)x(t) - Bu(t), Z_\gamma(t)x(t)) + h(-P_{U_0} B^* Z_\gamma(t)x(t)) - h(u(t))\} dt
\]

\[+ \gamma^2 \int_{t_0}^{t_f} \|w(t) - \gamma^{-2} D^* Z_\gamma(t)x(t)\|_W^2 dt + J_\gamma(t_0, 0; u, w) = 0,
\]

and thus

\[
\sup_{u} \rho(u, w) = \gamma^2 - \frac{\omega(u(t))}{\int_{t_0}^{t_f} \|w(t)\|_W^2 dt}
\]

where

\[
\omega(u(t)) = \int_{t_0}^{t_f} \{(BPu_0 B^* Z_\gamma(t)x(t) + Bu(t), Z_\gamma(t)x(t))
\]

\[+ h(-P_{U_0} B^* Z_\gamma(t)x(t)) + h(u(t))\} dt \geq 0
\]

and \(x\) satisfies

\[
\dot{x} = \left( A - (BPu_0 B^* Z_\gamma - \frac{1}{\gamma^2} DD^* Z_\gamma) \right)x + B(u - Pu_0 B^* Z_\gamma x), \quad x(t_0) = 0
\]

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Similar to (2.5.10) we have

(2.5.15) \[ \int_{t_0}^{t_f} \|x(t)\|_x^2 \, dt \leq c_1 \int_{t_0}^{t_f} \|u - P_{U_0} B^* Z x\|_U^2 \, dt \]

(2.5.13) and (2.5.15) imply that

\[ \inf_{M_{CL}} \sup_w \rho(u, w) < \gamma^2 \]

Note that

\[ \inf_{M_{CL}} \sup_w \rho(u, w) \geq \sup_w \inf_u \rho(u, w) \]

Therefore we can conclude that

\[ \inf_{M_{CL}} \sup_w \rho(u, w) = \sup_w \inf_u \rho(u, w) \]

**Remark 2.5.3.** Two extreme cases are:

(i) \( U_0 = \{0\} \) in which case \( P_{U_0} = 0 \);

(ii) \( U_0 = U \) in which case \( P_{U_0} = I \).

In both cases equation (2.5.8) is consistent with the known results in the finite-dimensional case (see [BB]). Another important application arises when \( U_0 \) is a finite-dimensional space, which is a realistic situation, arising due to implementation constraints.

**2.6. Examples**

**Example 1.** Consider the finite-dimensional (scalar) system

\[ \dot{x}(t) = -x(t) + u(t) + w(t), \quad u(t) \geq 0, \quad t \in (0, 1] \]

Here \( H = U = W = \mathbb{R} \) and \( Z = \mathbb{R} \times \mathbb{R} \); let the output be

\[ z = \begin{cases} 
\{x, u\} & x \geq 0 \\
\frac{1}{2} x, u \{ & x \leq 0 
\end{cases} \]

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Note that $z$ is not linear with respect to $x$, and in this case

$$g(x) = \begin{cases} x^2 & x \geq 0 \\ \frac{1}{2}x^2 & x \leq 0 \end{cases}$$

Let $h(u) = u^2 + I_{U_0}(u)$ where $U_0 = [0, \infty)$. Therefore equation (2.2.2.1) has the following form

$$\frac{d\varphi_{\gamma}(s, x)}{ds} - x\varphi_{\gamma} - \frac{1}{2}(\partial\varphi_{\gamma})^2 + \frac{1}{2}g(x) + \frac{\gamma^2}{2}(\partial\varphi_{\gamma})^2 = 0$$

$$\varphi_{\gamma}(1, h) = 0, \quad \forall h \in X$$

When $\frac{1}{\sqrt{3}} > \gamma > 0.45$, this equation admits the solution

$$\varphi_{\gamma}(t, x) = \begin{cases} \frac{1}{2}\{\gamma^2 - \sqrt{\gamma^2 - \gamma^4}\} \tan\left[\frac{(1-\gamma^2)(t-1)+\sqrt{\gamma^2 - \gamma^4}}{\sqrt{1-\gamma^2}}\right]x^2 & x \geq 0 \\ \{\frac{1}{4}(\gamma^2 - 1)^{-1}\gamma(-2\gamma + \sqrt{2}(1-3\gamma^2)^\frac{1}{2}} \tan\left[\frac{\gamma^{-1}\sqrt{1-3\gamma^2(t-1)+\sqrt{2}\tan^{-1}\left[\frac{\sqrt{\gamma^2}}{\sqrt{1-3\gamma^2}}\right]}}{\sqrt{2}}\right]\}x^2 & x \leq 0 \end{cases}$$

The optimal feedback control is

$$\hat{\mu}(t, x(t)) = \begin{cases} 0 & x \geq 0 \\ -\frac{1}{2}(\gamma^2 - 1)^{-1}\gamma(-2\gamma + \sqrt{2}(1-3\gamma^2)^\frac{1}{2}} \tan\left[\frac{\gamma^{-1}\sqrt{1-3\gamma^2(t-1)+\sqrt{2}\tan^{-1}\left[\frac{\sqrt{\gamma^2}}{\sqrt{1-3\gamma^2}}\right]}}{\sqrt{2}}\right]x & x \leq 0 \end{cases}$$

Numerical calculation yields that the optimum level of disturbance attenuation is approximately

$$\gamma^* \approx 0.442107.$$  

**Example 2.** Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ with regular boundary $\partial\Omega$. Consider the state equation:

$$(2.6.1) \quad \frac{\partial^2 x(t, \xi)}{\partial t^2}(t, \xi) = \Delta x(x, \xi) + (Bu(t, \cdot))(\xi) + (Dw(t, \cdot))(\xi), \text{ in } (0, T) \times \Omega$$
\[ x(t, \xi) = 0 \quad \text{on} \quad (0,T) \times \partial \Omega \]

\[ x(0, \xi) = x_0(\xi), \quad \text{in} \quad \Omega \]

\[ \frac{\partial x}{\partial t}(0, \xi) = x_1(\xi) \quad \text{in} \quad \Omega \]

where \( \Delta_\xi \) denotes the standard Laplacian operator. Set \( X = H^1_0(\Omega) \oplus L^2(\Omega) \) and \( U = W = L^2(\Omega) \) and denote by \( Y = \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} \) a generic element of \( X \). Define the inner product on \( X \) by

\[
\left( \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}, \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \right) = \int_{\Omega} (\nabla_\xi x^0 \cdot \nabla_\xi z^0 + x^1 z^1) d\xi
\]

Let \( \Lambda \) be the self-adjoint positive operator on \( L^2(\Omega) \), defined by

\[ D(\Lambda) = H^2(\Omega) \cap H^1_0(\Omega), \quad \Lambda x = -\Delta_\xi x \]

Then we have

\( (X, Z) = (\sqrt{\Lambda} x^0, \sqrt{\Lambda} z^0) + (x^1, z^1) \)

Define the linear operator \( A \) on \( H \):

\[
AY = \begin{bmatrix} 0 & 1 \\ -\Lambda & 0 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}, \quad \forall Y \in D(A)
\]

\[ D(A) = H^2(\Omega) \cap H^1_0(\Omega) \oplus H^1_0(\Omega) \]

We know that \( A \) is the infinitesimal generator of a contraction group in \( X \) because \( A^* = -A \)

(cf. Stone Theorem, see e.g. [Paz]). For simplicity, we may define \( B \in L(U, X) \) and \( D \in L(W, X) \) as

\[ Bu = \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad u \in U, \quad Dw = \begin{bmatrix} 0 \\ w \end{bmatrix}, \quad w \in W \]

Setting

\[ x^0(t) = x(t, \cdot), \quad x^1(t) = \frac{\partial x}{\partial t}(t, \cdot), \quad u(t) = u(t, \cdot), \quad w(t) = w(t, \cdot), \]

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the state equation (2.6.1) can be written as

\[ \dot{Y} = AY + Bu + Dw, \quad Y(0) = Y_0 \]

where

\[ Y(t) = \begin{bmatrix} x^0(t) \\ x^1(t) \end{bmatrix}, \quad Y_0 = \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} \]

A natural choice for the function \( g \) is:

\[
g(x(t, \cdot)) = \begin{cases} |\nabla \xi x(t, \cdot)|^2 + |\partial_\xi x(t, \cdot)|^2 & x(t, \cdot) \frac{\partial x}{\partial t}(t, \cdot) \geq 0 \\ |\nabla \xi x(t, \cdot)|^2 & x(t, \cdot) \frac{\partial x}{\partial t}(t, \cdot) \leq 0 \end{cases}
\]

Let the \( \gamma \)-parametrized cost function be

\[
J_\gamma(t, Y_0; u, w) = \int_t^T \int_\Omega (g(x(t, \xi)) + |u(t, \xi)|^2 - \gamma^2 |w(t, \xi)|^2) dtd\xi + \int_\Omega |\nabla \xi x(T, \xi)|^2 + |\frac{\partial x}{\partial t}(T, \xi)|^2 d\xi
\]

Introducing the value function:

\[
\varphi_\gamma(t, Y_0) = \sup \inf_{w u} J_\gamma(t, Y_0; u, w)
\]

\[
= \inf_{\mu \in M_{CL}} \sup_{w} J_\gamma(t, Y_0; u, w),
\]

it is not difficult to see that \( D(\varphi_\gamma) = X \) and \( Y_0 \rightarrow \partial \varphi_\gamma(t, Y_0) \) is linear. Therefore we may have

\[
\varphi_\gamma(t, Y_0) = (P_\gamma(t)Y_0, Y_0)_H, \quad P_\gamma(t) = \partial \varphi_\gamma(t, Y_0)
\]

where \( P_\gamma \) satisfies

\[
\dot{P}_\gamma(t) + \begin{bmatrix} 0 & -1 \\ \Lambda & 0 \end{bmatrix} P_\gamma + P_\gamma(t) \begin{bmatrix} 0 & 1 \\ -\Lambda & 0 \end{bmatrix} - P_\gamma(t) \begin{bmatrix} 0 & 0 \\ 0 & 1 - \gamma^{-2} \end{bmatrix} P_\gamma(t) 
\in \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \}
\]

\[
P_\gamma(T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
Let $P_1^i(t)$ and $P_2^i(t)$ be the solutions of the above, when respectively the first matrix and the second matrix on the right hand-side is taken. Such solutions exist when $\gamma > 1$ because the equation in these cases is the standard Riccati equation corresponding to the wave state equation ([BPDM], p. 145, Theorem 2.1). We can represent $P_\gamma^i$ as

$$P_\gamma^i = \begin{bmatrix} p_{11}^i & p_{12}^i \\ p_{21}^i & p_{22}^i \end{bmatrix}, \quad i = 1, 2,$$

where

$$p_{11}^i \in \mathcal{L}(H_0^1(\Omega)), \quad p_{12}^i \in \mathcal{L}(L^2(\Omega); H_0^1(\Omega)), \quad p_{21}^i \in \mathcal{L}(H_0^1(\Omega); L^2(\Omega)), \quad p_{22}^i \in \mathcal{L}(L^2(\Omega))$$

where in fact the following identities hold:

$$(p_{11}^i)^* = \Lambda p_{11}^i \Lambda^{-1}, \quad (p_{12}^i)^* = p_{21}^i \Lambda^{-1}, \quad (p_{21}^i)^* = \Lambda p_{12}^i, \quad (p_{22}^i)^* = p_{22}^i$$

For $\gamma > 1$, the optimal state-feedback control policy is

$$\mu(t, x(t, \xi)) = \begin{cases} -(p_{21}^i(t)x(t, \cdot))(\xi) + (p_{22}^i(t) \frac{\partial x}{\partial t}(t, \cdot))(\xi) & x(t, \xi) \frac{\partial x}{\partial t}(t, \xi) \geq 0 \\ -(p_{21}^i(t)x(t, \cdot))(\xi) + (p_{22}^i(t) \frac{\partial x}{\partial t}(t, \cdot))(\xi) & x(t, \xi) \frac{\partial x}{\partial t}(t, \xi) \leq 0 \end{cases}$$

and the worst disturbance (in state feedback form) is

$$\nu(t, x(t, \xi)) = \begin{cases} \gamma^{-2} \{ (p_{21}^i(t)x(t, \cdot))(\xi) + (p_{22}^i(t) \frac{\partial x}{\partial t}(t, \cdot))(\xi) \} & x(t, \xi) \frac{\partial x}{\partial t}(t, \xi) \geq 0 \\ \gamma^{-2} \{ (p_{21}^i(t)x(t, \cdot))(\xi) + (p_{22}^i(t) \frac{\partial x}{\partial t}(t, \cdot))(\xi) \} & x(t, \xi) \frac{\partial x}{\partial t}(t, \xi) \leq 0 \end{cases}$$

**Example 3.** Let $X = Z = U = W = L^2(\Omega)$ and $Q = \Omega \times (0, t_f)$, and consider the following system:

$$\frac{\partial y}{\partial t} = (\Delta + c)y + u + w \quad \text{in} \ (x, t) \in Q,$$

$$y = 0, \quad \text{in} \ \partial \Omega \times (0, t_f)$$

$$y(x, 0) = y_0(x) \quad \text{in} \ \Omega$$

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where $\Delta$ is the Laplacian operator with respect to $x \in \mathbb{R}^n$. We denote by $A$ the linear self-adjoint operator in $X$:

$$
\begin{align*}
A y &= \Delta y + cy, \quad \forall y \in D(A) \\
D(A) &= H^2(\Omega) \cap H_0^1(\Omega)
\end{align*}
$$

Hence (2.6.2) can be written in the abstract form (2.2.1.1). Let $K$ be a nontrivial closed convex cone in $X$ with vertex at the origin and $y_1 \in X$. The state constraint is $X_0 = y_1 + K$.

Let $U_0 := \{v \in U \mid y(t_f; v) \in X_0\}$, and

$$
\phi_0(y) = \int_\Omega |y(x) - y_1(x)|^2 dx + I_{X_0}(y)
$$

where

$$
I_{X_0}(y) = \begin{cases} 
0 & \text{if } y \in X_0 \\
+\infty & \text{if } y \notin X_0
\end{cases}
$$

Let the cost function be

$$
L_{y_0}(u, w) = \int_0^{t_f} \int_\Omega \{|y(t, x) - z_d(x, t)|^2 + |u(t, x)|^2\} dx dt + \phi_0(y(t_f))
$$

where $z_d \in L^2([0, t_f]; X)$. Note that the subdifferential of $\phi_0$ is

$$
\partial \phi_0(y) = y - y_1 + N_{X_0}(y)
$$

where $N_{X_0}(y)$ is the normal cone of $X_0$ at $y$. For a given $\gamma$, if there exists a controller $\mu$ and a nonnegative function $U(\cdot)$ such that

$$
L_{y_0}(\mu, w) \leq \epsilon^2 \int_0^{t_f} \int_\Omega |w(t, x)|^2 dx dt + U(y_0), \quad \forall (y_0, w) \in X \times L^2([0, t_f]; W)
$$

holds for $0 < \epsilon < \gamma$, then according to Theorem 2.3.1, the open-loop representation of the optimal controller is

$$
u(t) = -B^*p(t) = -p(t)$$
where \( p \in C([0, t_f]; X) \) satisfies

\[
\begin{align*}
\dot{p} &= -Ap + y^* - z_d \\
(p(t_f), y - y^*(t_f)) &\geq (y_1 - y^*(t_f), y - y^*(t_f)) \quad \forall y \in K \\
\dot{y}^* &= Ay^* + p - \frac{1}{\gamma^2} p \\
y^*(0) &= y_0
\end{align*}
\]

In fact (2.6.3) is equivalent to saying that \( y^*(t_f) = P_{X_0}(y_1 - p(t_f)) \), where \( P_{X_0} \) is defined by

\[
P_{X_0} : X \to X_0, \quad \inf_{z \in X_0} \|z - y\|_X = \|P_{X_0}(y) - y\|_X
\]

If we define

\[
\varphi_\gamma(s, y) = \sup_{L^2([0, t_f]; W)} \inf_{L^2([0, t_f]; U)} \int_s^{t_f} \int_{\Omega} \{ |y(t, x) - z_d(t, x)|^2 + |u(t, x)|^2 \} dx dt + \phi_0(y(t_f)) - \gamma^2 \int_s^{t_f} \int_{\Omega} |w(t, x)|^2 dx dt
\]

we know from Theorem 2.3.1 and Theorem 2.3.8 that \(-\frac{1}{2}p(t) \in \partial \varphi_\gamma(t, y(t))\), and \( \varphi_\gamma \) satisfies the equation

\[
\begin{align*}
\frac{\partial \varphi_\gamma}{\partial s}(s, y(s, x)) + \int_{\Omega} \nabla_x y(s, x) \cdot \nabla_x p(s, x) dx &= -\int_{\Omega} |(p(s, x))|^2 dx \\
&+ \int_{\Omega} |y(s, x) - z_d(s, x)|^2 dx + \frac{\gamma^{-2}}{4} \int_{\Omega} |p(s, x)|^2 dx = 0 \quad \forall y(s, \cdot) \in H_0^1(\Omega)
\end{align*}
\]

a.e. \( s \in (t_0, t_f) \)

\[
\varphi_\gamma(t_f, h) = \phi_0(h) \quad \forall h \in X
\]

If (2.6.4) admits a solution \( \varphi_\gamma \in \Phi \) for a given \( \gamma > 0 \), and (i)-(ii) in Theorem 2.4.3 are also satisfied for some \( \eta \in \partial \varphi_\gamma \), then the feedback \( H^\infty \)-optimal control is given by

\[
\mu(t, y(t, x)) = -\eta(t; y(t, x)) \quad a.e. x \in \Omega, \quad t \in [0, t_f],
\]
and the worst disturbance is (given in state feedback form)

\[ \nu(t; y(t, x)) = \frac{\gamma^2}{2} \eta(t; y(t, x)) \quad \text{a.e. } x \in \Omega, \quad t \in [0, t_f]. \]

Moreover, we have

\[
\int_0^{t'} \int_{\Omega} \{|y(t, x) - z_d(x, t)|^2 + |\mu(t; y(t, x))|^2\} \, dx \, dt + \int_{\Omega} |y(x) - y_1(x)|^2 \, dx \\
\leq \gamma^2 \int_0^{t'} \int_{\Omega} |w(t, x)|^2 \, dx \, dt + \varphi(0, y_0)
\]

for any \((y_0, w) \in X \times L^2([0, t_f]; W)\).

2.7. Conclusions

In this chapter, we have studied the \(H^\infty\)-optimal control problem for infinite-dimensional systems where the cost functions are not necessarily quadratic. By making use of the property that studying the \(H^\infty\)-optimal control problem is equivalent to studying the upper value of a differential game with kernel function \(J_\gamma(x_0; u, w) = L_{x_0}(u, w) - \gamma^2 \|w\|^2_2\), for \(\gamma > \gamma^0\) for some \(\gamma^0 > 0\), we have obtained a characterization of disturbance-attenuating controllers, parametrized by \(\gamma\).

We have studied here only the finite-horizon case, where the Arzela-Ascoli Theorem was used. In infinite-horizon case, since \([t_0, \infty)\) is not a compact set in \(\mathbb{R}\), further assumptions on the system are needed, such as \(A\) generating a compact \(C_0\)-semigroup. This extension will be considered in a future paper. Also under investigation are the cases where the state information is sampled, and where (finite) dimensionality constraints are imposed on the controller. Another line of research would be to consider extensions to the nonlinear dynamics case, in which case the upper or lower value functions will not be smooth (almost everywhere differentiable) as in the present problem. The Hamilton-Jacobi Isaacs (HJI)
partial differential equations associated with such systems generally do not admit classical solutions, and hence one way to approach these problems is via the concept of viscosity solutions. This would provide an appropriate framework in which to study solutions to HJI equations for nonlinear infinite-dimensional systems, and thus the associated disturbance attenuating controllers. For some representative prior work on this general topic, we refer the reader to [BH], [BB], [CIL], [CrLi 1-2], [ES], [FS], [L] and [Sor 1-2].

2.8. Appendix I

Let \( X, Y \) be real normed linear spaces, and \( f : X \to Y \). \( f \) is Fréchet differentiable at \( x \in X \) if and only if it is Gâteaux differentiable at \( x \) and the following limit

\[
\lim_{t \to 0} \frac{1}{t} [f(x + tu) - f(x)] = [Df(x)]u
\]

is uniform on the unit sphere of \( X \). If we denote the Fréchet derivative of \( f \) by \( f' \), then

\[
Df[x] = f'[x]
\]

**Proof.** Suppose that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( |t| < \delta \),

\[
\| \frac{1}{t} [f(x + tu) - f(x)] - [Df(x)]u \| < \varepsilon
\]

holds for all \( u \) on the unit sphere. Let \( v = tu \); when \( \|v\| < \delta \) we have

\[
\|f(x + v) - f(x) - [Df(x)]v\| < \varepsilon \|v\|
\]

Since \( Df(x) \in \mathcal{L}(X,Y) \), \( f'(x) \) exists and \( Df[x] = f'[x] \).

Conversely, assume that \( f \) is Fréchet differentiable at \( x \). Then for any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that if \( \|v\| < \delta \), we have

\[
\|f(x + v) - f(x) - f'(x)v\| < \varepsilon \|v\|
\]
For \( \|u\| = 1 \), and with \( \|t\| < \delta \), the above inequality yields

\[
\| \frac{1}{t} [f(x + tu) - f(x)] - f'(x)u \| < \varepsilon \|u\| = \varepsilon
\]

Therefore \( \frac{1}{t}[f(x + tu) - f(x)] \) is uniformly convergent to \( f'[x]u \). Clearly, it is also the Gâteaux derivative of \( f \) at \( x \). \( \square \)

2.9. Appendix II

We provide in this appendix, for the sake of completeness, precise statements for two results used in the main body of the chapter: the Arzela-Ascoli theorem and Gronwall's lemma (inequality):

**Arzela-Ascoli Theorem.** Let \( M \) be a compact metric space. If \( \mathcal{F} \) is a uniformly bounded equicontinuous family in \( C(M) \), then every sequence of functions in \( \mathcal{F} \) contains a uniformly convergent subsequence.

**Gronwall's Lemma.** Let

\[
u : [a, b] \rightarrow [0, \infty),
\]

\[
v : [a, b] \rightarrow \mathbb{R}^n,
\]

be continuous functions and let \( C \) be a constant. Then if

\[
v(t) \leq C + \int_a^t v(s)u(s)ds
\]

for \( t \in [a, b] \), it follows that

\[
v(t) \leq C \exp \left( \int_a^t u(s)ds \right)
\]

for \( t \in [a, b] \).
3. Finite-Dimensional Compensators for the $H^\infty$-Optimal Control of Infinite-Dimensional Systems via A Galerkin-Type Approximation

3.1. Introduction

Let $X$ be a real, separable Hilbert space. Consider the uncertain evolution equation on $X$ governed by

\begin{align}
\dot{x}(t) &= Ax(t) + Bu(t) + Dw(t) \\
x(0) &= x_0
\end{align}

(3.1.1)

Let $U$, $W$ be also real separable Hilbert spaces, and $B$ and $D$ be linear bounded operators from respectively $U$ and $W$ to $X$. $u$ is the control and $w$ is an unknown deterministic disturbance, with $u(t) \in U$ and $w(t) \in W$. $A$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on $X$, which we will henceforth refer to as the structure operator for system (3.1.1). The partial observation (also called noise-corrupted measurement) is given by

\begin{align}
y(t) &= Cx(t) + \eta(t)
\end{align}

(3.1.2)

where $C \in \mathcal{L}(X;Y)$, $Y$ being a Hilbert space, called the space of measurements, and $\eta \in W_1$ is another disturbance, modeling the measurement error. The controller is allowed to be only a causal function of the observation $y$. For the symmetry purposes, we write the performance index in terms of a second output

\begin{align}
y_1(t) &= Hx(t) + D_{12}u(t)
\end{align}

(3.1.3)
where $y_1 \in Y_1$, with $Y_1$ a real Hilbert space, $H \in \mathcal{L}(X; Y_1)$, and $D_{12} \in \mathcal{L}(U; Y_1)$ satisfying the following standard hypotheses:

\[(3.1.4) \quad D_{12}^* D_{12} = I, \quad D_{12}^* H = 0\]

Let the cost function corresponding to the output $y_1$ be

\[(3.1.5) \quad K_{x_0}(u, w) = \int_0^\infty (\|Hx(t)\|_{Y_1}^2 + \|u(t)\|_{U})dt\]

A natural class of infinite-dimensional compensators for (3.1.1) and (3.1.2), characterized by three maps $L \in \mathcal{L}(X; U)$, $M \in \mathcal{L}(X; X)$, $G \in \mathcal{L}(Y; X)$, is the following one

\[(3.1.6) \quad u(t) = Lz(t)\]

\[\quad \dot{z}(t) = (A + M)z(t) + Gy(t)\]

Here we do not lump $A$ and $M$ into a single operator, because $A$ is unbounded while $M$ is bounded. Let $w \in L^2(0, \infty; W)$ and $\eta \in L^2(0, \infty; W_1)$. The $H^\infty$-suboptimum design problem for (3.1.1), under (3.1.2) and (3.1.6), and with the cost function (3.1.5) is: Given $\gamma > 0$, find $L$, $M$ and $G$ such that

\[(3.1.7) \quad \sup_{w, \eta} \frac{K_0(u, w)}{\int_0^\infty (\|w(t)\|_{W}^2 + \|\eta(t)\|_{W_1}^2)dt} < \gamma^2\]

More precisely, we have the following definition:

**Definition 3.1.1.** We say that the $\gamma$-robustness property with observation (3.1.2) (with $\eta \equiv 0$) holds for system (3.1.1) and the cost function (3.1.5) if there exist $L$, $M$, and $G$ in (3.1.6) such that (3.1.7) holds.

It is well-known [BB] that in the finite-dimensional case, and when the dimension of the compensator is the same as the dimension of the state (i.e. $\dim(z) = \dim(x)$), under some
stabilizability and detectability conditions an explicit solution can be obtained for the triple $L, M, G$, in terms of two Riccati equations couped through a spectral radius condition. This result was subsequently generalized to infinite-dimensional spaces by Bensoussan and Bernhard (see [BeBe]). Before stating this extension, let us introduce the two associated infinite-dimensional Riccati equations

\[(3.1.8) \quad \Pi A + A^* \Pi - \Pi (BB^* - \frac{1}{\gamma^2} DD^*) \Pi + H^* H = 0\]
\[(3.1.9) \quad \Pi = \Pi^* \geq 0\]
\[(3.1.10) \quad A - (BB^* - \frac{1}{\gamma^2} DD^*) \Pi \text{ exponentially stable}\]

and

\[(3.1.11) \quad \Sigma A^* + A \Sigma - \Sigma (C^* C - \frac{1}{\gamma^2} H^* H) \Sigma + DD^* = 0\]
\[(3.1.12) \quad \Sigma = \Sigma^* \geq 0\]
\[(3.1.13) \quad A^* - (C^* C - \frac{1}{\gamma^2} H^* H) \Sigma \text{ exponentially stable}\]

Then we have [BeBe]:

**Theorem 3.1.2.** (Bensoussan and Bernhard, 1992) Assume that the pair $(A, D)$ is stabilizable and that the pair $(A, H)$ is detectable. Then the $\gamma$-robustness property with observation (3.1.2) holds for system (3.1.1), and under cost function $K_0(u, w)$, iff the Riccati equation (3.1.8) admits a solution $\Pi$ satisfying (3.1.9) - (3.1.10), and (3.1.11) admits a solution $\Sigma$ satisfying (3.1.12) - (3.1.13), and furthermore

\[(3.1.14) \quad I - \frac{1}{\gamma^2} \Pi \Sigma \text{ is invertible; } \Sigma (I - \frac{1}{\gamma^2} \Pi \Sigma)^{-1} \geq 0\]

An infinite-dimensional compensator that achieves the bound $\gamma^2$ in (3.1.7) is given by
(3.1.6) where

\( L = -B^*\Pi, \quad M = -(BB^* - \frac{1}{\gamma^2}DD^*)\Pi - \Gamma C^*C, \quad G = \Gamma C^* \)

and

\[ \Gamma = \Sigma(I - \frac{1}{\gamma^2}\Pi\Sigma)^{-1} \]

Even though this is a complete solution for the problem at hand, the fact that the compensator is infinite dimensional renders it impractical. There is therefore a need to develop a theory that will deliver finite-dimensional stabilizing compensators, which achieve a given level of disturbance attenuation. We will address this problem here when the measurement \( y \) is finite-dimensional.

The existence of finite-dimensional compensators for infinite-dimensional LQ-problems or for LQG-problems (linear-quadratic-gaussian problems) has been established in [Cu2], [Sa], [Sch1] and [GA] provided that the structure operator of the underlying system has a complete set of generalized eigenvectors; and their construction procedure is based on the eigenvectors of either the structure operator or the closed-loop structure operator obtained under state feedback. Ito has established in [It] the existence of finite-dimensional compensators for LQ-problems by using Galerkin-type approximations in which independent basis elements were used instead of the complete set of eigenvectors, making it possible for the results to be applied to general parabolic systems and hereditary differential systems.

The existence and design of finite-dimensional compensators for infinite-dimensional systems in an \( H^\infty \) framework was one of the open problems in \( H^\infty \)-optimal problems, posed by Curtain in 1990 (see [Cu1]). Another related paper is the one by Banks, Demetriou and Smith, who studied a two-dimensional structural acoustic model by using \( H^\infty \) periodic
control, which is approximated by a finite-dimensional compensator: pieoceramic actuators (see [BDS]). In the present chapter, we establish a general existence result for finite-dimensional $H^\infty$-optimal compensators (3.1.6) for (3.1.1) with finite-dimensional measurement (3.1.2), such that the robustness property (3.1.7) holds. More precisely, we study the evolution system

\begin{align}
(3.1.16) & \quad \dot{x}(t) = A x(t) + B u(t) + D w(t), \quad x(0) = x_0 \in X \\
(3.1.17) & \quad y(t) = C x(t) + \eta(t)
\end{align}

with finite-dimensional compensator

\begin{align}
(3.1.18) & \quad \dot{z}(t) = (A_c + M_c) z(t) + G_c y(t) \\
(3.1.19) & \quad u(t) = -L_c z(t)
\end{align}

where $u(t)$ is an $\mathbb{R}^m$-valued control function, the measurement space is $\mathbb{R}^p$ and $C \in \mathcal{L}(X, \mathbb{R}^p)$, $z(t) \in Z = \mathbb{R}^{n_c}$ and $A_c, M_c, G_c, L_c$ are matrices belonging appropriate spaces. The cost function is still expressed as:

\begin{align}
(3.1.20) & \quad K_{x_0}(u, w) = \int_0^\infty (\|H x(t)\|_{Y_1}^2 + \|u(t)\|_{R^m}) dt
\end{align}

We will show that there exists a finite-dimensional compensator (3.1.18)- (3.1.19) which stabilizes the infinite-dimensional system (3.1.16), and the $\gamma$-robustness property with noise measurement (3.1.7) holds, i.e.

\begin{align}
(3.1.21) & \quad \sup_{w, \eta} \frac{K_0(u, w)}{\int_0^\infty (\|w(t)\|_{W_1}^2 + \|\eta(t)\|_{W_1}^2) dt} < \gamma^2
\end{align}

provided that the $\gamma$-robustness property holds under the infinite-dimensional compensator (3.1.6).
The rest of this chapter is organized as follows. Section 2 provides assumptions and some preliminary results on asymptotic behavior and approximation of Riccati equations, proofs of which have been included in Appendix II. The existence of finite-dimensional compensators for the infinite-dimensional problem has been shown in section 3. Section 4 provides an example, section 5 includes some concluding remarks, and Appendix I contains a Duality Theorem which is used in section 3.

3.2. Assumptions and Some Preliminaries

Let $X^N$ be a sequence of finite-dimensional subspaces of $X$, and let $P^N$ denote the orthogonal projection$^3$ of $X$ onto $X^N$. Throughout this chapter, the following notation and conventions will be adopted, unless otherwise indicated: $<\cdot,\cdot>$ denotes the inner-product on $X$, $\mathcal{L}(X)$ denotes space of all bounded linear operators on $X$, and $A^*$ denotes the adjoint of the linear operator $A$,

$$\Sigma(X) = \{T \in \mathcal{L}(X) : \text{ } T \text{ is Hermitian}\}$$

$$\Sigma^+(X) = \{T \in \Sigma(X) : <Tx,x> \geq 0, \forall x \in X\}$$

and $B^N \in \mathcal{L}(U,X^N)$, $C^N \in \mathcal{L}(X^N,Y)$, $D^N \in \mathcal{L}(W,X^N)$, $H^N \in \mathcal{L}(X^N,Y_1)$, $\Pi^N \in \mathcal{L}(X^N)$ and $\Sigma^N \in \mathcal{L}(X^N)$.

**Definition 3.2.1.** A sequence $\{x_N\} \subset X^N$ converges to $x \in X$ if

$$\|P^N x - x_N\|_{X^N} \to 0 \text{ as } N \to \infty$$

$^3$Orthogonal projection means that the range and the null spaces of the projection are orthogonal.
**Definition 3.2.2.** Let \( X, Y \) be two Banach spaces. A sequence of linear operators \( A^N : X^N \to Y^N \) converges to an operator \( A : X \to Y \) if

\[
D(A) = \{ x : P^N x \in D(A^N), A^N P^N x \text{ converges in } Y_N \}
\]

and

\[
Ax = \lim_{N \to \infty} A^N P^N x \quad \text{for} \quad x \in D(A)
\]

We will denote this type of convergence by \( A^N \to A \).

**Definition 3.2.3.** We say that the Riccati equation

\[
\Pi A + A^* \Pi - \Pi (B B^* - \frac{1}{\gamma^2} D D^*) \Pi + H^* H = 0
\]

admits a solution \( \Pi \), if \( \Pi \in \Sigma^+(X) \) and \( \forall x, y \in D(A) \) the following holds

\[
< \Pi x, Ay > + < Ax, \Pi y > - < \Pi (B B^* - \frac{1}{\gamma^2} D D^*) \Pi x, y > + < H^* H x, y > = 0
\]

A similar definition applies to the Riccati equation

\[
\Sigma A^* + A \Sigma - \Sigma (C^* C - \frac{1}{\gamma^2} H^* H) \Sigma + D D^* = 0
\]

**Definition 3.2.4.** The pair \((A, B)\) in (3.1.6) is stabilizable iff there exists an operator \( F \in \mathcal{L}(X, U) \) such that \( A + BF \) is exponentially stable.

**Definition 3.2.5.** The pair \((A, H)\) (\(A\) in (3.1.6), \(H\) in (3.1.20)) is detectable iff there exists an operator \( G \in \mathcal{L}(Y_1, X) \) such that \( A + GH \) is exponentially stable.

**Assumptions:**

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(A1) For every $x \in X$, $e^{A_N t} P_N x$ converges strongly to $T(t)x$, $e^{A_N^* t} P_N x$ converges strongly to $T^*(t)x$, and the convergence is uniform in $t$ on bounded intervals of $[0, \infty]$. Here $T(t)$ is the $C_0$ semigroup generated by $A$, and $P_N$ is the orthogonal projection of $X$ onto a finite-dimensional space $X_N$.

(A2) For each integer $N > 0$, $A_N - (B_N B_N^* - \frac{1}{\gamma^2} D_N D_N^*) \Pi_N$ generates an exponentially stable semigroup on $X_N$, and

$$\|e^{(A_N - (B_N B_N^* - \frac{1}{\gamma^2} D_N D_N^*) \Pi_N) t} P_N \|_{\mathcal{L}(X, X_N)} \leq M_1 e^{-\omega_1 t}, \quad t \in [0, \infty[$$

for some $M_1 \geq 1$ and $\omega_1 > 0$, independent of $N$.

(A3) For each integer $N > 0$, $A_N^* - (C_N^* C_N - \frac{1}{\gamma^2} H^* H) \Sigma_N$ generates an exponentially stable $C_0$ semigroup on $X_N$, and

$$\|e^{(A_N^* - (C_N^* C_N - \frac{1}{\gamma^2} H^* H) \Sigma_N) t} P_N \|_{\mathcal{L}(X, X_N)} \leq M_2 e^{-\omega_2 t}, \quad t \in [0, \infty[$$

for some $M_2 \geq 1$ and $\omega_2 > 0$, independent of $N$.

(A4) For each $N > 0$ operator matrix

$$A_p^N = \begin{bmatrix} A + \frac{1}{\gamma^2} D D^* \Pi & BL_N \\ G_N C & A_N + M_N \end{bmatrix}$$

satisfies the spectrum-determined growth condition\(^4\), where

$$L_N = -B_N^* \Pi_N, G_N = \Gamma_N C_N^*, \Gamma_N = \Sigma_N (I - \frac{1}{\gamma^2} \Pi_N \Sigma_N)^{-1}$$

and

$$M_N = -(B_N^* B_N^* - \frac{1}{\gamma^2} D_N D_N^*) \Pi_N - \Gamma_N C_N^* C_N$$

---

\(^4\) Spectrum-determined growth condition means that

$$\sup \{ \text{Re} \lambda : \lambda \in \sigma(A_p) \} = \lim_{t \to \infty} \frac{1}{t} \ln \|e^{A_p t}\|$$

where $\sigma(A)$ denotes the spectrum of $A$.  

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(A5) For $P^N$ as given in (A1), define

$$B^N = P^N B, \quad C^N = C P^N, \quad D^N = P^N D, \quad H^N = H P^N$$

**Remark 3.2.6.** (1). In (A1), $e^{A_N t} P^N x$ converges strongly to $T(t)x$ means that

$$\|P^N T(t)x - e^{A_N t} P^N x\|_{X^N} \to 0 \quad \text{as} \quad N \to \infty$$

For $t = 0$ assumption (A1) implies that $P^N x \to x$ for each $x \in X$; thus we have the result that subspace $X^N$ approximates $X$.

(2). With $B^N = P^N B$ and $C^N = C P^N$, according to (A1) we have for each $u \in U$, $B^N u \to Bu$ and for each $x \in X$, $C^N x \to Cx$ as $N \to \infty$. Moreover, from the Banach-Steinhaus Theorem we know that

$$P^N \to I, \quad B^N \to B, \quad C^N \to C, \quad D^N \to D, \quad H^N \to H \quad \text{as} \quad N \to \infty$$

where the convergences are in corresponding operator norms.

(3). From assumption (A2), we know that the resolvent set of matrix

$$A^N - (B^N B^{N*} - \frac{1}{\gamma^2} D^N D^{N*})\Pi^N$$

contains the ray $(\omega_1, \infty)$ and

$$\|[(\lambda I - (A^N - (B^N B^{N*} - \frac{1}{\gamma^2} D^N D^{N*})\Pi^N))^{-1} \leq M_1/((\lambda + \omega_1), \quad \text{for} \quad Re\lambda > -\omega_1$$

A similar corresponding statement follows from assumption (A3).
Lemma 3.2.7. Suppose that for a given $\gamma > 0$ the Riccati equation

\[(3.2.1)\quad \Pi A + A^*\Pi - \Pi (BB^* - \frac{1}{\gamma^2}DD^*)\Pi + H^*H = 0\]

admits a solution $\Pi \in \Sigma^+ (X)$ such that $A - BB^* - \frac{1}{\gamma^2}DD^*\Pi$ is exponentially stable, and for each $N$

\[(3.2.2)\quad \Pi^N A^N + A^N*\Pi^N - \Pi^N (B^N B^N* - \frac{1}{\gamma^2}D^N D^N*)\Pi^N + H^N*H^N = 0\]

admits a solution $\Pi^N \in \Sigma^+ (X^N)$ which satisfies assumption (A2). Then under the assumptions (A1) and (A5), we have $\Pi^N \rightarrow \Pi$ as $N \rightarrow \infty$.

Proof. See Appendix II.

Lemma 3.2.8. Suppose that for a given $\gamma > 0$, the Riccati equation

\[(3.2.3)\quad \Sigma A^* + A\Sigma - \Sigma (C^*C - \frac{1}{\gamma^2}H^*H)\Sigma + DD^* = 0\]

admits a solution $\Sigma \in \Sigma^+ (X)$ such that $A^* - (C^*C - \frac{1}{\gamma^2}H^*H)\Sigma$ is exponentially stable, and

\[(3.2.4)\quad \Sigma^N A^N* + A^N\Sigma^N - \Sigma^N (C^N*C^N - \frac{1}{\gamma^2}H^N*H^N)\Sigma^N + D^N D^N* = 0\]

admits a solution $\Sigma^N \in \Sigma^+ (X)$ which satisfies assumption (A3). Then, under assumptions (A1) and (A5), we have $\Sigma^N \rightarrow \Sigma$ as $N \rightarrow \infty$.

Proof. See Appendix II.

The following lemma provides some properties of Riccati equations provided that the $\gamma$-robustness property holds.
Lemma 3.2.9. Assume that $(A, D)$ is stabilizable and that $(A, H)$ is detectable, and the \( \gamma \)-robustness property holds for system (3.1.16) with measurement (3.1.7) and cost function \( K_0 \). Then we have

(1) \( A - BB^*\Pi \) is exponentially stable, where \( \Pi \) is a solution of (3.1.8)–(3.1.10);

(2) \( A^* + \frac{1}{\gamma^2} \Pi DD^* - (C^*C - \frac{1}{\gamma^2} \Pi BB^*\Pi)\Gamma \) is exponentially stable, where \( \Gamma \) is

\[
\Gamma = \Sigma(I - \frac{1}{\gamma^2} \Pi \Sigma)^{-1}
\]

and \( \Sigma \) is the solution of (3.1.11)–(3.1.13).

(3) \( A^* + \frac{1}{\gamma^2} \Pi DD^* - C^*C\Gamma \) is exponentially stable.

(4) \( \Gamma \) is the unique solution of the following Riccati equation

\[
(3.2.5) \quad \Gamma(A^* + \frac{1}{\gamma^2} \Pi DD^*) + (A + \frac{1}{\gamma^2} DD^*\Pi)\Gamma - \Gamma(C^*C - \frac{1}{\gamma^2} \Pi BB^*\Pi)\Gamma + DD^* = 0
\]

(5) The system

\[
(3.2.6) \quad \dot{x} = (A^* + \frac{1}{\gamma^2} \Pi DD^*)x + C^*v + \Pi B \mu, \quad x(0) = 0
\]

\[
(3.2.7) \quad z = D^*x
\]

with cost function

\[
(3.2.8) \quad K_0(v, \mu) = \int_0^\infty (\|z\|_W^2 + \|v\|_Y^2)dt
\]

has the \( \gamma \)-robustness property, i.e. there exists \( \tilde{v} \in Y \) such that

\[
(3.2.9) \quad \sup_{\mu} \frac{\int_0^\infty (\|z\|^2 + \|\tilde{v}\|^2)dt}{\int_0^\infty \|\mu\|^2 dt} < \gamma^2.
\]

Proof. See Bensoussan and Bernhard [BeBe].
3.3. Main Existence Result and Its Proof

**Main Result.** Consider the uncertain evolution system (3.1.16) along with finite-dimensional partial measurement given (3.1.17) and cost function $K_{x_0}$ given by (3.1.20). Assume that the pair $(A,D)$ is stabilizable and the pair $(A,H)$ is detectable, and there exists an infinite-dimensional compensator for (3.1.16) such that the $\gamma$-robustness property holds with the observation (3.1.17) and cost function (3.1.20). Then, under assumptions (A1)-(A5), there exists a finite-dimensional compensator-based controller

$$u(t) = L^Nz(t)$$

$$\dot{z}(t) = (A^N + M^N)z(t) + G^Ny(t)$$

which stabilizes the uncertain system (3.1.16), and the $\gamma$-robustness property holds with the partial observation (3.1.17) and cost function (3.1.20).

We prove the result using a sequence of lemmas and theorems. First let us introduce

$$A^N = \begin{bmatrix} A & BL^N \\ G^NC & A^N + M^N \end{bmatrix}$$

where

$$L^N = -B^N\Pi^N, G^N = \Gamma^NC^N, \Gamma^N = \Sigma^N(I - \frac{1}{\gamma^2}\Pi^N\Sigma^N)^{-1}$$

and

$$M^N = -(B^ND^N - \frac{1}{\gamma^2}D^ND^{N*})\Pi^N - \Gamma^NC^{N*}C^N$$

An aim of this section is to show that there exists a sufficiently large $N$ such that $A^N$ is exponentially stable on $X \times X^N$. 70
Lemma 3.3.1. Assume that (A1), (A2) and (A5) are satisfied. Then for each $N$, \[ L^N = -B^N \Pi^N \in \mathcal{L}(X, \mathbb{R}^m) \]
and
\[ L^N \rightarrow L = -B^* \Pi \quad \text{as } N \rightarrow \infty \]
Furthermore, there exists an integer $N_1$ such that if $N \geq N_1$ and $\Re \lambda > -\omega_3$
\[ (\lambda I - (A + \frac{1}{\gamma^2} DD^* \Pi + BL^N))^{-1} \in \mathcal{L}(X) \]
and
\[ e^{(\lambda I - (A + \frac{1}{\gamma^2} DD^* \Pi + BL^N))t} \rightarrow e^{(\lambda I - (A + \frac{1}{\gamma^2} DD^* \Pi + BL^N))t} \quad \text{as } N \rightarrow \infty \]
where the convergence is uniform on bounded $t$-intervals.

Proof. The first assertion is obvious because by assumption (A1) we know that \[ B^N \rightarrow B^* \]
and by Lemma 3.2.7 we have $\Pi^N \rightarrow \Pi$; thus the conclusion follows. For the second assertion, let $S^N(t)$ be the $C_0$ semigroup generated by the operator $A + \frac{1}{\gamma^2} DD^* \Pi + BL^N$, and $S(t)$ be the $C_0$ semigroup generated by the operator $A + \frac{1}{\gamma^2} DD^* \Pi + BL$. Since
\[ A + \frac{1}{\gamma^2} DD^* \Pi + BL^N = A + \frac{1}{\gamma^2} DD^* \Pi + BL + B(L^N - L), \]
in view of the "Perturbation Theorem" of [CP], we have
\[ S^N(t) = S(t) + \int_0^t S(t-s)B(L^N - L)S^N(s)ds. \]
(3.3.1)

Since $A + \frac{1}{\gamma^2} DD^* \Pi + BL$ is exponentially stable, there exist $M_3 \geq 1$ and $\omega \geq 0$ such that
\[ \|S(t)\| \leq M_3 e^{-\omega t} \]
Thus we have

\[(3.3.2) \quad \|S^N(t)\| \leq M_3 e^{-\omega t} + \int_0^t M_3 e^{-\omega(t-s)}\|B\|L - L^N\|\|S^N(s)\|ds,\]

Gronwall’s lemma infers that

\[(3.3.3) \quad \|S^N(t)\| \leq M_3 e^{(-\omega + M_3\|B\|L - L^N\|)t}, \quad t \geq 0\]

Since \(L^N \rightarrow L\), by Hille-Yosida Theorem (see Theorem 1.5.3 of [Paz]) we know that there exists \(N_1 > 0\) such that when \(N > N_1\), \((\lambda I - (A + \frac{1}{\gamma^2}DD^*\Pi + BL^N))^{-1}\) is bounded in \(\mathcal{L}(X)\), this yields the second statement. To conclude the proof, note that

\[
(\lambda I - (A + \frac{1}{\gamma^2}DD^*\Pi + BL^N))^{-1} - (\lambda I - (A + \frac{1}{\gamma^2}DD^*\Pi + BL))^{-1}
\]

\[= (\lambda I - (A + \frac{1}{\gamma^2}DD^*\Pi + BL^N))^{-1}B(L^N - L)(\lambda I - (A + \frac{1}{\gamma^2}DD^*\Pi + BL))^{-1}
\]

and \(L^N \rightarrow L\), which lead to

\[
(\lambda I - (A + \frac{1}{\gamma^2}DD^*\Pi + BL^N))^{-1} \rightarrow (\lambda I - (A + \frac{1}{\gamma^2}DD^*\Pi + BL))^{-1} \quad \text{as} \quad N \rightarrow \infty
\]

By The Trotter-Kato Theorem (see Theorem 3.4.4 of [Paz]), we have the desired result. \(\square\)

**Remark 3.3.2.** Clearly, there exist \(\omega_3 \geq 0\) and \(N_3 > 0\) such that when \(N > N_3\), we have

\[
\|S^N(t)\| \leq M_3 e^{-\omega_3 t}, \quad t \geq 0,
\]

where both \(\omega_3\) and \(M_3\) are independent of \(N\).

**Lemma 3.3.3.** Let \(A^N, B^N, C^N, D^N\) and \(\Pi^N\) satisfy (A1),(A2), and (A3) for each \(N > 0\). Then for every \(x \in X\)

\[e^{(A^N - (B^NB^{N*} - \frac{1}{\gamma^2}D^ND^{N*})\Pi^N)t}P^Nx \rightarrow T_1(t)x
\]

uniformly on bounded intervals with respect to \(t\), where \(T_1(t)\) is the \(C_0\) semigroup generated by \(A - (BB^* - \frac{1}{\gamma^2}DD^*)\Pi\).

**Proof.** The proof is similar to that of the last part of Lemma 3.3.1.
Theorem 3.3.4. Let $A^N, B^N, C^N, D^N$ and $\Pi^N$ satisfy (A1),(A2) and (A5), and introduce

$$A_p^N = \begin{bmatrix} A + \frac{1}{\gamma^2} DD^* \Pi & BL^N \\ G^N C & A^N + M^N \end{bmatrix}$$

where $\Pi$ is the solution of (3.1.8) satisfying (3.1.9)- (3.1.10). Then, for sufficiently large $N$, there exists $\epsilon > 0$ which is independent of $N$ such that

$$\sigma(A_p^N) \subseteq (-\infty, -\epsilon]$$

where $\sigma(A_p)$ is the spectrum\(^5\) of $A_p$.

Proof. For a given $(h_1, h_2) \in X \times X^N$, consider the equation

(3.3.4) \hspace{1cm} (\lambda - A_p^N) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix},

which is equivalent to

(3.3.5) \hspace{1cm} \lambda v_1 - (Av_1 + \frac{1}{\gamma^2} DD^* \Pi v_1 + BL^N v_2) = h_1

(3.3.6) \hspace{1cm} \lambda v_2 - [G^N C v_1 + (A^N + M^N) v_2] = h_2

What we now show is that for any given $(h_1, h_2) \in X \times X^N$, when $N$ is sufficiently large, (3.3.5)- (3.3.6) admits a solution $(v_1, v_2)$ which is also in $X \times X^N$.

According to Lemma 3.3.1, there exists an integer $N_1 > 0$ such that when $\text{Re}\lambda > -\omega_3$,

$$\lambda I - (A + \frac{1}{\gamma^2} DD^* \Pi + BL^N)$$

is well-defined. Hence, (3.3.5) can be written as

(3.3.7) \hspace{1cm} v_1 = [\lambda I - (A + \frac{1}{\gamma^2} DD^* \Pi + BL^N)]^{-1}[BL^N(v_2 - v_1) + h_1]

\(^5\)The spectrum of an operator consists of three mutually exclusive parts: the point spectrum, the continuous spectrum, and the residual spectrum; for more details, see Kato[K1].
Similarly, since $A^N - (B^N B^N* - \frac{1}{\gamma^2} D^N D^N*) \Pi^N$ generates an exponentially stable semigroup by the hypothesis of the theorem and in view of Lemma 3.2.7, there exists $\lambda$, with $\text{Re}\lambda > -\omega_1$, for which (3.3.6) becomes:

(3.3.8)

$$v_2 = [\lambda I - (A^N - (B^N B^N* - \frac{1}{\gamma^2} D^N D^N*)) \Pi^N]^{-1} [G^N C(v_1 - v_2) + h_2]$$

Thus, if $\text{Re}\lambda > -\min(\omega_1, \omega_3)$ and $N > N_1$ we have

$$v_1 - v_2 = [\lambda I - (A + \frac{1}{\gamma^2} DD^* \Pi + BL^N)]^{-1} [BL^N(v_1 - v_2) + h_1]$$

$$\quad - [\lambda I - (A^N - (B^N B^N* - \frac{1}{\gamma^2} D^N D^N*)) \Pi^N]^{-1} [G^N C(v_1 - v_2) + h_2]$$

By some manipulation, we have

(3.3.9)

$$[\lambda I - (A^N + \frac{1}{\gamma^2} D^N D^N* \Pi^N - G^N C^N)](v_1 - v_2)$$

$$= [\lambda I - (A^N - (B^N B^N* - \frac{1}{\gamma^2} D^N D^N*) \Pi^N)](v_1 - v_2) + (B^N L^N + G^N C^N)(v_1 - v_2)$$

$$= [\lambda I - (A^N - (B^N B^N* - \frac{1}{\gamma^2} D^N D^N*) \Pi^N)]$$

$$\quad \times \left( -[\lambda I - (A + \frac{1}{\gamma^2} DD^* \Pi + BL^N)]^{-1} BL^N$$

$$\quad + [\lambda I - (A^N - (B^N B^N* - \frac{1}{\gamma^2} D^N D^N*) \Pi^N)]^{-1} B^N L^N \right)(v_1 - v_2)$$

$$+ [\lambda I - (A^N - (B^N B^N* - \frac{1}{\gamma^2} D^N D^N*) \Pi^N)] \left( [\lambda I - (A + \frac{1}{\gamma^2} DD^* \Pi + BL^N)]^{-1} h_1$$

$$- [\lambda I - (A^N - (B^N B^N* - \frac{1}{\gamma^2} D^N D^N*) \Pi^N)]^{-1} P^N h_1 \right)$$

$$+ P^N h_1 - h_2 + G^N (C^N - C)(v_1 - v_2)$$

By assumption (A3), for all $\lambda$ such that $\text{Re}\lambda > -\omega_2$, the matrix $[\lambda I - (A^N + \frac{1}{\gamma^2} D^N D^N* \Pi^N - G^N C^N)]$ is invertible, in view which the following equality holds

$$[\lambda I - (A^N + \frac{1}{\gamma^2} D^N D^N* \Pi^N - G^N C^N)]^{-1} [\lambda I - (A^N - (B^N B^N* - \frac{1}{\gamma^2} D^N D^N*) \Pi^N)]$$

$$= I + [\lambda I - (A^N + \frac{1}{\gamma^2} D^N D^N* \Pi^N - G^N C^N)]^{-1} (B^N B^N* \Pi^N - G^N C^N)$$
Thus for $\Re \lambda > -\min(\omega_1, \omega_2, \omega_3) := -\omega$ and $N > N_1$, (3.3.9) can be written as

\[(v_1 - v_2) - \varphi^N BL^N(v_1 - v_2) \]
\[+ [\lambda I - (A^N + \frac{1}{\gamma^2} D^N D^N* P^N - G^N C^N)]^{-1} G^N (C - C^N)(v_1 - v_2) \]
\[= [\lambda I - (A^N + \frac{1}{\gamma^2} D^N D^N* P^N - G^N C^N)]^{-1} (P^N h_1 - h_2) - \varphi^N h_1 \]

where $\varphi^N$ is

\[
\varphi^N = \left\{ I + [\lambda I - (A^N + \frac{1}{\gamma^2} D^N D^N* \Pi^N - G^N C^N)]^{-1} (B^N B^N* \Pi^N - G^N C^N) \right\} \\
\times \left\{ - [\lambda I - (A + \frac{1}{\gamma^2} D D^* \Pi + BL^N)]^{-1} \right. \\
\left. + [\lambda I - (A^N - (B^N B^N* - \frac{1}{\gamma^2} D^N D^N*)) \Pi^N)]^{-1} P^N \right\} 
\]

We show next that there exists $N'$ such that if $N > N'$

\[(3.3.11) \quad \| \varphi^N BL^N + [\lambda I - (A^N + \frac{1}{\gamma^2} D^N D^N* P^N - G^N C^N)]^{-1} G^N (C - C^N) \|_{C(X)} < 1 \]

which then says that (3.3.5), (3.3.6) are solvable for $\Re \lambda \geq 0$.

Again we denote by $S^N$ the $C_0$ semigroup generated by operator $A + \frac{1}{\gamma^2} D D^* \Pi + BL^N$.

Since both

\[ [\lambda I - (A + \frac{1}{\gamma^2} D D^* \Pi + BL^N)]^{-1} B = \int_0^\infty e^{-\lambda t} S^N(t) B dt, \]

and

\[ [\lambda I - (A^N + \frac{1}{\gamma^2} D^N D^N* \Pi^N + B^N B^N* \Pi^N)]^{-1} P^N B \]
\[= \int_0^\infty e^{-\lambda t} e^{(A^N + \frac{1}{\gamma^2} D^N D^N* \Pi^N + B^N B^N* \Pi^N)t} P^N B dt. \]

hold for $\Re \lambda \geq -\omega$, for any $T > 0$, we have

\[ [\lambda I - (A + \frac{1}{\gamma^2} D D^* \Pi + BL^N)]^{-1} \]

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\[- [\lambda I - (A^N + \frac{1}{\gamma^2} D^N D^N \ast \Pi^N - B^N B^N \ast \Pi^N)]^{-1} P^N B \]
\[ \leq \int_0^T e^{-\Re \lambda t} \| e^{(A^N + \frac{1}{\gamma^2} D^N D^N \ast \Pi^N - B^N B^N \ast \Pi^N) t} P^N B - S^N B \| dt \]
\[ + \left( M_1 \frac{e^{-(\Re \lambda + \omega_1) T}}{\Re \lambda + \omega_1} + M_3 \frac{e^{-(\Re \lambda + \omega_3) T}}{\Re \lambda + \omega_3} \right) \| B \|. \]

By assumption (A3) and the Principle of Uniform Boundedness [TL], for \( \Re \lambda \geq -\omega \) there is a constant \( c > 0 \) which is independent of \( N \) such that
\[ \|[\lambda I - (A^N + \frac{1}{\gamma^2} D^N D^N \ast \Pi^N - G^N C^N)]^{-1} (B^N B^N \ast P^N - G^N C^N)\| \leq c. \]

Denote \( c_L = \max_N \| I^N \| \) and choose \( T > 0 \) such that for \( \Re \lambda > -\omega \)
\[ c_{LC} \left( M_1 \frac{e^{-(\Re \lambda + \omega_1) T}}{\Re \lambda + \omega_1} + M_3 \frac{e^{-(\Re \lambda + \omega_3) T}}{\Re \lambda + \omega_3} \right) \| B \| \leq \frac{1}{4}. \]

On the other hand, as \( N \to \infty \) we have
\[ e^{(A^N + \frac{1}{\gamma^2} D^N D^N \ast \Pi^N - B^N B^N \ast \Pi^N) t} P^N B \to S(t)B, \]
\[ S^N(t)B \to S(t)B, \]
uniformly on \([0, T]\). Thus there exists \( N'' > 0 \) such that when \( N \geq N'' \), the following holds
\[ \int_0^T e^{-\Re \lambda t} \| e^{(A^N + \frac{1}{\gamma^2} D^N D^N \ast \Pi^N - B^N B^N \ast \Pi^N) t} P^N B - S^N B \| dt \leq \frac{1}{4c_{CL}}. \]

It then follows that for \( N \geq \max(N', N'') := \hat{N} \), \( \| \varphi^N B L^N \| \leq \frac{1}{2} \). Note that \( C^N \to C \) as \( N \to \infty \), and there exists \( \overline{N} > 0 \) such that when \( N > \overline{N} \)
\[ [\lambda I - (A^N + \frac{1}{\gamma^2} D^N D^N \ast P^N - G^N C^N)]^{-1} G^N (C - C^N) \| \mathcal{L}(X) \leq \frac{1}{2}. \]

Therefore for \( N > \max(\hat{N}, \overline{N}) \), (3.3.11) holds, and (3.3.5), (3.3.6) are solvable for sufficiently large \( N \) when \( \Re \lambda \geq -\omega \), which yields the conclusion. \( \square \)
Theorem 3.3.5. Under the assumptions of Theorem 3.3.4, let $A^N_p$ be given as in Theorem 3.3.4. Let

$$A = \begin{bmatrix} A + \frac{1}{\gamma^N} DD^* \Pi & -BB^* \Pi \\ \Gamma C^* C & A - (BB^* - \frac{1}{\gamma^N} DD^*) \Pi - \Gamma C^* C \end{bmatrix}$$

and $S(t)$ be the $C_0$ semigroup generated by $A$ on $X \times X$ with $D(A) = D(A) \times D(A)$. If there exists a $\lambda$ such that the operator $[\lambda I - A^N_p]^{-1}$ is uniformly bounded for all $N$, then

$$S^N(t) \rightarrow S(t) \quad \text{uniformly on bounded t-intervals as } N \rightarrow \infty$$

where $S^N(t)$ is the $C_0$ semigroup generated by $A^N_p$ on $X \times X$ with $D(A^N_p) = D(A) \times X^N$.

Proof. Introduce

$$Q^N = \begin{bmatrix} \frac{1}{\gamma^N} DD^* \Pi & -BB^N \Pi^N \\ \Gamma N C^N \ast C & -(B^N B^N - \frac{1}{\gamma^N} D^N D^N) \Pi^N - \Gamma N C^N \ast C^N \end{bmatrix}$$

and

$$Q = \begin{bmatrix} \frac{1}{\gamma^N} DD^* \Pi & -BB^* \Pi \\ \Gamma C^* C & -(BB^* - \frac{1}{\gamma^N} DD^*) \Pi - \Gamma C^* C \end{bmatrix}$$

Let $A = A_1 + Q$ and $A^N_p = A^N_{p1} + Q^N$. Since $S^N(t) \rightarrow S(t)$ for $t \geq 0$ is equivalent to show that for Re$\lambda > -\omega$

(3.3.12) \[ R(\lambda : A^N_p) \rightarrow R(\lambda : A) \]

i.e.

$$R(\lambda : A^N_p) \overline{P}^N x \rightarrow R(\lambda : A) x, \quad \forall x \in X$$

where $\overline{P}^N : X \times X \rightarrow X \times X^N$ is given by

$$\overline{P}^N = \begin{bmatrix} I & 0 \\ 0 & P^N \end{bmatrix}$$
Hence we next claim (3.3.12). First it is not difficult to verify that

\[
[\lambda I - (A_{p1}^N + Q^N)]^{-1} - [\lambda I - A_{p1}^N]^{-1}
\]

\[
= [\lambda I - (A_{p1}^N + Q^N)]^{-1} Q^N [\lambda I - A_{p1}^N]^{-1}
\]

and

\[
[\lambda I - (A_1 + Q)]^{-1} - [\lambda I - A_1]^{-1} = [\lambda I - (A_1 + Q)]^{-1} Q [\lambda I - A_1]^{-1}
\]

Hence we have (3.3.13)

\[
[\lambda I - (A_{p1}^N + Q^N)]^{-1} \overline{P}^N - [\lambda I - (A_1 + Q)]^{-1}
\]

\[
= [\lambda I - (A_{p1}^N + Q^N)]^{-1} Q^N ([\lambda I - A_{p1}^N]^{-1} \overline{P}^N - [\lambda I - A_1]^{-1})
\]

\[
- [\lambda I - (A_{p1}^N + Q^N)]^{-1} (\overline{P}^N Q - Q^N) [\lambda I - A_1]^{-1}
\]

\[
+ ([\lambda I - (A_{p1}^N + Q^N)]^{-1} \overline{P}^N - [\lambda I - (A_1 + Q)]^{-1}) Q [\lambda I - A_1]^{-1}
\]

\[
+ [\lambda I - A_{p1}^N]^{-1} \overline{P}^N - [\lambda I - A_1]^{-1}
\]

Reorganizing (3.3.13) yields

\[
\left( [\lambda I - (A_{p1}^N + Q^N)]^{-1} \overline{P}^N - [\lambda I - (A_1 + Q)]^{-1} \right) (I - Q [\lambda I - A_1]^{-1})
\]

\[
= \left( I + [\lambda I - (A_{p1}^N + Q^N)]^{-1} Q^N \right) \left( [\lambda I - A_{p1}^N]^{-1} \overline{P}^N - [\lambda I - A_1]^{-1} \right)
\]

\[
- [\lambda I - (A_{p1}^N + Q^N)]^{-1} (\overline{P}^N Q - Q^N) [\lambda I - A_1]^{-1}
\]

\[
\left( I - Q [\lambda I - A_1]^{-1} \right) [\lambda I - A_1]^{-1} = [\lambda I - A_1] [\lambda I - (A_1 + Q)]^{-1}
\]

we know that \([I + Q(\lambda I - A_1)^{-1}]^{-1}\) is bounded by the Closed Graph Theorem (see Theorem IV.5.7 of [TL]). Therefore we arrive at

\[
[\lambda I - (A_{p1}^N + Q^N)]^{-1} \overline{P}^N - [\lambda I - (A_1 + Q)]^{-1}
\]
\[\begin{align*}
&= \left\{ \left( I - [\lambda I - (A_{p1}^N + Q^N)]^{-1}Q^N \right) \left( [\lambda I - A_{p1}^N]^{-1}P^N - [\lambda I - A_1]^{-1} \right) \\
&\quad - [\lambda I - (A_{p1}^N + Q^N)]^{-1}(P^N Q - Q^N)[\lambda I - A_1]^{-1} \right\}[\lambda I - A_1][\lambda I - (A_1 + Q)]^{-1}
\end{align*}\]

By assumption (A1), we have
\[\left[ \lambda I - A_{p1}^N \right]^{-1}P^N \rightarrow [\lambda I - A_1]^{-1}, \quad P^N Q - Q^N \rightarrow 0, \quad \text{strongly as } N \to \infty\]

Therefore (3.3.12) holds as \( N \to \infty \), and this completes the proof of Theorem 3.3.5. \( \square \)

**Remark 3.3.6.** One sufficient condition for \( [\lambda I - A_{p1}^N]^{-1} \) to be uniformly bounded in \( N \) is the assumption (A5). Clearly if \( A \) is an analytic (or compact) \( C_0 \) semigroup, we also can have that \( [\lambda I - A_{p1}^N]^{-1} \) is uniformly bounded in \( N \).

**Theorem 3.3.7.** Under the assumptions of Theorem 3.3.4, let \( A_{p1}^N \) be given as in 3.3.4.

Recall that
\[ A^N = \begin{bmatrix} A & BL^N \\ \Gamma C & A^N + M^N \end{bmatrix} \]

If \( A_{p1}^N \) generates a stable semigroup, then \( A^N \) also generates a stable semigroup on \( X \times X^N \).

**Proof. Step 1.** Consider the system
\begin{align*}
\dot{x}_h &= (A + \frac{1}{\gamma^2}DD^*\Pi)x_h - BB^N*\Pi^Np_k + D\omega \\
\dot{p}_k^N &= (A^N - (B^N B^N* - \frac{1}{\gamma^2}D^N D^N*)\Pi^N - \Gamma^N C^N* C^N)p_k^N + \Gamma^N C^N* C x_h + \Gamma^N C^N* \eta \\
x_h(0) &= h \\
p_k^N(0) &= k^N
\end{align*}

We claim that for a given \( 0 < \delta < \gamma^2 \), there exists \( N_\delta > 0 \) such that when \( N \geq N_\delta \) the following estimate holds
\begin{align*}
\int_0^\infty \left( \|Hx_h\|^2 + \|B^N*\Pi^N p_k^N\|^2 - \gamma^2(\|\omega + \frac{1}{\gamma^2}D^*\Pi x_h\|^2 + \|\eta\|^2) \right) dt
\end{align*}

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\[ \leq -\delta^2 \int_0^\infty (\|w\|^2 + \|\eta\|^2) dt + C_0(\|h\|^2 + \|k^N\|^2). \]

First we denote the solution of (3.3.13) with initial condition \((0, 0)\) by \((x_0, p_0^N)\). It is easy to see that by evaluating \(\frac{d}{dt}(\Pi x_0, x_0)\) and integrating from 0 to \(\infty\), we have the following equality

(3.3.16)

\[ \int_0^\infty \left( \|H x_0\|^2 + \|B^N \Pi^N p_0^N\|^2 - \|B^* \Pi x_0 - B^N \Pi^N p_0^N\|^2 \right. \\
\left. + \gamma^2 (\|w\|^2 - \|w + \frac{1}{\gamma^2} D^* \Pi x_0\|^2) \right) dt = 0 \]

We then consider the following coupled (optimal) system

(3.3.17)

\[ \dot{x} = (A + \frac{1}{\gamma^2} DD^* \Pi) \bar{x} - B B^* \bar{\rho} + Dw \]

\[ \dot{\rho} = (A - (B B^* - \frac{1}{\gamma^2} DD^*) \Pi - \Gamma C^* C) \bar{\rho} + \Gamma C^* C \bar{x} + \Gamma C^* \eta \]

\[ \bar{x}(0) = 0 \]

\[ \bar{\rho}(0) = 0 \]

Letting \(\xi = \bar{x} - \bar{\rho}\), we have

(3.3.18)

\[ \dot{\xi} = (A + \frac{1}{\gamma^2} DD^* \Pi - CC^* \Pi) \xi + Dw - \Gamma C^* \eta, \quad \xi(0) = 0 \]

Note that (3.3.17) is in fact the dual system of (3.2.6)-(3.2.7) in Lemma 3.2.9, and thus by the Duality Theorem we have

(3.3.19)

\[ \gamma_1^2 := \sup_{w, \eta} \frac{\int_0^\infty \|B^* \xi\|^2 dt}{\int_0^\infty (\|w\|^2 + \|\eta\|^2) dt} = \sup_{w, \eta} \frac{\int_0^\infty \|B^* \Pi (\bar{x}(t) - \bar{\rho}(t))\|^2 dt}{\int_0^\infty (\|w\|^2 + \|\eta\|^2) dt} < \gamma^2 \]

Since \((x_h, p_h^N) \in L^2(0, \infty; X) \times L^2(0, \infty; X^N)\) and \((x, p) \in L^2(0, \infty; X) \times L^2(0, \infty; X)\), according to Theorem 3.3.5, letting \(\varepsilon = (\gamma^2 - \gamma_1^2)/2\), there exists \(N' > 0\) such that when
\[ N > N' \]

\[ \int_0^\infty \| B^* \Pi x_0 - B^* \Pi x \|^2 dt < \varepsilon, \quad \int_0^\infty \| B^* \Pi p - B^N \Pi N p^N \|^2 dt < \varepsilon, \]

which implies that when \( N > N' \)

\[
\sup_{w, \eta} \frac{\int_0^\infty \| B^* \Pi x_0(t) - B^N \Pi N p_0^N(t) \|^2 dt}{\int_0^\infty (\| w \|^2 + \| \eta \|^2) dt} < \gamma^2
\]

Hence we can find \( \delta < \gamma \) and \( N_\delta > N' \) such that for \( N \geq N_\delta \) we have

\[
(3.3.20) \quad \sup_{w, \eta} \frac{\int_0^\infty \| B^* \Pi x_0(t) - B^N \Pi N p_0^N(t) \|^2 dt}{\int_0^\infty (\| w \|^2 + \| \eta \|^2) dt} \leq \gamma^2 - \delta^2
\]

Hence (3.3.15), (3.3.20) yields that

\[
(3.3.21) \quad \int_0^\infty \left( \| H x_0 \|^2 + \| B^N \Pi N p_0^N \|^2 - \gamma^2 (\| w - \frac{1}{\gamma^2} D^* \Pi x_0 \|^2 + \| \eta \|^2) \right) \leq -\delta^2 \int_0^\infty (\| w \|^2 + \| \eta \|^2) dt.
\]

Denote \( x_1 = x_h - x_0, p_1^N = p_k^N - p_0^N \). Note that \( (x_1, p_1^N) \) depends only on \( h, k \) and not on \( w, \eta \) and clearly

\[
(3.3.22) \quad \| x_1 \| \leq c_1 \| h \|, \quad \| p_1^N \| \leq c_2 \| k \|
\]

Therefore replacing \( (x_0, p_0^N) \) by \( (x_h - x_1, p_k^N - p_1^N) \) in (3.3.21), we arrive at (3.3.15).

**Step 2.** Introduce the system

\[
(3.3.23) \quad \dot{x}_h = A \dot{x}_h - B B^N \Pi N \dot{p}_k^N
\]

\[
\dot{p}_k^N = (A^N - (B^N B^N* - \frac{1}{\gamma^2} D^N D^N*)\Pi N - \Gamma^N C^N* C^N) \dot{p}_k^N + \Gamma^N C^N* C \dot{x}_h
\]

\[ \dot{x}_h(0) = h \]

\[ \dot{p}_k^N(0) = k \]
We shall prove that there exists $N_1 > 0$ such that for $N \geq N_1$ we have
\[
\hat{x}_h, \hat{p}_k^N \in L^2(0, +\infty; X).
\]

For any $T > 0$, we define
\[
x_T(t) = \begin{cases} \hat{x}_h(t) & \text{if } t \leq T \\ x_{\hat{x}_h, \hat{p}_k^N}(t - T) & \text{if } t > T \end{cases}
\]
and
\[
p_T^N(t) = \begin{cases} \hat{p}_k^N(t) & \text{if } t \leq T \\ p_{\hat{x}_h, \hat{p}_k^N}(t - T) & \text{if } t > T \end{cases}
\]
where $(x_{\hat{x}_h, \hat{p}_k^N}, p_{\hat{x}_h, \hat{p}_k^N})$ is the solution of
\begin{equation}
\tag{3.3.24}
\begin{align*}
\dot{x} &= (A + \frac{1}{\gamma^2} DD^* \Pi)x - BB^N \Pi^N p^N \\
p^N &= (A^N - (B^N B^N - \frac{1}{\gamma^2} D^N D^N \Pi) \Pi^N - \Gamma^N C^N \Pi^N) p^N + \Gamma^N C^N C x \\
x(T) &= \hat{x}_h(T) \\
p^N(T) &= \hat{p}_k^N(T)
\end{align*}
\end{equation}

If we let
\[
\omega_T(t) = \begin{cases} -\frac{1}{\gamma^2} D^* \Pi \hat{x}_h(t) & \text{if } t \leq T \\ 0 & \text{if } t > T \end{cases}
\]
we can see that the pair $(x_T(\cdot), p_T^N(\cdot))$ is the solution of the system (3.3.14) corresponding to the perturbation $w(\cdot) = \omega_T(\cdot)$ and $\eta = 0$. Hence for $N \geq N_\delta$ by applying (3.3.15) we have
\begin{equation}
\tag{3.3.25}
\int_0^\infty \left( \|H x_T\|^2 + \|B^N \Pi^N p_T^N\|^2 - \gamma^2 \|w_T\|^2 + \frac{1}{\gamma^2} D^* \Pi x_T\|^2 \right) dt \\
\leq -\delta^2 \int_0^\infty \|w\|^2 dt + C_0(\|h\|^2 + \|k\|^2).
\end{equation}
Note that by the definition of $w_T$ we have

$$\int_T^\infty (\|HX_T\|^2 + \|B^*\Pi^N p_T^N\|^2 - \gamma^2 \|w_T + \frac{1}{\gamma^2} D^* \Pi x_T\|^2) dt$$

$$= \int_T^\infty (\|HX_{\hat{x}_T, \hat{p}_T^N}\|^2 + \|B^*\Pi^N p_{\hat{x}_T, \hat{p}_T^N}\|^2 - \gamma^2 \|\frac{1}{\gamma^2} D^* p_{\hat{x}_T, \hat{p}_T^N}\|^2) dt.$$ 

On the other hand, making use of (3.1.8) yields

(3.3.26)

$$\int_T^\infty (\|HX_T\|^2 + \|B^*\Pi^N p_T^N\|^2 - \gamma^2 \|w_T + \frac{1}{\gamma^2} D^* \Pi x_T\|^2) dt$$

$$= (\Pi \hat{x}_T, \hat{x}_T) + \int_T^\infty \|B^*\Pi x_{\hat{x}_T, \hat{p}_T^N} - B^*\Pi^N p_{\hat{x}_T, \hat{p}_T^N}\|^2 dt.$$ 

Therefore we deduce from (3.3.25) and (3.3.26) the estimate

(3.3.27)

$$\int_0^\infty (\|HX_T\|^2 + \|B^*\Pi^N p_T^N\|^2 - \gamma^2 \|w_T + \frac{1}{\gamma^2} D^* \Pi x_T\|^2) dt$$

$$= \int_0^T (\|HX_T\|^2 + \|B^*\Pi^N p_T^N\|^2) dt + (\Pi \hat{x}_T, \hat{x}_T)$$

$$+ \int_0^T \|B^*\Pi x_{\hat{x}_T, \hat{p}_T^N} - B^*\Pi^N p_{\hat{x}_T, \hat{p}_T^N}\|^2 dt$$

$$< -\delta^2 \int_0^\infty \|w_T\|^2 dt + C_0(\|h\|^2 + \|k\|^2)$$

(3.3.27) yields

$$\int_0^T \|D^* \Pi \hat{x}_h\|^2 dt \leq C_1(\|h\|^2 + \|k\|^2).$$

By letting $T$ go to infinity, we have $D^* \Pi \hat{x}_h \in L^2(0, +\infty; W)$. Note that $(\hat{x}_h, \hat{p}_k^N)$ appears as the solution of (3.3.14) with

$$w = -\frac{1}{\gamma^2} D^* \Pi \hat{x}_h, \text{ and } \eta = 0$$

Thus we have $\hat{x}_h, \hat{p}_k^N \in L^2(0, +\infty; X)$, and according to the Datko's Theorem (see Datko [Da]) $A^N$ is exponentially stable. This completes the proof of Theorem 3.3.7. \( \square \)
Corollary 3.3.8. For any choice of the matrices $A^N, L^N, G^N, M^N$, if $A_p^N$ defined in Theorem 3.3.4 satisfies the spectrum-determined growth condition, then there exists $N' > 0$ such that when $N > N'$, $A^N$ which is defined in Theorem 3.3.7 generates an exponentially stable semigroup.

Proof. The proof is quite straightforward. Note that according to Theorem 3.3.4, there exists an $N' > 0$ such that when $N > N'$ we have

$$\sigma(A_p^N) \subseteq (-\infty, -\varepsilon]$$

for some $\varepsilon > 0$ which is independent of $N$, and $A_p^N$ satisfies the spectrum-determined growth condition. Thus the growth constant $\omega_p$:

$$\omega_p = \lim_{t \to \infty} \frac{1}{t} \ln \|e^{A_p^N t}\|$$

is negative, and hence $A_p^N$ generates an exponentially stable $C_0$ semigroup. By Theorem 3.3.7, we have that when $N > N'$, $A^N$ is exponentially stable. \qed

Equipped with all the results above, we are now in a position to prove the main result stated at the beginning of this section.

Proof of Main Result. Let us view the system and its finite-dimensional compensator together as a coupled system, i.e.

\begin{align*}
\dot{x} &= Ax - BB^N\Pi^N p^N + Dw \\
\dot{p}^N &= (A^N - (B^N B^N* - \frac{1}{\gamma^2} D^N D^N*)\Pi^N - \Gamma^N C^N* C^N)p^N + \Gamma^N C^N* C x + \Gamma^N C^N* \eta
\end{align*}

$x(0) = 0$

$p^N(0) = 0$
By Corollary 3.3.8, there exists \( N_1 > 0 \) such that when \( N > N_1 \), the matrix operator
\[
\mathcal{A}^N = \begin{bmatrix} A & B L^N \\ G^N C & A^N + M^N \end{bmatrix}
\]
generates an exponentially stable \( C_0 \) semigroup; thus the coupled system (3.3.28)–(3.3.29) is an exponentially stable system. Let \( \dot{w} = w - \frac{1}{\gamma^2} D D^* \Pi x, \dot{w} \in L^2(0, \infty; W) \); then there exists \( N_2 \) such that \((x, p^N)\) satisfies (3.3.14) with \( h = 0, k = 0 \) for \( N > N_2 \), i.e.
\[
(3.3.30) \quad \int_0^\infty \left( \| H x \|^2 + \| B^N \Pi^N p^N \|^2 - \gamma^2 (\| \dot{w} \|^2 + \| \eta \|^2) \right) dt \\
\leq -\delta^2 \int_0^\infty (\| \dot{w} - \frac{1}{\gamma^2} D^* \Pi x \|^2 + \| \eta \|^2) dt
\]
which is equivalent to
\[
\sup_{\dot{w}, \eta} \frac{\int_0^\infty (\| H x \|^2 + \| B^N \Pi^N p^N \|^2) dt}{\int_0^\infty (\| \dot{w} \|^2 + \| \eta \|^2) dt} \\
\leq \gamma^2 - \delta^2 \inf_{\dot{w}, \eta} \frac{\int_0^\infty (\| \dot{w} \|^2 + \| \dot{w} - \frac{1}{\gamma^2} D^* \Pi x \|^2) dt}{\int_0^\infty (\| \dot{w} \|^2 + \| \eta \|^2) dt}
\]
Since \( \mathcal{A}_p^N \) is exponentially stable, we immediately have
\[
\inf_{\dot{w}, \eta} \frac{\int_0^\infty (\| \eta \|^2 + \| w - \frac{1}{\gamma^2} D^* \Pi x \|^2) dt}{\int_0^\infty (\| \dot{w} \|^2 + \| \eta \|^2) dt} > 0.
\]
Therefore (3.1.21) holds and the proof is complete. \( \Box \)

3.4. Example

Consider the scalar delay system
\[
(3.4.1) \quad \dot{x}(t) = ax(t) + bx(t - h) + u(t) + w(t), \quad t \geq 0
\]
\[x(0) = r \]
\[x(\theta) = f(\theta), \quad -h \leq \theta < 0\]
where $a, b, r$ and $h \in \mathbb{R}$ with $h > 0$, $f \in L^2([-h, 0]; \mathbb{R})$, $u, w \in L^2([0, \tau]; \mathbb{R})$ for every $\tau > 0$. Following Fabiano [Fab], by introducing the Hilbert space $X = M_2([-h, 0]; \mathbb{R}) := \mathbb{R} \oplus L^2([-h, 0]; \mathbb{R})$ with the inner product:

\[(3.4.2) \quad \left( \begin{bmatrix} r_1 \\ f_1 \end{bmatrix}, \begin{bmatrix} r_2 \\ f_2 \end{bmatrix} \right)_X := r_1 r_2 + |b| e^{-\xi r} \int_{-h}^{0} e^{-2\xi \theta} f_1(\theta) f_2(\theta) d\theta\]

where $\xi \in \mathbb{R}$, and letting the state $z(t) := [x(t), x(t + \cdot)]^T$, (4.1) can be formulated as an abstract system:

\[\dot{z}(t) = Az(t) + Bu + Dw, \quad z(0) = z_0 = [r, f(\cdot)]^T\]

where

\[A = \begin{bmatrix} r \\ f(\cdot) \end{bmatrix} = \begin{bmatrix} ar + bf(-h) \\ \frac{df}{d\theta}(\cdot) \end{bmatrix}, \quad B = D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\]

For simplicity, let the noise-corrupted measurement be $y(t) = x(t) + \eta(t)$, and the cost function be

\[K_{z_0} = \int_0^\infty (\|x\|^2 + \|u\|^2) dt\]

Note that in this case $B = D$ and $(A, D)$ is stabilizable and $(A, H)$ is detectable, and thus the Riccati equation (1.8)

\[(3.4.3) \quad \Pi A + A^* \Pi - \Pi (BB^* - \frac{1}{\gamma^2} DD^*) \Pi + H^* H = 0\]

where

\[H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\]

has a unique solution in $\Sigma^+(X)$ when $\gamma > 1^6$.

\^6$\gamma$ can in fact be pushed to lower values, but the precise bound (say $\gamma_c$) is difficult to compute for this problem. If a $\gamma \in (\gamma_c, 1)$ is picked, then the solution will not be unique even in $\Sigma^+(X)$, and we will have choose the minimal such solution.

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Now we use spline-based Galerkin approximation to construct the finite-dimensional compensator. Let $\theta_j^N = -j h / N$ for $j = 0, 1, \ldots, N$ be a partition of $[-h, 0]$. Define a piecewise linear function as follows

\[
e_0^N(\theta) = \begin{cases} \frac{N}{h}(\theta - \theta_1^N) & \text{if } \theta_1^N \leq \theta \leq 0 \\ 0 & \text{elsewhere} \end{cases}
\]

\[
e_N^N(\theta) = \begin{cases} -\frac{N}{h}(\theta - \theta_{N-1}^N) & \text{if } \theta_{N-1}^N \leq \theta \leq \theta_N^N \\ 0 & \text{elsewhere} \end{cases}
\]

and for $j = 1, 2, \ldots, N - 1$,

\[
e_j^N(\theta) = \begin{cases} \frac{N}{h}(\theta - \theta_{j-1}^N) & \text{if } \theta_{j-1}^N \leq \theta \leq \theta_j^N \\ \frac{N}{h}(\theta - \theta_{j+1}^N) & \text{if } \theta_{j+1}^N \leq \theta \leq \theta_j^N \\ 0 & \text{elsewhere} \end{cases}
\]

Let $\varphi_0^N = (1, e_0^N)$, $\varphi_j^N = (0, e_j^N)$ for $j = 1, 2, \ldots, N$. Define

\[X^N = \text{span}\{\varphi_0^N, \varphi_1^N, \ldots, \varphi_N^N\}\]

and space $V = \text{dom}(A)$ equipped with the graph norm. For $x = (\phi(0), \phi), y = (\psi(0), \psi) \in V$, define the bilinear form $\sigma : V \times V \to \mathbb{R}$ by

\[
\sigma(x, y) = [a\phi(0) + b\phi(-h)]\psi(0) + [b]e^{2s\theta} \int_{-h}^0 e^{-2\xi \theta} \phi'(\theta) \psi(\theta) d\theta
\]

Let $P^N : X \to X^N$ denote the orthogonal projection of $X$ onto $X^N$, and define $A^N : X^N \to X^N$ by

\[(A^N x, y) = \sigma(x, y) \quad \forall x, y \in X^N\]

Next we claim that

\[e^{A^N t} P^N x \to e^{At} x \quad \text{uniformly in bounded } t\text{-intervals} \quad \forall x \in X\]

It follows from [Fab] that, for any given $\psi \in H^3(-h, 0)$, letting $\psi_j(\theta) = \psi(\theta_j), j = 0, \ldots, N$, we have

\[\|\psi - \psi_j^N\|_{L^2(-h, 0)} \leq O\left(\frac{1}{N^2}\right), \quad \|\frac{d}{d\theta}(\psi - \psi_j^N)\|_{L^2(-h, 0)} \leq O\left(\frac{1}{N}\right)\]

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Let \( y = (\psi(0), \psi) \) and \( y^N = (\psi(0), \psi^N_I) \). Then for any \( x = (\phi(0), \phi) \in \text{dom}(A) \) we have

\[
|\sigma(x, y - y^N)| \leq c\|x\|_X \|y - y^N\|_Y \to 0
\]

It follows from Ito, Kappel and Salamon [IKS] that \( e^{A^N t} P^N x \to e^{A t} x \). Moreover the resolvent of \( A, R(\lambda; A) = [\lambda I - A]^{-1} \) is compact, and thus assumption (A1) and (A4) are satisfied.

The approximate Riccati equations are given by

\[
(3.4.4)
\Pi^N A^N + A^N \Pi^N - \Pi^N (B^N B^N* - \frac{1}{\gamma^2} D^N D^N*) \Pi^N + H^N* H^N = 0
\]

It follows from Ito's work (see Lemma 4.3 and Lemma 4.4 of [It]) that since \( (A, D) \) (again note here \( B = D \)) is stabilizable and \( (A, H) \) is detectable, for sufficiently large \( N, A^N, B^N \) is also stabilizable and \( A^N, H^N \) is also detectable. Thus for \( \gamma > 1 \) (3.4.4) has a unique solution for sufficient large \( N \). Moreover \( A^N - (B^N B^N* - \frac{1}{\gamma^2} D^N D^N*) \Pi^N \) is uniformly exponentially stable. Thus assumption (A2) is satisfied. By the Duality Theorem we can verify that assumption (A3) is fulfilled, in which the Riccati equation is

\[
\Sigma A^* + A \Sigma - \Sigma (C^* C - \frac{1}{\gamma^2} H^* H) \Sigma + D D^* = 0
\]

and its approximation is

\[
\Sigma^N A^N* + A^N \Sigma^N - \Sigma^N (C^N* C^N - \frac{1}{\gamma^2} H^N* H^N) \Sigma^N + D^N D^N* = 0
\]

Therefore the finite-dimensional compensator in \( X^N = \text{span}\{\varphi^N_j\} \) can be constructed as

\[
u^N(t) = -B^N* \Pi^N p^N(t)
\]

\[
\dot{p}^N = (A^N - (B^N B^N* + \frac{1}{\gamma^2} D^N D^N*) \Pi^N) p^N + \Sigma^N (I - \frac{1}{\gamma^2} \Pi^N \Sigma^N)^{-1} C^N* x(t)
\]
By the main result of the previous section, for any $\gamma > 1$ there exists a sufficiently large $N$, such that the $\gamma$-robustness property holds, i.e.

$$
\sup_{w, \eta} \frac{K_0(u^N, w)}{\int_0^\infty (\|w\|^2 + \|\eta\|^2) dt} < \gamma^2
$$

3.5. Concluding Remarks

In this chapter we have shown the existence of finite-dimensional compensators in the $H^\infty$-optimal control of infinite-dimensional systems by using a Galerkin-type approximation. This result automatically covers the case when the system structure operator $A$ has a complete set of generalized eigenvectors such as $A$ being a Riesz-spectral operator$^7$. In order to construct such a finite-dimensional compensator, we required that there exist an infinite-dimensional compensator with the $\gamma$-robustness property. This is quite natural because one cannot expect a finite-dimensional compensator to be able to achieve the $\gamma$-robustness property if there is no infinite-dimensional compensator to do so. From the proofs we can see that the finite-dimensional controller in fact converges to the infinite-dimensional controller as its order increases, and it can achieve the same $\gamma$-robustness as the infinite-dimensional controller as long as $\gamma > \gamma_c$, where $\gamma_c$ is the optimum level of disturbance attenuation for the system, which can be defined in a similar way as in the finite-dimensional case. The degree of the finite-dimensional controller depends on how much error the system can tolerate, which is measured by the difference $\gamma - \gamma_c$.

---

$^7$A is a Riesz-spectral operator means that it has simple eigenvalues and the corresponding eigenvectors form a Riesz basis: A sequence of vectors $\{\phi_n, n \geq 1\}$ in a Hilbert space $X$ such that (1) $\text{span}\{\phi_n\} = X$, (2) there exist constants $m, M$ such that $\forall N$

$$
m \sum_{n=1}^N ||\alpha_n||^2 \leq ||\sum_{n=1}^N \phi_n||^2 \leq M \sum_{n=1}^N ||\alpha_n||^2.
$$
The approach used in this chapter can be extended to the case when the finite-dimensional controller is only allowed to act on the boundary. Such an extension as well as outcomes of some numerical experiments will be reported elsewhere.

3.6. Appendix I

**Duality Theorem.** Consider Hilbert spaces $X, Y, Z$, identified with their duals. Let $A$ be the infinitesimal generator of a $C_0$ semigroup in $X$, and $B \in \mathcal{L}(Y, X), C \in \mathcal{L}(X, Z)$. With the triple $(A, B, C)$, associate a linear system

$$
\dot{x} = Ax + Bu; \quad x(0) = 0
$$

and a corresponding observation $Cx$. The system dual to this is defined as

$$
\dot{\xi} = A^*\xi + C^*v; \quad \xi(0) = 0
$$

and with a corresponding observation $B^*\xi$. Assume that $A$ is exponentially stable. Then we have

$$
\sup_u \int_0^\infty |Cx|^2 \, dt = \sup_{v} \int_0^\infty |B^*\xi|^2 \, dt
$$

*Proof.* See Bensoussan and Bernhard [BeBe].

3.7. Appendix II

We provide here a proof for Lemma 3.2.7, which is in the form of a sequence of propositions.

**Proposition 3.7.1.** The Riccati equation

$$(3.7.1) \quad \Pi A + A^*\Pi - \Pi(BB^* - \frac{1}{\gamma^2}DD^*)\Pi + H^*H = 0$$
has at most one solution \( \Pi \in \Sigma^+(X) \) such that \( A - (BB^* - \frac{1}{\gamma} DD^*)\Pi \) is exponentially stable.

**Proof.** Let \( \Pi_1, \Pi_2 \in \Sigma^+(X) \) be two solutions of (3.7.1) such that both \( A - (BB^* - \frac{1}{\gamma} DD^*)\Pi_1 \) and \( A - (BB^* - \frac{1}{\gamma} DD^*)\Pi_2 \) are exponentially stable. Consider the system

\[
\dot{x} = Ax + Bu + Dw, \quad x(0) = x_0
\]

Since \( \Pi_1, \Pi_2 \) satisfy (3.7.1), we have

\[
\frac{d}{dt} = (\Pi_1 - \Pi_2)x_0, x_0 >
= \gamma^2(\|w - \frac{1}{\gamma} D^*\Pi_2 x\|^2 - \|w - \frac{1}{\gamma} D^*\Pi_1 x\|^2) + \|u + B^*\Pi_2 x\|^2 - \|u + B^*\Pi_1 x\|^2
\]

By integrating this between 0 and \( T \), we obtain

\[
< (\Pi_1 - \Pi_2)x(T), x(T) > - < (\Pi_1 - \Pi_2)x_0, x_0 >
= \int_0^T \{ \gamma^2(\|w - \frac{1}{\gamma} D^*\Pi_1 x\|^2 - \|w - \frac{1}{\gamma} D^*\Pi_2 x\|^2) + \|u + B^*\Pi_2 x\|^2 - \|u + B^*\Pi_1 x\|^2 \} dt
\]

Setting \( u = -B^*(\frac{\Pi_1 + \Pi_2}{2}) x, w = \frac{1}{\gamma} D^* (\frac{\Pi_1 + \Pi_2}{2}) x \), we have

(3.7.2) \n < (\Pi_1 - \Pi_2)x_0, x_0 > = < (\Pi_1 - \Pi_2)x(T), x(T) >

Next we claim that the following system

(3.7.3) \n \dot{x} = Ax - BB^*(\frac{\Pi_1 + \Pi_2}{2}) x + \frac{1}{\gamma} DD^*(\frac{\Pi_1 + \Pi_2}{2}) x, \quad x(0) = x_0

is exponentially stable. Let \( \mathcal{A}_1 = \frac{1}{2}(A - BB^*\Pi_1 + \frac{1}{\gamma} DD^*\Pi_1), \mathcal{A}_2 = \frac{1}{2}(A - BB^*\Pi_2 + \frac{1}{\gamma} DD^*\Pi_2) \). Note both \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are exponentially stable. Thus by the generalized
Lyapunov's Theorem [Da] there exist two (unique) positive definite, self-adjoint bounded operators $P_1, P_2$ such that

$$< P_1 A_1 x, x > + < x, P_1 A_1 x > = -< x, x > \quad \forall x \in D(A)$$

and

$$< P_2 A_1 x, x > + < x, P_2 A_1 x > = -< x, x > \quad \forall x \in D(A)$$

It is known that $< x, y >_{e_1} := < x, P_1 y >, < x, y >_{e_2} := < x, P_2 y >$ define two equivalent inner products on $X$ such that

$$\alpha_1 \| x \|^2_X \leq \| x \|^2_{e_1} \leq \beta_1 \| x \|^2_X, \quad \alpha_1 > 0$$

$$\alpha_2 \| x \|^2_X \leq \| x \|^2_{e_2} \leq \beta_2 \| x \|^2_X, \quad \alpha_2 > 0$$

Thus from (2.3), for $x_0 \in D(A)$ we have

$$\frac{1}{2} \frac{d}{dt} x^2 = < \dot{x}, x > + < x, \dot{x} >$$

$$= < A_1 x, x > + < x, A_1 x > + < A_2 x, x > + < x, A_2 x >$$

$$\leq \alpha_1^{-1} (< A_1 x, x >_{e_1} + < x, A_1 x >_{e_1}) + \alpha_2^{-1} (< A_2 x, x >_{e_2} + < x, A_2 x >_{e_2})$$

$$= - (\alpha_1^{-1} + \alpha_2^{-1}) < x, x >$$

which implies that $x \in L^2(0, \infty; X)$. Hence (2.2) is exponentially stable.

Now in (2.2) letting $T \to \infty$ yields

$$< (\Pi_1 - \Pi_2) x_0, x_0 > = 0, \quad \forall x_0 \in X$$

which implies $\Pi_1 = \Pi_2$. □

A similar reasoning as above leads to the following result.
Proposition 3.7.2. The Riccati equation

\[ \Sigma A^* + A \Sigma - \Sigma (C^* C - \frac{1}{\gamma^2} H^* H) \Sigma + DD^* = 0 \]

has at most one positive solution \( \Sigma \in \Sigma^+(X) \) such that \( A^* - (C^* C - \frac{1}{\gamma^2} H^* H) \Sigma \) is exponentially stable.

Proposition 3.7.3. Suppose that the Riccati equation

(3.7.4) \[ \Pi A + A^* \Pi - \Pi (BB^* - \frac{1}{\gamma^2} DD^*) \Pi + H^* H = 0 \]

has a solution \( \Pi \in \Sigma^+(X) \) such that \( A - (BB^* - \frac{1}{\gamma^2} DD^*) \Pi \) is exponentially stable and that \( (A, H) \) is detectable. Then for any \( t_f > 0 \), the Riccati equation

(3.7.5) \[ \dot{\Pi}(t) + \Pi(t) A + A^* \Pi(t) - \Pi(t)(BB^* - \frac{1}{\gamma^2} DD^*) \Pi(t) + H^* H = 0 \]
\[ \Pi(t_f) = 0 \]

has a unique solution \( \Pi(t; t_f) \in \Sigma^+(X) \) on \([0, t_f]\) with property that \( \Pi(t; t_f) \leq \Pi \) and for all \( x \in X \) the limit

\[ \Pi x = \lim_{t_f \to \infty} \Pi(t; t_f)x \]

exists.

Proof. Consider the following optimization problem

\[ \varphi(s, x_0) = \sup_w \inf_u J_\gamma(x_0; u, w) = \sup_w \inf_u \int_s^{t_f} (\|Hx\|^2 + \|Bu\|^2 - \gamma^2 \|w\|) \, dt \]

where \( x \) is subject to

(3.7.6) \[ \dot{x}(s) = Ax(s) + Bu(s) + Dw(s), \quad x(t) = x_0 \]
Note that the conditions of this proposition guarantee that \( \varphi(t, x_0) \) is finite. According to Theorem 4.1 [XB1], we know that

\[
\varphi(t, x_0) = J_\gamma(x_0; u^*, w^*)
\]

where \( u^* = B^*p, w^* = -\frac{1}{\gamma}D^*p \) is subject to

\[
\begin{align*}
\dot{x} &= Ax + BB^*p - \frac{1}{\gamma^2}DD^*p, \quad x(t) = x_0 \\
\dot{p} &= -A^*p + H^*Hx, \quad p(t_f) = 0
\end{align*}
\]

Thus the operator \( x_0 \to -p = \partial \varphi(t, x_0) \) is linear and therefore self-adjoint on \( X \) (see example 2 in Chap. 2 of [BP1]). Moreover

\[
\varphi(t, x) = \langle P(t)x, x \rangle \quad \text{for} \quad x \in X, P(t) = \partial \varphi(t).
\]

Making use of (3.7.7) and (3.7.9), we have

\[
\frac{d}{ds} \langle P(s)x(s), x(s) \rangle = -\langle H^*Hx(s), x(s) \rangle - \langle P(s)BB^*P(s)x(s), x(s) \rangle \\
+ \frac{1}{\gamma^2}DD^*P(s)x(s), x(s) \rangle
\]

On the other hand from (3.7.8)

\[
\frac{d}{ds} \langle P(s)x(s), x(s) \rangle \\
= \langle \dot{P}(s)x(s), x(s) \rangle + \langle Ax(s), x(s) \rangle + \langle A^*x(s), x(s) \rangle \\
+ 2 \langle P(s)BB^*P(s)x(s), x(s) \rangle - \frac{2}{\gamma^2} \langle DD^*P(s)x(s), x(s) \rangle
\]

Therefore from the last two equalities we obtain

\[
\begin{align*}
\dot{P}(t) + P(t)A + A^*P(t) - P(t)(BB^* - \frac{1}{\gamma^2}DD^*)P(t) + H^*H &= 0 \\
P(t_f) &= 0
\end{align*}
\]
Thus, $P(t)$ solves (3.7.5). Let $\Pi(t; t_f) = P(t)$. From above discussion we have

\begin{equation}
< \Pi(0; t_f) x_0, x_0 > = \sup_w \inf_u J_{\gamma}^{t_f}(x_0; u, w).
\end{equation}

Note that $< \Pi(0; t_f) x_0, x_0 >$ is monotonically nondecreasing with increasing $t_f$, since the lower value of the game $J_{\gamma}^{t_f}(x_0; \cdot, \cdot)$ defined on $[0, t_f]$ cannot be larger than that of the one defined on a longer interval, $[0, t'_f]$, $t'_f > t_f$, as the maximizing player can always play zero control on the subinterval $[t_f, t'_f]$. Let $\hat{w} \in \mathcal{W}$ be picked to be zero beyond $t = t_f$.

Denoting all admissible feedback controls for (2.6) by $\mathcal{M}$, we have the following estimate:

\[< \Pi(0; t_f) x_0, x_0 > = \sup_w \inf_u J_{\gamma}^{t_f}(x_0; u, w) \]
\[\leq \inf_{\mu \in \mathcal{M}} \sup_w J_{\gamma}^{\infty}(x_0; u, \hat{w}) \]
\[\leq \inf_{\mu \in \mathcal{M}} \sup_w J_{\gamma}^{\infty}(x_0; u, w) = < \Pi x_0, x_0 > \]

By the Parallelogram Law in Hilbert space, we know that the limit $\lim_{t \to \infty} < \Pi^N(t) x, y >$ exists for any $x, y \in X$. Thus we can define a $R \in \Sigma^+(X)$ such that

\[\lim_{t \to \infty} < \Pi(t) x, y > = < Rx, y > \]

It can be verified by employing standard argument that $R$ in fact solves (2.4). Next we claim that $R = \Pi$, i.e, $\Pi(t; t_f)x \to \Pi x$ for all $x \in X$.

Since $A - (BB^* - \frac{1}{\gamma^2}DD^*) \Pi := K$ is exponentially stable, there exist $M > 0, \beta > 0$ such that

\[\|e^{Kt}\| \leq Me^{-\beta t} \]

Set $\Xi(t) = \Pi - \Pi(t; t_f)$; then $\Xi$ is the mild solution to the problem

\[\dot{\Xi} + K^* \Xi + \Xi K + \Xi(BB^* - \frac{1}{\gamma^2}DD^*) \Xi = 0,\]
\[\Xi(t_f) = 0\]

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which is equivalent to the equation

\[(\Pi - \Pi(t))x = \exp[(t_f-t)K^*] \Pi \exp[(t_f-t)K]x
+ \int_t^{t_f} \exp[(s-t)K^*] \Xi(s)(BB^* - \frac{1}{\gamma^2} DD^*) \Xi(s) \exp[(s-t)K]x \, ds\]

for any \(x \in X\). Since for any \(x \in X\), \(\{\Pi(t; t_f)x\}\) is bounded, by Uniform Boundedness Theorem, there exists a \(C > 0\) such that

\[(3.7.13) \quad \|\Pi(t; t_f)\| \leq C \quad \text{for} \quad 0 \leq t \leq t_f \leq \infty\]

Therefore \((3.7.12)\) and \((3.7.14)\) gives

\[\| (\Pi - \Pi(t; t_f))x \| \leq M^2 e^{-2\beta(t_f-t)} \|\Pi\| \|x\|
+ M^2 \int_t^{t_f} \exp[-\beta(s-t)] \| (\Pi - \Pi(t; t_f))x \| \|BB^* - \frac{1}{\gamma^2} DD^*\| (\|\Pi\| + C) e^{-\beta(s-t)} \, ds\]

By employing Gronwall's inequality, we obtain

\[\| (\Pi - \Pi(t; t_f))x \|
\leq M^2 e^{-2\beta(t_f-t)} \|\Pi\| \|x\| \exp(M^2 \int_t^{t_f} \|BB^* - \frac{1}{\gamma^2} DD^*\| (\|\Pi\| + C) e^{-\beta(s-t)} \, ds)
\leq M^2 e^{-2\beta(t_f-t)} \|\Pi\| \|x\| \exp(M^2 \|BB^* - \frac{1}{\gamma^2} DD^*\| (\|\Pi\| + C)/\beta)\]

which completes the proof. \(\square\)

Similarly, we have

**Proposition 3.7.4.** Suppose that the Riccati equation

\[\Pi^N A^N + A^N \Pi^N - \Pi^N (B^N B^N - \frac{1}{\gamma^2} D^N D^N^*) \Pi^N + H^N H^N = 0\]
has a symmetric positive definite solution $\Pi^N$ such that $A^N - B^N B^{N^*} + \frac{1}{\gamma^2} D^N D^{N^*}$ is exponentially stable. Then for any $t_f > 0$, the Riccati equation

$$\dot{\Pi}^N(t) + \Pi^N(t) A^N + A^{N^*} \Pi^N(t) - \Pi^N(t) (B^N B^{N^*} - \frac{1}{\gamma^2} D^N D^{N^*}) \Pi^N(t) + H^{N^*} H^N = 0$$

$$\Pi^N(t_f) = 0$$

has a unique symmetric positive definite solution $\Pi^N(t)$ with property that $\Pi^N(t) \leq \Pi$ for $t \in [0, \infty)$, and for all $x \in X$ the limit

$$\Pi^N x = \lim_{t \to \infty} \Pi^N(t)x$$

exists.

**Proposition 3.7.5.** Suppose that both Riccati equations

(3.7.14)

$$\Pi^N A^N + A^{N^*} \Pi^N - \Pi^N (B^N B^{N^*} - \frac{1}{\gamma^2} D^N D^{N^*}) \Pi^N + H^{N^*} H^N = 0 \quad N = 1, 2, \ldots,$$

and

(3.7.15)

$$\Pi A + A^{*} \Pi - \Pi (B B^{*} - \frac{1}{\gamma^2} D D^{*}) \Pi + H^{*} H = 0$$

have solutions $\Pi^N \in \Sigma^+(X^N)$, $\Pi \in \Sigma^+(X)$ such that $A^N - (B^N B^{N^*} - \frac{1}{\gamma^2} D^N D^{N^*}) \Pi^N$ and $A - (B B^{*} - \frac{1}{\gamma^2} D D^{*}) \Pi$ are both exponentially stable. For any $t_f > 0$ let $\Pi^N(t)$, $\Pi(t)$ be the solutions of the Riccati equations

$$\dot{\Pi}^N(t) + \Pi^N(t) A^N + A^{N^*} \Pi^N(t) - \Pi^N(t) (B^N B^{N^*} - \frac{1}{\gamma^2} D^N D^{N^*}) \Pi^N(t) + H^{N^*} H^N = 0$$

$$\Pi^N(t_f) = 0 \quad N = 1, 2, \ldots,$$

and

$$\dot{\Pi}(t) + \Pi(t) A + A^{*} \Pi(t) - \Pi(t) (B B^{*} - \frac{1}{\gamma^2} D D^{*}) \Pi(t) + H^{*} H = 0$$

$$\Pi(t_f) = 0$$

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respectively. Then

$$\Pi^N(t)x \to \Pi(t) \quad \text{uniformly in } t \text{ on } [0, t_f] \quad \text{as } N \to \infty$$

Proof. Consider the optimization problem

$$\sup_w \inf_u J^N_t(x_0^N; u, w) = \sup_w \inf_u \int_0^t (\|H^N x^N\|^2 + \|u\|^2 - \gamma^2 \|w\|^2) dt$$

where $x^N$ is subject to

$$\begin{align*}
\dot{x}^N &= A^N x + B^N u + D^N w \\
x^N(0) &= x_0^N
\end{align*}$$

(3.7.16) is equivalent to

$$x^N(t) = e^{A^N t}x_0^N + \int_0^t e^{A^N (t-s)}(B^N u(s) + D^N w(s))ds$$

Let us denote $L_2(0, t_f; X^N)$ by $X^N$, $L_2(0, t_f; U)$ by $U$, and $L_2(0, t_f; W)$ by $W$. Define $\mathcal{F}^N \in \mathcal{L}(X^N, X^N)$ by

$$(\mathcal{F}^N \phi)(s) = \int_0^t e^{A^N (t-s)}\phi(s)ds, \quad \phi \in X^N$$

By identifying $X^N, U$ with respective duals, it is easy to verify that

$$(\mathcal{F}^{N*} \phi)(s) = \int_t^U e^{A^{N*} (s-t)}\phi(s)ds, \quad \phi \in X^N$$

where $\mathcal{F}^{N*} \in \mathcal{L}(X^N, X^N)$. Next we write the cost function $J^N_\gamma$ as

(3.7.17)

$$J^N_\gamma (x_0^N; u, w) = \|H^N(e^{A^N}x_0^N + \mathcal{F}^N(B^N u + D^N w))\|_{X^N}^2 + \|u\|^2_U - \gamma^2 \|w\|^2_W$$

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For arbitrary but fixed \( w \in \mathcal{W} \), there exists a unique \( u \) which minimizes \( J^N_\gamma \) and this \( u \) is the unique solution of

\[
\delta_u J^N_\gamma(x^N_0; u, w)(v) = 0, \quad \forall v \in \mathcal{U}
\]

where \( \delta_u J^N_\gamma(x^N_0; u, w)(v) \) means the Gâteaux derivative of \( J^N_\gamma \) at \( u \), applied to \( v \). From the cost function (3.7.17), we have

\[
\begin{align*}
\delta J^N_\gamma(x^N_0; u, w)(v) &= 2 < H^N \ast H^N (e^{A^N} \cdot x^N_0 + \mathcal{F}(B^N u + D^N w)), \mathcal{F} B^N v >_{\mathcal{X}^N} + 2 < u, v >_u \\
&= 2 < T^N_1 u + T^N_2 w + T^N_3 x^N_0, v >
\end{align*}
\]

where

\[
T^N_1 = I + B^N \ast \mathcal{F} \ast H^N \ast H^N \mathcal{F} B^N \in \mathcal{L}(\mathcal{U}, \mathcal{U})
\]

\[
T^N_2 = B^N \ast \mathcal{F} \ast H^N \ast H^N \mathcal{F} D^N \in \mathcal{L}(\mathcal{W}, \mathcal{U})
\]

and

\[
T^N_3 = B^N \ast \mathcal{F} \ast H^N \ast H^N e^{A^N} \in \mathcal{L}(\mathcal{X}^N, \mathcal{U})
\]

According to (3.7.18), a necessary and sufficient condition that \( u \) be the optimal control for a fixed \( w \) is that

\[
u^w(t) = -(T^{-N}_1 (T^N_3 x^N_0 + T^N_2 w))(t), \quad a.e. \in [0, t_f]
\]

where we define \( T^{-N}_1 = (T^N_1)^{-1} \). Hence we have

\[
J^N_\gamma(x^N_0; u^w, w)
\]

\[
= \| H^N (e^{A^N} \cdot x^N_0 - \mathcal{F} B^N T^{-N}_1 T^N_3 x^N_0 + \mathcal{F} ((-B^N T^{-N}_1 T^N_2 + D^N)w)) \|_{\mathcal{X}^N}^2 \\
+ \| T^{-N}_1 (T^N_3 x^N_0 + T^N_2 w) \|_{\mathcal{U}}^2 - \gamma^2 \| w \|_{\mathcal{W}}^2
\]

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We now look at the problem of maximizing the expression $J^N_\gamma(x^N_0; u^w, w)$ for $w \in \mathcal{W}$. Note that although we have an LQ problem, the concavity is not a priori verified. However, Riccati equation (3.7.14) provides the $\gamma$-robustness for system (3.7.16) with cost function $J^N_\gamma$, and one can show that $w \rightarrow J^N_\gamma(x^N_0; u^w, w)$ is strictly concave. Thus the maximizer $w$ will satisfy the equation

\[(3.7.20) \quad \delta_w J^N_\gamma(x^N_0; u^w, w)(m) = 0, \quad \forall m \in \mathcal{W}\]

Direct calculation results in

\[(3.7.21) \quad \delta_w J^N_\gamma(x^N_0; u^w, w)(m) = 2 < T^N_1 w + T^N_2 x^N_0, m > \quad \forall m \in \mathcal{W}\]

where $T^N_1 \in \mathcal{L}(\mathcal{W}, \mathcal{W})$

\[T^N_1 = [\mathcal{F}^N(-B^N T^{-N} T^N_1 + D^N)] H^N H^N [\mathcal{F}^N(-B^N T^{-N} T^N_2 + D^N)] + T^N_2 H^N H^N T^{-N} T^N_2 - \gamma^2 I\]

and $T^N_2 \in \mathcal{L}(\mathcal{X}^N, \mathcal{W})$

\[T^N_2 = [\mathcal{F}^N(-B^N T^{-N} T^N_1 + D^N)] H^N H^N (e^{A^N} T^{-N} T^N_3 e^{A^N} - \mathcal{F}^N B^N T^{-N} T^N_3 e^{A^N}) + T^N_3 H^N H^N T^{-N} T^N_3 \]

Recalling the proof of first part of Proposition 3, we know that

\[< \Pi^N(0) x^N_0, x^N_0 > = \sup_{w} \inf_{u} J^N_\gamma(x^N_0; u, w)\]

Thus letting $x^N_0 = 0$, we have

\[\sup_{w} \inf_{u} J^N_\gamma(0; u, w) = 0\]
From (3.7.20) we have

\[ J^N_\gamma(0; u^w, w) = < T^N_1 w, w > \]

Since \( w \rightarrow J^N_\gamma(0; u^w, w) \) is strictly concave, \( w = 0 \) is the only maximizer of \( J^N_\gamma(0; u^w, w) \), which implies that \(-T^N_1\) is a positive definite operator of \( \mathcal{L}(\mathcal{W}, \mathcal{W}) \). Thus from (3.7.21) we have

\[ (3.7.22) \quad w = (-T^N_1)^{-1} T^N_2 x^N_0 \]

Hence by (3.7.18), (3.7.20) and (3.7.22), we have

\[ \sup_w J^N_\gamma(x^N_0; u^w, w) = < (\mathcal{K}^N \ast \mathcal{K}^N + T^N_3 T^{-N}_1 T^{-N}_1 T^N_3 \ast T^N_2 \ast (-T^N_1)^{-N} T^N_2) x^N_0, x^N_0 >_{\mathcal{X}^N} \]

where

\[ \mathcal{K}^N = H^N (e^{A^N} - \mathcal{J}^N B^N T^{-N}_1 T^N_3 \ast) \]

Therefore we obtain

\[ (3.7.23) \quad \Pi^N(0) x^N_0 = (\mathcal{K}^N \ast \mathcal{K}^N + T^N_3 T^{-N}_1 T^{-N}_1 T^N_3 \ast T^N_2 \ast (-T^N_1)^{-N} T^N_2) x^N_0 \]

Define \( \mathcal{F}, T_1, T_2, T_3 \) by

\[ ( \mathcal{F} \phi)(s) = \int_0^s e^{A(t-s)} \phi(s) ds, \quad \phi \in \mathcal{X} \]

\[ T_1 = I + B^* \mathcal{F}^* H^* H \mathcal{F} B \in \mathcal{L}(\mathcal{U}, \mathcal{U}) \]

\[ T_2 = B^* \mathcal{F}^* H^* H \mathcal{D} \in \mathcal{L}(\mathcal{W}, \mathcal{U}) \]

\[ T_3^* = B^* \mathcal{F}^* H^* H e^{A^t} \in \mathcal{L}(X, \mathcal{U}) \]

and \( T_1 \in \mathcal{L}(\mathcal{W}, \mathcal{W}) \) and \( T_2 \in \mathcal{L}(\mathcal{X}, \mathcal{W}) \) respectively by

\[ T_1 = [\mathcal{F}( - B T_1^{-1} T_2 + D )]^* H^* H [\mathcal{F}( - B T_1^{-1} T_2 + D )] \]

\[ + T_2^* T_1^{-1} T_1^{-1} T_2 - \gamma^2 I \]

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\[ T_2 = [\mathcal{F}(-BT_1^{-1}T_2 + D)]^* H^* H(e^{A^*} - \mathcal{F}BT_1^{-1}T_3^*) + T_3 T_1^{-*} T_1^{-1} T_3^* \]

Consider the optimization problem

\[ J_\gamma(x_0; u, w) = \sup \inf \int_0^{t_f} (\|x\|^2 + \|u\|^2 - \gamma^2 \|v\|^2) dt \]

where \( x \) is subject to

\[ \dot{x} = Ax + Bu + Dw, \quad x(0) = x_0 \]

Then following a discussion similar to the one that led to (3.7.23),

\[ \Pi(0)x_0 = (K^*K + T_3(T_1^{-1})^* T_1^{-1} T_3^* + T_2^*(-T_1)^{-1} T_2)x_0 \]

where

\[ K = H(e^{A^*} - \mathcal{F}BT_1^{-1}T_3^*) \]

By assumptions (A1)-(A5), as \( N \to \infty \)

\[ e^{A^N t} \to e^{At} \quad \text{uniformly on} \quad [0, t_f] \]

\[ e^{A^N^* t} \to e^{A^* t} \quad \text{uniformly on} \quad [0, t_f] \]

\[ B^N \to B \quad \text{strongly} \]

\[ D^N \to D \quad \text{strongly} \]

\[ H^N \to H \quad \text{strongly} \]

Let \( x_0^N = P^Nx_0 \), above discussion indicates that

\[ \Pi^N(0)P^Nx_0 \to \Pi(0)x_0 \quad \forall x_0 \in X \quad \text{as} \quad N \to \infty \]

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and this completes the proof of Proposition 5. □

Proof of Lemma 3.2.7. For each $\epsilon > 0$ and $x \in X$, according Propositions 3 and 4 there exists $t_f > 0$ such that

\[
\|\Pi(0; t_f)x - \Pi x\| < \epsilon
\]

\[
\|\Pi^N(0; t_f)P^N - \Pi^N P^N\| < \epsilon \quad \text{for each} \quad N = 1, 2, \ldots
\]

Note that

\[
\|\Pi^N P^N x - \Pi x\| \leq \|\Pi^N P^N x - \Pi(0; t_f)^N P^N x\| + \|\Pi(0; t_f)^N P^N x - \Pi^N(0; t_f)x\|
\]

\[
\|\Pi^N(0; t_f)x - \Pi(0; t_f)x\| + \|\Pi(0; t_f)x - \Pi x\|
\]

\[
\leq \epsilon\|x\| + \|\Pi^N(0; t_f)||P^N x - x\| + \|\Pi^N(0; t_f)x - \Pi(0; t_f)x\| + \epsilon
\]

Since $\|\Pi^N(0; t_f)||$ is uniformly bounded in $N$ by Uniform Boundness Theorem, and

\[
\Pi^N(0; t_f)x \rightarrow \Pi(0; t_f)x
\]

by Proposition 5, we thus have

\[
\Pi^N \rightarrow \Pi \quad \text{as} \quad N \rightarrow \infty
\]

and this completes the proof of Lemma 3.2.7. □

By using the Duality Theorem, Lemma 3.2.8 can be proven similarly.
4. The Generalized Riccati Equation for $H^\infty$-Optimal Control Problems on Hilbert Spaces: Finite-horizon Case

4.1. Introduction

As is well-known by now in $H^\infty$-optimal control of finite-dimensional systems, the following two Riccati equations play an important role in the computation of attainable disturbance attenuation levels under respectively open-loop and closed-loop state feedback information for the controller:

\begin{equation}
\dot{S}_\gamma(t) + A(t)^*S_\gamma(t) + S_\gamma(t)A(t) + \frac{1}{\gamma^2} S_\gamma(t)D(t)D(t)^*S_\gamma(t) + C(t)^*C(t) = 0 \\
S_\gamma(t_f) = Q_f
\end{equation}

and

\begin{equation}
\dot{Z}_\gamma(t) + A(t)^*Z_\gamma(t) + Z_\gamma(t)A(t) - Z_\gamma(t)B(t)B(t)^*Z_\gamma(t) + \frac{1}{\gamma^2} Z_\gamma(t)D(t)D(t)^*Z_\gamma(t) + C(t)^*C(t) = 0 \\
Z_\gamma(t_f) = Q_f
\end{equation}

Here $A, B, C, D$ and $Q_f$ are matrices of appropriate sizes, and $\gamma$ is the parameter characterizing the level of disturbance attenuation. It has been shown by Ba\c{s}ar and Bernhard [BB] that there are two critical parameter values $\hat{\gamma}_c$ and $\gamma_c$, with $\hat{\gamma}_c \geq \gamma_c$ such that when $\gamma > \hat{\gamma}_c$ (4.1.1) has a nonnegative-definite matrix solution, and when $\gamma > \gamma_c$ (4.1.2) also admits a nonnegative-definite matrix solution, while for $\gamma < \hat{\gamma}_c$ (respectively, $\gamma < \gamma_c$) (4.1.1) (respectively, (4.1.2)) ceases to have any solutions over the entire interval $[0, t_f]$.

Our objective here is to extend this result to infinite-dimensional systems, and obtain explicit expressions for these critical parameters so that the direct calculation of these
parameters become possible. We also discuss the connection between these two generalized Riccati equations and a two-person differential game, in which the original control becomes the control for minimizing player in the game formulation while the disturbance becomes the control for the maximizing player. Moreover, by using operator theory, we provide explicit, closed form solutions to (4.1.1) and (4.1.2) in general Hilbert spaces.

The rest of the chapter is organized as follows. Section 2 gives the definition of evolution operators which represent a larger class of infinite-dimensional system structures. Section 3 provides some $H^\infty$-optimal control backgrounds. In section 4, we discuss the relation between critical parameter values and Riccati equations (4.1.1) and (4.1.2). Two explicit expressions of the critical value are given in this section, and furthermore these two Riccati equations are solved and their solutions are expressed in terms of known operators. Section 5 discusses the connection between the critical values and a two person differential game, which is used to study the $H^\infty$-optimal control problems. Section 6 studies the approximation of Riccati equation (4.1.2), which is related to the $H^\infty$-optimal feedback control, and section 7 is the concluding remarks.

4.2. Evolution Operators

Let $H$ be a real, separable Hilbert space, and $\mathcal{L}(H)$ be the space of bounded linear operators mapping $H$ into itself.

Definition 4.2.1. A two parameter family of bounded linear operators $U(t, s) \in \mathcal{L}(H)$, $-\infty < t_0 \leq s \leq t \leq t_f < \infty$, on $H$ is called an evolution operator if the following two conditions are satisfied:
\( (i) \) \( U(s,s) = I, U(t,r)U(r,s) = U(t,s) \) for \( t_0 \leq s \leq r \leq t \leq t_f. \)

\( (ii) \) \( (t,s) \rightarrow U(t,s) \) is strongly continuous for \( t_0 \leq s \leq t \leq t_f. \)

We denote the Banach space of strongly measurable, essentially bounded functions from \((t_0,t_f)\) to \(L(H)\) by \(B_\infty(t_0,t_f;H,H)\). The following perturbation theorem is from [CP].

**Theorem 4.2.2.** Let \( U(\cdot,\cdot) \) be an evolution operator which is uniformly bounded, and let \( \Delta \) be in \( B_\infty(t_0,t_f;H,H) \). Then the operator integral equation

\[
T(t,s)x = U(t,s)x + \int_s^t U(t,\eta)\Delta(\eta)T(\eta,s)x\,d\eta, \quad x \in H
\]

has a unique solution \( T(\cdot,\cdot) \) in the class of strongly continuous bounded linear operators on \( H \). \( T(\cdot,\cdot) \) is an evolution operator and is called the perturbed evolution operator corresponding to the perturbation of \( U(\cdot,\cdot) \) by \( \Delta \). \( T(\cdot,\cdot) \) is also the unique solution of

\[
T(t,s)x = U(t,s)x + \int_s^t T(t,\eta)\Delta(\eta)U(\eta,s)x\,d\eta, \quad x \in H
\]

i.e., \( U(\cdot,\cdot) \) is the perturbed evolution operator corresponding to the perturbation of \( T(\cdot,\cdot) \) by \(-\Delta\). If \( U(\cdot,\cdot) \) is uniformly bounded by \( M \), we have

\[
\|T(t,s)\| \leq M \exp(M\|\Delta\|_{B_\infty}(t - s)).
\]

Throughout this chapter, the following notation will be adopted, unless otherwise indicated:

\[
\Sigma(H) = \{ T \in L(H) : T \text{ is hermitian} \}
\]

\[
\Sigma^+(H) = \{ T \in \Sigma(H) : (Tx,x) \geq 0, \forall x \in H \}
\]

The space \( \Sigma(H) \), endowed with the norm of \( L(H) \), is a real Banach space and \( \Sigma^+(H) \) is a cone in \( \Sigma(H) \). For any interval \( I \) in \( \mathbb{R} \), we shall denote by \( C(I;\Sigma(H)) \) the set of all
continuous mappings from \( I \) to \( \Sigma(H) \); and by \( C_s(I;\Sigma(H)) \) the set of \( C(I;\Sigma(H)) \) endowed with the topology of uniform convergence on compact subsets of \( I \). Let

\[
\mathcal{H}_t = L^2([t,t_f]; H), \mathcal{U}_t = L^2([t,t_f]; U), \mathcal{W}_t = L^2([t,t_f]; W), \mathcal{Z}_t = L^2([t,t_f]; Z)
\]

When \( t = 0 \), we will write \( \mathcal{H} \) for \( \mathcal{H}_0 \), \( \mathcal{U} \) for \( \mathcal{U}_0 \), \( \mathcal{W} \) for \( \mathcal{W}_0 \), and \( \mathcal{Z} \) for \( \mathcal{Z}_0 \). Define operator \( T_t \in \mathcal{L}(\mathcal{H}_t,\mathcal{H}_t), K_t \in \mathcal{L}(\mathcal{H}_t, H) \), and \( U_t \in \mathcal{L}(H, \mathcal{H}_t) \) by

\[
(4.2.3) \quad (T_t \phi)(s) = \int_t^s U(s, \eta) \phi(\eta) d\eta \quad \phi \in \mathcal{H}_t
\]

\[
(4.2.4) \quad K_t \phi = (T_t \phi)(t_f), \quad \phi \in \mathcal{H}_t
\]

and

\[
(4.2.5) \quad (U_t x)(s) = U(s, t) x \quad x \in H.
\]

Remark 4.2.3. It is not difficult to verify that

\[
(4.2.6) \quad (T_t^* \phi)(\tau) = \int_{\tau}^{t_f} U(\eta, \tau) \phi(\eta) d\eta, \quad \forall \phi \in \mathcal{H}_t'
\]

where \( \mathcal{H}_t' \) is the dual of \( \mathcal{H}_t \), and

\[
(4.2.7) \quad (K_t^* x)(s) = U(t_f, s)^* x, \quad x \in H
\]

\[
(4.2.8) \quad U_t^* \phi = \int_t^{t_f} U(\eta, t)^* \phi(\eta) d\eta, \quad \phi \in \mathcal{H}_t.
\]
4.3. The $H^\infty$-optimal Control Problem

We consider an evolution process defined by
\[(4.3.1)\]
\[x(t) = U(t,s)x_0 + \int_s^t U(t,\eta)(B(\eta)u(\eta) + D(\eta)w(\eta))d\eta \quad 0 \leq s \leq t \leq t_f\]
where $x(t) \in H$ is the state of the system, $u(t) \in L^2(0,t_f;U)$ is the control input where $U$ is a real Hilbert space, and $w(t) \in L^2(0,t_f;W)$ is the disturbance input where $W$ is also a real Hilbert space. $B \in \mathcal{B}_\infty(0,t_f;U,H)$, $B^* \in \mathcal{B}_\infty(0,t_f;H,U)$, $D \in \mathcal{B}_\infty(0,t_f;W,H)$ and $D^* \in \mathcal{B}_\infty(0,t_f;H,W)$. $U(\cdot,\cdot)$ is an evolution operator on $H$ associated with a family of unbounded operators $A(t), 0 \leq t \leq t_f$ such that when $x \in D(A(t))$
\[\frac{\partial}{\partial t}U(t,s)x = A(t)U(t,s)x, \quad \frac{\partial}{\partial s}U(t,s)x = -U(t,s)A(s)x\]
and $A(t) : D(A(t)) \subset H \rightarrow H$ generates a $C_0$ semigroup in $H$. Such assumptions on $U(\cdot,\cdot)$ hold in many problems, both parabolic and hyperbolic (see for instance A. Pazy [Paz], H. Tanabe [Tan], and A. Bensoussan et.al. [BPDM]). Let $Z$ be a real Hilbert space, and $z(t) \in Z$ is the to-be-control output defined by
\[(4.3.2)\]
\[z(t) = C(t)x(t) + D_{12}(t)u(t)\]
where $C \in \mathcal{B}_\infty(0,t_f;H,Z)$, $C^* \in \mathcal{B}_\infty(0,t_f,Z,H)$, $D_{12} \in \mathcal{B}_\infty(0,t_f;U,Z)$ and $D_{12}^* \in \mathcal{B}_\infty(0,t_f;Z,U)$. We introduce the quadratic cost function
\[(4.3.3)\]
\[L(u,w) = \int_0^{t_f} \|z(t)\|_Z^2 dt + < Q_{t_f}x(t_f),x(t_f) >_H\]
where $Q_f \in \Sigma^+(H)$. Under the standard hypothesis $D_{12}[C,D_{12}] = [0,I]$, (4.3.3) becomes
\[(4.3.4)\]
\[L(u,w) = \int_0^{t_f} (\|C(t)x(t)\|_Z^2 + \|u(t)\|_U^2) dt + < Q_{t_f}x(t_f),x(t_f) >_H\]

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For each fixed \( u \in L^2([0,t_f];U) \), \( L \) induces a mapping of \( L^2([0,t_f];W) \) into nonnegative reals, which we denote by \( \mathcal{F}_u \). The minimax (\( H^\infty \)-optimal) controller design problem in this context is to find a state-feedback controller \( \mu \) that minimizes the following index

\[
g(\mu) := \sup_{w} \{ \mathcal{F}_u^{\frac{1}{2}} / \| w \|_W \}
\]

whose optimum value we denote by \( \gamma_c \):

\[
\gamma_c := \inf_{\mu \in \mathcal{M}} g(\mu)
\]

The study of this problem is equivalent to study of a differential game with dynamics (4.3.1) and with kernel function:

\[
J_\gamma(x_0; u, w) = L(u, w) - \gamma^2 \| w \|_W^2;
\]

see Section 5 later for a further elucidation of this point.

4.4. Generalized Riccati Equations

In this section, we discuss some properties of the two generalized Riccati equations:

\[
\dot{S}_\gamma(t) + A(t)^*S_\gamma(t) + S_\gamma(t)A(t) + \frac{1}{\gamma^2} S_\gamma(t)D(t)D(t)^*S_\gamma(t) + C(t)^*C(t) = 0
\]

\[
S_\gamma(t_f) = Q_f
\]

and

\[
\dot{Z}_\gamma(t) + A(t)^*Z_\gamma(t) + Z_\gamma(t)A(t) - Z_\gamma(t)B(t)B(t)^*Z_\gamma(t)
\]

\[
+ \frac{1}{\gamma^2} Z_\gamma(t)D(t)D(t)^*Z_\gamma(t) + C(t)^*C(t) = 0
\]

\[
Z_\gamma(t_f) = Q_f
\]
Definition 4.4.1. For $\gamma > 0$, let $S_\gamma \in C_s([0, t_f], \Sigma(H))$ satisfy the integral equation

\[ S_\gamma(t)x = U(t_f, t)^*Q_{t_f}U(t_f, t)x + \int_t^{t_f} U(s, t)^*(C(s)^*C(s)
\]
\[ + \frac{1}{\gamma^2} S_\gamma(s)D(s)^*D(s)S_\gamma(s))U(s, t)x ds \quad \forall x \in H \]

Then $S_\gamma$ is called the integral (or mild) solution of the generalized Riccati equation (4.4.1).

Definition 4.4.2. For $\gamma > 0$, let $Z_\gamma \in C_s([0, t_f], \Sigma(H))$ satisfy the integral equation

\[ Z_\gamma(t)x = U(t_f, t)^*Q_{t_f}U(t_f, t)x + \int_t^{t_f} U(s, t)^*\left\{C(s)^*C(s)
\]
\[ - Z_\gamma(s)(B(s)B(s)^* - \frac{1}{\gamma^2} D(s)^*D(s)Z_\gamma(s))\right\}U(s, t)x ds \quad \forall x \in H \]

Then $Z_\gamma$ is called the integral (or mild) solution of the generalized Riccati equation (4.4.2).

Proposition 4.4.3. Riccati equations (4.4.1) and (4.4.2) have at most one integral solution each.

Proof. Suppose that (4.4.1) has two integral solutions $S_\gamma^1, S_\gamma^2$, and note the identity

\[ S_\gamma^1 DD^*S_\gamma^1 - S_\gamma^2 DD^*S_\gamma^2 = (S_\gamma^1 - S_\gamma^2)DD^*S_\gamma^2 + S_\gamma^1 DD^*(S_\gamma^1 - S_\gamma^2) \]

Since $H$ is separable, we have that $\|S_\gamma^1(\cdot) - S_\gamma^2(\cdot)\|_{\mathcal{L}(H)}$ is measurable. Thus by applying Gronwall's lemma to the following

\[ \|S_\gamma^1(t) - S_\gamma^2(t)\|_{\mathcal{L}(H)} \leq \int_t^{t_f} \left(\|S_\gamma^1 - S_\gamma^2\|_{\mathcal{L}(H)}\|D\|_{\mathcal{B}_\infty}\|D^*\|_{\mathcal{B}_\infty}\|S_\gamma^2\|_{\mathcal{L}(H)}
\]
\[ + \|S_\gamma^1\|_{\mathcal{L}(H)}\|D\|_{\mathcal{B}_\infty}\|D^*\|_{\mathcal{B}_\infty}\|S_\gamma^1 - S_\gamma^2\|_{\mathcal{L}(H)}\right)d\tau \]

we obtain that $S_\gamma^1 \equiv S_\gamma^2$. The proof for (4.4.2) is similar. □
**Proposition 4.4.4.** For a given $\gamma > 0$ suppose that the generalized Riccati equations (4.4.1) and (4.4.2) have integral solutions $S_\gamma$ and $Z_\gamma$, respectively, on interval $[0, t_f]$. Then we have

$$0 \leq Z_\gamma(t) \leq S_\gamma(t), \quad \forall t \in [0, t_f]$$

**Proof.** Let $V(t) = S_\gamma(t) - Z_\gamma(t)$. It can be verified that $V$ satisfies

$$V(t)x = \frac{1}{\gamma^2} \int_t^{t_f} T(s, t)^* V(s)D(s)^* D(s) V(s) T(s, t)x ds, \quad x \in H$$

where the evolution operator $T(\cdot, \cdot)$ is defined by

$$T(s, t)x = U(s, t)x + \int_t^s U(s, \eta) \frac{1}{\gamma^2} D(\eta) D(\eta)^* Z_\gamma(\eta) T(\eta, t)x d\eta$$

Thus we have $V(t) \geq 0, 0 \leq t \leq t_f$, and hence $0 \leq Z_\gamma(t) \leq S_\gamma(t), \forall t \in [0, t_f]$.

**Remark 4.4.5.** It can be observed that for given $\gamma_2 \geq \gamma_1 > 0$, if (4.4.1) has solutions $S_{\gamma_1}$ and $S_{\gamma_2}$ respectively, then we have

$$0 \leq S_{\gamma_1} \leq S_{\gamma_2}$$

Thus we can assert that if (4.4.1) has an integral solution for a $\gamma' > 0$, then it also has integral solutions for any $\gamma \geq \gamma'$. The same property holds for (4.4.2).

**4.4.1. Critical Value $\gamma_c$ for Maximization Problems**

In this section we consider the following maximization problem: for an arbitrary but fixed $u \in \mathcal{U}$; maximize

$$(4.4.1.1) \quad J_\gamma(x_0; u, w) = \int_0^{t_f} \|C(t)x(t)\|_Z^2 + \|u(s)\|_U^2 - \gamma^2 \|w(s)\|_W^2 dt + \langle Q_f x(t_f), x(t_f) \rangle_H$$

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where \( x(t) \) is given by (4.3.1). Let

\[
\hat{\gamma}_c^2 := \sup_{\|w\| = 1} \left\langle D^* (K_0^* Q_f K_0 + T_0^* C^* C T_0) D w, w \right\rangle_W
\]

\[= \|D^* (K_0^* Q_f K_0 + T_0^* C^* C T_0) D\|^{1/2}_{\mathcal{L}^2(W)}\]  

where \( K_0 \) and \( T_0 \) are as given in section 2. The following theorem shows that \( \hat{\gamma}_c \) is the critical value for the Maximization of \( J_\gamma \) over \( W \) with \( u \) fixed.

**Theorem 4.4.1.1.** If \( \gamma > \hat{\gamma}_c \), then for any \( x_0 \in H, \max_w J_\gamma(x_0; u, w) \) is finite. If \( \hat{\gamma}_c > 0 \) and \( \gamma < \hat{\gamma}_c \), \( \max_w J_\gamma(x_0; u, w) \) is unbounded for any \( x_0 \in H \).

**Proof.** By using the notations which is set up in section 2, we can write \( J_\gamma \) as

\[
J_\gamma(x_0; u, w) = \langle Q_f(U(t_f, 0)x_0 + K_0(Bu + Dw), U(t_f, 0)x_0 + K_0(Bu + Dw) > H
\]

\[
\|C(U_0 x_0 + T_0(Bu + Dw))\|_2^2 + \|u\|_U^2 - \gamma^2 \|w\|_W^2
\]

Since we are just interested in the quadratic terms, let us put all such terms together:

\[
J_\gamma(x_0; u, w) = \langle D^*(K_0^* Q_f K_0 + T_0^* C^* C T_0) D - \gamma^2 I \rangle w, w > W + g(w, x_0, u)
\]

where \( w \rightarrow g(w, x_0, u) \) is affine. If \( \gamma > \hat{\gamma}_c \), by the definition of \( \hat{\gamma}_c \) clearly the operator

\[D^*(K_0^* Q_f K_0 + T_0^* C^* C T_0) D\]

is nonnegative self-adjoint and satisfies

\[
\langle (D^*(K_0^* Q_f K_0 + T_0^* C^* C T_0) D) w, w > W \leq -(\gamma^2 - \hat{\gamma}_c^2) \|w\|_W^2
\]

Thus \( \max_w J_\gamma \) is finite because \( g(w, x_0, u) \) is linear growth in \( w \). While when \( \gamma < \hat{\gamma}_c \), according to the definition of \( \hat{\gamma}_c \) there exists \( w \) with \( \|w\|_W = 1 \) such that

\[
\langle (D^*(K_0^* Q_f K_0 + T_0^* C^* C T_0) D - \gamma^2 I) \dot{w}, \dot{w} > W > 0
\]

Letting \( w_k = k \dot{w} \), we have

\[
J_\gamma(x_0; u, w_k) \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty
\]

and this completes the proof. \( \square \)
Theorem 4.4.1.2. For fixed $u \in U$ and $x_0 \in H$, for $\gamma > \hat{\gamma}_c$, the mapping $w \to J_\gamma(x_0; u, w)$ is strictly concave.

Proof. Let

$$S_\gamma = D^*(K_0^*Q_fK_0 + T_0^*C^*CT_0)D - \gamma^2I$$

For $0 \leq \lambda \leq 1$, from the expression (4.4.1.1), we have

$$J_\gamma(x_0; u, \lambda w_1 + (1 - \lambda)w_2)$$

$$= \lambda J_\gamma(x_0; u, \lambda w_1) + (1 - \lambda)J_\gamma(0; u, w_2) - \lambda(1 - \lambda) < S_\gamma(w_1 - w_2), (w_1 - w_2) >_W$$

Since when $w_1 \neq w_2$, $< S_\gamma(w_1 - w_2), (w_1 - w_2) >_W > 0$, the strict concavity follows. □

Theorem 4.4.1.3. When $\gamma > \hat{\gamma}_c$, for a fixed $u \in U$, the maximizer of $J_\gamma(x_0; u, w)$ is given by

$$w = -S_\gamma^{-1}\{D^*[K_0^*Q_fU(t_f, 0) + T_0^*C^*CU_0]x_0 + D^*[K_0^*Q_fK_0 + T_0^*C^*CT_0]Bu\}$$

where $S_\gamma$ is given in Theorem 4.4.1.2.

Proof. By Theorem 4.4.1.2, we know that $J_\gamma(x_0, u, w)$ has a unique maximizer which is the (unique) solution of

$$(4.4.1.4) \quad \delta_w J_\gamma(x_0; u, w)(m) = 0, \quad \forall m \in W$$

where $\delta_w J_\gamma(x_0; u, w)(m)$ represents the Gâteaux derivative of $J_\gamma(x_0; u, \cdot)$ at $w$, applied to $m$. From (4.4.1.3), we have

$$\delta_w J_\gamma(x_0; u, w)(m) =$$

$$2 < S_\gamma w + D^*[K_0^*Q_fU(t_f, 0) + T_0^*C^*CU_0]x_0 + D^*[K_0^*Q_fK_0 + T_0^*C^*CT_0]Bu, m >_W,$$
where \( m \in \mathcal{W} \). In view of (4.4.1.4), we have the desired result. \( \square \)

The following theorem shows the relationship between the critical value \( \hat{\gamma}_c \) and the integral solution of the Riccati equation (4.4.1).

**Theorem 4.4.1.4.** If the Riccati equation (4.4.1) has an integral solution on \([0, t_f]\) for a given \( \gamma > 0 \), then \( \gamma \geq \hat{\gamma}_c \). Conversely, if \( \gamma > \hat{\gamma}_c \), then Riccati equation (4.4.1) admits an integral solution on \([0, t_f]\).

**Proof.** Suppose that the Riccati equation (4.4.1) has an integral solution \( S_\gamma \) on interval \([0, t_f]\). For a given \( x \in H \), \( u \in \mathcal{U} \) and \( w \in \mathcal{W} \), let \( x_n^0 \subset D(A(t)), 0 \leq t \leq t_f, u_n \subset C^1([0, t_f]; U) \) and \( w_n \subset C^1([0, t_f]; W) \) such that

\[
x_n^0 \rightarrow x_0, u_n(\cdot) \rightarrow u(\cdot), w_n(\cdot) \rightarrow w(\cdot) \quad \text{in } H
\]

Note that such sequences exist, since \( D(A(t)) \) is dense in \( H \) and spaces \( C^1([0, t_f]; U) \), \( C^1([0, t_f]; W) \) are dense in \( \mathcal{U} \) and in \( \mathcal{W} \), respectively. Let \( x_n \) be the mild solution of the equation:

\[
(4.4.1.5) \quad \dot{x}_n(t) = A(t)x_n(t) + B(t)u_n(t) + D(t)w_n(t)
\]

\[
x_n(0) = x_n^0 \in D(A(t))
\]

Note that in this case with \( u_n \in C^1([0, t_f]; U) \) and \( w_n \in C^1([0, t_f]; W) \), \( x_n(t) \) is differentiable by the standard argument. Let us introduce two backward Cauchy problems: For all \( t \in [0, t_f] \),

\[
(4.4.1.6) \quad \dot{r}(t) + (A^*(t) + \frac{1}{\gamma^2} S(t)D(t)D^*(t))r(t) + S(t)B(t)u(t) = 0
\]

\[
r(t_f) = 0
\]

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and
\[(4.4.1.7)\]
\[
\dot{r}_n(t) + (A^*(t) + \frac{1}{\gamma^2} S(t)D(t)D^*(t))r_n(t) + S(t)B(t)u_n(t) = 0
\]
\[
r_n(t_f) = 0
\]

Under the assumptions on $U(\cdot, \cdot)$ (see section 3), both (4.4.1.6) and (4.4.1.7) have well-defined unique solutions in $C([0, t_f]; H)$ with $r_n \to r$ uniformly as $n \to \infty$. In fact, it can be verified that the solution of (4.4.1.6) is almost everywhere differentiable on $[0, t_f[$. We may compute
\[
\frac{d}{ds}\{(S_n(s)x_n(s), x_n(s))_H + (r_n(s), x_n(s))_H + (x_n(s), r_n(s))_H\}
\]
\[
= -\gamma^2\|w(s) - \gamma^{-2}D(s)^*(S_n(s)x_n(s) + r_n(s))\|_W^2 + \|C(s)x(s)\|_Z^2 - \|D^*(s)r_n(s)\|_W^2
\]
\[
+ \gamma^2\|w(s)\|_W^2 + (r(s), B(s)u(s))_H + (B(s)u(s), r(s))_H
\]

Therefore by integrating the above identity and letting $n$ tend to infinity, we have
\[(4.4.1.8)\]
\[
J^*_\gamma(u, w, x_0) = (S(0)x_0, x_0)_H + (r(0), x_0)_H + (x_0, r(0))_H
\]
\[
+ \int_0^{t_f} \{\|u(s)\|_U^2 - \gamma^2\|w(s) - \gamma^{-2}D^*(s)(S(s)x(s) + r(s))\|_W^2 + \|D^*(s)r(s)\|_W^2 \} ds
\]
\[
+ \int_0^{t_f} \{(r(s), B(s)u(s))_H + (B(s)u(s), r(s))_H\} ds
\]

Hence
\[
\max_w J(u, w, x_0) = (S(0)x_0, x_0)_H + (r(0), x_0)_H + (x_0, r(0))_H
\]
\[
+ \int_0^{t_f} \{\|u(s)\|_U^2 - \|D^*(s)r(s)\|_W^2 \} ds + \int_0^{t_f} \{(r(s), B(s)u(s))_H + (B(s)u(s), r(s))_H\} ds
\]

Since every term in the above equality is well-defined, this yields
\[
\sup_w J^*_\gamma(x_0; u, w) < +\infty
\]

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and according to Theorem 4.4.1.1, we have $\gamma \geq \gamma_c$.

Conversely, if $\gamma > \gamma_c$, from Theorem 4.4.1.1 we know that $\max_u J_\gamma(x_0; u, w)$ is finite for arbitrary but fixed $u \in U$. Let $u = 0$ and $w^*$ be the maximizer of $J_\gamma(x_0; 0, w)$. From Theorem 4.4.1.3 the maximizer $w^*$ has the form

$$w^* = \frac{1}{\gamma^2} \{ D^*(K_0^*Q_fK_0 + T_0^*C^*CT_0)Dw^* + D^*(K_0^*Q_fU(t_f, 0) + T_0C^*CU_0)x_0 \}$$

Let us write the optimal trajectory by (under $\dot{w}^*$ as above, and with $u = 0$) as

$$x(t) = T(t, s)x, \quad s \leq t \leq t_f$$

Note that by (4.13) $w^*$ can be written as

$$w^*(t) = \frac{1}{\gamma^2} D^*(t)S(t)x(t)$$

where

$$S(t)x = U^*(t_f, t)Q_fT(t_f, t)x + \int_t^{t_f} U^*(\eta, t)C^*(\eta)C(\eta)T(\eta, t)x d\eta,$$

$$0 \leq t \leq t_f, x \in H$$

On the other hand, the operator $T(\cdot, \cdot)$ can be viewed as the perturbation of $U(\cdot, \cdot)$ by $-\frac{1}{\gamma^2} D(\cdot)D^*(\cdot)S(\cdot)$; thus it has the form

$$T(t, s)x = U(t, s)x + \int_s^t T(t, \eta)\frac{1}{\gamma^2} D(\eta)D^*(\eta)S(\eta)U(\eta, s)x d\eta, \quad x \in H$$

and from (4.4.1.10) we have

$$S(t)x = U^*(t_f, t)Q_fU(t_f, t)x$$

$$+ U^*(t_f, t)Q_f \int_t^{t_f} T(t_f, \eta)\frac{1}{\gamma^2} D(\eta)D^*(\eta)S(\eta)U(\eta, t)x d\eta$$

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\begin{align}
&+ \int_t^{t_f} U^*(\eta, t) C^*(\eta) C(\eta) U(\eta, t) x d\eta \\
&+ \int_t^{t_f} U^*(\eta, t) C^*(\eta) C(\eta) \int_\eta^n T(\eta, \xi) \frac{1}{\gamma^2} D(\xi) D^*(\xi) S(\xi) U(\xi, t) x d\xi d\eta
\end{align}

Note that $U^*(\tau, t) = (U(\tau, \eta) U(\eta, t))^* = U^*(\eta, t) U^*(\tau, \eta)$ for $0 \leq t \leq \tau \leq t_f$, and hence (4.4.1.12) becomes

\begin{align}
&U^*(t_f, t) Q_f \int_t^{t_f} T(t, \eta) \frac{1}{\gamma^2} D(\eta) D^*(\eta) S(\eta) U(\eta, t) x d\eta \\
&= \int_t^{t_f} U^*(\eta, t) U^*(t_f, \eta) Q_f T(t_f, \eta) \frac{1}{\gamma^2} D(\eta) D^*(\eta) S(\eta) U(\eta, t) x d\eta
\end{align}

Applying Fubini's theorem to interchange the order of the integral (4.4.1.14), yields

\begin{align}
&\int_t^{t_f} U^*(\eta, t) C^*(\eta) C(\eta) \int_\eta^n T(\eta, \xi) \frac{1}{\gamma^2} D(\xi) D^*(\xi) S(\xi) U(\xi, \eta) x d\xi d\eta \\
&= \int_t^{t_f} U^*(\eta, t) \int_\eta^n U^*(\xi, \eta) C^*(\xi) C(\xi) T(\xi, \eta) \frac{1}{\gamma^2} D(\eta) D^*(\eta) S(\eta) U(\eta, t) x d\xi d\eta
\end{align}

From (4.4.1.15) and (4.4.1.16), we have

\[ S(t)x = U^*(t_f, t) Q_f U(t_f, t)x + \int_t^{t_f} U^*(\eta, t) C^*(\eta) C(\eta) U(\eta, t) x d\eta \\
+ \int_t^{t_f} U^*(\eta, t) S(\eta) \frac{1}{\gamma^2} D(\eta) D^*(\eta) S(\eta) U(\eta, t) x d\eta \]

Up to now we have only shown that $S(\cdot) \in \mathcal{L}(0, t_f; H, H)$ and satisfies (4.4.3). Next we claim that $S(\cdot) \in \Sigma^+(H)$.

Let $x$ and $y$ be in $H$. From (4.4.1.11) we have

\begin{align}
&\langle S(t)x, y \rangle_H = \langle Q_f T(t_f, t)x, U(t_f, t)y \rangle_H \\
&\quad + \int_t^{t_f} \langle C(\eta) T(\eta, t)x, C(\eta) U(\eta, t)y \rangle_H d\eta
\end{align}
Note that (4.4.1.10) and operator $T(\cdot, \cdot)$ can be viewed as the perturbation of $U(\cdot, \cdot)$ by
\[ \frac{1}{\gamma^2} D(\cdot)D(\cdot)^* S(\cdot), \text{ i.e.,} \]
\[ T(t,s)x = U(t,s)x + \int_s^t U(t,\eta) \frac{1}{\gamma^2} D(\eta)D(\eta)^* S(\eta)T(\eta, s)x d\eta \quad \forall x \in H \]
Replacing $U(\cdot, \cdot)$ with $T(\cdot, \cdot)$ in (4.4.1.17), we have
\begin{align*}
(4.4.1.18) \quad < S(t)x, y >_H &= < Q_f T(t_f,t)x, T(t_f,t)y >_H \\
& - \int_t^{t_f} < Q_f T(t_f,t)x, U(t_f,\eta) \frac{1}{\gamma^2} D(\eta)D(\eta)^* S(\eta)T(\eta, t)y >_H d\eta \\
& + \int_t^{t_f} < C(\eta)T(\eta, t)x, C(\eta)T(\eta, t)y >_H d\eta \\
& - \int_t^{t_f} < C(\eta)T(\eta, t)x, \int_{\eta}^{t_f} C(\eta)U(\eta, \xi) \frac{1}{\gamma^2} D(\xi)D(\xi)^* S(\xi)T(\xi, t)y d\xi >_H d\eta
\end{align*}
Applying the Fubini's Theorem again to the last term and noting (4.4.1.11), we have:
\begin{align*}
& \int_t^{t_f} < C(\eta)T(\eta, t)x, \int_{\eta}^{t_f} C(\eta)U(\eta, \xi) \frac{1}{\gamma^2} D(\xi)D(\xi)^* S(\xi)T(\xi, t)y d\xi >_H d\eta \\
& = \int_t^{t_f} \left( \int_{\eta}^{t_f} U(\xi, \eta)^* C(\xi)^* C(\xi)T(\xi, \eta)T(\eta, t)x d\xi, \frac{1}{\gamma^2} D(\eta)D(\eta)^* S(\eta)T(\eta, t)y \right)_H d\eta \\
& = \int_t^{t_f} \left( [S(\eta) - U(t_f, \eta)^* Q_t, T(t_f, \eta)]T(\eta, t)x, \frac{1}{\gamma^2} D(\eta)D(\eta)^* S(\eta)T(\eta, t)y \right)_H d\eta
\end{align*}
Therefore (4.4.1.18) becomes
\begin{align*}
< S(t)x, y >_H &= < Q_f T(t_f,t)x, T(t_f,t)y >_H \\
& + \int_t^{t_f} < C(\eta)T(\eta, t)x, C(\eta)T(\eta, t)y >_H d\eta \\
& - \frac{1}{\gamma^2} \int_t^{t_f} < D(\eta)^* S(\eta)T(\eta, t)x, D(\eta)^* S(\eta)T(\eta, t)y >_H d\eta \\
& = < x, S(t)y >_H
\end{align*}
and we have that $S(\cdot)$ is self-adjoint. Clearly, $S(t)$ is nonnegative, thus $S(\cdot) \in \Sigma^+(H)$.
Therefore $S$ is the integral solution of (4.4.1).
Remark 4.4.1.5. In the case $\gamma = \gamma_c$, the Riccati equation (4.4.1) may or may not admit a solution over the interval $[0, t_f]$, which is determined by the maximization of a cost function. More precisely, we have the following corollary:

Corollary 4.4.1.6. The Riccati equation (4.4.1) admits a solution over the interval $[0, t_f]$ if and only if $\sup_w J^*_\gamma(x, 0, w)$ is bounded for all $x \in H$ and for all $t \in [0, t_f]$, where

$$ J^*_\gamma(x; 0, w) = < Q_f x(t_f), x(t_f) >_H + \int_t^{t_f} (\|C(s)x(s)\|^2_Y - \gamma^2 \|w(s)\|^2_W)ds $$

Proof. From the first part of the proof in Theorem 4.4.1.4, we know that if (4.4.1) admits a solution, $\sup_w J_\gamma(x_0, u, w)$ is bounded for all $x_0 \in H$ and all $t \in [0, t_f]$. Conversely, if $\sup_w J^*_\gamma(x_0, 0, w)$ is bounded for all $x_0 \in H$ and all $u \in U$, then (4.4.1) admits a solution from the second part of proof in Theorem 4.4.1.4. □

Corollary 4.4.1.7. (Representation formula of $S_\gamma$) If $\gamma > \gamma_c$, Riccati equation (4.4.1) admits an integral solution $S_\gamma$ over the interval $[0, t_f]$, which can be expressed as

(4.4.1.19) $S_\gamma(t)x = U(t_f, t)^*Q_fU(t_f, t)x + U_t^*C^*CU_t x$

$$ - [K_t^*Q_fU(t_f, t) + T_t^*C^*CU_t]^*DS^{-1}D^*[K_t^*Q_fU(t_f, t) + T_t^*C^*CU_t] $$

where $T_t, K_t, U_t$ have been defined in section 2, and

(4.4.1.20) $S_\gamma = D^*[K_t^*Q_fK_t + T_t^*C^*CT_t]D - \gamma^2 I$

Proof. Introduce cost function

$$ J^*_\gamma(x; 0, w) = < Q_f x(t_f), x(t_f) >_H + \int_t^{t_f} (\|C(s)x(s)\|^2_Y - \gamma^2 \|w(s)\|^2_W)ds $$

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By Theorem 4.4.1.3, for fixed $x \in H$ the maximizer of $J_{\gamma}^t(x; 0, w)$ is

$$w^*= -S_{\gamma}^{-1}D^*[\mathcal{K}_i^*Q_fU(t_f, t) + T_t^*C^*CU_i]x$$

(4.4.1.21)

Direct calculation on $J_{\gamma}^t$, making use of (4.4.1.20), yields that

(4.4.1.22)

$$J_{\gamma}^t(x; 0, w^*) = ||Q_f^\frac{1}{2}(U(t_f, t)x + \mathcal{K}_t Dw^*)||_H^2 + ||C(U_t x + T_t Dw^*)||_H^2 - \gamma^2 ||w^*||_{\mathcal{W}_t}^2$$

$$= <U(t_f, t)^*Q_fU(t_f, t)x, x>_H + <U_t^*C^*CU_t x, x>_{\mathcal{H}_t}$$

$$+ 2 <[U(t_f, t)^*Q_f\mathcal{K}_t + U_t^*C^*CT_t]Dw^*, x>_H$$

$$- <[U(t_f, t)^*Q_f\mathcal{K}_t + U_t^*C^*CT_t]D(S_{\gamma}^{-1})^*D^*\mathcal{K}_t^*Q_f\mathcal{K}_t Dw^*, x>_H$$

$$- <[U(t_f, t)^*Q_f\mathcal{K}_t + U_t^*C^*CT_t]D(S_{\gamma}^{-1})^*D^*T_t^*C^*CT_t Dw^*, x>_H$$

$$+ \gamma^2 <[U(t_f, t)^*Q_f\mathcal{K}_t + U_t^*C^*CT_t]D(S_{\gamma}^{-1})^*w^*, x>_H$$

$$= <U(t_f, t)^*Q_fU(t_f, t)x, x>_H + <U_t^*C^*CU_t x, x>_H$$

$$+ <[U(t_f, t)^*Q_f\mathcal{K}_t + U_t^*C^*CT_t]Dw^*, x>_H$$

$$= <U(t_f, t)^*Q_fU(t_f, t)x, x>_H + <U_t^*C^*CU_t x, x>_H$$

$$- <[U(t_f, t)^*Q_f\mathcal{K}_t + U_t^*C^*CT_t]DS_{\gamma}^{-1}D^*[\mathcal{K}_t^*Q_fU(t_f, t) + T_t^*C^*CU_t]x, x>_H$$

$$= :=\Xi(t)x, x>_H$$

Clearly $\Xi(t)^* = \Xi(t)$. From the second part of proof of Theorem 4.4.1.4, we know that

$$\sup_{w} J_{\gamma}^t(x; 0, w) = (S_{\gamma}(t)x, x)_H$$

Hence we have

(4.4.1.23) $$<[S_{\gamma}(t) - \Xi(t)]x, x>_H = 0, \quad \forall x \in H$$
which implies that for every $t \in [0, t_f]$

$$R(S_\gamma(t) - \Xi(t))^\perp = H$$

Since $N([S(t) - \Xi(t)]^\perp) = R(S(t) - \Xi(t))^\perp$, we hence have

$$N([S_\gamma(t) - \Xi(t)]^\perp) = H$$

Therefore, $S_\gamma(t)^* \equiv \Xi(t)^*$, and thus $S_\gamma(t) \equiv \Xi(t)$ because both $S_\gamma$ and $\Pi$ are self-adjoint.

The proof is thus complete. \hfill \square

**Remark 4.4.1.8.** $S_\gamma$ can be expressed in terms of integrands as follows:

\begin{equation}
S_\gamma(t)x = U(t_f, t)^* Q_f U(t_f, t)x + \int_t^{t_f} U(s, t)^* C(s)^* C(s) U(s, t)x ds \\
- U(t_f, t)^* Q_f \int_t^{t_f} U(t_f, \eta) D(\eta) S_\gamma^{-1}(\eta) D(\eta)^* \\
\times \left[ U(t_f, \eta)^* Q_f U(t_f, \eta) + \int_\eta^{t_f} U(s, \eta)^* C(s)^* C(s) U(s, \eta) x ds \right] d\eta \\
- \int_t^{t_f} U(\eta, t)^* C(\eta)^* C(\eta) U(\eta, \xi) D(\xi) S_\gamma^{-1}(\xi) D(\xi)^* \\
\times \left[ U(t_f, \xi)^* Q_f U(t_f, \xi) + \int_\xi^{t_f} U(s, \xi)^* C(s)^* C(s) U(s, t) x ds \right] d\xi d\eta
\end{equation}

By using (4.4.1.10), (4.4.1.21) and Fubini's theorem, (4.4.1.24) can be shown to be equivalent to (4.4.3), the integral solution of (4.4.1).

4.4.2. Critical Value $\gamma_c$ for the Maximin Problems

In this section, we consider the lower value of the game defined by kernel $J_\gamma$, where

$$J_\gamma(x_0; u, w) = \int_0^{t_f} ||C(t)x(t)||_2^2 + ||u(s)||_U^2 - \gamma^2 ||w(s)||_W^2 dt + <Q_f x(t_f), x(t_f)>_H$$
and establish the critical value for the lower value of the game. For later reference, let us define some operators first:

\[(4.4.2.1) \quad T_1 = I + B^* T_0^* C^* C T_0 B + B^* K_0^* Q_f K_0 B \in \mathcal{L}(\mathcal{U}, \mathcal{U}) \]

\[(4.4.2.2) \quad T_2 = B^* T_0^* C^* C T_0 D + B^* K_0^* Q_f K_0 D \in \mathcal{L}(\mathcal{W}, \mathcal{U}) \]

\[(4.4.2.3) \quad T_3^* = B^* T_0^* C^* C U_0 + B^* K_0^* Q_f U(t_f, 0) \in \mathcal{L}(H, \mathcal{U}) \]

and \( S \in \mathcal{L}(\mathcal{W}, \mathcal{W}) \)

\[(4.4.2.4) \quad S = [T_0(-BT_1^{-1}T_2 + D)]^* C^* C [T_0(-BT_1^{-1}T_2 + D)] + T_2 T_1^{-1} T_1^{-1} T_2^* + [K_0(-BT_1^{-1}T_2 + D)]^* Q_f [K_0(-BT_1^{-1}T_2 + D)] \]

where \( T_0, K_0, U_0 \) are defined in section 2. Define

\[ \gamma_c^2 := \sup_{\|w\|=1} <S w, w>_\mathcal{W} = \|S^\frac{1}{2}\|_{\mathcal{L}(\mathcal{W})}^2 \]

We are now ready to state the following:

**Theorem 4.4.2.1.** If \( \gamma > \gamma_c \), the lower value of the game \( \sup_w \inf_u J_\gamma(x_0; u, w) \) is finite for all \( x_0 \in H \). If \( \gamma_c > 0 \) and \( \gamma < \gamma_c \), the lower value of the game \( \sup_w \inf_u J_\gamma(x_0; u, w) \) is unbounded for any \( x_0 \in H \).

**Proof.** By using the notation introduced in section 2, we write the cost function \( J_\gamma \) as

\[(4.4.2.5) \quad J_\gamma(x_0; u, w) = <Q_f(U(t_f, 0)x_0 + K_0(Bu + Dw)), U(t_f, 0)x_0 + K_0(Bu + Dw)>_H \]

\[\|C(U_0 x_0 + T_0(Bu + Dw))\|_H^2 + \|u\|_U^2 - \gamma^2 \|w\|_\mathcal{W}^2 \]

Since for arbitrary but fixed \( w \in \mathcal{W} \), \( u \to J_\gamma(x_0; u, w) \) is strictly convex, there exists a unique \( u^w \) which minimizes \( J_\gamma \) and this \( u^w \) is the unique solution of

\[(4.4.2.6) \quad \delta_u J_\gamma(x_0; u, w)(v) = 0, \quad \forall v \in \mathcal{U} \]

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where $\delta_u J_\gamma(x_0; u, w)(v)$ is the Gâteaux derivative of $J_\gamma$ at $u$, applied to $v$. From (4.4.2.5), we have

$$
\delta_u J_\gamma(x_0; u, w)(v) = 2 < Q_f(U(t_f, 0)x_0 + K_0(Bu + Dw)), K_0 Bu >_H \\
+ 2 < C(U_0x_0 + T_0(Bu + Dw)), CT_0 Bu >_H + 2 < u, v >_U
$$

By identifying $H$ and $U$, and therefore $\mathcal{H}_t$ and $\mathcal{U}_t$, with respective duals, we may write above (4.4.2.7) as

$$
\delta_u J_\gamma(x_0; u, w)(v) = 2 < T_1 u + T_2 w + T_3^* x_0, v >_U
$$

Since $T_1$ is invertible, by (4.4.2.6) the optimal control $u^w$ for a fixed $w$ is given by

$$
u^w(t) = - T_1^{-1}(T_2 w + T_3^* x_0)(t), \quad a.e. \quad in \quad [0, t_f]
$$

By (4.4.2.8), the cost function (4.4.2.5) becomes

$$
J_\gamma(x_0; u^w, w) = (Q_f(U(t_f, 0)x_0 + K_0[B(- T_1^{-1}(T_2 w + T_3^* x_0)) + Dw]), \\
U(t_f, 0)x_0 + K_0[B(- T_1^{-1}(T_2 w + T_3^* x_0)) + Dw])>_H \\
+ \|C(U(t_f, 0)x_0 + T_0[B(- T_1^{-1}(T_2 w + T_3^* x_0)) + Dw])\|_2^2 \\
+ \|- T_1^{-1}(T_2 w + T_3^* x_0)\|_U^2 - \gamma^2 \|w\|_W^2
$$

$$
= < (S - \gamma^2 I)w, w >_W + \hat{g}(x_0, w)
$$

where $w \rightarrow \hat{g}(x_0, w)$ is affine. Similar to the proof of Theorem 4.4.1.1, if $\gamma > \gamma_c$, then

$$
< (S - \gamma^2 I)w, w >_W \leq (\gamma^2 - \gamma_c^2) \|w\|_W^2
$$

while since $\hat{g}$ is linear growth in $w$, $\max_w J_\gamma(x_0; u^w, w)$ is bounded for any $x_0 \in H$. When $\gamma < \gamma_c$, there exists at least one $\hat{w} \in W$ with $\|\hat{w}\|_W = 1$ such that

$$
< (S - \gamma^2 I)\hat{w}, \hat{w} >_W > 0
$$

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Hence

\[ J_\gamma(x_0; u^k w, k\hat{w}) \to \infty, \quad \text{as } k \to \infty \]

This completes the proof. \( \square \)

**Theorem 4.4.2.2.** Let \( \gamma > \gamma_c \). Then the mapping \( w \to J_\gamma(x_0; u^w, w) \) is strictly concave.

**Proof.** Similar to Theorem 4.4.1.2.

**Theorem 4.4.2.3.** For a given \( \gamma > 0 \), if (4.4.2) admits an integral solution, then \( \gamma \geq \gamma_c \).

Conversely, if \( \gamma > \gamma_c \) then (4.4.2) admits an integral solution.

**Proof.** Let the initial condition of (4.3.1) be in \( D(A(t)) \), and \( x(t) \) be the trajectory under control \( u \) and disturbance \( w \). By standard regularization and method of completion of squares, we have

\[
\frac{d}{dt}(Z_\gamma x, x) = \|u + B^* Z_\gamma x\|^2_U - \gamma^2 \|w - \gamma^{-2} D^* Z_\gamma x\|^2_W \\
- \|C x\|^2_Z - \|u\|^2_U + \gamma^2 \|w\|^2_W \quad \text{a.e. on } [0, t_f]
\]

Thus, \( \forall (u, w) \in U \times W \)

\[ J_\gamma(x_0; u, w) = (Z_\gamma(0)x_0, x_0) + \|u + B^* Z_\gamma x\| - \gamma^2 \|w - \gamma^{-2} D^* Z_\gamma x\|^2 \]

By letting \( u = -B^* Z_\gamma x \), we have

\[ \sup_{w} \inf_{u} J_\gamma(x_0; u, w) \leq (Z_\gamma(0)x_0, x_0) < \infty \]

Hence by Theorem 4.4.2.1, we obtain \( \gamma \geq \gamma_c \).

When \( \gamma > \gamma_c \), it follows from Theorem 4.4.2.1 that

\[ \inf_{u} J_\gamma(x_0; u, w) = J_\gamma(x_0; u^w, w) \]

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where

\[(4.4.2.9) \quad u^w(t) = -T_1^{-1}(T_2w + T_3^*x_0)(t), \quad \text{a.e.} \quad [0, t_f]\]

Since \( w \to J_\gamma(x_0; u^w, w) \) is strictly concave, there exists a unique maximizer \( w^* \) which satisfies the equation

\[\delta_w J_\gamma(x_0; u^w, w) = 0\]

Direct calculation yields

\[(4.4.2.10) \quad \delta_w J_\gamma(x_0; u^w, w)(m) = <S_\gamma w + Gx_0, m>_W \quad m \in W\]

where \( G \in \mathcal{L}(H, W) \)

\[(4.4.2.11) \quad G = [\kappa_0(-B T_1^{-1} T_2 + D)]^* Q_f[U(t_f, 0) + \kappa_0 B(-T_1)^{-1} T_3^*] + T_2^*(T_1^{-1})^* T_1^{-1} T_3^*\]

\[+ [T_0(-B T_1^{-1} T_2 + D)]^* C^* C[U_0 + T_0 B(-T_1)^{-1} T_3^*]\]

and \( S_\gamma = S - \gamma^2 I \), \( S \) is given by (4.4.2.4). Thus we have

\[(4.4.2.12) \quad S_\gamma w^* = -Gx_0.\]

Let \( x^*(t) \) be the optimal trajectory which corresponds to \( (u, w) = (u^*, w^*) \) and \( x^*(t) = T(t, s)x, 0 \leq s \leq t \leq t_f \). By using (4.4.2.9) and (4.4.2.12), \( (u^*, w^*) \) can be represented as

\[(4.4.2.13) \quad u^* = -B^* T_0 C^* C T_0 B u^* - B^* \kappa_0 Q_f \kappa_0 B u^* - (T_2 w^* + T_3^* x_0)\]

\[(4.4.2.14) \quad w^* = \frac{1}{\gamma^2} (S w^* + Gx_0)\]

By the definitions of \( T_1, T_2, T_3 \) and \( S \), (4.4.2.13) and (4.4.2.14) can be further written as

\[(4.4.2.15) \quad u^*(t) = -B(t)^* Z(t)x(t)\]

\[(4.4.2.16) \quad w^*(t) = \frac{1}{\gamma^2} D(t)^* Z(t)x(t)\]

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where $Z(t)$ satisfies

$$Z(t)x = U^*(t_f, t)Q_fT(t_f, t)x + \int_t^{t_f} U^*(\eta, t)C^*(\eta)C(\eta)T(\eta, t)x d\eta, 0 \leq t \leq t_f, x \in H$$

Note that the optimal system $x(t) = T(t, s)x$ can be viewed as $T(\cdot, \cdot)$ being the perturbed evolution operator corresponding to the perturbation of $U(\cdot, \cdot)$ by $(BB^* - \frac{1}{\gamma^2} DD^*)Z$. Thus, for $x \in H$,

$$T(t, s)x = U(t, s)x + \int_s^t T(t, \eta)(B(\eta)B^*(\eta) - \frac{1}{\gamma^2} D(\eta)D^*(\eta))Z(\eta)U(\eta, s)x d\eta.$$ 

Substitute $T(t, s)$ into (4.4.2.17), to obtain

$$Z(t)x = U^*(t_f, t)Q_fU(t_f, t)x + U^*(t_f, t)Q_f \int_t^{t_f} T(t_f, \eta)(B(\eta)B^*(\eta) - \frac{1}{\gamma^2} D(\eta)D^*(\eta))Z(\eta)U(\eta, t)x d\eta$$

$$+ \int_t^{t_f} U^*(\eta, t)C^*(\eta)C(\eta)U(\eta, t)x d\eta$$

$$+ \int_t^{t_f} U^*(\eta, t)C^*(\eta)C(\eta) \int_t^\eta T(\eta, \xi)(B(\xi)B^*(\xi) - \frac{1}{\gamma^2} D(\xi)D^*(\xi))Z(\xi)U(\xi, t)x d\xi$$

Note that $T^*(\tau, t) = (T(\tau, \eta)T(\eta, t))^* = T^*(\eta, t)T^*(\tau, \eta)$. Thus

$$U^*(t_f, t)Q_f \int_t^{t_f} T(t_f, \eta)(B(\eta)B^*(\eta) - \frac{1}{\gamma^2} D(\eta)D^*(\eta))Z(\eta)U(\eta, t)x d\eta$$

$$= \int_t^{t_f} U^*(\eta, t)U^*(t_f, \eta)Q_fT(t_f, \eta)(B(\eta)B^*(\eta) - \frac{1}{\gamma^2} D(\eta)D^*(\eta))Z(\eta)U(\eta, t)x d\eta$$

By using Fubini's theorem, we know that

$$\int_t^{t_f} U^*(\eta, t)C^*(\eta)C(\eta) \int_t^\eta T(\eta, \xi)(B(\xi)B^*(\xi) - \frac{1}{\gamma^2} D(\xi)D^*(\xi))Z(\xi)U(\xi, t)x d\xi$$

$$= \int_t^{t_f} U^*(\eta, t) \int_t^\eta U^*(\xi, \eta)T(\xi, \eta)(B(\eta)B^*(\eta) - \frac{1}{\gamma^2} D(\eta)D^*(\eta))Z(\eta)U(\eta, t)x d\xi d\eta$$

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By (4.4.2.19) and (4.4.2.20), (4.4.2.18) becomes

\[
Z(t)x = U^*(t_f,t)Q_f U(t_f,t)x + \int_t^{t_f} U^*(\eta,t)C^*(\eta)C(\eta)U(\eta,t)x d\eta \\
+ \int_t^{t_f} U^*(\eta,t)Z(\eta)(B(\eta)B^*(\eta) - \frac{1}{\gamma^2} D(\eta)D^*(\eta))Z(\eta)U(\eta,t)x d\eta
\]

Next we claim that \( Z \in \Sigma^+(H) \). For any \( x, y \in H \), by (4.4.2.17) we have

(4.4.2.21)

\[
< Z(t)x, y >_H = < Q_f T(t_f,t)x, U(t_f,t)y >_H \\
+ \int_t^{t_f} < C(\eta)T(\eta,t)x, C(\eta)U(\eta,t)y >_H d\eta
\]

Since from (4.4.2.15), (4.4.2.16), and (4.3.1), evolution operator \( T(\cdot, \cdot) \) can be viewed as the perturbation of \( U(\cdot, \cdot) \) by \(- (BB^* - \frac{1}{\gamma^2} DD^*)Z\), we have

\[
T(t,s)x = U(t,s)x - \int_s^t U(t,\eta)(B(\eta)B(\eta)^* - \frac{1}{\gamma^2} D(\eta)D(\eta)^*)Z(\eta)T(\eta,s)x d\eta
\]

Following the proof of last part of Theorem 4.4.1.4, we can obtain

\[
< Z(t)x, y >_H = < Q_f T(t_f,t)x, T(t_f,t)y >_H \\
+ \int_t^{t_f} < C(\eta)T(\eta,t)x, C(\eta)T(\eta,t)y >_H d\eta \\
- \int_t^{t_f} < B(\eta)T(\eta,t)S(\eta)x, B(\eta)S(\eta)T(\eta,t)y >_H d\eta \\
+ \frac{1}{\gamma^2} \int_t^{t_f} < D(\eta)^*S(\eta)T(\eta,t)x, D(\eta)^*S(\eta)T(\eta,t)y >_H d\eta \\
= < x, Z(t)y >_H
\]

The nonnegativity is obvious, and thus \( Z(\cdot) \in \Sigma^+(H) \). Therefore \( Z \) is the integral solution of (4.4.2).  \( \square \)

**Remark 4.4.2.4.** According to Proposition 4.4.4, we have \( \hat{\gamma}_c \geq \gamma_c \). Another observation is that letting \( \gamma_c^{t_f} \) and \( \gamma_c^{t_f'} \) be the critical values of the Riccati equation (4.4.2) on \([0, t_f]\)
and \([0, t_f']\) with \(t_f' \geq t_f\), then we have

\[
\gamma_c^{t_f'} \geq \gamma_c^{t_f'}
\]

which means that \(\{\gamma_c^{t_f'}\}\) is nondecreasing with respect to terminal time \(t_f\). This is important when we consider the case \(t_f \to \infty\).

**Corollary 4.4.2.5.** The Riccati equation (4.4.2) admits an integral solution if and only if the lower value of the game \(J^t_{\gamma}(x; u, w)\) is finite for any \(x \in H\) and \(t \in [0, t_f]\), where

\[
J^t_{\gamma}(x; u, w) = \langle Q_f x(t_f), x(t_f) \rangle_H + \int_t^{t_f'} \|C(s)x(s)\|_H^2 + \|u(s)\|_U^2 + \gamma^2 \|w(s)\|_V^2 ds
\]

**Proof.** The proof is straightforward. By Theorem 4.4.2.3, we have that if (4.4.2) admits an integral solution, then the lower value of the game is \(< Z_{\gamma}(t)x, x >_H\), thus it is finite for any \(x_0 \in H\) and \(t \in [0, t_f]\). Conversely, under the assumption of \(\sup_w \inf_u J^t_{\gamma}(x; u, w) < \infty\) for all \(t \in [0, t_f]\), by using the same argument as in the proof of second part of Theorem 4.4.2.3, the conclusion follows. \(\square\)

**Corollary 4.4.2.6.** (Representation formula of \(Z_{\gamma}\)) If \(\gamma > \gamma_c\), Riccati equation (4.4.2) admits an integral solution \(Z_{\gamma}\) on the interval \([0, t_f]\), which has the form:

\[
Z_{\gamma}(t)x = [U(t_f, t) - \kappa_t B T_1^{-1} T_3^*]^* Q_f [U(t_f, t) - \kappa_t B T_1^{-1} T_3^*] x

+ [U_t - \kappa_t B T_1^{-1} T_3^*]^* C^* C [U_t - \kappa_t B T_1^{-1} T_3^*] x - G^* S_{\gamma}^{-1} G x, \quad \forall x \in H,
\]

where

\[
S_{\gamma} = [\kappa_t (-B T_1^{-1} T_2 + D)]^* C^* C [\kappa_t (-B T_1^{-1} T_2 + D)] + T_2^* T_1^{-1} T_1^{-1} T_2

+ [\kappa_t (-B T_1^{-1} T_2 + D)]^* Q_f [\kappa_t (-B T_1^{-1} T_2 + D)] - \gamma^2 I
\]

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\[ G = [\mathcal{K}_t(-BT_1^{-1}T_2 + D)]^*Q_f[U(t_f, t) + \mathcal{K}_tB(-T_1)^{-1}T_3^*] + T_2^*(T_1^{-1})^*T_3^* \\
+ [T_1(-BT_1^{-1}T_2 + D)]^*C^*C[U_t + T_1B(-T_1)^{-1}T_3^*] \]

where \( T_t, \mathcal{K}_t, U_t \) are defined in (4.2.9) - (4.2.5), and \( T_1, T_2, T_3 \) are defined in (4.4.2.1) - (4.4.2.3) by replacing the corresponding lower scripts 0 with \( t \).

**Proof.** Introduce cost function

\[ J^t_\gamma(x; u, w) = <Q_f x(t_f), x(t_f)>_H + \int_t^{t_f} \|C(s)x(s)\|^2_2 + \|u(s)\|^2_T - \gamma^2 \|w(s)\|^2_W \, ds \]

From (4.4.2.9), we know that

\[ \inf_{u} J^t_\gamma(x; u, w) = J_\gamma(x; u^w, w) \]

where

\[ u^w(s) = -T_1^{-1}(T_2w + T_3^* x)(s), \quad a.e. [t, t_f] \]

Note that

\[ J^t_\gamma(x; u^w, w) = <Q_f(U(t_f, t)x + \mathcal{K}_t(Bu^w + Dw)), U(t_f, t)x + \mathcal{K}_t(Bu^w + Dw)>_H \]

\[ \|C(U_t x + T_t(Bu^w + Dw))\|_{Z_t}^2 + \|u^w\|_{U_t}^2 - \gamma^2 \|w\|_{W_t}^2 \]

Letting \( x = 0 \), we have

\[ \inf_{u} J^t_\gamma(0; u, w) = J^t_\gamma(0; u^w, w) = <S_\gamma w, w>_W \]

From Theorem 4.4.2.3, we know that

\[ \sup_{w} \inf_{u} J^t_\gamma(0; u, w) = 0 \]
Since \( w \rightarrow \inf_u J^t_\gamma(0; u, w) \) is strictly concave, \( w = 0 \) is the only maximizer of the function \( \inf_u J^t_\gamma(0; u, w) \), thus \( -S_\gamma \) is positive definite operator of \( L(W) \). Hence, according to (4.4.2.12), the maximizer of \( \inf_u J^t_\gamma(x; u, w) \) is given by

\[
w^* = -S_\gamma^{-1}Gx
\]

where \( G \) is given by (4.4.2.11). We next compute \( J^t_\gamma(x; u^w, w^*) \). For later convenience, we denote by \( \Lambda_1, \Lambda_2, \Lambda_3 \), respectively

(4.4.2.25)

\[
\Lambda_1 = \mathcal{K}_t(-BT_1^{-1}T_2 + D), \quad \Lambda_2 = C[T_t(-BT_1^{-1}T_2 + D)], \quad \Lambda_3 = T_1^{-1}T_2
\]

By using (4.4.2.25), (4.4.2.23), and (4.4.2.24), we have

\[
J^t_\gamma(x; u^w, w^*) \\
= \|Q^1_1(U(t_f, t) - \mathcal{K}_t BT_1^{-1}T_3^*) - \Lambda_1 S_\gamma^{-1}Gx\|_H^2 + \|C(U_t - C[T_t BT_1^{-1}T_3^*] - \Lambda_2 S_\gamma^{-1}Gx\|_{W_t}^2 \\
+ \|\Lambda_3 S_\gamma^{-1}G - T_1^{-1}T_3^*\|_W_t - \gamma^2 \|S_\gamma^{-1}Gx\|_{W_t}^2 \\
= \|Q^1_1(U(t_f, t) - \mathcal{K}_t BT_1^{-1}T_3^*)x\|_H^2 + \|C(U_t - T_t BT_1^{-1}T_3^*)x\|_{Z_t}^2 - <Gx, S_\gamma^{-1}Gx>_W_t \\
= <\Theta(t)x, x>_H
\]

where

\[
\Theta(t)x = [U(t_f, t) - \mathcal{K}_t BT_1^{-1}T_3^*]Q_1[U(t_f, t) - \mathcal{K}_t BT_1^{-1}T_3^*]x \\
+ [U_t - T_t BT_1^{-1}T_3^*]C*C[U_t - T_t BT_1^{-1}T_3^*]x - G*S_\gamma^{-1}Gx
\]

From Theorem 4.4.2.3, we know that

\[
\sup \inf_{w, u} J^t_\gamma(x; u, w) = <Z_\gamma(t)x, x>_H
\]

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Thus we have

$$< [Z_{\gamma}(t) - \Theta(t)]x, x >_H = 0 \quad \forall x \in H, \quad \forall t \in [0, t_f]$$

Since both $Z_{\gamma}$ and $\Theta$ are self-adjoint, following the same argument as Corollary 4.4.1.7, we have $Z_{\gamma} \equiv \Theta$ on $[0, t_f]$, and thus the proof is complete. \qed

Remark 4.4.2.7. If we let $B \equiv 0$, formula (4.4.2.22) becomes the same as the representation formula of $S_{\gamma}$ given by (4.4.1.19). In the case $\gamma = \gamma_e$, Riccati equation (4.4.2) may or may not have solution, which depends on whether the range of $G$ is contained in the domain of $S_{\gamma}^{-1}$ for all $t \in [0, t_f]$. In fact one can show that if the range of $G$ is contained in the domain of $S_{\gamma}^{-1}$ for all $t \in [0, t_f]$, the lower value of the game $J_{\gamma}^L(x; u, w)$ is bounded for any $x \in H$, and thus (4.4.2) admits a solution by Corollary 4.4.1.6, and the solution is given by (4.4.2.22). Similar argument can be used for (4.4.1).

4.5. The Differential Game and Its Saddle Point Solution

In this section, we establish the connection between the two critical values $\hat{\gamma}_c$, $\gamma_e$ and the soft-constrained differential game defined by the kernel:

$$J_{\gamma}(x_0; u, w) = \int_0^{t_f} \|C(t)x(t)\|_Z^2 + \|u(t)\|_U^2 - \gamma^2 \|w(t)\|_V^2 dt + < Q_f x(t_f), x(t_f) >_H$$

**Theorem. (Open-Loop Policy)** If $\gamma > \hat{\gamma}_c$, then the differential game $J_{\gamma}$ admits a unique saddle-point solution, given by

(4.5.1) \quad u^*(t) = -B(t)^*Z_{\gamma}(t)x^*(t)

(4.5.2) \quad w^*(t) = \frac{1}{\gamma^2} D(t)^*Z_{\gamma}(t)x^*(t)

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where $Z_\gamma$ is the integral solution of (4.4.2) and $x^*$ is the corresponding state trajectory, generated by

\[(4.5.3)\quad x^*(t) = T(t, 0)x_0\]

where evolution operator $T(\cdot, \cdot)$ satisfies

\[T(t, s)x = U(t, s)x + \int_s^t U(t, \eta)(B(\eta)B^*(\eta) - \frac{1}{\gamma^2} D(\eta)D^*(\eta)Z_\gamma(\eta))T(\eta, s)x d\eta\]

Moreover

\[J_\gamma(x_0; u^*, w^*) = (Z_\gamma(0)x_0, x_0)\]

If $\gamma < \gamma_c$, then the upper value of the game is unbounded for any $x_0 \in H$.

**Proof.** Since (4.2.1) has a mild solution, we know that (4.2.8) also has a bounded mild solution according to Theorem 4.4.4. Hence $Z_\gamma$ is well defined over $[0, t_f]$. We are going to show that for $x_0 \in H$ the following holds $\forall (u, w) \in U \times W$

\[J_\gamma(x_0; u^*, w) \leq J_\gamma(x_0; u^*, w^*) \leq J_\gamma(x_0; u, w^*)\]

Let us first show that $\min_u J_\gamma(x_0; u, w^*) = J(x_0; u^*, w^*)$. Since $\gamma > \gamma_c$, we know from Theorem 4.4.1.1 that

\[
\sup_u \inf_w J_\gamma(x_0; u, w) < \infty,
\]

and thus

\[(4.5.4)\quad \inf_u J_\gamma(x_0; u, w^*) \leq \sup_u \inf_w J_\gamma(x_0; u, w) < \infty\]

On the other hand, we have

\[(4.5.5)\quad J_\gamma(x_0; u, w^*) \geq \|u\|_U^2 - \gamma^2 \|w^*\|_W^2\]

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and \( u \to J_\gamma(x_0; u, w^*) \) is strictly convex. Hence (4.5.4) and (4.5.5) imply that there exists a unique \( \hat{u} \in \mathcal{U} \) such that

\[
\min_u J_\gamma(x_0; u, w^*) = J_\gamma(x_0; \hat{u}, w^*)
\]

Next we claim that \( \hat{u} = u^* \). Since \( \hat{u} \) satisfies equation

\[
\delta_u J_\gamma(x_0; u, w^*)(v) = 0, \quad \forall v \in \mathcal{U}
\]

which is equivalent to \( \hat{u} \) satisfying (4.4.2.13), we have

\[
(4.5.6) \quad \hat{u} = -B^*T_0C^*CT_0B\hat{u} - B^*K_0Q_fK_0B\hat{u} - T_1^{-1}(T_2w^* + T_3^*x_0)
\]

Note that \( w^* = \frac{1}{\gamma^2}D^*Z_{\gamma}x^* \) by (4.5.2), and by (4.5.1) \( u^* \) in fact solves (4.5.6) by Theorem 4.4.2.3. The uniqueness of minimizer gives that \( \hat{u} = u^* \).

Next we claim that \( \max_w J_\gamma(x_0; u^*, w) = J_\gamma(x_0; u^*, w^*) \). Again since \( \gamma > \gamma_c \), according to Theorem 4.4.1.1, we have that

\[
\sup_w J_\gamma(x_0; u^*, w) < \infty
\]

and \( w \to J_\gamma(x_0; u^*, w) \) is strictly concave. Hence there exists a unique \( \hat{w} \in \mathcal{W} \) such that

\[
\sup_w J_\gamma(x_0; u^*, \hat{w}) = J_\gamma(x_0; u^*, \hat{w})
\]

and this \( \hat{w} \) satisfies the equation

\[
\delta_w J_\gamma(x_0; u^*, w)(m) = 0, \quad m \in \mathcal{W}
\]

On the other hand, form Theorem 4.4.2.3, we know that

\[
\delta_w J_\gamma(x_0; u^*, w^*)(m) = 0, \quad m \in \mathcal{W},
\]

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which implies $\hat{w} = w^*$ by the uniqueness of the maximizer.

If $\gamma < \gamma_c$, note that

$$\inf_u \sup_w J_{\gamma}(x_0; u, w) \geq \sup_w \inf_u J_{\gamma}(x_0; u, w)$$

and $\sup_w \inf_u J_{\gamma}(x_0; u, w) = +\infty$ by Theorem 4.4.1.1; therefore the upper value of $J_{\gamma}$ is unbounded. $\square$

Let us assume that the controller has access to instantaneous perfect measurements of the state of system (4.3.1). We therefore want to allow control of the form $u(t) = \mu(x(t))$ for a large class of $\mu$'s. We denote by $\mathcal{M}$ the class of all applications from $H$ into $U$ that are such that the integral equation

(4.5.7)

$$x(t) = U(t, s)x_0 + \int_s^t U(t, \eta)(B(\eta)\mu(x(\eta)) + D(\eta)w(\eta))d\eta \quad 0 \leq s \leq t \leq t_f$$

has a solution in $C(0, t_f; H)$ for all $x_0 \in H$ and for all $w \in \mathcal{W}$. We are now in a position to state the following result.

**Theorem 4.5.1. (Closed-Loop Policy)** If $\gamma > \gamma_c$, then the value function

(4.5.8)

$$\varphi(x_0) = \inf_{\mathcal{M}} \sup_{\mathcal{W}} J_{\gamma}(u, w, x_0) = \sup_{\mathcal{W}} \inf_{\mathcal{M}} J_{\gamma}(u, w, x_0)$$

admits a unique saddle-point solution, given by

(4.5.9)

$$\mu^*(t, x(t)) = -B^*(t)Z_{\gamma}(t)x(t)$$

(4.5.10)

$$\nu^*(t, x(t)) = \frac{1}{\gamma^2} D(t)^*Z_{\gamma}(t)x(t)$$

Moreover

$$J_{\gamma}(x_0; \mu^*, \nu^*) = (Z_{\gamma}(0)x_0, x_0)$$

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If $\gamma < \gamma_c$, then both the upper and lower value are unbounded for any $x_0 \in H$.

**Proof.** For $\gamma > \gamma_c$ we know that the Riccati equation (4.4.2) has a well-defined integral solution over the integral $[0, t_f]$. By using standard regularization and the completion of squares, we can get $\forall (u, w) \in U \times W$

\begin{equation}
J_\gamma(x_0; u, w) = (Z_\gamma(0)x_0, x_0)_H \\
+ \|u + B^*Z_\gamma x\|^2_U - \gamma^2\|w - \gamma^{-2}D^*Z_\gamma x\|^2_W
\end{equation}

It is obvious that $\forall (u, w) \in U \times W$ we have

$$J_\gamma(x_0; \mu^*, w) \leq J_\gamma(x_0; \mu^*, \nu^*) \leq J_\gamma(x_0; u, \nu^*)$$

Thus the conclusion follows.

If $\gamma < \gamma_c$, note that

$$\sup_w \inf_{\mu \in M} J_\gamma(x_0; u, w) = \sup_w \inf_u J_\gamma(x_0; u, w) \leq \sup_w J_\gamma(x_0; \mu, w), \quad \forall \mu \in M$$

and thus

$$\sup_w \inf_u J_\gamma(x_0; u, w) \leq \inf_{\mu \in M} \sup_w J_\gamma(x_0; \mu, w)$$

By Theorem 4.4.2.1, we have $\sup_w \inf_u J_\gamma(x_0; u, w) = +\infty$, and hence both upper and lower values are unbounded for any $x_0 \in H$. \qed

**Remark 4.5.2.** The relationship between the minimax disturbance attenuation of $H^\infty$-optimal control problem and the differential game $J_\gamma$ can now be seen. For $\gamma > \gamma_c$, under the choice of the feedback controller given by (4.5.9), the upper value of the game is finite according to Theorem 4.5.1. In particular, $\varphi(0) = 0$, which gives $\max_w J_\gamma(0; \mu^*, w) \leq 0$. 

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Thus we have

\[
(4.5.12) \quad \frac{\int_0^{t_f} (\|C x(t_f)\|^2 + \|\mu^*\|^2) dt + < Q_f x(t_f), x(t_f) >_H}{\int_0^{t_f} \|w\|^2_W dt} < \gamma^2
\]

which implies the $H^\infty$-optimal controller design, and that $\gamma_c$ is the smallest bound so that (4.5.12) can hold.

4.6. An Approximation

From a computational view point, we generally need to know conditions under which the solutions of a sequence of Riccati equations converge to the solution of a given Riccati equation. This happens in particular when we design a finite-dimensional compensator for an infinite-dimensional optimal control problems; the Riccati equations for the finite-dimensional compensator usually are ordinary differential equations, and the approximation to an infinite-dimensional controller is based upon these Riccati equations. Such convergence has been proven for LQ problems involving particular types of differential equations (see, for instance, Ito [It], Gibson [G2], Lions [L1], [L3], Banks and Kunish [BK] and Curtain [Cu2]). In $H^\infty$-optimal control problems, the critical value is also needed, so that the robustness (4.5.12) can be maintained during the approximation process. This is what is undertaken in this section, where we study the approximation of the generalized Riccati equation (4.4.2).

Suppose that $\{U_\varepsilon\}$ is a sequence of evolution operators on $H$ and that $\{B_\varepsilon(\cdot)\}$, $\{C_\varepsilon(\cdot)\}$, $\{D_\varepsilon(\cdot)\}$, and $Q^\varepsilon_{t_f}$ are sequences of operators in the spaces of $B_\infty(0,t_f;U,H)$, $B_\infty(0,t_f;H,Y)$, $B_\infty(0,t_f;W,H)$, and $\mathcal{L}(H)$, respectively. Suppose that

\[
(4.6.1) \quad U_\varepsilon(t,s)x \rightarrow U(t,s)x \quad \text{strongly, } 0 \leq s \leq t \leq t_f
\]
\[
(4.6.2) \quad U_\varepsilon(t, s)^* x \to U(t, s)^* x \quad \text{strongly, } 0 \leq s \leq t \leq t_f \\
(4.6.3) \quad B_\varepsilon(t) u \to B(t) u \quad \text{strongly, a.e.,} \\
(4.6.4) \quad B_\varepsilon(t)^* x \to B(t)^* x \quad \text{strongly, a.e.,} \\
(4.6.5) \quad C_\varepsilon(t) x \to C(t) x \quad \text{strongly, a.e.,} \\
(4.6.6) \quad C_\varepsilon(t)^* y \to C(t)^* y \quad \text{strongly, a.e.,} \\
(4.6.7) \quad D_\varepsilon(t) w \to D(t) w \quad \text{strongly, a.e.,} \\
(4.6.8) \quad D_\varepsilon(t)^* x \to D(t)^* x \quad \text{strongly, a.e.,} \\
(4.6.9) \quad Q_\varepsilon^* x \to Q_f x \quad \text{strongly,}
\]

as \( \varepsilon \to \infty \), where \( x \in H \), \( u \in U \), \( w \in W \), and \( y \in Y \). We also assume that \( \|U_\varepsilon(t, s)\|_{L^*(H)} \), \( \|B_\varepsilon\|_{S_\infty} \), \( \|C_\varepsilon\|_{S_\infty} \), \( \|D_\varepsilon\|_{S_\infty} \), and \( \|Q_\varepsilon^*\| \) are uniformly bounded in \( \varepsilon \), \( t \), and \( s \). For the following sequence of Riccati equations:

\[
(4.6.10) \quad Z^\varepsilon(t) x = U_\varepsilon(t_f, t)^* Q_\varepsilon^* U_\varepsilon(t_f, t) x + \int_t^{t_f} U_\varepsilon(s, t)^* \left\{ C_\varepsilon^*(s) C_\varepsilon(s) \\
- Z^\varepsilon(s)(B_\varepsilon(s) B_\varepsilon(s)^* - \frac{1}{\gamma} D_\varepsilon^*(s) D_\varepsilon(s)) Z^\varepsilon(s) \right\} U_\varepsilon(s, t) x ds \quad \forall x \in H
\]

the critical values are

\[
\gamma^2_{\varepsilon } := \|(S^\varepsilon)^{1/2}\|_{L^2(W)}^2
\]

where

\[
S^\varepsilon = [T^\varepsilon_0 (-B_\varepsilon(T^\varepsilon_1)^{-1} T^\varepsilon_2 + D_\varepsilon)]^* C_\varepsilon^* C_\varepsilon [T^\varepsilon_0 (-B_\varepsilon(T^\varepsilon_1)^{-1} T^\varepsilon_2 + D_\varepsilon)]^* \\
+ (T^\varepsilon_2)^*(T^\varepsilon_1)^{-1}(T^\varepsilon_1)^{-1} T^\varepsilon_2 + [K^\varepsilon_0 (-BT_1^{-1} T_2 + D)]^* [K^\varepsilon_0 (-B_\varepsilon(T^\varepsilon_1)^{-1} T^\varepsilon_2 + D_\varepsilon)]
\]

and \( T^\varepsilon_1, T^\varepsilon_2, T^\varepsilon_3 \) are as defined in the beginning of section 4.2, by replacing all terms with corresponding \( \varepsilon \)-terms, and \( T^\varepsilon_0, K^\varepsilon_0 \) are defined in the section 2 by replacing all terms with
corresponding $\varepsilon$-terms. Under the uniform boundedness assumptions, we have $\gamma^M := \sup_{\varepsilon} \gamma_{\varepsilon} < \infty$. Thus for $\gamma > \gamma^M$, the sequence of Riccati equations (4.6.10) have unique integral solutions. Let

$$ J^\varepsilon_\gamma(x_0^\varepsilon; u, w) = \langle Q^\gamma \bar{x}_\varepsilon(t_f), \bar{x}_\varepsilon(t_f) \rangle_H + \int_0^{t_f} (\| C_\varepsilon x_\varepsilon \|^2_Z + \| u \|^2_Y - \gamma^2 \| w \|^2_W) d\eta $$

where we assume that $x_0^\varepsilon \to x_0$. According to Theorem 4.4.2.3, the lower value solution of $J^\varepsilon_\gamma$ is given by

$$ w^\varepsilon = (S^\varepsilon_\gamma)^{-1} G^\varepsilon x_0^\varepsilon, \quad u^\varepsilon = -(T^\varepsilon_1)^{-1}(T^\varepsilon_2 w^\varepsilon + (T^\varepsilon_3)^* x_0^\varepsilon) $$

where $G^\varepsilon$ is as defined in section 4.2 by replacing all terms with $\varepsilon$-terms. Note that $(T^\varepsilon_1)^{-1}$ and $(S^\varepsilon_1)^{-1}$ are also uniformly bounded in $t, s, \varepsilon$, and

$$(T^\varepsilon_1)^{-1} - T_1^{-1} = (T^\varepsilon_1)^{-1}(T_1^\varepsilon - T_1)T_1^{-1}, \quad (S^\varepsilon_1)^{-1} - S_1^{-1} = (S^\varepsilon_1)^{-1}(S_1^\varepsilon - S_1)S_1^{-1}$$

Thus by (4.6.1)-(4.6.9), we have

$$ w^\varepsilon \to S^{-1}_1 G x_0 := w^*(t) \quad \text{strongly, a.e. and in } L^2(0, t_f; W) $$

$$ u^\varepsilon(t) \to -T_1^{-1}(T_2 w^* + T_3^* x_0) := u^*(t) \quad \text{strongly, a.e. and in } L^2(0, t_f; U) $$

and

$$ x_\varepsilon(t) = U_\varepsilon(t, 0)x_0^\varepsilon + \int_0^t U_\varepsilon(t, \eta)(B_\varepsilon(\eta)u^\varepsilon(\eta) + D_\varepsilon(\eta)w^\varepsilon(\eta)) d\eta $$

$$ \longrightarrow x^*(t) = U(t, 0)x + \int_0^t U(t, \eta)(B(\eta)u^*(\eta) + D(\eta)w^*(\eta)) d\eta $$

strongly, pointwise and in $L^2(0, t_f; H)$

and

(4.6.11)

$$ \sup_{w} \inf_{u} J^\varepsilon_\gamma(x_0^\varepsilon; u, w) = J^\varepsilon_\gamma(x_0^\varepsilon; u^\varepsilon, w^\varepsilon) \to J_\gamma(x_0; u^*, w^*) = \sup_{w} \inf_{u} J_\gamma(x_0; u, w) $$

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Since we have the strong convergence for $U^*_\varepsilon(\cdot, \cdot)$, $Q^*_\varepsilon$, $C_\varepsilon(\cdot)$, and $T_\varepsilon(\cdot, \cdot)$, (4.4.2.5) implies that $Z^\varepsilon_\gamma$ also has strong convergence. Applying Theorem 4.5.1, we know that $(u^\varepsilon, w^\varepsilon)$ and $(u^*, w^*)$ can be represented in feedback forms; therefore we have

\[ u^\varepsilon(t) = -B_\varepsilon(t)^*Z^\varepsilon_\gamma(t)x_\varepsilon(t) \rightarrow -B(t)^*Z_\gamma x(t) = u^*(t), \]

strongly, a.e. and in $L^2(0, t_f; U)$

\[ w^\varepsilon(t) = \frac{1}{\gamma_\varepsilon^2} D(t)^*Z^\varepsilon_\gamma(t)x_\varepsilon(t) \rightarrow \frac{1}{\gamma^2} D(t)^*Z_\gamma x(t) = w^*(t), \]

strongly, a.e. and in $L^2(0, t_f; W)$

Usually we cannot have $\gamma_{ce} \rightarrow \gamma_c$ (otherwise it would imply that $S^\varepsilon$ converges to $S$ in norm), but a weaker convergence holds:

\[ \limsup_{\varepsilon \rightarrow \infty} \gamma_{ce} = \gamma_c \]

which can be easily verified by (4.6.11).

We summarize the above discussion in the following theorem.

**Theorem 4.6.1.** Let (4.6.1)–(4.6.9) and the uniform boundedness condition hold, and $\gamma^M$ be defined as

\[ \gamma^M := \sup_{\varepsilon} \{\gamma_{ce}\} \]

Then we have:

1. For $\gamma > \gamma^M$, let the solutions of (4.6.10) be $\{Z^\varepsilon_\gamma\}$. Then

\[ Z^\varepsilon_\gamma(t)x \rightarrow Z_\gamma(t)x \quad \text{strongly, pointwise and in} \quad L^2(0, t_f; H), \]

(4.6.12)
where $Z_\gamma$ is the integral solution of (4.4.2).

(2) Let the pairs $(u^\varepsilon, w^\varepsilon)$ and $(u^*, w^*)$ be, respectively, such that

$$J_\gamma(x_0; u^*, w^*) = \sup_w \inf_u J_\gamma(x_0; u, w), \quad J_\gamma^\varepsilon(x_0^\varepsilon; u^\varepsilon, w^\varepsilon) = \sup_w \inf_u J_\gamma^\varepsilon(x_0^\varepsilon; u, w)$$

Then,

(4.6.13)\hspace{1cm} u^\varepsilon(t) = -B_\varepsilon(t)^*Z_\gamma^\varepsilon(t)x_\varepsilon(t) \to -B(t)^*Z_\gamma x(t) = u^*(t), \text{ strongly, a.e. and in } L^2(0, t_f; U)

(4.6.14)\hspace{1cm} w^\varepsilon(t) = \frac{1}{\gamma^2} D(t)^*Z_\gamma^\varepsilon(t) \to \frac{1}{\gamma^2} D(t)^*Z_\gamma x(t) = w^*(t), \text{ strongly, a.e. and in } L^2(0, t_f; W)

(3) Let $x_\varepsilon(t)$ and $x^*(t)$ be the optimal trajectories corresponding to $(u^\varepsilon, w^\varepsilon)$ and $(u^*, w^*)$ respectively. Then

(4.6.15)\hspace{1cm} x_\varepsilon(t) \to x^*(t) \text{ strongly, a.e. and in } L^2(0, t_f; H)

(4) When $\gamma > \gamma^M$, we have

$$\sup_w \inf_u J_\gamma^\varepsilon(x_0^\varepsilon; u, w) \to \sup_w \inf_u J_\gamma(x_0; u, w)$$

and

$$\inf_{\mu \in \mathcal{M}} \sup_w J_\gamma^\varepsilon(x_0^\varepsilon; u, w) \to \inf_{\mathcal{M}} \sup_w J_\gamma(x_0; u, w)$$

(5) Let $\gamma_c$ be the critical value of (4.4.2). Then

$$\lim_{\varepsilon \to \infty} \gamma_{c\varepsilon} = \gamma_c$$

(6) If $B(\cdot), B(\cdot)^*, C(\cdot), C(\cdot)^*, D(\cdot), \text{ and } D(\cdot)^*$ are piecewise strongly continuous (i.e., possess a finite number of discontinuities), and (4.6.1)-(4.6.9) are uniformly convergent, then (4.6.12)-(4.6.15) are also uniformly convergent.

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4.7. Concluding Remarks

In this chapter, we have shown that there are two critical parameter values $\gamma_c$ and $\tilde{\gamma}_c$ which determine whether two generalized Riccati equations have solutions in a real, separable Hilbert space. These two parameters are in fact the open-loop disturbance attenuation bounds and the state-feedback minimax disturbance attenuation bounds for the $H^\infty$-optimal control problems associated with a class of time-variant evolution processes in a Hilbert space, driven by disturbances also from Hilbert spaces. Since explicit expressions are given, the computation of these two parameters now become possible. This is particularly important in the $H^\infty$-optimal controller design because one in general wants to know what the achievable best performance of a system is under uncertainties.

We can in fact drop the assumption that the evolution operator $U(\cdot, \cdot)$ is associated with a family of infinitesimal generators of $C_0$ semigroup as Gibson did in [G2]. The reason we did not do that is because such an assumption already covers a larger class of infinite-dimensional systems, and moreover with this assumption the proof of Theorem 4.4.1.4 and Theorem 4.4.2.3 become easier, and more transparent.

It follows from Remark 4.4.2.4 that the critical values of (4.4.2) are monotonically nondecreasing as the terminal time $t_f$ increases. In a follow-up paper, we actually define the critical values of algebraic Riccati equations by letting $t_f \to \infty$, under the appropriate stabilizability and detectability assumptions.
5. $H^\infty$-Optimal Control Under Sampled State Measurements

5.1. Introduction

As we know, in practice, output information is not always available as a continuous stream. In fact, output information is often only available at discrete time instants. Hence a feedback scheme which only uses sampled data is an attractive choice from a practical viewpoint. It is this reason which motivates us to study sampled feedback controls. To date, some related work have been done by Chammas and Leondes [CL 1-4] for finite dimensional case, and by Tarn, Zeng and Zavgren for infinite dimensional case, which, however, is not related to $H^\infty$-optimal problems. They used sampled feedback for time invariant linear systems—the so called periodic output feedback control [TZZ]. Earlier work on the $H^\infty$-optimal control problem has dealt primarily with the continuous state/output measurement case in finite dimensional case (see [Fr],[DGKF]), or in infinite-dimensions (see Keulen [Ke], Peters and Curtain [KPC], Barbu [B 1-2], McMillian and Triggiani [MT 1-3]). For the $H^\infty$ problem with sampled measurements in finite dimensions, the characterization and computation has been studied by Bamieh and Pearson [BaPe], Chen and Francis [CF], and Kabamba and Hara [KH], among others.

Başar introduced time-varying sampled feedback control scheme for finite-dimensional linear systems in 1991 (see [B]). In that work, the following linear system has been studied

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + D(t)w(t)$$

$$y = C(t)x(t)$$

by using sampled feedback scheme

$$u(t) = \mu(t, x_{[0,k]}), \quad t_k \leq t < t_{k+1}$$
to deal with the $H^\infty$-optimal problem. Here $x_{\{0,k\}}$ denotes the sequence of sampled states $x(t_0), x(t_1), \ldots, x(t_k)$, $t_1, t_2, \ldots, t_k$ is a sequence of sampling times, and $A(t), B(t), C(t)$ are matrices of the appropriate sizes. The control scheme can be thought of as a hybrid of open-loop and closed-loop control in the following sense: the control is a closed-loop control at the sampling times and it is an open-loop time-varying control between the sampling times. The control function $u(\cdot)$ has no information concerning the state of the system on the open intervals $(t_k, t_{k+1})$, other than its value at $x(t_k)$. A complete solution to the finite-horizon $H^\infty$-optimal control problem formulated above as well as for the time-invariant system for the infinite-horizon case has been obtained in [B].

In this chapter we extend the previous results on finite-dimensional spaces to infinite-dimensional spaces, covering both time-varying linear systems on finite horizon and time-invariant linear systems on infinite horizon. Furthermore, we show that using sampled-data state measurements, it is possible to attain attenuation levels achievable by continuous feedback, provided of course that the number of samples are sufficiently large. We devise a procedure for the computation of the optimum disturbance attenuation level in the sampled-data case and obtain an expression for the optimum controller when the desired level is larger than the optimum under continuous perfect state measurements. We then study the limiting behaviors of the finite-horizon $H^\infty$-optimal controller and the attenuation bound as $t_f \to \infty$, and also discuss the choice of the 'optimum' sampling times.

The rest of the chapter is organized as follows. Section 2 provides problems formulation and assumptions as well as some preliminaries related to two generalized Riccati equations. In section 3, we provide some connection between the optimization of a differential game and the two generalized Riccati equations. In section 4 we discuss the saddle-point proper-
ties of a differential game under the open-loop policy and closed-loop policy, respectively.

The main results on sampled-data feedback control of $H^\infty$-optimal control are given in section 5 and section 6. Section 5 discusses the finite-horizon case, while section 6 is for the infinite-horizon case. One example is provided in section 7 to illustrate the theory. Section 8 provides concluding remarks, and the chapter ends with an Appendix which provides proof of a result used in the proof of Theorem 5.4.1.

5.2. Preliminaries

5.2.1. Problem formulation and assumptions

Let $H$ be a real, separable Hilbert space, and $\mathcal{L}(H)$ be the space of bounded linear operators mapping $H$ into itself. The following is the definition of an evolution operator which is used throughout this chapter.

**Definition 5.2.1.1.** A two parameter family of bounded linear operators $U(t,s) \in \mathcal{L}(H)$, where $-\infty < t_0 \leq s \leq t \leq t_f < \infty$, on $H$ is called an evolution operator if the following two conditions are satisfied:

(i) $U(s,s) = I, U(t,r)U(r,s) = U(t,s)$ for $t_0 \leq s \leq r \leq t \leq t_f$.

(ii) $(t,s) \rightarrow U(t,s)$ is strongly continuous for $t_0 \leq s \leq t \leq t_f$.

We denote the Banach space of strongly measurable, essentially bounded functions from $(0,t_f)$ to $\mathcal{L}(H)$ by $B(0,t_f; H,H)$. We consider the dynamical system governed by the evolution process

(5.2.1.1) 

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)(B(s)u(s) + D(s)w(s))ds \quad 0 < t \leq t_f$$

(5.2.1.2) 

$$z(t) = C(t)x(t) + D_{12}(t)u(t);$$

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where \( x(t) \in H \) is the state of the system, and \( U(\cdot, \cdot) \) is an evolution operator on \( H \). \( u(t) \in U \) is the control input, \( w(t) \in W \) is the disturbance input, and \( z(t) \in Z \) is the to-be-controlled output, where \( U, W, \) and \( Z \) are also real separable Hilbert spaces. \( B \in B_\infty(0, t_f; U, H), B^* \in B_\infty(0, t_f; H, U), D \in B_\infty(0, t_f; W, H), D^* \in B_\infty(0, t_f; H, W), C \in B_\infty(0, t_f; H, Z), C^* \in B_\infty(0, t_f; Z, H), D_{12} \in B_\infty(0, t_f; U, Z) \), and \( D_{12}^* \in B_\infty(0, t_f; Z, U) \).

Let us first assume that the controller can access instantaneous perfect measurements of the state of the system (5.2.1.1), i.e. \( u(t) = \mu(x(t)) \) for a large class of \( \mu \)'s. We denote by \( \mathcal{M} \) the class of all applications from \( H \) to \( U \) such that the integral equation:

\[
x(t) = U(t, 0)x_0 + \int_0^t U(t, s)(B(s)\mu(x(s)) + D(s)w(s))ds
\]

has a solution in \( C(0, t_f; H) \) for all \( x_0 \in H \) and for all \( w \in L^2(0, t_f; W) \). We introduce the quadratic cost function

\[
(5.2.1.3) \quad L(u, w) = \int_0^{t_f} \|z(t)\|_Z^2 dt + (Q_f x(t_f), x(t_f))_H
\]

where \( Q_f \in \mathcal{L}(H) \) is Hermitian and nonnegative. Under the standard hypothesis that \( D_{12}[C, D_{12}] = [0, I] \), (5.2.1.3) becomes

\[
(5.2.1.4) \quad L(u, w) = \int_0^{t_f} (\|C(t)x(t)\|_Z^2 + \|u(t)\|_2^2)dt + (Q_f x(t_f), x(t_f))_H
\]

For each fixed \( \mu \in \mathcal{M} \), \( L \) induces a mapping of \( L^2([0, t_f]; W) \) into nonnegative reals which we denote by \( \mathcal{F}_\mu \). The minimax (\( H^\infty \)-optimal) controller design problem is to find a controller \( \mu \) that minimizes the following index

\[
(5.2.1.5) \quad g(\mu) := \sup_w \{ \mathcal{F}_\mu^1 / \|w\|_{L^2(0, t_f; W)} \}
\]

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or for a given desired attenuation level $\gamma_{des}$ to find $\mu$ such that

\begin{equation}
(5.2.1.6) \quad g(\mu) := \sup_w \left\{ \mathcal{F}_u^\frac{1}{2}/\|w\|_{L^2(0,t_f; W)} \right\} < \gamma_{des}
\end{equation}

Note that if the controller space were restricted to comprise only linear maps, then $g(\mu)$ would be the operator norm of the (weighted) gain from the disturbance to the output $x$.

We make the further assumptions on the evolution operator given in (5.2.1.1):

(i) for each $t \in [0, t_f]$ $A(t) : D(A(t)) : D(A(t)) \subset H \to H$ generates a $C_0$ semigroup in $H$, and

$$
\frac{\partial}{\partial t} U(t, s) x = A(t) U(t, s) x, \quad \frac{\partial}{\partial s} U(t, s) x = -U(t, s) A(s) x
$$

for $\forall x \in D(A(t))$; $0 \leq s \leq t \leq t_f$.

(ii) $U(\cdot, \cdot)^*$ is strongly continuous mapping:

$$
U(\cdot, \cdot)^* : \{ (t, s) \in \mathbb{R}^2 : t \geq s \} \to \mathcal{L}(H).
$$

(iii) $U_{A_n}(t, s)x \to U(t, s)x, \forall x \in H$ uniformly on the bounded sets of $\{ (t, s) \in \mathbb{R}^2 : t \geq s \}$, where $U_{A_n}(t, s)$ is the evolution operator generated by the Yosida approximations of $A(t)$.

Before we move to the next section, let us introduce some notations:

$$
\Sigma(H) = \{ T \in \mathcal{L}(H) : T \text{ is Hermitian} \}
$$

$$
\Sigma^+(H) = \{ T \in \Sigma(H) : (Tx, x) \geq 0, \forall x \in H \}
$$

The space $\Sigma(H)$, endowed with the norm of $\mathcal{L}(H)$, is a real Banach space and $\Sigma^+(H)$ is a cone in $\Sigma(H)$. For any interval $I$ in $\mathbb{R}$, we shall denote by $C(I; \Sigma(H))$ the set of all continuous mappings from $I$ to $\Sigma(H)$, by $C_c(I; \Sigma(H))$ the set of $C(I; \Sigma(H))$ endowed with
the topology of uniform convergence on compact subsets of $I$, and by $C_u(I; \Sigma(H))$ the set of $C(I; \Sigma(H))$ endowed with the topology of operator norm. Throughout this chapter, $J_\gamma$ denotes the soft-constrained cost function:

$$J_\gamma(x_0; u, w) = \int_0^{t_f} \left\{ \|C(t)x(t)\|_Z^2 + \| u(t) \|_U + \frac{1}{\gamma^2} \| w(t) \|_W^2 \right\} dt + (Q_f x(t_f), x(t_f))_H$$

and

$$\mathcal{U} := L^2(0, t_f; U), \quad \mathcal{W} := L^2(0, t_f; W)$$

5.2.2. Two generalized Riccati equations

First we introduce the definition of solutions of two generalized Riccati equations of relevance to the problem at hand:

$$\dot{S}_\gamma(t) + A(t)^* S_\gamma(t) + S_\gamma(t) A(t) + \frac{1}{\gamma^2} S_\gamma(t) D(t) D(t)^* S_\gamma(t) + C(t)^* C(t) = 0 \quad (S)$$

$$S_\gamma(t_f) = Q_f$$

and

$$\dot{Z}_\gamma(t) + A(t)^* Z_\gamma(t) + Z_\gamma(t) A(t) - Z_\gamma(t) B(t) B(t)^* Z_\gamma(t)$$

$$+ \frac{1}{\gamma^2} Z_\gamma(t) D(t) D(t)^* Z_\gamma(t) + C(t)^* C(t) = 0 \quad (Z)$$

$$Z_\gamma(t_f) = Q_f$$

**Definition 5.2.2.1.** For $\gamma > 0$, let $S_\gamma \in C_s([0, t_f], \Sigma(H))$ satisfy the integral equation (5.2.2.1)

$$S_\gamma(t)x = U(t_f, t)^* P_f U(t_f, t)x + \int_t^{t_f} U(s, t)^* (C(s)^* C(s)$$

$$+ \frac{1}{\gamma^2} S_\gamma(s) D(s)^* D(s) S_\gamma(s)) U(s, t)x ds \quad \forall x \in H$$

Then $S_\gamma$ is called the mild solution of the generalized Riccati equation (S).
**Definition 5.2.2.2.** For $\gamma > 0$, let $Z_\gamma \in C_\delta([0, t_f], \Sigma(H))$ satisfy the integral equation

\[
Z_\gamma(t)x = U(t_f, t)^* P_{t_f} U(t_f, t)x + \int_t^{t_f} U(s, t)^* \left\{ C(s)^* C(s) - Z_\gamma(s)(B(s)B(s)^* - \frac{1}{\gamma^2} D(s)D(s)^*)S_\gamma(s) \right\} U(s, t)x ds \quad \forall x \in H
\]

Then $Z_\gamma$ is called the mild solution of the generalized Riccati equation (Z).

**5.2.3. Properties of the generalized Riccati equations (S) and (Z)**

The following result will be needed in the sequel.

**Proposition 5.2.3.1.** Riccati equation (S) (respectively, (Z)) has at most one integral solution.

**Proof.** See Proposition 4.4.3 of Chapter 4.

**Proposition 5.2.3.2.** For a given $\gamma > 0$ suppose that both generalized Riccati equations (S) and (Z) admit mild solutions $S_\gamma, Z_\gamma$ respectively on the interval $[0, t_f]$. Then we have

\[
0 \leq Z_\gamma(t) \leq S_\gamma(t), \quad \forall t \in [0, t_f]
\]

**Proof.** See Proposition 4.4.4 of Chapter 4.

**Proposition 5.2.3.3.** The generalized Riccati equation (S) admits a unique mild solution $S \in C_u([\tau, t_f]; \Sigma(H))$ with $\tau$ satisfying the bound

\[
0 < t_f - \tau \leq \min\left\{ \frac{\|Q_f\|_{\mathcal{L}(H)}}{\|C\|_{B_\infty}^2 + \frac{\gamma^2}{\gamma^2} \|D\|_{B_\infty}^2}, \frac{\alpha \gamma^2}{2r M_{t_f}^2 \|D\|_{B_\infty}^2}, \frac{t_f}{2} \right\}
\]

where

\[
M_{t_f} = \sup\{\|U(t, s)\|; 0 \leq s \leq t \leq t_f\}, \quad r = 2M_{t_f}^2 \|Q_f\|_{\mathcal{L}(H)}, \quad 0 < \alpha < 1
\]
and

$$
\lim_{n \to \infty} S_n = S \text{ in } C_s([\tau, t_f]; \Sigma(H))
$$

where $S_n$ is the solution to equation (5) with $A(t)$ replaced by $A_n(t)$.

**Proof.** Note that

$$
t_f - \min\left\{ \frac{\|Q_f\|}{\|C\|_1^2 + \frac{r^2}{\gamma^2} \|D\|_1^2}, \frac{\alpha}{2rM_{t_f}^2 \|D\|_1^2}, \frac{t_f}{2} \right\} > 0
$$

Choose $r > 0$ such that (2.9) is true and consider a ball with radius $r$ in space $C_u([\tau, t_f]; \Sigma(H))$:

$$
B_{r, r} = \{ F \in C_u([\tau, t_f]; \Sigma(H)) : \|F\| \leq r \}
$$

Introduce transformations $T$ and $T_n$, both mapping $C_u([\tau, t_f]; \Sigma(H))$ into itself as:

$$
T(S)(t)x = U(t_f, t)^* Q_f U(t_f, t)x
$$

$$
+ \int_t^{t_f} U^*(s, t)(C(s)^* C(s) + \frac{1}{\gamma^2} S(s)D(s)D(s)^* S(s)) U(s, t)x ds
$$

and

$$
T_n(S)(t)x = U_{A_n}^*(t_f, t) Q_f U_{A_n}(t_f, t)x
$$

$$
- \int_t^{t_f} U_{A_n}^*(s, t)(C(s)^* C(s) + \frac{1}{\gamma^2} S(s)D(s)D(s)^* S(s)) U_{A_n}(s, t)x ds
$$

Let $S \in B_{r, r}$; then we have

$$
\|T(S)(t)x\|_H \leq M_{t_f}^2 \left\{ \|Q_f\|_{\Sigma(H)} + (t_f - \tau)(\|C\|_1^2 + \frac{r^2}{\gamma^2} \|D\|_1^2) \right\}
$$

$$
\leq 2M_{t_f}^2 \|Q_f\| \leq r\|x\|
$$

Similarly

$$
\|T_n(S)(t)x\|_{\Sigma(H)} \leq r\|x\|
$$
It follows that

\[ \|T(S)(t)\| \leq r \quad \|T_n(S)(t)\| \leq r \quad \forall t \in [\tau, t_f] \quad n \in N, \quad S \in B_{r,r}, \]

and \( T \) and \( T_n \) map \( B_{r,r} \) into \( B_{r,r} \). For \( S_1, S_2 \in B_{r,r} \), we have

\[
T(S_1)(t)x - T(S_2)(t)x
= \frac{1}{\gamma^2} \int_t^{t_f} U^*(s,t)([S_1DD^*(S_2 - S_1) + (S_2 - S_1)DD^*S_2](s)U(s,t)x)ds
\]

It follows that

\[
\|T(S_1)(t) - T(S_2)(t)\| \leq 2rM_t^2 \gamma^{-2}(t_f - \tau)\|D\|^2_2\|S_1 - S_2\|
\leq \alpha\|S_1 - S_2\|
\]

Similarly

\[
\|T_n(S_1)(t) - T_n(S_2)(t)\| \leq 2rM_t^2 \gamma^{-2}(t_f - \tau)\|D\|^2_2\|S_1 - S_2\|
\leq \alpha\|S_1 - S_2\|
\]

Thus \( T \) and \( T_n \) are \( \alpha \)-contraction in \( B_{r,r} \). Hence \( T \) and \( T_n \) have unique fixed points in \( C_u([\tau, t_f]; \Sigma(H)) \). However generally we do not have

\[
S_n \to S \quad \text{in} \quad C_u([\tau, t_f]; \Sigma(H))
\]

but the following weaker result holds:

\[
S_n \to S \quad \text{in} \quad C_s([\tau, t_f]; \Sigma(H))
\]

due to (iii) of assumptions given in section 2.1. This completes the proof. □
Corollary 5.2.3.4. For a given $\gamma > 0$ there exists $\tau_{\min} : 0 \leq \tau_{\min} < t_f$ such that the generalized Riccati equation (S) admits a unique mild solution on $[\tau_{\min}, t_f]$. Furthermore, if $\tau_{\min} > 0$, then

$$\lim_{t \downarrow \tau_{\min}} \| S_{\gamma}(t) \|_{\mathcal{L}(H)} = +\infty$$

Proof.

From what we have proved in Proposition 5.2.3.3, it follows that if $S_{\gamma}$ is a mild solution of (S) on the interval $[\tau, t_f]$ it can be extended to the interval $[\tau - \delta, t_f]$ with $\delta > 0$, by defining on $[\tau - \delta, \tau], S_{\gamma}(t) = \overline{S}_{\gamma}(t)$ where $\overline{S}_{\gamma}(t)$ is the solution of the integral equation

$$\overline{S}_{\gamma}(t)y = U(\tau, t)^* S_{\gamma}(\tau) U(\tau, t)y + \int_{\tau}^{t} U(s, t)^* \{ C(s)^* C(s)$$

$$+ \frac{1}{\gamma^2} \overline{S}_{\gamma}(s) D(s) D(s)^* \overline{S}_{\gamma}(s) \} U(s, t)y ds \quad \forall y \in H$$

Moreover, $\delta$ depends only on $\| S_{\gamma}(\tau) \|$ in addition to $\| C \|_{B_\infty}, \| D \|_{B_\infty}, M_{t_f}, \alpha$ and $\gamma$.

Let $[\tau_{\min}, t_f]$ be the maximal interval of existence of the mild solution $S_{\gamma}$ of (S). If $\tau > 0$ then $\lim_{t \downarrow \tau_{\min}} \| S_{\gamma}(t) \|_{\mathcal{L}(H)} = +\infty$ since otherwise there is a sequence $t_n \downarrow \tau_{\min}$ such that $\| S_{\gamma}(t_n) \|_{\mathcal{L}(H)} \leq C$ for all $n$. This would imply by what we have just proved that for each $t_n$, near enough to $\tau_{\min}$, $S_{\gamma}$ can be extended to $[t_n - \delta, t_f]$ where $\delta > 0$ is independent of $t_n$ and hence $S_{\gamma}$ can be extended beyond $\tau_{\min}$ contradicting the definition of $\tau_{\min}$. □

For the generalized Riccati equation (Z) we have a similar result which we state without a proof.

Proposition 5.2.3.5. For a given $\gamma > 0$, there exists $\tau_{\min} : 0 \leq \tau_{\min} < t_f$ such that the generalized Riccati equation (Z) has a unique mild solution $Z_{\gamma}$ on interval $[\tau_{\min}, t_f]$.

Furthermore if $\tau_{\min} > 0$, then

$$\lim_{t \downarrow \tau_{\min}} \| Z_{\gamma}(t) \|_{\mathcal{L}(H)} = +\infty$$
In fact, there are two critical values which can determine if (S) and (Z) possess mild solutions over the interval \([0, t_f]\). Before we state these two critical values, we introduce some notation. Let

\[ \mathcal{H}_t = L^2([t, t_f]; H), \mathcal{U}_t = L^2([t, t_f]; U), \mathcal{W}_t = L^2([t, t_f]; W), \mathcal{Z}_t = L^2([t, t_f]; \mathcal{Z}) \]

When \( t = 0 \), we denote \( \mathcal{H}_0 \) by \( \mathcal{H} \), \( \mathcal{U}_0 \) by \( \mathcal{U} \), \( \mathcal{W}_0 \) by \( \mathcal{W} \), and \( \mathcal{Z}_0 \) by \( \mathcal{Z} \). Introduce operators \( T_t \in \mathcal{L}(\mathcal{H}_t, \mathcal{H}_t) \) and \( \mathcal{K}_t \in \mathcal{L}(\mathcal{H}_t, H) \) as

\[
(T_t \phi)(s) = \int_t^s U(s, \eta) \phi(\eta) d\eta \quad \phi \in \mathcal{H}_t
\]

\[
\mathcal{K}_t \phi = (T_t \phi)(t_f), \quad \phi \in \mathcal{H}_t
\]

and

\[
U_0 x_0 = U(t_f, 0) x_0, \quad x_0 \in H
\]

The first critical value is defined as

\[
(5.2.3.2) \quad \gamma^S_c := \sup_{\|w\|_W = 1} \langle D^*(\mathcal{K}_0^* Q_f \mathcal{K}_0 + T_0^* C^* C T_0) D w, w \rangle^\frac{1}{2}_W
\]

\[
= \|[D^*(\mathcal{K}_0^* Q_f \mathcal{K}_0^* + T_0^* C^* C T_0) D]^{\frac{1}{2}}\|_W
\]

We now introduce some other operators:

\[
T_1 = I + B^* T_0^* C^* C T_0 B + B^* \mathcal{K}_0^* Q_f \mathcal{K}_0 B \in \mathcal{L}(\mathcal{U}, \mathcal{U})
\]

\[
T_2 = B^* T_0^* C^* C T_0 D + B^* \mathcal{K}_0^* Q_f \mathcal{K}_0 D \in \mathcal{L}(\mathcal{W}, \mathcal{U})
\]

\[
T_3^* = B^* T_0^* C^* C U_0 + B^* \mathcal{K}_0^* Q_f U(t_f, 0) \in \mathcal{L}(H, \mathcal{U})
\]

and \( S \in \mathcal{L}(\mathcal{W}, \mathcal{W}) \)

\[
S = [T_0(-B T_1^{-1} T_2 + D)]^* C^* C [T_0(-B T_1^{-1} T_2 + D)] + T_2^* T_1^{-1} T_1^{-1} T_2
\]

\[
+ [\mathcal{K}_0(-B T_1^{-1} T_2 + D)]^* [\mathcal{K}_0(-B T_1^{-1} T_2 + D)]
\]

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The second critical value is defined as:

\[(5.2.3.3) \quad \gamma^Z_c := \sup_{\|w\|=1} <Sw,w>_{\mathcal{W}}^{\frac{1}{2}} = \|S^{\frac{1}{2}}\|_{\mathcal{L}(\mathcal{W})}\]

Now we are ready to state following:

**Proposition 5.2.3.6.** Let \(\gamma^S_c, \gamma^Z_c\) be defined as above. Then,

1. The generalized Riccati equation \((S)\) admits a mild solution on the interval \([0,t_f]\) if \(\gamma > \gamma^S_c\) and only if \(\gamma \geq \gamma^S_c\).

2. The generalized Riccati equation \((Z)\) admits a mild solution on the interval \([0,t_f]\) if \(\gamma > \gamma^Z_c\) and only if \(\gamma \geq \gamma^Z_c\).

**Proof.** See Theorem 4.4.1.4 and Theorem 4.4.2.3 of Chapter 4.

5.3. Optimization of the differential game \(J_\gamma\)

5.3.1. Maximization of \(J_\gamma\) over \(\mathcal{W}\) for fixed \(u\)

We consider the following maximization problem: for a given fixed but arbitrary \(u \in L^2([0,t_f];U)\); maximize over \(L^2([0,t_f];W)\):

\[J_\gamma(x_0;u,w) = \int_0^{t_f} \|C(t)x(t)\|^2_Z + \|u(s)\|^2_U - \gamma^2\|w(s)\|^2_W + (Q_f x(t_f),x(t_f))_H dt\]

where \(x(t)\) is the solution (5.2.1.1).

**Theorem 5.3.1.1.** If \(\gamma > \gamma^S_c\), for arbitrary but fixed \(u \in U\) and for any \(x_0 \in H\), the maximization \(\max_w J_\gamma(x_0;u,w)\) is finite. If \(\gamma^S > 0\) and \(\gamma < \gamma^S_c\), \(\max_w J_\gamma(x_0;u,w)\) is unbounded for all \(x_0 \in H\).

**Proof.** See Theorem 4.4.1.1 of Chapter 4.
Theorem 5.3.1.2. For a given $\gamma > 0$, if $(S)$ admits a mild solution over the interval $[0,t_f]$, then $\max_w J_\gamma(x_0; u, w)$ is bounded for any $x_0 \in H$ and any $u \in \mathcal{U}$.

Proof. See Theorem 4.4.1.4 of Chapter 4.

5.3.2. Maximin of $J_\gamma$ over $\mathcal{U}$ and $\mathcal{W}$

Theorem 5.3.2.1. If $\gamma > \gamma_c^Z$, the lower value of the game $J_\gamma(x_0; u, w)$ is finite for any $x_0 \in H$. If $\gamma < \gamma_c^Z$, the lower value of the game $J_\gamma(x_0; u, w)$ is unbounded for any $x_0 \in H$.

Proof. See Theorem 4.4.2.1 of Chapter 4.

5.4. A differential game and its saddle point

Theorem 5.4.1. (Open-Loop Policy) Suppose that $\gamma > \gamma_c^S$. Then, the differential game $J_\gamma(x_0; u, w)$ admits a unique saddle-point solution over $L^2([0,t_f]; U) \times L^2([0,t_f]; W)$, given by

\begin{align}
(5.4.1) \quad \hat{u}(t) &= -B(t)^* Z_\gamma(t)x(t) \\
(5.4.2) \quad \hat{w}(t) &= \frac{1}{\gamma^2} D(t)^* Z_\gamma(t)x(t)
\end{align}

where $Z_\gamma$ is the mild solution of $(Z)$ and $x$ is the corresponding state trajectory, generated by

\[ x(t) = \Phi(t,0)x_0 \]

where the evolution operator $\Phi(\cdot, \cdot)$ satisfies

\[ \Phi(t, s)x = U(t,s)x + \int_s^t U(t, \eta) \left( B(\eta)B(\eta)^* - \frac{1}{\gamma^2} D(\eta)D(\eta)^* Z_\gamma(\eta) \right) \Phi(\eta, s)x d\eta \]
Moreover

\[(5.4.3) \quad J_\gamma(x_0; u^*, w^*) = (Z_\gamma(0)x_0, x_0)\]

Proof. Since \( \gamma > \gamma^S_c \), we know that (S) admits a mild solution according to Proposition 5.2.3.6 (1), and thus (Z) also admits a mild solution by Proposition 5.2.3.2. Hence \( Z_\gamma \) is well defined over \([0, t_f]\). We are going to show that for \( x_0 \in H \) the following holds

\[\forall (u, w) \in L^2([0, t_f]; U) \times L^2([0, t_f]; W)\]

\[J_\gamma(x_0; \hat{u}, w) \leq J_\gamma(x_0; \hat{u}, \hat{w}) \leq J_\gamma(x_0; u, \hat{w})\]

Let us first show that \( \min_u J_\gamma(x_0; u, \hat{w}) = J(x_0; \hat{u}, \hat{w}) \). Note \( J_\gamma(x_0; \cdot, \hat{w}) \) is strictly convex and coercive, hence has a unique minimizing solution \( \hat{u} \). In view of standard results, \( \hat{u} = -B(t)^*p(t) \) satisfies the Hamiltonian system

\[(5.4.4) \quad \dot{x}(t) = A(t)x(t) - B(t)B(t)^*p(t) + \frac{1}{\gamma^2} D(t)D(t)^*x(t), x(0) = x_0\]

\[(5.4.5) \quad \dot{p}(t) = -A(t)^*p(t) - C(t)^*C(t)x(t), p(t_f) = Q_f x(t_f)\]

and (5.4.4) and (5.4.5) are in the "mild" sense. It is not difficult to see that \((x^o, p) = (x, Z_\gamma x)\) satisfies this Hamiltonian system (Verification of this observation would involve replacing \( A(t) \) by \( A_n(t) \), the Yosida approximations of \( A(t) \), and show that \((x_n, Z_\gamma^n x_n)\) is the strong solution of \( x_n(t) = \Phi_n(t, 0)x_0 \) for \( x_0 \in D(A(t)) \), and letting \( n \to \infty \). Therefore we have

\[\hat{u}(t) = -B(t)^*Z_\gamma x(t) = \hat{u}(t)\]

This gives

\[(Z_\gamma(0)x_0, x_0)_H = J_\gamma(x_0; \hat{u}, \hat{w}) \leq J_\gamma(x_0; u, \hat{w})\]
Now we verify the other side of the inequality. By our assumption that $\gamma > \gamma_{c}^{S}$, we know that $J_{\gamma}(x_{0}; \hat{u}, \cdot)$ has a unique finite maximizing solution $\bar{w} \in L^{2}([0, t_{f}]; W)$. Note that $J_{\gamma}(x_{0}; u^{*}, \cdot)$ is Gâteaux differentiable and

$$\delta_{w}J_{\gamma}(x_{0}; \hat{u}, w) = D(t)^{*} Z_{\gamma}(t)x(t) - \gamma^{2} w$$

(for a proof, see the Appendix.) Hence we have

$$\bar{w}(t) = \frac{1}{\gamma^{2}} D(t)^{*} Z_{\gamma}(t)x(t) = \hat{w}(t)$$

Therefore $\bar{w}(t) = \hat{w}(t)$ and we have

$$J_{\gamma}(x_{0}; \hat{u}, w) \leq J_{\gamma}(x_{0}; \hat{u}, \bar{w})$$

Thus $(\hat{u}, \bar{w})$ is a saddle point solution of $J_{\gamma}(x_{0}; u, w)$ and the proof is complete. □

**Theorem 5.4.2.** (Closed-loop policy) Suppose that $\gamma > \gamma_{c}^{Z}$, then the differential game $J_{\gamma}(u, w, x_{0})$ admits a unique feedback saddle-point solution over $\mathcal{M} \times \mathcal{W}$, given by

$$\mu^{*}(t, x(t)) = -B(t)^{*} Z_{\gamma}(t)x(t)$$  \hspace{1cm} (5.4.6)

$$\nu^{*}(t, x(t)) = \frac{1}{\gamma^{2}} D(t)^{*} Z_{\gamma}(t)x(t)$$  \hspace{1cm} (5.4.7)

Moreover

$$J_{\gamma}(x_{0}; \mu^{*}, \nu^{*}) = (Z_{\gamma}(0)x_{0}, x_{0})_{H}$$  \hspace{1cm} (5.4.8)

**Proof.** By using Yosida approximations and the completion of squares we have:

$$J_{\gamma}(x_{0}; u, w) = (Z_{\gamma}(0)x_{0}, x_{0})_{H} + \|u + B^{*} Z_{\gamma}x\|_{\hat{u}}^{2}$$

$$- \gamma^{2} \|w - \gamma^{-2} D^{*} Z_{\gamma}x\|_{\hat{W}}^{2}$$  \hspace{1cm} (5.4.9)

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It follows that \( \forall (\mu, w) \in \mathcal{M} \times \mathcal{W} \) the following holds

\[
J_\gamma(x_0; \mu^*, w) \leq J_\gamma(x_0; \mu^*, \nu^*) \leq J_\gamma(x_0; \mu, \nu^*)
\]

This completes the proof. \( \square \)

Remark 5.4.3. There is a significant difference between Theorem 5.3.1.1 and Theorem 5.3.1.2 even though the saddle point of the game \( J_\gamma(x_0; u, w) \) has the same form, because we in fact have \( \gamma^S_c \geq \gamma^Z_c \) according to Proposition 5.2.3.2. Thus the condition of Theorem 5.4.1 is stronger than the one of Theorem 5.4.2.

5.5. Finite-horizon optimum design under sampled state measurements

Let \( t_0, t_1, \ldots, t_{N-1} \) denote \( N \) sampling times, such that \( 0 = t_0 < t_1 < \cdots < t_{N-1} < t_f \).

Introduce \( N \) operator Riccati evolution equations as follows:

\[
\begin{align*}
\dot{S}_n(t) + A(t)^* S_n(t) + S_n(t)A(t) + C(t)^* C(t) + \frac{1}{\gamma^2} S_n(t)D(t)D(t)^* S_n(t) &= 0 \\
S_n(t_{n+1}) &= Z_\gamma(t_{n+1}) \\
t_n \leq t < t_{n+1}, \quad n = N - 1, N - 2, ..., 0.
\end{align*}
\]

and the corresponding critical values are:

\[
\begin{align*}
\gamma^S_n := \sup_{\|w\|=1} \left\langle D^*(\mathcal{K}_n^* Z_\gamma(t_{n+1}) \mathcal{K}_n + T_n^* C^* C T_n)Dw, w \right\rangle_{\mathcal{W}}^{\frac{1}{2}} \\
&= \|[D^*(\mathcal{K}_n^* Z_\gamma(t_{n+1}) \mathcal{K}_n^* + T_n^* C^* C T_n)D]^{\frac{1}{2}}\|_{\mathcal{W}} \\
(T_n \phi)(s) &= \int_{t_n}^{s} U(s, \eta)\phi(\eta) d\eta, \quad t_n \leq s < t_{n+1} \quad \phi \in \mathcal{H}_t \\
\mathcal{K}_n \phi &= (T_n \phi)(t_{n+1}), \quad \phi \in \mathcal{H}_t.
\end{align*}
\]

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A sampled feedback control scheme is defined by

\[ u(t) = K(t)x(t_k) \quad t_k \leq t < t_{k+1}, k = 0, 1, \ldots, K - 1 \]

where \( K(t) \in B_\infty(0, t_f; \mathcal{L}(H, U)) \). For future reference let us denote the collection of all linear sampled-data feedback controllers with \( N \) sampling intervals by \( M_s^N \). The following theorem is now a generalization of Theorem 2.1 of [B].

**Theorem 5.5.1.** Suppose that \( \gamma > \gamma_c^2 \). Then we have:

(i) Under \( M_s^N \) there exists a unique strongly time consistent saddle-point controller \( \mu^s_\gamma \in M_s^N \) for all \( x_0 \in H \) if \( \gamma > \sup_n \{\gamma_c^{S_n}\} \) and only if \( \gamma \geq \sup_n \{\gamma_c^{S_n}\} \).

(ii) If \( \gamma > \sup_n \{\gamma_c^{S_n}\} \), then the unique strongly time consistent saddle-point controller is given by

\[ \mu^s_\gamma(t, x(t_n)) = -B(t)^*Z_\gamma(t)\Phi_\gamma(t, t_n)x(t_n) \quad t_n \leq t < t_{n+1} \]

where evolution operator \( \Phi_\gamma(t, s) \) satisfies

\[ \Phi(t, s)x = U(t, s)x + \int_s^t U(t, \eta)\left( B(\eta)B(\eta)^* - \frac{1}{\gamma^2} D(\eta)D(\eta)^* Z_\gamma(\eta)\right)\Phi(\eta, s)x d\eta \]

Moreover we have

\[ \min_{M_s^N} \max_{L^2([0, t_f]; W)} J_\gamma(x_0; u, w) = \max_{L^2([0, t_f]; W)} \min_{M_s^N} J_\gamma(x_0; u, w) = (Z_\gamma(0)x_0, x_0) \]

(iii) If there exists \( n \) such \( \gamma_c^{S_n} < \gamma \), then for any \( x_0 \in H \) the upper value of the game \( J_\gamma(x_0; u, w) \) becomes unbounded under sampled-data controllers from \( M_s^N \), even though it is bounded under the state feedback controller.

(iv) If \( \gamma < \gamma_c^2 \), then for any \( x_0 \in H \), \( J_\gamma(x_0; u, w) \) does not admit a saddle point under any information pattern for the controller; moreover, there exists a sequence \( \{w_n\} \subset \)
\[ L^2([0, t_f]; W) \] such that

\[ J_\gamma(x_0; \cdots, w_n) \to +\infty \quad n \to \infty. \]

**Proof.** (i) Recall that

\[ J_\gamma(x_0; u, w) = \int_0^{t_f} (\|C(t)x(t)\|^2_Z + \|u(t)\|^2_U - \gamma^2\|w(t)\|^2_W)dt + (Q_f x(t_f), x(t_f))_H \]

For \( 0 \leq n \leq N - 1 \) denote

\[ J^n_\gamma(x_0; u, w) = \int_{t_n}^{t_{n+1}} (\|C(t)x(t)\|^2_Z + \|u(s)\|^2_U - \gamma^2\|w(t)\|^2_W)dt \]

\[ + (Z_\gamma(t_{n+1})x(t_{n+1}), x(t_{n+1}))_H \]

If \( \gamma > \sup_n \{\gamma_{c^n} \} \), we know that the quantity \( \sup_w J^n_\gamma(x(t_k); u, w) \) is finite for each \( 0 \leq n \leq N - 1 \) according to Theorem 5.3.1.1; moreover,

\[ \inf_u \sup_w J^n_\gamma(x(t_n); u, w) = (Z_\gamma(t_n)x(t_n), x(t_n))_H \]

Introduce

\[ \nu^n_\gamma(t, x(t_n)) = \frac{1}{\gamma^2} D(t)^* Z_\gamma(t) \Phi_\gamma(t, t_n)x(t_n), \quad t_n \leq t < t_{n+1}, \quad n = 0, 1, \ldots, N - 1 \]

For each subinterval \( t_n \leq t \leq t_{n+1} \), by Theorem 5.4.1, we know that \( (\mu^n_\gamma(t), \nu^n_\gamma(t)) \) is the unique saddle point solution of \( J^n_\gamma(x(t_n); u, w) \). Since

(5.5.4)

\[ J_\gamma(x_0; u, w) = \sum_{n=0}^{N} J^n_\gamma(x(t_n); u, w) - \sum_{n=0}^{N-1} (Z_\gamma(t_{n+1})x(t_{n+1}), x(t_{n+1}))_H \]

this yields

\[ J_\gamma(x_0; \mu^n_\gamma, \nu^n_\gamma) = (Z_\gamma(0)x_0, x_0)_H \]

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According to the saddle point property, we have

\[(5.5.5) \quad J^n_\gamma(x(t_n); \mu_{\gamma}^*, \nu_\gamma^*) \leq J^n_\gamma(x(t_n); \mu_{\gamma}^*, \nu_\gamma^*) = (Z_\gamma(t_n)x(t_n), x(t_n))_H \leq J^n_\gamma(x(t_n); u, \nu_\gamma^*)\]

for all \((u, w) \in L^2([t_n, t_{n+1}]; U) \times L^2([t_n, t_{n+1}]; W)\). According to Theorem 5.4.1, for each \(1 \leq n \leq N\)

\[(5.5.6) \quad J^n_\gamma(x(t_n); \mu_{\gamma}^*, \nu_\gamma^*) \leq (Z_\gamma(t_n)x(t_n), x(t_n))_H\]

Thus from (5.5.4) we have

\[J_\gamma(x_0; \mu_{\gamma}^*, \nu_\gamma^*) \leq (Z_\gamma(0)x_0, x_0)_H = J_\gamma(x_0; \mu_{\gamma}^*, \nu_\gamma^*), \quad \forall w \in \mathcal{W}\]

On the other hand, from the Theorem 5.4.1 again, we know that

\[J^n_\gamma(x(t_n); u, \nu_\gamma^*) \geq (Z_\gamma(t_n)x(t_n), x(t_n))_H\]

Hence

\[(5.5.7) \quad J_\gamma(x_0; u, \nu_\gamma^*) \geq (Z_\gamma(0)x_0, x_0)_H = J_\gamma(x_0; \mu_{\gamma}^*, \nu_\gamma^*), \quad \forall u \in \mathcal{U}\]

Therefore we have

\[J_\gamma(x_0; \mu_{\gamma}^*, \nu_\gamma^*) \leq J_\gamma(x_0; \mu_{\gamma}^*, \nu_\gamma^*) \leq J_\gamma(x_0; u, \nu_\gamma^*)\]

This implies that \((\mu_{\gamma}^*, \nu_\gamma^*)\) is a saddle point of the game \(J_\gamma(x_0; u, w)\).

Next we claim that if the game \(J_\gamma(x_0; u, w)\) admits a saddle point solution \((\mu_{\gamma}^*, \nu_\gamma^*)\), then we have \(\gamma \geq \sup_n \{\gamma_n^S\}\). Since \((\mu_{\gamma}^*, \nu_\gamma^*)\) is a saddle-point solution of \(J_\gamma(x_0; u, w)\), we have

\[(5.5.8) \quad J_\gamma(x_0; \mu_{\gamma}^*, \nu_\gamma^*) \leq J_\gamma(x_0; \mu_{\gamma}^*, \nu_\gamma^*) < +\infty\]

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From (5.5.4) and (5.5.8), we have

\[
\sup_w \left\{ \sum_{n=0}^{N} J_{\gamma}^n(x(t_n); u, w) \right\} = \sup_w \left\{ J_{\gamma}(x_0; u, w) + \sum_{n=0}^{N-1} (Z_{\gamma}(t_{n+1})x(t_{n+1}), x(t_{n+1}))_H \right\} < +\infty
\]

Hence for each \( n \), \( \sup_w J^n(x(t_n); u, w) \) is finite, which implies that \( \gamma \geq \gamma_c^{S_n} \) by Theorem 5.3.1.1. Thus this completes the proof of part (i).

(ii) Since \( \gamma > \sup_n \{\gamma_c^{S_n}\} \), the saddle point solution of \( J_{\gamma}(x_0; u, w) \) exists. The conclusion follows from part (i).

(iii) If there exists a \( \gamma > 0 \) such that \( \gamma < \gamma_c^{S_n} \), by Theorem 5.3.1.1, we know that \( \sup_w J_{\gamma}(x(t_n); u, w) \) is unbounded. Note that from (5.5.4)

\[
\sup_w J_{\gamma}(x_0; u, w) \geq \sup_w \left\{ \sum_{n=0}^{N} J_{\gamma}^n(x(t_n); u, w) \right\}
\]

which implies that the upper value of the game \( J_{\gamma}(x_0; u, w) \) is unbounded for any \( x_0 \in H, u \in M^N \). But since \( \gamma > \gamma_c^Z \), according to Theorem 5.4.2, the upper value of the game \( J_{\gamma}(x_0; u, w) \) is finite under the state feedback policy \( M \).

(iv) Since \( \gamma < \gamma_c^Z \), according to Theorem 5.3.2.1, we have that \( \sup_{w} \inf_{u} J_{\gamma}(x_0; u, w) \) is unbounded for any \( x_0 \in H \). Given any class \( \mathcal{P} \) of control policies \( \mathcal{P} \), note the inequality

\[
\sup_{w} \inf_{u} J_{\gamma}(x_0; u, w) \leq \inf_{\mu \in \mathcal{P}} \sup_{w} J_{\gamma}(x_0; \mu, w)
\]

Thus the upper value of \( J_{\gamma}(x_0; u, w) \) over \( \mathcal{P} \) is unbounded, of control policies, and the proof of Theorem 5.5.1 is complete. □

Remark 5.5.2. Theorem 5.5.1(ii) holds even when \( \gamma = \sup_n \{\gamma_{nc}\} \) provided that the value of the corresponding game is finite over \( M^N_s \).
**Remark 5.5.3.** From (5.5.1), we can see that $S^n$ tends to $Z$ (the mild solution of (Z)) as the number of sampling times increases indefinitely, which implies that

$$\liminf_{n \to \infty} \gamma^S_n = \gamma^Z_c$$

Therefore, for a given $\gamma > \gamma^Z_c$, there exists an $N > 0$ such that for $n > N$, $\gamma > \gamma^S_n$, which implies that a sampled-data controller exists by Theorem 5.5.1 (i) and (ii). This brings us to the following theorem.

**Theorem 5.5.4.** Suppose that given $\gamma > \gamma^Z_c$ the sampling intervals are chosen such that (5.5.9)

$$0 < t_{n+1} - t_n \leq \min \left\{ \frac{\|Z(\tau_{k+1})\|_{\mathcal{L}(H)}}{\|C\|_{\mathcal{B}_\infty}^2 + \frac{r^2}{\gamma^2} \|D\|_{\mathcal{B}_\infty}^2}, \frac{\alpha \gamma^2}{2 r_n M_t^2 \|D\|_{\mathcal{B}_\infty}^2} \right\}, 0 < \alpha < 1$$

where

$$M_t = \sup \{\|U(t,s)\|; 0 \leq s \leq t \leq t_f\}, r_n = 2 M_t^2 \|Z(t_{n+1})\|_{\mathcal{L}(H)}, n = 0, 1, 2, ..., N - 1$$

Then, with $x_0 = 0$ the sampled-data controller

$$\mu^*(t, x(t_n)) = -B(t)^* Z(t) \Phi(t, t_n) x(t_n), \quad t_n \leq t < t_{n+1}$$

attains attenuation level $\gamma$, i.e.

$$\sup_w \{\{F_{\mu^*}\}^{1/2}/\|w\|_W\} < \gamma$$

where $F_{\mu^*} : \mathcal{W} \to \mathbb{R}$ is induced by

$$L(u, w) = \int_0^{t_f} \left\{ \|C(t)y(t)\|_{\mathcal{H}}^2 + \|u(t)\|_{\mathcal{U}}^2 \right\} dt + (Q_f x(t_f), x(t_f))_H$$

**Proof.** Note that with the sampling times picked as in (5.5.9), all $N$ equations in (5.5.1) have mild solutions on their corresponding intervals due to Proposition 5.2.3.3. Now we apply Theorem 5.5.1 (i) and (ii), and have

$$J_\gamma(x_0; \mu^*_\gamma, w) \leq (Z_\gamma x_0, x_0) \equiv 0, \quad \forall w \in L^2([0, t_f]; \mathcal{W})$$
which yields
\[ F_{\mu^2} / \| w \|_{\mathcal{W}}^2 \leq \gamma^2, \forall w \in \mathcal{W} \]

This completes the proof. \( \Box \)

For time-invariant systems with \( Q_f = 0 \), we can replace (5.5.1) by a single generalized Riccati equation. Letting

\[ t_s := \max\{ t_{n+1} - t_n : n = 0, 1, ..., N - 1 \} \]

and formulating the following Riccati equation

(5.5.10) \[ \dot{S}_\gamma(t) + A^* S_\gamma(t) + S_\gamma(t) A + C^* C + \frac{1}{\gamma} S_\gamma(t) DD^* S_\gamma(t) = 0 \]

\[ S(t_s) = Z_\gamma(0) \]

its critical value is

\[ \gamma_c^{S_{t_0,t_s}} = \sup_{\| w \| = 1} \left( D^* (K_s^* Z_{\gamma}(0) K_s + T_s^* C^* C T_s) D w, w \right)_{\mathcal{W}}^{\frac{1}{2}} \]

\[ = \| [D^* (K_s^* Z_{\gamma}(0) K_s + T_s^* C^* C T_s) D]^{\frac{1}{2}} \|_{L(\mathcal{W})}^2 \]

where

\[ (T_s \phi)(\tau) = \int_0^\tau U(s, \eta) \phi(\eta) d\eta, \quad 0 \leq \tau \leq t_s, \phi \in \mathcal{H} \]

and

\[ K_s \phi = (T_s)(t_s), \quad \phi \in \mathcal{H} \]

We now have the following corollary.

**Corollary 5.5.5.** Suppose that (5.2.1.1) is time-invariant and \( Q_f = 0 \) in (5.2.1.3).

Given \( \gamma > 0 \) with \( \gamma > \gamma_c^Z \), we have:
(i) If $\gamma > \gamma_c^{S_{0,t+1}}$, then there exists a unique saddle-point controller which is as given in Theorem 5.5.1(ii).

(ii) Theorem 5.5.1(i), (ii) still holds when we replace $\sup_n \{\gamma_c^{S_n}\}$ by $\gamma_c^{S_{0,t+1}}$.

(iii) If the sampling intervals satisfy the bound:

\[ 0 < t_{n+1} - t_n \leq \min\left\{ \frac{\|Z_\gamma(0)\|_C}{\|C\|_{\infty}} + \frac{\alpha^2}{\gamma^2} \|D\|_{\infty}, \frac{\alpha^2}{2 r M_{t_f}^2 \|D\|_{\infty}^2} \right\} \]

where $r = 2 M_{t_f}^2 \|Z_\gamma(0)\|_C$, then Theorem 5.5.4 still holds when we replace (5.5.9) by (5.5.11).

**Proof.** Note in the time-invariant case and with $Q_f = 0$, we have

\[ Z_\gamma(0) \geq Z_\gamma(t), \quad \forall t \in [0, t_f] \]

Using time-invariance, rewrite (5.5.10) as

\[ \dot{S}_n + A^* S_n + S_n A + C^* C + \frac{1}{\gamma^2} S_n D D^* S_n = 0 \]

\[ S_n(t_{n+1} - t_n) = Z_\gamma(t_{n+1} - t_n) \]

\[ 0 < t < t_{n+1} - t_n, \quad n = N - 1, N - 2, \ldots, 0 \]

Since $Z_\gamma(0) \geq Z_\gamma(t_{n+1} - t_n)$ for all $0 \leq n \leq N$, for each $n$ we have

\[ S_n(t_{n+1} - t_n - t) \leq S_\gamma(t_s - t), \quad \text{for} \quad 0 \leq t \leq t_{n+1} - t_n \leq t_s \]

This means that

\[ \gamma_c^{S_{0,t+1}} \geq \sup_n \{\gamma_c^{S_n}\} \]

Therefore, by Theorem 5.5.4, there exists unique saddle-point controller which is given by Theorem 5.5.1(ii). Part (ii) and (iii) follow the fact that

\[ \gamma > \gamma_c^{S_{0,t+1}} \geq \sup_n \{\gamma_c^{S_n}\} \]

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and thus the corresponding assumptions needed are satisfied. \qed

**Remark 5.5.6.** If the sampling times are uniform, i.e.,

\[ t_{n+1} - t_n = t_s, \quad n = 0, 1, \ldots, N - 1 \]

we can see that

\[ \sup_n \{ \gamma_c^{S_n} \} = \gamma_c^{S_n} = \gamma_c^{S_{[0,t_s]}} \]

Thus, under the same sampled rate (i.e. \( N \) fixed), uniform sampling times will have smaller disturbance attenuation bound than one under nonuniform sampling times.

**An algorithm for sampled-data controller design in the finite-horizon case.**

Given a desired attenuation level \( \gamma > 0 \):

**Step 1.** Check if \( \gamma > \gamma^Z \), that is to see if (Z) admits a mild solution over interval \([0, t_f]\).

If not, stop; \( \gamma \) has to be picked larger.

**Step 2.** Compute \( \|C\| = \sup_{[0,t_f]} \|C(t)\|_{L(H,Z)}, \|D\| = \sup_{[0,t_f]} \|D(t)\|_{L(W,Z)}, M_{t_f} = \sup \{ \|U_A(t,s)\|; 0 \leq s \leq t \leq t_f \} \), where \( U_A(t,s) \) is the evolution operator associated with \( A(t) \).

**Step 3.** Construct a finite time sequence \( \{t_n\}_{0}^{N} \) such that (5.5.9) holds. If the time sequence is given and (5.5.9) does not hold, increase the sampled rate such that (5.5.9) can hold. This can be done as long as \( \gamma > \gamma^Z \) according to Remark 1.5.3.

**Step 4.** Compute \( \Phi(t,s) \) which is the evolution operator satisfying

\[
\Phi(t,s)x = U(t,s)x + \int_{s}^{t} U(t,\eta) \left( B(\eta)B(\eta)^* - \frac{1}{\gamma^2} D(\eta)D(\eta)^* Z_\gamma(\eta) \right) \Phi(\eta,s)x \, d\eta
\]

**Step 5.** At the current time \( t_n \), measure the sampled state \( x(t_n) \).

**Step 6.** Construct the feedback sampled-data controller on the interval \([t_n, t_{n+1}]\) as follows:

\[
\mu^*_\gamma(t, x(t_n)) = -B(t)^* Z_\gamma(t) \Phi(t, t_n) x(t_n)
\]
Step 7. At time $t_{n+1}$, goes back to step 5, and repeat this process until $t_n = t_f$.

5.6. Infinite-horizon optimum design under sampled state measurements

In this section, we consider time-invariant version of system of (5.2.1.1), i.e. with

$$A(t) \equiv A, B(t) \equiv B, C(t) \equiv C, D(t) \equiv D,$$

and $Q_f = 0$. Consider the following stationary Riccati equation:

$$A^* Z_\gamma + Z_\gamma A + C^* C - Z_\gamma (B B^* - \frac{1}{\gamma^2} D D^*) Z_\gamma = 0 \quad (5.6.1)$$

**Definition 5.6.1.** We say that $Z_\gamma \in \Sigma^+(H)$ is a solution of (5.6.1) if

$$ (Z_\gamma x, Ay) + (Ax, Z_\gamma y) - (B^* Z_\gamma x, B^* Z_\gamma y) + \frac{1}{\gamma^2} (D^* Z_\gamma x, D^* Z_\gamma y) + (C x, C y) = 0 \quad (5.6.2) $$

for all $x, y \in D(A)$.

The next result is due to Keulen, Peters and Curtain ([KPC] Theorem 2.2).

**Lemma 5.6.2.** (Keulen, Peters and Curtain 1993) Assume that $(A, B)$ is exponentially stabilizable and that $(A, C)$ is exponentially detectable. Then, for a given $\gamma > 0$,

$$ \frac{\int_0^\infty (\|Cx\|^2 + \|u\|^2_v) dt}{\int_0^\infty \|w\|^2_y dt} < \gamma^2 \quad (5.6.3) $$

holds if and only if (5.6.1) has a solution $Z_{\min}^\infty$ and

$$ A - BB^* Z_{\min}^\infty + \frac{1}{\gamma^2} DD^* Z_{\min}^\infty \quad (5.6.4) $$

is exponentially stable. In this case

$$ \sup_{w} \inf_{u} J_\gamma(x_0; u, w) = (Z_{\min}^\infty x_0, x_0) \quad (5.6.5) $$

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where
\[ J_\gamma(x_0; u, w) = \int_0^\infty (\|Cx\|^2_Z + \|u\|^2_Y - \gamma^2\|w\|^2_W)dt \]

Let \( \{t_n\} \) be an infinite sequence and there exist constants \( m, M \) such that
\[ m \leq \tau_n := t_{n+1} - t_n \leq M \]

Introduce
\[ t_s = \sup_{n=0,1,...} \{\tau_n\} \]

Assume that (5.5.1) has a solution \( Z_{\text{min}}^\infty \) such that (5.5.4) is exponentially stable, and introduce the Riccati equation
\[ (5.6.6) \quad \dot{S} + A^*S + SA + C^*C + \frac{1}{\gamma^2} SDD^* = 0 \]
\[ S(t_s) = Z_{\text{min}}^\infty \]

whose critical value is
\[ \gamma_c^{S[0,t_s]} = \|D^*(K_s^*Z_{\text{min}}^\infty K_s + T_s^*C^*CT_s)D\|^\frac{1}{2} \|\mathcal{L}(Y)\]  

where
\[ (T_s\phi)(\tau) = \int_0^\tau e^{A(\tau-\eta)}\phi(\eta)d\eta, \quad 0 \leq \tau \leq t_s \quad \phi \in \mathcal{H} \]

and
\[ K_s\phi = (T_s\phi)(t_s). \]

A sample feedback control scheme is defined by
\[ u(t) = K(t)x(t_n) \quad t_n \leq t < t_{n+1}, \quad n = 1, 2, \ldots, \]

where \( K \in \mathcal{B}_\infty(0, \infty; H, H) \). We denote by \( M_s \) the collection of all linear sampled-data controllers. Now we are ready to state the following:
Theorem 5.6.3. Suppose that \((A, B)\) is exponentially stabilizable and that \((A, C)\) is exponentially detectable.

(i) Under \(M_*\) the game \(J_\gamma(x_0; u, w)\) has equal finite upper and lower values if \(\gamma > \gamma_{c}^{S_{0, 1}}\) and only if \(\gamma \geq \gamma_{c}^{S_{0, 1}}\).

(ii) If \(\gamma > \gamma_{c}^{S_{0, 1}}\), letting \((u, w) = (\mu_\gamma^\infty, \nu_\gamma^\infty)\), where

\[
\mu_\gamma^\infty(t, x(t_n)) = -B^*Z_\min^\infty T(t - t_n)x(t_n), \quad t_n \leq t < t_{n+1}
\]

\[
\nu_\gamma^\infty(t, x(t_n)) = \frac{1}{\gamma^2} D^*Z_\min^\infty T(t - t_n)x(t_n), \quad t_n \leq t < t_{n+1}
\]

where the \(C_0\) semigroup \(T(t)\) is generated by

\[
A - BB^*Z_\min^\infty + \frac{1}{\gamma^2} DD^*Z_\min^\infty
\]

then the state \(x(t)\) is exponentially stable, i.e. \(\exists M(\gamma) > 0, \theta > 0\) such that

\[
\|x(t)\|_H \leq M(\gamma)e^{-\theta t}\|x_0\|_H, \quad \forall t \geq 0
\]

(iii) If \(\gamma > \gamma_{c}^{S_{0, 1}}\), then the value of the game is \((Z_\min^\infty x_0, x_0)_H\), that is

\[
\sup_{w}\inf_{M_*} J_\gamma(x_0; u, w) = \inf_{M_*}\sup_{w} J_\gamma(x_0; u, w) = (Z_\min^\infty x_0, x_0)_H
\]

Proof. (i) Let us introduce

\[
J_\gamma^n(x(t_n); u, w) = \int_{t_n}^{t_{n+1}} (\|Cx(t)\|_Z + \|u(t)\|_U - \gamma^2\|w(t)\|_W^2)dt
\]

If \(\gamma > \gamma_{c}^{S_{0, 1}}\), according to Theorem 5.3.1.1, we know that

\[
\sup_{w}\{J_\gamma^n(x(t_n); u, w) + (Z_\min^\infty x(t_{n+1}), x(t_{n+1}))\} < +\infty
\]
Moreover

\[ \inf_u \sup_w \{ J^n_\gamma(x(t_n); u, w) + (Z^\infty_{\min} x(t_{n+1}), x(t_{n+1}))_H \} = (Z^\infty_{\min} x(t_n), x(t_n))_H \]

Hence for arbitrary large integer \( N > 0 \) we have

\[ \inf_M \sup_w \sum_{n=0}^N J^n_\gamma(x_0; u, w) \leq (Z^\infty_{\min} x_0, x_0)_H \]

Note that

\[ J_\gamma(x_0; u, w) = \sum_{n=0}^\infty J_\gamma(x(t_n); u, w) \]

and hence

\[ \inf_M \sup_w J_\gamma(x_0; u, w) \leq (Z^\infty_{\min} x_0, x_0)_H \]

On the other hand, we know that

\[ \sup_w \inf_M J_\gamma(x_0; u, w) \leq \sup_w \inf_M J_\gamma(x_0; u, w) \]

because \( M_{CL} \supseteq M_s \). By Lemma 5.6.2 we have

\[ \sup_w \inf_M J_\gamma(x_0; u, w) = (Z^\infty_{\min} x_0, x_0)_H \]

This implies that

\[ \inf_M \sup_w J_\gamma(x_0; u, w) = \sup_w \inf_M J_\gamma(x_0; u, w) = (Z^\infty_{\min} x_0, x_0)_H \]

Conversely, if the game has equal finite upper and lower values under \( M_s \) this implies on \([0, t_s]\)

\[ \sup_w \{ J^{[0,t_s]}_\gamma(x_0; u, w) + (Z^\infty_{\min} x(t_s), x(t_s))_H \} < \infty \]
and thus $\gamma \geq \gamma_c^{S_{0,1}}$ by Theorem 5.3.1.1. This completes the proof of part (i).

(ii) Consider the dynamic system

$$\dot{x} = Ax + B\mu^\infty_\gamma + D\nu^\infty_\gamma, \quad x(0) = x_0$$

The state at the sampling time $t_{n+1}$ can be expressed by

$$x(t_{n+1}) = e^{A(t_{n+1} - t_n)}x(t_n) + \int_{t_n}^{t_{n+1}} e^{A(t_{n+1} - s)}(B\mu^\infty_\gamma + D\nu^\infty_\gamma)ds$$

$$= \left\{ e^{A(t_{n+1} - t_n)} \right. $$

$$+ \left. \int_{t_n}^{t_{n+1}} e^{A(t_{n+1} - s)}( - B^*Z^\infty_{\min}T(s - t_n) + \frac{1}{\gamma^2}D^*Z^\infty_{\min}T(s - t_n)ds) \right\}x(t_n)$$

$$= T(t_{n+1} - t_n)x(x_n)$$

$$= T(t_{n+1} - t_n)T(t_n - t_{n-1})\cdots T(t_1 - t_0)x_0$$

$$= T(t_{n+1})x_0$$

Since $T(t)$ is an exponentially stable semigroup generated by the operator of (5.6.4) by Lemma 5.6.2, there exists $M > 0, \theta > 0$ such that

$$\|T(t)\| \leq Me^{-\theta t}$$

Thus we have

$$\|T(t_{n+1})\| \leq Me^{-\theta t_{n+1}}$$

This implies from (5.6.11) that

$$\|x(t_{n+1})\|_H \leq Me^{-\theta t_{n+1}}x_0$$

Therefore we have $x(t_{n+1}) \to 0$ as $n \to \infty$. Given any $t$, $x(t)$ can be expressed as

$$x(t) = e^{A(t-t_n)}x(t_n) + \int_{t_n}^{t} e^{A(t-s)}(B\mu^\infty_\gamma + D\nu^\infty_\gamma)ds$$

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\[
\begin{align*}
&= \left\{ e^{A(t-t_n)} + \int_{t_n}^{t} e^{A(t-s)} \left( - B^* Z_{\min}^\infty T(s-t_n) + \frac{1}{\gamma^2} D^* Z_{\min}^\infty T(s-t_n) ds \right) \right\} x(t_n) \\
&\quad + \int_{t_n}^{t} e^{A(t-s)} \left( - B^* Z_{\min}^\infty T(s-t_n) + \frac{1}{\gamma^2} D^* Z_{\min}^\infty T(s-t_n) ds \right) x(s) ds \\
&\quad + \int_{t_n}^{t} e^{A(t-s)} \left( - B^* Z_{\min}^\infty T(s-t_n) + \frac{1}{\gamma^2} D^* Z_{\min}^\infty T(s-t_n) ds \right) \nu(s) ds.
\end{align*}
\]

Because \(0 \leq t - t_n \leq t\) there exists a constant \(C(\gamma) > 0\) such that

\[
\|e^{A(t-t_n)} + \int_{t_n}^{t} e^{A(t-s)} \left( - B^* Z_{\min}^\infty T(s-t_n) + \frac{1}{\gamma^2} D^* Z_{\min}^\infty T(s-t_n) ds \right) \| \leq C(\gamma)
\]

Therefore, making use of (5.6.12), we arrive at

\[
\|x(t)\|_H \leq C(\gamma)\|x(t_n)\|_H \leq C(\gamma) Me^{-\theta t_n} \|x_0\|_H \\
\leq C(\gamma) Me^{\theta t} e^{-\theta t} \|x_0\|_H = M(\gamma) e^{-\theta t} \|x_0\|_H
\]

and this completes the proof of (ii).

(iii) Note that according to Lemma 5.6.2, we have

\[
\inf_{M, w} \sup_{\mathcal{M}, w} J(\gamma(x_0; u, w)) \geq \inf_{M, w} \sup_{\mathcal{M}, w} J(x_0; u, w)
\]

\[
= \sup_{w} \inf_{u} J(x_0; u, w) = (Z_{\min}^\infty x_0, x_0)_H
\]

(5.6.13)

and from the proof of part (i), we have for any \(N > 0\)

\[
\inf_{M, w} \sup_{\mathcal{M}, w} \sum_{n=0}^{N} J_n(\gamma(x_0; u, w)) = \sum_{n=0}^{N} J_n(x_0; \mu_\gamma^\infty, \nu_\gamma^\infty)
\]

\[
= (Z_{\min}^\infty x_0, x_0)_H - (Z_{\min}^\infty x(t_{n+1}), x(t_{n+1}))_H \\
\leq (Z_{\min}^\infty x_0, x_0)_H
\]

Hence, (5.6.13) and (5.6.14) yields

\[
\inf_{M, w} \sup_{\mathcal{M}, w} J(x_0; u, w) = \sup_{w} \inf_{M, w} J(x_0; u, w) = (Z_{\min}^\infty x_0, x_0)_H
\]

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This completes the proof of part (iii).  

Remark 5.6.4. As in Remark 5.5.2, Theorem 5.6.3(ii), (iii) hold even if $\gamma = \gamma_{c}^{S[0,t_{1}]}$, provided that the value of the corresponding game is finite over the policy space $M_{s}$.

As in Theorem 5.5.4, we have

**Theorem 5.6.5.** Let $t_{f} > 0$ be sufficiently large, and

(5.6.15)

$$
0 < t_{s} \leq \min\{\frac{\|Z_{\min}^{\infty}\|_{\mathcal{L}(H)}}{\|C\|_{\mathcal{B}_{\infty}} + \frac{r^{2}}{r_{t}}\|D\|_{\mathcal{B}_{\infty}}^{2}}, \frac{\alpha \gamma^{2}}{2rM_{t}^{2}\|D\|_{\mathcal{B}_{\infty}}}, t_{f}/2\}, 0 < \alpha < 1
$$

where

$$
M_{t} = \sup\{\|e^{At}\|_{\mathcal{L}(H)}; 0 \leq t \leq t_{f}\}, r = 2M_{t}^{2}\|Z_{\min}^{\infty}\|
$$

If $x_{0} = 0$ the sampled-data feedback controller

(5.6.16)

$$
\mu_{\gamma}^{\infty}(t, x(t_{n})) = -D^{*}Z_{\min}^{\infty}T(t - t_{n})x(t_{n}) \quad t_{n} \leq t < t_{n+1}
$$

attains this attenuation level $\gamma$, i.e.

(5.6.17)

$$
\sup_{w}\{|F_{\mu_{\gamma}^{\infty}}|^{\frac{1}{2}}/\|w\|_{\mathcal{W}}\} < \gamma
$$

**Proof.** The proof is straightforward. Under (5.6.15), (5.6.6) has a mild solution and applying Theorem 5.6.3(ii), (5.6.16) follows. (5.6.17) follows from Theorem 5.6.3(iii) with $x_{0} = 0$.  

An algorithm for the sampled-data controller design with uniform sampled rate in infinite-horizon case.

Given a desired attenuation level $\gamma > 0$. Assume that (5.6.1) admits a solution $Z_{\min}^{\infty}$ such that

$$
A - BB^{*} + \frac{1}{\gamma^{2}}DD^{*}
$$
is exponentially stable.

**Step 1.** Choose a sufficiently large \( t_f \) (here sufficiently large \( t_f \) means that \( t_f \) compares to the desired sampled rate.) and compute

\[
\|C\| = \|C\|_{\mathcal{L}(H, Z)}, \|D\| = \|D\|_{\mathcal{L}(W, Z)}, M_{t_f} = \sup\{\|T_A(t)\|; 0 \leq t \leq t_f\}
\]

Where \( T_A(t) \) is the \( C_0 \) semigroup generated by \( A \).

**Step 2.** Determine the maximum sampled rate \( t_s \) such that (5.6.15) holds. Make sure that the sampled rate sequence \( \{t_n\} \) satisfies

\[
\sup_{n=0,1,2,\ldots} \{t_{n+1} - t_n\} \leq t_s
\]

**Step 3.** Compute the \( C_0 \) semigroup \( T(t) \) generated by operator

\[
A - BB^* + \frac{1}{\gamma^2} DD^*
\]

**Step 4.** At the current time \( t = t_n \), measure the sampled state \( x(t_n) \).

**Step 5.** Construct the feedback sampled controller

\[
\mu_\gamma^\infty(t, x(t_n)) = -B^* \varphi^\infty_{\min} T(t - t_n)x(t_n) \quad t_n \leq t < t_{n+1}
\]

**Step 6.** At time \( t_{n+1} \) goes back to step 4, and repeat this process until \( t_n \to \infty \).

5.7. Example

A **Parabolic system with disturbance.**

Let \( \Omega \) be an open bounded set of \( \mathbb{R}^n \) with regular boundary \( \partial \Omega \). Consider the disturbance-driven linear parabolic system

\[
\frac{\partial x}{\partial t} (t, \xi) = \Delta_x x(t, \xi) + u(t, \xi) + w(t, \xi), \text{ in } (0, t_f] \times \Omega
\]

\[x(t, \xi) = 0 \text{ in } (0, t_f] \times \partial \Omega\]

\[x(0, \xi) = x_0(\xi) \text{ in } \Omega\]
We choose $H = U = W = Z = L^2(\Omega)$ as space of states, controls, disturbances and outputs. We denote by $A$ the linear self-adjoint operator in $H$:

$$Ax = \Delta_\xi x, \forall x \in D(A)$$

$$D(A) = H^2(\Omega) \cap H^1_0(\Omega).$$

From Poincaré's inequality, we know that there exists $\delta > 0$ such that

$$(-Ax, x) = (\langle -A \rangle^{\frac{1}{2}} x, \langle -A \rangle^{\frac{1}{2}} x) = \|\langle -A \rangle^{\frac{1}{2}} x\|^2 \geq \delta \|x\|^2$$

Hence $A$ is the infinitesimal generator of an analytic, contraction semigroup $T(t)$ in $H$.

Since $\Omega$ is bounded there exists a complete orthonormal system in $L^2(\Omega)$, \{\(e_k\)\}, and a sequence of real numbers

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq .... \rightarrow +\infty$$

such that

$$(-A)e_k = \lambda_k e_k, k = 1, 2, ....$$

Thus $T(t)$ may be expressed as

$$T(t)x = \sum_{k=0}^{\infty} e^{-\lambda_k t}(x, e_k)e_k$$

The kernel function we consider here is, in finite-horizon,

$$J_\gamma(x_0; u, w) = \int_0^t \int_{\Omega} \{ |x(t, \xi)|^2 + |u(t, \xi)|^2 - \frac{1}{\gamma^2} |w(t, \xi)|^2 \} d\xi dt + \int_\Omega |x(t_f, \xi)|^2 d\xi$$

and in infinite-horizon

$$J_\gamma(x_0; u, w) = \int_0^\infty \int_{\Omega} \{ |x(t, \xi)|^2 + |u(t, \xi)|^2 - \frac{1}{\gamma^2} |w(t, \xi)|^2 \} d\xi dt$$
The Riccati equation (S) in this case is

\[(5.7.5) \quad \dot{S}_\gamma(t) + AS_\gamma + S_\gamma A + \frac{1}{\gamma^2} S_\gamma(t) S_\gamma(t) + I = 0 \]

\[S_\gamma(t_f) = I \]

and the Riccati equation (Z) is

\[(5.7.6) \quad \dot{Z}_\gamma(t) + AZ_\gamma + Z_\gamma A - Z_\gamma(t)(1 - \frac{1}{\gamma^2})Z_\gamma(t) + I = 0 \]

\[Z_\gamma(t_f) = I \]

Here \(I\) is the identity map of \(H\). Clearly when \(\gamma \geq 1\), \((5.7.6)\) admits a mild solution on \([0, t_f]\) for any \(t_f > 0\); hence the critical value of \((5.7.6)\) \(\gamma_c^Z < 1\).

(i) Finite-horizon design.

Let the desired attenuation level \(\gamma\) be \(\gamma_{des} = 1\). It is straightforward to see that \((5.7.6)\) has the following mild solution:

\[Z_{\gamma=1}(t)x = T^*(t)T(t)x + \int_0^t T^*(t-s)T(t-s)xds \quad \forall x \in H \]

and hence we have

\[\|Z_{\gamma=1}(t)\| \leq 1 + t_f \]

due to the contraction of \(T(t)\) (i.e. \(\|T(t)\|_{L(H)} \leq 1\)). We estimate the uniform sampling rate \(\{t_n\}_{n=0}^N\).

\[0 < t_{n+1} - t_n \leq \frac{1 + t_f}{1 + 2(1 + t_f)^2} \]

and hence we can see the relationship between sampling rate and final terminal time. Some calculations show the following:

\(t_f = 1, \implies t_s = 0.2222\)
\[ t_f = 2, \implies t_s = 0.1578 \]
\[ t_f = 5, \implies t_s = 0.0821 \]
\[ t_f = 10, \implies t_s = 0.0452 \]

In fact if we make use of the fact that \( T(t) \) is exponentially stable and
\[
(5.7.7) \quad \|T(t)\| \leq e^{-\lambda_1 t}
\]
we may get a larger sampling interval \( t_s \). In this case, \( t_s \) depends on \( \lambda_1 \) which is determined by \( \Omega \). We will carry this out when we deal with the infinite-horizon case. The sampled feedback controller for this parabolic system is
\[
(5.7.8) \quad \mu_{\gamma=1}^s(t, x(t_k, \xi)) = -Z_{\gamma=1}(t)T(t - t_k)x(t_k, \xi)
\]
\[
= T^*(t)T(t)(t - t_k)x(t_k, \xi) + \int_{t_k}^{t} T^*(t - s)T(t - s)T(t - t_k)x(t_k, \xi)ds
\]
\[
= T(3t - t_k)x(t_k, \xi) + \int_{t_k}^{t} T(3t - 2s - t_k)x(t_k, \xi)ds
\]
\[
= \sum_{k=0}^{+\infty} e^{-\lambda_k(3t - t_k)}(x(t_k), e_k)e_k + \int_{t_k}^{t} \left\{ \sum_{k=0}^{+\infty} e^{-\lambda_k(3t - 2s - t_k)}(x(t_k), e_k)e_k \right\}ds
\]
\[ t_k \leq t < t_{k+1} \]

(ii) Infinite-horizon design.

In this case, the algebraic Riccati equation is
\[
(5.7.9) \quad A^* Z_\gamma + Z_\gamma A - Z_\gamma (1 - \frac{1}{\gamma^2}) Z_\gamma + I = 0
\]

Since \((A, B)\) is exponentially stabilizable and \((A, C)\) is exponentially detectable, \((5.7.9)\) admits a unique positive solution in \( \Sigma^+(H) \) when \( \gamma > 1^8 \). We still assume that the desired attenuation level is \( \gamma_{des}^\infty = 1 \). That \((5.7.9)\) admits a unique positive solution implies that

\[ ^8 \text{Because if } \gamma > 1 \text{ (5.7.9) is in fact the algebraic Riccati equation for the linear quadratic optimal control problem.} \]
it is possible to achieve this goal by using sampled-data feedback controller. Since $T(t)$ is exponentially stable, it is easy to see that

$$(5.7.10) \quad Z_{\min}^\infty x = \sum_{k=0}^\infty \frac{1}{2\lambda_k} (x, e_k)e_k$$

Thus if we use uniform sampling rate $t_s$, we have the bound:

$$(5.7.11) \quad 0 < t_s \leq \min \left\{ \frac{\|Z_{\min}^\infty\|}{1 + 4\|Z_{\min}^\infty\|^2}, \frac{\alpha}{4\|Z_{\min}^\infty\|} \right\}, 0 < \alpha < 1$$

Since the eigenvalues $\lambda_k, k \in N$, and the eigenvectors $e_k \in H$ are related to $\Omega$, the sampling rate is also affected by $\Omega$. For example, if $\Omega = (0, 1)$ then we have

$$\lambda_k = k^2 \pi^2, \quad e_k = \sqrt{2} \cos(k\pi \xi), \quad 0 < \xi < 1$$

By using (5.7.9), we have

$$\|Z_{\min}^\infty\|_{\mathcal{L}(H)} \leq 1.64393$$

Thus we can estimate the uniform sampling rate $t_s$, and using (5.7.10) we find that $t_s = 0.1499$. By using this sampling rate, we can achieve the attenuation level $\gamma_{des} = 1$. The sampling interval is quite small because this is just the "first" estimate. In fact, we can increase this sampling rate by a careful study of (5.6.6) for this example. Also it is not difficult to see that as $\gamma$ increases, $t_s$ can be increased. The sampled-data feedback controller in the infinite-horizon case is

$$(5.7.12) \quad \mu(t, x(t_k) = -Z_{\min}^\infty T(t - t_k)x(t_k)$$

where $Z_{\min}^\infty$ is given by (7.11) and $T(t)$ is given by (5.7.2).
5.8. Concluding Remarks

In this chapter, we have provided $H^\infty$-optimal sampled-data controller for general finite-horizon time-varying evolution systems and infinite-horizon time-invariant evolution systems in infinite-dimensional spaces. The approach can be extended to the case that the available state information for controllers is in sampled form obtained from the boundary only. Result on this extension will be reported in the next chapter.

5.9. Appendix

Here we provide a proof of

$$\delta_w J_\gamma(x_0; \hat{u}, w) = D(t)^* Z_\gamma(t)x(t) - \gamma^2 w$$

which appears in Theorem 5.4.1.

Proof: Define

$$\varphi(w) = \gamma^2 \|w\|^2_2 - \bar{\varphi}(w), w \in L^2([0, t_0]; W)$$

where

$$\bar{\varphi}(w) = \int_0^{t_f} (\|C(t)x^w(t)\|^2_Z + \|\hat{u}(t)\|^2_Z) dt$$

and $(x^w, \hat{u})$ satisfies

$$\dot{x}^w(t) = A(t)x^w(t) + B(t)\hat{u}(t) + D(t)w(t), t \in (0, t_f]$$

$x^w(0) = x_0$

$$\hat{u}(t) = -B(t)^* Z_\gamma(t) \Phi(t, 0)x_0$$
We have therefore

\[
(\delta_w \varphi(w), \bar{w}) = \\
\int_0^{t_f} \left( ((C(t)x(t), C(t)z(t)) + (C(t)z(t), C(t)x(t))) \right) + (Q_f x(t_f), z(t_f)) + (z(t_f), Q_f x(t_f)), \forall \bar{w} \in L^2([0, t_f]; W)
\]

where \( z \) satisfies

\[
\dot{z}(t) = A(t)z(t) + D(t)\bar{w}(t)
\]

\( z(0) = 0 \)

Note that

\[
\int_0^{t_f} (C(t)^*C(t)x(t), z(t))dt = \int_0^{t_f} (p(t), D(t)\bar{w}(t))dt - (Q_f x(t_f), z(t_f))
\]

where \( x(t) = \Phi(t, 0)x_0 \) and \( p \) is defined in (5.4.5). This gives

\[
(\delta_w \varphi(w), \bar{w}) = \int_0^{t_f} ((p(t), D(t)\bar{w}(t)) + (D(t)\bar{w}(t), p(t)))dt
\]

and for all \( \bar{w} \in L^2([0, t_f]; W) \)

\[
(\delta_w \varphi(w), \bar{w}) = \int_0^{t_f} (\gamma^2(w, \bar{w}) + \gamma^2(\bar{w}, w) - (p(t), D(t)\bar{w}(t)) - (D(t)\bar{w}(t), p(t)))dt
\]

Hence we have

\[
\delta_w \varphi(w) = \gamma^2 w - D^*(t)p(t)
\]

Since we already know from (5.4.5) that

\[
p(t) = Z_\gamma(t)x(t) = Z_\gamma(t)\Phi(t, 0)x_0
\]

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we have that

$$\delta_w J_\gamma(x_0; \hat{u}, w) = D(t)^* Z_\gamma(t)x(t) - \gamma^2 w$$

and consequently

$$\hat{w}(t) = \frac{1}{\gamma^2} D(t)^* Z_\gamma(t)x(t) = \frac{1}{\gamma^2} D(t)^* Z_\gamma(t)\Phi(t, 0)x_0.$$
6. $H^\infty$ Boundary Control of Parabolic Systems with Sampled State Measurements

6.1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with regular boundary $\partial \Omega$, and denote the control space by $U = L^2(\partial \Omega)$ and the state space by $Y = L^2(\Omega)$. Consider the following parabolic system with disturbance-perturbed dynamics:

\begin{equation}
  y_1(t, x) = A(t, x; D)y_1(t, x) + B_2(t, x)w(t, x) \quad \text{in } (0, t_f) \times \Omega
\end{equation}
\begin{equation}
  B(t, x; D)y(t, x) = u(t, x) \quad \text{in } (0, t_f) \times \partial \Omega
\end{equation}
\begin{equation}
  y(0, x) = y_0(x) \quad \text{in } \Omega
\end{equation}
\begin{equation}
  z(t, x) = C(t, x)y(t, x) + D_{12}(t, x)u(t, x) \quad \text{in } (0, t_f) \times \Omega
\end{equation}

Here the function $u \in L^2(0, t_f; U)$ is the control, $w \in L^2(0, t_f; W)$ is a deterministic disturbance, $y(t) \in Y$ is the state of the system, and $z(t) \in Z$ is the to-be-controlled output (where $Z$ is another separable Hilbert space). In a typical parabolic boundary-value problem, $A(t, x; D)$ is a differential operator of order $2m$ with smooth coefficient functions of the variable $x$ in $\bar{\Omega}$ and Hölder continuous with respect to time $t$, and $B(t, x; D)$ consists of $m$ boundary differential operators of order $\leq 2m - 1$ with smooth coefficients on $\partial \Omega$.

In second-order parabolic systems, for example, the boundary condition can be either the Dirichlet boundary condition

\begin{equation}
  y(t, x) = u(t, x), \quad 0 \leq t \leq t_f, x \in \partial \Omega
\end{equation}

or the oblique derivative boundary condition

\begin{equation}
  \sum_{i=1}^{n} \beta_i(t, x)y_{x_i}(t, x) + \gamma(t, x)y(t, x) = u(t, x), \quad 0 \leq t \leq t_f, x \in \partial \Omega
\end{equation}
We introduce the quadratic cost function

\[ L(u, w) = \int_0^{t_f} \|z(t)\|_Z^2 dt + (P_{t_f} y(t_f), y(t_f))_Y \]

where \( P_{t_f} \) is assumed to satisfy the abstract hypotheses listed in the next section. For each fixed \( u \in L^2([0, t_f]; U) \), \( L \) induces a mapping of \( L^2([0, t_f]; W) \) into nonnegative reals which we denote by \( \mathcal{F}_u \). The minimax (\( H^\infty \)-optimal) controller design problem is to find a controller \( u \) (usually a feedback controller) that minimizes the following index

\[ g(u) := \sup_w \{ \mathcal{F}_u^{1/2} / \|w\|_2 \} \]

and the minimax (\( H^\infty \)-suboptimal) controller design problem is for a given desired attenuation level \( \gamma_{des} \) to find a controller \( u \) such that

\[ g(u) = \sup_w \{ \mathcal{F}_u^{1/2} / \|w\|_2 \} < \gamma_{des} \]

Note that if the controller space were restricted to comprise only linear maps, then \( g(\mu) \) would be the operator norm of the (weighted) gain from the disturbance to the output \( z \).

As shown in [BB], studying (6.1.6) is equivalent to studying the upper value of a zero-sum differential game with cost function

\[ \bar{L}_\gamma(u, w) = L(u, w) - \gamma^2 \int_0^{t_f} \|w(t)\|_W^2 dt \]

where \( \gamma \) is a positive parameter.

Boundary control of systems governed by partial differential equations is one of the most important problems in control theory. In the case of parabolic equations, both variational and semigroup methods have been successfully applied. Early papers treating boundary control problems are due to Balakrishnan ([Ball]) and Washburn ([Wa]), who developed
further an old idea of Fattorini ([Fa]). Simplifications and refinements through domains with fractional powers were introduced by Triggiani ([Tr]). Keulen, McMillian and Triggiani, Barbu were among the first to study the $H^\infty$ boundary control problems involving particular type of partial differential equations ([Ke], [MT 1-3], [B2]). They both treated the Riccati equations arising in boundary control problems in an abstract framework so that the technique involved in the study of the Riccati equations as well as the representation formulas can be avoided. This treating method is quite neat from a theoretical point of view. However, it is not convenient in a practical sense because when we deal with $H^\infty$-optimal control problem (or linear quadratic regulator problems), we eventually need to construct the optimal controller. Hence a concrete form for the optimal controller is more important from a practical point of view.

In this chapter we extend the sampled feedback scheme presented in the previous chapter further to the boundary control problem for general parabolic system (6.1.1). The major differences between those work of $H^\infty$-optimal boundary control problems we mentioned before and the work of this chapter are: (i) the underlying system in this chapter is time-variant, and those of all previous works were time-invariant, (ii) we only deal with the finite-horizon case, and thus stability is not our major concern. Our major interest here is that how long the sampling time should be such that a desired level of robustness can be maintained. The basic approach to the problem here is a blend of semigroup operator theory and differential game theory, with a particular differential game associated with the disturbance attenuation problem. In order to provide a closed-form expression for the optimal controller, we study the special of generalized Riccati equations arising in boundary $H^\infty$-optimal control problems for the parabolic systems. Because only sample
state information is available, we need to study two generalized Riccati equations, one describing the open-loop performance and the other one used to construct the optimal control for the closed loop case. The major difficulty here is that the optimal controller is related to an unbounded operator with a fractional power. We have shown that in parabolic systems the Riccati equation corresponding to the closed-loop performance has enough regularity to overcome this difficulty under suitable assumptions so that the optimal controller is in $L^2$ space. The current chapter seems to be the first one to address and resolve the question of optimum (time-varying) intersample behavior in the context of $H^\infty$ optimization for boundary-value problems. We provide a sampled feedback scheme, which only uses sampled data, is capable of achieving desired attenuation level for which usually continuous feedback is needed. We devise a procedure for the computation of the optimum disturbance attenuation level and obtain an expression for the optimum controller when the desired level is larger than that under continuous perfect measurements.

This chapter consists of seven sections. Section 2 is devoted to developing a suitable approach to initial boundary value problems based on the theory of evolution operators, in view of its application to boundary control problems. Section 2.1 contains the basic assumptions for the parabolic systems. In section 2.2, we introduce two generalized Riccati equations, and study the existence and regularization of the solutions of these two equations. Here we follow the approach of [Fl1] based on a direct solution of the Riccati equation arising in the LQR problem. Section 3 discusses the existence of solutions to (6.1.1) under open-loop and closed-loop controls. Section 4 investigates the $H^\infty$ problem associated with (6.1.1) when both open-loop and closed-loop controls are used. In section 5, we derive a sampled-data $H^\infty$ controller, and elaborate on the connection be-
tween sampled rate and related Riccati equations. An example is given in section 6 to illustrate the application. The chapter ends with the concluding remarks of section 7, and an Appendix.

We conclude this section by listing some notation.

Throughout the chapter we denote

\[ \Sigma = \partial \Omega \times ]0, t_f[, \quad Q = \Omega \times ]0, t_f[ \]

and, following [LM], we define, with \( r, s \) being two nonnegative real numbers

\[ H^{r,s}(Q) = H^0(0, t_f; H^r(\Omega)) \cap H^s(0, t_f; H^0(\Omega)) \]

which is a Hilbert space with the norm

\[ \{ \int_0^{t_f} \| y(t) \|_{H^r(\Omega)}^2 dt + \| y \|_{H^s(0, t_f; H^0(\Omega))}^2 \}^{1/2} \]

where \( H^r \) is Sobolev space as defined in [LM]. Let \( X, Y \) be a Hilbert space, and set:

\[ \mathcal{L}(Y) := \text{space of bounded linear operators } T : Y \to Y \]

\[ \Sigma(Y) := \text{space of self-adjoint operators } T \in \mathcal{L}(Y), \text{endowed with the norm of } \mathcal{L}(Y) \]

\[ \Sigma^+(Y) := \text{space of self-adjoint operators } T \in \mathcal{L}(Y) \text{ which are positive, i.e.} \]

\[ (T x, x)_Y \geq 0 \text{ for each } x \in Y \]

\[ C_s([a, b], \mathcal{L}(Y)) := \text{space of operator-valued functions } T(\cdot) : [a, b] \to \mathcal{L}(Y) \]

which are strongly continuous

\[ C_u([a, b], \mathcal{L}(Y)) := \text{space of operator-valued functions } T(\cdot) : [a, b] \to \mathcal{L}(Y) \]

which is endowed with the norm \( \| T \| = \sup_{t \in [a, b]} \| T(t) \|_{\mathcal{L}(Y)} \)
\( C^0([a, b], \mathcal{L}(Y)) \) := space of functions which are \( \vartheta \)-Hölder continuous

\( \mathcal{B}_\infty(0, t_f; X, Y) := \) space of strongly measurable, essentially bounded functions : \((0, t_f) \rightarrow \mathcal{L}(X, Y)\)

\( T^* := \) adjoint operator of \( T \) whenever it exists

6.2. Problem formulation and hypotheses.

6.2.1. Abstract formulation

We invoke the following standard assumptions on parabolic systems:

1. \( \mathcal{A}(t, x; D) \) is properly elliptic in \( \Omega \) (cf. [LM], Vol.1, Chapter 2, Def. 1.2),

2. \( \mathcal{B}(t, x; D) = \{ B_j(t, x; D) \}_{1 \leq j \leq m} \) is a normal system on \( \partial \Omega \) (cf. [LM], Vol.1, Chapter 2, Def. 1.4),

3. \( \mathcal{A}(t, x; D) \) and \( \mathcal{B}(t, x; D) \) satisfy the conditions given in [T] (Chapter 3, Thm 3.8.1) for \( \theta = \pi \).

For each \( t \in [0, t_f] \) we define \( \{ \mathcal{A}(t) \}_{[0, t_f]} \) as the realization in \( L^2(\Omega) \) of the operator \( \mathcal{A}(t, x; D) \) with homogeneous boundary conditions determined by \( \mathcal{B}(t, \cdot, D) \) as follows

\[
D_{\mathcal{A}(t)} := \{ y \in H^{2m}(\Omega) : B_j(t, \cdot, D)y = 0 \text{ on } \partial \Omega, 1 \leq j \leq m \},
\]

\[
\mathcal{A}(t)y := \mathcal{A}(t, \cdot; D)y, \forall y \in D_{\mathcal{A}(t)}
\]

Its adjoint is given by

\[
D_{\mathcal{A}^*(t)} = \{ v \in H^{2m}(\Omega) : C_j(t, x; D)v = 0 \text{ on } \partial \Omega, 1 \leq j \leq m \}
\]

\[
\mathcal{A}^*(t)v = \mathcal{A}(t, \cdot; D)v
\]
where \( \{C_j(t, x; D)\}_{1 \leq j \leq m} \) is the adjoint system of \( \{B_j(t, \cdot; D)\}_{1 \leq j \leq m} \). A family of linear operators \( \{U(t, s) : 0 \leq s \leq t \leq t_f\} \subset \mathcal{L}(Y) \) is said to be an evolution operator if

(a) \( U(t, s)U(s, r) = U(t, r), \quad U(s, s) = I, \quad t \leq r \leq s \leq t \leq t_f, \)

(b) \( (t, s) \rightarrow U(t, s) \) is strongly continuous for \( 0 \leq s \leq t \leq t_f. \)

According to the nature of parabolic systems (1)-(3) and from a control point of view, it is natural to introduce the following assumptions:

(A1) \( \{A(t)\}_{t \in [0, t_f]} \) is a family of infinitesimal generators of analytic semigroups in \( Y \) and there exists a \( \lambda_0 > 0 \) such that \( \lambda_0 \in \rho(A(t)) \) for all \( t \in [0, t_f] \) and

\[
\|((\lambda + \lambda_0)I - A(t))^{-1}\| \leq \frac{M}{1 + |\lambda|}, \text{ for } \text{Re}\lambda \geq 0, t \in [0, t_f]
\]

(A2) \( \{U(t, s)\}_{0 \leq s \leq t \leq t_f} \) is the strongly continuous evolution operator in \( Y \).

(A3) The operator-valued function \( (t, s) \rightarrow U(t, s)^* \) belongs to \( C_s([0, t_f], \mathcal{L}(Y)) \); moreover, for each \( \eta \in [0, 1] \) and \( (t, s) \in \Delta, U(t, s)^* \in \mathcal{L}(Y, D[\lambda_0I - A(s)^*]^{\eta}) \), the map \( (t, s) \rightarrow [\lambda_0I - A(s)^*]^{\eta}U(t, s)^* \) is strongly measurable and for \( \forall (t, s) \in \Delta \)

\[
\frac{\partial}{\partial s} U(t, s)^* = -A(s)^* U(t, s)^*
\]

and \( \forall \eta, \mu \in [0, 1] \)

\[
\|[(\lambda_0I - A(s)^*)^{\eta}U(t, s)^*][\lambda_0I - A(t)^*]^{-\mu}\| \leq M_{\eta\mu}[(t - s)^{\mu - \eta} + 1]
\]

(A4) \( \{G(t)\}_{t \in [0, t_f]} \) is a family of operators in \( \mathcal{L}(U, Y) \) such that there exits \( \alpha \in [0, 1] \) with the following properties: \( G(t) \in \mathcal{L}(U, D[\lambda_0I - A(t)^*]^\alpha) \) for each \( t \in [0, t_f] \) and the map \( t \rightarrow [\lambda_0I - A(t)]^\alpha G(t) \) belongs to \( L^\infty(0, t_f; \mathcal{L}(U, Y)) \).

(A5) There exist constants \( L \) and \( 0 < \alpha \leq 1 \) such that

\[
\|(A(t) - A(s))(\lambda_0 - A(\tau))^{-1}\| \leq L|t - s|^\alpha, \text{ for } s, t, \tau \in [0, t_f]
\]
Remark 6.2.1.1. (i) If \( U = L^2(\partial \Omega) \) and \( Y = L^2(\Omega) \), assumption (A4) is a minimal standard assumption to guarantee that one can define the map \( G : [0, t_f] \times L^2(\partial \Omega) \to L^2(\Omega) \) as \( G(t)u := \Phi \) where \( \Phi \) is the unique solution of the problem:

\[
\lambda_0 \Phi - A(t, \cdot, D)\Phi = 0 \text{ in } \Omega.
\]

\[
B(t, \cdot, D)\Phi = u \text{ on } \partial \Omega.
\]

In the case of (6.1.2), such a map (called Dirichlet map) exists when \( \alpha \in ]0, \frac{1}{4}[, \) and in the case of (6.1.3) the map (called Neumann map) is well-defined when \( \alpha \in ]0, \frac{3}{4}[, \) (cf. [LM] or [AFT]).

(ii) Assumptions (A3) and (A4) indicate that for \( \forall (t, s) \in \Delta \) (cf. [AFT])

\[
\|[\lambda_0 I - A(s)^*]^{1-\alpha} U(t, s)^* [\lambda_0 I - A(s)]^\alpha G(s)\|_{L(U, Y)} \leq c(t - s)^{\alpha - 1}
\]

This is a sufficient condition for the existence of a (mild) solution to (6.1.1) a result that we will prove later in Section 3.

(iii) Assumptions (A1) and (A5) generally hold in parabolic systems; they imply that:

\[
\frac{\partial}{\partial t} U(t, s) = A(t)U(t, s), \text{ for } 0 \leq t \leq t_f,
\]

\[
\| \frac{\partial}{\partial t} U(t, s) \|_{L(Y)} = \| A(t)U(t, s) \|_{L(Y)} \leq \frac{C}{t - s} \text{ for } t > s
\]

(cf. Lemma 6.1, Chapter 7 of [Paz]).

Under the above assumptions, we introduce a representation formula for state \( y(t) \) as follows:
**Definition 6.2.1.2.** For given \( u \in L^2(0, t_f; U) \), \( w \in L^2(0, t_f; W) \) and \( B_2 \) is in the space of \( B_\infty(0, t_f; W, Y) \), the following expression

\[
y(t) = U(t, 0)y_0 + \int_0^t U(t, s)B_2(s)w(s)ds \\
+ \int_0^t [(\lambda_0 I - A(s)^*)^{1-\alpha}U(t, s)^*][\lambda_0 I - A(s)]^\alpha G(s)u(s)ds
\]

is called the representation formula for (6.1.1).

**Remark 6.2.1.3.** Under assumption (A3), the representation formula (6.2.1.1) makes sense in view of Remark 6.2.1.1 (ii). We will see that it provides in fact a unique (mild) solution of (6.1.1).

**Theorem 6.2.1.4.**

1. If \( u \in L^2(0, t_f; U) \) and \( w \in L^2(0, t_f; W) \), then (2.1) defines a function \( y \in L^2(0, t_f; Y) \), and \( \exists c > 0 \) such that

\[
\|y\|_{L^2(0, t_f; Y)} \leq c \{\|y_0\|_Y + \|u\|_{L^2(0, t_f; U)} + \|w\|_{L^2(0, t_f; W)}\}
\]

2. If \( u \in L^p(0, t_f; U) \) for some \( p > 1/\alpha \), and \( w \in L^2(0, t_f; W) \), then \( y \in C([0, t_f]; Y) \) and \( \exists c > 0 \) such that

\[
\|y\|_{C([0, t_f]; Y)} \leq c \{\|y_0\|_Y + \|u\|_{L^p(0, t_f; U)} + \|w\|_{L^2(0, t_f; W)}\}
\]

**Proof.** (1) This result follows from assumptions (A3), (A4) and Young's inequality.

(2) For convenience, we let

\[
\overline{B}(t, s) := [(\lambda_0 I - A(s)^*)^{1-\alpha}U(t, s)^*][\lambda_0 I - A(s)]^\alpha G(s)
\]

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If \( p > 1/\alpha \), by Remark 6.2.1.1(ii) we have, for \( 0 \leq \tau < t \leq t_f \),

\[
\| \int_\tau^t \mathcal{B}(t, s)u(s)ds \|_Y \leq \left( \int_\tau^t c(t-s)^{-(\alpha-\alpha p/(p-1))}ds \right)^{(p-1)/p} \| u \|_{L^p(\tau, t_f; U)} \\
\leq c \frac{p-1}{\alpha p-1} (t-\tau)^{\alpha-1/p} \| u \|_{L^p(\tau, t_f; U)}
\]

which implies \( y \in L^p(0, t_f; Y) \) by (A2). Next we claim that actually \( y \) is in the space of \( C([0, t_f]; Y) \). Let \( t_0 \in [0, t_f] \). We know that \( \forall \varepsilon > 0 \) there exists \( \delta > 0 \) such that when \( |t - t_0| \leq \delta \) we have

\[
\| \int_{t_0-\delta}^t \mathcal{B}(t, s)u(s)ds \|_Y \leq \varepsilon
\]

Thus we get

\[
\| y(t) - y(t_0) \|_Y \leq \| U(t, 0)y_0 - U(t_0, 0)y_0 \|_Y + M_0 \varepsilon
\]

Note that \( t \to U(t, 0) \) is strongly continuous; hence the result follows. \( \square \)

6.2.2. Generalized Riccati evolution equations

Recall that the cost function adopted is

\[
L(u, w) = \int_0^{t_f} \| z(t) \|^2 dt + (P_{t_f}y(t_f), y(t_f))_Y
\]

for which we now invoke the following conditions:

(A6) \( C \in B_\infty(0, t_f; Y, Z), B_2 \in B_\infty(0, t_f; W, Y); \)

(A7) \( D_{12}(t)[C(t), D_{12}(t)] = [0, I]; \)

(A8) \( P_{t_f} \in \Sigma^+(Y) \), and if \( \alpha \leq \frac{1}{2} \) there exists \( \beta \in ]\frac{1}{2} - \alpha, \frac{1-\alpha}{2} [ \) such that \( P_{t_f} \) is in the space of \( \mathcal{L}(Y, D[\lambda_0I - A(t_f)^*]) \) and

\[
[\lambda_0I - A(t_f)^*]^{\beta} P_{t_f} [\lambda_0I - A(t_f)]^{\beta}
\]

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is bounded, where $\alpha$ satisfies assumption (A4).

Remark 6.2.2.1. From Theorem 6.2.1.4 we know that in general $y \in L^2(0, t_f; Y)$, hence the term $(P(t_f, y(t_f), y(t_f)))_Y$ may not be well defined. Assumption (A8) is called the symmetric regularity condition which implies that the operator

$$[\lambda_0 I - A(t_f)]^\beta P(t_f, [\lambda_0 I - A(t_f)]^\beta$$

can be extended to an operator $L \in L(Y)$ (cf. [FL 1-2]). Therefore we have the following proposition which is analogous to that of the [AFT] which is without disturbance.

Proposition 6.2.2.2. Define a mapping $C(0, t_f; U) \times C(0, t_f; W) \rightarrow \mathbb{R}$ by

$$\begin{equation}
(u, w) \rightarrow (P(t_f, y(t_f), y(t_f)))_Y
\end{equation}$$

(6.2.2.1)

Then it is locally uniformly continuous with respect to the topology of space $L^2(0, t_f; U) \times L^2(0, t_f; W)$, and hence it can be extended to $L^2(0, t_f; U) \times L^2(0, t_f; W)(= \mathcal{U} \times \mathcal{W})$.

Proof. Let $L$ be the bounded extension to $Y$ of the operator

$$[\lambda_0 I - A(t_f)]^\beta P(t_f, [\lambda_0 I - A(t_f)]^\beta$$

we have by (6.2.1.1), (A2), (A3), and (A5)

$$(P(t_f, y(t_f), y(t_f)))_Y$$

$$= \|L^{1/2}[\lambda_0 I - A(t_f)]^{-\beta} y(t_f)\|_Y^2$$

$$\leq \|L^{1/2}\|_{L(Y)}^2 \left\{ \|\lambda_0 I - A(t_f)]^{-\beta} U(t_f, 0)y_0\|_Y + \int_0^{t_f} \|\lambda_0 I - A(t_f)]^{-\beta} U(t_f, s)B_2(s)\|_Y + \int_0^{t_f} \|[(\lambda_0 I - A(s))^{1-\alpha} U(t_f, s)^*[\lambda_0 I - A(t_f)]^{-\beta}]^* [\lambda_0 I - A(s)]^\alpha G(s)u(s)\|_Y \text{d}s \right\}^2$$

$$\leq c \left\{ \|y_0\|_Y^2 + \|w\|_W^2 + \left[ \int_0^{t_f} \left[ 1 + (t_f - s)^{\beta + \alpha - 1} \right] \|u(s)\|_U \text{d}s \right]^2 \right\}$$

$$\leq c \left\{ \|y_0\|_Y^2 + \|w\|_W^2 + \left( t_f^{-\beta + 2\alpha - 1} + 2t_f^{\alpha + \beta} + t_f \right) \|u\|_U^2 \right\}$$

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Hence if \((u_i, w_i) \in C([0, t_f], U) \times C([0, t_f], W), i = 1, 2\) and \(y_1, y_2\) are corresponding functions (2.1) with initial state \(y_0\) we have

\[
\begin{align*}
|(P_{t_f} y_1(t_f), y_1(t_f))_Y - (P_{t_f} y_2(t_f), y_2(t_f))_Y| & \\
& \leq \|L^{1/2} [\lambda_0 I - A(t_f)]^{-\beta} y_1(t_f)\|_Y + L^{1/2} [\lambda_0 I - A(t_f)]^{-\beta} y_2(t_f)\|_Y \\
& \cdot \|L^{1/2} [\lambda_0 I - A(t_f)]^{-\beta} [y_2(t_f) - y_1(t_f)]\|_Y \\
& \leq c \{\|y_0\|_Y + \sum_{i=1}^2 (\|u_i\|_U + \|w_2\|_W)\} (\|u_2 - u_1\|_U + \|w_2 - w_1\|_W^2)
\end{align*}
\]

Therefore, the mapping defined by (6.2.2.1) is locally uniformly continuous on the space \(L^2(0, t_f; U) \times L^2(0, t_f; W)\). □

Now we introduce the solution definitions of two generalized Riccati equations, associated with the differential game with cost (6.1.7):

(6.2.2.2)

\[
\dot{S}_\gamma(t) + A(t)^* S_\gamma(t) + S_\gamma(t) A(t) + \frac{1}{\gamma^2} S_\gamma(t) B_2(t) B_2(t)^* S_\gamma(t) + C(t) C(t)^* = 0
\]

\[
S(t_f) = P_{t_f}
\]

and

(6.2.3.3)

\[
\dot{Z}_\gamma(t) + A(t)^* Z_\gamma(t) + Z_\gamma(t) A(t) + \frac{1}{\gamma^2} Z_\gamma(t) B_2(t) B_2(t)^* Z_\gamma(t) + C(t) C(t)^* \\
- [(\lambda_0 I - A(t)^*)^{1-\alpha} Z_\gamma(t)]^* K(t) (\lambda_0 I - A(t)^*)^{1-\alpha} Z_\gamma(t) = 0
\]

where

\[
K(s) := [\lambda_0 I - A(s)]^\alpha G(s) [((\lambda_0 I - A(s))^\alpha G(s))^*]
\]

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Definition 6.2.2.3. For $\gamma > 0$, let $S_\gamma \in C_s([0, t_f], \Sigma(Y))$ satisfy integral equation

\begin{equation}
S_\gamma(t)y = U(t_f, t)^* P_t U(t_f, t)y + \int_t^{t_f} [U(s, t)^*(C^*(s)C(s) + \frac{1}{\gamma^2} S_\gamma(s)B_2(s)B_2(s)^*S_\gamma(s))U(s, t)y] \, ds \quad \forall y \in Y
\end{equation}

Then $S_\gamma$ is called the mild solution of the generalized Riccati equation (6.2.2.2).

Definition 6.2.2.4. For $\gamma > 0$, let $Z_\gamma \in C_s([0, t_f], \Sigma(Y))$ satisfy integral equation

\begin{equation}
Z_\gamma(t)y = U(t_f, t)^* P_t U(t_f, t)y + \int_t^{t_f} U(s, t)^* \left\{ C(s)^* C(s) - [\lambda_0 I - A(s)^*]^{1-\alpha} Z_\gamma(s) \right\}^* \cdot K(s)[\lambda_0 I - A(s)^*]^{1-\alpha} Z_\gamma(s) + \frac{1}{\gamma^2} Z_\gamma(s)B_2(s)B_2(s)^*Z_\gamma(s) \right\}U(s, t)y \, ds
\end{equation}

where

\[ K(s) := [\lambda_0 I - A(s)]^\alpha G(s)[(\lambda_0 I - A(s))^\alpha G(s)]^* \]

Then $Z_\gamma$ is called the mild solution of the generalized Riccati equation (6.2.2.3).

Remark 6.2.2.5. (i) From the definition 6.2.2.3, we can see immediately that

\[ S_\gamma \in C_s([0, t_f], \Sigma^+(Y)) \]

provided that $S_\gamma$ is the mild solution of (6.2.2.2). From the Definition 6.2.2.4, using similar argument as that in Proposition 6.2.2.12, we have

\[ Z_\gamma \in C_s([0, t_f], \Sigma^+(Y)) \]

(ii) In the $H^\infty$ optimum problems, (6.2.2.2) is used for open loop and (6.2.2.3) for closed loop (cf. [BB],[B],[XB1]). For the case $B_2(t) \equiv 0$, Acquistapace, Flandoli, and Terreni discussed the integral equation of (6.2.2.3)(cf. [AFT], [AT]).

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We now introduce the approximate version of (6.2.2.2):

\[
\dot{S}^n(\gamma(t)) + A^n_\gamma(t)S^n_\gamma(t) + S^n_\gamma(t)A_n(t) + \frac{1}{\gamma^2}S^n_\gamma(t)B_2(t)B_2^*(t)S^n_\gamma(t) + C(t)C^*(t) = 0
\]

\[S_n(t_f) = Q_f\]

where \(A_n(t) = n^2R(n, A(t)) - nI\) is the Yosida approximation of \(A(t)\) and \(R(n, A(t))\) is the resolvent of \(A(t)\). Without loss of generality we may assume that

\[\lim_{n \to \infty} U_{A_n}(t, s)x = U_A(t, s)x, \forall x \in Y \text{ uniformly on } \Omega\]

where \(U_{A_n}(t, s)\) is the evolution operator generated by \(A_n(t)\). We provide here some facts on the generalized Riccati equations (6.2.2.2), (6.2.2.3) which we need later.

**Proposition 6.2.2.6.** Given \(\gamma > 0\), there exists \(\tau_{\min} : 0 \leq \tau_{\min} < t_f\) such that the generalized Riccati equations (6.2.2.2), (6.2.2.7) have unique mild solutions \(S_\gamma, S^n_\gamma\) on \([\tau_{\min}, t_f]\), respectively, and

\[\lim_{n \to \infty} S^n_\gamma = S_\gamma \text{ in } C_s([\tau_{\min} + \varepsilon, t_f], \Sigma(Y)), 0 < \varepsilon < t_f - \tau_{\min}\]

the above limit exists uniformly in \(t\) but it is dependent on \(\gamma\) and \(\varepsilon\). Moreover, if \(\tau_{\min} > 0\) then

\[\lim_{t \downarrow \tau_{\min}} \|S_\gamma(t)\|_{C(Y)} = +\infty.\]

**Proof.** The proof is similar to the one given in the previous chapter. Denote

\[M_{t_f} = \sup\{\|U_A(t, s)\|; 0 \leq s \leq t \leq t_f\}, \quad r = 2M^2_{t_f}\|P_{t_f}\|, \quad 0 < \alpha < 1\]

For convenience, we denote

\[\|C\|_{\mathcal{B}_\infty(0, t_f; Y, Z)} = \|C\|_\infty \text{ and } \|B_2\|_{\mathcal{B}_\infty(0, t_f; W, Y)} = \|B_2\|_\infty.\]

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Let \( \tau \) be such that

\[
0 < t_f - \tau < \min\{ \frac{\|P_{t_f}\|}{\|C\|_\infty^2 + \frac{1}{\gamma^2 \tau^2} \|B_2\|_\infty^2}, \frac{\gamma^2 \alpha}{2r M_{t_f}^2 \|B_2\|_\infty^2 r^2} \}
\]

Consider a ball with radius \( r \) in \( C_u([\tau, t_f], \Sigma(Y)) \):

\[
B_{r, \tau} = \{ F \in C_u([\tau, t_f]; \Sigma(Y)) : \|F\| \leq r \}
\]

Define mapping \( T, T_n \) which map \( C_u([\tau, t_f]; \Sigma(Y)) \) into itself as:

\[
T(S)(t)x = U(t_f, t)^* P_{t_f} U(t_f, t)x \\
+ \int_t^{t_f} U(s, t)^* (C^*(s)C(s) + \frac{1}{\gamma^2} S(s)B_2(s)B_2^*(s)S(s))U(s, t)x ds
\]

and

\[
T_n(S)(t)x = U_{A_n}(t_f, t)^* P_{t_f} U_{A_n}(t_f, t)x \\
+ \int_t^{t_f} U_{A_n}^*(s, t)(C^*(s)C(s) + \frac{1}{\gamma^2} S(s)B_2(s)B_2^*(s)S(s))U_{A_n}(s, t)x ds
\]

Let \( S \in B_{r, \tau} \); then we have

\[
\|T(S)(t)x\|_Y \leq M_{t_f}^2 \{ \|P_{t_f}\|_{\mathcal{L}(Y)} + (t_f - \tau)(\|C\|_\infty^2 + \frac{\tau^2}{\gamma^2} \|B_2\|_\infty^2) \}
\]

\[
\leq 2M_{t_f}^2 \|P_{t_f}\|_{\mathcal{L}(Y)} \leq r\|x\|
\]

Similarly

\[
\|T_n(S)(t)x\|_Y \leq r\|x\|
\]

It follows that

\[
\|T(S)(t)\|_{\Sigma(Y)} \leq r \quad \|T_n(S)(t)\|_{\Sigma(Y)} \leq r \quad \forall t \in [\tau, t_f] \quad n \in N \quad S \in B_{r, \tau}
\]

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$T$ and $T_n$ map $B_{r,r}$ into $B_{r,r}$. For $S_1, S_2 \in B_{r,r}$ we have

$$T(S_1(t)x - T(S_2)(t)x = \frac{1}{\gamma^2} \int_t^{t_f} U^*(s,t)[(S_1B_2B_2^*(S_2 - S_1) + (S_2 - S_1)B_2B_2^*S_2](s)U(s,t)x ds$$

It follows that

$$\|T(S_1)(t) - T(S_2)(t)\|_{\Sigma(Y)} = \frac{2rM^2_{t_f}}{\gamma^2}(t_f - \tau)\|B_2\|_{\infty}^2\|S_1 - S_2\|_{\Sigma(Y)}$$

$$\leq \alpha \|S_1 - S_2\|_{\Sigma(Y)}$$

Similarly

$$\|T_n(S_1)(t) - T_n(S_2)(t)\|_{\Sigma(Y)} = \frac{2rM^2_{t_f}}{\gamma^2}(t_f - \tau)\|B_2\|_{\infty}^2\|S_1 - S_2\|_{\Sigma(Y)}$$

$$\leq \alpha \|S_1 - S_2\|_{\Sigma(Y)}$$

Thus $T$ and $T_n$ are $\alpha$-contraction in $B_{r,r}$. Hence $T$ and $T_n$ have unique fixed points in $C_u([\tau, t_f]; \Sigma(Y))$. However generally we do not have

$$S_n \to S \quad \text{in} \quad C_u([\tau, t_f]; \Sigma(Y))$$

but the weaker result holds:

$$S_n \to S \quad \text{in} \quad C_s([\tau, t_f]; \Sigma(Y))$$

due to our assumption on $U_{A_n}$.

From what we have just proved it follows that if $S_\gamma$ is a mild solution of (6.2.2.2) on the interval $[\tau, t_f]$ it can be extended to the interval $[\tau - \delta, t_f]$ with $\delta > 0$ by defining on $[\tau - \delta, \tau], S_\gamma(t) = \overline{S}_\gamma(t)$ where $\overline{S}_\gamma(t)$ is the solution of the integral equation

$$\overline{S}_\gamma(t)y = U(\tau, t)^*S_\gamma(\tau)U(\tau, t)y + \int_t^\tau U(s, t)^*\{C(s)^*C(s)$$

$$+ \frac{1}{\gamma^2} \overline{S}_\gamma(s)B_2(s)B_2^*(s)\overline{S}_\gamma(s)}U(s, t)y ds \quad \forall y \in Y$$

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Moreover, $\delta$ depends only on $\|S_\gamma(\tau)\|$ in addition to $\|C\|_\infty$, $\|D\|_\infty$, $M_{t_f}$, $\alpha$ and $\gamma$.

Let $[\tau_{\min}, t_f]$ be the maximal interval of existence of the mild solution $S_\gamma$ of (6.2.2.2). If $\tau > 0$ then $\lim_{t \downarrow \tau_{\min}} \|S_\gamma(t)\|_{L(Y)} = +\infty$ since otherwise there is a sequence $t_n \downarrow \tau_{\min}$ such that $\|S_\gamma(t_n)\|_{L(Y)} \leq C$ for all $n$. This would imply by what we have just proved that for each $t_n$, near enough to $\tau_{\min}$, $S_\gamma$ can be extended to $[t_n - \delta, t_f]$ where $\delta > 0$ is independent of $t_n$ and hence $S_\gamma$ can be extended beyond $\tau_{\min}$ contradicting the definition of $\tau_{\min}$. Hence the proof of Proposition 6.2.2.6 is complete. \qed

Remark 6.2.2.7. We can see from Proposition 6.2.2.6 that given any $\gamma > 0$, (6.2.2.1) always has a local solution near $t_f$. This fact will play a role when we deal with sampled measurements later. Another important fact is that for $\gamma_1 \leq \gamma_2$

$$S_{\gamma_1}(t) \geq S_{\gamma_2}(t) \geq 0, \forall t \in [0, t_f]$$

provided that $S_{\gamma_1}(t)$, $S_{\gamma_2}(t)$ are the mild solutions of (6.2.2.2) when $\gamma = \gamma_1$ and $\gamma = \gamma_2$ respectively.

Since for each $t \in [0, t_f]$, $A(t)$ is an infinitesimal generator of analytic semigroup in $Y$, Proposition 6.2.2.6 can be sharpened as follows:

**Proposition 6.2.2.8.** If $S_\gamma \in C_s([0, t_f]; \Sigma(Y))$ is the mild solution of (6.2.2.3), then $S_\gamma(t)$ is differentiable on $]0, t_f[$ and belongs to $C^\infty([0, t_f - \epsilon]; \Sigma(Y)), \forall \epsilon \in ]0, t_f[$.

**Proof.** We sketch the proof in three steps.

Step 1. Let $F$ be defined by

$$F(t)x = \int_t^{t_f} U(s, t)^* \left\{ C(s)C(s)^* + \frac{1}{\gamma^2} S_\gamma(s)B_2(s)^*B_2(s)S_\gamma(s) \right\} U(s, t)x \, ds$$

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We claim that for any $\alpha \in [0, 1]$, $F \in C^\alpha([0, t_f]; \Sigma(Y))$ and there exists a constant $C_{t_f} > 0$ such that

$$|F(t)x - F(r)x| \leq C_{t_f} |t - r|^{\alpha} \{\|C\|_\infty^2 + \frac{1}{\gamma^2} \|S\|^2_{\mathcal{L}([0, t_f]; \Sigma(Y))} \|B_2\|_\infty^2 \} \|x\|_Y$$

for all $t, r \in [0, t_f]$ with $0 < t < r \leq t_f$.

We note that

$$F(t)x - F(r)x = \int_t^r U(s, t)^* \{C(s)C(s)^* + \frac{1}{\gamma^2} S_\gamma(s)B_2(s)^*B_2(s)S_\gamma(s)\} U(s, t)xds$$

$$+ \int_r^{t_f} \int_r^t A(s)^* U(s, \sigma)^* \{C(s)C(s)^* + \frac{1}{\gamma^2} S_\gamma(s)B_2(s)^*B_2(s)S_\gamma(s)\} U(s, \sigma)xdsd\sigma$$

$$+ \int_r^{t_f} \int_r^t U(s, t)^* \{C(s)C(s)^* + \frac{1}{\gamma^2} S_\gamma(s)B_2(s)^*B_2(s)S_\gamma(s)\} U(s, \sigma)A(\sigma)xdsd\sigma$$

and we that

$$\|A(s)^*U(s, \sigma)^*\|_{\mathcal{L}(Y)} \leq M_1 (s - \sigma)^{-1}, \quad \|U(s, \sigma)A(\sigma)\|_{\mathcal{L}(Y)} \leq M_2 (s - \sigma)^{-1}$$

Further noting that

$$\int_r^{t_f} ds \int_t^r (s - \sigma)^{-1} d\sigma = \int_r^{t_f} ds \int_{s-t}^{s-r} \frac{ds}{\sigma}$$

$$\leq \int_r^{t_f} (s-r)^{-\alpha} ds \int_{s-t}^{s-r} \sigma^{\alpha-1} d\sigma$$

$$\leq \alpha^{-1} (r-t)^\alpha \int_r^{t_f} (s-r)^{-\alpha} ds$$

the conclusion of step 1 follows.

Step 2. Let $G$ be defined as

$$G(t)x = \int_t^{t_f} U(s, t)^* F(s)U(s, t)xds$$

Then $G \in C^1([0, t_f]; \Sigma(Y))$ and

$$G'(t) = -A(t)^* G(t) - G(t)A(t) - F(t)$$
Since $F$ is Hölder continuous on $[0, t_f]$, the result follows by Theorem 7.1, Chapter 5 of Pazy [Paz].

Step 3. Let $S_{\gamma}$ be the mild solution of (6.2.2.2). Since $U(t_f, t)^*P_{t_f}U(t_f, t)$ is analytic in $t$ for $t \in [0, t_f[$, from step 1 we have $S_{\gamma} \in C^\alpha([0, t_f - \varepsilon]; \Sigma(Y))$ for $0 < \varepsilon < t_f$, $0 < \alpha < 1$. Moreover for all $x \in Y$ and $t \leq t_f - \varepsilon$,

$$S_{\gamma}(t)x = U(t_f - \varepsilon, t)^*S_{\gamma}(t_f - \varepsilon)U(t_f - \varepsilon, t)x + \int_t^{t_f - \varepsilon} U(s, t_f - \varepsilon)^*V(s)U(s, t_f - \varepsilon)xds$$

From step 2, it follows that

$$S_{\gamma} \in C^1([0, t_f - 2\varepsilon]; \Sigma(Y)), 0 < 2\varepsilon < t_f$$

which implies $V \in C^1([0, t_f - 2\varepsilon]; \Sigma(Y))$. Again, by the identity

$$S_{\gamma}(t)x = U(t_f - 2\varepsilon, t)^*S_{\gamma}(t_f - 2\varepsilon)U(t_f - 2\varepsilon, t)x$$

$$+ \int_t^{t_f - 2\varepsilon} U(s, t_f - 2\varepsilon)^*V(s)U(s, t_f - 2\varepsilon)xds$$

with $t \geq 2\varepsilon$, it follows that $P \in C^2([0, t_f - 3\varepsilon]; \Sigma(Y)), 0 \leq 3\varepsilon \leq t_f$. By repeating this argument several times we find $S_{\gamma} \in C^\infty([0, t_f - \varepsilon]; \Sigma(Y))$, for any $\varepsilon > 0$, as required. □

From Proposition 6.2.2.6, we can see that Riccati equation (6.2.2.2) not always admits a mild solution over the entire interval $[0, t_f]$. However, under the assumptions on $A$, $C$, $B_2$, and $P_{t_f}$, there exits a nonnegative value $\hat{\gamma}_c < +\infty$ such that when $\gamma > \hat{\gamma}_c$, (6.2.2.2) has a mild solution over $[0, t_f]$ from Chapter 4. Such result can be stated as follows:

**Theorem 6.2.2.9.** Define operators $T$ and $K$ as follows:

$$(T\phi)(s) = \int_0^s U(s, \eta)\phi(\eta)d\eta, \phi \in L(0, t_f; Y)$$

$$(K\phi)(s) = (T)(t_f), \phi \in L(0, t_f; Y)$$
Introduce a (critical) value

\[(6.2.2.8) \quad \hat{\gamma}_c := \sup_{\|w\|=1} \left\langle B_2^*(K^*P_tK + T^*C^*CT)B_2w, w \right\rangle_{L(0,t_f;W)}\]

Then, \((6.2.2.2)\) has a mild solution on \([0,t_f]\) if \(\gamma > \hat{\gamma}_c\) and only if \(\gamma \geq \hat{\gamma}_c\).

The following proposition establishes some regularity of \(Z_\gamma\) which will be used for closed loop feedback later.

**Proposition 6.2.2.10.** For fixed \(\gamma > 0\), there exists an interval \([\tau, t_f]\) and a unique function \(Z_\gamma \in C([\tau, t_f]; \Sigma(Y))\) such that:

(i) \([\lambda_0I - A(\cdot)^*]^{1-\alpha}Z_\gamma(\cdot)\) is well defined and strongly measurable from \([\tau, t_f]\) into \(\mathcal{L}(Y)\),

(ii) \(\|\lambda_0I - A(t)^*\|_{\mathcal{L}(Y)} \leq c(\tau, \gamma)(t - \tau)^{-\theta}\), for all \(t \in [\tau, t_f]\), \(\theta = \max\{0, 1 - \alpha - 2\beta\}\)

(iii) \(Z_\gamma\) is the solution of \((6.2.2.3)\) in \([\tau, t_f]\).

**Proof.** Let \(\tau \in [0, t_f]\) and denote by \(B_\gamma(\tau, t_f)\) the Banach space of all strongly measurable functions \(Q : [\tau, t_f] \rightarrow \mathcal{L}(Y)\) such that

\[\|Q\|_{B_\gamma(\tau, t_f)} = \sup_{\tau \leq t \leq t_f} (t_f - t)^\theta \|Q(t)\|_{\mathcal{L}(Y)} < \infty\]

For \(Q \in B_\gamma(\tau, t_f)\), define

\[
\Gamma_\gamma(Q)(t) := [\lambda_0I - A(t)^*]^{1-\alpha}U(t_f, t)^*P_tU(t_f, t)
+ \int_t^{t_f} [\lambda_0I - A(t)^*]^{1-\alpha}U(s, t)^*\left\{C(s)^*C(s) - Q(s)^*K(s)Q(s)
+ \frac{1}{\gamma^2}Q^*(s)[\lambda_0 - A(s)]^{\alpha-1}B_2(s)B_2^*(s)[\lambda_0 - A(s)^*]^{\alpha-1}Q(s)\right\}U(s, t)ds \quad t \in [\tau, t_f]
\]

where

\[K(s) := [\lambda_0I - A(s)]^\alpha G(s)[(\lambda_0I - A(s))^\alpha G(s)]^*
\]
By assumption (A3), we know that there exists a constant $M_{1-\alpha,0}$ such that

\[(6.2.2.9) \quad \|[\lambda_0 I - A(t)^*]^{1-\alpha}U(s,t)\|_{\mathcal{L}(Y)} \leq \frac{M_{1-\alpha,0}}{(s-t)^{1-\alpha}}\]

By assumption (A8), we know that $\|[\lambda_0 I - A(t_f)^*]^{2\beta}P_{t_f}\|_{\mathcal{L}(Y)}$ is well defined. Now, using assumptions (A2), (A4), (A.6), and (6.2.2.9) we have

$$
\|\Gamma_r(Q)(t)\|_{\mathcal{L}(Y)} \\
\leq M_{1-\alpha,2\beta}[1 + (t_f - t)^{2\beta+\alpha-1}]\|[\lambda_0 I - A(t_f)^*]^{2\beta}P_{t_f}\|_{\mathcal{L}(Y)}\|U(t_f, t)\|_{\mathcal{L}(Y)} \\
+ M_{1-\alpha,0} \int_t^{t_f} (s-t)^{\alpha-1}\{\|C^*(s)C(s)\|_{\mathcal{L}(Y)} \\
+ \frac{M^2}{\gamma^2}\|B_2(s)B_2^*(s)\|_{\mathcal{L}(Y)}(t_f - s)^{-2\theta}\|Q\|^2_{B_3(\tau, t_f)} \\
+ \|K(s)\|_{\mathcal{L}(Y)}(t_f - s)^{-2\theta}\|Q\|^2_{B_2(\tau, t_f)}\} \|U(s, t)\|_{\mathcal{L}(Y)}ds \\
\leq c[(t_f - t)^{-\theta} + (t_f - t)^{\alpha} + (t_f - t)^{\alpha-2\theta}\|Q\|^2_{B_3(\tau, t_f)}], \quad \forall t \in [\tau, t_f[ \]

Thus we have $\Gamma_r(Q) \in B_\gamma(\tau, t_f)$ and

$$
\|\Gamma_r(Q)(t)\|_{B_\gamma(\tau, t_f)} \leq c_1 + c_2(t_f - \tau)^{\alpha-\theta}\|Q\|^2_{B_3(\tau, t_f)}.
$$

Next, we claim that $\Gamma_r$ is a contraction on the ball

$$
B_\gamma(\tau, t_f; r) := \{Q \in B_\gamma(\tau, t_f) : \|Q\|_{B_\gamma(\tau, t_f)} \leq r\}
$$

for a suitable $r > 0$. Let $Q_1, Q_2 \in B_\gamma(\tau, t_f; r)$. We have as before

$$
\|\Gamma_r(Q_1)(t) - \Gamma_r(Q_2)(t)\|_{\mathcal{L}(Y)} \\
\leq c(1 + \gamma^{-2}) \int_t^{t_f} (s-t)^{\alpha-1}[\|Q_1(s)\|_{\mathcal{L}(Y)} + \|Q_2(s)\|_{\mathcal{L}(Y)}]\|Q_1(s) - Q_2(s)\|_{\mathcal{L}(Y)}ds \\
\leq c(1 + \gamma^{-2})(t-\tau)^{\alpha-2\theta}r\|Q_1(s) - Q_2(s)\|_{\mathcal{L}(Y)}, \forall t \in [\tau, t_f[,
$$

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By choosing appropriate \( r \) and \( \tau \), \( \Gamma_\tau \) maps \( B_\gamma(\tau, t_f; r) \) into itself and is a contraction. Hence there exists a unique fixed point \( Q \) in \( B_\gamma(\tau, t_f; r) \) such that \( \Gamma_\tau(Q) = Q \) in \([\tau, t_f]\). Therefore \( Z_\gamma := [\lambda_0 I - A(\cdot)^*]^{\alpha-1}Q(\cdot) \) is a well defined operator function that satisfies (i)-(iii). The property \( Z_\gamma \in C_s([\tau, t_f]; \mathcal{L}(Y)) \) follows from (2.6), whereas the property \( Z_\gamma(t) \in \Sigma(Y) \) is a consequence of the fact that \( Z_\gamma(\cdot)^* \) is also in \( C_s([\tau, t_f]; \mathcal{L}(Y)) \) and satisfies (i)-(iii), so that \( Z_\gamma(t)^* \equiv Z_\gamma(t) \) in \([\tau, t_f]\). \( \square \)

**Corollary 6.2.2.11.** Suppose that generalized Riccati equation (6.2.2.3) has a mild solution \( Z_\gamma \) on \([0, t_f]\) for given \( \gamma > 0 \). Then

\[
(6.2.2.10) \quad \|[\lambda_0 I - A(t)^*]^{1-\alpha}Z_\gamma(t)\|_{\mathcal{L}(Y)} \leq c_\gamma(t_f - t)^{2\beta + \alpha - 1}, \quad \forall t \in [0, t_f],
\]

where \( \beta \) is defined in assumption (A8).

**Proof.** Since (6.2.2.3) admits a mild solution \( Z_\gamma \) over the interval \([0, t_f]\), by 6.2.2.4, we have

\[
Z_\gamma(t) \leq U(t, t_f)^*P_{t_f}U(t_f, t) + \int_t^{t_f} U(s, t)^* \left\{ C(s)^*C(s) + \frac{1}{\gamma^2}Z_\gamma(s)B_2(s)B_2(s)^*Z_\gamma(s) \right\} U(s, t)ds
\]

By the assumption (A3), (A8), and the hypothesis of the proposition, we have

\[
\|[\lambda_0 I - A(t)^*]^{1-\alpha}Z_\gamma(t)\|_{\mathcal{L}(Y)}
\leq M_{1-\alpha,2\beta}[1 + (t_f - t)^{2\beta + \alpha - 1}]\|[\lambda_0 I - A(t_f)^*]^{2\beta}P_{t_f}\|_{\mathcal{L}(Y)}\|U(t_f, t)\|_{\mathcal{L}(Y)}
+ M_{1-\alpha,0} \int_t^{t_f} (s - t)^{\alpha - 1} \left\{ \|C(s)^*C(s)\|_{\mathcal{L}(Y)} + \frac{C}{\gamma^2}\|B(s)B(s)^*\|_{\mathcal{L}(Y)} \right\} ds
\leq c_\gamma(t_f - t)^{2\beta + \alpha - 1}
\]

and thus the proof is complete. \( \square \)
Proposition 6.2.2.12. Suppose that there exists an interval \([\tau, t_f]\) such that both generalized Riccati equations (6.2.2.2) and (6.2.2.3) have mild solutions \(S_\gamma, Z_\gamma\) respectively on \([\tau, t_f]\). Then we have

\[ S_\gamma(t) \geq Z_\gamma(t), \quad \forall t \in [\tau, t_f] \]

Proof. Since both \(S_\gamma, Z_\gamma\) are in \(C_s([\tau, t_f]; \Sigma^+(Y))\), we know that \(S_\gamma x, Z_\gamma x\) are differentiable on \([\tau, t_f]\) when \(x \in D(A(t)), 0 \leq t \leq t_f\). Define an operator function \(\Pi(t)\) on \([\tau, t_f]\) as

\[ \Pi(t)x = S_\gamma(t)x - Z_\gamma(t)x \]

We assume first that \(B_2(t)\) is bounded on \([\tau, t_f]\), and thus \(\frac{1}{\tau_f} B_2(t) B_2(t)^\ast S_\gamma(t)\) is also bounded on \([\tau, t_f]\). By Theorem 2.3, Chapter 5 of [Paz], we know that there exists an evolution operator \(\Psi(t, s)\) associated with \(A(t) + \frac{1}{\tau_f} B_2(t) B_2(t)^\ast S_\gamma(t)\). Now we can easily verify that \(\Pi(t)\) satisfies the integral equation

\[ (6.2.2.11) \quad \Pi(t)x = \int_t^{t_f} \Psi(s, t)^\ast \Pi(s)V(s)^\ast E(s)E(s)^\ast V(s)\Pi(s)\Psi(s, t)x \, ds, \quad t \in [\tau, t_f] \]

where \(E(s) = [\lambda_0 I - A(s)]^\alpha G(s), V(s) = [\lambda_0 I - A(s)^\ast]^{1-\alpha} Z_\gamma(s)\). Since \(D(A(t))\) is dense in \(Y\), we have \(\Pi \in C_s([\tau, t_f]; \Sigma^+(Y))\), and hence \(\Pi \geq 0\). This gives

\[ S_\gamma(t) \geq Z_\gamma(t), \quad \forall t \in [\tau, t_f] \]

But this still holds when \(B_2 \in B_\infty([\tau, t_f]; W, Y)\) because (6.2.2.11) is allowed to fail on a set of measure zero. \(\square\)

6.3. Existence of a solution to (6.1.1) under open-loop and closed-loop controllers

In this section, we will establish the connection between the solution of (6.1.1) and an
associated representation formula. Concerning existence and uniqueness, we invoke the result of Theorem 6.1 of [LM] in which a "compatibility" condition is required.

Proposition 6.3.1. Let $w \in L^2(Q)$, $y_0 \in H^m(\Omega)$

$$g_j \in H^{2m-m_j-1/2, (2m-m_j-1/2)/2m}(\Sigma), \ 0 \leq j \leq m-1,$$

and let the following compatibility relations hold:

$$(6.3.1) \quad B_j(0, x, D)y_0 = g_j(x, 0), x \in \partial \Omega \text{ if } m_j < m - \frac{1}{2}.$$

Then there exists a unique function $y$ in $H^{2m,1}(Q)$ satisfying

$$y_t(t, x) = A(t, x; D)y + B_2w(t)$$

$$B_j(t, x; D)y(t, x) = g_j(t, x), 0 \leq j \leq m-1, x \in \partial \Omega$$

$$y(0, x) = y_0$$

Theorem 6.3.2. Given $y_0 \in H^m(\Omega)$, $w \in \mathcal{W}$.

$$u \in (H^{2m-m_j-1/2, (2m-m_j-1/2)/2m}(\Sigma))^N, N \leq 2m-1$$

and suppose that the compatibility condition (6.3.1) holds. Then the solution $y(t)$ of (6.1.1) is given by (6.2.1.1).

Proof. By Proposition 6.3.1, we know that (6.1.1) has a unique solution $y \in H^{2m,1}(Q)$. Note that $y(t, \cdot) - G(t)u(t, \cdot) \in D_{A(t)}$ and $A(t)[y(t, \cdot) - G(t)u(t, \cdot)] = A(t, \cdot, D)y(t, \cdot)$. Define

$$\varphi(s) := (y(s, \cdot), U(t, s)^*v)_{L^2(\Omega)}, \forall v \in D_{A^*(t)} \quad s \in [0, t].$$
By assumption (A3), we may compute

\[ \varphi'(s) = (D_{s}(e^{\lambda_{0} - s}y(s, \cdot)), e^{\lambda_{0} s}U(t, s)^{*}v) + (y(s, \cdot), [\lambda_{0} - A(s)]^{*}U(t, s)^{*}v) \]

\[ = -([\lambda_{0}I - A(s)]^{\dag}y(s, \cdot) - B_{2}(s)w(s, \cdot), U(t, s)^{*}v) \]

\[ + (y(s, \cdot) - G(s)u(s, \cdot), [\lambda_{0}I - A(s)]^{*}U(t, s)^{*}v) \]

\[ + (G(s)u(s, \cdot), [\lambda_{0}I - A(s)]^{*}U(t, s)^{*}v) \]

\[ = (B_{2}(s)w(s), e^{\lambda_{0} s}U(t, s)^{*}v) + (G(s)u(s, \cdot), [\lambda_{0}I - A(s)]^{*}U(t, s)^{*}v) \]

By assumption (A4), we may have

\[ \varphi'(s) = (B_{2}(s)w(s), e^{\lambda_{0} s}U(t, s)^{*}v) \]

\[ (((\lambda_{0}I - A(s))^{\alpha}G(s)u(s, \cdot), [\lambda_{0}I - A(s)]^{1-\alpha}U(t, s)^{*}v) \]

\[ = (B_{2}(s)w(s), e^{\lambda_{0} s}U(t, s)^{*}v) \]

\[ + (((\lambda_{0}I - A(s))^{1-\alpha}U(t, s)^{*})^{\alpha}[\lambda_{0}I - A(s)]^{\alpha}G(s)u(s, \cdot), v), s \in ]0, t[, \]

By integrating above equality, we have

\[ (y(t, \cdot), v)_{L^{2}(\Omega)} = (U(t, 0)y_{0}(\cdot), v) + (\int_{0}^{t} U(t, s)B_{2}(s)w(s)ds, v) \]

\[ + (\int_{0}^{t} ((\lambda_{0}I - A(s))^{1-\alpha}U(t, s)^{*})^{\alpha}[\lambda_{0}I - A(s)]^{\alpha}G(s)u(s, \cdot)ds, v)_{L^{2}(\Omega)} \]

Since \( D_{A_{\cdot}(t)} \) is dense in \( L^{2}(\Omega) \), we get (6.2.1.1). \( \square \)

**Corollary 6.3.3.** Theorem 6.3.2 still holds for any \( y_{0} \in L^{2}(\Omega) \) and \( u \in L^{2}(\Sigma) \). In this case, \( y \) is in \( L^{2}(Q) \).

**Proof.** Since by assumptions (A3), (A4), we have

\[ \|y(t, \cdot)\|_{Y} \leq c\{\|y_{0}\|_{Y} + \int_{0}^{t} ||U(t, s)B_{2}(s)w(s)||_{Y} ds + \int_{0}^{t} (t - s)^{\alpha - 1} ||u(s, \cdot)||_{U} ds \} \]

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this implies
\[ \|y(t, \cdot)\|_Y \leq c \{ t_f \|y_0\|_Y + \|w\|_W \} + \frac{t_f^{2\alpha}}{\alpha^2} \|u\|_U^2 \}
\]

Recalling that \( U = L^2(\partial \Omega) \) and \( U = L^2(0, t_f; U) \), the conclusion follows.

**Theorem 6.3.4.** Suppose that the generalized Riccati equation (6.2.2.2) admits a mild solution \( Z_\gamma \) over the interval \([0, t_f]\). Consider the following closed loop equation
\[ (6.3.2) \]
\[ y(t) = U(t, 0) y_0 + \int_0^t U(t, s) \frac{1}{\gamma^2} B_2(s) B_2^*(s) Z_\gamma(s) y(s) ds \]
\[ - \int_0^t [(\lambda_0 I - A(s))^{1-\alpha} U(t, s)^*]^{*} E(s) E^*(s) V(s) y(s) ds \]

where
\[ (6.3.3) \]
\[ E(s) = [\lambda_0 I - A(s)]^{\alpha} G(s) \]
\[ (6.3.4) \]
\[ V(s) = [\lambda_0 I - A^*(s)]^{1-\alpha} Z_\gamma(s) \]

Assume that (A1)-(A5) hold. Then there exists a unique solution \( y \in C([0, t_f]; Y) \) of the closed-loop system (6.3.2).

**Proof.** Consider the integral equation
\[ (6.3.5) \]
\[ \Phi_\gamma(t, s) y = U(t, s) y + \frac{1}{\gamma^2} \int_s^t U(t, r) B_2(r) B_2^*(r) Z_\gamma(r) \Phi_\gamma(r, s) y dr \]
\[ - \int_s^t [(\lambda_0 I - A(r)^*)^{1-\alpha} U(t, r)^*]^{*} E(r) E^*(r)^* V(r) \Phi_\gamma(r, s) y dr \]

Fix \( s \in [0, t_f] \). If \( \Phi_\gamma(\cdot, s) y \in C([s, t_f]; Y) \), then
\[ V(\cdot) \Phi_\gamma(\cdot, s) = [\lambda_0 I - A^*(\cdot)]^{1-\alpha} Z_\gamma(\cdot) \Phi_\gamma(\cdot, s) y \in L^p(s, t_f; Y) \]

for some \( p > 1/\alpha \) by Corollary 6.2.2.11. Introduce the mapping
\[ \Gamma : C([s, t_f]; Y) \mapsto C([s, t_f]; Y) \]

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by

\begin{align*}
\Gamma(\Phi)(t) &= U(t, s)\Phi(t) + \int_{0}^{t} U(t, r) \frac{1}{\gamma^2} B_2(r) B^*_2(r) Z_\gamma(r) \Phi(r, s) yds \\
&\quad - \int_{s}^{t} \left[ [\lambda_0 I - A(r)]^{1-\alpha} U(t, r)^* \right]^* E(r) E^*(r) V(r) \Phi(r, s) ydr
\end{align*}

By assumption (A3), Corollary 6.2.2.11, and using the Hölder estimate, for $\Phi_1, \Phi_2$ to be in the space $C([s, t_f]; Y)$, we have

\begin{align*}
\|\Gamma(\Phi_1)(t) - \Gamma(\Phi_2)(t)\|_Y &\leq \int_{s}^{t} \|U(t, r) \frac{1}{\gamma^2} B_2(r) B^*_2(r) Z_\gamma(r)\|_{L(Y)} \|\Phi_1(r) - \Phi_2(r)\|_Y dr \\
&\quad + \int_{s}^{t} \|\left[ [\lambda_0 I - A(r)]^{1-\alpha} U(t, r)^* \right]^* E(r) E^*(r) V(r)\|_{L(Y)} \|\Phi_1(r) - \Phi_2(r)\|_Y dr \\
&\leq \left\{ \frac{c_1}{\gamma^2} (t - s) + c_2 \int_{s}^{t} (t - r)^{\alpha-1} (t_f - r)^{\alpha+2\beta-1} dr \right\} \|\Phi_1 - \Phi_2\|_{C([0, t]; Y)} \\
&= \left\{ \frac{c_1}{\gamma^2} (t - s) + c_2 (t_f - t)^{2\alpha+2\beta-1} \int_{t}^{t_f} \xi^{\alpha-1} (1 + \xi)^{\alpha+2\beta-1} d\xi \right\} \|\Phi_1 - \Phi_2\|_{C([0, t]; Y)} \\
&\leq \left\{ \frac{c_1}{\gamma^2} (t - s) + c_2 (t_f - t)^{2\alpha+2\beta-1} \int_{t}^{t_f} \xi^{2\alpha+2\beta-2} d\xi \right\} \|\Phi_1 - \Phi_2\|_{C([0, t]; Y)} \\
&= \left\{ \frac{c_1}{\gamma^2} (t - s) + c'_2 (t - s)^{2\alpha+2\beta-1} \right\} \|\Phi_1 - \Phi_2\|_{C([0, t]; Y)}
\end{align*}

Let $t_f - s = \varepsilon$. Thus, if $\varepsilon$ is small, $\Gamma$ is a contraction and (6.3.6) has a unique solution in $C([t_f - \varepsilon, t_f]; Y)$. Note that this argument can be repeated in $[t_f - 2\varepsilon, t_f - \varepsilon]$ and so on, yielding the conclusion. $\square$

6.4. A related differential game and its saddle point

6.4.1. Maximization of $J_\gamma$ over $W$ for fixed $u$

In this section, we consider the maximization problem $\sup_w J_\gamma(y_0; u, w)$ with $u$ arbitrary but fixed. $J_\gamma(y_0; u, w)$ is given by

\begin{equation*}
J_\gamma(y_0; u, w) = \int_{0}^{t_f} \left\{ \|C(t)y(t)\|_Y^2 + \|u(t)\|_U^2 - \frac{1}{\gamma^2} \|w(t)\|_W^2 \right\} dt + (P_{t_f}, y(t_f), y(t_f))\gamma
\end{equation*}
where \( y(t) \) is given by (6.2.1.1).

**Theorem 6.4.1.1.** Let \( \gamma_c \) be given as in section 2 of this chapter. Then if \( \gamma > \gamma_c \), for arbitrary but fixed \( u \in \mathcal{U} \), \( J_\gamma(y_0; u, w) \) has a finite supremum in \( w \) for all \( y_0 \) in \( Y \). Conversely, if \( \gamma < \gamma_c \), for arbitrary but fixed \( u \in \mathcal{U} \), \( \sup_w J_\gamma(y_0; u, w) \) is unbounded for any \( y_0 \in Y \).

**Proof.** See Theorem 4.4.1.1 and Theorem 4.4.1.4 of Chapter 4.

### 6.4.2. Solutions of \( J_\gamma \) under open-loop and closed-loop policies

**Theorem 6.4.2.1.** (Open-Loop Policy) Let \( \gamma \) be given as in section 2 of this chapter and \( \gamma > \gamma_c \). Then, the differential game \( J_\gamma(x_0; u, w) \) has a unique saddle-point solution, given by

\[(6.4.2.1) \quad u^o(t) = -\left[[\lambda_0 I - A(t)]^\alpha G(t)\right]^*\left[[\lambda_0 I - A^*(t)]^{1-\alpha} Z_\gamma(t)\right]y(t)\]

\[(6.4.2.2) \quad w^o(t) = \frac{1}{\gamma^2} B^*_2(t) Z_\gamma(t)y(t)\]

where \( Z_\gamma \) is the solution of (6.2.2.2) and \( y \) is the solution of the closed-loop equation (6.3.2).

Moreover

\[ \varphi_\gamma(y_0) = (Z_\gamma(0)y_0, y_0)_Y \]

**Proof.** Since \( \gamma > \gamma_c \), it follows from section 3 that (6.2.2.2) admits a mild solution, and hence (6.2.2.3) also has a solution according to Proposition 6.2.2.12. Thus \( Z_\gamma \) is well defined over \([0, t_f]\). We need to show that for \( y_0 \in Y \) and \( \mathcal{V}(u, w) \in \mathcal{U} \times \mathcal{W} \), the following holds

\[ J_\gamma(y_0; \hat{u}, w) \leq J_\gamma(y_0; \hat{u}, \hat{w}) \leq J_\gamma(y_0; u, \hat{w}) \]
We prove the left inequality first, i.e. \( \min_u J_\gamma(y_0; u, \hat{w}) = J_\gamma(y_0, \hat{u}, \hat{w}) \). Note that function \( J_\gamma(y_0; \cdot, \hat{w}) \) is strictly convex and coercive, and thus has a unique minimum solution \( \hat{u} \). By Theorem 6.3.2, we know that \( y \) can be expressed in the form of (6.2.1.1). Using standard results, \( \hat{u}(t) = -\left[ [\lambda_0 I - A(t)]^\alpha G(t) \right]^* [\lambda_0 I - A^*(t)]^{1-\alpha} p(t) \) and \( p(t) \) satisfies the Hamiltonian system (cf. Theorem 1.1, chapter 4 [BP1])

\[
\begin{align*}
\dot{y}^\alpha(t) &= A(t) y^\alpha(t) - E(t) [\lambda_0 I - A^*(t)]^{1-\alpha} [E(t) [\lambda_0 I - A(t)]^{1-\alpha}]^* p(t) \\
&\quad + \frac{1}{\gamma^2} B_2(t) B_2^*(t) y(t) \\
\dot{p}(t) - A^*(t) p(t) &= -C^*(t) C(t) y^\alpha(t) \\
y^\alpha(0) = y_0; p(t_f) = P_t y^\alpha(t_f)
\end{align*}
\]  

where (6.4.2.3) and (6.4.2.4) holds in “mild sense”. Clearly, \( (y, Z_\gamma y) \) is the solution of (6.4.2.3), (6.4.2.4) by Theorem 6.3.4 and Definition 6.2.2.4. Therefore we have

\[
\hat{u}(t) = -\left[ [\lambda_0 I - A(s)]^\alpha G(s) \right]^* [\lambda_0 I - A^*(s)]^{1-\alpha} Z_\gamma y(t) = \hat{u}(t)
\]

This gives

\[
(Z_\gamma(0)y_0, y_0) = J_\gamma(y_0; \hat{u}, \hat{w}) \leq J_\gamma(y_0; u, w)
\]

Next we prove the right-hand side inequality. Note that by our assumption that \( \gamma > \gamma_c \), and hence \( J_\gamma(y_0; u, \cdot) \) has a unique finite maximum \( \hat{w} \in \mathcal{W} \). Since \( J_\gamma(y_0; \hat{u}, \cdot) \) is Gâteaux differentiable, and (see the Appendix)

\[
\delta_w J_\gamma(y_0; \hat{u}, \hat{w}) = 2B_2^*(s)Z_\gamma(t)y(t) - 2\gamma^2 \hat{w}(t),
\]

we have

\[
\hat{w}(t) = \frac{1}{\gamma^2} B_2(t)^* Z_\gamma(t)y(t) = \hat{w}(t)
\]

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Thus
\[ J_\gamma(y_0; \hat{u}, \hat{w}) \leq J_\gamma(y_0; \hat{u}, \hat{w}) \]

Therefore \((\hat{u}, \hat{w})\) is the unique saddle point of \(J_\gamma(y_0; u, w)\). Uniqueness follows from the uniqueness of solutions to the minimization and maximization problems, because of ordered interchangeability of multiple saddle-point equilibria. This completes the proof of Theorem 6.4.2.1. \(\square\)

Let us assume that the controller has access to instantaneous perfect measurement of the state of system (6.1.1) from the boundary. We denote \(\mathcal{M}\) the class of all such applications from \(Y\) to \(U\), and have the following:

**Theorem 6.4.2.2. (Closed-Loop Policy)** Suppose that (6.2.2.3) admits a mild solution over the interval \([0, t_f]\). Then the differential game

\[ \varphi_\gamma(y_0) = \inf_{\mathcal{M}} \sup_{\mathcal{W}} J_\gamma(x_0; u, w) \]

has a saddle-point solution, given by

\[ \mu^*(t, y(t)) = -[\lambda_0 I + A(t)]^*G(t)[\lambda_0 I + A^*(s)]^{-\alpha}Z_\gamma(t)y(t) \]

\[ \nu^*(t, y(t)) = \frac{1}{\gamma^2}B_2^*(t)Z_\gamma(t)y(t) \]

where \(Z_\gamma\) is the solution of (6.3.2). Moreover

\[ \varphi_\gamma(y_0) = (Z_\gamma(0)y_0, y_0)_Y \]

**Proof.** Let \(u \in L^2(0, t_f; U), w \in L^2(0, t_f; W)\), and let \(y\) be the solution to (6.2.1.1). Let \(\{u_k\}\) and \(\{w_k\}\) be sequences in \(W^{1,2}(0, t_f; U)\) and \(W^{1,2}(0, t_f; W)\), respectively, such that
\( u_k \to u \) in \( u \in L^2(0,t_f;U) \), and \( u_k \to u \) in \( w \in L^2(0,t_f;W) \). Now we compute the derivative

\[
\frac{d}{ds}(Z_\gamma(s)y_k(s),y_k(s))_Y
\]

and, integrating from 0 to \( t_f \) and letting \( k \) tend to infinity, we obtain

\[
J_\gamma(y_0;u,w) = (P_{t_f}y_0,y_0)
+ \int_0^{t_f} \|u(s) + E(s)^*V(s)y(s)\|_U^2 ds
- \int_0^{t_f} \|w(s) - \frac{1}{\gamma^2}B_2^*(s)Z_\gamma(s)y(s)\|_W^2 ds
\]

where \( E, V \) are defined in (6.3.3), (6.3.4). Thus the conclusion follows. \( \square \)

6.5. Solution under sampled feedback

Recall that for a given pair \((u, w) \in U \times W\) the solution of (6.1.1) can be written as

(6.5.1)

\[
y(t) = U(t,0)y_0 + \int_0^t U(t,s)B_2(s)w(s)ds
+ \int_0^t [(\lambda_0 I - A(s)^*)^{1-\alpha}U(t,s)^*][\lambda_0 I - A(s)]^\alpha G(s)u(s)ds
\]

Let \( t_1, \ldots, t_N \) denote \( N \) sampling times, such that \( 0 = t_1 < \cdots < t_{N-1} < t_N = t_f \). We shall consider (6.5.1) in conjunction with a sampled state feedback control of the form

(6.5.2)

\[
u(t) = K(t)y(t_n), \quad t_n \leq t < t_{n+1}, n = 1, 2, \ldots, N - 1
\]

where

\[
K \in L^2(0,t_f;\mathcal{L}(Y,U))
\]

Introduce

\[
\mathcal{M}_s^N = \{u \in U : u(t) = K(t)x_n, t_n \leq t < t_{n+1}, n = 1, 2, \ldots, N - 1, K \in L^2(0,t_f;\mathcal{L}(Y,U)) \quad x_n \in Y \}
\]
We will refer to $\mathcal{M}_s^N$ the set of sampled controllers with $N$ sampling times. From Theorem 6.3.4, it is not difficult to see that the closed loop solution $y(t)$ in (6.3.2) can be written as

\begin{equation}
(6.5.4) \quad y(t) = \Phi(t,s)y(s), \quad 0 \leq s \leq t \leq t_f
\end{equation}

Suppose that the generalized Riccati equation (6.2.2.3) admits a mild solution over the entire interval $[0, t_f]$. Introduce $N$ backward Riccati equations as follows:

\begin{equation}
(6.5.5) \quad \dot{S}_n(t) + A(t)^*S_n(t) + S_n(t)A(t) + C(t)^*C(t) + \frac{1}{\gamma^2}S_n(t)B_2(t)B_2(t)^*S_n(t) = 0
\end{equation}

\[ S_n(t_{n+1}) = Z_\gamma(t_{n+1}) \]

\[ t_n \leq t < t_{n+1}, n = N - 1, N - 2, ..., 1. \]

For each $n$, by the section 2 of this chapter the corresponding (critical) value is

\begin{equation}
(6.5.6) \quad \hat{\gamma}_{nc} := \sup_{\|w\|=1} \left( B_2^*(\mathcal{K}_n Z_\gamma(t_{n+1}) \mathcal{K}_n + T_n^*C^*C T_n) B_2 w, w \right)_{L(0,t_f;W)}
\end{equation}

where

\[(T_n\phi)(s) = \int_{t_n}^{s} U(s,\eta)\phi(\eta)\,d\eta, \quad t_n \leq s \leq t_{n+1}, \phi \in L(0,t_f;Y)\]

\[(\mathcal{K}_n\phi)(s) = (T_n)(t_{n+1}), \phi \in L(0,t_f;Y)\]

Now we are ready to state the main theorem in this section.

**Theorem 6.5.1.** Given $\gamma > 0$, suppose that (6.2.2.3) admits a mild solution on $[0,t_f]$.

Then:

(i) Under $\mathcal{M}_s^N$ there exists a unique strongly time consistent saddle-point controller $\mu^*_s$ if $\gamma > \max_n \{\hat{\gamma}_{nc}\}$ and only if $\gamma \geq \max_n \{\hat{\gamma}_{nc}\}$. 

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(ii) If \( \gamma > \max_n \{ \bar{\gamma}_{nc} \} \), then the unique strongly time consistent saddle-point controller given by

\[
\mu^*(t, y(t_n)) = -[(\lambda_0 I - A(t))^\alpha G(t)]^*[(\lambda_0 I - A(t))^\beta]^1 - \alpha Z_\gamma(t) \Phi(t, t_n) y(t_n)
\]

\[t_n \leq t < t_{n+1}, n = 1, 2, ..., N - 1\]

where evolution operator \( \Phi(t, s) \) is given by (6.3.5). Moreover we have

\[
\min_{\mathcal{M}^N} \max_{\mathcal{W}} J_\gamma(y_0; u, w) = \max_{\mathcal{W}} \min_{\mathcal{M}^N} J_\gamma(y_0; u, w) = (Z_\gamma(0) y_0, y_0)
\]

(iii) If there exists \( n \) such that \( \gamma < \bar{\gamma}_{nc} \), then the upper value of the game \( J_\gamma(y_0; u, w) \) is unbounded under sampled state feedback controllers.

Proof. (i) Recall that

(6.5.7)

\[
J_\gamma(y_0; u, w) = \int_0^{t_f} \left\{ ||C(t) y(t)||_Y^2 + ||u(t)||_U^2 - \gamma^2 ||w(t)||_W^2 \right\} dt
\]

\[+ (P_t, y(t_f), y(t_f))_Y\]

For \( 1 \leq n \leq N - 1 \), introduce

(6.5.8)

\[
J_\gamma^n(y_{t_n}; u, w) = \int_{t_n}^{t_{n+1}} \left\{ ||C(t) y(t)||_Y^2 + ||u(t)||_U^2 - \gamma^2 ||w(t)||_W^2 \right\} dt
\]

\[+ (Z_\gamma(t_{n+1}) y(t_{n+1}), y(t_{n+1}))_Y\]

Assume that \( \gamma > \max_n \{ \bar{\gamma}_{nc} \} \). By Theorem 6.4.2.1, we know that for each \( 1 \leq n \leq N \),

\[
\sup_w J_\gamma^n(y(t_n); u, w) \text{ is finite and }
\]

(6.5.9)

\[
\sup_w \inf_u J_\gamma^n(y(t_n); u, w) = \inf_u \sup_w J_\gamma^n(y(t_n); u, w) = (Z_\gamma(t_n) y(t_n), y(t_n))_Y
\]

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If we introduce

\begin{equation}
\nu_\gamma^* = \frac{1}{\gamma^2} B_2(t)^* Z_\gamma(t) \Phi_\gamma(t, t_n) y(t_n), \quad t_n \leq t < t_{n+1}
\end{equation}

we know that for \( t_n \leq t \leq t_{n+1} \) \((\mu_\gamma^*, \nu_\gamma^*)\) is the unique saddle-point of \( J_\gamma^n(y(t_n); u, w) \) by Theorem 6.4.2.1. Hence \( \forall (u, w) \in L^2(t_n, t_{n+1}; U) \times L^2(t_n, t_{n+1}; W) \) we have

\begin{equation}
J_\gamma^n(y(t_n); \mu_\gamma^*, w) \leq J_\gamma^n(y(t_n); \mu_\gamma^*, \nu_\gamma^*) \leq J_\gamma^n(y(t_n); u, \nu_\gamma^*)
\end{equation}

Note that

\begin{equation}
J_\gamma(y_0; u, w) = \sum_{n=1}^N J_\gamma^n(y(t_n); u, w) - \sum_{n=1}^{N-1} (Z_\gamma(t_{n+1})y(t_{n+1}), y(t_{n+1}))
\end{equation}

Clearly

\begin{equation}
J_\gamma(y(t_n); \mu_\gamma^*, \nu_\gamma^*) = (Z_\gamma(t_n)y(t_n), y(t_n))
\end{equation}

Since for each \( 1 \leq n \leq N \)

\begin{equation}
J_\gamma^n(y(t_n); \mu_\gamma^*, w) \leq (Z_\gamma(t_n)y(t_n), y(t_n))
\end{equation}

We have

\begin{equation}
J_\gamma(y_0; \mu_\gamma^*, w) \leq (Z_\gamma(0)y_0, y_0) = J_\gamma(y_0; \mu_\gamma^*, \nu_\gamma^*) \quad \forall w \in W
\end{equation}

On the other hand we know that

\begin{equation}
J_\gamma^n(y(t_n); u, \nu_\gamma^*) \geq (Z_\gamma(t_n)y(t_n), y(t_n))
\end{equation}

Hence

\begin{equation}
J_\gamma(y_0; u, \nu_\gamma^*) \geq (Z_\gamma(0)y_0, y_0) = J_\gamma(y_0; \mu_\gamma^*, \nu_\gamma^*) \quad \forall u \in U
\end{equation}
and thus that \((µ^*_γ, ν^*_γ)\) is the unique saddle-point solution of \(J_γ(y₀; u, w)\).

Next we claim that when there exists a saddle-point controller \(µ^*_γ\) under \(M^N_s\), we have \(γ ≥ \max_n\{γ_{nc}\}\). If there would exist an \(n\) such that \(γ < γ_{nc}\), according to Theorem 6.4.1.1, we have that \(\sup_w J^N_γ(y(t_n); u, w)\) is unbounded. Let \((µ^*_γ, ν^*_γ)\) be a saddle-point solution of \(J_γ(y₀; u, w)\) under \(M^N_s\). Note that

\[
J_γ(y₀; µ^*_γ, ν^*_γ) = \sum_{n=1}^{N} J^n_γ(y(t_n); µ^*_γ, ν^*_γ) - \sum_{n=1}^{N-1} (Z_γ(t_{n+1})y(t_{n+1}), y(t_{n+1}))_Y
\]

and for each \(n\), \(J^n_γ(y(t_n); µ^*_γ, ν^*_γ) ≥ 0\) by Theorem 6.4.2.1. Since \(γ_{nc} < γ\), we have \(\sup_w J^N_γ(y(t_n); µ^*_γ, w) = +∞\). Note that \(\sum_{n=1}^{N-1} (Z_γ(t_{n+1})y(t_{n+1}), y(t_{n+1}))_Y\) is finite. Thus we arrive at

\[
J_γ(y₀; µ^*_γ, ν^*_γ) = \sum_{n=1}^{N} \sup_w J^n_γ(y(t_n); µ^*_γ, w) = +∞
\]

which contradicts the hypothesis, this completes the proof of part(i).

(ii) Part(ii) follows from part(i) and Theorem 6.4.2.1.

(iii) Since

\[
\inf_{M^N_s} \sup_w J_γ(y₀; u, w) ≥ \sup_w \inf_u J_γ(y₀; u, w) ≥ \sup_{\sup_w} \inf_u J_γ(y₀; u, w)
\]

and \(\sup_w \inf_u J_γ(y₀; u, w)\) is unbounded from part(i). The result follows, and the proof of Theorem 6.5.1 is complete. □

As we know from the Proposition 6.2.2.6, each equation in (6.5.6) has a solution near its terminal value. Thus, for any desired level of attenuation, the sampled time interval can be estimated beforehand, as delineated in the following Theorem 6.5.2.

**Theorem 6.5.2.** Given \(γ > 0\), suppose that (6.2.2.3) has a mild solution on \([0, t_f]\) and the sampling intervals satisfy the bound

\[
0 < t_{n+1} - t_n ≤ \min\left\{ \frac{\|Z_γ(t_{n+1})\|_Y}{\|C\|_∞ + \frac{2r}{r^2} \|B_2\|_∞}, \frac{γ^2α}{2rM_t^2 \|B_2\|_∞^2} \right\}
\]

(6.5.18)
where

\[ M_{t_f} = \sup\{\|U(t, s)\|; 0 \leq s \leq t \leq t_f\}, r = 2M_{t_f}^2\|P_{t_f}\|, 0 < \alpha < 1 \]

Then, with \( y_0 = 0 \) the sampled-data controller

\[
\mu_\gamma^s(t, y(t_n)) = -[(\lambda_0 I - A)^\alpha G(t)\]^{\star}[\lambda_0 I - A^\star]^{1-\alpha}Z_\gamma(t)\Phi_\gamma(t, t_n)y(t_n)
\]

\[ t_n \leq t < t_{n+1}, n = 1, 2, ..., N - 1 \]

attains the attenuation level \( \gamma \), i.e.

\[
\sup_w \{\mathcal{F}_{\gamma}^{\frac{1}{2}} / \|w\|_W\} < \gamma
\]

where \( \mathcal{F}_\mu : \mathcal{W} \to \mathbb{R} \) is induced by

\[
L(u, w) = \int_0^{t_f} \{\|C(t)y(t)\|_Y^2 + \|u(t)\|_U^2\} dt + (P_{t_f}y(t_f), y(t_f))_Y
\]

**Proof.** With the sampling sub-intervals given by (6.5.18), all \( N \) equations of (6.5.6) admit mild solutions, by Proposition 6.2.2.6. Now, applying Theorem 6.5.1, and have

\[
J_\gamma(y_0; \mu_\gamma^s, w) \leq (Z_\gamma(0)y_0, y_0) \equiv 0, \quad \forall w \in \mathcal{W}
\]

which yields

\[
\mathcal{F}_{\mu_\gamma^s} < \gamma^2 \|w\|_W^2, \quad \forall w \in \mathcal{W}
\]

This completes the proof of Theorem 6.5.2. \( \Box \)

**Remark 6.5.3.** The bound estimate for the sampled intervals as given by (6.5.18) is conservative, and can definitely be improved upon, but at the expense of complexity.

**Remark 6.5.4.** Since all terms on the right-hand side of (6.5.18) are bounded, this means that we can always construct a sampled feedback scheme \( M_{a_s}^N \) to achieve the attenuation
level γ, by choosing N big enough provided that this γ is picked such that (6.2.2.3) has a mild solution on [0, t_f].

Note that (6.5.6) consists of a family of generalized Riccati equations, and sampled feedback scheme requires all these equations to have mild solutions on their corresponding sampling intervals. For the time-invariant case we can replace (6.5.6) by a single generalized Riccati equation. Letting

\[ t_s := \max \{ t_{n+1} - t_n : n = 1, \ldots, N - 1 \}, \]

now consider the following Riccati equation

\[ \dot{S}(t) + A^*S(t) + S(t)A + C^*C + \frac{1}{\gamma^2}S(t)B_2B_2^*S(t) = 0 \]
\[ S(t_s) = Z_\gamma(0) \]

Note that the (critical) value of (6.5.19) is

\[ \bar{\gamma}_{sc} := \sup_{\|w\|=1} \left\langle B_2^*(\mathcal{K}_s^*Z_\gamma(t_s)\mathcal{K}_s + T_s^*C^*CT_s)B_2 w, w \right\rangle_{L(0,t_f;W)} \]

where

\[ (T_s\phi)(\tau) = \int_0^\tau e^{A(\tau-\eta)}\phi(\eta)d\eta, \quad 0 \leq \tau \leq t_s, \phi \in L(0,t_f;Y) \]
\[ (\mathcal{K}_s\phi)(\tau) = (T_n)(t_s), \phi \in L(0,t_f;Y) \]

Then we have:

**Corollary 6.5.5.** Assume that (6.1.1) is time-invariant. Given γ > 0, suppose that the generalized Riccati equation (6.2.2.3) admits a mild solution on [0, t_f] with Q_f = 0

(i) If γ > \bar{\gamma}_{sc}, then there exists unique strongly time consistent saddle-point controller, as given by Theorem 6.5.2 (ii).
(ii) If the sampling intervals satisfy the bound

\begin{equation}
0 < t_{n+1} - t_n \leq \min\left\{ \frac{\|Z_\gamma(0)\|_{\mathcal{L}(Y)}}{\|C\|_{\mathcal{L}(Y)}} + \frac{\gamma^2}{\gamma^2}, \frac{\gamma^2}{2rM^T_\gamma B_2^T B_2^\gamma} \right\}
\end{equation}

Theorem 6.5.2 still holds, with (6.5.18) replaced by (6.5.20).

\textbf{Proof.} (i) Since in the time-invariant case, we have

\[ Z_\gamma(0) \geq Z_\gamma(t_n), \quad n = 1, 2, ..., N - 1 \]

and also (6.5.6) can be rewritten as

\[ \dot{S}_n + A^* S_n + S_n A + C^* C + \frac{1}{\gamma^2} S_n B_2 B_2^* S_n = 0 \]

\[ S_n(t_{n+1} - t_n) = Z_\gamma(t_{n+1}) \]

\[ 0 < t < t_{n+1} - t_n, \quad n = N - 1, N - 2, ..., 1 \]

it is not difficult to see that \( S_n(t) \leq S(t) \) for \( t_n \leq t \leq t_{n+1} \), which yields that \( \hat{\gamma}_{sc} \geq \hat{\gamma}_{nc} \).

This means that (6.5.19) has mild solution on \([0, t_s]\) implies that all equations in (6.5.6) have mild solutions on their corresponding intervals by Proposition 6.2.2.6. Thus, by Theorem 6.5.1, there exists a unique saddle-point controller, which is the one given in a Theorem 6.5.1(i).

(ii) Since the bound (6.5.20) implies that (6.5.19) has a mild solution on the interval \([0, t_s]\), the result follows. \( \square \)

\textbf{An algorithm for the sampled-data controller design}

Given a desired attenuation level \( \gamma > 0 \):

\textbf{Step 1.} Check to see if (6.2.2.3) has a mild solution on \([0, t_f]\). If not, stop because in this case, we can not even use the state feedback given in (6.3.2) to achieve this level.
Step 2. Compute \( \|C\|_{\infty}, \|B_2\|_{\infty}, M_t = \sup\{\|U(t, s)\|_{\mathcal{L}(Y)}; 0 \leq s \leq t \leq t_f\}. \)

Step 3. Construct a finite time sequence \( \{t_n\}_0^N \) and \( \|Z(t_n)\|_{\mathcal{L}(Y)} \) such that (6.5.18) holds. If the time sequence is given and (6.5.18) does not hold, increase the sampled rate until (6.5.18) holds. This can be done as long as (6.2.2.3) has a mild solution under such \( \gamma \) according to Remark 6.5.4.

Step 4. Compute \( \Phi_\gamma(t, s) \), which is given in (6.5.4).

Step 5. At the current time \( t_n \), measure the sampled state \( y(t_n) \).

Step 6. Construct the feedback sampled controller on interval \([t_n, t_{n+1}]\) as follows:

\[
\mu_\gamma^a(t, y(t_k)) = -[(\lambda_0 I - A)^a G(s)]^* [\lambda_0 I - A^*]^{1 - a} Z_\gamma(t) \Phi_\gamma(t, t_n) y(t_n)
\]

Step 7. At time \( t = t_n \), goes back to step 5, and repeat this process until \( t_n = t_f \).

Remark 6.5.6. One of the difficulties in this algorithm is in step 4. Usually it is almost impossible to directly compute \( \Phi_\gamma(t, s) \) because this is related to the solution of the system (6.1.1). Recall that when we proved Theorem 6.3.4, a standard fixed point theorem based on contraction was used there. Hence recursive approximation would be a possible scheme to compute \( \Phi(t, s) \).

6.6. Example

Boundary control of heat equation. Let \( Y = L^2(0, 1) \) and \( U = W = \mathbb{C} \). Consider a metal bar of length 1 that is perfectly isolated at one end and temperature is controlled at the other end \( \xi = 1 \). This can be formulated by the heat equation with boundary control action:

\[
(6.6.1) \quad \frac{\partial y}{\partial t}(t, \xi) = \frac{\partial^2 y}{\partial \xi^2}(t, \xi) + w(t, \xi)
\]

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\[ \frac{\partial y}{\partial \xi}(t, 0) = 0 \quad \frac{\partial y}{\partial \xi}(t, 1) = u(t) \]

\[ y(\xi, 0) = y_0(\xi) \]

In this case the boundary operator \( B : L^2(0, 1) \to \mathbb{C} \) is

(6.6.2) \[ B y = \frac{dy}{d\xi}(1), \]

\[ D(B) = \{ y \in L^2(0, 1) | y \text{ and } \frac{dy}{d\xi} \text{ are absolutely continuous,} \]
\[ \frac{d^2y}{d\xi^2} \in L^2(0, 1) \text{ and } \frac{dy}{d\xi}(0) = 0 \} \]

and we can define

(6.6.3) \[ A = \frac{d^2}{d\xi^2} \]

\[ D_A = \{ y \in L^2(0, 1) | y \text{ and } \frac{dy}{d\xi} \text{ are absolutely continuous,} \]
\[ \frac{d^2y}{d\xi^2} \in L^2(0, 1) \text{ and } \frac{dy}{d\xi}(0) = 0 = \frac{dy}{d\xi}(1) \} \]

Clearly, \( A \) is the infinitesimal generator of analytic semigroup \( T(t) \) in \( Y \) and

\[ T(t) = \sum_{n=0}^{\infty} e^{\rho_n t} \langle \cdot, \phi_n \rangle \phi_n \]

where \( \rho_n = -n^2 \pi^2, n \geq 0, \phi_0 = 1 \text{ and } \phi_n(\cdot) = \sqrt{2} \cos(n \pi \cdot), n \geq 1. \) It is easy to see that in this case if we choose \( \lambda_0 = 1 \) then

\[ G(t)u(t) \equiv Gu(t) = \frac{1}{e^\xi - e^{-\xi}} (e^\xi + e^{-\xi})u(t) := \Phi(\xi, t) \]

because

\[ \Phi(\xi, t) - \frac{d^2}{d\xi^2} \Phi(\xi, t) = 0 \text{ in } (0, 1) \]
\[ B\Phi(\xi, t) = u(t) \text{ on } \xi = 1 \]

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As proved in Lions and Magenes [LM], we have \( G \in \mathcal{L}(U; H^{\frac{3}{4}}(0,1)) \); thus \( G \in \mathcal{L}(U; D([1 - A]^\alpha)) \) for any \( \alpha \in ]0, \frac{3}{4}[1 \). To avoid the computation of fractional powers, we use regularization for this simple problem. We first assume that \( \frac{du}{dt} \in L^p([0,t_f]; U) \) for some \( p \geq 1 \). The mild solution of (6.6.1) is given by

\[
y(t) = Gu(t) - T(t)Gu(0) + T(t)y_0 + \int_0^t T(t-s)w(s)ds - \int_0^t T(t-s)G\dot{u}(s)ds + \int_0^t T(t-s)\phi_0 u(s)ds
\]

We may compute (6.6.4) term by term. We first see

\[
T(t-s)w(s) = \sum_{n=0}^{\infty} e^{\rho_n t}(w, \phi_n)\phi_n
\]

and

\[
\int_0^t T(t-s)1u(s)ds = \int_0^t T(t-s)\phi_0 u(s)ds = \int_0^t u(s)ds\phi_0
\]

and also

\[
\int_0^t T(t-s)G\dot{u}(s)ds
\]

\[
= \int_0^t \sum_{n=0}^{\infty} e^{\rho_n (t-s)}(G, \phi_n)\phi_n\dot{u}(s)ds
\]

\[
= \sum_{n=0}^{\infty} \int_0^t e^{\rho_n (t-s)}(G, \phi_n)\phi_n\dot{u}(s)ds
\]

\[
= \sqrt{2} \sum_{n=0}^{\infty} \int_0^t e^{\rho_n (t-s)} \frac{(-1)^n}{1 + n^2\pi^2} \dot{u}(s)ds\phi_n
\]

\[
= \sqrt{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{1 + n^2\pi^2} \left\{ [u(t) - u(0)e^{-n^2\pi^2t}] - n^2\pi^2 \int_0^t e^{-n^2\pi^2(s-t)}u(s)ds \right\} ds\phi_n
\]

Furthermore, \( G \) has the expansion

\[
Gu = \sum_{n=0}^{\infty} (G, \phi_n)\phi_n u = \sqrt{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{1 + n^2\pi^2} u\phi_n
\]
and

\[(6.6.9) \quad T(t)G u = \sum_{n=0}^{\infty} (G, \phi_n) \phi_n u = \sqrt{2} \sum_{n=0}^{\infty} e^{-n^2 \pi^2 t} \frac{(-1)^n}{1 + n^2 \pi^2} u \phi_n \]

Combining (6.6.4), (6.6.5), (6.6.6), (6.6.7), (6.6.8) and (6.6.9), it follows that the mild solution of (6.6.1) is given by

\[(6.6.10) \quad y(t) = T(t)y_0 + \sqrt{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{1 + n^2 \pi^2} u(t) \phi_n - \sqrt{2} \sum_{n=0}^{\infty} e^{-n^2 \pi^2 t} \frac{(-1)^n}{1 + n^2 \pi^2} u(0) \phi_n \]

\[+ \int_0^t \sum_{n=0}^{\infty} e^{-n^2 \pi^2 (t-s)} (w, \phi_n) d\phi_n + \int_0^t u(s) d\phi_0 \]

\[- \sqrt{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{1 + n^2 \pi^2} \left\{ [u(t) - u(0) e^{-n^2 \pi^2 t}] - n^2 \pi^2 \int_0^t e^{-n^2 \pi^2 (t-s)} u(s) ds \right\} d\phi_n \]

\[= T(t)y_0 + \sum_{n=0}^{\infty} \int_0^t e^{-n^2 \pi^2 (t-s)} (w, \phi_n) d\phi_n + \int_0^t u(s) d\phi_0 \]

\[+ \sum_{n=1}^{\infty} \sqrt{2} \frac{(-1)^n n^2 \pi^2}{1 + n^2 \pi^2} \int_0^t e^{-n^2 \pi^2 (t-s)} u(s) d\phi_n \]

The mild solution of (6.6.1) is defined assuming that \(u \in L^p([0, 1]; u)\), for some \(p \geq 1\), but in fact (6.6.10) is well-defined for every \(u \in L^2([0, 1]; U)\), since

\[| \int_0^t e^{-n^2 \pi^2 (t-s)} u(s) ds |^2 \leq \frac{1 - e^{-2n^2 \pi^2 t}}{2n^2 \pi^2} \int_0^t |u(s)|^2 ds. \]

Recall from our representation formula and density theorem that, for any \(\alpha \in \{0, \frac{1}{2}\}\) (cf. [LM]) we may have

\[[(1 - A^*)^{1-\alpha} T(t-s)^*]^{\alpha} [1 - A]^{\alpha} G = \phi_0 + \sum_{n=1}^{\infty} \sqrt{2} \frac{(-1)^n n^2 \pi^2}{1 + n^2 \pi^2} e^{-n^2 \pi^2 (t-s)} \phi_n \]

or

\[1 - A^* G = \phi_0 + \sum_{n=1}^{\infty} \sqrt{2} \frac{(-1)^n n^2 \pi^2}{1 + n^2 \pi^2} \phi_n \]
and \([1 - A^*]^{1 - \alpha}[1 - A]^\alpha G\) belongs to \(L(U, (D(A)^{1 - \alpha})')\). Let the game cost function be

\[
J_\gamma(y_0; u, w) = \int_0^{t_f} \int_0^1 |y(\xi, t)|^2 d\xi dt + \int_0^{t_f} |u(t)|^2 dt + \int_0^1 |y(\xi, t_f)|^2 d\xi \\
- \frac{1}{\gamma^2} \int_0^{t_f} \int_0^1 |w(\xi, t)| d\xi dt
\]

We will now construct the state feedback controller and the sampled feedback controller. First we claim that when \(\gamma \geq 1\), (6.2.2.3) has a mild solution on \([0, t_f]\) for any \(t_f > 0\). Note that \(T(t) = e^{At}\) is a semigroup of contraction and from (6.2.2.5) we have

\[
Z_\gamma(t) := e^{A^*(t_f-t)}e^{A(t_f-t)} \\
+ \int_t^{t_f} e^{A^*(s-t)} \{I - [(1 - A^*)^{1 - \alpha}Z_\gamma(s)]^* K[1 - A^*]^{1 - \alpha}Z_\gamma(s) - \frac{1}{\gamma^2} Z_\gamma(s) Z_\gamma(s)\} e^{A(s-t)} ds
\]

where

\[
K := [1 - A]^{\alpha} G[(1 - A)^{\alpha} G]^*
\]

We let \(\gamma\) be large enough such that

(6.6.11) \(q^2 := \|T(t_f)[1 - A]^{1 - \alpha} K[(1 - A)^*]^{1 - \alpha} T(t_f)^*\|_{\mathcal{L}(Y)} \geq \frac{1}{\gamma^2}\)

and we have

\[
\|Z_\gamma(t)\|_{\mathcal{L}(Y)} \leq \|e^{A^*(t_f-t)} e^{A(t_f-t)}\|_{\mathcal{L}(Y)} \\
+ \int_t^{t_f} \{1 - \|Z_\gamma(s)[1 - A]^{1 - \alpha} K[1 - A^*]^{1 - \alpha} Z_\gamma(s)\|_{\mathcal{L}(Y)} + \frac{1}{\gamma^2} \|Z_\gamma(s)\|^2_{\mathcal{L}(Y)}\} ds \\
\leq 1 + \int_t^{t_f} \{1 - (\|T(t_f)[1 - A]^{1 - \alpha} K[1 - A^*]^{1 - \alpha} T(t_f)^*\|_{\mathcal{L}(Y)} + \frac{1}{\gamma^2} \|Z_\gamma(s)\|^2_{\mathcal{L}(Y)}\} ds \\
= 1 + \int_t^{t_f} \{1 - (q^2 - \frac{1}{\gamma^2}) \|Z_\gamma(s)\|^2_{\mathcal{L}(Y)}\} ds \\
\leq 1 + t_f - t
\]
Thus $Z_\gamma(t)$ is bounded on $[0, t_f]$ when $\gamma \geq \frac{1}{\epsilon}$. We compute $\varrho$ by using (6.6.11), which yields

$$\varrho = \| \phi_0 + \sum_{n=1}^{\infty} \sqrt{2} \frac{(-1)^n n^2 \pi^2}{1 + n^2 \pi^2} e^{-n^2 \pi^2 t_f} \phi_n \| = \left( 1 + \sum_{n=1}^{\infty} \left( \frac{2n^2 \pi^2}{1 + n^2 \pi^2} \right)^2 e^{-2n^2 \pi^2 t_f} \right)^{\frac{1}{2}}$$

and this implies when $\gamma \geq 1$, that (6.2.2.3) has a mild solution. If we use the state feedback, then the unique feedback saddle-point solution for the differential game

$$\varphi(y_0) = \inf_{\mathcal{M}} \sup_{\mathcal{W}} J_\gamma(y_0; u, w)$$

is given by

$$\mu^*(t, y(t)) = -[(1 - A)^{\alpha} G]^*(1 - A^*)^{1 - \alpha} Z_\gamma(t)y(t)$$

$$\nu^*(t, y(t)) = \frac{1}{\gamma^2} Z_\gamma(t)y(t)$$

according to Theorem 6.4.2.2. Next, we construct the sampled feedback controller. Suppose that the desired attenuation level is $\gamma_{des} = 1$. By the above discussion, we know that (6.2.2.3) has a mild solution on $[0, t_f]$ for $\gamma > \gamma_{des} = 1$. According to Theorem 6.5.2, we know that the sampling rate $\{t_n\}$ need to satisfy the bound:

$$0 < t_{n+1} - t_n \leq \min \left\{ \frac{\|Z_\gamma(t_{n+1})\|_{\mathcal{L}(Y)}}{1 + 4\gamma^{-2} \|Z_\gamma(t_{n+1})\|_{\mathcal{L}(Y)}^2}, \frac{\gamma^\alpha}{4\|Z_\gamma(t_{n+1})\|_{\mathcal{L}(Y)}} \right\}$$

If we use uniform sampling rate, and note $\|Z_\gamma(t)\|_{\mathcal{L}(Y)} \leq 1 + t_f$, let $t_f = 10$, we can have

$$t_{n+1} - t_n = 0.099, \quad 1 \leq n \leq 102$$

The corresponding sampled-data feedback controller is

$$\mu_\gamma^*(t, y(t_n)) = -[(1 - A)^{\alpha} G(s)]^*[1 - A^*]^{1 - \alpha} Z_\gamma(t)\Phi_\gamma(t, t_n)y(t_n)$$

$$t_n \leq t < t_{n+1}, n = 1, 2, \ldots, 102$$
Under this sampled feedback controller, the desired attenuation level $\gamma_{des} = 1$ is achieved, i.e. (setting $y_0 = 0$)

$$\sup_w \{ \mathcal{F}_{x, \gamma}^\frac{1}{2} / \|w\|_W \} < 1$$

where $\mathcal{F}_{x, \gamma}^\frac{1}{2} : W \rightarrow \mathbb{R}$ is induced by

$$L(u, w) = \int_0^{10} \int_0^1 |y(\xi, t)|^2 d\xi dt + \int_0^{10} \|u(\gamma(s))\|_2^2 ds + \int_0^1 |y(\xi, 10)|^2 d\xi$$

One observation here is that the sampling rate increases as attenuation level increases (i.e. $\gamma$ decreases). For example, in this case ($t_f = 10$), $t_s = t_{n+1} - t_n$ we have

$\gamma = 10 \Rightarrow t_s = 5$, and the total sampled time = 2

$\gamma = 5 \Rightarrow t_s = 2$, and the total sampled time = 5

$\gamma = 3 \Rightarrow t_s = 0.826$, and the total sampled time = 13

$\gamma = 2 \Rightarrow t_s = 0.385$, and the total sampled time = 26

$\gamma = 1 \Rightarrow t_s = 0.099$, and the total sampled time = 102

6.7. Concluding Remarks

In this chapter, we have studied the finite-horizon $H^\infty$ optimal control problem for general time-varying parabolic systems with boundary sampled state feedback. The only available information for the controller is the sampled state information, obtained from the boundary of the system. This latter fact that information comes from the boundary only, and in sampled form, has necessitated the development of a new theory, requiring rather subtle analysis.

We have considered here only the finite-time horizon case, as the infinite horizon case requires a separate analysis. There, stability is an important issue and further assumptions
have to be imposed on the system such as exponentially detectability, finite cost condition, etc. The corresponding result for this case will be reported elsewhere.

6.8. Appendix

In this Appendix, we provide a proof for the expression

$$\delta_w J_\gamma(x_0; \hat{u}, w) = B_2(t)^* Z_\gamma(t)y(t) - \gamma^2 w$$

which appears in the proof of Theorem 6.4.2.1.

**Proof:** Denote \( \varphi \) by

$$\varphi(w) = 2\gamma^2 \|w\|_2^2 - 2\tilde{\varphi}(w), w \in L^2([0, t_0]; W)$$

where

$$\tilde{\varphi}(w) = \int_0^{t_f} (\|C(t)y^w(t)\|_{Y}^2 + \|\hat{u}(t)\|_{U}^2)dt$$

and \((y^w, \hat{u})\) satisfies

$$\dot{y}^w(t) = A(t)y^w(t) + [(\lambda_0 I - A(s))^1-\alpha[\lambda_0 I - A(s)]^\alpha G(t)\hat{u}(t) + B_2(t)w(t), t \in (0, t_f]$$

$$y^w(0) = y_0$$

We have therefore

$$(\delta_w \tilde{\varphi}(w), \bar{w}) =$$

$$\int_0^{t_f} \langle (C(t)y^w(t), C(t)z(t)) + (C(t)z(t), C(t)y^w(t)) \rangle$$

$$+ \langle P_t y^w(t_f), z(t_f) \rangle + \langle z(t_f), P_t y^w(t_f) \rangle, \forall \bar{w} \in L^2([0, t_f]; W)$$
where \( z \) satisfies

\[
\dot{z}(t) = A(t)z(t) + B_2(t)\bar{w}(t)
\]

\( z(0) = 0 \)

Note that

\[
\int_0^{t_f} \langle C(t)^* C(t) y(t), z(t) \rangle dt = \int_0^{t_f} \langle p(t), B_2(t)\bar{w}(t) \rangle dt - \langle P_y(t_f), z(t_f) \rangle
\]

where \((y, p)\) is defined in (6.3.2) and (6.4.2.4). This yields

\[
(\delta_w \phi(\hat{w}), \bar{w}) = \int_0^{t_f} ((p(t), B_2(t)\bar{w}(t)) + (B_2(t)\bar{w}(t), p(t))) dt
\]

and for all \( \bar{w} \in L^2([0, t_f]; W) \)

\[
(\delta_w \varphi(\hat{w}), \bar{w}) = 2 \int_0^{t_f} (\gamma^2 (\hat{w}, \bar{w}) - (p(t), B_2(t)\bar{w}(t)))
\]

Hence we have

\[
\delta_w \varphi(\hat{w}) = 2\gamma^2 \hat{w} - 2B_2(t)^* p(t)
\]

Since we already know from (6.4.2.3), (6.4.2.4) that

\[
p(t) = Z_\gamma(t)y(t)
\]

we have

\[
\hat{w}(t) = \frac{1}{\gamma^2} B_2(t)^* Z_\gamma(t)y(t)
\]

and at \( w = \hat{w} \), this leads to

\[
\delta_w J_\gamma(x_0; \hat{u}, \hat{w}) = 2B_2(t)^* Z_\gamma(t)y(t) - 2\gamma^2 \hat{w}
\]
7. Viscosity Solutions of A Class of Coupled Partial Differential Equations Arising in the Optimal Control of Piecewise Deterministic Nonlinear Systems with Controlled Transitions

7.1. Introduction

This chapter studies viscosity solutions for systems of coupled Hamilton-Jacobi-Bellman (HJB) equations of the following type, where $\Omega$ is a subset of $\mathbb{R}^n$ with nonempty interior:

(I) Finite horizon:

(7.1.1) $$\beta V(t, x, i) + \sup_{u^1, u^2} \left[ - A^{(u^1, u^2)} V(t, x, i) - L(x, i, u^1) \right] = 0 \quad \text{in } (0, t_f] \times \Omega$$

$$V(t_f, x, i) = g(x, i) \quad \text{on } \Omega$$

(II) Infinite horizon:

(7.1.2) $$\beta V(x, i) + \sup_{u^1, u^2} \left[ - G^{(u^1, u^2)} V(x, i) - L(x, i, u^1) \right] = 0 \quad \text{in } \Omega$$

Here, the operators $A$ and $G$ are defined as follows:

$$A^{(u^1, a)} V(t, x, i) := \frac{\partial V(t, x, i)}{\partial t} + D_x V(t, x, i) F(x, u^1, i) + \sum_{j \in S} \lambda_{iaj} V(t, x, j)$$

and

$$G^{(u^1, a)} V(x, i) := D_x V(x, i) F(x, u^1, i) + \sum_{j \in S} \lambda_{iaj} V(x, j),$$

with

$$F(x, u^1, i) := f(x, i) + B(x, i) u^1, \quad L(x, i, u^1) := Q(x, i) + < u^1, R(x, i) u^1 >$$

where the roles of various terms introduced will be clarified shortly, with precise technical conditions given in the next section. The coupled PDE's (7.1.1) and (7.1.2) are the dynamic programming equations for the following optimum stochastic control problem with
piecewise deterministic dynamics: Consider a system described by piecewise continuous dynamics that are nonlinear in the state and linear in the piecewise continuous control $u^1$:

\[(7.1.3)\]
\[
\frac{dx}{dt}(t) = f(x(t), \theta(t)) + B(x(t), \theta(t))u^1(t)
\]
\[x(0) = x_0\]

where $x \in \mathbb{R}^n$, $x_0$ is a fixed (known) initial state, $u^1$ is a control, taking values in $U_1 = \mathbb{R}^r$, and $\theta(t)$ is a controlled, continuous time Markov process, taking values in a finite state space $\mathcal{S}$, of cardinality $s$. Transitions from state $i \in \mathcal{S}$ to $j \in \mathcal{S}$ occur at a rate controlled by a second controller, who chooses at time $t$ an action $u^2(t)$ from a finite set $U_2(i)$ of actions available at state $i$. Let $U_2 := \bigcup_{i \in \mathcal{S}} U_2(i)$. The controlled rate matrix (of transitions within $\mathcal{S}$) is

\[
\Lambda = \{\lambda_{i,a,j}\}, \quad i, j \in \mathcal{S}, \quad a = u^2(t) \in U_2(i)
\]

where henceforth we drop the "commas" in the subscripts of $\lambda$. The $\lambda_{iaj}$'s are real numbers such that for any $i \neq j$, and $a \in U_2(i)$, $\lambda_{iaj} \geq 0$, and for all $a \in U_2(i)$ and $i \in \mathcal{S}$, $\lambda_{iai} = -\sum_{j \neq i} \lambda_{iaj}$. Fix some initial state $i_0$ of the controlled Markov chain $\mathcal{S}$, and the final time $t_f$ (which may be infinite). Consider the class of policies $\mu_k \in \mathcal{U}_k$ for controller $k$, $k = 1, 2$, whose elements (taking values in $\mathcal{U}_k$) are of the form

\[(7.1.4)\]
\[
u^k(t) = \mu^k(t, x_{[0,t]}, \theta_{[0,t]}), \quad t \in [0, t_f).
\]

Here, $\nu^k$ is taken to be piecewise continuous in its first argument, and piecewise Lipschitz continuous in its second argument. Define $\mathcal{X} = \mathbb{R}^n \times \mathcal{S}$ to be the combined state space of the system and $\mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_2$ to be the class of multi-strategies $\nu := (\nu^1, \nu^2)$. Define a
running cost $L : \mathcal{X} \times U_1 \to \mathbb{R}$ as

$$L(x, i, u^1) = Q(x, i) + \langle u^1, R(x, i)u^1 \rangle,$$

where $Q$ and $R$ are generally nonlinear (and non-quadratic) in $x$, which are also assumed to be positive.

To any fixed initial state $(x_0, i_0)$ and a multi-strategy $\mu \in \mathcal{U}$, there corresponds a unique probability measure $P^\mu_{x_0, i_0}$ on the canonical probability space of the states and actions of the players, equipped with the standard Borel $\sigma$-algebra. Denote by $E^\mu_{x_0, i_0}$ the expectation operator corresponding to $P^\mu_{x_0, i_0}$.

For each fixed initial state $(x_0, i_0)$, multi-strategy $\mu \in \mathcal{U}$, and a finite horizon of duration $t_f$, the discounted (expected) cost function is defined as

$$J_\beta(x_0, i_0, \mu; t_f) := E^\mu_{x_0, i_0}\left\{g(x(t_f), \theta(t_f))e^{-\beta t_f} + \int_0^{t_f} e^{-\beta t}L(x(t), \theta(t), u^1(t))dt\right\} \tag{7.1.5}$$

where $g$ is a terminal cost function, and $\beta \geq 0$ is the discount factor. For $t_f$ infinite, a corresponding discounted cost function is defined as:

$$J_\beta(x_0, i_0, \mu) := E^\mu_{x_0, i_0}\left\{\int_0^{\infty} e^{-\beta t}L(x(t), \theta(t), u^1(t))dt\right\} \tag{7.1.6}$$

We further denote the cost-to-go from any time-state pair $(t; x, i)$, under a multi-strategy $\mu \in \mathcal{U}$ by

$$J_\beta(t; x, i, \mu; t_f) := E^\mu_{x, i}\left\{g(x(t_f), \theta(t_f))e^{-\beta(t_f-t)}
+ \int_t^{t_f} e^{-\beta(\tau-t)}L(x(\tau), \theta(\tau), u^1(\tau))d\tau\right\}$$

$^9$Here, $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product.
and

\[ J_{\beta}(t; x, i, \mu) := E^\mu_{x, i} \left\{ \int_t^\infty e^{-\beta(t-\tau)} L(x(\tau), \theta(\tau), u^1(\tau)) d\tau \right\} \]

The optimum value functions are then defined by:

\[ V(t; x, i; t_f) = \inf_{\mu^1 \in U_1} \inf_{\mu^2 \in U_2} J_{\beta}(t; x, i, \mu; t_f) \quad \text{finite-horizon case} \]

\[ V(t; x, i; +\infty) = \inf_{\mu^1 \in U_1} \inf_{\mu^2 \in U_2} J_{\beta}(t; x, i, \mu) \quad \text{infinite-horizon case} \]

Now by using dynamic programming\(^{10}\), it is not difficult to arrive at the two coupled HJB equations (7.1.1) and (7.1.2), corresponding to the finite and infinite horizon cases, respectively, provided that these equations admit unique viscosity solutions on \( \Omega \)—a concept that will be introduced in precise terms later in Section 2. It is a verification of the latter that is the main goal of this chapter.

**Remark 7.1.1.** If we assume that \( R(x, i) \) is (strictly) positive definite for all \( x \) and \( i \), then (7.1.1) can be written as

\[
\begin{align*}
- \frac{\partial V(t, x, i)}{\partial t} + \beta V(t, x, i) & \leq D_x V(t, x, i), f(x, i) > - \inf_{j \in S} \{ \sum_{a} \lambda_{iaj} V(t, x, j) \} \\
- Q(x, i) + \frac{1}{4} < D_x V(t, x, i), B(x, i) R^{-1}(x, i) B(x, i)^T D_x V(t, x, i) > = 0 \\
V(t_f, x, i) = g(x, i),
\end{align*}
\]

with the associated minimizing controller \( \mu^1 \) being\(^{11}\):

\[
\mu^1(x(t), i) = - R^{-1}(x(t), i) B^T(x(t), i) D_x V(t, x(t), i)
\]

---

\(^{10}\)For background on the approach that can be used here, see [FS]; see also section 3 of this chapter.

\(^{11}\)Since \( D_x V \) may not be well defined, this expression is quite informal at this point. It will be made more precise in section 5.
Under the same condition, (7.1.2) can be written as

\begin{equation}
\beta V(x, i) - < D_z V(x, i), f(x, i) > - \inf_a \{ \sum_{j \in s} \lambda_{ij} V(x, j) \} \\
- Q(x, i) + \frac{1}{4} < D_z V(x, i), B(x, i)R^{-1}(x, i)B(x, i)^T D_z V(x, i) >= 0
\end{equation}

with the corresponding minimizing controller \( \mu^1 \) in this case being

\[ \mu^1(x(t), i) = -R^{-1}(x(t), i)B^T(x(t), i)D_z V(x(t), i). \]

For future use, we introduce the Hamiltonian for both finite and infinite horizon cases by \( \bar{H}(x, r, p) = \{ \bar{H}_i(x, r, p) \}_{i=1}^s \) where \( \bar{H}_i \) is given by

\begin{equation}
\bar{H}_i(x, r, p) = \beta r_i - \inf_a \{ \sum_{j \in s} \lambda_{ij} r_j \} + H_i(x, p)
\end{equation}

where \( r = (r_1, ..., r_s) \), and

\begin{equation}
H_i(x, p) = - < p, f(x, i) > - Q(x, i) + \frac{1}{4} < p, B(x, i)R^{-1}(x, i)B(x, i)^T p >.
\end{equation}

HJB equations are nonlinear partial differential equations, and it is well-known that in general such equations do not admit global classical solutions. Furthermore, the value functions of optimal control problems (and differential games as well) are not differentiable in general, and they are not even continuous for some classes of problems (e.g. cf [Sor1],[XB4],[FQ]). The theory of viscosity solutions, first introduced by M.G. Crandall and P.L. Lions, provides a convenient framework to study HJB equations. The questions of existence and uniqueness for viscosity solutions of HJB equations have been studied by a number of authors, and in particular, by Crandall and Lions [CrLi 1-2], Lions [L], Crandall [Cr2], Evans [E], Ishii[Is], Crandall, Evans and Lions [CEL], Fleming and Soner.
[FS], Tataru [T 1-2], Soravia [Sor2], Ball and Helton [BH], McEneaney [M 1], and Evans and Souganidis [ES].

The class of HJB equations (7.1.1) and (7.1.2) to be studied in this chapter are different from the ones treated in all these earlier references, first because they are (linearly) coupled, and second because the Hamiltonian does not satisfy the “structure condition”, which is:

\[(7.1.11)\quad |\hat{H}_i(x, r, p) - \hat{H}_i(y, r, q)| \leq m_R(|p - q| + |x - y|)\]

where \((x, p), (y, q) \in \mathbb{R}^n \times B_R(0)\) and \(m_R\) is a modulus function (c.f. [Cr 1-2]). When the control space is bounded, the structure condition (7.1.11) generally guarantees that HJB equations admit Lipschitz continuous viscosity solutions (see e.g., [Cr1], [FS], [M2], [Son]).

Clearly the boundedness assumption on the control space could be overly restrictive, for example, in the linear-quadratic regulator problem (with piecewise deterministic dynamics), the control cannot be bounded a priori, and hence (7.1.11) does not hold. Other assumptions that are commonly made [FS] to replace (7.1.11) also will result in restrictions which are not natural in the framework adopted here. Moreover since the Hamiltonian is also a coupled system, the monotonicity property with respect to \(r\) no longer holds in general, i.e. \(\hat{H}(x, r, p)\) is not montone with respect to \(r\). Without this property and (7.1.11), the standard comparison theorem does not apply ([Cr1] or [Cr2]), and the proof of uniqueness becomes more challenging.

In this chapter, we generalize the standard comparison theorem to cover coupled HJB equations where the Hamiltonian does not satisfy the structure condition and the monotonicity property, but instead, it has a quadratic structure, which corresponds to a large class of nonlinear systems in the form (7.1.3). We prove the existence of solutions to (7.1.1) and (7.1.2), under assumptions quite different from the standard ones. We also discuss the
derivation of an explicit optimal control law.

The structure of the chapter is as follows. In section 2.1, we provide the necessary assumptions on the HJB equations (7.1.1) and (7.1.2). Section 2.2 provides the definition of a viscosity solution for coupled HJB equations (7.1.1) and (7.1.2). In section 3, we show the existence of viscosity solutions to (7.1.1) and (7.1.2) by using dynamic programming and a variation of Dynkin's formula. The generalized comparison theorem for coupled HJB equations (7.1.1) and (7.1.2) are provided in section 4. The optimal control law is discussed in Section 5. Finally, in section 6, we apply the results obtained to a scalar linear-quadratic problem.

7.2. Assumptions and Definitions

7.2.1. Assumptions

(A1) For each $i$, $f$ is an $n$-vector, and there exists a constant $L_f \geq 0$ such that

$$\sup_i \{|f(x,i) - f(y,i)| \leq L_f |x-y|, \quad x \in \mathbb{R}^n$$

(A2) For each $i$, $B(x,i)$ is an $n \times r$ matrix, and

$$\sup_i |B(x,i)| \leq C_{b1}, \quad \sup_i |B(x,i) - B(y,i)| \leq C_{b2}|x-y|, \quad \forall x, y \in \mathbb{R}^n$$

for some $C_{b1}, C_{b2} < \infty$.

(A3) For each $i$, $Q(\cdot,i) : \mathbb{R}^n \to [0, +\infty)$, with

$$0 \leq \sup_i Q(x,i) \leq C_q|x|^2, \quad \forall x \in \mathbb{R}^n$$

(A4) For each $i$, $R(x,i)$ is an $n \times n$ matrix with $R(x,i) = R(x,i)^T > 0, \forall x \in \mathbb{R}^n$, and

$$\sup_i |R(x,i)| \leq C_r, \quad \sup_i |R(x,i) - R(y,i)| \leq C'_r|x-y|, \quad x, y \in \mathbb{R}^n$$
for some $C_r, C'_r > 0$, and there exists $L_R > 0$ such that

$$\sup_i |R^{-1}(x, i) - R^{-1}(y, i)| \leq L_R |x - y|, \quad \forall x, y \in \mathbb{R}^n$$

(A5) For $i \neq j$, $0 \leq \lambda_{ij} \leq C_\lambda$, and $\lambda_{ii} + \sum_{j \neq i} \lambda_{ij} \equiv 0$, $1 \leq i \leq K$, where $C_\lambda$ is a positive constant.

(A6) For each $i$ and any $g(\cdot, i) : \mathbb{R}^n \to [0, \infty)$,

$$\sup_i |g(x, i)| \leq (1 + C_g) |x|^2,$$

$$\sup_i |g(x, i) - g(y, i)| \leq C'_g (1 + |x| + |y|) |x - y|$$

for all $x, y \in \mathbb{R}^n$, where $C_g, C'_g$ are positive constants.

(A7) $\beta$ is a nonnegative constant.

(A8) For any $z \in \mathbb{R}^n$, there exists a nondecreasing function $\omega : \{0\} \cup \mathbb{R}^+ \to \{0\} \cup \mathbb{R}^+$ such that $\omega(0) = 0, \lim_{r \to +\infty} \omega(r)/r = +\infty$ such that

$$< z, B(x, i) R^{-1}(x, i) B(x, i)^* z > \geq \omega(|z|), \quad \forall x \in \mathbb{R}^n, \forall i \in S.$$

Remark 7.2.1.1. When

$$f(x, i) = A(i)x, \quad B(x, i) = B(i), \quad Q(x, i) = x^T Q(i) x, \quad R(x, i) = R(i)$$

and $g(x, i) = x^T Q_{ti} x$, where $A(i), B(i), Q(i), R(i)$ and $Q_{ti}(i)$ are appropriate dimensional matrices which are dependent only on $i$, then assumptions (A1)-(A4) and (A6) are automatically satisfied.

Remark 7.2.1.2. By assumptions (A8) and (A4), we can see that for fixed $x \in \mathbb{R}^n$

$$H_i(x, p) \to +\infty \quad \text{as} \quad |p| \to +\infty.$$
Throughout the chapter, the following conventions will be adopted, unless otherwise indicated: (1) \(u^2\) and \(a\) are used interchangeably to denote the second control; (2) by an abuse of notation \(\mu^2(t)\) will be used to denote \(\mu^1(x(t), \theta(t))\).

### 7.2.2. Definitions

**Definition 7.2.2.1.** Let \(\overline{V}\) be a vector function

\[
\overline{V} = (V(\cdot, \cdot, 1), V(\cdot, \cdot, 2), \ldots, V(\cdot, \cdot, s)) : ([0, t_f] \times \Omega)^s \to (\mathbb{R}^n)^s
\]

(a) \(\overline{V}\) is a **viscosity subsolution** of (7.1.1), if for any \(i, V(\cdot, \cdot, i)\) is upper semicontinuous and

\[
\beta \Phi(t_0, x_0, i) + \sup_{\mu^1, \mu^2} \left[ -A(\mu^1, \mu^2) \Phi(t_0, x_0, i) - L(x_0, i, \mu^1) \right] \leq 0 \quad \text{on} \quad \Omega
\]

\[
\Phi(t_f, x, i) \leq g(t_f, x, i) \quad \text{on} \quad \Omega
\]

whenever \(\Phi(\cdot, \cdot, i) \in C^1([0, t_f] \times \Omega)\) is such that \(V(t, x, i) - \Phi(t, x, i)\) attains a local maximum at \((t_0, x_0)\) with \(\Phi(t_0, x_0, j) = V(t_0, x_0, j)\) for each \(j \in S\);

(b) \(\overline{V}\) is a **viscosity supersolution** of (7.1.1) if for any \(i, V(\cdot, \cdot, i)\) is lower semicontinuous and

\[
\beta \Phi(t_0, x_0, i) + \sup_{\mu^1, \mu^2} \left[ -A(\mu^1, \mu^2) \Phi(t_0, x_0, i) - L(x_0, i, \mu^1) \right] \geq 0 \quad \text{on} \quad \Omega
\]

\[
\Phi(t_f, x, i) \geq g(t_f, x, i)
\]

whenever \(\Phi(\cdot, \cdot, i) \in C^1([0, t_f] \times \Omega)\) is such that \(V(t, x, i) - \Phi(t, x, i)\) attains a local minimum at \((t_0, x_0)\) with \(\Phi(t_0, x_0, j) = V(t_0, x_0, j)\) for each \(j \in S\);

(c) \(\overline{V}\) is a **viscosity solution** of (7.1.1) if \(\overline{V}\) satisfies both (a) and (b).

**Remark 7.2.2.** \(V(t, x, i) - \Phi(t, x, i)\) attains a local maximum (respectively, minimum) at \((t_0, x_0), i = i_0\) means that there exist \(\eta_1, \eta_2 > 0\) and a subset \(S \subseteq S, i_0 \in S\), such that
when \((t, x) \in [B(t_0, \eta_1) \times B(x_0, \eta_2)] \cap [[0, t_f] \times \Omega]\) and \(i \in S\), we have

\[
V(t, x, i) - \Phi(t, x, i) \geq V(t_0, x_0, i_0) - \Phi(t_0, x_0, i_0)
\]

(respectively, \(V(t, x, i) - \Phi(t, x, i) \leq V(t_0, x_0, i_0) - \Phi(t_0, x_0, i_0)\))

The notion of a viscosity solution for (7.1.2) can be introduced analogously.

### 7.3. The Existence of Viscosity Solution

The following is just the dynamic programming principle, and we provide a proof for reader’s convenience.

**Proposition 7.3.1.** For any \(0 \leq t \leq \tau \leq t_f\), and with \(x(t) = x, \theta(t) = i\),

\[
V(t; x, i; t_f) = \inf_{\mu \in \mathcal{U}} E_{x, i}^\mu \left\{ \int_t^\tau e^{-\beta(s-t)} L(x(s), \theta(s), \mu(s)) ds + e^{-\beta(\tau-t)} V(\tau; x(\tau), \theta(\tau); t_f) \right\}
\]

**Proof.** Given any \(\delta > 0\), it follows from the definition of the value function that there exists a controller, say \(\hat{\mu} = (\hat{\mu}^1, \hat{\mu}^2) \in \mathcal{U}\), such that

\[
E_{x(\tau), \theta(\tau)}^{\hat{\mu}} \left\{ \int_\tau^{t_f} e^{-\beta(s-\tau)} L(x(s), \theta(s), \hat{\mu}(s)) ds + g(x, \theta(t_f)) e^{-\beta(t_f-\tau)} \right\} \leq V(\tau; x(\tau), \theta(\tau); t_f) + \delta
\]

where, of course, only the restriction of \(\hat{\mu}\) to \([\tau, t_f]\) is relevant. Now considering the restriction of \(\hat{\mu}\) to \([t, t_f]\), we have

\[
V(t; x, i; t_f) \leq J_\beta(t; x, i, \mu; t_f)
\]

\[
= E_{x, i}^\mu \left\{ \int_t^{t_f} e^{-\beta(s-t)} L(x(s), \theta(s), \hat{\mu}(s)) ds + g(x, \theta(t_f)) e^{-\beta(t_f-t)} \right\}
\]
\[ = E_{x,i}^{\hat{\mu}} \left\{ \int_t^T e^{-\beta(s-t)} L(x(s), \theta(s), \hat{\mu}^1(s)) ds \\
+ e^{-\beta(t-t_f)} E_{x(t_f), \theta(t_f)}^{\hat{\mu}} \left[ \int_\tau^{t_f} e^{-\beta(s-\tau)} L(x(s), \theta(s), \hat{\mu}^1(s)) ds + g(x(t_f), \theta(t_f)) e^{-\beta(t_f-\tau)} \right] \right\} \\
\leq E_{x,i}^{\hat{\mu}} \left\{ \int_t^T e^{-\beta(s-t)} L(x(s), \theta(s), \hat{\mu}^1(s)) ds + e^{-\beta(t-t_f)} V(\tau; x(\tau), \theta(\tau); t_f) \right\} + \delta \]

where the last line has followed from (7.3.2). On the other hand, we can choose an admissible control \( \hat{\mu} = (\hat{\mu}^1, \hat{\mu}^2) \), such that

\[ \delta + V(t; x, i; t_f) \geq J_\beta(t; x, i, \hat{\mu}; t_f) \]

\[ = E_{x,i}^{\hat{\mu}} \left\{ \int_t^{t_f} e^{-\beta(s-t)} L(x(s), \theta(s), \hat{\mu}^1(s)) ds + g(x(t_f), \theta(t_f)) e^{-\beta(t_f-t)} \right\} \\
= E_{x,i}^{\hat{\mu}} \left\{ \int_t^T e^{-\beta(s-t)} L(x(s), \theta(s), \hat{\mu}^1(s)) ds + \int_t^{t_f} e^{-\beta(s-t)} L(x(s), \theta(s), \hat{\mu}^1(s)) ds \\
+ g(x(t_f), \theta(t_f)) e^{-\beta(t_f-t)} \right\} \\
= E_{x,i}^{\hat{\mu}} \left\{ \int_t^T e^{-\beta(s-t)} L(x(s), \theta(s), \hat{\mu}^1(s)) ds + e^{-\beta(t-t_f)} J_\beta(\tau; x(\tau), \theta(\tau), \hat{\mu}; t_f) \right\} \\
\geq E_{x,i}^{\hat{\mu}} \left\{ \int_t^T e^{-\beta(s-t)} L(x(s), \theta(s), \hat{\mu}^1(s)) ds + e^{-\beta(t-t_f)} V(\tau; x(\tau), \theta(\tau); t_f) \right\} \]

Hence,

\[ V(t; x, i; t_f) \]

\[ \geq E_{x,i}^{\hat{\mu}} \left\{ \int_t^T e^{-\beta(s-t)} L(x(s), \theta(s), \hat{\mu}^1(s)) ds + e^{-\beta(t-t_f)} V(\tau; x(\tau), \theta(\tau); t_f) \right\} - \delta \]

Since \( \delta \) was arbitrary, the conclusion follows. \( \square \)

**Remark 7.3.2.** In the infinite horizon case, the dynamic programming principle can be stated as follows:

For any \( t > 0 \)

\[ V(x, i) = \inf_{\mu \in U} E_{x,i}^\mu \left\{ \int_0^t e^{-\beta s} L(x(s), \theta(s), \mu^1(s)) ds + e^{-\beta t} V(x(t), \theta(t)) \right\} \]

The following proposition is a variant of Dynkin's formula.
Proposition 7.3.3. Define $\Phi : [0,t_f] \times \mathbb{R}^n \times S \rightarrow \mathbb{R}$ such that the partial derivatives $\Phi_t, \Phi_{x_i}, i = 1, \ldots, n$ are continuous. Then, for $x(t) = x$ and $\theta(t) = i$,

$$E_{x,i}^\mu \Phi(s, x(s), \theta(s)) - \Phi(t, x, i) = E_{x,i}^\mu \int_t^s A^{(\mu^1, \mu^2)}(r, x(r), \theta(r))dr$$

where $A^{(\mu^1, \mu^2)}$ is defined as

(7.3.3) $$A^{(u^1, u^2)}(t, x, i) := \Phi_t(t, x, i) + \Phi_x(t, x, i)F(x, u^1, i) + \sum_{j \in S} \lambda_{iaj} \Phi(t, x, j)$$

with $u^1 = \mu^1(x, i), u^2 = \mu^2(x, i)$.

**Proof.** See Appendix B of [FS]

**Remark 7.3.4.** Suppose that the value function $V$ is differentiable with respect to $x$ and $t$. By Proposition 7.3.1, we have

(7.3.4) $$0 = \inf_{\mu \in U(t)} E_{x,i}^\mu \left\{ \int_t^s e^{-\beta(s-r)}L(x(s), \theta(s), \mu^1(s))ds + e^{-\beta(t-t_f)}V(\tau; x(\tau), \theta(\tau); t_f) - V(t; x, i; t_f) \right\}$$

Since

(7.3.5) $$e^{-\beta(t-t_f)}V(\tau; x(\tau), \theta(\tau); t_f) - V(t; x, i; t_f)$$

$$= e^{-\beta(t-t_f)}[V(\tau; x(\tau), \theta(\tau); t_f) - V(t; x, i; t_f)] - [1 - e^{-\beta(t-t_f)}]V(t; x, i; t_f)$$

using this in (7.3.4), and applying Proposition 7.3.3, we arrive at

(7.3.6) $$0 = \inf_{\mu^1 \in U_1} \inf_{\mu^2 \in U_2} E_{x,i}^\mu \left\{ \int_t^s e^{-\beta(s-r)} \left[ L(x(s), \theta(s), \mu^1(s)) + A^{(\mu^1, \mu^2)}(s, x(s), \theta(s); t_f) \right]ds \right.$$ 

$$- [1 - e^{-\beta(t-t_f)}]V(t; x, i; t_f) \right\}$$

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Dividing (7.3.6) by \( \tau - t \), and letting \( \tau \downarrow t \), we get the coupled first order partial differential equations (7.1.1). In general, the value function \( V \) is not differentiable with respect to either \( x \) or \( t \), and hence we would like to explore the connection between the value functions \( \{V(\cdot, \cdot, i, t_f)\}_{i=1}^n \) and the coupled HJB equations (7.1.1) in the viscosity sense. This brings us to the following:

**Theorem 7.3.5.** Under the assumptions (A1)-(A5), for each \( i \), let

\[
V(t; x, i, t_f) = \inf_{\mu \in U} E_{x, i}^{\mu} \left\{ g(x(t_f), \theta(t_f)) e^{-\beta(t_f - t)} \right. \\
+ \left. \int_t^{t_f} e^{-\beta(\tau - t)} L(x(\tau), \theta(\tau), \mu^1(\tau)) d\tau \right\}.
\]

If \( V(\cdot, \cdot, i, t_f) \in C([0, t_f] \times \mathbb{R}^n) \), then \( \overline{V} = \{V(t; x, i, t_f)\}_{i \in S} \) is a viscosity solution of (7.1.1).

**Proof.** Suppose that to the contrary \( \{V(t; x, i, t_f)\}_{i \in S} \) is not a viscosity supersolution of (7.1.1). Then there would exist at least one \( i_0 \in S \), a pair \( (t_0, x_0) \in [0, t_f] \times \mathbb{R}^n \) and a function \( \Phi : [0, t_f] \times \mathbb{R}^n \times S \rightarrow \mathbb{R}^n \) such that \( V(\cdot, \cdot, i_0, t_f) - \Phi(\cdot, \cdot, i_0) \) has a local minimum. But there exists an \( \varepsilon > 0 \) such that

\[
\beta \Phi(t_0, x_0, i_0) + \sup_{u^1, u^2} \left[ -A^{(u^1, u^2)} \Phi(t_0, x_0, i_0) - L(x_0, i_0, u^1) \right] \leq -\varepsilon
\]

(7.3.7)

For any admissible control \( \mu = (\mu^1, \mu^2) \), we consider the following system:

\[
\frac{dx}{dt}(t) = f(x(t), \theta(t)) + B(x(t), \theta(t))\mu^1(t)
\]

\[
x(t_0) = x_0
\]

\[
\theta(t_0) = i_0
\]

Note that \( x(t) \rightarrow x_0 \) as \( t \downarrow t_0 \). Hence for sufficiently small \( t \),

\[
\beta \Phi(t, x(t), \theta(t)) - A^{(u^1, u^2)} \Phi(t, x(t), \theta(t)) - L(x(t), \theta(t), \mu^1) \leq -\varepsilon
\]

(7.3.9)
Multiply both sides of (7.3.9) by $e^{-\beta(t-t_0)}$. Noting that $L \geq 0$, integrating both sides from $t_0$ to $t$, and applying Proposition 7.3.3, we have

\begin{equation}
(7.3.10) \quad \Phi(t_0, x_0, i_0) - E_{x_0, i_0}^\mu \{ e^{-\beta(t-t_0)} \Phi(t, x(t), \theta(t)) \} \leq -\varepsilon(t - t_0)
\end{equation}

Since $(t_0, x_0)$ is a local minimizer of $V(\cdot, i_0; t_f) - \Phi(\cdot, i_0)$,

\begin{equation}
(7.3.11) \quad V(t_0; x_0, i_0; t_f) - V(t; x(t), \theta(t); t_f) \leq \Phi(t_0, x_0, i_0) - \Phi(t, x(t), \theta(t))
\end{equation}

Therefore we arrive at

\begin{equation}
(7.3.12) \quad V(t_0; x_0, i_0; t_f) - E_{x_0, i_0}^\mu \{ e^{-\beta(t-t_0)} V(t; x(t), \theta(t); t_f) \} \leq -\varepsilon(t - t_0)
\end{equation}

which contradicts the statement of Proposition 7.3.1 (where we let $t = t_0$, and $\tau = t$).

For the subsolution case, let $(t_0, y_0) \in [0, t_f] \times \mathbb{R}^n, j_0 \in \mathcal{S}$ and $\psi(\cdot, \cdot, i) \in C^1([0, t_f] \times \mathbb{R}^n)$ be such that $(t_0, y_0)$ is a local maximizer of $V - \psi$, and $V(t_0; y_0, j_0; t_f) - \psi(t_0, y_0, j_0) = 0$.

By Proposition 7.3.1, for any $\mu = (\mu^1, \mu^2) \in \mathcal{U}$ and $t > t_0$,

\begin{equation}
(7.3.13) \quad \psi(t_0, y_0, j_0) = V(t_0; y_0, j_0; t) \leq E_{y_0, j_0}^\mu \left\{ \int_{t_0}^{t} e^{-\beta s} L(x(s), \theta(s), \mu^1(s)) ds + e^{-\beta t} V(t; x(t), \theta(t); t) \right\}
\end{equation}

Observing that \{ $x(t, \theta(t), y_0, \mu^1)$ \} is continuous in $t$, when $t > t_0$ is sufficiently close to $t_0$, by Proposition 7.3.3, (7.3.13) becomes

\begin{equation}
(7.3.14) \quad E_{y_0, j_0}^{\mu^1} \left\{ \frac{e^{-\beta(t-t_0)}}{t-t_0} \int_{t_0}^{t} -\mathcal{A}(\mu^1, \mu^2) \psi(r; x(r, \theta(r), y_0, \mu), \theta(r)) dr \right\}
+ \left( \frac{1 - e^{-\beta(t-t_0)}}{t-t_0} \right) \psi(t_0, y_0, j_0)
\leq E_{y_0, j_0}^{\mu^1} \left\{ \frac{1}{t-t_0} \int_{t_0}^{t} e^{-\beta s} L(x(s, \theta(s), y_0, \mu), \theta(s), \mu(s)) ds \right\}
\end{equation}

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Taking limits when \( t \downarrow t_0 \), observing that the integrands are continuous in \( s \) or \( r \) and 
\[ x(t_0, j_0, y_0, \mu^1) = y_0, \] we obtain

\[(7.3.15) \quad \beta \psi(t_0, y_0, j_0) - A(\mu^1, \mu^2) \psi(t_0, y_0, j_0) - L(y_0, j_0, \mu^1) \leq 0\]

Since \((\mu^1, \mu^2)\) was arbitrary, we have

\[(7.3.16) \quad \beta \psi(t_0, y_0, j_0) + \sup_{\mu^1, \mu^2} \left[ -A(\mu^1, \mu^2) \psi(t_0, y_0, j_0) - L(y_0, j_0, \mu^1) \right] \leq 0,\]

that is to say, \( \bar{V} \) is a viscosity subsolution of (7.1.1) and this completes the proof. \( \square \)

**Remark 7.3.6.** In Theorem 7.3.5, we did not need \( R \) to be strictly positive definite as required by Assumption (A4). (A4) will be needed, however, for uniqueness.

We provide below two sufficient conditions for \( V \) to be continuous.

**Theorem 7.3.7.** When the control space \( \mathcal{U}_1 \) is compact, and Assumptions (A1)-(A6) hold, for each \( i \), we have \( V(\cdot; \cdot; i; t_f) \in C([0, t_f]; \mathbb{R}^n) \).

**Proof.** Suppose that \( x \) and \( y \) are solutions of (7.1.3) with initial conditions \( x(s) = x_0 \) and \( y(s) = y_0 \), respectively. Since \( \mathcal{U}_1 \) is compact, we may assume that there exists a constant \( M_{\mu^1} \) such that \( |\mu^1| \leq M_{\mu^1} \) for any \( \mu^1 \in \mathcal{U}_1 \). By assumptions (A1)-(A5), we have

\[(7.3.17) \quad |x(t) - y(t)| \leq |x_0 - y_0| + (L_f + C_{b_2} M_{\mu^1}) \int_s^t |x(\tau) - y(\tau)| d\tau\]

Using Gronwall's inequality, one can see that, for given \( \varepsilon > 0 \), there is \( \delta > 0 \) such that \( |x_0 - y_0| < \delta \) implies \( |x(t) - y(t)| < \varepsilon \). Noting that \( L \) and \( g \) are locally Lipschitz, it follows from the definition of \( V \) that

\[(7.3.18) \quad |V(s; x_0, i; t_f) - V(s; y_0, i; t_f)\]
\[ \leq \sup_{\mu \in \mathcal{U}} \left| E\left\{ e^{-\beta(t_f-s)}[g(x(t_f), \theta(t), i) - g(y(t_f), \theta(t), i)] + \int_{s}^{t_f} e^{-\beta(\tau-s)}[L(x(\tau), \theta(\tau), \mu^1) - L(y(\tau), \theta(\tau), \mu^1)]d\tau \right\} \right| \]
\[ \leq F(R)|x_0 - y_0|, \quad \text{where} \quad R < \infty \]

for some locally bounded function \( F \). This leads to continuity of \( V \) with respect to the space variable \( x_0 \). Now we show the continuity of \( V \) with respect to the time variable. Let \( 0 \leq s_1 \leq s_2 \leq t_f \), and \( \mu^1_0 \in \mathcal{U}_1 \). By (A1)-(A6), \( L(x, i, \mu^1_0) < \infty \) for all \( x \in \mathcal{X} \) and \( i \in S \).

Consider function \( v : [s_1, t_f] \to \mathcal{U} \) defined by
\[ v(t) = \begin{cases} 
\mu_0 & s_1 \leq t < s_2 \\
\mu^* & s_2 \leq t \leq t_f 
\end{cases} \]
where \( \mu^* \in \mathcal{U} \) is such that
\[ V(s_2; x_0, i; t_f) = J(s_2; x_0, i, \mu^*; t_f). \]

Note that (7.3.19)
\[ V(s_1; x_0, i; t_f) - V(s_2; x_0, i; t_f) \leq E_{x,i}^{\mu_0} \left\{ \int_{s_1}^{s_2} e^{-\beta(\tau-s_1)} L(x_{s_1}^{\mu_0}(\tau), \theta(\tau), \mu^1_0) d\tau \right\} + E_{x,i}^{\mu^*} \left\{ \int_{s_2}^{t_f} e^{-\beta(\tau-s_2)} \left( L(x_{s_2}^{\mu^*}(\tau), \theta(\tau), \mu^*(\tau)) - L(x_{s_2}^{\mu^*}(\tau), \theta(\tau), \mu^*(\tau)) \right) d\tau \right\} + e^{-\beta(t_f-s_1)} E_{x,i}^{\mu^*} \left\{ g(x_{s_1}(t_f), \theta(t_f)) \right\} - e^{-\beta(t_f-s_2)} E \left\{ g(x_{s_2}^{\mu^*}(t_f), \theta(t_f)) \right\} \]

On the other hand, \( x_{s_1}^{v}(t) \) can be viewed as the solution of
\[ \frac{dx_{s_1}^{v}(t)}{dt} = f(x_{s_1}^{v}(t), \theta(t)) + B(x_{s_1}^{v}(t), \theta(t))v(t) \]
\[ x_{s_1}^{v}(s_2) = x_{s_1}^{\mu_0}(s_2) \]

Hence, we have (7.3.20)
\[ |x_{s_1}^{v}(t) - x_{s_2}^{\mu^*}(t)| \leq |x_{s_1}^{\mu_0}(s_2) - x_0| + (L_f + M_{\mu_1}C_{b_2}) \int_{s_2}^{t} |x_{s_1}^{v}(\tau) - x_{s_2}^{\mu^*}(\tau)| d\tau \]

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By assumption (A1)

(7.3.21)

\[ |x^{\mu_0}_{s_1}(s_2) - x_0| \leq \int_{s_1}^{s_2} |f(x^{\mu_0}_{s_1}(\tau), \theta(\tau)) + B(x^{\mu_0}_{s_1}(\tau), \theta(\tau))_{\mu_0} d\tau \]

\[ \leq \int_{s_1}^{s_2} [L_f \| x^{\mu_0}_{s_1}(\tau) - x_0 \| + |f(x_0, \theta(\tau))| + C_{b_1} \| \mu_0 \|] d\tau \]

Thus by Gronwall's inequality, we have

(7.3.22)

\[ |x^{\mu_0}_{s_1}(s_2) - x_0| \leq C e^{s_2 - s_1}(s_2 - s_1) \]

where \( C \) is a constant. Again note that \( g \) and \( L \) are locally Lipschitz; therefore

(7.3.23)

\[ |V(s_1; x_0, i; t_f) - V(s_2; x_0, i; t_f)| \leq C_{x_0} |s_1 - s_2| \]

where \( C_{x_0} \) is independent of \( s_1 \) and \( s_2 \), which completes our proof. \( \square \)

Remark 7.3.8. From the above proof we can see that for each \( i \), \( V(\cdot, \cdot, i) \) is uniformly locally Lipschitz in \( x \). From elementary analysis, we know that given \( R > 0 \), there exists \( K_R < \infty \) such that

\[ |V(t; x, i; t_f) - V(t; y, i; t_f)| \leq K_R |x - y|, \quad \forall (t, x), (t, y) \in [0, t_f] \times \{ x \in \mathbb{R}^n : |x| \leq R \} \]

\[ \| \nabla V(t; \cdot, i; t_f) \|_{L^\infty(\{|x| \leq R\})} \leq K_R \quad \forall t \in [0, t_f]. \]

This fact was used to prove the uniqueness of viscosity solutions for the system considered in [M1] by assuming the control space to be compact. We, however, can not make use of this fact because here the control space is allowed to be unbounded.

Theorem 7.3.9. Let \( f(x, i) = A(i)x, Q(x, i) = x^T Q(i)x \), where \( A(i) \) and \( Q(i) \) are \( n \times n \) matrices, and let \( B, R \) be independent of \( x \), and \( g(x, i) = x^T Q_f(i)x \). Then we have for each \( i \), \( V(\cdot, \cdot, i) \in C([0, t_f]; \mathbb{R}^n) \).
Proof. It is not difficult to see that if $f$, $B$ are linear with respect to $x$, $L(\cdot, i, \cdot)$ is convex jointly in $(x, \mu)$ and $g$ is convex with respect to $x$, then $J(t; \cdot, i, \cdot; tf)$ is convex jointly in $(x, \mu)$ for each $i \in S$. This implies that $V(t; \cdot, i; tf)$ is convex in $\mathbb{R}^n$, and thus is continuous with respect to the space variable $x$ (in fact, it is locally Lipschitz in $x$, for a proof, see [XB3]). Now following the proof of Theorem 7.3.7 by using the fact that $B$ is independent of $x$ (thus the condition on boundedness of $\mathcal{U}_1$ can be removed), we arrive at the desired result. \(\square\)

For the infinite horizon case, an argument similar to that used in the proof of Theorem 7.3.5 leads to the following result, for general case:

**Theorem 7.3.10.** Under assumptions (A1)-(A6), for each $i \in S$, let

$$V(x, i) = \inf_{\mu \in U} E^\mu \mathbb{E}_x \left\{ \int_0^\infty e^{-\beta \tau} L(x(\tau), \theta(\tau), \mu^1(\tau)) d\tau \right\}$$

Then, if $V \neq +\infty$ and $V(\cdot, i) \in C(\mathbb{R}^n)$, $V = \{V(x, i)\}_{i=1}^S$ is a viscosity solution of (7.1.2).

### 7.4. Uniqueness of the Viscosity Solution

In this section, we show that the viscosity solution of (7.1.1) is unique, and when $\beta > 0$ the viscosity solution of (7.1.2) is also unique. For the finite horizon case, suppose that $\bar{V}$, $\bar{W}$ are a viscosity supersolution and a viscosity subsolution, respectively, of (7.1.1) on $Q_{t_f}^\Omega = [0, t_f] \times \Omega$, and introduce $\bar{V}, \bar{W}$ as

$$\bar{V}(t, x) = \{V(t, x, 1), V(t, x, 2), \ldots, V(t, x, s)\}$$

$$\bar{W}(t, x) = \{W(t, x, 1), W(t, x, 2), \ldots, W(t, x, s)\}$$

Furthermore, we assume that

\[(7.4.1)\quad \bar{W} \leq \bar{V} \quad \text{on} \quad (\{t = t_f\} \times \Omega) \cup ([0, t_f] \times \partial \Omega).\]
and adopt the convention that \( \overline{W} \leq \overline{V} \) on \( Q_t^\Omega \) if and only if for each \( i \), \( W(t,x,i) \leq V(t,x,i) \) for any \( (t,x) \in Q_t^\Omega \). \( \Omega \) may be all of \( \mathbb{R}^n \), in which case the boundary of \( \Omega \) (denote by \( \partial \Omega \)) is empty. The statements below are similar in form to the corresponding lemmas in McEneaney [M1], and Crandall and Lions [CrLi 1-2].

**Lemma 7.4.1.** Let \( V, W \) be as above and assume that (A1)-(A8) hold. Let \( R < \infty \) and introduce a function \( \Lambda \in C^1(Q_t^\Omega) \), such that \( \Lambda \geq 0, \Lambda(t,\cdot) = 0 \) if \( |x| \geq R \), and

\[
-\Lambda + \beta \Lambda > 0 \quad \text{on} \quad (\text{supp}\Lambda)^o \cap (Q_t^\Omega)^o.
\]

where the superscript \( o \) indicates interior. Then \( \overline{W} \geq \overline{V} \) on \((\text{supp}\Lambda) \cap Q_t^\Omega\).

**Proof.** Suppose that

\[
M_i^i = \Lambda(t_0,x_0)[W(t_0,x_0,i_0) - V(t_0,x_0,i_0)]
= \max_{Q_t^\Omega} \Lambda(t,x)[W(t,x,i) - V(t,x,i)] > 0,
\]

since otherwise result has already been established.

Let a function \( \Phi^{\epsilon,\delta} : [0,t_f] \times \Omega \times [0,t_f] \times \Omega \times I_K \rightarrow \mathbb{R}^n \) be

\[
\Phi^{\epsilon,\delta}(t,x,s,y,i) = \Lambda(s,y)W(t,x,i) - \Lambda(t,x)V(s,y,i) - \frac{1}{2\epsilon}|x-y|^2 - \frac{1}{2\delta}|t-s|^2
\]

Since \( \Phi^{\epsilon,\delta} \) is upper semicontinuous, \( \Lambda \) has a compact support and \( S \) is a finite set, there exist \( (t_\delta,x_\delta,s_\delta,y_\delta,k_{\epsilon,\delta}) \in Q_t^\Omega \times Q_t^\Omega \times S \) such that

\[
\Phi^{\epsilon,\delta}(t_\delta,x_\delta,s_\delta,y_\delta,k_{\epsilon,\delta}) = \max_{Q_t^\Omega \times Q_t^\Omega \times S} \Phi^{\epsilon,\delta}(t,x,s,y,i)
\]

We prove (7.4.2) in several steps.
1. In this step, we establish

\[(7.4.6) \quad \frac{1}{2\varepsilon} |x_\varepsilon - y_\varepsilon|^2 \to 0 \quad \text{as} \quad \varepsilon \downarrow 0, \quad \frac{1}{2\delta} |t_\delta - s_\delta|^2 \to 0 \quad \text{as} \quad \delta \downarrow 0\]

Let \( M^{\varepsilon, \delta} = \Phi^{\varepsilon, \delta}(t_\delta, x_\varepsilon, s_\delta, y_\varepsilon, k_\varepsilon, \delta), \) consider \( 0 < \varepsilon_2 \leq \varepsilon_1 \) and \( 0 < \delta_2 \leq \delta_1, \) then

\[
M^{\varepsilon_1, \delta_1} - \left( \frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1} \right) \left| \frac{x_{\varepsilon_2} - y_{\varepsilon_2}}{2} \right|^2 - \left( \frac{1}{\delta_2} - \frac{1}{\delta_1} \right) \left| \frac{t_{\delta_2} - s_{\delta_2}}{2} \right|^2 \\
\geq \Phi^{\varepsilon_1, \delta_1}(t_{\delta_2}, x_{\varepsilon_2}, s_{\delta_2}, y_{\varepsilon_2}, k_{\varepsilon_2}, \delta_2) - \left( \frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1} \right) \left| \frac{x_{\varepsilon_2} - y_{\varepsilon_2}}{2} \right|^2 - \left( \frac{1}{\delta_2} - \frac{1}{\delta_1} \right) \left| \frac{t_{\delta_2} - s_{\delta_2}}{2} \right|^2 \\
= \Lambda(s_{\delta_2}, y_{\varepsilon_2})W(t_{\delta_2}, x_{\varepsilon_2}, i) - \Lambda(t_{\delta_2}, x_{\varepsilon_2})V(s_{\delta_2}, y_{\varepsilon_2}, i) - \frac{1}{2\varepsilon_1} |x_{\varepsilon_2} - y_{\varepsilon_2}|^2 \\
- \frac{1}{2\delta_2} |t_{\delta_2} - s_{\delta_2}|^2 - \left( \frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1} \right) \left| \frac{x_{\varepsilon_2} - y_{\varepsilon_2}}{2} \right|^2 - \left( \frac{1}{\delta_2} - \frac{1}{\delta_1} \right) \left| \frac{t_{\delta_2} - s_{\delta_2}}{2} \right|^2 \\
= \Phi^{\varepsilon_1, \delta_1}(t_{\delta_2}, x_{\varepsilon_2}, s_{\delta_2}, y_{\varepsilon_2}, k_{\varepsilon_2}, \delta_2) = M^{\varepsilon_2, \delta_2}
\]

Hence, we can see that \((\varepsilon, \delta) \mapsto M^{\varepsilon, \delta}\) is nondecreasing. Let \( \varepsilon_1 = 2\varepsilon, \varepsilon_2 = \varepsilon \) and \( \delta_1 = \delta_2 = \delta; \) then

\[(7.4.7) \quad M^{2\varepsilon, \delta} - M^{\varepsilon, \delta} \geq \frac{1}{2\varepsilon} \frac{|x_\varepsilon - y_\varepsilon|^2}{2}\]

Note that \( M^{2\varepsilon, \delta} - M^{\varepsilon, \delta} \to 0 \) as \( \varepsilon \downarrow 0. \) Thus the first part of \((7.4.6)\) holds. For the second part, the proof is similar.

2. Since \( \Lambda \) has compact support and \((7.4.6)\) holds, there exist sequences \( \{\varepsilon_n\} \) and \( \{\delta_m\} \)
which converge to zero such that

\[x_{\varepsilon_n} \to \hat{x}, y_{\varepsilon_n} \to \hat{x}, \quad \text{as} \quad n \to \infty, \quad t_{\delta_m} \to \hat{t}, s_{\delta_m} \to \hat{t} \quad \text{as} \quad m \to \infty\]

where \((\hat{t}, \hat{x}) \in Q^\Omega_{t_0}. \) In fact it is easy to see that \( \hat{x} = x_0, \) \( \hat{t} = t_0. \) Note that by our assumption \((t_0, x_0) \in (\text{supp}\Lambda)^0 \cap (Q^\Omega_{t_0})^0. \) Therefore, for sufficiently large \( n \) and \( m, \) we have \((t_{\delta_m}, x_{\varepsilon_n}), (s_{\delta_m}, y_{\varepsilon_n}) \in (\text{supp}\Lambda)^0 \cap (Q^\Omega_{t_0})^0. \) By noting that \( S \) is a finite set, without loss of generality, we may assume \( k_{\varepsilon_n, \delta_m} = k_0. \)

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3. Since

\[ W(t, x, k_0) - \frac{1}{\Lambda(s_{\delta_m}, y_{\varepsilon_n})} \left[ \Lambda(t, x)V(t_{\delta_m}, x_{\varepsilon_n}, k_0) + \frac{1}{2\varepsilon_n} |x - y_{\varepsilon_n}|^2 + \frac{1}{2\delta_m} |t - s_{\delta_m}|^2 \right] \]

attains its maximum at \((t, x) = (t_{\delta_m}, x_{\varepsilon_n})\), by the definition of viscosity subsolution, we have

\[ \Lambda(t_{\varepsilon_n}, x_{\varepsilon_n})V(t_{\delta_m}, x_{\varepsilon_n}, k_0) + \frac{(t_{\delta_m} - s_{\delta_m})}{\delta_m} \]

\[ + \beta W(t_{\delta_m}, x_{\varepsilon_n}) - \inf_{u \in \mathcal{U}} \left\{ \sum_{j \in \mathcal{S}} \lambda_{k_0, a_j} W(t_{\delta_m}, x_{\varepsilon_n}, j) \right\} \]

\[ + H_i \left( x_{\varepsilon_n}, \frac{\Lambda_x(t_{\delta_m}, x_{\varepsilon_n})V(t_{\delta_m}, x_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(s_{\delta_m}, y_{\varepsilon_n})} \right) \leq 0 \]

Similarly,

\[ V(s, y, k_0) - \frac{1}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})} \left[ \Lambda(s, y)V(t_{\delta_m}, x_{\varepsilon_n}, k_0) - \frac{1}{2\varepsilon_n} |x_{\varepsilon_n} - y|^2 - \frac{1}{2\delta_m} |t_{\delta_m} - s|^2 \right] \]

has a minimum at \((s_{\delta_m}, y_{\varepsilon_n})\). Note that \(W(\cdot, \cdot, k_0)\) is a supersolution, which results in

\[ \Lambda(s_{\delta_m}, y_{\varepsilon_n})W(t_{\delta_m}, x_{\varepsilon_n}, k_0) + \frac{(t_{\delta_m} - s_{\delta_m})}{\delta_m} \]

\[ + \beta V(s_{\delta_m}, y_{\varepsilon_n}) - \inf_{u \in \mathcal{U}} \left\{ \sum_{j \in \mathcal{S}} \lambda_{k_0, a_j} V(s_{\delta_m}, y_{\varepsilon_n}, j) \right\} \]

\[ + H_i \left( y_{\varepsilon_n}, \frac{\Lambda_y(s_{\delta_m}, y_{\varepsilon_n})W(t_{\delta_m}, x_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})} \right) \geq 0 \]

Fix \(\varepsilon_n\) and let \(m \to \infty\); (7.4.9) and (7.4.11) imply that the sequence \(\{(t_{\delta_m} - s_{\delta_m})/\delta_m\}\) is bounded, thus there exists a converging subsequence, which we still denote by \(\{(t_{\delta_m} - s_{\delta_m})/\delta_m\}\). By (7.4.9) and assumption \((A8)\), we have that \((x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n\) is also bounded.

Note that

\[ H \left( x_{\varepsilon_n}, \frac{\Lambda_x(t_{\delta_m}, x_{\varepsilon_n})V(t_{\delta_m}, x_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(s_{\delta_m}, y_{\varepsilon_n})} \right) \]
\[-H\left(\varepsilon_n, \frac{\Lambda y(s_{\delta_m}, y_{\varepsilon_n})W(t_{\delta_m}, x_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(t_{\delta_m}, x_{\varepsilon_n})}\right)\]
\[\leq C\left\{L_f + C_{b2}|R^{-1}(x_{\varepsilon_n})B^T(x_{\varepsilon_n})| + L_R|B(x_{\varepsilon_n})||B^T(y_{\varepsilon_n})|\right\}\]
\[+ C_{b2}|R^{-1}(y_{\varepsilon_n})B^T(y_{\varepsilon_n})|\left\{\frac{|x_{\varepsilon_n} - y_{\varepsilon_n}|^2}{\varepsilon_n} + L_Q(R)|x_{\varepsilon_n} - y_{\varepsilon_n}|\right\}\]

Hence let \(m \to \infty\) in both (7.4.9) and (7.4.11), and substract (7.4.11) from (7.4.9), and let
\(n \to \infty\). Thus this yields
(7.4.13)
\[-\frac{\Lambda(t_0, x_0)[W(t_0, x_0, k_0) - V(t_0, x_0, k_0)]}{\Lambda(t_0, x_0)} + \beta[W(t_0, x_0, k_0) - V(t_0, x_0, k_0)]\]
\[+ \inf_{u_2} \left\{ \sum_{j \in S} \lambda_{k_0} a_j V(t_0, x_0, j) \right\} - \inf_{u_2} \left\{ \sum_{j \in S} \lambda_{k_0} a_j W(t_0, x_0, j) \right\} \leq 0\]

4. Here we establish the inequality
(7.4.14)
\[\inf_{u_2} \left\{ \sum_{j \in S} \lambda_{k_0} a_j W(t_0, x_0, j) \right\} - \inf_{u_2} \left\{ \sum_{j \in S} \lambda_{k_0} a_j V(t_0, x_0, j) \right\} \leq 0\]

Note that by (7.4.3), we have
(7.4.15)
\[\Lambda(t_0, x_0)[W(t_0, x_0, i) - V(t_0, x_0, i)] \leq \Lambda(t_0, x_0)[W(t_0, x_0, k_0) - V(t_0, x_0, k_0)] \quad \forall i \in S\]

By using assumption (A5), and (7.4.15), we have
(7.4.16)
\[\inf_{u_2} \left\{ \sum_{j \in S} \lambda_{k_0} a_j W(t_0, x_0, j) \right\} - \inf_{u_2} \left\{ \sum_{j \in S} \lambda_{k_0} a_j V(t_0, x_0, j) \right\}\]
\[= \inf_{u_2} \left\{ \sum_{j \neq k_0} \lambda_{k_0} a_j W(t_0, x_0, j) + \lambda_{k_0} a_{k_0} W(t_0, x_0, k_0) \right\} - \inf_{u_2} \left\{ \sum_{j \neq k_0} \lambda_{k_0} a_j V(t_0, x_0, j) + \lambda_{k_0} a_{k_0} V(t_0, x_0, k_0) \right\}\]
\[= \inf_{u_2} \left\{ \sum_{j \neq k_0} \lambda_{k_0} a_j [W(t_0, x_0, j) - W(t_0, x_0, k_0)] \right\} - \inf_{u_2} \left\{ \sum_{j \neq k_0} \lambda_{k_0} a_j [V(t_0, x_0, j) - V(t_0, x_0, k_0)] \right\}\]
\[\leq \sup_{u_2} \left\{ \sum_{j \neq k_0} \lambda_{k_0} a_j [W(t_0, x_0, j) - W(t_0, x_0, k_0) - V(t_0, x_0, j) + V(t_0, x_0, k_0)] \right\}\]
\[\leq 0\]
5. (7.4.3), (7.4.13) and (7.4.16) yield that

\[[7.4.17] \quad -\Lambda(t, x_0) + \beta \Lambda(t_0, x_0) \leq 0\]

which contradicts the assumption of the lemma. Therefore

\[[7.4.18] \quad \max_{Q_{t_j}^\Omega \times Q_{t_j}^\Omega} \Lambda(t, x)[W(t, x, i) - V(t, x, i)] \leq 0 \quad \forall i \in S\]

and this completes the proof of Lemma 7.4.1. \(\Box\)

Now we are ready to state the following comparison theorem:

**Theorem 7.4.2.** Under Assumptions (A1)-(A7), if (7.4.1) holds, then we have

\[[7.4.19] \quad \overline{W} \leq \overline{V} \quad \text{on} \quad Q_{t_j}^\Omega\]

**Proof.** We are interested in finding \(\Lambda\) such that the conditions of Lemma 7.4.1 will be satisfied. A natural choice for the function \(\Lambda\) is:

\[
\Lambda(t, x) = \begin{cases} 
\exp\left(\frac{R^2}{|x|^2 - R^2} + (\beta - 1)t\right), & |x| < R \\
0, & |x| \geq R
\end{cases}
\]

Suppose that there were \((t_0, x_0, i_0) \in Q_{t_j}^\Omega\) such that

\[[7.4.20] \quad W(t_0, x_0, i_0) > V(t_0, x_0, i_0)\]

Let \(R > |x_0|\), and \(\Lambda\) as above. Clearly, (7.4.2) is satisfied under the choice of the \(\Lambda\). Applying Lemma 7.4.1, we know that (7.4.20) could not hold. Therefore (7.4.19) must be true. \(\Box\)

Under our assumptions, the comparison theorem leads to the uniqueness of viscosity solution of (7.1.1).
Corollary 7.4.3. Let \( \overline{V}, \overline{W} \) be two viscosity solutions of (7.1.1) with boundary and terminal conditions

(7.4.21) \[ V(t, x, i) = W(t, x, i) = \varphi(t, x, i) \quad \text{on} \quad [0, t_f] \times \partial \Omega \]

(7.4.22) \[ V(t_f, x, i) = W(t_f, x, i) = g(x, i) \quad \text{on} \quad \Omega \]

Under Assumptions (A1)-(A7), we have

(7.4.23) \[ \overline{V} = \overline{W} \quad \text{on} \quad [0, t_f] \times \Omega \]

Remark 7.4.4. In the case \( \Omega = \mathbb{R}^n \), the boundary condition (7.4.21) is no longer there, and we only consider the terminal condition (7.4.22).

For the infinite horizon case, when \( \beta > 0 \), we have a similar result:

Theorem 7.4.5. Suppose that \( \beta > 0 \). Assume that both \( \overline{V}_1 \) and \( \overline{V}_2 \) are viscosity solutions of (7.1.2). If

(1) \( \Omega = \mathbb{R}^n \), or

(2) \( V_1(x, i) = V_2(x, i) = g(x, i) \) on \( \partial \Omega \).

then \( \overline{V}_1 = \overline{V}_2 \) on \( \Omega \).

Proof. Since the proof is similar to the finite-horizon case, we only give an outline of the proof of part (1). Let function \( \Lambda \) be defined as

\[ \Lambda(x) = \begin{cases} \exp \left\{ \frac{R^2}{|x|^2 - R^2} \right\}, & |x| < R \\ 0, & |x| \geq R \end{cases} \]

Assume that there exist \( x_0 \in \mathbb{R}^n \) and \( i_0 \in S \) such that \( V_1(x_0, i_0) - V_2(x_0, i_0) > 0 \). Let \( R \) be sufficiently large so that \( R > |x_0| \). Let a function \( \Phi^\varepsilon : \Omega \times \Omega \times I_K \to \mathbb{R}^n \) be

(7.4.24) \[ \Phi^\varepsilon(x, y, i) = \Lambda(y) V_1(x, i) - \Lambda(x) V_2(y, i) - \frac{1}{2\varepsilon} |x - y|^2 \]
Since $A$ has a compact support and $S$ is a finite set, there exist $(x_\varepsilon, y_\varepsilon, k_\varepsilon) \in \mathbb{R}^n \times \mathbb{R}^n \times S$ such that

\begin{equation}
\Phi(x_\varepsilon, y_\varepsilon, k_\varepsilon) = \max_{\mathbb{R}^n \times \mathbb{R}^n \times S} \Phi(x, y, i)
\end{equation}

1. As in Lemma 7.4.1, we have

\begin{equation}
\frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2 \to 0 \quad \text{as} \quad \varepsilon \downarrow 0.
\end{equation}

2. Since $\{x_\varepsilon\}, \{y_\varepsilon\} \in supp(A)$, there exists a sequence $\varepsilon_n \downarrow 0$ such that

\begin{equation}
x_{\varepsilon_n} \to x_0, \quad y_{\varepsilon_n} \to x_0
\end{equation}

3. Since

\begin{equation}
V_1(x, k_0) - \frac{1}{\Lambda(y_{\varepsilon_n})} \left[ \Lambda(x)V_2(y_{\varepsilon_n}, k_0) + \frac{1}{2\varepsilon_n}|x - y_{\varepsilon_n}|^2 \right]
\end{equation}

attains its maximum at $x = x_{\varepsilon_n}$, by the definition of viscosity subsolution, we have

\begin{equation}
\beta V_1(x_{\varepsilon_n}, k_0) - \inf_{u^2} \left\{ \sum_{j \in S} \lambda_{k_0, a_j} V_1(x_{\varepsilon_n}, j) \right\}
\end{equation}

\[ + H_i \left( x_{\varepsilon_n}, \frac{\Lambda'(x_{\varepsilon_n})V_2(x_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(y_{\varepsilon_n})} \right) \leq 0 \]

Similarly,

\begin{equation}
V_2(y, k_0) - \frac{1}{\Lambda(x_{\varepsilon_n})} \left[ \Lambda(y)V_1(x_{\varepsilon_n}, k_0) - \frac{1}{2\varepsilon_n}|x_{\varepsilon_n} - y|^2 \right]
\end{equation}

has a minimum at $(y_{\varepsilon_n})$. Thus

\begin{equation}
\beta V_2(y_{\varepsilon_n}, k_0) - \inf_{u^2} \left\{ \sum_{j \in S} \lambda_{k_0, a_j} V_2(y_{\varepsilon_n}, j) \right\}
\end{equation}

\[ + H_i \left( y_{\varepsilon_n}, \frac{\Lambda'(y_{\varepsilon_n})V_1(y_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(x_{\varepsilon_n})} \right) \geq 0 \]
Inequality (7.4.29) and assumption (A8) imply that the sequence \( \{(x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n\} \) is bounded. Note that

\[
H_i\left(x_{\varepsilon_n}, \frac{\Lambda'(x_{\varepsilon_n})V_2(x_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(y_{\varepsilon_n})}\right)
- H\left(y_{\varepsilon_n}, \frac{\Lambda'(y_{\varepsilon_n})V_1(y_{\varepsilon_n}, k_0) + (x_{\varepsilon_n} - y_{\varepsilon_n})/\varepsilon_n}{\Lambda(x_{\varepsilon_n})}\right)
\leq C\left[L_f + C_{b2}|R^{-1}(x_{\varepsilon_n})B^T(x_{\varepsilon_n}) + L_R|B(x_{\varepsilon_n})|^2 + C_{b2}|R^{-1}(y_{\varepsilon_n})B^T(y_{\varepsilon_n})|\right] \frac{|x_{\varepsilon_n} - y_{\varepsilon_n}|^2}{\varepsilon_n} + L_Q(R)|x_{\varepsilon_n} - y_{\varepsilon_n}|
\]

Subtracting (7.4.31) from (7.4.29), and letting \( n \to \infty \), yields

\[
\beta(V_1(x_0, k_0) - V_2(x_0, k_0)) \leq \inf_{u^2} \left\{ \sum_{j \in \mathcal{S}} \lambda_{k_0a_j} V_1(x_0, j) \right\} - \inf_{u^2} \left\{ \sum_{j \in \mathcal{S}} \lambda_{k_0a_j} V_2(y_0, j) \right\}
\]

4. Following the same lines as in proof of the Lemma 7.4.1,

\[
\inf_{u^2} \left\{ \sum_{j \in \mathcal{S}} \lambda_{k_0a_j} V_1(x_0, j) \right\} - \inf_{u^2} \left\{ \sum_{j \in \mathcal{S}} \lambda_{k_0a_j} V_2(y_0, j) \right\} \leq 0
\]

5. In view of the assumption \( V_1(x_0, i_0) - V_2(x_0, i_0) > 0 \), and (7.4.25), we have

\[
\beta(V_1(x_0, i_0) - V_2(x_0, i_0)) \leq 0,
\]

which contradicts the hypothesis that \( \beta > 0 \). Therefore we arrive at

\[
V_1(x, i) \leq V_2(x, i), \quad \forall x \in \mathbb{R}^n, \forall i \in \mathcal{S}
\]

The reverse inequality can be shown similarly. \( \square \)

7.5. Optimal Control Law

We discuss in this section derivation of the optimal control law when the value function is only continuous with respect to \( x \) or \( t \). We first introduce the notions of a superdifferential and a subdifferential of a continuous function.
Definition 7.5.1. Let \( V \in C([0,t_f] \times \mathbb{R}^n) \) and \( (t,x) \in [0,t_f] \times \mathbb{R}^n \). Then:

(a) The superdifferential, \( D^+ V(t,x) \), of \( V \) at \( (t,x) \) is

\[
D^+ V(t,x) = \left\{ (q,p) \in \mathbb{R}^{n+1} : \lim_{(s,y) \to (t,x)} \sup \frac{V(s,y) - V(t,x) - q(s-t) - p \cdot (y-x)}{|s-t| + |x-y|} \leq 0 \right\}
\]

(b) The subdifferential, \( D^- V(t,x) \), of \( V \) at \( (t,x) \) is

\[
D^- V(t,x) = \left\{ (q,p) \in \mathbb{R}^{n+1} : \lim_{(s,y) \to (t,x)} \sup \frac{V(s,y) - V(t,x) - q(s-t) - p \cdot (y-x)}{|s-t| + |x-y|} \geq 0 \right\}
\]

Remark 7.5.2. When \( V \) is differentiable at \( (t,x) \), we have

\[
D^+ V(t,x) = D^- V(t,x) = \left\{ \left( \frac{\partial}{\partial t} V(t,x), \nabla_x V(t,x) \right) \right\}
\]

If \( V \) is only differentiable with respect to \( t \), then

\[
D^+ V(t,x) = \left\{ \left( \frac{\partial}{\partial t} V(t,x), \bullet \right) \right\}, \quad D^- V(t,x) = \left\{ \left( \frac{\partial}{\partial t} V(t,x), \bullet \right) \right\}.
\]

Lemma 7.5.3. \( \{ V(\cdot, i; t_f) \}_{i=1}^S \) is a viscosity subsolution of (7.1.1) in \([0,t_f] \times \mathbb{R}^n\) if and only if for each \( i \in S \)

\[
- q + \beta V(t,x,i; t_f) - \inf_{\mu^2} \left\{ \sum_{j \in S} \lambda_{ij} V(t,x,j; t_f) \right\} + H_i(x,p) \leq 0
\]

\( \forall (q,p) \in D^+ V(t,x,i; t_f), (t,x) \in [0,t_f] \times \mathbb{R}^n \), where \( H_i \) is defined by (7.1.9).

Similarly \( \{ V(\cdot, i; t_f) \}_{i=1}^S \) is a viscosity supersolution of (7.1.1) in \([0,t_f] \times \mathbb{R}^n\) if and only if

\[
- q + \beta V(t,x,i; t_f) - \inf_{\mu^2} \left\{ \sum_{j \in S} \lambda_{ij} V(t,x,j; t_f) \right\} + H_i(x,p) \geq 0
\]
\( \forall (q, p) \in D^- V(t; x, i; t_f), (t, x) \in [0, t_f] \times \mathbb{R}^n. \)

**Proof.** We only prove the first part, as the proof of the second part is similar.

Suppose that (7.5.3) holds. Let \( \varphi(\cdot, \cdot, i) \subset C^1([0, t_f] \times \mathbb{R}^n) \) be such that \((t_0, x_0)\) is a local maximizer of \( V(\cdot, \cdot, i; t_f) - \varphi(\cdot, \cdot, i) \) for some \( i \) with \( V(t_0; x_0, k; t_f) = \varphi(t_0, x_0, k), k = 1, \ldots, s. \)

Since \( \varphi(\cdot, \cdot, i) \in C^1([0, t_f] \times \mathbb{R}^n) \), it yields

\[
\varphi(t, x, i) = \varphi(t_0, x_0, i) + \varphi_t(t_0, x_0, i)(t - t_0) + \varphi_x(t_0, x_0, i)(x - x_0) \\
+ o(|t - t_0|) + o(|x - x_0|).
\]

Hence for \((t, x)\) sufficiently close to \((t_0, x_0)\),

\[
V(t; x, i; t_f) \leq \varphi(t_0, x_0, i) + \varphi_t(t_0, x_0, i)(t - t_0) + \varphi_x(t_0, x_0, i)(x - x_0) \\
+ o(|t - t_0|) + o(|x - x_0|)
\]

Let \( p = \varphi_x(t_0, x_0, i), q = \varphi_t(t_0, x_0, i) \). Then, (7.5.3) implies that

(7.5.5)

\[-\varphi_t(t_0, x_0, i) + \beta \varphi (t_0, x_0, i) - \inf_{\mu^2} \{ \sum_{j \in S} \lambda_{iaj} \varphi(t_0, x_0, j) \} + H_i(x_0, \varphi_x(t_0, x_0, i)) \leq 0 \]

Thus \( \{V(\cdot, \cdot, i)\}_{i=1}^{s} \) is a viscosity subsolution of (7.1.1).

Conversely, let \((q, p) \in D^+ V(t; x, i; t_f)\). When \((s, y)\) is sufficiently close to \((t, x)\), we have

(7.5.6)

\[
V(s; y, i; t_f) \leq V(t; x, i; t_f) + q(s - t) + p(y - x) + o(|s - t|) + o(|y - x|)
\]

Introduce the test function

(7.5.7)

\[
\varphi(s, y, i) = V(t; x, i; t_f) + q(s - t) + p \cdot (y - x) + g_1(|s - t|) + g_2(|x - y|)
\]

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where \( g_i : (0, \infty) \to (0, \infty), i = 1, 2 \) are nondecreasing functions such that \( g_i(r) = o(r) \), \( Dg_i(r) \big|_{r=0} = 0 \) (For construction of such functions, see [Cr1] or [FS]). Hence by such choice of \( g_1, g_2 \), we can see that in fact \( (t, x) \) is a strict maximizer of \( V(\cdot, \cdot, i) - \varphi(\cdot, \cdot, i) \), and \( \varphi_x(t, x, i) = p, \varphi_t(t, x, i) = q \), as a result of which (7.5.3) holds by the definition of viscosity subsolution. \( \square \)

**Definition 7.5.4.** An **admissible feedback controller** \( u^1(t) = \mu^1(t, x) \) for system (7.1.3) is a (nonlinear, multivalued) mapping \( F : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^r \) such that \( \mu^1(t, x) = F(t, x) \) and the following differential inclusion

\[
\frac{dx}{dt}(t) \in f(x(t), \theta(t)) + B(x(t), \theta(t))F(t, x(t))
\]

\[ x(0) = x_0 \]

\[ \theta(0) = i_0 \]

has at least one solution. Solution to differential inclusions (7.5.8) are understood in the Carathéodory sense, i.e., absolutely continuous functions verifying (7.5.8) almost everywhere.

**Theorem 7.5.5.** Assume that (7.1.1) has a viscosity solution. For each \( i \) \((0 \leq i \leq s)\), let \( \{v_n(\cdot, \cdot, i)\} \subset C^{\infty}_b(\mathbb{R}^{n+1}) \) be such that \( v_n(t, x, i) \) uniformly converges to \( V(t; x, i; t_f) \) on each compact subset of \( \mathbb{R}^n \). Suppose that:

1. there exists a set \( Fx \subset \{p(t, x, i) : (\cdot, p) \in D^-V(t; x, i; t_f)\} \) such that

\[
\mu^1(x, i) = -R^{-1}(x, i)B^T(x, i)Fx
\]

is an admissible feedback controller for system (7.1.3);
For $p \in Fx$, we have:

(7.5.9)

$$\inf_n \left[ \int_0^{t_f} \left< p(t, x, i) - \frac{\partial}{\partial x} v_n(t, x, i), \dot{x}(t, i, \mu^1) \right> \right] dt \geq 0 \quad \forall i \in S$$

Then $\mu^1$ is an optimal control for system (7.1.3) under cost function (7.1.5).

**Proof.** Note that

(7.5.10)

$$V(t; x, i; t_f) = \inf_{\mu \in \mathcal{U}} E_{x, i}^\mu \left\{ g(x(t_f), \theta(t_f)) e^{-\beta(t_f-t)} \right. + \int_t^{t_f} e^{-\beta(\tau-t)} L(x(\tau), \theta(\tau), \mu^1(\tau)) d\tau \}
$$

and the right hand side of (7.5.10) is absolutely continuous with respect to $t$ for each fixed $x$. Thus $V$ is almost differentiable with respect to $t$ on $[0, t_f]$. According to Remark 7.5.2 and Lemma 7.5.3, we have

(7.5.11)

$$-\frac{\partial}{\partial t} V(t; x, i; t_f) + \beta V(t; x, i; t_f) - \inf_{\mu^2} \left\{ \sum_{j \in S} \lambda_{iaj} V(t; x, j; t_f) \right\} + H_i(x, p) \geq 0
$$

\text{a.e. on } [0, t_f]

\forall (\cdot, p) \in D^- V(t; x, i; t_f), (t, x) \in [0, t_f] \times \mathbb{R}^n$. Let $u^1$ be an admissible control of (1.3).

Then, there exists $R > 0$ such that $\max\{ |x(x_0, t, i, u^1)| : t \in [0, t_f], i \in S \} \leq R$. Let

(7.5.12)

$$B_R := \{ x \in \mathbb{R}^n ; |x| \leq R \}
$$

Given any $N > 0$ there is $\hat{N}$ such that for $n > \hat{N}$

(7.5.13)

$$|v_n(t, x, i) - V(t; x, i; t_f)| < \frac{1}{N}, \quad \forall i \in S \quad (t, x) \in [0, t_f] \times B_R
$$

By (7.5.11) and assumption (A5), we have

(7.5.14)

$$0 \leq -\frac{\partial}{\partial t} V(t; x, i; t_f) + \beta V(t; x, i; t_f) - \inf_{\mu^2} \left\{ \sum_{j \in S} \lambda_{iaj} V(t; x, j; t_f) \right\} + H_i(x, p)$$

$$\leq -\frac{\partial}{\partial t} V(t; x, i; t_f) + \beta v_n(t, x, i) + \frac{\beta}{N} - \inf_{\mu^2} \left\{ \sum_{j \in S} \lambda_{iaj} v_n(t, x, j) \right\} + H_i(x, p)
$$

$$\leq -\frac{\partial}{\partial t} V(t; x, i; t_f) + \beta v_n(t, x, i) + \frac{\beta}{N} - \inf_{\mu^2} \left\{ \sum_{j \in S} \lambda_{iaj} v_n(t, x, j) \right\} + H_i(x, p)$$

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except on a subset of \([0, t_f]\) of measure zero. According to the definition of \(H_i\), (7.5.14) can be written as

\[
(7.5.15) \quad -\frac{\partial}{\partial t}V(t; x, i; t_f) \geq L(x(t, i, u^1), u^1) - |R^{\frac{1}{2}}(x, i)u + R^{-\frac{1}{2}}B^*(x, i)p(t, x, i)|^2 - \beta v_n(t, x, i) + \inf_{\mu^2} \left\{ \sum_{j \in S} \lambda_{iaj}v_n(t, x, j) \right\} + \langle p(t, x, i), \dot{x}(t, x, u^1) \rangle - \frac{\beta}{N}
\]

According to Proposition 7.3.3, we know that

\[
E_{x, i}e^{-\beta t_f}v_n(t_f, x(t_f), \theta(t_f)) - v_n(0, x, i) = \int_0^{t_f} A^{(\mu^1, \mu^2)}e^{-\beta \tau}v_n(\tau, x(\tau), \theta(\tau))d\tau
\]

Integrate (7.5.15) from 0 to \(t_f\), and let \(N \to \infty\). Then on each compact subset of \(\mathbb{R}^n\), noting assumption (7.5.9), (7.5.15) becomes

\[
(7.5.16) \quad V(0; x, i; t_f) \geq E_{x, i}^{\mu^1} \left\{ g(x, \theta(t))e^{-\beta t_f} + \int_0^{t_f} \left[ e^{-\beta \tau}L(x(t, \theta(t), u^1(t)), \theta(t), u^1(t)) - |R^{\frac{1}{2}}(x, \theta(t))u^1(t) + R^{-\frac{1}{2}}B^*(x, \theta(t))p(t, x, \theta(t))|^2 \right] dt \right\}
\]

Proposition 7.3.1 implies that when

\[
(7.5.17) \quad \mu^{1*}(t, x, i) = -R^{-1}(x, i)B^*(x, i)p(t, x, i)
\]

we have

\[
V(0; x, i; t_f) = E_{x, i}^{\mu^{1*}} \left\{ g(x, \theta(t))e^{-\beta t_f} + \int_0^{t_f} \left[ e^{-\beta \tau}L(x(t, \theta(t), \mu^{1*}(t)), \theta(t), \mu^{1*}(t)) \right] dt \right\} = \inf_{\mu \in \mathcal{U}} E_{x, i}^{\mu^1} \left\{ g(x, \theta(t))e^{-\beta t_f} + \int_0^{t_f} \left[ e^{-\beta \tau}L(x(t, \theta(t), \mu^1(t)), \theta(t), \mu^1(t)) \right] dt \right\}
\]

and this completes the proof of Theorem 7.5.5. □
Remark 7.5.6. When \( V(\cdot, i; t_f) \) is differentiable with respect to \( x \), condition (7.5.9) is automatically satisfied. When \( V(\cdot, i; t_f) \) is convex in \( x \), one can show that

\[
\mu_1(t, x, i) = -R^{-1}(x, i)B^T(x, i)(p(t, x, i); (\cdot, p) \in D^- V(t; x, i; t_f))
\]

is an optimal admissible feedback controller, for more detail, c.f. [XB3].

Remark 7.5.7. Under the assumption of Theorem 7.5.5, it is not difficult to see that an optimal second controller exists, and is given by

\[
\mu_2^*(t, x, i) = \arg \min_{a \in U^2} \left\{ \sum_{j \in S} \lambda_{ij} V(t; x, j; t_f) \right\},
\]

where the minimum exists because the set \( U^2 \) is finite and \( x \) belongs to a compact subset of \( \mathbb{R}^n \).

For the infinite horizon case, we introduce the definition of Stochastic Stabilizability of (7.1.3).

Definition 7.5.8. We say that the system (7.1.3) is \( \beta \)-stochastically stabilizable if, for all finite \( x_0 \in \mathbb{R}^n \) and \( i_0 \in S \) with \( x(0) = x_0, \theta(0) = i_0 \), there exists an admissible feedback control \( \mu(t, x(t), \theta(t)) \) such that

\[
\lim_{t_f \to \infty} E_{x,i} \int_0^{t_f} [e^{-\beta t} L(x(t, \theta(t), \mu), \theta(t), \mu)] dt < +\infty
\]

We now have the following Theorem:

Theorem 7.5.9. Assume that both (7.1.1) and (7.1.2) have a viscosity solutions. For each \( i, (0 \leq i \leq s) \), let \( \{v_n(\cdot, i)\} \subset C^\infty_0(\mathbb{R}^n) \) be such that \( v_n(x, i) \) uniformly converges to \( V(0; x, i; +\infty) \) on each compact subset of \( \mathbb{R}^n \). Suppose that:

(1) there exists a set \( F_x \subset \{p(x, i); (\cdot, p) \in D^- V(0; x, i; +\infty)\} \) such that

\[
\mu_1 = -R^{-1}(x, i)B^*(x, i)F_x
\]
is an admissible feedback control for system (7.1.3);

(2) For \( p \in Fx \), we have:

\[
\inf \mathbb{E}^n \int_0^{+\infty} \left< p(x, i) - \frac{\partial}{\partial x} v_n(x, i), \dot{x}(t, i, \mu^1) \right> dt \geq 0 \quad \forall i \in S
\]

Then, \( \mu^1 \) is a \( \beta \)-stochastically stabilizable control for system (7.1.3) and an optimal control under the cost function (7.1.6).

**Proof.** According to Theorem 7.3.10, we know that \( V(0; \cdot, i; +\infty) \) is the viscosity solution of (7.1.2). For any \( t_f \in (0, +\infty) \), \( V(0; \cdot, i; +\infty) \) is also the (steady-state) solution of the Cauchy problem

\[
\beta V(t, x, i) + \sup_{u^1, u^2} \left[ -A(u^1, u^2) V(t, x, i) - L(x, i, u^1) \right] = 0 \quad \text{in } \mathbb{R}^n \times (0, t_f]
\]

\[
V(t_f, x, i) = V(0; x, i; +\infty) \quad \text{on } \mathbb{R}^n
\]

Now application of Theorem 7.5.5, yields that

\[
\mu^{1*}(x, i) = -R^{-1}(x, i)B^T(x, i)\{p(x, i); p \in Fx \subset D^- V(0; x, i; +\infty)\}
\]

is the optimal feedback controller in the infinite-horizon case, and for any \( t_f > 0 \)

\[
V(0; x, i; +\infty) \geq E_{x, i} \int_0^{t_f} [e^{-\beta t} L(x(t, \theta(t), \mu^{1*}), \theta(t), \mu^{1*})] dt
\]

Thus, this completes the proof of Theorem 7.5.9. \( \square \)

### 7.6. A Simple Example

As an example, we consider a simple case: scalar linear-quadratic problem. Let \( n = 1 \), and

\[
f(x, i) = A(i)x, B(x, i) = B(i), Q(x, i) = Q(i)x^2, R(x, i) = R(i)
\]
and $g(x,i) = Q_{t_f}(i)x^2$. By remark 2.1, all assumptions (A1)-(A7) hold. According to Theorems 7.3.5, 7.3.9, we know that (7.1.1) admits a unique viscosity solution $V$. Moreover, $V(t;\cdot,i;t_f)$ is convex and Lipschitz (c.f. [XB3]). One can show that $x \rightarrow p(t,x,i) = \partial V(t;x,i;t_f)$ is linear, thus self-adjoint on $\mathbb{R}$ (see e.g. [BP1]), and therefore

\begin{equation}
V(t;x,i;t_f) = P(t,i)x^2, \forall x \in \mathbb{R}, \quad \partial V(t;x,i;t_f) = 2P(t,i)x
\end{equation}

where $P(t,i)$ is a scalar for each $i \in S, t \in [0,t_f]$. Substituting (7.6.2) into (7.1.1) or (7.1.7), we obtain (with $x \neq 0$):

\begin{equation}
\begin{aligned}
& -P_t(t,i) + \beta P(t,i) - 2A(i)P(t,i) - Q(i) \\
& + P^2(t,i)B^2(i)R^{-1}(i) - \inf_{u \in \mathcal{S}} \sum_{j \in S} \lambda_{iaj}P(t,j) = 0 \quad \text{a.e. on } [0,t_f]
\end{aligned}
\end{equation}

\begin{equation}
P(t_f,i) = Q_{t_f}(i), \quad i = 1, \ldots, s
\end{equation}

The solution in (7.6.3) now is in the sense that $P(t,i)$ is absolutely continuous and satisfies (7.6.3) almost everywhere on $[0,t_f]$. The existence of solution of (7.6.3) has been shown by Theorem 7.3.9. It is clear that (7.5.9) in Theorem 7.5.5 holds in this case. Therefore the optimal feedback control $\mu = (\mu^1, \mu^2)$ is given by

\begin{equation}
\mu^1^*(t,x,i) = -R^{-1}(i)B(i)P(t,i)x
\end{equation}

\begin{equation}
\mu^{2^*}(t,i) = f_i(t)
\end{equation}

where $f_i \in \mathcal{U}_2(i)$ is such that

\begin{equation}
f_i(t) = \text{argmin}\{\sum_{j \in S} \lambda_{iaj}P(t,j), a \in \mathcal{U}_2\}
\end{equation}

Similarly, in the infinite horizon case, if $V(0;x,i;+\infty) < +\infty$ for each fixed $x \in \mathbb{R}$, then $V(0;\cdot,i;+\infty)$ solves (7.1.2) in the viscosity sense by Theorem 7.3.10. As per the discussion
above, we know that there exists a scalar \( P(i) \), for each \( i \in S \), such that

\[
V(0; x, i; +\infty) = P(i)x^2
\]

(7.6.7)

where \( P(i) \) satisfies the algebraic coupled equations

\[
\beta P(i) - 2A(i)P(i) - Q(i) + P^2(i)B^2(i)R^{-1}(i) - \inf_{u^2} \sum_{j \in S} \lambda_{iaj} P(j) = 0
\]

(7.6.8)

One can show that (7.6.8) has a solution if and only if system (7.1.3) is \( \beta \)-stochastically stabilizable. In this case, the optimal control \( \mu = (\mu_1, \mu_2) \) is given by

\[
\mu^1(x, i) = -R^{-1}(i)B(i)P(i)x
\]

(7.6.9)

\[
\mu^2(t, i) = f_i
\]

(7.6.10)

where \( f_i \in U_2(i) \) is such that

\[
f_i = \arg \min \{ \sum_{j \in S} \lambda_{iaj} P(j), a \in U_2 \}.
\]

(7.6.11)

One more comment in this example is that by using Ascoli-Arzela Theorem we can find that as \( t_f \to +\infty \) and \( Q_f \equiv 0 \)

\[
V(0; \cdot, i; t_f) \to V(0; \cdot, i; +\infty), \text{ uniformly on compact subset of } \mathbb{R}.
\]

(7.6.12)

\[
7.7. \text{ Concluding Remarks}
\]

In this chapter, we have shown the existence of a viscosity solution to a set of coupled HJB equations. We have also generalized the standard comparison theorem from a single HJB equation to a set of coupled HJB equations and have obtained uniqueness of the viscosity solution for the coupled system. By using the viscosity solutions obtained, an
optimal controller can be constructed under appropriate assumptions. Since the auxiliary Markov process had finite state, it is quite easy to see the existence of the second optimal controller (that enters the transition rate matrix). Once the viscosity solutions are obtained (say by numerical method), we can construct both optimal controllers by using the results of this chapter. When the Markov process has an infinite state, then it will be an interesting but challenging research topic to define the viscosity solution in precise terms for such resulting infinite number of coupled partial differential equations, to prove existence, and to explore construction of an optimal controller.
8. Viscosity SuperSolutions of A Class of Hamilton-Jacobi-Isaacs Equations Arising in Nonlinear $H^\infty$ Control

8.1. Introduction

Consider the nonlinear $H^\infty$ control problem with cost functional

\begin{equation}
J_\gamma(t, x : u, w) = \frac{1}{2} \int_t^{t_f} \{g(x(s)) + h(u(s)) - \gamma^2 |w(s)|^2\} ds + \psi(x(t_f))
\end{equation}

where $x(\cdot)$ is the solution of the input-affine problem

\begin{equation}
\dot{x}(s) = A(x(s)) + B_1(x(s))u(s) + B_2(x(s))w(s), \quad t \leq s \leq t_f
\end{equation}

$x(t) = x$.

Here, $u(\cdot)$ is the control variable with values in $\mathbb{R}^p$, $w$ is a reference or disturbance with values in $\mathbb{R}^l$ and $x$ is the state vector with values in $\mathbb{R}^n$. We assume that $A, B_1, B_2$ are matrix functions on $\mathbb{R}^n$, of appropriate sizes, $g : \mathbb{R}^n \to [0, +\infty)$, and $h : \mathbb{R}^l \to [0, +\infty)$; more precise conditions on $A, B_1, B_2, g, h$ and $\psi$ will be given later in section 8.2. If $t_f = +\infty$, we set $\psi \equiv 0$.

The Hamilton-Jacobi-Isaacs equations associated with this problems in the finite and infinite horizons are (cf. [BB]), respectively,

(I) Finite horizon, i.e. $t_f < +\infty$:

\begin{equation}
\frac{\partial V(s, x)}{\partial s} + g(x) + H(s, x, \nabla_x V(s, x)) = 0
\end{equation}

$V(t_f, x) = \psi(x)$

where

\begin{equation}
H(s, x, \nabla_x V) = \inf_{u} \sup_{w} \left\{ [A(x) + B_1(x)u + B_2(x)w] \nabla_x V(s, x) + h(u) - \gamma^2 |w|^2 \right\}
\end{equation}

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(II) Infinite horizon, i.e. \( t_f = +\infty \):

\[
(8.1.5) \quad g(x) + H(x, \nabla_x V(x)) = 0
\]

where

\[
(8.1.6) \quad H(x, \nabla_x V) = \inf_{u} \sup_{w} \left\{ [A(x) + B_1(x)u + B_2(x)w] \nabla_x V(x) + h(u) - \gamma^2 |w|^2 \right\}
\]

**Remark 8.1.1.** Note that in both finite and infinite horizon cases, the Isaacs' condition is satisfied since the control \( u \) and disturbance \( w \) appear separately in system (1.2) and cost function (1.1).

As standard in nonlinear \( H^\infty \) control, we are interested in, for a given open subset \( \mathcal{O} \subset \mathbb{R}^n \), finding a state feedback controller \( u(t) = \mu(x(t)) \) such that system (8.1.2) has \( L_2 \)-gain for some prescribed \( \gamma > 0 \); that is for any \( w \in \mathcal{W} \) the following holds, with \( x(0) = x_0 \in \mathcal{O} \):

\[
(8.1.7) \quad \text{finite horizon case:} \quad \int_0^{t_f} \{g(x(s)) + h(u(s))\} ds + \psi(x(t_f)) \leq \gamma^2 \int_0^{t_f} |w(s)|^2 ds + \varphi(x_0)
\]

\[
(8.1.8) \quad \text{infinite horizon case:} \quad \int_0^t \{g(x(s)) + h(u(s))\} ds \leq \gamma^2 \int_0^t |w(s)|^2 ds + \varphi(x_0) \quad \forall t > 0
\]

where \( \varphi : \mathcal{O} \subset \mathbb{R}^n \to [0, \infty) \) is a nonnegative function. If there is more than one function \( \varphi \) which satisfies (8.1.7) or (8.1.8) above, we would like to pick the smallest one.

Inequalities (8.1.7) and (8.1.8) are related to the existence of an energy or storage function (see e.g. [J]) \( \Theta(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^1 \cup \{0\} \). That is, for a fixed measurable control \( \mu(\cdot) \) and any measurable disturbance \( w(\cdot) \), if for any \( t_2 \geq t_1 \geq 0 \), \( \Theta \) satisfies

\[
(8.1.9) \quad \Theta(t_2, x(t_2, \mu(\cdot), w(\cdot))) - \Theta(t_1, x(t_1, \mu(\cdot), w(\cdot)))
\]

\[
\leq \int_{t_1}^{t_2} \left( \gamma^2 |w(s)|^2 - g(x(s, \mu(s), w(s))) - h(\mu(s)) \right) ds
\]

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where \( x(\cdot, \mu(\cdot), w(\cdot)) \) denotes the solution of (8.1.2) corresponding to \( \mu(\cdot) \) and \( w(\cdot) \), then \( \Theta \) is called a storage function associated with the controller \( \mu(\cdot) \). Note that if \( \Theta \) is nonnegative and \( \Theta(0, x_0) = \varphi(x_0) \), (8.1.9) becomes (8.1.8), and in the finite horizon case, moreover if \( \Theta(t_f, x(t_f)) = \psi(x(t_f)) \), (8.1.9) becomes (8.1.7).

The nonlinear \( H^\infty \) control problem as formulated above has been intensively studied in recent years, with one focus of attention being the existence of solutions to the HJI equation. It is already known that for nonlinear problems, the HJI equation may not admit a classical solution (e.g. [BB], [FS], [BH], [M1],[M2], [Sor1]), and hence one has to direct attention to weaker notions of a solution, and in particular the viscosity solution (or supersolution)[CrLi 1-2]. Viscosity solutions have been investigated during the last decade in various contexts, including optimal control, differential games (see [L], [ES]), and control of systems governed by partial differential equations (see e.g. [T 1-2], [KS]). In the context of nonlinear \( H^\infty \) control problems, several authors have studied the existence of value or storage functions under various assumptions. Among these are James [J], who used continuous storage (or energy) functions associated with nonlinear systems for the study of dissipative systems, originally studied by Willems [Wi 1-2](see also [HM]), and Soravia [S], who showed that for a wide class of nonlinear systems the smallest nonnegative continuous viscosity solution is the available storage. Most of this prior work has dealt with continuous viscosity solutions.

In this chapter, we consider a wider class of storage functions associated with the nonlinear system (8.1.2), which are only lower semicontinuous. In fact, in various applications, to require the storage function (or value function) to be continuous could be quite restrictive. In the recent work of McEneaney ([M 1-2]), the continuity of the value function requires

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that either the control space be compact or additional assumptions be imposed on the cost function. Simple examples can be devised to show that the value function need not be continuous; one such example is given later in section 5 (see Example 5.2) where the value function is only lower semicontinuous. Accordingly, one objective of this chapter is to establish conditions under which the HJI equation of the nonlinear $H^\infty$ control problem of the type formulated above admits only a viscosity supersolution. A second objective is to establish connections between such viscosity supersolutions and feedback controller design.

The remainder of the chapter is structured as follows. In section 2, we first consider the system (8.1.2) without the control term. In this case, the HJI equation reduces to a HJB equation with only a maximization (for the disturbance) appearing in the Hamiltonian. It is shown that the value function is a lower semicontinuous viscosity supersolution of this HJB equation. We also compare the HJI and HJB equations and show that the viscosity supersolution of the HJB equation also constitutes a viscosity supersolution to the HJI equation with the control term present. In section 3, we study the original problem, as a deterministic differential game with the controller representing the minimizing player and the disturbance representing the maximizing player. We show that if value function is finite then it will be a viscosity supersolution of the corresponding HJI equation. Conversely, if the HJI equation has a viscosity supersolution then it will be a storage function associated with a certain feedback control. As an application, in section 4 we also discuss the case in which the control is constrained. Section 5 provides two examples to illustrate the results of the previous sections.
8.2. Preliminaries and Assumptions

We start this sub-section by recalling the definition of viscosity solutions of the nonlinear, first-order, partial differential equations of the type (8.1.3) and (8.1.5) in the lower semicontinuous case. We take \( \Omega \) as an open set in \( \mathbb{R}^n \).

**Definition 8.2.1.** The lower (respectively, upper) semicontinuous function \( V : [0, t_f] \times \Omega \to \mathbb{R} \) is a viscosity supersolution (respectively, subsolution) of

\[
\frac{\partial V}{\partial t}(t, x) + g(x) + H(x, \nabla_x V(t, x)) = 0 \quad \text{in} \quad (0, t_f] \times \Omega,
\]

\[
V(t_f, x) = \psi(x) \quad \text{on} \quad \{t_f\} \times \Omega
\]

if for all \( \phi \in C^1(\bar{Q}_T) \) (\( Q_{t_f} = \Omega \times (0, t_f] \)) and \((t_0, x_0) \in \text{argmin}_{(t, x) \in Q_T} (V - \phi)\) (respectively, \((t_0, x_0) \in \text{argmax}_{(t, x) \in Q_{t_f}} (V - \phi)\)), we have

\[
\phi_t(t_0, x_0) + g(x_0) + H(t_0, x_0, \phi_x(t_0, x_0)) \leq 0
\]

\[V(x, t_f) \geq \psi(x) \text{ in } \{t_f\} \times \Omega
\]

(respectively,

\[
\phi_t(t_0, x_0) + g(x_0) + H(t_0, x_0, \phi_x(t_0, x_0)) \geq 0
\]

\[V(x, t_f) \leq \psi(x) \text{ in } \{t_f\} \times \Omega
\]

)

\( V \) is a viscosity solution of (8.2.1) if \( V \) is both a viscosity supersolution and a viscosity subsolution.

**Definition 8.2.2.** The lower (respectively, upper) semicontinuous function \( V : \Omega \to \mathbb{R} \) is a viscosity supersolution (respectively, subsolution) of

\[
g(x) + H(x, \nabla_x V(x)) = 0
\]

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if for all $\varphi \in C^1(\Omega)$ and $x_0 \in \text{argmin}_{x \in \Omega}(V - \varphi)$ (respectively, $x_0 \in \text{argmax}_{x \in \Omega}(V - \varphi)$),
we have
\[(8.2.5)\]
\[g(x_0) + H(x_0, \varphi_x(x_0)) \leq 0, \quad (\text{resp., } g(x_0) + H(x_0, \varphi_x(x_0)) \geq 0).\]

$V$ is a viscosity solution of (8.2.4) if $V$ is both a viscosity supersolution and a viscosity subsolution.

**Assumptions**

(A1) $A(x) = A_0 x + G(x)$, where $A_0$ is a constant matrix and $G : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^1$ function with locally Lipschitz continuous derivative, such that $G(0) = 0$.

(A2) There exists a constant $b_1 > 0$ such that
\[|B_1(x) - B_1(y)| \leq b_1|x - y| \quad \forall x, y \in \mathbb{R}^n\]

(A3) There exists a constant $b_2 > 0$ such that
\[|B_2(x) - B_2(y)| \leq b_2|x - y| \quad \forall x, y \in \mathbb{R}^n\]

(A4) $g : \mathbb{R}^n \to [0, +\infty]$ is lower semicontinuous, and for any sequence $\{x_n\} \subset \mathbb{R}^n$, $\{g(x_n)\}$ bounded implies that $\{x_n\}$ is bounded.

(A5) $h : \mathbb{R}^p \to [0, +\infty]$ is a proper convex function with $h(0) = 0$, and there exits a nondecreasing continuous function $\omega : [0, +\infty) \to [0, +\infty)$ with properties $\omega(0) = 0, \lim_{r \to +\infty} \omega(r^2)/r = +\infty$ such that
\[h(u) \geq \omega(|u|^2).\]

(A6) $\psi : \mathbb{R}^n \to \mathbb{R}^+$ is convex, locally Lipschitz.

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Remark 8.2.3. Assumption (A1) will prove to be convenient when we consider the linearization of the nonlinear system around $x = 0$; but naturally it does not bring in any loss of generality.

Throughout this chapter, we adopt the following notation:

$$
\mathcal{U} = L^2_{loc}(\mathbb{R}^+, \mathbb{R}^p), \quad \mathcal{W} = L^2_{loc}(\mathbb{R}^+, \mathbb{R}^l)
$$

8.3. Control-free System

In this section, we consider system (8.1.2) when $u(t) \equiv 0$, i.e.

(8.3.1) \quad \dot{x}(s) = A(x(s)) + B_2(x(s))w(s), \quad t \leq s \leq t_f

$$
\begin{align*}
\quad x(t) &= x
\end{align*}
$$

For the infinite-horizon case, the Hamilton-Jacobi-Bellman (HJB) equation associated with the corresponding maximization problem is:

(8.3.2) \quad g(x) + H(x, \nabla_x V(x)) = 0

where

(8.3.3) \quad H(x, p) = \sup_w \{[A(x) + B_2(x)w]p - \gamma^2|w(s)|^2\}

In fact, by assumption (A1), (8.3.2) can be rewritten as

(8.3.4) \quad g(x) + A(x)^T \nabla_x V_\gamma(x) + \frac{1}{4\gamma^2} \nabla_x V_\gamma(x)^T B_2(x)^T B_2(x) \nabla_x V_\gamma(x) = 0,

Let the value function be

(8.3.5) \quad V_\gamma(x) = \sup_{w \in \mathcal{W}} \sup_{t_f < \infty} \int_t^{t_f} \left( g(x(s)) - \gamma^2|w(s)|^2 \right) ds

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with \( x(t) = x \).

**Stability Assumption:** \( A_0 \) is a Hurwitz matrix.

**Remark 8.3.1.** Let \( \sigma(A_0) \) be the spectrum of \( A_0 \) and

\[
\lambda_0 = -\sup \{ \Re \lambda : \lambda \in \sigma(A_0) \}
\]

Then the stability assumption implies that \( \lambda_0 > 0 \), and there exists \( M \geq 0 \) such that

\[
|e^{A_0 t}x| \leq Me^{-\lambda_0 t}|x| \quad \forall x \in \mathbb{R}^n.
\]

It is a standard result that under the stability assumption the undisturbed system

\[
\dot{x}(t) = A_0 x(t) + G(x(t))
\]

\( x(0) = x_0 \)

is asymptotically stable when the initial condition \( x(0) = x_0 \) belongs to some neighborhood of the origin in \( \mathbb{R}^n \).

**Remark 8.3.2.** The results presented below remain true if the Stability Assumption is replaced by the assumption that the undisturbed system is locally Liapunov stable to the origin.

Let \( \mathcal{O} \) be a given open subset of \( \mathbb{R}^n \). Given \( \gamma > 0 \), we are interested in the existence of a nonnegative function \( \varphi : \mathcal{O} \subset \mathbb{R}^n \to [0, \infty) \) such that the following dissipation inequality holds:

\[
\int_0^t g(x(s))ds \leq \gamma^2 \int_0^t |w(s)|^2 ds + \varphi(x), \quad \forall t > 0, \quad \forall x \in \mathcal{O}, \quad \forall w \in \mathcal{W}
\]

The following theorem provides a conclusive answer to this question.
Theorem 8.3.3. Suppose that $\mathcal{O} \subset \overline{\text{dom}(g)}$. Then given a $\gamma > 0$, (8.3.8) holds if and only if $V_\gamma(x) < \infty$ for all $x \in \mathcal{O}$. Moreover, $V_\gamma$ is a lower semicontinuous storage function.

Proof. The first assertion is obvious. Therefore we only prove that $V_\gamma$ is lower semicontinuous. Let $x_n \to x, x \in \text{dom}(V_\gamma)$; then we claim that

\begin{equation}
\lim_{n \to \infty} \inf V_\gamma(x_n) = V_\gamma(x)
\end{equation}

For every $t \geq 0$ and $w \in L^2(0,t;W)$, clearly

\begin{equation}
V_\gamma(x_n) \geq \int_0^t (g(x(s,x_n,w)) - \gamma^2|w(s)|^2)ds
\end{equation}

Note that $x(s,x_n,w)$ is bounded by assumption (A4). Thus there exists a subsequence of \{x(s,x_n,w)\} which converges to $\{x(s,x,w)\}$ in $\mathbb{R}^n$. Therefore $\forall t > 0$ and $w \in W$, from Fatou's lemma we deduce

\begin{equation}
\liminf_{n \to \infty} V_\gamma(x_n) \geq \int_0^t (g(x(s,x,w)) - \gamma^2|w(s)|^2)ds
\end{equation}

which completes the proof. □

Lemma 8.3.4. Under the stability assumption, for any $t > 0$, we have

\begin{equation}
V_\gamma(x) \geq \sup_{w \in L^2([0,t];W)} \left\{ \int_0^t g(x(t)) - \gamma^2|w(t)|^2 dt + V_\gamma(x(t,x,w)) \right\}
\end{equation}

Proof. This is a standard dynamic programming result. □

Theorem 8.3.5. If $V_\gamma < +\infty$ over $\overline{\text{dom}(g)}$, then $V_\gamma$ solves (8.3.4) in the viscosity supersolution sense in $\overline{\text{dom}(g)}$.

Proof. Suppose that the conclusion were not true. Then there would exist a function $\varphi \in C^1(\mathbb{R}^n)$ such that $V_\gamma - \varphi$ attains a local minimum at $x_0 \in \overline{\text{dom}(g)}$ and there exists
\( \varepsilon > 0 \) such that

\[
(8.3.13) \quad -A(x_0)^T \varphi'(x_0) - \frac{1}{4\gamma^2} \varphi'(x_0)^T B_2(x_0)^T B_2(x_0) \varphi'(x_0) < g(x_0) - \varepsilon
\]

Define

\[
(8.3.14) \quad \dot{y}(t) = A(y(t)) + B_2(y(t)) w(t)
\]

\[ y(0) = x_0 \]

Note that \( y(t) \to x_0 \) as \( t \downarrow 0 \). Hence for sufficiently small \( t \),

\[
(8.3.15) \quad -A(y(t))^T \varphi'(y(t)) - \frac{1}{4\gamma^2} \varphi'(y(t))^T B_2(y(t))^T B_2(y(t)) \varphi'(y(t)) \leq g(y(t)) - \varepsilon
\]

Integrating (8.3.15), we obtain for sufficiently small \( t \):

\[
(8.3.16) \quad \varphi(y(t)) \geq \varphi(x_0)
\]

\[
\geq \varepsilon t + \int_0^t \left\{ \gamma^2 |w(s)|^2 - g(y(s)) \right\} ds - \gamma^2 \int_0^t |w(s) - \frac{1}{2\gamma^2} B_2(y(s)) \varphi'(y(s))|^2 ds
\]

Hence

\[
(8.3.17) \quad V_\gamma(y(t)) - V_\gamma(x_0) \geq \varphi(y(t)) - \varphi(x_0)
\]

\[
\geq \varepsilon t + \int_0^t \left\{ \gamma^2 |w(s)|^2 - g(y(s)) \right\} ds - \gamma^2 \int_0^t |w(s) - \frac{1}{2\gamma^2} B_2(y(s)) \varphi'(y(s))|^2 ds
\]

and thus

\[
\sup_w \left\{ \int_0^t \left( g(y(s)) - \gamma^2 |w(s)|^2 \right) ds + V_\gamma(y(t)) \right\}
\]

\[
\geq \sup_w \left\{ V_\gamma(x_0) - \gamma^2 \int_0^t |w(s) - \frac{1}{2\gamma^2} B_2(y(s)) \varphi'(y(s))|^2 ds \right\} = V_\gamma(x_0)
\]

which contradicts the statement of Lemma 8.3.4. \( \square \)
Theorem 8.3.6. If (8.3.4) admits a viscosity supersolution \( V_\gamma \), then \( V_\gamma \) is also a viscosity supersolution of

\[
g(x) + A(x)^T \nabla_x V_\gamma(x) - h^*(-B_1(x)^T \nabla_x V_\gamma(x)) + \frac{1}{4\gamma^2} \nabla_x V_\gamma(x)^T B_2(x)^T B_2(x) \nabla_x V_\gamma(x) = 0
\]

(8.3.18)

where \( h^* \) is the conjugate function of \( h \).

Proof. Since \( h(0) = 0, h^*(\cdot) \geq 0 \). Let \( V_\gamma \) be a viscosity supersolution of (8.3.4) and \( \varphi \in C^1(\mathbb{R}^n) \) with \( V_\gamma - \varphi \) having a local minimum at \( x_0 \in \mathbb{R}^n \). Then we know that

(8.3.19)

\[
g(x_0) + A(x_0)^T \varphi'(x_0) + \frac{1}{4\gamma^2} (\varphi'(x_0))^T B_2(x_0)^T B_2(x_0) \varphi'(x_0) \leq 0.
\]

Clearly, in this case we also have

(8.3.20)

\[
g(x_0) + A(x_0)^T \varphi'(x_0) - h^*(-B_1(x_0)^T \varphi'(x_0)) + \frac{1}{4\gamma^2} (\varphi'(x_0))^T B_2(x_0)^T B_2(x_0) \varphi'(x_0) \leq 0.
\]

Hence, \( V_\gamma \) is also a viscosity supersolution of (8.3.18). \( \square \)

Remark 8.3.7. An important observation here is that (8.3.2) (or equivalently (8.3.4)) is the HJI equation corresponding to (8.1.1) - (8.1.2), but without control. If we set the feedback controller as (which, at this point, is only an informal expression to be rigorized later in section 3.1):

\[
\mu(x(t)) = \partial h^*(-B_1(x(t)) \nabla_x V_\gamma(x))
\]

then we can see that (8.3.18) is the HJI equation corresponding to (8.1.1) - (8.1.2) with this control. Theorem 8.3.6 implies that for a given attenuation level \( \gamma > 0 \) systems with this feedback controller will provide no worse performance than systems with control set to zero.

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8.4. Systems With Control

**Definition 8.4.1.** (Finite horizon case) Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$.

1. An “admissible state feedback controller” for system (8.1.2) is a (nonlinear, multivalued) mapping $F: [0, t_f] \times \mathbb{R}^n \rightarrow \mathbb{R}^p (= U)$ such that $u \in Fx$ and for every $w \in L^2([0, t_f]; \mathbb{R}^l)(= \mathcal{W})$ and $x_0 \in \mathcal{O}$, the following differential inclusion

$$\dot{x}(s) \in A(x(s)) + B_1(x(s))F(x(s)) + B_2(x(s))w(s), \quad x(0) = x_0 \quad 0 \leq t \leq t_f$$

has at least one solution;

2. Given an admissible state feedback controller $u$, if there exists a nonnegative function $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^+$ such that for any $w \in \mathcal{W}$ we have

$$\int_0^{t_f} \{g(x(s)) + h(u(s))\} ds \leq \gamma^2 \int_0^{t_f} |w(s)|^2 ds + \varphi(x_0)$$

then the controller is said to be an “admissible $\gamma$-attenuation state feedback controller”.

**Definition 8.4.2.** (Infinite horizon case) Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$.

1. An “admissible state feedback controller” for system (8.1.2) is a (nonlinear, multivalued) mapping $F: \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^p (= U)$ such that $u \in Fx$ and for every $w \in L^2_{loc}(\mathbb{R}^+; \mathbb{R}^l)(= \mathcal{W})$ and $x_0 \in \mathcal{O}$, the following differential inclusion

$$\dot{x}(s) \in A(x(s)) + B_1(x(s))F(x(s)) + B_2(x(s))w(s), \quad s > 0 \quad x(0) = x_0$$

has at least one solution;

2. Given an admissible state feedback controller $u$, if there exists a nonnegative function $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^+$ such that for any $t > 0$ and any $w \in \mathcal{W}$ we have

$$\int_0^t \{g(x(s)) + h(u(s))\} ds \leq \gamma^2 \int_0^t |w(s)|^2 ds + \varphi(x_0)$$

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then the controller is said to be an "admissible $\gamma$-attenuation state feedback controller".

**Remark 8.4.3.** Solution to differential inclusions (8.4.3) are understood in the Carathéodory sense, i.e., absolutely continuous functions verifying (8.4.3) almost everywhere.

### 8.4.1. Finite Horizon Case

We denote by $\mathcal{M}_{CL}$ the class of admissible feedback controls for system (8.1.2) and define the value function:

$$V_\gamma(t,x) = \inf_{\mathcal{M}_{CL}} \sup_{\mathcal{W}} J_\gamma(t,x; \mu, w)$$

**Theorem 8.4.1.1.** Suppose that state feedback control

$$\mu(x(t)) = \partial h^*\left(-B_1(x(t))^T \hat{p}(t,x(t))\right)$$

is an admissible feedback control, where

$$(\cdot, \hat{p}) \in D^- V_\gamma(t,x) = \{(q(t,x), p(t,x)) \in \mathbb{R}^{n+1} : \liminf_{(s,y) \to (t,x)} \frac{V_\gamma(s,y) - V_\gamma(t,x) - (s-t)p(y-x)}{|s-t| + |y-x|} \geq 0\}$$

Then, for this controller to be an admissible $\gamma$-attenuation feedback control, it is necessary that the following HJI equation

(8.4.1.1)

$$V_\gamma(t,x) + g(x) + A(x)^T \nabla_x V_\gamma(t,x) - h^*(-B_1(x)^T \nabla_x V_\gamma(t,x))$$

$$+ \frac{1}{4\gamma^2} \nabla_x V_\gamma(t,x)^T B_2(x) B_2(x)^T \nabla_x V_\gamma(t,x) = 0$$

$$V_\gamma(t_f,x) = \psi(x)$$

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have a nonnegative viscosity supersolution $V_\gamma$ with property $\text{dom}(V_\gamma) \supset \overline{\text{dom}(g)}$.

**Remark 8.4.1.2.** $D^{-}V_\gamma(t, x)$ is convex for every $(t, x) \in [0, t_f] \times \mathbb{R}^n$. Moreover if $V_\gamma$ is differentiable at $(t, x)$ we have

$$D^{-}V_\gamma(t, x) = \left\{ \left( \frac{\partial}{\partial t} V_\gamma(t, x), D_x V_\gamma(t, x) \right) \right\}$$

(cf. [FS]) and the feedback controller becomes

$$(8.4.1.2) \quad \mu(x(t)) = \partial h^* \left( - B_1(x(t))^T D_x V_\gamma(t, x(t)) \right)$$

**Theorem 8.4.1.3.** Suppose that the following four conditions hold:

1. HJI equation (8.4.1.1) has a nonnegative viscosity supersolution $V_\gamma$ with the property that for any $t \in [0, t_f]$, $\text{dom}(V_\gamma(t, \cdot)) \supset \overline{\text{dom}(g)}$;

2. There exists a pair $(\hat{q}, \hat{p}) \in D^{-}V_\gamma$ such that $\hat{p}(t, x) \geq \psi(x), x \in \Omega$;

3. the feedback controller

$$\mu(x(t)) = \partial h^* \left( - B_1(x(t))^T \hat{p}(t, x(t)) \right)$$

is an admissible feedback control;

4. There exists a function $\varphi \in C^1([0, t_f] \times \mathbb{R}^n)$ such that

$$\varphi \geq 0 \quad \forall (t, x) \in [0, t_f] \times \text{dom}(V_\gamma), \quad \nabla \varphi = (\hat{p}, \hat{q}) \quad \text{a.e.} x \in \text{dom}(V).$$

Then, the state feedback controller $\mu$ is an admissible $\gamma$-attenuation feedback control, i.e. we have

$$\int_0^{t_f} \{ g(x(t)) + h(\mu(x(t))) \} dt \leq \gamma^2 \int_0^{t_f} |w(t)|^2 dt + \varphi(x_0) \quad \forall w \in \mathcal{W}, \quad x_0 \in \mathcal{O}$$

Theorem 8.4.1.1 and Theorem 8.4.1.3 will be established through the following series of lemmas.
Lemma 8.4.1.4. For any $0 \leq s \leq \tau \leq t_f$

\begin{equation}
V_\gamma(s, x) \geq \inf_{\mathcal{M}_{CL}} \sup_{\mathcal{W}} \left\{ \int_s^\tau L(t, x, \mu, w)dt + V_\gamma(\tau, x(\tau)) \right\}
\end{equation}

where

\[ L(s, x, \mu, w) = g(x) + h(\mu) - \gamma^2|w|^2. \]

**Proof.** The proof is standard.

Lemma 8.4.1.5. Assume that $V_\gamma(x, t_f) \geq \psi(x), x \in \Omega$. $V_\gamma$ is a viscosity supersolution of (8.4.1.1) with $(t, x) \in [0, t_f] \times \Omega$ if and only if

\begin{equation}
\hat{q}(t, x) + g(x) + A(x)^T\mathbf{\hat{p}}(t, x) - h^*(-B_1(x)^T\mathbf{\hat{p}}(t, x))
\end{equation}

\[ + \frac{1}{4\gamma^2}\mathbf{\hat{p}}(t, x)^TB_2(x)B_2(x)^T\mathbf{\hat{p}}(t, x) \leq 0 \]

for any $(\hat{q}, \mathbf{\hat{p}}) \in D^-V_\gamma(t, x), (t, x) \in [0, t_f] \times \Omega$.

**Proof.** Suppose that (8.4.1.4) holds. Let $\phi \in C^1([0, t_f] \times \mathbb{R}^n)$ be such that $(t_0, x_0)$ be a local minimizer of $V_\gamma - \phi$ with $V_\gamma(t_0, x_0) = \phi(t_0, x_0), (t_0, x_0) \in [0, t_f] \times \Omega$. Since $\phi \in C^1([0, t_f] \times \mathbb{R}^n)$, it gives

\[ \phi(t, x) = \phi(t_0, x_0) + \phi_t(t_0, x_0)(t - t_0) + \phi_x(t_0, x_0)(x - x_0) + o(|t - t_0|) + o(|x - x_0|) \]

Hence there exists an $\delta > 0$ such that

\[ V(t, x) \geq \phi(t_0, x_0) + \phi_t(t_0, x_0)(t - t_0) + \phi_x(t_0, x_0)(x - x_0) + o(|t - t_0|) + o(|x - x_0|) \]

with $|t - t_0| + |x - x_0| < \delta$. Let $\mathbf{\hat{p}} = \phi_x(t_0, x_0), \hat{q} = \phi_t(t_0, x_0)$. According to the definition of $D^-V_\gamma$ (given in Theorem 8.4.1.1) and (8.4.1.4), we have

\begin{equation}
\phi_t(t_0, x_0) + g(x_0) + A(x_0)^T\phi_x(t_0, x_0) - h^*(-B_1(x_0)^T\phi_x(t_0, x_0))
\end{equation}

\[ + \frac{1}{4\gamma^2}\phi_x(t_0, x_0)^TB_2(x_0)B_2(x_0)^T\phi_x(t_0, x_0) \leq 0 \]

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Therefore $V_\gamma$ is a viscosity supersolution of (8.4.1.1) with $(t, x) \in [0, t_f] \times \Omega$.

Conversely, suppose that $V_\gamma$ is a viscosity supersolution of (8.4.1.1). Let $(\hat{q}, \hat{p}) \in D^- V_\gamma(t, x), (t, x) \in [0, t_f] \times \Omega$. As $(s, y)$ closes to $(t, s)$ sufficiently, we have

$$V_\gamma(s, y) \geq V_\gamma(t, x) + q(s - t) + p(y - x) + o(|s - t|) + o(|y - s|)$$

Define test function

$$\phi(s, y) := V_\gamma(t, x) + q(s - t) + p(y - x) + g_{1(1-t)} + g_{2(|y - s|)}$$

where $g_i : (0, \infty) \to (0, \infty), i = 1, 2$ are nondecreasing functions such that

$$g_i(r) = o(r), \quad D g_i(r)|_{r=0} = 0$$

(the construction for these functions, c.f. [C1] or [FS]). Therefore $(t, x)$ is in fact a strict minimizer of $V - \phi$, and $\phi(t, x) = \hat{p}, \phi(t, x) = \hat{q}$, and (8.4.1.4) holds by the definition of viscosity supersolution. □

**Lemma 8.4.1.6.** If there exists an admissible $\gamma$-attenuation feedback control for system (8.1.2), any lower semicontinuous storage function $\Theta$ associated with this control is a viscosity supersolution of (8.4.1.1) provided that $\Theta \geq \psi$.

**Proof.** Let $\phi \in C^1([0, t_f] \times \mathbb{R}^n)$ and $\Theta$ be a lower semicontinuous storage function which satisfies the hypothesis of the theorem. Suppose that $\Theta - \phi$ achieves a local minimum at $(t_0, x_0) \in [0, t_f] \times \mathbb{R}^n$ and $x(t)$ is the trajectory of (8.1.2) with $x(t_0) = x_0, t \geq t_0$, under the given controller and the fixed $w(t) = \frac{1}{2\gamma} B_2(x_0)^T \phi_x(t, x_0)$. Thus for $t - t_0$ sufficiently small, we have

$$\Theta(t, x(t)) - \phi(t, x(t)) \leq \Theta(t, x(t)) - \phi(t, x(t))$$

(8.4.1.6)
Since $\Theta$ is a storage function, (8.4.1.6) yields $\forall w \in \mathcal{W}$

\[(8.4.1.7)\]
\[
\phi(t, x(t)) - \phi(t_0, x_0) \leq \frac{1}{t - t_0} \int_{t_0}^{t} (\gamma^2 |w(s)|^2 - g(x(s)) - h(u(s)))ds
\]

Letting $t \downarrow 0$, (8.4.1.7) gives

\[(8.4.1.8)\]
\[
\phi_t(t_0, x_0) + \phi_x(t_0, x_0)(A(x_0) + B_1(x_0)u(t_0) + B_2(x_0)w(t_0))
\]
\[
\leq \gamma^2 |w(t_0)|^2 - g(x_0) - h(u(t_0))
\]

Note that (8.4.1.8) can be rewritten as

\[(8.4.1.9)\]
\[
\phi_t(t_0, x_0) + g(x_0) + A(x_0)^T \phi_x(t_0, x_0) + (B_1(x_0)u(t_0))^T \phi_x(t_0, x_0) + h(u(t_0))
\]
\[
- (B_2(x_0)u)^T \phi_x(t_0, x_0) + \frac{1}{4\gamma^2} \phi_x(t_0, x_0)^T B_2(x_0)^T B_2(x_0) \phi_x(t_0, x_0)
\]
\[
- \gamma^2 (w(t_0) - \frac{1}{2\gamma^2} B_2(x_0)^T \phi_x(t_0, x_0))^2 \leq 0
\]

Since

\[h(u) + h^*(-B_1^T \phi_x) \geq (-B_1 u)^T \phi_x,
\]

we thus arrive at

\[
\phi_t(t_0, x_0) + g(x_0) + A(x_0)^T \phi_x(t_0, x_0) - h^*(-B_1^T (x_0) \phi_x)
\]
\[
+ \frac{1}{4\gamma^2} \phi_x(t_0, x_0)^T B_2(x_0) B_2(x_0)^T \phi_x(t_0, x_0) \leq 0
\]

and this completes the proof. $\Box$

**Remark 8.4.1.7.** Lemma 8.4.1.6 implies that the function

\[\Phi(t, x) = \Theta(t, x) + C; \quad \text{where } C \text{ is a nonnegative constant}\]
is also a viscosity supersolution of (8.4.1.1) provided that $\Theta$ is a viscosity supersolution of (8.4.1.1). Another observation is that $\Theta$ is also a viscosity supersolution of

$$V_{\gamma}(t, x) + g(x) + A(x)^T \nabla_x V_{\gamma}(t, x) - h^*(-B_1^T(x) \nabla_x V_{\gamma}(t, x)) = 0$$

which corresponds to the case without disturbance.

**Lemma 8.4.1.8.** Suppose that there exists a nonnegative storage function associated with an admissible state feedback control such that (8.1.9) holds. Then

$$V_{\gamma}(t, x) = \inf_{M_{CL}} \sup_{W} J_{\gamma}(t, x; u, w)$$

solves (8.4.1.1) in the viscosity supersolution sense in $\text{dom}(V_{\gamma}(t, \cdot)), \forall t \in [0, t_f]$.

**Proof.** Suppose that $V(\cdot, \cdot)$ were not a viscosity supersolution of (8.4.1.1). Then there exist a function $\phi \in C([0, t_f] \times \mathbb{R}^n)$, a point $(t_0, x_0) \in [0, t_f] \times \mathbb{R}^n$ such that $V - \phi$ has a local minimum at $(t_0, x_0)$ and an $\varepsilon > 0$ such that

(8.4.1.10)

$$\phi_t(t_0, x_0) + g(x_0) + A(x_0)^T \phi_x(t_0, x_0) - h^*(-B(x_0)^T \phi_x(t_0, x_0))$$

$$+ \frac{1}{4\gamma^2} \phi_x(t_0, x_0)^T B_2(x_0)^T B_2(x_0) \phi_x(t_0, x_0) \geq \varepsilon$$

Using the fact that for any $u \in \mathcal{U}$

(8.4.1.11)

$$h(u) + h^*(-B_1^T \phi_x) \geq (-B_1 u)^T \phi_x$$

we have

(8.4.1.12)

$$\phi_t(t_0, x_0) + g(x_0) + A(x_0)^T \phi_x(t_0, x_0) + h(u(t_0)) + (B_1(x_0) u(t_0))^T$$

$$+ \frac{1}{4\gamma^2} \phi_x(t_0, x_0)^T B_2(x_0)^T B_2(x_0) \phi_x(t_0, x_0) \geq \varepsilon$$

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Moreover
\[(8.4.1.13)\]
\[\phi(t_0, x_0) + A(x_0)T \phi(x(t_0), x_0) + h(u(t_0)) + (B_1(x_0)u(t_0))^T \phi(x(t_0), x_0) + (B_2(x_0)w(t_0))^T \phi(x(t_0), x_0) \geq \varepsilon - g(x_0) - h(u(t_0)) + \gamma^2 |w(t_0)|^2\]

Since \(g, h\) are lower semicontinuous, there exists a \(\delta > 0\) such that when \(t \in [t_0, \delta]\) the above inequality still holds. For arbitrary but fixed \(w \in W\), introduce \(\dot{u}\) such that
\[\inf_{\dot{u}} \int_{t_0}^{t} \{g(x(s)) + h(u(s)) - \gamma^2 |w(s)|^2\} ds = \int_{t_0}^{t} \{g(x(s)) + h(\dot{u}(s)) - \gamma^2 |w(s)|^2\} ds\]

Letting
\[(8.4.1.14)\]
\[\dot{x}(t) = A_0 x(t) + B_1(x(t))\dot{u}(t) + B_2(x(t))w(t)\]
\[x(t_0) = x_0\]

we arrive at
\[(8.4.1.15)\]
\[\frac{d\phi(t, x(t))}{dt} \geq \varepsilon - g(x(t)) - h(\dot{u}(t)) + \gamma^2 |w(t)|^2 \quad 0 \leq t < \delta\]

Therefore
\[(8.4.1.16)\]
\[\phi(x(t)) - \phi(x_0) \geq \varepsilon (t - t_0) + \int_{t_0}^{t} (\gamma^2 |w(s)|^2 - g(x(s)) - h(\dot{u}(s))) ds\]

Note that \((t_0, x_0)\) is a local minimum of \(V_\gamma - \phi\); thus there exists a \(\delta' > 0\) such that when \(0 < t < \min\{\delta, \delta'\}\)
\[(8.4.1.17)\]
\[\phi(t, x(t)) - \phi(t_0, x_0) \leq V_\gamma(t, x(t)) - V_\gamma(t_0, x_0)\]

\[(8.4.1.16)\] and \[(8.4.1.17)\] imply for any \(0 < t < \min\{\delta, \delta'\}\),
\[(8.4.1.18)\]
\[V_\gamma(t, x(t)) - V_\gamma(t_0, x_0) \geq \varepsilon (t - t_0) + \int_{t_0}^{t} (\gamma^2 |w(s)|^2 - g(x(s)) - h(u(s))) ds\]

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By Lemma 8.4.1.4, and (8.4.1.17), we have

$$V(t_0, x_0) \geq \inf_{\mathcal{M}_{GL}} \sup_{W} \left\{ \int_{t_0}^{t} L(x(s), \mu(x(s)), w(s)) ds + V(t_0, x_0) + \phi(t, x(t)) - \phi(t_0, x_0) \right\}$$

where $L(x, \mu, w) = g(x) + h(\mu) - \gamma^2 |w|^2$, which gives

(8.4.1.19)

$$\phi(t, x(t)) - \phi(t_0, x_0) \leq \sup_{\mathcal{M}_{GL}} \inf_{W} \int_{t_0}^{t} ( - L(x(s), \mu(x(s)), w(s))) ds$$

$$\leq \sup_{\mathcal{M}_{GL}} \int_{t_0}^{t} ( - L(x(s), \mu(x(s)), w(s))) ds$$

$$= \int_{t_0}^{t} \{ \gamma^2 |w(s)|^2 - g(x(s)) - h(\tilde{u}(s)) \} ds$$

Combining (8.4.1.16) and (8.4.1.19) results in

$$\varepsilon(t - t_0) \leq 0,$$

which implies that $V_\gamma$ should be a viscosity supersolution of (8.4.1.1). □

The next theorem provides a sufficient condition for the control to be an admissible feedback controller.

**Theorem 8.4.1.9.** Suppose that $G$ is Lipschitz with Lipschitz constant $G_{\text{Lip}}$, and $B_1$ satisfies

$$|B_1(x)| \leq b_1^c, \quad \forall x \in \mathbb{R}^n$$

where $b_1^c$ is a constant.

If for every $t \in [0, t_f]$, $V_\gamma(t, \cdot)$ is convex, then

(8.4.1.20) \hspace{1cm} \mu(x(t)) = \partial h^*(-B_1(x(t))^T \partial_x V_\gamma(t, x(t)))

is an admissible feedback control for the nonlinear system (8.1.2).

**Proof.** We define $V_\gamma^\lambda : [0, t_f] \times \mathbb{R}^n \to \overline{\mathbb{R}}$ and $h^*_\lambda : \mathbb{R}^p \to \mathbb{R}$ as follows

(8.4.1.21) \hspace{1cm} V_\gamma^\lambda(t, x) = |x - J^\nu_\lambda x|^2 / 2\lambda + V_\gamma(J^\nu_\lambda x)

(8.4.1.22) \hspace{1cm} h^*_\lambda(x) = |x - J^h_\lambda x|^2 / 2\lambda + h^*(J^h_\lambda x)$

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where \( \lambda > 0 \), \( J_{\lambda}^u = (I + \lambda \partial V_{\gamma})^{-1} \) and \( J_{\lambda}^h = (I + \lambda \partial h^*)^{-1} \) are nonexpansive single-valued mappings from \( \mathbb{R}^n \), \( \mathbb{R}^p \) to \( D(\partial V_{\gamma}) \), \( D(\partial h^*) \) respectively, where \( D(\partial V_{\gamma}) \), \( D(\partial h^*) \) denote the valid domains for \( \partial V_{\gamma} \) and \( \partial h^* \). Note that \( V_{\gamma}^\lambda \) and \( h_{\lambda}^* \) are differentiable with respect to \( x \), and \( \nabla_x V_{\gamma} = (\partial V_{\gamma})_\lambda = \lambda^{-1}(I - J_{\lambda}^u) \), \( \nabla_x h_{\lambda}^* = (\partial h^*)_\lambda = \lambda^{-1}(I - J_{\lambda}^h) \) are Lipschitzian. For arbitrary but fixed \( w \in \mathcal{W} \) we consider the approximation equation

\[
\dot{x}_\lambda = A_0 x_\lambda + G(x_\lambda) + B_1(x_\lambda) \nabla_x h_{\lambda}^* \left( -B_1(x_\lambda)^T \nabla_x V_{\gamma}^\lambda(t, x_\lambda) \right) + B_2(x_\lambda)w
\]

\( x_\lambda(0) = x_0 \)

which is equivalent to the integral equation

\[
x_\lambda(t) = e^{A_0 t} x_0 + \int_0^t e^{A_0(t-s)} \left\{ G(x_\lambda(s)) + B_2(x_\lambda(s))w \right. \\
+ B_1(x_\lambda(s)) \nabla_x h_{\lambda}^* \left( -B_1(x_\lambda(s))^T \nabla_x V_{\gamma}^\lambda(s, x_\lambda(s)) \right) \right\} ds := (Q x_\lambda)(t)
\]

Set \( X = C([0, t_f], \mathbb{R}^n) \). Introduce

\[
M_{t_f} = \max \{|e^{A_0 t}| : 0 \leq t \leq t_f\}, \quad K_1 = 2M_{t_f}|x_0|
\]

We start by showing that under assumption (A1)-(A3), (8.4.1.23) has a unique solution \( x_\lambda \) on an interval \([0, t_1] \) whose length is bounded below by

\[
\delta(|x_0|) = \min \left\{ 1, \frac{|x_0|}{K_1 (G_{\text{Lip}} + \lambda^{-2}(b_1^2) + b_2\|w\|\mathcal{W})} \right\}
\]

The mapping \( Q \) defined by (8.4.1.24) maps the ball of radius \( K_1 \) centered at 0 of \( C([0, t_1] : \mathbb{R}^n) \) into itself. This follows from the estimate

\[
|Q(x_\lambda)(t)| \leq M_{t_f} |x_0| + M_{t_f} \left\{ (G_{\text{Lip}} + \lambda^{-2}(b_1^2)K_1 + b_2\|w\|\mathcal{W}) \right\} t
\]

\[
\leq 2M_{t_f} |x_0| = K_1
\]

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It follows from the nonexpansive property of \( J^\rho_\lambda, J^h_\lambda \) and assumptions \((A2), (A3)\) by induction that

\[
\|Q^n(x^1_\lambda) - Q^n(x^2_\lambda)\| \leq \frac{M_{tf}C^n_{t1}}{\lambda^{2n}n!}\|x^1 - x^2\|, \quad n = 1, 2, \ldots
\]

where

\[
C = \lambda^{-2}(b_1^cK_1 + (b_1^c)^2 + b_1b_1^cK_1)
\]

and thus \(Q\) has a unique fixed point \(x_\lambda\) in \(C([0, t_1] : \mathbb{R}^n)\) which is the solution of \((8.4.1.23)\) for each \(\lambda > 0\).

From what we have just proved it follows that if \(x_\lambda\) is a solution of \((8.4.1.23)\) on the interval \([0, t_1]\) it can be extended to the interval \([0, t_1 + \delta]\) with \(\delta > 0\) by defining on \([t_1, t_1 + \delta]\), \(x_\lambda(t) = z_\lambda(t)\) where \(z_\lambda(t)\) is the solution of

\[
\dot{z}_\lambda = A_0z_\lambda + G(z_\lambda) + B_1(z_\lambda)\nabla_x h_\lambda^*( - B_1(z_\lambda)^T \nabla_x V_{\gamma}^\lambda(t, z_\gamma)) + B_2(z_\lambda)w
\]

\[z_\lambda(t_1) = x_\lambda(t_1)\]

Note the inequality:

\[
|z_\lambda(t)| \leq M_{tf} |x_\lambda(t_1)| + M_{tf}b_2^c|w| + M_{tf}(G_{Lip} + (\frac{b_1^c}{\lambda})^2) \int_{t_1}^{t} |z_\lambda(s)|ds
\]

Since constants \(M_{tf}, b_2^c, \lambda, G_{Lip}, x_0\) above are independent of \(t\), in view of Gronwall’s lemma \(z_\lambda\) is uniformly bounded, and thus by repeating the previous step, \((8.4.1.23)\) has a unique solution \(x_\lambda\) on \([0, t_f]\). Since \(x_\lambda\) is uniformly bounded, there exists a subsequence \(\{x_{\lambda_n}\}\) of \(\{x_\lambda\}\) which is uniformly convergent to a continuous function \(x = x(t)\) on \([0, t_f]\) as \(\lambda_n \to 0\). It is easy to see that \(\nabla_x V_\gamma^\lambda\) is also uniformly bounded, and there exists a subsequence \(\nabla_x V_\gamma^{\lambda_n}\) and \(p \in \partial_x V_\gamma\) such that \(\nabla_x V_\gamma^{\lambda_n} \to p\) pointwise and \(p \in \partial_x V_\gamma\). Note

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that
\[ \nabla_x h_{\lambda_n}^* \left( -B_1(x_{\lambda_n})^T \nabla_x V_{\gamma_n}(t, x_\gamma) \right) - \nabla_x h_{\lambda_n}^* \left( -B_1(x_{\lambda_n})^T p \right) \to 0 \]
and thus there exists \( \mu \in \partial h^* \) such that
\[ \nabla_x h_{\lambda_n}^* \left( -B_1(x_{\lambda_n})^T p \right) \to \mu \quad \text{as} \quad \lambda_n \to 0. \]
It is readily follows that \( x(t) \) is a unique solution of (8.1.2) with
\[ \mu(x(t)) \in \partial h^* \left( -B_1(x(t))^T \partial V_{\gamma}(t, x(t)) \right). \]
and a fixed \( w \in \mathcal{W} \). \( \square \)

8.4.2. Infinite Horizon Case

In this case, the value function is
\[ V_{\gamma}(x) = \inf_{\mathcal{M}_{CL}} \sup_{\mathcal{W}} \sup_{t_f < \infty} J_{\gamma}(0, x; u, w) \]
As a counterpart of Theorem 8.4.1.1, we state the following theorem without proof (the proof is similar to that of Theorem 8.4.1.1).

**Theorem 8.4.2.1.** There exists an admissible \( \gamma \)-attenuation feedback control
\[ \mu(x(t)) = \partial h^* \left( -B_1(x(t))^T \hat{p}(x(t)) \right) \]
where
\[ \hat{p} \in D^{-} V_{\gamma}(x) = \{ p \in \mathbb{R}^n, \lim_{y \to x} \inf_{y \to x} \frac{V_{\gamma}(y) - V_{\gamma}(x) - p(y-x)}{|y-x|} \geq 0 \} \]
for system (8.1.2) only if the HJI equation
(8.4.2.1)
\[ g(x) + A(x)^T \nabla_x V(x) - h^*(-B_1(x)^T \nabla_x V_{\gamma}(x)) + \frac{1}{4 \gamma^2} \nabla_x V_{\gamma}(x)^T B_2(x)^T B_2(x) \nabla_x V_{\gamma}(x) = 0 \]
admits a supersolution \( V_{\gamma} \) with property \( \text{dom}(V_{\gamma}) \supset \text{dom}(g) \).
Lemma 8.4.2.2. \( V_\gamma \) is a supersolution of (8.4.2.1) with \( x \in \Omega \) if and only if
\[
g(x) + A(x)^T p(x) - h^*(-B_1(x)^T p(x)) + \frac{1}{4\gamma^2} p(x)^T B_2(x)^T B_2(x)p(x) \leq 0
\]
holds for any \( p \in D^- V_\gamma(x), x \in \Omega \).

\[\text{Proof.} \quad \text{The proof is similar to the one given in Lemma 8.4.1.5, and we omit the detail.}\]

The following theorem provides a sufficient condition for stabilization of the nonlinear system (8.1.2).

Theorem 8.4.2.3. Assume that

1. HJI equation (8.4.2.1) has a nonnegative supersolution \( V_\gamma \) with property
\[
\text{dom}(V_\gamma) \supset \text{dom}(g);
\]

2. The state feedback controller
\[
\mu(x) = \partial h^*(-B_1(x)^T p(x))
\]
is an admissible state feedback control;

3. There exists a function \( \varphi \in C^1(\mathbb{R}^n) \) such that
\[
\varphi \geq 0 \quad x \in \text{dom}(V_\gamma), \quad \nabla_x \varphi = p, \quad \text{a.e.} x \in \text{dom}(V) \subset \mathbb{R}^n
\]

4. \( g(x(\cdot)) \in L^1(\mathbb{R}^+; \mathbb{R}) \) implies \( x \in L^2(\mathbb{R}^+; \mathbb{R}^n) \).

Then with the feedback controller (8.4.2.3), the worst-case system
\[
\dot{x}^* = A x^* + B_1(x^*) \partial h^*(-B_1(x^*)^T p(x^*)) + \frac{1}{\gamma^2} B_2(x^* B_2(x^*)^T p(x^*)
\]

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is globally asymptotically stable, i.e.

\[ x^* \in C(\mathbb{R}^+; \mathbb{R}^n) \cap L^2(\mathbb{R}^+; \mathbb{R}^n); \quad \lim_{t \to \infty} x^*(t) = 0. \]

**Remark 8.4.2.4.** Assumption (2) of Theorem 8.4.2.3 says that system (8.4.2.5) is dissipative with respect to the supply rate

\[ \gamma^2 B_2(x^*)^T B_2(x^*) p(x^*) - g(x^*) - h(\mu(x^*(t))). \]

**Proof of Theorem 8.4.2.3.** By Lemma 8.4.2.2, and the hypothesis (3) of the theorem, we have

\[ g(x) + A(x)^T \nabla_x \varphi(x) - h(-B_1(x)^T \nabla_x \varphi(x)) + \frac{1}{4\gamma^2} \nabla_x \varphi(x)^T B_2(x)^T B_2(x) \nabla_x \varphi(x) \leq 0 \quad a.e. \quad x \in \text{dom}(V_\gamma) \]

This inequality implies that

\[ \frac{d\varphi(x^*(t))}{dt} \leq \gamma^2 p(x^*(t))^T B_2(x^*(t))^T B_2(x^*(t)) p(x^*(t)) - g(x^*(t)) - h(\mu(x^*(t))) \]

This gives for any \( t > 0 \)

\[ \varphi(x_0) - \varphi(x^*(t)) \geq \int_0^t \{ g(x^*(s)) + h(\mu(x^*(s))) - \gamma^2 p(x^*(s))^T B_2(x^*(s))^T B_2(x^*(s)) p(x^*(s)) \} ds \]

hypothesis (3) and (8.4.2.8) yields

\[ V(x^*(t)) + \int_0^t L(s, x^*, p) ds \leq \varphi(x_0), \quad \forall t > 0 \]

where

\[ L(s, x^*, p) = g(x^*(s)) + h(\mu(x^*(s))) - \gamma^2 p(x^*(s))^T B_2(x^*(s))^T B_2(x^*(s)) p(x^*(s)). \]
Thus from (8.4.2.9) we get

\[ \int_0^t g(x^*(s))ds \leq C, \quad \forall t > 0 \]

Hypothesis (2) implies that \( x^* \in C(\mathbb{R}^+, \mathbb{R}^n) \); hence with hypothesis (4) we have

\[ x^* \in C(\mathbb{R}^+, \mathbb{R}^n) \cap L^2(\mathbb{R}^+, \mathbb{R}^n) \quad \text{and} \quad \lim_{t \to \infty} x^*(t) = 0 \]

as claimed. The proof of Theorem 8.4.2.3 is thus complete. \( \square \)

**Theorem 8.4.2.5.** Under the assumptions of Theorem 8.4.2.3, for any \( w \in \mathcal{W} \) and system trajectory generated by

(8.4.2.10)

\[ \dot{x}(t) = Ax(t) + B_1(x(t))\partial h^*(-B_1(x(t))^T p(x(t))) + B_2(x(t))w, \quad x(0) = x_0 \]

we have for any \( t > 0 \)

(8.4.2.11)

\[ \int_0^t \{g(x(s)) + h(\mu(x(s)))\}ds \leq \gamma^2 \int_0^t |w(s)|^2ds + \varphi(x_0) \]

**Proof.** Similar to the proof of Theorem 8.4.2.3, we have

\[ g(x) + A(x)^T \nabla_x \varphi(x) - h^*(-B_1(x)^T \nabla_x \varphi(x)) \]

\[ + \frac{1}{4\gamma^2} \nabla_x \varphi(x)^T B_2(x)^T B_2(x) \nabla_x \varphi(x) \leq 0 \quad \text{a.e.} \quad x \in \text{dom}(V) \]

which yields

(8.4.2.12)

\[ \frac{d\varphi(x(t))}{dt} \leq -\gamma^2(w - \frac{1}{2\gamma^2}B_2(x(t))\nabla_x \varphi(x(t))^2 + \gamma^2 |w(t)|^2 - g(x(t)) - h(\mu(x(t))) \]

Since \( \varphi > 0 \), integrating (8.4.2.12), we can arrive at (8.4.2.11). \( \square \)
Corollary 8.4.2.6. Suppose that (8.4.2.1) has a solution $V_\gamma$ of class $C^1$ and the state feedback controller

$$\mu(x) = \partial h^*(-B_1(x)^T \nabla_x V_\gamma)$$

is an admissible $\gamma$-attenuation state feedback controller. Then $V_\gamma$ is a storage function.

Proof. With control $\mu$, we have the following system

$$\dot{x} = Ax + B_1(x)\partial h^*(-B_1(x)^T \nabla_x V_\gamma) + B_2(x)w, \quad x(0) = x_0.$$ 

Using (8.4.2.1), we get

$$\frac{d}{dt} V_\gamma(x(t)) = \nabla_x V_\gamma \dot{x}(t)$$

$$= \nabla_x V_\gamma(A(x(t)) + B_1(x(t))\partial h^*(-B_1(x(t))^T \nabla_x V) + B_2(x(t))w(t))$$

$$= -g(x(t)) - h(\mu(x(t))) + \gamma^2 |w(t)|^2 - \gamma^2 |w(t) - \frac{1}{2\gamma^2} B_2^T(x(t))|^2$$

$$\leq -g(x(t)) - h(\mu(x(t))) + \gamma^2 |w(t)|^2.$$ 

Integrating the above inequality on any interval $[t_1, t_2]$, we have

$$V(x(t_2)) \leq V(x(t_1)) + \int_{t_1}^{t_2} \{\gamma^2 |w(t)|^2 - g(x(t)) - h(\mu(x(t)))\} dt$$

which shows that $V_\gamma$ is a storage function. \qed

If the system (8.1.2) is linear, time-invariant, a stronger result can be obtained as follows:

Theorem 8.4.2.7. If the system (8.1.2) is linear, time-invariant, i.e.

$$(8.4.2.13) \quad \dot{x}(t) = A^0 x(t) + B_1^0 u(t) + B_2^0 w(t), \quad x(0) = x_0$$

where $A^0, B_1^0, B_2^0$ are constant matrices of appropriate sizes. Assume that:

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(1) \( g \) is convex, lower semicontinuous and there exists an \( \alpha > 0 \) such that
\[
g(x) \geq \alpha |x|^2, \quad \forall x \in \mathbb{R}^n
\]

(2) HJI equation (8.4.2.1) has a nonnegative viscosity supersolution \( V_\gamma \) with property that
\[
\text{dom}(V_\gamma) \supset \text{dom}(g); \quad \text{Then}
\]
(8.4.2.14) \[
\mu(x) = \partial h^*\left(-\left(B_1^0\right)^T p(x)\right)
\]
is an admissible \( \gamma \)-attenuation feedback control. Moreover \( V_\gamma \) is a storage function.

Proof. Since \( V_\gamma : \mathbb{R}^n \rightarrow [0, \infty) \) is convex, (8.4.2.14) is an admissible feedback control by Theorem 8.4.1.9. Note that \( V_\gamma \) is convex, lower semicontinuous, and hence is continuous in \( \text{dom}(V_\gamma) \). Moreover \( V_\gamma \) is right differentiable on \( \text{Intdom}(V_\gamma) \)(cf. [Au]). Therefore from Lemma 8.4.2.2, we have
(8.4.2.15) \[
\frac{dV_\gamma(x(t))}{dt} \leq -\gamma^2 (w - \frac{1}{2\gamma^2} B_2(x(t)) p(x(t)))^2 + \gamma^2 |w(t)|^2 - g(x(t)) - h(\mu(x(t)))
\]
and integrating this on any interval \([t_1, t_2] \) we get
(8.4.2.16) \[
V_\gamma(x(t_2)) \leq V_\gamma(x(t_1)) + \int_{t_1}^{t_2} \gamma^2 |w(s)|^2 - g(x(s)) - h(\mu(x(s))) ds
\]
Since \( V_\gamma \) is nonnegative, this completes the proof. \( \square \)

8.5. An Application to Problems with Control Constraints

In this section, we consider system (8.1.2) with control constraints. Let \( U_0 \) be a closed convex subset of control space \( U \). Then, (8.1.2) can be written as
\[
\dot{x}(s) = A(x(s)) + B_1(x(s)) u(s) + B_2(x(s)) w(s), \quad u(t) \in U_0
\]
\[
x(t) = x.
\]

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Let $I_{U_0}$ be the indicator function for the set $U_0$, i.e.

$$I_{U_0}(u) = \begin{cases} 
0 & \text{if } u \in U_0 \\
+\infty & \text{if } u \notin U_0
\end{cases}$$

We consider, for the finite $t_f$, the cost function

(8.5.1)

$$J_\gamma(t, x; u, w) = \int_t^{t_f} \{ g(x(s)) + |u(s)|^2 - \gamma^2 |w(s)|^2 \} ds + \psi(x(t_f))$$

and if $t_f = +\infty$, we set $\psi \equiv 0$. In order to incorporate this problem into an unconstrained optimization, we modify (8.5.1) into

(8.5.2)

$$J_\gamma(t, x; u, w) = \int_t^{t_f} \{ g(x(s)) + |u(s)|^2 + I_{U_0}(u) - \gamma^2 \|w(s)\|_{L^2(t, t_f)}^2 \} ds + \psi(x(t_f)).$$

Clearly, if $u \notin U_0$, $J(t, x; u, w) \equiv +\infty$. Define $P_{U_0} : U \to U_0$ by $P_{U_0}u = u_0 \in U_0$, where $u_0$ is defined by

$$\inf_{v \in U_0} |v - u| = |u_0 - u|$$

If we let $h(u) = |u|^2 + I_{U_0}(u)$ then the conjugate function of $h$ is given by

$$h^*(u) = \frac{1}{4}(|u|^2 - |u - P_{U_0}u|^2).$$

Therefore our previous result can be restated as follows:

**Theorem 8.5.1.** Assume that:

1. the HJI equation

(8.5.3)

$$V_{\gamma t}(t, x) + g(x) + A(x)^T \nabla_x V_{\gamma}(t, x) - 2(-B_1(x)^T \nabla_x V_{\gamma}(t, x))^T P_{U_0}((-B_1(x)^T \nabla_x V_{\gamma}(t, x))$$

$$+ (P_{U_0}(-B_1(x)^T \nabla_x V_{\gamma}(t, x)))^T P_{U_0}((-B_1(x)^T \nabla_x V_{\gamma}(t, x)))$$

$$+ \frac{1}{4\gamma^2} \nabla_x V_{\gamma}(t, x)^T B_2(x) B_2(x)^T \nabla_x V_{\gamma}(t, x) = 0$$

$$V_{\gamma}(t_f, x) = \psi(x)$$

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admits a nonnegative viscosity supersolution \( V_\gamma \) with the property that \( \text{dom}(V_\gamma) \supset \text{dom}(g) \);

(2) there exists a pair \( \hat{p}, \hat{q} \in D^-V_\gamma \) such that \( \hat{p}(t_f, x) \geq \psi(x) \);

(3) the state feedback controller

\[
\mu(x) = U_0(-B_1(x)^T \hat{p}(t, x))
\]

is an admissible feedback control (see Definition 8.4.2);

(4) there exists a function \( \varphi \in C^1([0, t_f] \times \mathbb{R}^n) \) such that

\[
\varphi \geq 0 \quad \forall (t, x) \in [0, t_f] \times \text{dom}(V_\gamma), \quad \nabla \varphi = (\hat{p}, \hat{q}) \quad \text{a.e.} \ x \in \text{dom}(V_\gamma).
\]

Then, the state feedback controller \( \mu \) is a \( \gamma \)-attenuation feedback control, i.e. we have

\[
\int_0^{t_f} \{g(x(t)) + h(\mu(x(t)))\} dt \leq \gamma^2 \int_0^{t_f} |w(t)|^2 dt + \varphi(x), \quad \forall w \in \mathcal{W}, \ x \in \mathcal{O}
\]

**Theorem 8.5.2.** Suppose that:

(1) the following HJI equation

\[
(8.5.4) \quad g(x) + A(x)^T \nabla_x V_\gamma(x) + 2(-B_1(x)^T \nabla_x V_\gamma(x))^T U_0(-B_1(x)^T \nabla_x V_\gamma(x)) + U_0(-B_1(x)^T \nabla_x V_\gamma(x))^T U_0(-B_1(x)^T \nabla_x V_\gamma(x)) \\
+ \frac{1}{4\gamma^2} \nabla_x V_\gamma(x)^T B_2(x) B_2(x)^T \nabla_x V_\gamma(x) = 0
\]

has a nonnegative viscosity supersolution \( V_\gamma \) with the property \( \text{dom}(V_\gamma) \supset \text{dom}(g) \);

(2) the feedback controller

\[
\mu(x) = U_0(-B_1(x)^T \hat{p}(x))
\]

is an admissible feedback control;
(3) there exists a function \( \varphi \in C^1(\mathbb{R}^n) \) such that

\[
\varphi \geq 0 \quad x \in \text{dom}(V_\gamma), \quad \nabla_x \varphi = \hat{\rho}, \quad \text{a.e.} \ x \in \text{dom}(V_\gamma) \subset \mathbb{R}^n.
\]

Then, \( \mu \) is a \( \gamma \)-attenuation feedback controller. Moreover, the system associated with this control \( \mu \) and worst-case disturbance,

\[
\dot{x} = A(x) + B_1(x)P_{U_0}(-B_1(x)^T \hat{\rho}(x)) + \frac{1}{\gamma^2} B_2(x)B_2^T(x)\hat{\rho}(x),
\]

is asymptotically stable for \( x_0 \in \mathcal{O} \).

**Remark 8.5.3.** If we let \( U_0 = \{0\} \), then (8.5.3), (8.5.4) correspond to the HJI equations of (8.1.1)-(8.1.2) in finite and infinite horizon cases respectively, and (8.5.4) corresponds in fact to (8.3.3) in section 2.1.

**Remark 8.5.4.** In the infinite horizon case, if \( U_0 \equiv \{0\} \) and \( A(x) = A_0 x \) (i.e. \( G \equiv 0 \)) then a necessary condition for (8.5.4) to have a minimal, nonnegative solution is that \( A_0 \) must have at least one negative eigenvalue.

Now let us consider the case when system (8.5.1) is a time-invariant linear system

\[
\dot{x}(t) = A^0 x(t) + B_1^0 u(t) + B_2^0 w(t), \quad u(t) \in U_0
\]

\[
x(0) = x_0
\]

Suppose that \( g(x) = x^T Q x, \psi(x) \equiv 0, U_0 \) is a subspace of \( U \), and (8.5.3) has a nonnegative viscosity supersolution \( V_\gamma \) (it is in fact a viscosity solution). It is not difficult to see that \( V_\gamma \) in fact is a (unique) classical solution and \( V_\gamma(t,x) = x^T Z_\gamma(t)x \) where \( Z_\gamma \) satisfies

\[
\dot{Z}_\gamma(t) + A^T Z_\gamma(t) + Z_\gamma(t)A - 2Z_\gamma(t)B_1 P_{U_0} B_1^T Z_\gamma(t)
\]

\[
+ Z_\gamma(t)(B_1 P_{U_0}^T P_{U_0} B_1^T + \frac{1}{\gamma^2} B_2 B_2^T) Z_\gamma(t) + Q = 0
\]

\[
Z_\gamma(t_f) = 0
\]
If there exists a constant $C$ such that

$$Z_\gamma(t) \leq C \quad \forall t > 0$$

then $\lim_{t \to \infty} Z_\gamma(t)$ exists. Define

$$Z_\gamma^\infty = \lim_{t \to \infty} Z_\gamma(t).$$

Then $V_\gamma(x) = x^T Z_\infty x$ is the minimal, nonnegative solution of (8.5.4). More precisely, $Z_\infty$ is the minimal, nonnegative solution of

$$Z_\gamma + A^T Z_\gamma + Z_\gamma A - 2 Z_\gamma B_1 P U_0 B_1^T Z_\gamma + Z_\gamma (B_1 P U_0 P U_0 B_1^T + \frac{1}{\gamma^2} B_2 B_2^T) Z_\gamma + Q = 0$$

In the finite horizon case the optimal feedback control and the worst disturbance are given by, respectively

(8.5.5) \quad \mu(x(t)) = -P U_0 B_1^T Z_\gamma(t) x(t)

(8.5.6) \quad \nu(x(t)) = \gamma^{-2} B_2^T Z_\gamma(t) x(t)

while in the infinite-horizon case are

(8.5.7) \quad \mu(x(t)) = -P U_0 B_1^T Z_\gamma^\infty x(t)

(8.5.8) \quad \nu(x(t)) = \gamma^{-2} B_2^T Z_\gamma^\infty x(t)

with $\lim_{t \to \infty} x(t) = 0$ by Theorem 8.4.2.3. Some further analysis on this problem leads to the following theorem:

**Theorem 8.5.5.** Let $x_0 = 0$ and

$$\rho(u, w) = \sup_{t > 0} \frac{\int_0^t \{ |x(s)|^2 + |u(s)|^2 + I_{U_0}(u(s)) \} \, ds}{\int_0^t |w(s)|^2 \, ds}.$$
Given $\gamma > 0$,

$$\sup_{w} \inf_{u} \rho(u, w) < \gamma^2$$

if and only if (8.5.1) admits a minimal, nonnegative, continuous viscosity solution. In this case the admissible $\gamma$-attenuation feedback controller is given by (8.5.5). Moreover

$$\sup_{w} \inf_{u} \rho(u, w) < \gamma^2 \implies \inf_{M_{CL}} \sup_{w} \rho(u, w) = \sup_{w} \inf_{u} \rho(u, w) < \gamma^2.$$ 

Proof. The proof is similar to Corollary 4.2 of [XB3].

8.6. Examples

1. A bilinear system with control constraints. Consider the bilinear (scalar) dynamics

(8.6.1) \[ \dot{x}(t) = (u(t) + w(t))x(t), \quad x(0) = x_0 \]

and the cost function

$$J_\gamma(x; u, w) = \int_{0}^{\infty} \{x^2(t) + u^2(t) + I_{U_0}(u(t)) - \gamma^2 w^2(t)\} dt$$

Note that in this case there is trivially an admissible feedback controller. The associated HJI equation is

(8.6.2) \[ -x^2 + h^*(-V_x x) - \frac{1}{4\gamma^2} V_x^2 x^2 = 0 \]

and

$$h^*(u) = \frac{1}{4}(u^2 - (u - P_{U_0} u)^2)$$

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First let us look at the particular control constraint \( U_0 = \{0\} \), which corresponds to the control free case of section 2.1. In this case \( h^* \equiv 0 \) and (5.2) becomes

\[
(8.6.3) \quad -x^2 - \frac{1}{4\gamma^2} V_x^2 x^2 = 0
\]

Clearly, for any \( x \neq 0 \) (5.3) has no viscosity supersolution. Suppose that the control constraint is \( u \geq 0 \), and thus \( U_0 = [0, +\infty) \). Then the HJI equation is

\[
(8.6.4) \quad x^2 + \frac{1}{4} \{(V_x x + P_{U_0}(-V_x x))^2 - (V_x x)^2\} + \frac{1}{4\gamma^2} V_x^2 x^2 = 0
\]

It is easy to verify that (8.6.4) has a \textit{nonnegative minimal} viscosity supersolution if and only if \( x \leq 0 \), which means that in order for the attenuation level to be \( \gamma > 1 \), the initial condition must be negative. In this case the nonnegative minimal viscosity supersolution of (8.6.4) is

\[
(8.6.5) \quad V_\gamma(x) = -\frac{2\gamma}{\sqrt{\gamma^2 - 1}} x.
\]

The disturbance attenuating feedback control is given by

\[
\mu^\infty(x(t)) = \begin{cases} 
0 & \text{if } x(t) \geq 0 \\
-\frac{\gamma}{\sqrt{\gamma^2 - 1}} x(t) & \text{if } x(t) \leq 0
\end{cases}
\]

Now we change the constraint to \(-a \leq u \leq a\), i.e. \( U_0 = [-a, a] \) where \( a > 0 \). When \( \gamma > 1 \) the HJI equation (8.6.4) has a viscosity supersolution on the interval \(-\frac{a}{2} \sqrt{\gamma^2 - 1} \leq x \leq \frac{a}{2} \sqrt{\gamma^2 - 1}\):

\[
V_\gamma(x) = \begin{cases} 
\frac{2\gamma}{\sqrt{\gamma^2 - 1}} |x| & \text{if } |x| \leq a \frac{\sqrt{\gamma^2 - 1}}{2\gamma} \\
G(x) & \text{if } x \geq a \frac{\sqrt{\gamma^2 - 1}}{2\gamma} \\
H(x) & \text{if } x \leq -a \frac{\sqrt{\gamma^2 - 1}}{2\gamma}
\end{cases}
\]

where

\[
G'(x) = \frac{a\gamma^2 - \gamma \sqrt{a^2 \gamma^2 - a^2 - 4x^2}}{x}, \quad H'(x) = -\frac{a\gamma^2 - \gamma \sqrt{a^2 \gamma^2 - a^2 - 4x^2}}{x}
\]
The controller and the disturbance obtained using this viscosity supersolution are:

$$
\mu^\infty_\gamma(x) = \begin{cases} 
-\frac{1}{\sqrt{\gamma^2 - 1}}|x| & \text{if } -a\frac{\sqrt{\gamma^2 - 1}}{2\gamma} \leq x \leq a\frac{\sqrt{\gamma^2 - 1}}{2\gamma} \\
\frac{1}{2}(a\gamma^2 - \gamma\sqrt{a^2\gamma^2 - a^2 - 4x^2}) & \text{if } a\frac{\sqrt{\gamma^2 - 1}}{2\gamma} \leq x \leq a\frac{\sqrt{\gamma^2 - 1}}{2}\gamma \\
\frac{1}{2}(a\gamma^2 - \gamma\sqrt{a^2\gamma^2 - a^2 - 4x^2}) & \text{if } -a\frac{\sqrt{\gamma^2 - 1}}{2\gamma} \leq x \leq -a\frac{\sqrt{\gamma^2 - 1}}{2}\gamma 
\end{cases}
$$

$$
\nu^\infty_\gamma(x) = \begin{cases} 
-\frac{1}{\gamma\sqrt{\gamma^2 - 1}}|x| & \text{if } -a\frac{\sqrt{\gamma^2 - 1}}{2\gamma} \leq x \leq a\frac{\sqrt{\gamma^2 - 1}}{2\gamma} \\
\frac{1}{2\gamma}(a\gamma - \sqrt{a^2\gamma^2 - a^2 - 4x^2}) & \text{if } a\frac{\sqrt{\gamma^2 - 1}}{2\gamma} \leq x \leq a\frac{\sqrt{\gamma^2 - 1}}{2}\gamma \\
\frac{1}{2\gamma}(a\gamma - \sqrt{a^2\gamma^2 - a^2 - 4x^2}) & \text{if } -a\frac{\sqrt{\gamma^2 - 1}}{2\gamma} \leq x \leq -a\frac{\sqrt{\gamma^2 - 1}}{2}\gamma 
\end{cases}
$$

It is not difficult to see that for any $x_0 \in [-a\frac{\sqrt{\gamma^2 - 1}}{2\gamma}, a\frac{\sqrt{\gamma^2 - 1}}{2\gamma}]$, the worst-case system dynamics described by

$$
\dot{x}^*(t) = (\mu^\infty(x^*(t)) + \nu^\infty(x^*(t)))x^*(t)
$$

has a unique solution

$$
x^* \in C(\mathbb{R}^+; \mathbb{R}) \cap L^2(\mathbb{R}^+; \mathbb{R}); \quad \lim_{t \to \infty} x^*(t) = 0
$$

**Remark 3.6.1.** In this example we can see the connections between the initial condition, control constraint, and the desired attenuation level. Increasing $\gamma$ results in a larger range of values of initial conditions, while making the control more restrained will limit the initial data.

2. **A case with only lower semicontinuous value function.** We still consider the bilinear dynamics as example 5.1, but the cost function is now

$$
J_\gamma(x; u, w) = \int_0^\infty \{g(x(t)) + u^2(t) - \gamma^2w^2(t)\}dt
$$

where

$$
g(x) = \begin{cases} 
(2x + 1)^2 & \text{if } x > 1 \\
x^2 & \text{if } x \leq 1
\end{cases}
$$
Note that $g$ is lower semicontinuous. The corresponding HJI equation is

\[(8.6.6) \quad -g(x) + \frac{1}{4}(1 - \frac{1}{\gamma^2})V_x^2x^2 = 0\]

When $\gamma > 1$, the nonnegative minimal viscosity supersolution of (8.6.6) is

\[V_{\gamma}(x) = \begin{cases} \frac{2\gamma}{\sqrt{\gamma^2 - 1}}(2 + \ln x) & \text{if } x > 1 \\ \frac{2\gamma}{\sqrt{\gamma^2 - 1}}|x| & \text{if } x \leq 1 \end{cases}\]

which is lower semicontinuous. The controller obtained using this viscosity supersolution is

\[\mu^\infty_{\gamma}(x) = \begin{cases} -\frac{2\gamma}{\sqrt{\gamma^2 - 1}} & \text{if } x > 1 \\ -\frac{2\gamma}{\sqrt{\gamma^2 - 1}}|x| & \text{if } x \leq 1 \end{cases}\]

The worst disturbance is

\[\nu^\infty_{\gamma}(x) = \begin{cases} \frac{2}{\gamma\sqrt{\gamma^2 - 1}} & \text{if } x > 1 \\ \frac{2}{\gamma\sqrt{\gamma^2 - 1}}|x| & \text{if } x \leq 1 \end{cases}\]

It is now easy to see that the corresponding worst-case system trajectory

\[\dot{x}(t) = (\mu^\infty_{\gamma}(x(t)) + \nu^\infty_{\gamma}(x(t)))x(t)\]

is globally asymptotically stable.

8.7. Concluding Remarks

In this chapter, we have studied the existence of viscosity supersolutions to HJI equations that arise in a class of nonlinear $H^\infty$ control problems. We have shown that under a general cost function (not necessarily continuous), lower semicontinuous value functions (or more general lower semicontinuous storage functions) exist. The notion of an admissible $\gamma$-attenuation state feedback controller has been introduced, and it has been shown that

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such a controller exists when the value function is convex. If the convexity property does not hold, then verifying existence of such a controller stands out as a challenging problem for the future. A possible approach here is to use the Continuous Selection Theorem or the Measurable Selection Theorem[De]. Another approach is to use the Viability Theorem [AF] in which case additional assumptions may be needed. Moreover one can exhibit the relationship between storage functions, and viscosity and epicontingent supersolutions in the lower semicontinuous case(see [FQ]).
9. Conclusions and Future Challenges

9.1. Summary of the thesis

In Chapter 2, we have developed an $H^\infty$-optimal control theory for infinite-dimensional systems with general cost functions with the standard quadratic cost function being a special case. Thus the theory developed in this chapter covers the standard results of $H^\infty$ problems in finite- and infinite-dimensions. The result on the existence of finite-dimensional compensators for infinite-dimensional $H^\infty$ problems provided in Chapter 3 is an important step in the control of real distributed parameter systems with uncertainties, because in infinite-dimensions one can never observe the entire state and thus it is necessary to control (or stabilize) by (finite-dimensional) output feedback.

As we mentioned before, the key issue in $H^\infty$-optimal control problems is to determine the critical values, which actually are the minimax disturbance attenuation levels. Such critical values for linear time-varying evolution systems in finite-horizon have been explicitly given in Chapter 4, which now make numerical calculations for these values possible. Hybrid control is also an important controller in practice because output information is not always a continuous stream, and is often only available at discrete time instants. Chapter 5 provides sampled-data schemes for $H^\infty$-optimal control for general linear evolution systems, and presents such hybrid controllers. Chapter 6 gives another set of sampled-data schemes for $H^\infty$-optimal control for general linear parabolic systems where hybrid controllers only act on the boundary.

Viscosity solution theory produces a fundamental basis in which continuous value functions (not necessarily differentiable) can be used to construct feedback controllers. Chapter
7 provides the existence and uniqueness of viscosity solutions for coupled nonlinear partial differential equations, and the results also cover the uncoupled cases. Chapter 8 removes the requirement for value functions to be continuous, and allows for the construction of $H^\infty$-optimal controllers by value functions which are only lower semicontinuous.

9.2. Possible extensions

Results in Chapter 2 can be generalized to the infinite-horizon case by placing additional assumptions on the system operator $A$. For example, by assuming that $A$ generates a compact analytic $C_0$ semigroup, a similar proof as in this chapter can be carried out because the compactness and analytical property of $C_0$ semigroup can replace the Arzela-Ascoli Theorem which was used in this Chapter. Making the assumption that $A$ is just a dissipative operator is also possible, but this leads to a much more challenging research problem because one then has to deal with the Hamilton-Jacobi-Isaacs evolution equation in the viscosity sense, which is further discussed in the next section.

There are two not only interesting but also useful extensions of results of Chapter 3, which are: (1) the finite-dimensional compensator applies controls only at the system boundaries, (2) the finite-dimensional compensator is a hybrid controller. Such extensions will lead to easier implementation of the control law and will lead to more practical results, in particular when the level of complexity of the underlying systems increase.

Chapter 6 can be viewed as a generalization of Chapter 5. In Chapter 6, we did not provide the expression for the critical value for the boundary-value problem because the generalized Riccati equation involved an unbounded operator with fractional power $\alpha$, which is not uniquely determined. The generalized Riccati equation admits a unique
solution only under the assumption that \( \alpha \) is pre-specified, and fixed. Thus a further study will be needed to investigate whether a critical value can be defined to be independent of \( \alpha \). If not, how to choose an appropriate \( \alpha \) would become a major issue for \( H^\infty \) boundary control problems.

The uniqueness of the super-viscosity solution of Chapter 8 can be proven by following the argument used to prove the generalized comparison theorem of Chapter 7 under certain additional assumptions (for establishing the inequality of other direction). A major difficulty in Chapter 8 is how to ascertain that a designed controller is an admissible control for a system. A traditional method is to use various fixed point theorems, which however is quite restrictive here because fixed point theorems require either the mappings to be contractive or the mappings to be on compact sets (uniqueness can not be guaranteed in this case in general). Such restrictions result in directly assuming that a design is admissible control, as in most of the current works on the topic (see, for instance, [Sor], [FQ], [KS]), and thus the applications become questionable. A possible way to alleviate this difficulty we believe is to use the Continuous Selection Theorem or the Measurable Selection Theorem (see [De]). Continuous Selection Theorem only requires the (multivalued) mapping to be lower semicontinuous and its range to be convex, and Measurable Selection Theorem concerns with more general topological spaces, keeping the condition of lower semicontinuity. It will be significant if one can find necessary and sufficient conditions for the solvability of the HJI equation after a corresponding optimal admissible controller has been selected and substituted back into the HJI equation.
9.3. Future challenges

The study of $H^\infty$-optimal control problems in infinite-dimensional spaces requires dealing with evolutionary or stationary Hamilton-Jacobi-Isaacs equations which are obtained from dynamic programming arguments. When systems are linear and cost functions are quadratic, the HJI evolution equations become evolutionary or stationary generalized (nonstandard) Riccati evolution equations. More precisely, let $H$ be a real separable infinite-dimensional Hilbert space and $A$ be an unbounded, linear operator controlling the semigroup generated by $-A$. Then the evolutionary and stationary generalized Riccati evolutionary equations have the following forms:

$$V_{t+} < Ax, \nabla V > + F(t, x, \nabla V) = 0 \quad \text{in } (0, T] \times H$$  \hspace{1cm} (E)

and

$$< Ax, \nabla V > + F(x, \nabla V) = 0 \quad \text{in } H$$  \hspace{1cm} (S)

where the solutions $V(t, x)$ and $V(x)$ are real-valued functions defined on $[0, T] \times H$ and $H$, respectively, with $T > 0$ given, and $F(t, x, p)$ and $F(x, p)$ being real-valued continuous functions satisfying some regularity assumptions. Without any further assumptions on $A$, such as $A$ generating a $C_0$ analytic semigroup or a $C_0$ differential semigroup, smooth solutions usually cannot be obtained. Introducing solutions of (E) and (S) in the viscosity sense is quite different from the one in the finite dimensional cases because of the presence of the unbounded terms $< Ax, \nabla V >$. By assuming $A$ to be a linear densely defined maximal monotone operator, Crandall and Lions have interpreted (E) and (S) in the viscosity sense when the solutions may be expected to be weakly sequentially continuous and the viscosity notions can handle the terms $< Ax, \nabla V >$. Tataru studied (E) and (S)
in general Banach spaces by introducing an equivalent definition of a viscosity solution, with the requirement on $A$ being an $m$-accretive operator. However, there are a number of well-known equations which may not fit into the above framework because the operator $A$ is usually not dissipative. Some examples are the Schrödinger's equation, Korteweg-de Vries equation, and hereditary differential equation. Thus a broader notion of a viscosity solution will be expected when $A$ is merely an infinitesimal generator of a $C_0$ semigroup. The major difficulty is that in order to consider viscosity solution of, for example, (S), we would typically consider maxima and minima of functions $V - \varphi$, where $\varphi$ is differentiable, and evaluate the equation at these points with $\nabla V$ replaced by $\nabla \varphi$. By contrast, in the theory of non-evolutionary case, the expression $< Ay, \nabla \varphi(y) >$ is not defined in general.

In order to have this expression well defined, Crandall and Lions have made use of the fact that $< Ax, x > \geq 0$ for any $x$ in accordance with the maximal monotonicity of $A$. Tataru's notion is essentially the same as Crandall and Lions' where $m$-accretivity must be used. In spite of the limitation we mentioned above, there is still considerable room for an $H^{\infty}$-optimal control theory to be developed under the assumption that $A$ is a maximal monotone operator. We have so far seen only one paper applying this notion, which is the one by Kocan and Soravia [KS], where they have assumed $A$ to be a maximal monotone operator on a Hilbert space.

In the past few years, geometric theory has proved to be an effective means of analysis and design of nonlinear control systems (governed by ordinary differential equations) as it was in the past for the Laplace transform, complex variable theory and linear algebra in relation to linear systems. Synthesis problems of longstanding interest such as disturbance decoupling, noninteracting control, output regulation, and the shaping of the input-output
response, can be dealt with relative ease by using differential geometric methods. On the other hand, because of its complexity and sensitivity to certain variations, the evolution of a nonlinear system cannot be predicted by mere computations, be it analytical or numerical. Such computations do not offer a satisfactory solution, even if they produce a feasible one; nonlinear phenomena are global and there is a need for a more geometrical description of the phenomena which could provide the proper guidelines for the computation. The limits of the computational methods have been pointed out by Poincaré in his classical work on differential equations; he showed the need to marry analytic and geometric methods, and although he was concerned with asymptotic analytic methods, this also applies to numerical methods since the difficulty is inherent in the problem. In the finite-dimensional robustness control, Trentelman and Stoorvogel used special geometric co-ordinate decompositions and results from almost disturbance decoupling to obtain the stability radius, and such a result was then applied to a singular $H^\infty$-problem (see [ST], [Cu1]). The stability radius provides an important index to design a feedback stabilizing controller. But since the geometric theory in infinite-dimensions is much less complete (see [Zw]), the result mentioned above has not been fully generalized to infinite-dimensional systems as yet. Therefore, developing geometric theory in infinite-dimensions is believed to be also important in control theory, including $H^\infty$-optimal control theory as well as related computational problems.

An other interesting but challenging research topic for $H^\infty$-problems in infinite-dimensions is to investigate the relationship between state and frequency domain formulations of the $H^\infty$-problems. In finite-dimensions, the state-space approach is usually more attractive because of its numerical advantages, while this will not always be the case in infinite-dimensions. In general, transfer functions of distributed parameter systems are
transcendental functions in the Laplace transform variable $s$, such as time delay, stiffness or damping coefficient of a beam, which results in unpleasant state-space realizations. Such effects on the control design, however, can be studied easily in the frequency domain approach. Therefore, while the state-space $H^\infty$-problems presents some challenging problems for LQ-exponents, frequency domain approach may offer suitable solutions, in particular, from the computational view point.

There are some other interesting open problems in infinite-dimensions for $H^\infty$-problems, such as $H^\infty$-filter problems, parameter identification problems for uncertain plants, singular $H^\infty$-problems, \ldots. We are not going to discuss these issues here, and the research on these problems will be carried on in the near future.

◊ ◊ ◊ ◊ ◊

To end this thesis, I would like to quote a few sentences from Richard Wollheim in his article *Thread of Life*: "It is perfectly true, as philosophers say, that life must be understood backwards. But they forget the other proposition, that it must be lived forwards... And if one thinks over the proposition it becomes more and more evident that life can never really be understood in time simply because at no particular moment can I find the necessary resting-place from which to understand it - backwards" (Harvard University Press, Cambridge, Mass. 1984).
References


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[Tad] G. Tadmor, Worst-case design in the time domain. The maximum principle and the standard $H^\infty$-problem, MCSS 3 (1990), 301-324.


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PUBLICATIONS:


3. *Finite-Dimensional Compensators of $H^\infty$-Optimal Control for Infinite-Dimensional Systems via A Galerkin-Type Approximation*, Report of Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, August 1996. (co-authored with T. Başar)


H-INFINITY CONTROL IN INFINITE-DIMENSIONAL SPACES AND RELATED PARTIAL DIFFERENTIAL EQUATIONS

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This thesis furthers the $H^\infty$-optimal control theory for infinite-dimensional systems and develops new results on related partial differential equations.

The thesis consists of nine chapters, and begins with a general introduction to $H^\infty$-optimal control problems for infinite-dimensional systems, and to the corresponding theory of (viscosity or operator) solution for Hamilton-Jacobi-Isaacs equations (HJI). The second chapter studies the $H^\infty$ Control of a class of infinite-dimensional linear systems with nonlinear outputs. By using a differential game-theoretic approach, a set of necessary and sufficient conditions are obtained for the existence of a state-feedback controller under which a given $H^\infty$ bound (on disturbance attenuation) is achieved. Characterization of such a controller is given, and the result is applied to a disturbance attenuation problem with control constraints.

The third chapter demonstrates the existence of finite-dimensional compensators in connection with the $H^\infty$-optimal control of infinite-dimensional linear time-invariant systems. The approach adopted uses a Galerkin-type approximation, where there is no requirement for the system operator to have a complete set of eigenvectors. It is shown that if there exists an infinite-dimensional compensator satisfying the $H^\infty$ robustness property, then a finite-dimensional compensator exists and achieves the same level of robustness. The order
of the finite-dimensional compensator depends on the requirement of robustness which is measured by the difference between the desired attenuation level and the critical value of the system, which is inherent to a given system. A procedure for constructing finite-dimensional compensators based on approximate solution of the optimal regulator and optimal observer via Galerkin-type approximations is provided.

A relationship between two generalized Riccati equations and two critical parameter values, which are related to the attenuation of $H^\infty$-optimal control problems associated with a class of time-variant linear evolution systems on infinite-dimensional spaces, is established in chapter 4. Explicit formulas for calculating these two critical parameters as well as the closed-form solutions of these two generalized Riccati equations are given. The connection between these two parameters and a two-person differential game is also investigated. The approximation of the infinite-dimensional Riccati equations is also considered.

The fifth chapter deals with the $H^\infty$-optimal control problem in infinite dimensional linear systems when only sampled values of the state are available. For finite-horizon time-varying systems the optimum controller is characterized in terms of the solution of a particular generalized Riccati evolution equation, with the optimum performance determined by a family of generalized Riccati evolution equations. For the infinite-horizon time-invariant problem the optimum controller is characterized in terms of the solution of a particular generalized Riccati evolution equation, and the performance is determined in terms of the conditions of a single generalized Riccati equation, defined on the longest sampling interval. It is shown that when the sampling interval is chosen appropriately the sampled feedback controller adopted can achieve the desired attenuation level for which usually regular feedback controller is needed. Moreover an upper bound on the sampling
interval is estimated to guarantee desired performance. Two algorithms are presented for designing sampled-data controllers, one for the finite-horizon case, and the other one for the infinite-horizon case.

The sixth chapter is considered to be a further study of the topics of the fifth chapter, when only sampled values of the state are available, with control acting on the boundary. Under appropriate assumptions the existence and uniqueness of an optimal disturbance attenuating controller are proven. Properties of the solutions of the associated generalized Riccati equations are also studied. The problem is formulated in a differential game framework by associating a zero-sum differential game with the original disturbance attenuation control problem. The minimizing player's minimax strategy in this game corresponds to the optimal controller in the disturbance attenuation problem, which is linear and is characterized in terms of the solution of a particular generalized Riccati evolution equation. The optimum achievable performance is determined by the condition of existence of a solution to another family of generalized Riccati evolution equations. The formulation allows for the control to be time-varying between two consecutive sampling times, and in this respect this chapter presents optimum choices for these waveforms as functions of sampled values of the state. A relationship is obtained between the length of the longest sampling interval and the performance of the derived controllers, and it is shown that as the former goes to zero the latter converges to the optimum performance obtained under continuous feedback information.

In the seventh chapter, two Hamilton-Jacobi-Bellman (HJB) equations corresponding to nonlinear control of piecewise deterministic systems are considered in which the control could be unbounded. Since these two HJB equations are coupled partial differential
equations, their Hamiltonian structures are different from the standard one. Existence and uniqueness results have been obtained in the sense of viscosity solutions. Explicit control structures are also provided.

The eighth chapter studies the solvability of HJI equations that arise in finite and infinite-horizon nonlinear $H^\infty$ control problems where the system is affine in the control and the disturbance, while the cost function is not necessarily continuous in the state and the control. In each case the existence of viscosity supersolutions is proven, under the assumption that the value function is finite. A result on global asymptotic stability of closed-loop system under the $H^\infty$ controller and the worst-case disturbance is obtained.

The thesis ends with chapter 9, which provides a summary of contents and contributions of the thesis, and discusses possible extensions of the results obtained, as well as some future challenges.