WAVE TAILORING IN ELASTIC AND ELASTOPLASTIC GRANULAR SYSTEMS

BY

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DISSERTATION

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Abstract

This dissertation studies wave propagation in granular media with the objective of developing stress wave tailoring applications. Two mechanisms for wave tailoring are investigated: the first part focuses on energy dissipation in elasto-plastic granules and the second studies tunable wave propagation in elastic granular lattices. We start by developing a unified contact law for elasto-plastic granules of distinct sizes and material properties using quasi-static finite element simulations. Extensive numerical studies are then conducted on the dynamics of elastic and elasto-plastic granular chains under a wide range of loading conditions and models are developed for predicting the key quantities. Compared to their elastic counterparts, elasto-plastic chains exhibited distinct features like rapid decay of waves, formation and merging of wave trains, yielding of contact points, etc. Then we quantify key impact properties of 3D granular packings and compare with 3D continuum media. Scaling laws for dissipation are derived from first principles and verified numerically for both the media.

In the second part of this dissertation, we develop systems for tunable wave propagation by exploiting the intrinsic nonlinearity of Hertzian contact in elastic granular lattices. We design a granular lattice of spheres packed in a cylindrical tube whose response can be varied from near solitary waves to rapidly decaying waves by applying external precompression. The designs are demonstrated using numerical simulations and the trends are explained by an asymptotic analysis. We also designed energy filters and band gap systems tunable by external control using lattices of spheres and cylinders subjected to impact and harmonic loadings. Finally, we introduce the concept of wave tailoring by altering the network topology in granular lattices. The designs are demonstrated using a combination of modeling, numerical simulations and experiments. Good agreement is obtained between them, illustrating the feasibility of our designs for practical applications.
To my parents, for their love and support.
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Chapter 1

Introduction

1.1 Motivation: Wave tailoring

Designing materials and systems for controlling energy from dynamic loadings finds diverse applications and continues to be an active area of research. Real world examples range from impact protection, blast wave resistance to those systems experiencing high pressure load fluctuation and energy harvesting devices. There have primarily been two approaches to designing these material systems. The first is to develop new materials including high strength alloys, composites, shock absorbing polymers, etc. The second more recent approach is to build metamaterials from conventional materials by exploiting specific geometric properties, thus resulting in features not achievable by conventional materials. This approach has led to developing systems with exotic properties like negative stiffness [1], negative Poisson’s ratio [2], specified bandgaps, etc, which enable to build new applications and devices for dynamics applications. A key challenge is to develop systems to control or tailor the wave energy from an external source.

Granular media are a class of metamaterials, which have distinct properties compared to their continuum counterparts owing to their discreteness and nonlinear inter-particle interactions. They have been widely studied over the past few decades and have demonstrated potential in diverse fields ranging from dynamic stress wave mitigation by dissipation at point contacts [3], acoustic filtering and rectification by exploiting bifurcations and band gaps [4], granular grippers activated by jamming [5], etc. Nesterenko [6] in his seminal work predicted the existence of solitary waves in chains of ordered granules subjected to impact loading. Since then, a number of different configurations of ordered granular chains have been studied and a wide variety of physical phenomena have been reported [7]. These unique features
have the potential to be useful for shock disintegration and other wave tailoring applications.

We are interested in the potential of these granular materials to dissipate energy for use as impact protectors on systems subjected to high loads, of the order of a few kilonewtons. Most of the prior studies have been conducted on elastic granules, which are typically modeled by spheres. They are characterized by a point contact or a small area of contact, which leads to high stress concentrations in the contact regions. In real systems subjected to high loads, these stress concentrations lead to plastic deformations, and the effects of plasticity become significant. Hence, it is crucial to develop a fundamental understanding of the role of plasticity in wave propagation in granular media, which will form the basis of designing systems for stress wave tailoring applications.

We are also interested in designing wave tailoring systems tunable by a simple, robust external control, again when subjected to high loads, which is of practical significance. In the last decade, granular crystals have been used for novel applications including acoustic lenses, acoustic diodes and rectifiers, etc by exploiting the nonlinearity of Hertzian elastic contact [8]. These have limited regime of applicability due to plasticity limits and are generally hard to control reliably. Recently, preconditioning [9] has been demonstrated as a means to achieve solitary waves of high amplitudes (a few kN) without causing plastic dissipation. This paves the way for diverse applications using elastic contact nonlinearity even at high loads.

1.2 Literature review

1.2.1 Granular media

With mechanical properties distinct from their solid and fluid counterparts [10], granular media have been an active research area in the past few decades. These systems are typically composed of regular or irregular contacting granules [11, 12, 13], and have been extensively modeled as deformable spheres. Two broad approaches have been adopted in modeling granular media: discrete element methods and continuum-based hydrodynamic approaches [14]. Impact of large projectiles on granular media have primarily been modeled
by the first approach [15, 16, 17, 18], where the time scales of individual collisions become relevant. The primary focus of these studies is to understand the effect of the size and velocity of the impactor on the penetration depth. The role of friction and structural disorder is also of key interest in these studies, where the media exhibits signatures of both solid and fluid behavior. Saeki [19] studied the effect of granular particle size and impact velocity on impact damping in a vibrating system. Tanaka et al. [20] studied 2D granular packings subjected to impact by a spherical projectile and characterized the contact forces experienced at the bottom of a plate. In [21], the authors studied dissipation in viscoelastic granular collisions using the Hertz-Kuwubara-Kono model and determined scaling of coefficient of restitution and collision time for these particles. Potapov and Campbell [22] studied breakage of granules by varying impact velocity and particle size. The second approach [23, 24, 25] is useful for studying phenomena which occur over much larger time scales, compared to time scales of individual collisions. In the present work, our focus is on the dynamics of ordered granular systems and we provide a comprehensive literature review on these systems in the remainder of this section.

### 1.2.2 Elastic granular systems

Nesterenko [6] predicted and numerically demonstrated the existence of solitary waves, which are compact non-dispersive stress waves in a chain of identical elastic granules in contact under zero prestress. The system was modeled as point masses connected by nonlinear springs with stiffness derived from the Hertzian contact law. Using a long wavelength approximation, he predicted a travelling wave solution for the strain $\xi(x, t)$:

$$
\xi(x, t) = \left( \frac{5 V^2}{4 c^2} \right)^2 \cos^4 \left( \frac{\sqrt{10}}{5a} (x - Vt) \right),
$$

(1.1)

where $V$ is the wave speed, $a$ is the granule diameter and $c$ is a material property. This wave solution has a spatial width of about $5a$, while its wave speed is amplitude dependent and matches well with numerical simulations. Solitary waves have since been observed experimentally [11, 26] for chains subjected to impact velocities small enough to avoid plastic deformations.
The travelling wave solutions have been refined by Chatterjee [27] following an asymptotic approach on the exponent in the Hertzian contact law and by Sen et al. [28], who introduced a series solution with the coefficients obtained by fitting numerical solutions.

The wave structure of these chains is also sensitive to the loading duration [29, 30], with longer loading times leading to qualitatively different behaviors like wave trains [31] and shock fronts [32]. Wave trains in contacting elastic granules have been observed in various studies, starting from the works of Lazaridi and Nesterenko [31]. Job et al. [29] observed wave trains in granular chains subjected to impulse loadings in two cases: when the mass of the striker is large, and when there is a change in radii along the chain. Sokolow et al. [33] conducted similar studies and noted the absence of a simple relationship between the amplitudes of various solitary waves in the wave train. Molinari and Daraio [32] studied the characteristics of periodic granular chains subjected to a constant velocity at one end, and characterized the quasi-steady response of the system using a homogenized equation and a traveling wave solution.

Porter et al. [34] investigated the existence of solitary waves in heterogeneous dimer and trimer chains, and determined the wave width, speed and forces for various material combinations. Harbola et al. [35] studied pulse propagation in dimer chains using a binary collision model and obtained analytic expressions for the key wave propagation properties. Jayaprakash et al. [36, 37] studied wave propagation in dimer granular systems and reported the existence of new kinds of solitary waves for certain mass ratios. Manjunath et al. [38] also observed a similar family of solitary waves in periodic 2-D lattices subjected to plane loading. Both studies utilize Sen’s solution of solitary waves [28] in homogeneous chains and develop solutions in other systems using asymptotic analysis. In 2D, Awasthi et al. [13] studied wave propagation in elastic square packings with intruders and observed distinct wave patterns depending on the relative material properties of large and small beads.

There have also been studies on the interaction of solitary waves in granular chains with boundaries, starting from the work of Sen et al. [39], who modeled and studied experimentally the effect of solitary waves reflecting off a rigid wall. Daraio et al. [40] studied interaction of waves at the interface of granular chain with a linear elastic medium. Using a time-delay system model to
simulate wave propagation in the linear elastic medium, the authors studied the effect of properties of a single and a composite medium in contact with a granular chain.

The dynamics of these chains is quite distinct under a static precompression load and their long wavelength approximation leads to the KdV equation whose solitary wave has a different structure than (1.1). Daraio et al. [41] studied the effect of precompression on granular chains and demonstrated that their peak force and wave speed vary with the level of static compression. Herbold et al. [42] studied pulse propagation in diatomic periodic chains under precompression and characterized the effect of acoustic frequency band-gaps on wave propagation.

1.2.3 Dispersion and dissipation

While solitary waves form and travel unattenuated through certain granular systems under special conditions, wave attenuation by dispersion and dissipation leads to a very different behavior in many granular systems. In recent years, the potential for impact wave mitigation by dissipation [43, 44, 45], inter-particle friction [46] and randomness in properties [47, 48] have been demonstrated in these systems. Wave dispersion has been observed in chains having varying radii, due to impedance mismatch along the chain [29, 49, 34] which causes wave reflection at each contact. Melo et al. [50] experimentally demonstrated the effect of wave propagation in tapered chains, where the leading wave accelerates and decreases in amplitude. Sen et al. [7, 51] studied the dynamics of a chain of spheres with uniformly varying radii.

Dissipation has primarily been modeled by adding a viscoelastic term [43, 44, 45] to the contact law, with the form of viscoelastic term differing in various works. The dissipation is associated with the relative motion between particles. Rosas et al. [43] conducted the first numerical study in this direction and observed a two-wave structure when a dissipative term is added to the Hertzian contact force. Vergara [45] modeled dissipation in granular chains by adding a viscoelastic term and a term proportional to the square of the beads’ relative velocities to the equations of motion. In [44], Carretero-Gonzalez et al. modeled dissipation by adding a power law of relative velocities to the Hertzian law, with the power law exponent determined
from experiments to fit the simulation data. A key observation from that study is the existence of secondary waves below a critical exponent.

Most of the aforementioned studies have been conducted for impact velocities small enough to avoid yielding. As noted in [52, 53], when these systems are subjected to high forces, stress concentration at contacts lead to permanent deformations, and the effects of plasticity become significant. To study plasticity effects, we need an appropriate contact law between elastoplastic granules.

1.2.4 Elastoplastic contact laws

Contact between elasto-plastic spheres has also been studied in the literature to develop models for use in discrete element methods [54] to model flow of granular particles. Several models have been proposed to extend Hertz’s classical theory of elastic contact [8] and account for plastic deformations in contacting spheres. One of the most widely used models for force-displacement in elastic-perfectly-plastic contacting spheres is due to Thornton [55]. However, as recently shown by Wang et al. [53], while the Thornton model captures key features of the loading and unloading responses, it substantially underpredicts the experimental force-displacement behavior. The problem of two identical contacting spheres is equivalent to that of a sphere pressing against a rigid flat surface, and the latter problem has been extensively studied in the literature.

One of the early examples in this line of work is due to Abbot and Firestone [56], who assumed fully plastic conditions, and thus that the contact area is the area intercepted by the translation of the rigid flat surface. The contact force is obtained by multiplying the contact area with the average contact pressure, which is assumed to be the hardness. Zhao et al. [57] developed an elasto-plastic model which interpolates smoothly between the elastic (Hertzian) and fully plastic model in a transition region. Vu-Quoc [58] assumed that the contact area can be additively decomposed into elastic and plastic components, and developed an implicit relation between force and displacement using finite element simulations. Kogut and Etsion [59] and Jackson and Green [60] performed extensive finite element simulations and constructed empirical models for contact force-displacement response. It
should be noted that the models developed by Vu-Quoc [58] and Kogut [59] operate in displacement ranges much smaller than those considered in wave tailoring applications of interest in the present study. Li et al. [61] obtained contact laws for spheres undergoing very large deformations using a material point method. Wang and co-workers conducted static and dynamic experiments to obtain contact force-displacement data for identical [53] and dissimilar [62] spheres undergoing plastic deformations. Additionally, most of the above models focus only on the loading response. There is at present a paucity of accurate models for contacting spheres of dissimilar materials, which can operate in loading, unloading and reloading regimes over large displacements.

1.2.5 Applications: Materials design

Numerous applications have been demonstrated in the past decade with ordered packings of granular lattices and we present their overview along with the key physical phenomena. Heterogeneous periodic and non-periodic ordered arrangements of granular beads have been investigated to design chains for specific objectives. Nesterenko et al. [49] studied the characteristics of wave reflection at the interface of two granular chains with distinct radii. Hong [63] demonstrated the potential of these chains to be used as impact protectors by showing universal power decay in granular confinements at the interfaces between distinct kinds of granules. Job et al. [64] studied the effect of a single intruder granule with a different radius in an otherwise homogeneous chain, demonstrating energy localization at the intruder. A part of the incident energy is trapped as localized oscillations, whose frequency spectrum is shifted depending on the mass of this intruder and on the incident wave, showing its potential in wave mitigation and sound trapping in 3D granular assemblies. Using these concepts, Daraio et al. [65] demonstrated energy trapping in composite granular media, trapping and disintegrating high amplitude waves in softer particles into weaker separated pulses. Finally, Fraternali et al. [66] designed granular protectors using genetic algorithms to minimize the force transmitted to the end of the chain by optimizing the locations of intruders.

Boechler et al. [4] designed systems for tunable acoustic rectification us-
ing statically precompressed granular chains having light mass defects. The design exploits a bifurcation phenomena that occurs at large displacements when the nonlinear terms become significant. Acoustic lenses [67] have been developed using arrays of precompressed 1-D disks, which allow for focusing of acoustic pulses into linear elastic media. Boechler et al. [68] demonstrated the existence of breathers in dimer chains subjected to frequency loading, thus showing that these systems have potential to be used as energy harvesting devices. Granular packings are also finding potential applications as waveguides because they exhibit a range of nonlinear phenomena such as band gaps and nonlinear refraction [69]. Recently, Gusev and Tournat [70] demonstrated the design of waveguides in buried subsurface channels using ordered granular packings, utilizing change in contact stiffness with depth.

1.3 Thesis objectives and organization

The primary objective of this dissertation is to study ordered granular systems with the intention of subjecting them to high impact loads. The first part involves conducting a systematic study of the fundamental role of plasticity on wave propagation in these granular systems. A contact law is derived and the salient features of wave propagation in elastio-perfectly plastic contacting granules is studied. The dissipation characteristics of 3D granular packings and their continuum counterparts are compared.

The second part involves designing granular systems for wave tailoring: modifying the response by an external control. The development of preconditioning allows us to use elastic systems for high loading conditions. In this part, the objective is not to dissipate the energy, but to control it in a desired way in a robust manner.

This dissertation is thus concerned with illustrating two key mechanisms. The first one is dissipation due to plasticity in granular systems, which is the subject of study in Chapters 2 to 5. The second one focuses on wave tailoring by external control in elastic systems, which can be utilized for high amplitude forces by preconditioning. Chapters 6 to 8 are devoted to study this effect. The chapters in this dissertation are thus organized as follows:

- Chapter 2 presents the finite element simulations of two contacting
elastic-perfectly-plastic half-spheres, from which a contact law is derived. This is used to conduct dynamic simulations on chain of granules and the salient features of elasto-plastic waves is studied. This work resulted in the journal article: “Wave propagation in elasto-plastic granular systems. Granular Matter, 15, 747-758, 2013.”

- In Chapter 3, we study the effect of an intruder granule, having a different property on the wave propagation in an elastoplastic chain. We develop and validate a general contact law for elasto-plastic spheres of distinct materials. This contact law is used to conduct dynamic simulations of granular chain in a Split Hopkinson Pressure Bar setup. This work resulted in the journal article: “Impact response of elasto-plastic granular chains containing an intruder particle. Journal of Applied Mechanics, 2014.”

- Chapter 4 presents a systematic study of the effect of loading duration on elastic and elasto-plastic chains. The key physics is studied and observed to be very distinct for short and long loading times for both the chains. Unified scaling laws are derived for the key physical quantities of the system. This work resulted in the journal article: “Characterization of wave propagation in elastic and elastoplastic granular chains. Physical Review E, 89(1), 012204, 2014.”

- Chapter 5 presents a study of dissipation characteristics in densely packed 3D granular media and continuum media for an elastic-perfectly-plastic material. Unified scaling laws for the two systems are derived analytically and verified with numerical simulations. This work resulted in the journal article: “Impact response of elasto-plastic granular and continuum media: A comparative study. Mechanics of Materials, 73, 38-50, 2014.”

- In Chapter 6, wave tailoring by applying external precompression is introduced on a system of big and small spheres confined in a cylindrical tube. Numerical simulations are conducted to illustrate the phenomena and the results are explained by an asymptotic analysis. This work resulted in the journal article: “Wave tailoring by precompression in confined granular systems. Physical Review E, 90(4), 042204, 2014.”
• In Chapter 7, the wave tailoring capabilities by external compression is studied in a system of spheres and cylinders, which is easier to realize in practical applications. The system is studied under both impact and harmonic loadings and their tunability is illustrated.

• In Chapter 8, wave tailoring by altering the network topology in granular lattices is introduced. The lattice here also consists of spheres and cylinders confined between movable plates and the system is tunable under impact loads. The novel designs are demonstrated by a combination of modeling, numerical simulations and experiments.

• Finally, Chapter 9 summarizes the key contributions of this dissertation and gives an overview of potential future research directions.
Chapter 2

Elastoplastic contact law and dynamic simulations

This chapter models systematically the effect of plasticity in wave propagation in granular chains by developing a contact force-displacement law and incorporating it into a dynamic model of wave motion in a 1-D chain of elastic-perfectly-plastic spheres. In the first part of this chapter, finite element simulations are used to develop force-displacement laws for a wide range of material properties and a large relative displacement range, under both loading and unloading conditions. This law, first derived for identical spheres in contact, is then extended to incorporate spheres of distinct radii made of the same material. The contact force-displacement law is then used to study the dynamic response of uncompressed spherical beads in contact, analyze the distinct characteristics of plastic waves, and characterize the decay in peak force and wave velocity. That aspect of the study includes the development of scaling laws for energy dissipation of the system.

2.1 Contact Force-Displacement Model

The objective of this section is to construct a model to compute the contact force for a given relative displacement between the centers of two contacting spheres. Detailed finite element simulations are performed to obtain the force-displacement solution for two rate-independent elastic-perfectly-plastic spheres in contact, and this solution is compared with experimental data [53] and with the Thornton model [55]. The limitations of the Thornton model are presented, and a new model having a wider applicability is constructed based on the finite element solutions.
2.1.1 Finite Element Simulation

The boundary value problem of two elastic-perfectly-plastic spheres under contact is presented schematically in Fig. 2.1(a), together with a close-up of the mesh in the vicinity of the contact surface. To extract the relation between the contact force and the relative displacement between the center of the spheres, two half-spheres in contact are considered in the problem setup. A static finite element analysis is performed to extract the force-displacement data for a rate-independent material, based on the assumption that the time scale of dynamic wave propagation across the chain is much larger than that of elastic wave propagation within the sphere [6].

One end of a half sphere is fixed, while a uniform vertical displacement is imposed on the entire flat surface of the other half-sphere. The flat surfaces of both the half-spheres are free to move in the horizontal direction, while the curved surfaces are traction free. Since the spheres in this first set of simulations are identical, the bottom sphere could be replaced by a rigid flat surface. However, we use two half-spheres since spheres of dissimilar radii are later modeled using the same problem set-up.

The finite element analysis software ABAQUS is used to solve the above problem. First-order axisymmetric quadrilateral (CAX4R) and triangular (CAX3) elements are used with the master-slave contact algorithm at the contact surfaces. The half-spheres are meshed with approximately 72,000 elements, with a fine layer of structured mesh along the contact surface, and an unstructured mesh in the remaining domain. To match the experimental results described later, the radii of the spheres are chosen to be $3/8$” (4.76 mm). The Young’s modulus ($E = 115$ GPa) and Poisson ratio ($\nu = 0.30$) are chosen to correspond to brass, although the results are presented hereafter in a non-dimensionalized way. Finite element simulations are performed for a range of yield strength values from 400 to 1500 MPa. Coulomb friction is introduced between the contact surfaces with a friction coefficient 0.30 to keep the solution symmetric with respect to the contact surface. In the absence of friction, the numerical solution deviates from symmetry and becomes unstable at large relative displacements. Large deformation static solutions are performed with loading and unloading cycles as shown in Fig. 2.1(b). In that figure, the prescribed displacement $\alpha$ is normalized by $R^*$, which is the
Figure 2.1: Problem description. Figure 2.1(a) shows a schematic of the axisymmetric problem with boundary conditions. A close-up of the mesh at the contact surface is also shown, emphasizing the fine structured mesh used in the vicinity of the contact surface. Figure 2.1(b) shows the variation of prescribed displacement over the loading history used in the simulation.
effective radius of the two spherical surfaces in contact given by

\[ R^* = \frac{R_1 R_2}{R_1 + R_2}, \] (2.1)

where \( R_1 \) and \( R_2 \) are the radii of the two contacting surfaces. The total contact force \( F \) is computed by summing the reaction forces acting on all the nodes at the top or bottom flat surfaces of the half-spheres.

2.1.2 Force-Displacement Model

**Loading**

The force-displacement data extracted from finite element solution is normalized as follows. The displacement \( \alpha \) between the sphere centers is normalized by \( \alpha_y \), the displacement at the onset of yielding given by [55]

\[ \alpha_y = \frac{\pi^2}{4} \left( \frac{p_y}{E^*} \right)^2 R^*. \] (2.2)

In (2.2), \( E^* \) is the effective modulus of the solids in contact, defined as

\[ \frac{1}{E^*} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} = \frac{2(1 - \nu^2)}{E}, \]

and \( p_y \) is the contact yield stress, which is related to the material yield strength \( \sigma_y \) by \( p_y = 1.60 \sigma_y \) for materials with Poisson ratio \( \nu = 0.30 \) [8]. This relation is found to be consistent with our numerical solutions, where yielding starts when the peak pressure on the contact surface reaches \( p_y \). The contact force \( F \) is normalized by \( F_y \), the force at the onset of yield obtained from Hertzian law [8] as

\[ F_y = \frac{4}{3} E^* R^{3/2} \alpha_y^{3/2}. \] (2.3)

Figure 2.2 shows the normalized force-displacement data extracted from the finite element solution for \( \sigma_y = 550 \) MPa. The monotonic curve starting from the origin corresponds to plastic loading, while the steeper curves originating from this loading curve correspond to unloading and reloading. Also shown in the figure is the Hertzian solution, which is much steeper, and deviates from
the elasto-plastic solution for small values of contact force, emphasizing the key role played by plasticity. For the material properties used in this study, the force required to cause yielding, \( F_y \), defined in (2.3) is quite small and is around 5 N, which corresponds to unity in Fig. 2.2. Due to residual plastic deformations for contact forces exceeding \( F_y \), a displacement is observed when the force goes to zero during unloading. Furthermore, subsequent reloading follows the unloading curve until the force level at which unloading started, implying that the unloading and reloading are elastic.

The Thornton model [55] predicts the following linear variation of force with relative displacement

\[
F = F_y + \pi \sigma_y R^* (\alpha - \alpha_y) \quad (2.4)
\]

in plastic loading, and it assumes the following power law in unloading and elastic reloading:

\[
F = \frac{4}{3} E^* R_p^{1/2} (\alpha - \alpha_p)^{3/2}, \quad (2.5)
\]

where

\[
R_p^* = \frac{4 E^*}{3 F_{\text{max}}} \left( \frac{2 F_{\text{max}} + F_y}{2 \pi \sigma_y} \right)^{3/2} \quad (2.6)
\]

is the effective radii at contact during unloading [55], \( F_{\text{max}} \) is the contact force at the onset of unloading and \( \alpha_p \) is the residual displacement, obtained by satisfying equations (2.4) and (2.5) at the onset of unloading. The Thornton model is shown for comparison in Fig. 2.2. Though qualitatively similar to the finite element solution, it clearly under-predicts the contact force during plastic loading.

To explain this discrepancy, one should examine in detail the two key contributions to the contact force \( F = \bar{p}A \), that is, the average contact pressure \( \bar{p} \) and the contact area \( A \). Figure 2.3 shows the variation of contact pressure (traction stress in the axial direction) along the contact surface. The numerical solution shows the pressure to rise beyond \( p_y \) and reach a maximum value of about 2.5 \( \sigma_y \). Similar trends have been observed by Jackson and Green [60], who related the contact pressure to the hardness of the material. For large relative displacements, the pressure is nearly constant along the contact surface, allowing characterization of the surface contact pressure by an average value \( \bar{p} \).

Figure 2.4 shows the variation of the normalized average pressure (\( \bar{p}/\sigma_y \))
Figure 2.2: Force-displacement response of two contacting elastic-perfectly-plastic spheres for the loading profile in Fig. 2.1(b). The monotonic curve starting from the origin corresponds to plastic loading, while the steeper curves correspond to elastic unloading and reloading. Also shown are the elastic (Hertzian) solution and the prediction provided by the Thornton model [55] for the same system and loading conditions.

Figure 2.3: Pressure distribution along the contact surface for identical elasto-plastic spheres, evaluated using FEA for ten values of the applied relative displacement $\alpha$. For high $\alpha$, the pressure is almost uniform over the contact surface. The dotted line shows the pressure distribution predicted by the Thornton model for $\alpha/R^* = 0.20$. 
Figure 2.4: Average contact pressure ($\bar{p}$) versus applied displacement for 8 values of the normalized yield stress ($\gamma = \sigma_y / E^*$). The variation of normalized average pressure with displacement seems to follow a single law for different yield strengths. For small displacements, there is a transition from the elastic Hertzian solution to the almost constant value observed for large relative displacements.
with applied relative displacement $\alpha$ for multiple values of material yield strengths, with $\gamma = \sigma_y/E^\ast$. As shown there, the average pressure increases sharply to a constant value as $\alpha$ increases and we thus construct a simple model that transitions between the two regimes. At the onset of yielding, the pressure distribution along the contact surface is elliptic [8], as described by the Hertzian solution and the average surface pressure is evaluated as

$$\bar{p} = \frac{1}{\pi a^2} \int_A p_y \sqrt{1 - \left(\frac{r}{a}\right)^2} dA = \frac{2}{3} p_y = 1.07\sigma_y, \quad (2.7)$$

where $a$ and $A$ are the contact radius and area, respectively. An exponential correction in the transition regime from elastic to larger displacement fully plastic regime leads to the following relation:

$$\frac{\bar{p}}{\sigma_y} = c_1 - (c_1 - 1.07) \exp (-c_2(\tilde{\alpha} - 1)), \quad (2.8)$$

where $c_1 = 2.48$, $c_2 = 0.098$ and $\tilde{\alpha} = \alpha/\alpha_y$ is the normalized displacement. Relation (2.8) is shown as a dashed curve in Fig. 2.4.

To complete the force-displacement relation in loading, we now characterize the evolution of the contact area $A$ with the applied displacement $\alpha$. The contact radius $a$ can be extracted numerically by capturing the point along the contact surface where the pressure $p$ drops to zero. The contact area ($A = \pi a^2$) is normalized by the contact area at the onset of yielding $A_y$, obtained from Hertzian contact law as

$$A_y = \pi a_y^2 = \pi R^\ast \alpha_y, \quad (2.9)$$

where $R^\ast$ and $\alpha_y$ are given by (2.1) and (2.2), respectively. Figure 2.5 presents the contact area distribution for different values of yield strength $\sigma_y$, showing the existence of a single ‘master’ curve. For all displacements, the Thornton model [55] assumes a linear variation of contact area with displacement following the Hertzian law. The numerical results obtained in this study show that the contact area indeed varies almost linearly, but with a different slope than that assumed by the Thornton model. For small displacements after the onset of yielding, the contact area follows a power law in the transition
Displacement ($\alpha/\alpha_y$)

Normalized area ($A/A_y$)

$\gamma = 6.33e^{-3}$

$\gamma = 7.91e^{-3}$

$\gamma = 9.5e^{-3}$

$\gamma = 1.11e^{-3}$

$\gamma = 1.27e^{-3}$

$\gamma = 1.42e^{-2}$

$\gamma = 1.82e^{-2}$

$\gamma = 2.37e^{-2}$

Thornton model

Eqn. (2.10)

Thornton model

Equation (2.10) is shown as a dashed curve in Fig. 2.5.

Figure 2.5: Contact area versus displacement for eight values of yield stress ($\gamma = \sigma_y/E^*$). When normalized by their respective values at the onset of yield, the area-displacement curves collapse to a single curve.

The Thornton model [55] assumes that the maximum contact pressure does not increase beyond $p_y$, the maximum contact pressure along the surface at the onset of yield. However, the numerical solution (Fig. 2.3) shows the pressure increasing beyond $p_y$ and reaching a maximum of about $2.5\sigma_y$. The difference in the plastic loading regime between the numerical solution and Thornton model thus arises from two factors: the contact pressure is about 2.31 times the surface contact pressure assumed in the Thornton model, and the contact area is about 2.1 times the contact area assumed in the Thornton model. Recently, Wang et al. [53] used a modified Thornton model, where the contact yield stress is related to the yield strength as $p_y = f\sigma_y$, where $f$ is a fitting parameter derived from matching a Thornton model curve with the maximum load levels attained by experiments. Thus, a combination of
Scaled displacement \((\alpha - \alpha_R)/(\alpha_{\text{max}} - \alpha_R)\) 

Contact force \((F/F_{\text{max}})\)

\[\frac{\alpha_{\text{max}}}{R} = 0.02\]
\[\frac{\alpha_{\text{max}}}{R} = 0.04\]
\[\frac{\alpha_{\text{max}}}{R} = 0.06\]
\[\frac{\alpha_{\text{max}}}{R} = 0.08\]
\[\frac{\alpha_{\text{max}}}{R} = 0.10\]
\[\frac{\alpha_{\text{max}}}{R} = 0.12\]
\[\frac{\alpha_{\text{max}}}{R} = 0.14\]
\[\frac{\alpha_{\text{max}}}{R} = 0.16\]
\[\frac{\alpha_{\text{max}}}{R} = 0.18\]
\[\frac{\alpha_{\text{max}}}{R} = 0.20\]

Figure 2.6: Unloading force-displacement curves obtained for different onsets of unloading characterized by the maximum previously achieved value of displacement \(\alpha_{\text{max}}\) and force \(F_{\text{max}}\).

these two effects gives a factor close to 4.7, the parameter used by Wang et al. [53] in the modified Thornton model. Furthermore, both our data-fitted model and the Thornton model are linear for large displacements, and hence a modified Thornton model [53] gives a good fit in the loading regime.

Unloading

The unloading response of the contacting spheres is characterized by the displacement \(\alpha_{\text{max}}\) and corresponding force \(F_{\text{max}}\) at the onset of unloading. Since the unloading and subsequent reloading curves overlap (cf. Fig. 2.2), unloading is a completely elastic process. Let \(\alpha_R\) be the residual displacement at zero contact force. For a given unloading curve, the displacement can be scaled as

\[\alpha_s = \frac{\alpha - \alpha_R}{\alpha_{\text{max}} - \alpha_R},\]  

(2.11)

while the force is scaled by \(F_{\text{max}}\). As shown in Fig. 2.6, the normalized unloading contact curves obtained numerically for ten values of \(\alpha_{\text{max}}\) collapse to one curve. Unloading curves for other yield strength values also show the same behavior. Thus a single law suffices to describe the unloading response.
in the form
\[
\frac{F}{F_{\text{max}}} = \left( \frac{\tilde{\alpha} - \tilde{\alpha}_R}{\tilde{\alpha}_{\text{max}} - \tilde{\alpha}_R} \right)^n, \tag{2.12}
\]
where \(n\) is the unloading exponent, and all displacements have been normalized by \(\alpha_y\). To compute the force during unloading using (2.12), we need to determine the residual displacement \(\alpha_R\) for a given displacement at the onset of unloading \(\alpha_{\text{max}}\). As shown in Fig. 2.7, the variations of residual displacement \(\tilde{\alpha}_R = \alpha_R/\alpha_y\) with maximum displacement \(\tilde{\alpha}_{\text{max}}\) for different yield strengths all follow a single ‘master’ curve. A linear fit captures the variation well at larger displacements, but it deviates for small displacements. There is a transition regime between zero residual displacement at the onset of yield and the linear variation regime at large displacements. Again, an exponential correction is added to capture the variation in this transition regime:
\[
\tilde{\alpha}_R = \frac{\alpha_R}{\alpha_y} = c_6 \tilde{\alpha}_{\text{max}} - c_7 + (c_7 - c_6) \exp \left( -c_8 (\tilde{\alpha}_{\text{max}} - 1) \right), \tag{2.13}
\]
where \(c_6 = 0.95\), \(c_7 = 25.94\) and \(c_8 = 0.015\). The exponent for all yield strength values and for all unloading points in the plastic regime is found to vary between 1.30 and 1.42. We therefore use a constant value of \(n = 1.35\) and
Figure 2.8: Comparison between the numerically obtained force-displacement data (symbols) with the model described by equations (2.14), (2.16) and (2.17).

observe that this value fits the finite element solution data with reasonable accuracy for all regimes and for a wide range of yield strength values. It should be noted that this value is slightly inferior to the value \( n = 1.5 \) assumed in the Thornton model (2.5).

**Verification and Validation**

The complete force-displacement law for the plastic loading regime is thus

\[
F = \sigma_y A_y (2.48 - 1.41 \exp(-0.098(\hat{\alpha} - 1))) \hat{A},
\]

(2.14)

with \( \hat{A} \) given by

\[
\hat{A} = \begin{cases} 
\hat{\alpha}^{1.14}, & \text{if } \hat{\alpha} < 177.6, \\
(2.37\hat{\alpha} - 59.96), & \text{otherwise}
\end{cases}
\]

(2.15)

for the plastic loading regime and

\[
F = F_{\text{max}} \left( \frac{\alpha - \alpha_R}{\alpha_{\text{max}} - \alpha_R} \right)^{1.35}
\]

(2.16)
for unloading and elastic reloading regimes, with the residual displacement \( \alpha_R \) given by

\[
\alpha_R = 0.95\alpha_{\text{max}} - 25.94\sigma_y + 25.0\sigma_y \exp (-0.015 [\tilde{\alpha}_{\text{max}} - 1]). \tag{2.17}
\]

Figure 2.8 shows the comparison of numerical results to the model described by (2.14)-(2.17) for materials of yield strength \( \sigma_y = 400 \) MPa, 500 MPa and 800 MPa. The model matches the finite element solution accurately for small and moderate displacements, while for large displacements, the force is slightly over-predicted. The maximum error in the model compared to the finite element solutions is found to be about 6%. The force predicted by the model starts to deviate from the numerical solution for large displacements due to the average contact pressure falling from the constant value, as seen in Fig. 2.4, and this is attributed to the effect of change in curvature at large deformations. Jackson et al. [60] report similar trends, where the contact pressure reaches a peak value before progressively decreasing.

Figure 2.9 shows a comparison of the data fitted model for brass with \( \sigma_y = 550 \) MPa with experimental data for quasi-static loading of two brass
Figure 2.10: Force-displacement data extracted from finite element solutions for spheres of different radii in contact. For small and moderate displacements, the normalized curves overlap, for large displacements, the entire smaller sphere yields, and the force-displacement curves begin to deviate.

There is some disagreement between the experimental data and our model, arising due to an uncertainty of ±50 MPa in the actual value of yield strength $\sigma_y$ used in experiments. The low slope of the loading curve at small displacements for the experimental curve is attributed to the transition regime from elastic to plastic behavior for brass, in contrast to the elastic-perfectly-plastic model assumed for finite element simulations.

2.1.3 Contact law for spheres of different radii

Finite element simulations with contacting spheres of different sizes are also performed, with the radii ratio ranging from 1.5 to 4. The problem schematic and the boundary conditions are the same as in Fig. 2.1(a). The same prescribed displacement (shown in Fig. 2.1(b)) is applied to all systems, and the force-displacement response is shown in Fig. 2.10. The curves completely overlap for small to moderate displacements. At large displacements, the loading curves for dissimilar radii begin to deviate from those for identical spheres. This is attributed to two phenomena: at higher displacements, the entire smaller sphere becomes plastic and this leads to the material becoming softer. Similar to the effect of average contact pressure decreasing for iden-
tical spheres, there may be geometric softening causing the loading curve to deviate from the model and from identical spheres finite element solution for large relative displacements. Thus the force-displacement model (Eqns. (2.14–2.17)) developed earlier for spheres of same radius can be directly used for spheres of distinct radii in contact, after appropriate normalizations.

2.2 Wave propagation in elasto-plastic granular chains

2.2.1 Problem statement

The contact law developed in the previous section (Eqns. (2.14)-(2.17)) is now used to study the response of a semi-infinite chain of elastic-perfectly-plastic spherical beads subjected to a force impulse at one end. The chain is represented by a series of point masses connected by nonlinear springs with stiffness defined by the force-displacement contact law. The equation of motion of the $i^{th}$ bead along the chain can be written as

$$m_i \ddot{u}_i = F_{i-1,i} (\alpha_{i-1,i}) - F_{i,i+1} (\alpha_{i,i+1}),$$

(2.18)

with $F_{i,j}$ denoting the contact force between spheres $i$ and $j$ and

$$\alpha_{i-1,i} := \begin{cases} u_{i-1} - u_i, & \text{if } u_{i-1} > u_i \\ 0, & \text{otherwise}. \end{cases}$$

(2.19)

For the first bead, the equation of motion is

$$m_1 \ddot{u}_1 = f(t) - F_{1,2} (\alpha_{1,2}),$$

(2.20)

where $f(t)$ denotes the applied impulse excitation load and is given by

$$f(t) = f[P, T](t) = \begin{cases} P \sin \left( \frac{\pi t}{T} \right), & \text{if } 0 \leq t \leq T \\ 0, & \text{otherwise}. \end{cases}$$

(2.21)

For a given amplitude $P$ and time $T$ of loading, the total impulse imparted to the first bead is $I = 2PT/\pi$. A fourth-order Runge Kutta solver is used to
Table 2.1: Force and impulse applied to the granular chain.

<table>
<thead>
<tr>
<th>Amplitude, $P$ (N)</th>
<th>Impulse, $I$ (Ns)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 \times 10^5$</td>
<td>$2 \times 10^4$</td>
</tr>
<tr>
<td>$5 \times 10^5$</td>
<td>$1 \times 10^4$</td>
</tr>
<tr>
<td>$2 \times 10^4$</td>
<td>$2 \times 10^3$</td>
</tr>
</tbody>
</table>

Figure 2.11: Force history at four contacts along the elastic (dotted curves) and elasto-plastic (solid curves) chains. In the elastic case, a single solitary wave of constant amplitude propagates, while the plastic chain is characterized by a wave train of decaying amplitude.

simulate the dynamic response of the system with a time step of $5 \times 10^{-9}$s. The solutions are presented nondimensionally, with the wave velocity $v$ and impulse $I$ being nondimensionalized as:

$$\tilde{v} = v \tau / R^*$$

$$\tilde{I} = I / E^* R^* \tau^2$$

where $\tau$ is the intrinsic time-scale associated with the elastic sphere system

$$\tau = \sqrt{\frac{\pi \rho R^2}{4 E^*}}.$$  \hspace{1cm} (2.24)

2.2.2 General characteristics of waves

The tests listed in Table 2.1 are performed on brass spheres having density $8500 \text{ kg/m}^3$, Young’s modulus $115 \text{ GPa}$, Poisson ratio 0.30, yield strength
Figure 2.12: *x*-t plot of compressive force at each contact point along an elastic and elasto-plastic chain. The elastic response is plotted shifted from the origin, and consists of a single solitary wave propagating at a constant speed and amplitude. In contrast, wave trains form and the wavefront speed and peak force decrease with distance in the elasto-plastic case.

550 MPa and diameter 4.763 mm (3/8″). Dynamic simulations are performed on both elastic and elasto-plastic chains, with elastic chains modeled by the Hertz contact model for all forces, and the time evolution of the peak force at each contact point is studied for both cases. Figure 2.11 shows the time history of force acting at a few contact points for $P = 20$ kN and $T = 10 \mu$s and Fig. 2.12 shows the *x*-t plot of force at the contact points over the entire simulation. To avoid overlap, the elastic response is plotted shifted from the origin, starting from the twentieth contact. In an elastic chain, a solitary wave forms in the first few beads and propagates unattenuated at a constant speed along the chain, with the peak force remaining constant as seen in Figs. 2.11 and 2.12.

In the elasto-plastic case, multiple waves pass through each contact point as wave trains form and waves interact. The amplitude of the leading wave also decreases as the wave progresses along the chain due to dissipation. The figure also shows that in the plastic case, the maximum force at a contact point does not always correspond to the first peak force experienced by the bead. The force amplitude and the speed of the leading wave decrease rapidly, as the leading wave operates in the plastic loading regime of the beads. As seen in Fig. 2.12, a continuous stream of trailing secondary waves form behind the leading wave. These trailing waves primarily operate in the elastic loading regime as their peak contact force is lower than the previously attained maximum contact force due to the passage of the leading wave. At some contacts, the trailing waves cause dissipation when their peak force
Figure 2.13: Force decay along the chain consisting of two regimes. In the first regime, the force decay is exponential. The trailing waves catch up soon, and the second regime having an inverse power law decay rate starts. The inset shows the response of a longer chain on a log-log scale.

exceeds the maximum force attained by the leading wave. The wave speed is proportional to the slope of the force-displacement curve, and the elastic unloading part has a higher slope than the plastic loading part at high forces. Thus, trailing waves with a high enough force amplitude travel faster than the leading wave and eventually merge with the leading wave at the front. This wave merging leads to an increase in amplitude of the wave and thus the peak force at contact points does not vary monotonically with distance along the chain.

2.2.3 Force and velocity characterization

The peak force decays rapidly along the elasto-plastic chain and Fig. 2.13 shows that the decay rate has two distinct regimes. In the first regime, which comprises of the first few contacts, the energy dissipation is due to a single decaying wave traveling down the chain, and the decay rate of the amplitude of this leading wave with distance along the chain is exponential. The single
leading wave travels some distance until the faster moving trailing waves merge with it, and the second regime starts. In the second regime, wave interaction and merging occurs and the dissipation at a contact happens due to multiple waves. The peak force is observed to be inversely proportional to the distance along the chain in this regime. To characterize the decay rate of the first leading wave, we consider a loading case with a peak loading amplitude and loading time of 2000 N and 10 $\mu$s respectively. For this loading, the first trailing wave merges with the leading wave at the 21$^{\text{st}}$ contact point. Using this solution, the following characterization for the peak force decay of the leading wave is obtained:

$$\frac{F_{\text{max}}}{E^* R^*^2 \tilde{I}} = a \exp \left( -b x / R^* \right), \quad (2.25)$$

where $a = 3.74 \times 10^{-2}$ and $b = 0.256$. For the second regime, the following relation is obtained for the peak force variation along the chain:

$$\frac{F_{\text{max}}}{E^* R^*^2 \tilde{I}} = \frac{C}{x / R^*}, \quad (2.26)$$

where $C = 0.268$. Figure 2.13 shows the peak force variation and the exponential and inverse law curves for three values of the applied impulse.

We now study the response of a long chain subjected to different loading amplitudes $P$ and impulses $I$ (listed in Table 2.1) and characterize the peak force and wave velocity. The top curves in Fig. 2.14 show the peak force along the chain for different loading amplitudes, but with the same initial input impulse. When the loading time $T$ is short, the responses are observed to be almost identical for systems subjected to the same initial impulse. Since waves merge in the second regime, the location of the peak contact force along the chain varies discontinuously with time. To characterize the wave velocity of elasto-plastic waves, the notion of a leading wave velocity is introduced as the time taken for the leading front or disturbance to travel between two adjacent contacts. The lower curves in Fig. 2.14 show the leading wave velocity for chains subjected to different loading amplitudes and a fixed impulse. Once again, the curves overlap and the response is independent of loading amplitude.

Figure 2.13 shows the peak force decay for systems subjected to different impulses $I$, but with the same loading amplitude $P$. When normalized by
2.2.4 Energy dissipation

The dissipation along the chain, both at an individual contact level and for the entire chain, is characterized using the peak force relations developed above. At a contact point attaining a peak force $F_{\text{max}}$, the dissipated en-
Figure 2.15: Wave velocity along the chain. When normalized by $\tilde{I}^{1/6}$, the responses for distinct input impulses collapse to a single curve away from the first few beads. The inset shows the variation on a log-log scale and the two distinct propagation regimes. In the second regime, the velocity variation follows a power law with exponent $-1/5$.

Figure 2.16: Total dissipation versus distance. The simulation results are compared with the predicted results obtained from (2.28), and they are in good agreement.
ergy $E_{\text{dis}}$ is equal to the difference in areas under the loading and unloading force-displacement curves for this value of $F_{\text{max}}$. The energy dissipation at a contact is given by the expression

$$
\tilde{E}_{\text{dis}} = \frac{1}{F_y\alpha_y} \int_0^\alpha (F_{\text{load}} - F_{\text{unload}}) \, d\alpha
$$

$$
= \begin{cases} 
\frac{1}{2} \tilde{F} (\tilde{\alpha} - \tilde{\alpha}_R) & \text{if } \alpha_0 \leq \tilde{\alpha} \\
\frac{5}{16} \sigma_y \left( \frac{\tilde{c}_3}{\tilde{c}_3 + 1} \right) \tilde{\alpha}^{\tilde{c}_3 + 1} - 1 & \text{if } 1 < \tilde{\alpha} < \alpha_0 \\
0 & \text{otherwise,}
\end{cases}
$$

(2.27)

where the constants $c_i$ are as defined in Section 2.1 and $\Gamma(\cdot, \cdot)$ is the Gamma function. The total energy dissipated in the system upto location $x_N$, as described by Eqns. (2.28-2.30) is given by

$$
\frac{4E_{\text{tot}}}{F_y\alpha_y} = \begin{cases} 
d_0 (\tilde{x}_N - \tilde{x}_t) + d_1 a \tilde{I} b (e^{-b \tilde{x}_1} - e^{-b \tilde{x}_N}) \\
+2d_2 \left( a \tilde{I} \right)^2 b (e^{-2b \tilde{x}_1} - e^{-2b \tilde{x}_N}) & \text{if } \tilde{x}_N < \tilde{x}_t \\
E_t + d_0 (\tilde{x}_N - \tilde{x}_t) + d_1 C \tilde{I} \ln \left( \frac{x_N}{x_t} \right) \\
-2d_2 \left( C \tilde{I} \right)^2 \left( \frac{1}{\tilde{x}_N} - \frac{1}{\tilde{x}_t} \right) & \text{otherwise,}
\end{cases}
$$

where $\tilde{x} = x/R^*$ and $E_t = d_0 (\tilde{x}_t - \tilde{x}_1) + d_1 a \tilde{I} b (e^{-b \tilde{x}_1} - e^{-b \tilde{x}_t}) + 2d_2 \left( a \tilde{I} \right)^2 b (e^{-2b \tilde{x}_1} - e^{-2b \tilde{x}_t})$ is the energy dissipated by the contact points in the first regime. Here $\alpha = \alpha_{\text{max}}$ is the maximum relative displacement attained at the contact in Eqns. (2.27) and (2.28).
Equation (2.27) gives the energy dissipation expression as a function of the maximum displacement $\alpha_{\text{max}}$ for the model constructed in the previous section. To compute the energy dissipation along the entire chain using the peak force decay rates (Eqns. (2.25) and (2.26)), a simple approximation is constructed relating the energy dissipation and force at a contact. Since the contact force varies linearly with displacement for large displacements in the loading regime (Eqns. (2.14) and (2.15)), the area under the loading curve can be approximated by a quadratic function of the force. For unloading, since the residual displacement $\alpha_R$ varies linearly with the displacement at the start of unloading $\alpha_{\text{max}}$ for large displacements (Eqn. (2.17)), the area under the unloading curve also varies quadratically with force. Hence the energy dissipation at a given contact can be approximated by a quadratic function:

$$\tilde{E}_{\text{dis}} = \frac{E_{\text{dis}}}{F_y \alpha_y} = d_0 + d_1 \tilde{F}_{\text{max}} + d_2 \tilde{F}_{\text{max}}^2,$$

(2.28)

where $\tilde{F}_{\text{max}} = F_{\text{max}}/F_y$, $d_0 = 1.23 \times 10^{-4}$, $d_1 = -3.63 \times 10^{-7}$ and $d_2 = 1.86 \times 10^{-8}$ for the model described by (2.14)-(2.17). This approximation is found to be in good agreement with the exact expression given by (2.27).

Using these relations, we can evaluate the energy dissipated at a contact located distance $\tilde{x} = x/R^*$ along the chain, after all the waves have passed. In the first regime, the exponential force decay rate (2.25) leads to

$$\tilde{E}_{\text{dis}} = d_0 + d_1 a \tilde{I} \exp (-b \tilde{x}) + d_2 \left(a \tilde{I} \exp (-b \tilde{x})\right)^2,$$

(2.29)

while in the second regime, the spatial variation of the energy dissipation is given by

$$\tilde{E}_{\text{dis}} = d_0 + d_1 C \frac{\tilde{I}}{\tilde{x}} + d_2 \left(\frac{C \tilde{I}}{\tilde{x}}\right)^2,$$

(2.30)

using the inverse force decay rate (2.26). As apparent in Fig. 2.13, the second regime starts from about the seventh contact point. In the numerical simulations performed in this study, the second regime is observed to start between the sixth and tenth contact points. Therefore, to compute the dissipation at any contact point along the chain, the transition between the first and second regimes is fixed at an average value of $\tilde{x}_t = x_t/R^* = 28$, that is, the seventh contact point.

The total energy dissipated $E_{\text{tot}}$ along a chain is obtained by summing the
dissipation at each contact point as:

\[ E_{\text{tot}} = \sum_{i=1}^{N} E_{\text{dis}}(x_i) \approx \frac{1}{2R} \int_{x_1}^{x_N} E_{\text{dis}} dx. \]  

(2.31)

The above integral is evaluated using Eqns. (2.29) and (2.30) and the expression is provided in Eqn. (2.28). Figure 2.16 shows the total energy dissipation along the chain for different impulses, obtained from numerical simulations and the predicted energy dissipation using relation (2.31). For lower input impulses, the total energy dissipated along the chain is described accurately by the above equations while, for higher impulses, the error is less than 6%. This error is due to the model assuming the transition from exponential to inverse decay regime for peak force to always happen at a fixed point for all impulses. A large fraction of the energy is dissipated in the first regime, where the forces are high and the decay rate is exponential.

2.3 Conclusions

In the first part of this chapter, finite element analyses of contacting elastic-perfectly-plastic spheres of the same material, having the same or distinct radii have been performed. Simulations for a wide range of material properties and for a large displacement range have been conducted. Scaling relations and numerical fits have been extracted to develop a 9-parameter model and describe the contact force-displacement relation for loading, unloading and reloading of elasto-plastic spheres.

In the second part, Hertzian law and new elasto-plastic contact model have been used to perform dynamic simulations on semi-infinite elastic and elasto-plastic granular chains, with a force impulse applied at one end. In an elastic chain, a solitary wave propagates unchanged, while in an elasto-plastic chain, the wave amplitude decreases due to dissipation, and there is formation and interaction of wave trains. For small loading times, the peak force and wave velocity along the chain depend only on the input impulse and the scaling laws for these quantities have been determined. The force decay has two distinct regimes, the first regime has an exponential decay rate due to the dissipation associated with the leading wave, and the second regime has an inverse decay rate, during merging and interaction of multiple waves.
Finally, the dissipation in these elasto-plastic chains has been characterized for different impulses.
Chapter 3

Impact response of elasto-plastic granular chains containing an intruder particle

This chapter focuses on the effect of intruder particles on the dynamics of elasto-plastic granular chains when subjected to impact loads. In the first part of this chapter, contact laws for dissimilar elastic-perfectly-plastic spherical granules are developed using finite element simulations. They are systematically normalized, with the normalizing variables determined from first principles, and a unified contact law for heterogeneous spheres is constructed and validated. In the second part, dynamic simulations are performed on granular chains placed in a Split Hopkinson Pressure Bar setup. An intruder particle having different material properties is placed in an otherwise homogeneous granular chain. The position and relative material property of the intruder is shown to have a significant effect on the energy and peak transmitted force down the chain. Finally, the key nondimensional material parameter that dictates the fraction of energy transmitted in a heterogeneous granular chain is identified.

3.1 Contact law

The granules are modeled as elastic-perfectly-plastic spheres, with no rate dependence. To model wave propagation in heterogeneous granular chains composed of this type of spheres, a contact force-displacement law is required between two contacting spheres of same or distinct materials. In the first part of this section, appropriate normalizations are derived for a unified contact law between distinct materials starting from the governing equations. Then, the force-displacement contact response for a wide range of material combinations is extracted from finite element simulations, and a single unified law is developed, based on the normalizations. The model is validated with quasi-static experiments between two contacting half spheres of different material
combinations.

3.1.1 Normalized contact law

In [52], numerical results were presented for contacting spheres made of identical materials over a wide range of yield strengths and size ratios. It was observed that normalizing the force-displacement data led to the curves for all these distinct cases collapsing to a single curve. Here we demonstrate the validity of this normalization starting from the governing equations and considering two spheres of different materials and sizes in contact. Let \((E_i, \nu_i, \sigma_{yi}, R_i)\) be the Young’s moduli, Poisson’s ratios, yield strengths and radii of the two contacting spheres, denoted by subscripts \(i = 1, 2\), and let \((u, \epsilon, \sigma)\) denote the displacement, strain and stress fields associated with the contact problem. The governing equation for the static continuum problem in the absence of body forces is \(\nabla \cdot \sigma = 0\). Let \(f(\sigma, \sigma_y) = 0\) be the von Mises yield criteria with yield function \(f\), and let \(C\) be the elastic constitutive tensor. Assuming small deformations and strains, the constitutive law in incremental form for an elastic-perfectly-plastic material following this yield criterion with associative Levy-Saint Venant flow rule is given by \(\dot{\sigma} = C^{ep}\dot{\epsilon}\) with [71]

\[
C^{ep} = \begin{cases} 
    C & \text{if } \gamma = 0, \\
    \left(\frac{(3\nu + 1)E}{3(1 - 2\nu)(1 + \nu)}\right)I \otimes I + \\
    \left(\frac{E}{1 + \nu}\right)\left(I - \frac{1}{3}I \otimes I - n \otimes n\right) & \text{if } \gamma > 0,
\end{cases}
\]

where \(I\) and \(\mathbb{I}\) are the second- and fourth-order identity tensors, respectively, \(n = \text{dev}(\sigma)/\|\text{dev}(\sigma)\|\) and \(\gamma = n : \dot{\epsilon}\) is the consistency parameter. The key observation here is that \(C^{ep}\) is a linear function of the Young’s modulus \(E\).

Let \(\alpha\) be the relative displacement between the sphere centers and let \(R^* = R_1 R_2/(R_1 + R_2)\) be the effective radius between the two spheres in contact. The solution is expressed in a cylindrical coordinate system with origin at the initial point of contact, the \(z\)-axis directed normal to the contact surface and \(r\) denoting the radial coordinate. Displacement and traction continuity
conditions at the contact interface then respectively take the form [8]

\[ u_z^1(r) + u_z^2(r) = \alpha - \frac{v^2}{2R^*}, \quad \sigma_1 e_z = \sigma_2 e_z, \quad (3.2) \]

where, once again, subscripts 1 and 2 correspond to the two contacting spheres. The Poisson’s ratio is kept fixed at 0.30, which is representative of many metallic materials of interest in this work. Without loss of generality, assuming \( \sigma_{y1} \leq \sigma_{y2} \), the normalized problem is expressed in terms of the yield strength of the softer material, \( \sigma_y = \min(\sigma_{y1}, \sigma_{y2}) = \sigma_{y1} \) and the ratio of the yield strengths, \( \eta = \sigma_{y2}/\sigma_{y1} \). We now introduce the following normalizations for the coordinate system \( \mathbf{x} = (r \cos \theta, r \sin \theta, z) \), displacement and stress fields:

\[ \tilde{\mathbf{x}} = \mathbf{x} \frac{\sigma_y}{E^*}, \quad \tilde{\mathbf{u}}_i = \mathbf{u}_i \left( \frac{\sigma_y}{E^*} \right)^2 \left( \frac{E^*}{E_i} \right), \quad \tilde{\sigma}_i = \frac{\sigma_i}{\sigma_y}, \quad (3.3) \]

where \( E^* = \left(1 - \nu_1^2/E_1 + 1 - \nu_2^2/E_2 \right)^{-1} \) is the effective stiffness of the two contacting materials.

These normalizations allow us to write the entire set of governing equations and boundary conditions in terms of dimensionless variables, independent of the physical dimensions and material properties. The only variable entering this system is \( \eta = \sigma_{y2}/\sigma_{y1} \). The displacement \( \alpha_y \) and contact force \( F_y \) at the onset of yield are given by [8]

\[ \alpha_y = \frac{\pi^2}{4} \left( \frac{1.6\sigma_y}{E^*} \right)^2 R^* = B \left( \frac{\sigma_y}{E^*} \right)^2 R^*, \quad (3.4) \]

\[ F_y = \frac{\pi^3}{6} \left( \frac{1.6\sigma_y}{E^*} \right)^3 E^* R^2 = C \frac{\sigma_y^3 R^2}{E^*^2}, \quad (3.5) \]

where \( B \) and \( C \) are numerical constants. The contact force is obtained from the stress field by

\[ F = \int_A \sigma_{zz} dA = \frac{\sigma_y^3 R^2}{E^*^2} \int_{\tilde{A}} \tilde{\sigma}_{zz} d\tilde{A}, \quad (3.6) \]

and the non-dimensional force is defined to be

\[ \tilde{F} = \frac{F}{F_y} = \frac{1}{C} \int_A \tilde{\sigma}_{zz} d\tilde{A}. \quad (3.7) \]
Figure 3.1: von Mises stress contours between steel (top) and brass (bottom) spheres. Though both spheres have a large plastic yield volume, the plastic deformations are entirely in the softer brass sphere, similar to the experimental observations by Wang et al. [62].

The relative displacement between the two spheres is computed far from the point of contact, and its non-dimensional form is defined by

$$\tilde{\alpha} = \frac{\alpha}{\alpha_y} = \frac{u_{z1} (z \to \infty) - u_{z2} (z \to \infty)}{\alpha_y} \approx \frac{\bar{u}_{z2} (z = -R) - \bar{u}_{z1} (z = R)}{A}.$$  

From the above discussion, we note that the contact force-displacement law for any two contacting spheres of possibly different sizes, after normalizing by \((F_y, \alpha_y)\) as shown above, depends on a single parameter \(\eta = \sigma_{y2}/\sigma_{y1}\), and is independent of the specific values of material parameters of both materials. This implies that a single contact law suffices to describe the behavior for a given value of \(\eta\). In the remainder of this section, the above normalizations are verified for two spheres of different materials using finite element simulations.

3.1.2 Finite element simulations

Similar to the approach followed in Sec. 2.1.1, finite element analyses of contacting spheres are conducted to extract an elasto-plastic contact force-
displacement law. The bottom sphere has material properties with Young’s modulus \( E_2 = 115 \text{ GPa} \), Poisson’s ratio \( \nu_2 = 0.30 \) and yield strength \( \sigma_{y2} = 550 \text{ MPa} \) (representative of brass). The Poisson’s ratio of the top sphere is kept fixed at \( \nu_1 = 0.30 \), while its Young’s modulus and yield strength are varied from \( E_1 = 15 \text{ GPa} \) to \( E_1 = 1150 \text{ GPa} \) and from \( \sigma_{y1} = 100 \text{ MPa} \) to perfectly elastic material \( (\sigma_{y1} \to \infty) \). Both spheres have radius \( R = 4.76 \text{ mm} \).

The finite element analyses are conducted in ABAQUS with the same mesh (refined in the contact area) as that used in [52]. An axisymmetric domain is considered with the boundary conditions and loading-unloading cycles as shown in Fig. 3.1. The flat surface of the bottom sphere is constrained to move horizontally, while the vertical component of displacement is prescribed on the flat surface of the top sphere and the curved surfaces are traction free.

Figure 3.1 shows the von Mises stress contours for steel (top) and brass (bottom) spheres in contact \( (\alpha/R^*=0.08) \), with yield strengths of 700 MPa and 550 MPa, respectively. It is observed that, though a significant volume of material yields plastically in both spheres, the deformation is almost entirely in the softer material (brass). Similar trends have been observed in both quasi-static and dynamic experiments on half spheres [62] where all the plastic deformations take place in the softer material (i.e., material with lower yield strength).

From finite element simulations, the net contact force at each step of the prescribed displacement is computed by summing the reaction forces on the top sphere. Figure 3.2(a) shows the force-displacement data for a small sample set of the conducted simulations. The force-displacement data are then non-dimensionalized by normalizing the relative displacement and contact force with \( \alpha_y \) and \( F_y \) (Eqns. (3.4) and (3.5)) , respectively, which are the displacement and contact force at the onset of yield of the lower strength material. As noted above, these are functions of the yield strength of the weaker material, and the effective stiffness and radii of contacting spheres. Figure 3.2(b) shows this non-dimensional force-displacement data. The loading and unloading curves collapse to a single curve for low and moderate displacements, where the assumptions of small strain deformation are valid. Also shown in this figure is the force-displacement curve given by the model developed in Sec. 2.1. Although a small set of numerical results are presented here, the normalized force-displacement data for several combinations of material properties over the entire range is found to coincide well with
Figure 3.2: (Top) Force-displacement data for two contacting spheres with distinct material properties varying over a wide range. (Bottom) When normalized appropriately, the distinct curves collapse to a single curve, showing that a unified model is able to describe the contact force-displacement behavior for a wide range of material combinations.
the model. This model, in normalized form, for the plastic loading regime is given by

\[ \tilde{F} = \frac{F}{F_y} = \frac{\sigma_y \pi R^2 \alpha_y}{F_y} (2.48 - 1.41 \exp(-0.098[\tilde{\alpha} - 1])) \tilde{A}, \]

\[ \tilde{A} = \begin{cases} \tilde{\alpha}^{1.14}, & \text{if } \tilde{\alpha} < 177.6 \\ (2.37\tilde{\alpha} - 59.96), & \text{otherwise}, \end{cases} \]  

(3.9)

where \( \tilde{\alpha} = \alpha/\alpha_y \). In unloading and elastic reloading regimes with displacement \( \alpha_{\text{max}} \) and force \( F_{\text{max}} \) at the start of unloading, the contact force is given by

\[ \tilde{F} = \left( \frac{\tilde{\alpha} - \tilde{\alpha}_R}{\tilde{\alpha}_{\text{max}} - \tilde{\alpha}_R} \right)^{1.35}, \]

\[ \tilde{\alpha}_R = 0.95\tilde{\alpha}_{\text{max}} - 25.94 + 25.0 \exp(-0.015[\tilde{\alpha}_{\text{max}} - 1]), \]  

(3.10)

with \( \alpha_R \) being the residual displacement and \( \tilde{\alpha}_R = \alpha_R/\alpha_y \). From Fig. 3.2(b), it is clear that a single contact law describes the force-displacement behavior for any two elastic-perfectly-plastic contacting spheres having size, stiffness or yield strength mismatch. Hence the contact law is identical even for distinct values of the yield strength ratio \( \eta \) and this gives a unified law for any two elastic-perfectly-plastic spheres of distinct materials and sizes.

### 3.1.3 Validation

Next, the model described by Eqns. (3.9) and (3.10) is validated by comparing with experiments performed on spherical particles made of different materials [62]. The experimental results for three different pairs (brass alloy 260 - stainless steel 302, brass alloy 260 - aluminum alloy 2017 and stainless steel 302 - aluminum alloy 2017) are compared with the proposed model in Fig. 3.3. As apparent there, good agreement is achieved, indicating that the contact model can capture a wide range of material combinations. The difference between experimental measurements and model predictions is attributed to the assumption of an elastic-perfectly-plastic material model.

Although all simulations in the present work have been conducted with both spheres having identical radii, it is noted that the finite element analyses
results on spheres made of the same material but having distinct sizes follow the same normalized law. Furthermore, the scaling laws derived earlier in the section indicate that contact response of spheres with distinct sizes also follows the same law. Thus, the proposed model [52] derived for identical contacting materials can be used to accurately predict the force-displacement behavior of two spheres having different physical properties and sizes.

3.2 Dynamic simulations

3.2.1 Problem setup and numerical method

The contact law developed in the previous section is now used to study wave propagation in heterogeneous granular chains. The effect of a single intruder location on the force and energy transmission in a granular chain has been studied experimentally in [72], using a split-Hopkinson bar setup, as sketched in Fig. 3.4. To validate our dynamic simulations, we first describe the numerical procedure to simulate the dynamics of a granular chain in contact with an elastic bar. This numerical procedure is then used to systematically study the effect of material properties and location of an intruder in an otherwise homogeneous granular chain, which is in contact with an elastic medium, a
Figure 3.4: Schematic of a Split Hopkinson Pressure Bar experiment used to study force transmission through a heterogeneous granular chain [72].

The granular chain is modeled as a nonlinear spring mass system with the granules as point masses and the contacts between them modeled as nonlinear springs with a constitutive response described by (3.9) and (3.10). The equations of motion of the beads are the same as Eqn. (2.18) in the previous chapter.

The bar is modeled as a linearly elastic material with density $\rho_b$, cross-section area $a_b$ and stiffness $E_b$, and its displacement field $u_b(x, t)$ is governed by the equation

$$\rho_b \frac{\partial^2 u_b}{\partial t^2} = E_b \frac{\partial^2 u_b}{\partial x^2}. \quad (3.11)$$

The interaction between the bar and contacting granule is also represented by the nonlinear spring following the model described in the previous section, with $R^* = R$ and $\sigma_y = \sigma^\text{gran}_y$. The bar is represented using linear (2-node) finite elements, together with a central difference explicit time-marching scheme, while a Runge-Kutta time stepping scheme is used for the granules.

3.2.2 Validation

The experimental setup is briefly described here; more details on the experiments can be found in [72]. The diameters of the incident and transmitter bars are all 12.7 mm. A striker with same diameter and a length of 152.4 mm is used to generate an incident loading pulse of approximately 100 µs. A 14 bead granular chain with 13 brass alloy 260 beads and 1 aluminum alloy 2017 intruder bead (8th bead from the incident bar) is sandwiched between
the incident and transmitted bar. All beads are of radius $R = 4.76$ mm. The set of incident, reflected and transmitted strain signals measured from the strain gauges on the incident and transmitted bars are used to calculate the transmitted force ratio (peak transmitted force/peak incident force) and transmitted energy ratio (total transmitted energy/total incident energy). For the numerical simulations, the incident signal measured by the strain gauge is applied as a traction boundary condition at the strain gauge location of the incident bar (Fig. 3.4). The time step $\Delta t$ is chosen to be $2 \times 10^{-9} \text{s}$ and the above numerical formulation is validated by comparing with experimental measurements.

Figure 3.5(a) shows a comparison of the force at the incident and transmitted gauges recorded for the transmitted and reflected waves. Most of the energy input is reflected back into the bar, and this response is captured well by the numerical simulation. The transmitted wave is also predicted reasonably well, with Fig. 3.5(b) showing some difference in a zoomed-in scale for the transmitted force. This difference between the numerical and experimental results in Fig. 3.5(b) is attributed to the errors in the specification of the granular material properties and the contact law, which results in a progressive accumulation of error along the chain. It is also noted that, though the contact law is derived from quasi-static finite element analysis, it compares well with dynamic experimental data and can be used in dynamic studies. This is similar to the observations made in [62], since brass is rate insensitive and stainless steel is not expected to yield in this case.

3.2.3 Effect of intruder

Having validated the contact law and the dynamic solver involving a granular chain in a SHPB setup, we now study the effect of intruder granules on the dynamics of these systems. We remark here that the SHPB setup, where the granular chain is confined between two bars, was intended as a validation of the numerical procedure with experiments. However, to understand the dynamics of these granular chains, we consider a simpler setup, where the loading is directly applied on the granular chain. Indeed, the design of real granular systems would involve finite chains in contact with a continuum media. Here, the medium at the end of the chain is taken to be a linear
Figure 3.5: (Top) Comparison of experimental data with numerical simulation for transmitted and reflected force signals, extracted from a split Hopkinson pressure bar setup. (Bottom) A close-up view of the transmitted signal in the bar.
elastic bar, having a sufficiently high cross-sectional area and yield strength to resist plastic deformation. This setup is in tune with other works, which either consider fixed or other boundary conditions [39], and in contrast with the investigations of wave propagation in infinite elasto-plastic granular systems [30, 52]. Also, in contrast to [40] where the authors work with short bars at the end of granular chains, the bar here is taken to be sufficiently long so that there is no effect of reflection from its free end. To design heterogeneous chains of granular beads, it is essential to first understand the dynamics in the presence of a single intruder under a variety of loading conditions and intruder material properties. To this end, numerical simulations are performed with the system subjected to loading times much longer than the timescale of wave propagation through the chain.

Figure 3.6(a) shows a schematic of the problem configuration: a finite granular chain with a single intruder is subjected to a step load of amplitude $P$ for duration $T$. The objective in the present work is to study the amount of energy transmitted to the bar and its peak force $F_p$ when the granular chain is impacted. The simpler case of impact on a homogeneous chain is presented here to illustrate many of the key features of the phenomena. Figure 3.6(b) shows the time evolution of contact forces across a long homogeneous chain in contact with a semi-infinite elastic bar. There are multiple waves interacting between the granules. A wave may operate either in the loading or unloading part of the force-displacement law at a contact, and these are correspondingly referred to as the loading and unloading waves. At initial times, a loading wave propagates through the chain. As the loading stops, an unloading wave is initiated and its collision with the leading wave results in a complex pattern of interacting waves. The wave speed is proportional to the slope of the force-displacement contact law. Therefore, the unloading wave (U) has a higher speed than the loading wave (L), as evident from the $xt$-diagram, where the unloading waves are observed to have lower slope. Figure 3.6(b) also illustrates that the wave speed of elastic wave in the bar (B) is much higher than either waves in the granular chain. Indeed, this is consistent with our assumption of dynamic equilibrium attained in each of the granules as the waves pass through them, which allow for treating the granules as point masses. We observe that the beads eventually move to the left, and there is no contact with the bar, after the unloading wave has traversed the chain almost 3 times. The dynamics of heterogeneous chains also have
these features, in addition to wave reflection at the intruder interface due to impedance mismatch.

The heterogeneous system under study consists of a chain of 19 brass beads \((E = 115 \text{ GPa}, \nu = 0.30, \sigma_y = 550 \text{ MPa} \text{ and } \rho = 8500 \text{ kg/m}^3)\) and a single intruder bead. Its left end is free and is impacted while the right end is in contact with an elastic bar. Two intruder materials are considered: a softer intruder made of pure Aluminum \((E = 67 \text{ GPa}, \nu = 0.30, \sigma_y = 150 \text{ MPa} \text{ and } \rho = 2700 \text{ kg/m}^3)\) and a harder intruder made of steel \((E = 210 \text{ GPa}, \nu = 0.3, \sigma_y = 700 \text{ MPa} \text{ and } \rho = 7850 \text{ kg/m}^3)\). All the beads have a radius \(R = 4.76 \text{ mm}\). The chain is subjected to a step load of amplitude \(P = 10 \text{ kN}\) for a duration \(T = 100 \mu\text{s}\) and the response of the system is studied by varying the location of the intruder bead. Figure 3.7 shows the peak transmitted force, \(F_p\) (normalized by \(P\)) and the total energy (strain and kinetic energy) in the bar after all the waves have passed in the granular chain and the beads have moved to the left. Also shown are corresponding values for the pure brass, steel and aluminum granular chains for the same loading conditions.

The presence of a steel intruder has little effect on the transmitted force or energy, while these quantities vary significantly with the position of the aluminum intruder. When the intruder is placed further down the chain, both the peak force and the transmitted energy decrease. A soft intruder has the effect of increasing the loading time and decreasing the amplitude of the incident wave. Since the total loading time on the bar is constant (almost twice the time taken for an unloading wave to traverse the whole chain, after which all the beads move to the left) and the peak force decreases down the chain, the transmitted force and energy to the bar also decrease as the intruder is moved further down the chain. Numerical simulations show that the energy dissipated increases and both the kinetic energy of granules and energy transmitted decrease as the intruder is moved down the chain. When the intruder is located in the initial part of the chain, the transmitted energy increases and then decreases, while the peak force also shows some variation. This is attributed to inertial effects and waves interacting with the boundary while the loading is still on.

These results show that the presence of a single intruder in a granular chain can significantly alter the transmitted energy and peak force if the intruder is softer than the rest of the chain, by additional yielding and by wave reflection. It is also noted that these quantities can be lower than the
Figure 3.6: (Top) Schematic illustrating the problem setup for numerical investigation of effect of intruder. A force $P$ (with duration $T$) is applied on the left end, and the peak transmitted force along with energy transmitted to the bar are computed for various intruders. (Bottom) $xt$ diagram of contact forces and net section forces along the bar and a homogeneous granular chain. The colliding wavefronts give rise to additional wavefronts resulting in a complex pattern.
Figure 3.7: Variation of peak transmitted force (dashed) and total energy (solid) with the position of a single intruder in a brass chain, along with the values for monodisperse chains (indicated by the markers along the vertical axes). These quantities are sensitive to the location in the case of a soft (Al) intruder, while they remain almost constant for a hard (steel) intruder.

corresponding values for a chain composed purely of beads of the intruder material (pure Al in this case). The key reason for this variation is the pulse broadening due to the impedance mismatch at the intruder interface. This results in a larger fraction of energy dispersed in the form of kinetic energy of the beads. Although the dissipation at a softer intruder is higher, we note that this impedance mismatch in a heterogeneous chain results in lower energy getting transmitted to the bar compared to the case where the entire granular chain is composed of the softer intruder material. Thus impedance mismatch between the two media, leading to energy reflected back, is identified as the dominant mode, as opposed to plastic dissipation, of impact resistance in these heterogeneous granular chains for the loading conditions considered here. We also remark here that the position at which the maximum force is attained can vary with the loading time.

3.2.4 Normalization variables

Finally, the key material properties of the intruder that influence energy transmission in granular chains are identified by conducting a systematic
parametric study of a chain in contact with a bar. The material properties of the single intruder in a chain are varied, one at a time, over a large range, and the transmitted energy to the bar is computed. A chain of 20 elastoplastic beads in contact with an elastic bar is considered, with the bar having a Young’s modulus $E_b = 200$ GPa, Poisson’s ratio $\nu_b = 0.30$, density $\rho_b = 8100$ kg/m$^3$ and cross sectional area $a_b = 127$ mm$^2$. As before, the length of the bar is chosen to be sufficiently large so that the reflection of transmitted waves from its free end does not affect the solution at the interface. The granule material properties are chosen to correspond to steel, with material properties $E = 210$ GPa, $\nu = 0.3$, $\sigma_y = 700$ MPa, $\rho = 7850$ kg/m$^3$ and radius $R = 4.76$ mm. The intruder is located at the 10th position, with three of its material parameters chosen as the same as those of the granules in the chain, while a single parameter is varied over a range. A constant force $P = 10$ kN is prescribed on the first bead for a time duration $T = 100$ $\mu$s.

Figure 3.8(a) shows the total energy $e_T$ transmitted to the bar for the case of the chain with intruder, normalized by the energy $e_T^0$ associated with the homogeneous chain. The subscripts $i$ and 0 indicate the intruder and homogeneous chain beads’ properties, respectively. The transmitted energy in an intruder chain is almost constant, even as the intruder’s Young’s modulus is varied over four orders of magnitude. The transmitted energy ratio increases with decreasing yield strength, and with increasing radii and density. These trends can be explained by considering the relative timescale of motion of the intruder and the homogeneous material. Based on a first-order approximation of the contact law in the equation of motion in the granular chain, the acceleration of the bead scales with its displacement as $\ddot{u} \sim (\sigma_y/\rho D^2)u$. This relation suggests that the characteristic timescale of wave propagation through a bead of specified material and size is $\tau^* = D\sqrt{\rho/\sigma_y}$. Figure 3.8(b) recasts the data presented in Fig. 3.8(a) by scaling the properties on the $x$-axis according to the functional dependence of $\tau^*$ on these properties. As apparent in that figure, this transformation leads to the curves associated with the yield stress ($\sigma_{yi}$), radius ($R^*$) and density ($\rho^*$) collapsing onto a single curve. The small deviations arise due to nonlinear dependence of the contact law on the material parameters, while the timescale $\tau^*$ was derived with the assumption of linearity of the contact law in both loading and unloading regimes.

From a physical point of view, when the timescale characterizing the mo-
Figure 3.8: (a) Ratio of energy $e_T$ transmitted to the bar in a chain with intruder to that in a homogeneous chain $e_T^0$ as a function of the intruder’s material properties. The energy ratio shows little variation with the stiffness of the intruder, while it increases with decreasing density, radius and increasing yield strength. (b) When normalized appropriately, the distinct curves almost collapse to a single curve, indicating the existence of a single nondimensional parameter governing the fraction of energy transmitted.
tion of the intruder is much larger, the leading wave encounters an interface having a higher impedance. This is similar to the leading wave encountering an interface like a bar, which leads to a buildup of forces behind it, as shown previously in region 1 in Fig. 3.6(b). Both the duration of this buildup and the increase in force magnitude depend on the relative timescales $\tau^*$ of the intruder and homogeneous beads. On the other hand, when this timescale of the intruder is shorter, there is no corresponding buildup of forces behind the interface, and hence only a small variation is observed, which is due to waves reflected at the interface. Hence, the change in transmitted energy due to the presence of the intruder is solely a function of the ratio of $\tau^*$ for the intruder and the granular chain materials.

3.3 Conclusions

A numerical study of the effect of an intruder on the dynamics of elastoplastic granular chains has been conducted. In the first part of this study, normalized contact laws have been derived for any two elastic-perfectly-plastic spheres of distinct materials and sizes. The normalizations have been derived from the governing equations and are verified using finite element simulations. After appropriate normalizations, a unified contact law has been obtained and validated using quasi-static experiments, showing good agreement.

In the second part, the unified contact law has been used to perform dynamic simulations of a granular chain in contact with a linear elastic bar. The effect of an intruder in a granular chain has been analyzed by varying its position. The peak transmitted force and energy to the bar have been found to depend on the location of the intruder when it is softer than the homogeneous chain material, while it has little sensitivity to position when the intruder is of a harder material. Finally, the key non-dimensional material parameter which dictates the extent of energy transmitted was identified by physical arguments and has been verified by systematic numerical simulations. This work is intended to be the foundation for designing real granular systems to mitigate or tailor impact waves, in energy regimes where the materials deform plastically.
In this chapter, the dynamics of monodisperse elastic and elasto-plastic granular chains under a wide range of loading conditions is studied, and two distinct response regimes are identified in each of them. In the first part of this chapter, wave propagation in elastic granular chains subjected to short and long duration force impulses is studied systematically. A short loading duration leads to a single solitary wave propagating down the chain, while a long loading duration leads to the formation of a train of solitary waves. A simple model is developed to predict the peak force and wave velocity for any loading duration and amplitude. In the second part of this chapter (Section 4.2), the peak contact force along an elasto-plastic chain is characterized for a wide range of loading conditions. Here, wave trains form even for short loading times due to a mechanism distinct from that in elastic chains. A model based on energy balance predicts the decay rate and transition point between the two decay regimes. For long loading durations, loading and unloading waves propagate along the chain, and a model is developed to predict the contact force and particle velocity.

4.1 Elastic chains

4.1.1 Problem setup

In this section, the dynamic response of a semi-infinite monodisperse elastic granular chain subjected to a force impulse at one end is studied. The granular chain is composed of spherical contacting beads. The dynamic response of this system can be modeled as a spring-mass system, with the spheres modeled as point masses, and the contact between them being represented by nonlinear springs. These nonlinear springs follow a contact law based on
Hertzian theory for spherical surfaces in contact [8]. The equation of motion of bead $i$ within the chain with displacement $u_i$ is the same as Eqn. (2.18).

For the first sphere, the equation of motion is

$$m_1 \ddot{u}_1 = -F_{1,2}(\alpha_{1,2}) + f(t),$$

(4.1)

where $f(t)$ is the external force applied to the left of the first sphere and is given by

$$f(t) = \begin{cases} \frac{A}{\tau} & \text{if } 0 < t \leq T, \\ 0 & \text{otherwise}. \end{cases}$$

(4.2)

For a given amplitude $A$ and loading time $T$, the total impulse is thus $I = AT$.

A fourth-order Runge Kutta scheme is used to solve the system of coupled ordinary differential equations with a time step of $5 \times 10^{-9} \text{ s}$. In all simulations, the radii of spheres are taken to be 4.76 mm, and the material properties are chosen to correspond to brass, with density $\rho = 8500 \text{ kg/m}^3$, Young’s modulus $E = 115 \text{ GPa}$ and Poisson’s ratio $\nu = 0.30$. Let $E^* = E/2(1 - \nu^2)$ and $R^* = R/2$ be the effective stiffness and effective radii of two identical contacting spheres [8]. The solutions are presented hereafter in a non-dimensional form, with the force and impulse normalized as

$$\tilde{F} = \frac{F}{E^*R^2\tau},$$

(4.3)

$$\tilde{I} = \frac{I}{E^*R^2\tau},$$

(4.4)

where $\tau$ is the intrinsic time-scale associated with the elastic sphere system, given by Eqn. (2.24). The key variables of interest are the peak contact force and wave velocity along the chain for a wide range of loading conditions.

### 4.1.2 Single solitary wave vs. wave trains

To demonstrate the effect of loading times, the elastic chain is subjected to different load amplitudes $A$, with the total impulse kept fixed at $I = 0.40 \text{ Ns}$. The loading times for the two cases are 1 $\mu$s ($T/\tau = 1.29$) and 100 $\mu$s ($T/\tau = 1.29 \times 10^2$), respectively. Similar to the observations in [29], where a small (large) impact mass leads to a shorter (longer) loading time, for a fixed input impulse, a short loading time results in a single solitary
Figure 4.1: Transition from single solitary wave to wave train regime. The solid curve denotes the critical loading time above which wave trains form. This critical loading time follows a $-1/6$ power law with input amplitude. The symbols indicate numerical simulation values.

wave, while a long loading time results in a train of solitary waves down the chain [31]. The leading solitary wave has the highest amplitude since the wave velocity depends on the force amplitude with $1/6$ power [6].

A systematic study is conducted to determine the transition from a single solitary wave to a wave train. The loading amplitude $A$ is kept fixed while the loading time $T$ is increased from a low value until a transition time is attained when the response signal breaks into a train of solitary waves down the chain. The symbols in Fig. 4.1 show this transition time, obtained numerically for various loading amplitudes. Two distinct regimes exist and a power law with exponent $-1/6$ describes the transition. This implies that for any loading time $T$ (or amplitude $A$), there is a transition amplitude $A$ (or loading time $T$) below which a single solitary wave forms and above which, train of solitary waves form.

We examined the contact force evolution with time at a few initial contacts along the chain for both the short $T = 1$ $\mu$s and long $T = 100$ $\mu$s duration loading cases. In the long loading duration case, the force evolution at a contact steepens down the chain to form multiple distinct solitary waves, while in the short loading duration case, the contact time increases down the chain to form a single solitary wave. Due to the dependence of the wave speed on the force amplitude, there is an inherent time scale $\tau_s$ of loading
at a contact for a solitary wave, which is a function of the amplitude. If the loading time at a contact is less than the time scale $\tau_s$ corresponding to the loading amplitude, the contact time increases and the amplitude decreases down the chain until a solitary wave forms, for which the contact time equals the inherent time scale $\tau_s$ corresponding to the loading amplitude at the contact. Similarly, if the contact time is longer than the time scale $\tau_s$ for the corresponding loading amplitude, the amplitude increases, the contact time decreases and the wave breaks down the chain until a leading solitary wave forms. Thus, for a fixed input impulse, if the time of contact along the chain exceeds $\tau_s$, the response breaks into wave trains, each having shorter contact time, and the peak force increases at contacts down the chain. On the other hand, if the time of contact is less than $\tau_s$, the contact time increases and the peak force decreases as the wave progresses down the chain. The system has an inherent tendency to form solitary waves by appropriately adjusting the peak force and time of contact. Similar arguments have been made by Job et al. [29], who observed a transition depending on the size of the striker impacting the chain. Since the wave velocity scales with peak force amplitude as $F_p^{1/6}$, the inverse of time scale $\tau_s$ would also follow the same scaling law. Thus the transition time to wave trains $T$, which is the contact time at the first contact, also follows a $-1/6$ power law with loading amplitude, as seen earlier in Fig. 4.1.

4.1.3 Model

In this section, a model is developed to predict the peak contact force and leading wave velocity down an infinitely long chain, subjected to any constant loading amplitude and impulse. For sufficiently short loading times, the amplitude $F_p$ of the solitary wave propagating down the chain depends only on the input impulse $I$. This amplitude can be evaluated by considering the limit of an infinite force acting for zero time. Indeed, the limiting case is equivalent to the problem of an initial velocity $v_0$ prescribed to the first sphere. Through dimensional arguments [7], the peak force of the solitary wave scales with initial velocity as $F_p \propto v_0^{6/5}$ and, from numerical experi-
ments, the solution to this problem is determined to be
\[ F_p = 0.719 \left( m^3 E^* R^* v_0^6 \right)^{1/5} = 0.719 \left( \frac{E^* R^* f_0^6}{m^3} \right)^{1/5}. \]  
(4.5)

Next, the response of the system is analyzed for large loading times. In this case, a train of solitary waves of decreasing amplitude traverse down the chain. Since the wave with the highest amplitude is the leading wave, the peak contact force down an infinite chain is the force due to the leading solitary wave. To determine this peak force, consider a chain subjected to a long loading time of \( T = 10 \text{ ms} \) and \( A = 40 \text{ N} \). The contact force and sphere velocity distribution at time \( t = 100 \text{ ms} \) are plotted in Fig. 4.2, showing two regimes: a constant force and velocity regime near the point of loading, and a solitary wave-like regime at the leading edge. It is noted that there is a transition between the two regimes and the solitary waves at the leading front have not fully formed and started detaching yet. The waves at the leading front start to detach after loading stops, and we observe from numerical experiments that the peak force of leading solitary wave is almost equal to the peak force at the leading front of the wave sufficiently down the chain. The solid curve in Fig. 4.2 shows the peak contact force, and although it continues to increase as noted in [6] for a related problem, the rate of increase is small and the peak force can be considered constant.

To determine the upper bound on the peak force, a simplified model is considered based on two regimes: a leading regime consisting of solitary waves, and a trailing regime having a constant force and velocity with a sharp transition between them. In the trailing regime, the spheres move with a constant velocity \( v \), and hence the bead displacement is given by
\[ u(x,t) = U(x) + \left( t - \frac{x}{c_t} \right) v, \quad x < c_t t, \]  
(4.6)

where \( c_t \) is the speed of the transition point and \( U(x) \) is the bead displacement after the leading regime has passed \( x \). Noting that the virial due to the above expression and in a solitary wave are constant, the kinetic \( K \) and potential energy \( P \) are related by the virial theorem as \( 4K = 5P \), which is found to be in excellent agreement with numerical simulations.

Let the leading front be moving with a velocity \( c \) and the solitary wave
Figure 4.2: Contact force and bead velocity snapshots along the chain for a long loading time. Two regimes are seen: at the left end, the contact force and sphere velocity are constant, while at the right end, there are distinct waves. The solid curve in the top figure denotes the peak force achieved at every contact.

at the front have a corresponding peak contact force \( F_p \). To evaluate the velocity of the spheres in the constant regime, we assume a quasi-steady state of the front, wherein adjacent contacts have the same force histories, but shifted by the time taken for the front to propagate a bead diameter. This leads to the following relation between the contact forces \( F_L \) and \( F_R \) acting on the left and right of a sphere, respectively:

\[
F_R(t) = F_L \left( t - \frac{2R}{c} \right).
\]  

(4.7)

Integrating the equation of motion of the sphere and noting that \( F_R(t) = F_L(t) = A \) when the bead is in the constant regime leads to the following relation for the bead velocity:

\[
\int_{0}^{\infty} m \frac{dv}{dt} dt = \int_{0}^{\infty} (F_L - F_R) dt
\]

\[
\Rightarrow mv = \frac{2RA}{c}.
\]  

(4.9)

Finally, to evaluate the speed of the leading front, an energy balance in a time interval \( \Delta t \) leads to

\[
Av\Delta t = \Delta P + \Delta K = \frac{9}{4} \Delta P.
\]  

(4.10)
Assuming that the force distribution and hence the energy in the solitary wave regime remain constant, all the input energy in this time interval goes to the constant force regime. Since the contact force is constant in the first regime, Eqn. (4.10) becomes

\[ A v \Delta t = \frac{9}{4} \left( \frac{2A^{5/3}}{5k^{2/3}} \right) \frac{c \Delta t}{2R}, \tag{4.11} \]

where the bracketed term is the potential energy at a contact having a force amplitude \( A \) and \( k = 4E^* \sqrt{R^*} / 3 \) is the constant defining the elastic contact. Using (4.9) leads to the following expression for an upper bound of the front velocity:

\[ c^u = \sqrt{\frac{10A^{1/3}k^{2/3}D^2}{9m}}. \tag{4.12} \]

To evaluate a lower bound, consider the contact force acting on a spring between beads \( i - 1 \) and \( i \), given by Hertz law as:

\[ F = k (u_i - u_{i-1})^{3/2}. \tag{4.13} \]

Differentiating (4.13), the relative velocity is given by

\[ \dot{u}_{i-1} - \dot{u}_i = \frac{d}{dt} \left( \frac{F}{k} \right)^{2/3}. \tag{4.14} \]

Integrating (4.14) between \( t = 0 \) and \( t \rightarrow \infty \), and again using quasi-steady considerations, i.e., \( v_i(t) = v_{i-1}(t - 2R/c) \) leads to

\[ \frac{2\nu R}{c} = \left( \frac{A}{k} \right)^{2/3}. \tag{4.15} \]

Combining (4.9) and (4.15), we get a lower bound of velocity \( c \) of the front

\[ c^l = \sqrt{\frac{A^{1/3}k^{2/3}D^2}{m}}. \tag{4.16} \]

The lower and upper values of wave speed given by the above expressions are \( c^l = 668.0 \text{ m/s} \) and \( c^u = 704.0 \text{ m/s} \). Using Nesterenko’s solution [6] for solitary wave, we get the upper and lower bound on peak force values, \( F^u_p = 107.2 \text{ N} \) and \( F^l_p = 78.15 \text{ N} \). Comparing to the numerical values of
Figure 4.3: (Top) Peak contact force for varying amplitudes and loading times, with each curve corresponding to a fixed impulse. (Bottom) When normalized appropriately, the response collapses to a single curve. Two regimes are observed: the constant force corresponds to a single solitary wave, while the other regime corresponds to wave trains.
wave velocity $c = 687.2\text{ m/s}$ and peak force $F_p = 92.7\text{ N}$, we note a 3% and a 15% error in wave velocity and peak force, respectively. This difference arises due to the assumption that the entire energy input goes to the first regime. However, in numerical simulations, a careful observation reveals that the energy per unit length of chain is higher in the solitary wave regime than in the constant force regime, and there is a transfer of energy between the two regimes. Hence, the potential energy expression used in Eqn. (4.11) is slightly lower than the total potential energy input to the system, and thus it is clear that the predicted force and wave velocity are indeed upper bounds on the exact values.

For long loading times leading to the formation of wave trains, the above analysis shows that the peak force of the leading solitary wave scales linearly with the input amplitude ($F_p \sim c^6 \sim A$), while in the single solitary wave regime, dimensional arguments show that the peak force scales with impulse as $F \sim I^{6/5}$. Figure 4.3(a) shows the variation of peak contact force ($\bar{F}_p = F_p/E^*R^2$) with loading amplitude ($\bar{A} = A/E^*R^2$) down a long chain for distinct impulses. Two response regimes are observed: in the wave train regime, the peak force increases with amplitude of input force and is independent of $I$, while in the single solitary wave regime, the peak force is independent of the loading amplitude. Indeed, as observed in Fig. 4.3(b), the appropriate normalizations lead to the distinct responses collapsing to a single ‘master’ curve. This normalized curve allows to predict the peak contact force down a long elastic chain for any impulse, amplitude and duration of step loading. For short loading times, the response is only a function of the form of the total impulse and does not depend on the form of the loading function, while for long loading times, the peak force is a function of loading amplitude only and independent of the loading duration or impulse.

4.2 Elasto-plastic chains

We now turn our attention to wave propagation in elastic-perfectly-plastic granular chains for a wide range of loading durations. Two force-displacement models are considered here: the first is a simple bilinear model following Walton and Braun [73], and the second is a more accurate elasto-plastic contact model described in Sec. 2.1. Both models are shown in Fig. 4.4. The sim-
Figure 4.4: Two contact force-displacement laws for elasto-plastic spheres. The simpler bilinear model is defined by 2 parameters: stiffness in plastic loading $k$ and ratio of residual to maximum displacement, $\beta = 1 - \frac{k}{k_u}$. $F_y$ and $\alpha_y$ denote the contact force and relative displacement at the onset of yield.

The simpler bilinear model allows for the capture of the key contributions of the material-induced dissipation on the dynamic response of the elasto-plastic chain, while the second, more complex contact model provides a more accurate quantitative description of the system, especially with regard to the elastic unloading and reloading. Two parameters define the bilinear model: the stiffness, $k$, associated with plastic loading and the unloading coefficient $\beta < 1$, which is the ratio of the residual displacement upon complete unloading to the maximum displacement previously achieved, i.e., the displacement at the start of unloading. The contact stiffness during unloading is the slope of the unloading curve $k_u = k/(1 - \beta)$. The energy dissipated at a contact for which a peak force $F$ has been attained is the difference in areas between the associated loading and unloading curves and is given by

$$E_{\text{dis}} = \int_0^{\alpha_{\text{max}}} (F_{\text{load}} - F_{\text{unload}}) \, d\alpha = \frac{F^2}{2} \left( \frac{1}{k} - \frac{1}{k_u} \right) = \frac{\beta}{2k(1 - \beta)} F^2,$$

where $\alpha_{\text{max}}$ is the maximum displacement corresponding to the force $F$. The plastic loading stiffness is set as $k = 18.5\sigma_y R^*$, with $\sigma_y$ being the yield
strength, to match the stiffness in plastic loading for large forces in the model described in Sec. 2.1. The dynamic response of a semi-infinite chain is studied for short and long duration loading times since, as demonstrated hereafter, the wave characteristics and contact forces along the chain are very different for these two loading conditions.

4.2.1 Short loading times

Wave characteristics

In this section, the contact force along the chain and its decay rate are characterized for the simpler, bilinear contact law. As shown in Sec. 2.2, for short loading times, elasto-plastic chains have distinct characteristics including wave merging and interaction, and rapid decay in peak force and energy dissipation along the chain. In that study, two regimes of spatial force decay were observed: exponential decay over the first few beads, followed by an inverse decay. Furthermore, that study showed that, after appropriate normalizations, the dynamic response of elasto-plastic systems is identical for any impulse and short loading times.

The second regime of force decay starts when the trailing waves begin to cause dissipation at a contact, i.e., when the peak force due to the trailing wave is higher than the force previously attained due to the leading wave. Figure 4.5 shows the contact force and bead velocity history of the first few beads in an elasto-plastic chain ($\beta = 0.80$) subjected to an impulse $I = 0.1$ Ns for a loading time $T = 1 \mu s$. For an elastic system subjected to such an impulse, the first few beads end up moving to the left following the initial impact events [27]. However, in the presence of plastic dissipation, we observe in Fig. 4.5(b) that the beads continue to move to the right. Due to higher stiffness and hence higher wave speeds during unloading, the decrease in contact force during unloading is much steeper than the increase during plastic loading (Fig. 4.5(a)). This effect causes a lower impulse to be transmitted to the bead on the right. The momentum imparted to a bead, $mv = \int (F_L - F_R) dt$ is thus positive, and the bead has a net residual velocity to the right after the leading wave has passed. Since there is energy dissipation at each contact, the net impulse transferred through the beads also
Figure 4.5: Time evolution of the contact force and bead velocity for the first few beads in an elasto-plastic chain. Secondary waves behind the leading wave are caused by collisions of beads in free flight.
decreases along the chain. Indeed, the residual velocity of beads and contact forces also decrease down the chain. As apparent in Fig. 4.5(b), the beads are initially in free flight and then collide, leading to secondary waves. This is in sharp contrast to the creation of secondary waves in elastic chains caused by wave steepening, as described previously in section 4.1.2. Collisions between beads behind the leading wave cause contact forces, which can produce additional dissipation. To determine the point at which the transition happens, we first derive the spatial dependence of the peak contact force and residual velocity along the chain.

**Exponential decay regime**

In the first regime, a leading wave travels down the chain, causing an exponential decay in peak contact force. To derive the expression of that decay for the case of a bilinear elasto-plastic contact law, we use an energy balance between the instants at which two adjacent contacts attain their respective peak force:

$$E^{\text{dis}}_{\Delta t} = E^{\text{tot}}_t - E^{\text{tot}}_{t+\Delta t},$$

where $E^{\text{dis}}_{\Delta t}$ is the energy dissipated in a time interval $\Delta t$ and $E^{\text{tot}}_t$ is the sum of kinetic and potential energies over the entire chain at time $t$.

We now construct approximations for the various components of Eqn. (4.18) as the leading wave propagates down the chain. At a contact, the potential energy is the area under the unloading curve, and is a quadratic function of the force. Figure 4.6 shows the contact forces at the time instant when the second contact reaches its peak force. Let us assume that the leading wave has a self-similar structure, i.e., that the force at the contact ahead $F_a$ is a fraction of the peak contact force ($F_a = \eta F$). Due to the bilinear nature of the contact law and the assumption of self-similar structure of the leading wave, the wave velocity, the time taken by the wave to traverse a bead and thus the contact time $T_c$ on a bead are constant. From dimensional arguments, the bead velocity $v_m$ at the instant of peak contact force scales linearly with $F$, as $v_m/T_c \sim (F - F_a) \sim F$ and thus $v_m \sim F$. Similarly, the free flight residual velocity $v_r$ of the bead to the left of contact also scales as the peak contact force.

To compute the total energy of the leading wave in the exponential decay
Figure 4.6: Force decay of the first few beads for short loading times. The peak force obtained numerically (circles) is well predicted by the analytical model (solid curve). Also shown is the leading wave profile (histogram) at the instant the second contact attains its peak force.

Figure 4.7: Spatial variation of peak contact force along the chain for 3 distinct unloading parameters $\beta$. The symbols denote numerical results, while the solid and dashed curves, respectively, show the decay predicted by the model (Eqn. (4.28)) and the peak force due to secondary waves. Their intersection point (dotted vertical arrow) is the start of transition from exponential to power law decay regimes. The circles on the $x$-axis denote the transition points obtained numerically.

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regime, the velocities of the spheres supporting this wave and the contact forces between them (shown by histograms in Fig. 4.6) are determined in terms of the peak contact force $F_p$. All expressions are derived at the time instant a contact attains the peak force. The contact associated with the peak force and the contacts ahead operate only in the plastic loading regime and hence the contact forces and velocity of spheres are functions of $k$ and $m$ only and do not depend on the unloading law of the elasto-plastic material. The expressions for $F_p$, the force on second contact $F_2$ and the velocity of the first two spheres $v_m$ are derived by considering a series of linear spring mass system having mass $m$ and stiffness $k$, respectively, and with left end of the first mass subjected to an impulse $I$. They are given by

$$v_m = 0.482 \frac{I}{m}, \quad (4.19)$$

$$F_p = 0.721 \sqrt{\frac{k}{m} I}, \quad (4.20)$$

$$F_2 = 0.199 \sqrt{\frac{k}{m} I}. \quad (4.21)$$

The trailing wave velocity depends on the unloading coefficient $\beta$. The following relation is obtained for the trailing velocity $v_t$:

$$v_t = v_m - 0.599 \sqrt{1 - \beta \frac{I}{m}}, \quad (4.22)$$

and it is found to be in excellent agreement with the numerical values. Using the above relations, the maximum $v_m$ and trailing $v_t$ velocities of the two spheres and the contact force $F_2$ are expressed in terms of the peak contact force $F_p$ in the leading wave.

Using the procedure described in the preceding paragraph, the bead velocity $v$, contact force ahead $F_a$ and residual velocity of the bead $v_r$ are related to the peak force $F$ by

$$F_a = \eta F, \quad v_m = \gamma F, \quad v_r = \omega F, \quad (4.23)$$

where $\gamma, \eta, \omega$ are functions of material properties.

Though the leading wave spans 4 bead diameters and 3 contacts (Fig. 4.6), the contributions of potential energy due only to the two contacts having higher forces are considered to compute the total energy. Similarly the ve-
locity \(v_m\) of two beads whose contact force is maximum are considered for kinetic energy, while the remaining bead velocities and contact force, having small values, are neglected. When the force at a contact is maximum, the relative velocity of the beads at that contact is zero. The total kinetic \(K\) and potential \(P\) energy components due to the leading wave are thus

\[
K \approx 2 \frac{1}{2} m v_m^2 = m \gamma^2 F^2, \quad P \approx \frac{1}{2 k_u} (F^2 + \eta^2 F^2). \tag{4.24}
\]

Let the peak contact force at two adjacent contacts distance \(x\) and \(x + dx\) be \(F\) and \(F + dF\), respectively, and let \(\Delta t\) be the time interval between the times when the two adjacent contacts attain their respective peak forces. The energy dissipated in the time interval \(\Delta t\) as the peak force is attained at two adjacent contacts is

\[
E_{\text{dis}}^\Delta = \frac{1}{2} \left( \frac{1}{k_u} - \frac{1}{k} \right) \left[ F^2 + (1 + \eta^2) (2F dF + dF^2) \right]. \tag{4.25}
\]

Substituting these expressions into the energy balance (Eqn. (4.18)), and accounting for the kinetic energy due to residual velocity \(v_r\) leads to

\[
\left( (dF)^2 + 2F dF \right) \left[ \frac{1 + \eta^2}{2k} + m \gamma^2 \right] + F^2 \left[ \frac{1}{2} \left( \frac{1}{k} - \frac{1}{k_u} \right) + \frac{1}{2} m \omega^2 \right] = 0. \tag{4.26}
\]

Noting that energy dissipation takes place at discrete locations along the chain separated by a distance \(D\), we obtain the following differential equation:

\[
\left[ \left( \frac{dF}{dx} \right)^2 + 2DF \frac{dF}{dx} \right] \left[ \frac{1 + \eta^2}{2k} + m \gamma^2 \right] + F^2 \left[ \frac{1}{2} \left( \frac{1}{k} - \frac{1}{k_u} \right) + \frac{1}{2} m \omega^2 \right] = 0, \tag{4.27}
\]

whose solution is

\[
F(x) = \exp \left[ -\frac{x}{D} + \frac{x}{D} \sqrt{1 - D \frac{(\frac{1}{k} - \frac{1}{k_u} + m \omega^2)}{(\frac{1 + \eta^2}{k} + 2m \gamma^2)}} \right]. \tag{4.28}
\]
Figure 4.6 shows the comparison between the numerical solution (symbols) and approximation (Eqn. (4.28)) for the peak force along the initial part of the chain. The difference between the numerical solution and analytical approximation arises due to the assumption of self-similar wave structure, whereas in numerical simulations, the bead velocity and force at the contact ahead also exhibit small variations with time of loading. The difference between the predicted peak contact force and that obtained numerically is found to be around 10%.

Transition and inverse decay regime

The exponential decay regime ends when the contact force associated with trailing secondary waves is higher than that attained by the leading wave. As observed earlier (Fig. 4.5), secondary waves are caused by collisions of beads in free flight. From Eqn. (4.23) and Fig. 4.5(b), we observe that the residual velocities of beads behind the leading wave scale with the peak force, and hence decrease down the chain. Eventually, these beads collide causing secondary waves, which can be modeled as independent collisions between beads. To determine the peak force evolution of these secondary waves, we first determine the peak contact force between two beads in free flight having relative velocity $\Delta v$ and operating in the elastic reloading regime of the force-displacement law. For the bilinear law, following Johnson [8], the relative displacement $\alpha$ satisfies the equation:

$$m\ddot{\alpha} + 2k_0\alpha = 0,$$

and the peak contact force between them is given by

$$F_s = k_0\alpha = \sqrt{k_0m/2\Delta v}.$$  

(4.30)

When two beads of equal masses following a non-dissipative contact law collide, it is well known that their velocities interchange. Collisions between the beads in free flight lead to the contact forces in secondary waves, whose magnitude scales with the difference in velocities $\Delta v$. Since the residual velocity decreases down the chain ($v_i = \omega F$), this leads to a corresponding increase of the contact forces due to collisions, and a simple model is con-
Figure 4.8: Bead velocity distribution along the chain at three time instants for a short loading time, $T = 1\mu s$. In the inverse regime, the velocity distribution is almost uniform.

Structures to predict the peak force due to these collisions. Assuming that the collisions occur in order and are independent, the peak force due to secondary waves, based on Eqn. (4.23), is given by

$$F_{\text{s}i} = \sqrt{\frac{k_u m}{2} \Delta v^i} = \sqrt{\frac{k_u m \Delta F^i}{\omega}}, \quad (4.31)$$

where $\Delta F^i$ is the difference in peak force attained at the first contact and the $(i + 1)^{\text{th}}$ contact.

The solid and dashed curves in Fig. 4.7 respectively show the approximate spatial variation of the leading (4.28) and secondary (4.31) waves for three distinct values of unloading coefficient $\beta$. The corresponding numerical values of the peak force are denoted by symbols. The intersections between these two sets of curves (emphasized by the dotted vertical arrows) provide estimates of the transition points between exponential and inverse decay regimes. For these three values of $\beta$, the numerical simulations give these transition points to be at the third, fifth and ninth contact points, respectively, and are shown circled on the $x$-axis in Fig. 4.7. These transition points are relatively well predicted by the model, with the difference due primarily to the assumption behind the prediction of the peak force of the leading wave.
As demonstrated in [52], the peak force decays inversely with distance in the second regime. Figure 4.8 shows the sphere velocity distribution along the chain in the inverse decay regime at three time instants for the bilinear material model with $\beta = 0.90$. As apparent there, the velocity distribution is almost uniform along the chain. Furthermore, from numerical simulations, we observe that most of the energy is associated with the kinetic energy of the spheres along the chain while the total potential energy remains very small. A simple model based on energy balance is then constructed assuming that the potential energy is negligible, and that the velocity of spheres is uniform along the chain. The energy dissipated $E_{\text{dis}}$ as the wave travels a distance $\Delta x$ is then the difference in total kinetic energy

$$E_{\text{dis}}(\Delta x) = \frac{1}{2}mv^2\bar{x} - \frac{1}{2}m(v + \Delta v)(\bar{x} + \Delta \bar{x}),$$

(4.32)

where $\bar{x} = x/D$ is the number of beads in a distance $x$. Using Eqn. (4.17), assuming quasi-steady conditions (Eqn. (4.9)) in the time interval $\Delta t$ and noting that the velocity in the loading regime $c = \sqrt{k/m}$ for the bilinear law, Eqn. (4.32) simplifies to

$$\left(\frac{1}{k} - \frac{1}{k_u}\right)F^2 \Delta x = -\frac{F^2}{k} \Delta x - \frac{2F}{x} \Delta x \Delta F.$$  

(4.33)

Replacing the difference operators by differentials leads to

$$\frac{dF}{dx} + \left(\frac{1 + \beta}{2}\right)\frac{F}{x} = 0,$$

(4.34)

and its solution is given by

$$F = \left(\frac{a}{x}\right)^{(1+\beta)/2},$$

(4.35)

where $a$ is a constant of integration. For values of $\beta$ close to unity, the force is thus predicted to decay as an inverse power law with distance along the chain.
4.2.2 Long loading times

Finally, the behavior of elasto-plastic chains subjected to impulses with long loading times and constant loading force is studied, and expressions for the force and velocity of the system are developed. A monodisperse elasto-plastic chain following a bilinear force-displacement law with unloading coefficient $\beta = 0.80$ is subjected to the loading conditions in Eqn. (4.2) with loading time $T = 2.5 \text{ ms}$ and amplitude $A = 8 \text{kN}$. We first discuss the qualitative features of the wave structure, and then construct a model for predicting the peak contact forces and particle velocities.

Figure 4.9 shows the distribution of contact force and bead velocity at three different time instants, along with the distribution of peak contact force attained along the chain. The distributions are seen to be qualitatively distinct at the three instants, having uniform values in certain regions with sharp transitions (wavefronts) along the chain. At early times ($t = 2 \text{ ms}$ in Fig. 4.9), when the chain is still under external loading, a single wavefront travels to the right. After the loading stops ($t = 3 \text{ ms}$), a trailing unloading wave travels from the left end. The beads behind this trailing wave have a lower velocity than the beads ahead of it, and the contacts have zero force behind the trailing wavefront. As the force reduces from the constant value $F_1$ to zero in the contacts traversing the trailing wavefront, these contacts operate in the steeper unloading part of the contact force-displacement law. Thus, this wavefront has a higher velocity and eventually collides with the leading wave. After the collision of wavefronts, there is a decrease in the peak force of the leading wave and two wavefronts move in opposite directions, as shown by the results, corresponding to time $t = 5.2 \text{ ms}$. The wavefront traveling to the left is reflected at the free end and again travels to the right with the same amplitude, until it collides again with the leading wavefront. This process continues indefinitely, with the trailing wavefront being reflected from the free end and colliding with the leading wavefront. The leading wavefront always operates in the plastic loading part of the force-displacement law, while the trailing wavefront always operates in the elastic unloading and reloading part. The amplitudes decrease progressively with each collision, as the energy gets distributed over a longer part of the chain. It is observed that the peak force in the first few contacts is higher than the constant value attained later. The peak force reaches a higher value than the amplitude $A$. 

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Figure 4.9: Force and velocity distribution along the chain for long loading time at three time instants. At $t = 2\text{ ms}$, a single wave moves to the right, while at $t = 3\text{ ms}$, two waves are moving to the right. Finally, at $t = 5.2\text{ ms}$, two waves are moving in opposite directions.
due to inertial effects in these contacts. In the remainder of this section, expressions for the steady contact forces and velocity of beads along the chain are derived and the point of transition when the wavefronts collide and the contact force drops is determined. Here, we present the analysis until the end of the first collision, although the procedure we use here can be applied to determine the amplitudes after subsequent interactions.

Along an elasto-plastic chain following the bilinear law, the leading wave travels with a speed \( c = D\sqrt{k/m} \) as it operates in the plastic loading regime of the force-displacement law. To evaluate the velocity of the beads, consider a bead down the chain, in contact with two beads, which exert contact forces \( F_L(t) \) and \( F_R(t) \) from the left and right side, respectively. Consider a time instant just after the leading wave has crossed this bead. The bead has attained a constant velocity \( v_1 \), and the left and right contacts have a constant force \( F_1 \). Similar to the elastic case, assuming a quasi-steady response between two adjacent contacts, i.e., \( F_R(t) = F_L(t - c/2R) \), and integrating the equation of motion until this time instant leads to the following relation for the bead velocity:

\[
mv_1 = \frac{2F_1 R}{c}. \tag{4.36}
\]

After loading stops, a trailing wave starts from the first bead and merges with the leading wave. Since the beads in contact operate in the unloading regime as the trailing wave passes, the speed of the trailing wave \( c_u \) is given by the slope of the unloading curve in the force-displacement law, and it is higher than the leading wave velocity \( c \). The trailing wave causes the contact forces to go down to zero behind it, and the particle velocities go down to \( v_t \). Again, integrating the equation of motion until a time instant just after the passage of this trailing wave gives the bead velocity in this region as \( v_t = v_1 - 2F_1 R/c_u \). The transition distance \( x_t \) and time \( T_t \) when the trailing wave merges with the leading wave are given by

\[
x_t = T \left( \frac{1}{c} - \frac{1}{c_u} \right)^{-1}, \quad T_t = x_t/c. \tag{4.37}
\]

The merging of leading and trailing waves leads to a decrease in the peak contact force and velocity of the beads, and two wavefronts move in opposite directions. The wavefront moving to the right operates in the plastic loading regime, while the wavefront moving to the left operates in the elastic reload-
Table 4.1: Comparison of predicted values with numerical simulations for material with $\beta = 0.80$.

Solving the above system leads to the following expressions for the peak force and bead velocity:

$$F_2 = \frac{mv_{t}}{D} \left( \frac{1}{c} + \frac{1}{{c}_u} \right)^{-1}, \quad v_2 = v_t \left( 1 + \frac{c}{{c}_u} \right)^{-1}.$$  (4.39)

Finally, it is noted that although the peak contact force in the first regime is a function of the unloading parameter $\beta$, the steady state force $F_1$ decreases gradually to reach the input amplitude $A$. Table 4.1 presents a comparison of various variables computed using the above expressions with values extracted from numerical simulations for a loading amplitude $A = 8 \text{kN}$ and impulse $I = 20 \text{Ns}$. A good agreement is observed. This model can easily be extended to predict the peak force, wave and particle velocity along the chain for any impulse and long times of loading, beyond the first interaction of wavefronts.
4.3 Conclusions

In this chapter, a systematic study has been performed on wave propagation in monodisperse semi-infinite elastic and elasto-plastic chains subjected to short and long duration loadings, and expressions for the force and velocity of the wave have been determined using simple models and appropriate approximations. In the elastic case, the response is a function of the input impulse and time of loading. Two distinct regimes have been observed and identified, with the peak force of the leading solitary wave determined for both regimes. The analytical predictions have been found to be in good agreement with numerical simulations.

In elasto-plastic granular chains, the mechanism of wave train formation is very distinct from their elastic counterpart. A simple bilinear model has been shown to capture the key phenomena associated with a more complex non-linear model. The peak force decay rates in the exponential regime have been derived based on the bilinear model. Furthermore, the transition point has been predicted by determining forces due to secondary waves and compared with numerical simulations. The response is distinct for long loading times and a wave structure having a quasi-steady front has been observed. The structure of the wave has been characterized using the quasi-steady approximation, and a simple model has been constructed to predict these quantities. The model predicts the contact force, wave and particle velocity with good accuracy.
Chapter 5

Impact response of granular and continuum systems

A comparative study of the impact response of three-dimensional ordered granular sphere packings and continuum half-spaces made of elastic-perfectly-plastic materials is conducted in this chapter. In the first part, impact simulations on 3D granular packings of spheres and continuum half-spaces composed of elastic-perfectly-plastic material are performed. The general characteristics of wave propagation in the two systems are studied. In the second part, the results are non-dimensionalized using appropriate scaling laws for both systems. Scaling laws with respect to material properties are derived for energy dissipation and are validated numerically. Finally, the plastic zone volume and dissipation are compared for the granular packing and continuum system, and the specific impact properties of both systems are derived.

5.1 Problem description

5.1.1 Granular system

In 3D packings of identical spheres, face-centered cubic (FCC) and hexagonal close packed (HCP) systems have the highest packing density possible [74]. These two systems are ordered packings with a density of 0.7404, and have parallel planes of hexagonal packings stacked on top of each other. In this chapter, impact on lattices of identical granules arranged in FCC packing are studied with the direction of impact being normal to the stacked hexagonal planes. Figure 5.1(a) shows the schematic of the impact problem on a semi-infinite granular FCC packing. An initial velocity is prescribed on a granule at the free top surface, and the response of the system is studied.

The dynamic response of the granular packing is modeled by a network of point masses connected by nonlinear springs, with the spring stiffness
Figure 5.1: Schematic of granular and continuum impact problems. In the granular case (a), an initial velocity is prescribed on a sphere in a semi-infinite lattice, while in the continuum case (b), a rigid impactor impacts the elasto-plastic half-space. The top surface is free in both systems.

defined by a contact law. The granules are modeled as rate-independent elastic-perfectly-plastic spheres, interacting with the contact law developed in Sec. 2.1. The contact force $F$ and relative displacement $\alpha$ are normalized by the contact force $F_y$ and relative displacement $\alpha_y$ at the onset of yield. The expressions for $\alpha_y$ and $F_y$ are derived from the von Mises yield criterion [8] and are given by Eqns. (2.2) and (2.3), respectively.

Since the wave speeds in the granular media are much smaller than the material wave speeds, the granular system can be represented by point masses. Let $m_i$, $u_i$, and $x_i$ respectively denote the mass, displacement and position of granule $i$ and let $d_{ij}$ be the unit normal vector in the direction of line joining centers of granules $i$ and $j$, given by

$$d_{ij} = \frac{x_i - x_j}{|x_i - x_j|}.$$  \hspace{1cm} (5.1)

The equations of motion of a granule $i$ contacting $N_i$ granules are then

$$m_i \ddot{u}_i = \sum_{j=1}^{N_i} F_{i,j} (\alpha_{i,j}) d_{ij},$$  \hspace{1cm} (5.2)

with $F_{ij}$ denoting the contact force between granules $i$ and $j$, and $\alpha_{ij}$, the
relative normal displacement between the two particles:

\[
\alpha_{ij} = \begin{cases} 
(u_i - u_j) \cdot d_{ij}, & \text{if } (u_i - u_j) \cdot d_{ij} > 0, \\
0, & \text{otherwise}.
\end{cases}
\] (5.3)

For comparison between the discrete and continuum systems, the material properties are chosen to correspond to brass, with a Young’s modulus \( E = 115 \text{ GPa} \), Poisson’s ratio \( \nu = 0.30 \), yield strength \( \sigma_y = 550 \text{ MPa} \) and density \( \rho = 8500 \text{ kg/m}^3 \), while the bead diameter is chosen as \( D = 0.01 \text{ m} \). However, simulations are carried out for a range of material properties and scaling laws for all material properties are derived later. The results are also presented non-dimensionally after appropriate normalizations. This work is restricted to the study of semi-infinite granular packings, with the lattices being infinite in \( x \) and \( y \) directions and semi-infinite in the \( z \) direction. For numerical simulations, a lattice of dimensions \( 72 \times 72 \times 30 \) is considered, i.e., 30 layers in the \( z \) direction of \( 72 \times 72 \) spheres in hexagonal packing in the \( xy \)-plane, as the quantities of interest (dissipation, size of plastic zone) do not vary as the size of packings are increased further. The top surface \( z = 0 \) is free, while the lower surface \( z = z_0 \) has rigid wall boundary conditions, and the granules are free to move at the boundaries of the \( xy \)-planes. An initial velocity is imposed on a bead at the center of the top surface, and the response of the system is studied. The open source software LAMMPS [75] is used simulate the problem, using the velocity Verlet time stepping algorithm with a time step of \( 2 \times 10^{-9} \text{ s} \).

### 5.1.2 Continuum system

To compare the response of the granular media with continuum, impact simulations are also performed on elastic-perfectly-plastic half-spaces. Similar to the granular case, simulations are performed for a wide range of material properties to study the variation of response with respect to these properties, and the results are also presented below in a non-dimensional form. An axisymmetric domain is considered, and a rigid impactor impacts the half-space with a specified velocity. The von Mises yield criterion is adopted to describe the onset of plastic behavior. The diameter of the impactor is taken to be the same as that of the particles in the discrete system, i.e., \( D = 0.01 \text{ m} \),
while its mass and velocity are varied to get different energy inputs. The finite element software ABAQUS Explicit is used to perform the continuum simulations, and the domain is meshed with 62500 four-noded first-order axisymmetric (CAX4R) elements. The master-slave contact algorithm is used to simulate a frictionless and non-adhesive contact between the impactor and half-space. The top surface and lateral surfaces of the elasto-plastic block are traction free, while the bottom surface is fixed. The block has a diameter $d = 0.1 \text{ m}$ and a height $h = 0.1 \text{ m}$, with these dimensions again chosen to be sufficiently large so that the waves reflected from the boundaries do not influence the plastic zone size, and total amount of plastic dissipation, which are the key quantities of interest in this study. The block is initially at rest and the impactor, having a flat surface, impacts the block with a prescribed initial velocity. The impactor is tapered at the ends to prevent numerical instabilities, and the finite element solutions are found to be independent of taper radius for sufficiently small taper radii, in the range $0.05 - 0.2 \text{ mm}$. For the impact velocities studied in this work, finite deformation effects are assumed to be negligible.

5.2 Typical impact response

Figure 5.2 shows a unit FCC cell, and the impact direction OD with a Miller index (111) is indicated by the dotted arrow. A 2D hexagonal packing, when subjected to a point load on a free surface has two principal directions [76] along which the contact forces are maximum in the initial layers. Similarly, in 3D, from symmetry considerations of the FCC packing, there are three principal directions along which waves primarily propagate from the impact point. These directions (OA, OB, OC) are shown by solid arrows in Fig. 5.2 and have indices (110), (101) and (010). They lie in three planes (shaded in Fig. 5.2) having hexagonal packings, and these planes (OAB, OCA, OBC) are referred to as principal planes of the packing in this work. Furthermore, due to the symmetry in the FCC packing and direction of loading, the solution to a normal point impact loading will be periodic about the impact axis with a period of $\pi/6$.

Figure 5.3 shows the distribution of normalized peak contact forces along the principal plane OAB. As mentioned previously, the solution is identical
Figure 5.2: Schematic of a FCC packing. The dotted arrow shows the impact direction, while the three solid arrows denote the principal directions along which the contact forces are maximum.

on all three principal planes due to symmetry. The contact forces are maximum along the principal directions and decay rapidly away from the point of impact (set at the origin). Figure 5.3(b) shows the peak contact force distribution on the plane OCD, which is orthogonal to the principal plane OAB. The peak forces are higher in the principal direction, and the force field is three-dimensional. The waves have similar characteristics to those observed in [52], with a single leading wave having the peak amplitude in the first few layers, and wave trains merging beyond the initial few layers. In the initial few layers, the wave is highly directional, with the peak forces in planes normal to the impact direction OD (i.e., planes parallel to ABC) lying where the principal directions (OA, OB, OC) intersect these planes. Beyond the first few layers, the waves become fully elastic as the force levels decrease below force required to cause yielding. The wavefront becomes planar, i.e., the peak forces are uniform across a constant $z-$plane due to waves merging. Each sphere is in contact with twelve spheres, and the decay in forces is due to both plastic dissipation and dimensionality, which causes the distribution of wave energy at the contacts.

Next, we turn our attention to the continuum system, where the wave propagation is axisymmetric, in contrast to the directional nature of the wave
Figure 5.3: (a) Peak contact forces in the principal plane OAB, and (b) in the plane OCD, which is orthogonal to the principal plane OAB. The contact forces decay rapidly away from the point of impact due to dimensionality and plastic dissipation.
profile in granular media. Figure 5.4 shows the von Mises stress contours on the block at time $t = 0.16 \times 10^{-4}$ s. As expected, the plastic zone (red region in the figure) is localized near the impact zone. This is in contrast to the dissipation contours in granular media (discussed later in Fig.5.9), where the plastic zone is spread out over a large volume. Beyond the zone of plastic yield, longitudinal and shear elastic waves propagate into the medium, causing no further energy dissipation.

5.3 Scaling laws: Granular impact

5.3.1 Scaling laws and normalized relations

In this section, the contact force-displacement model (2.14–2.17) and governing equations of motion are normalized and scaling relations with respect to various material properties and problem parameters are obtained. For a contact force $F$ associated with a relative displacement $\alpha$, the normalized...
displacement $\tilde{\alpha}$ and force $\tilde{F}$ are defined by

$$\tilde{\alpha} = \frac{\alpha}{\alpha_y} = \left(\frac{2}{1.6\pi} \frac{E^*}{\sigma_y}\right)^2 \frac{\alpha}{R^*}, \quad (5.4)$$

$$\tilde{F} = \frac{F}{\sigma_y \pi R^* \alpha_y} = \left(\frac{4}{1.6^2 \pi^3}\right) \left(\frac{E^*^2}{R^*^2 \sigma_y^3}\right) F. \quad (5.5)$$

Equation (2.14) then takes the following non-dimensional form:

$$\tilde{F} = (c_1 + c_2 \exp [\tilde{\alpha} - 1]) (c_4 \tilde{\alpha} + c_5), \quad (5.6)$$

where $c_i$ are numerical constants independent of material properties.

The unloading part of force-displacement law can also be nondimensionalized. Let $F_{\text{max}}$ and $\alpha_R$ be the peak force and residual displacement for a displacement $\alpha_{\text{max}}$ at the onset of unloading, with $F_{\text{max}}$ given by the plastic loading law, and $\alpha_R$ (Eqn. 2.17) normalized as

$$\tilde{\alpha}_R = \frac{\alpha_R}{\alpha_y} = c_6 \tilde{\alpha}_{\text{max}} - c_7 + (c_7 - c_6) \exp (\tilde{\alpha}_{\text{max}} - 1).$$

After normalization, the force during unloading (2.16) can be written as

$$\tilde{F} = \tilde{F}_{\text{max}} \left(\frac{\tilde{\alpha} - \tilde{\alpha}_R}{\tilde{\alpha}_{\text{max}} - \tilde{\alpha}_R}\right)^n. \quad (5.7)$$

From Eqn. (5.6), $\tilde{F}_{\text{max}}$ and hence Eqn. (5.7) does not depend on material parameters.

Finally, the elastic loading part, following the Hertzian contact law, is expressed in a nondimensional form as

$$F = \frac{4}{3} E^* \sqrt{R^*} \alpha^{3/2} = \frac{4}{3} E^* \sqrt{R^*} \alpha_y^{3/2} \left(\frac{\alpha}{\alpha_y}\right)^{3/2} = \frac{4}{3} \left(\frac{1.6\pi}{2}\right)^3 \frac{R^*^2 \sigma_y^3}{E^*^2} \left(\frac{\alpha}{\alpha_y}\right)^{3/2},$$

which leads to

$$\tilde{F} = 1.6 \left(\frac{2}{3}\right) \tilde{\alpha}^{3/2}. \quad (5.8)$$

In the governing equation (5.2) for a bead, the number of contacts $N$ depend on the geometry of the packing and the location of the bead. However, the nondimensional form of equations that are derived below are identical for all packings and locations of bead, i.e., the normalizations and scaling laws

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are independent of the packing geometry. Let us start by normalizing the displacement and time as

\[ \tilde{\mathbf{u}} = \frac{\mathbf{u}}{\alpha_y}, \quad (5.9) \]

\[ \tau = \sqrt{\frac{3\sigma_y}{4\rho R^*}} t. \quad (5.10) \]

The normalized velocity \( \tilde{\mathbf{v}} \) is then given by

\[ \tilde{\mathbf{v}} = \frac{d\tilde{\mathbf{u}}}{d\tau} = \frac{2}{\sqrt{3}} \left( \frac{2}{1.6\pi} \right) \frac{E^{*2}}{\sigma_y^{5/2}} \frac{\sqrt{\rho}}{d\mathbf{u}/dt}. \quad (5.11) \]

The impact on a granular system can then be described by the following non-dimensional initial value problem:

\[ \frac{d^2\tilde{\alpha}}{d\tau^2} = \sum_N \tilde{F}(\mathbf{d}_{ij}) \mathbf{d}_{ij}, \quad (5.12) \]

\[ \tilde{\mathbf{v}}(\tau = 0, \tilde{\mathbf{x}} = 0) = \tilde{\mathbf{v}}_0 = (0 0 \tilde{\mathbf{v}}_0)^T. \quad (5.13) \]

To derive scaling laws for dissipation, the dissipated and input energies are expressed in terms of the aforementioned non-dimensional variables. Energy dissipation at a contact is the difference in areas under the loading and unloading curves after complete unloading, i.e., after all the waves have passed:

\[ e_{\text{dis}} = \int_0^{\alpha_{\text{max}}} (F_L - F_U) \, d\alpha = \pi \left( \frac{1.6\pi}{2} \right) \frac{R^{*3}\sigma_y^{5}}{E^{*4}} \int_0^{\tilde{\alpha}_{\text{max}}} \left( \tilde{F}_L - \tilde{F}_U \right) \, d\tilde{\alpha}. \quad (5.14) \]

The input energy is the kinetic energy associated with the prescribed initial velocity on the center bead:

\[ e_{\text{in}} = \frac{1}{2} m \mathbf{v}_0 \cdot \mathbf{v}_0 = \pi \left( \frac{1.6\pi}{4} \right) \left( \frac{R^{*3}\sigma_y^{5}}{E^{*4}} \right) \tilde{\mathbf{v}}_0 \cdot \tilde{\mathbf{v}}_0. \]

The normalized energy input and dissipation, independent of material prop-
As described earlier, the reference granular configuration has brass spheres of diameter 0.01 m. To analyze the impact of various parameters defining the problem, the diameter, density, stiffness and yield strength are varied, one at a time from the reference configuration. Figure 5.5(a) shows the variation of fraction of energy dissipated with energy input for the different granular material properties. Since smaller spheres have a smaller contact force $F_y$ at the onset of yield, they undergo more total dissipation for the same energy input. Similarly, materials with higher yield stress or lower Young’s modulus require more force for onset of plastic yielding, and hence, undergo less plastic dissipation for a given energy input. These trends are observed in Fig. 5.5(a) and are consistent with the scaling laws derived above. As predicted by the scaling laws, it is observed that varying the density of granular material has no effect on total dissipation. Figure 5.5(b) shows the same results on normalized axes, with the normalization of the input energy derived from Eqn. (5.15). As observed there, the curves for distinct materials and sizes collapse on a single curve, confirming the nondimensionalization derived earlier.

In [52], it was observed that the peak contact force $F_{\text{max}}(x)$ at any point $x$ along the chain scales linearly with input impulse $I$. This observation has been verified to hold in the FCC packings considered here. Furthermore, as also shown in [52], the dissipation at a contact can be approximated by a quadratic function of peak contact force \( \bar{\epsilon}_{\text{dis}} = d_0 + d_1 \bar{F}_{\text{max}} + d_2 \bar{F}_{\text{max}}^2 \). Since the input impulse is directly proportional to the impact velocity, the dissipation at a contact is a quadratic function of the impact velocity $v_0$. The total dissipation in the packing is obtained by summing the dissipation at
Figure 5.5: (a) Variation of fraction of energy dissipated with input energy for different material properties. (b) Master curve obtained with the normalization of the input energy given by Eqn. (5.15).
Figure 5.6: (a) Variation of dissipation with impact velocity: comparison between numerical values and the quadratic approximation given by Eqn. 5.17. (b) Simulated and predicted values of fraction of energy dissipated with energy input.
contacts, and can be expressed as

\[ \tilde{e}_{\text{dis}} = \sum_i \tilde{e}_{\text{dis}}^i = \sum_i \left[ d_0 + d_1 \tilde{F}_{\text{max}}(x_i) + d_2 \left( \tilde{F}_{\text{max}}(x_i) \right)^2 \right] \]  

\[ = D_0 + D_1 \tilde{v}_0 + D_2 \tilde{v}_0^2. \]  

(5.16) (5.17)

Since the dissipation at every contact is a quadratic function of impact velocity \(v_0\), the total dissipation \(\tilde{e}_{\text{dis}}\) is also a quadratic function of \(v_0\), and the quantities \((D_0, D_1, D_2)\) are solely dependent on the geometry of the packing. These quantities are evaluated from numerical simulations without knowledge of the spatial variation \(F_{\text{max}}(x)\), and they are found to be: \(D_0 = 9.45 \times 10^3\), \(D_1 = -3.04 \times 10^3\) and \(D_2 = 5.00 \times 10^3\) by curve fitting. Figure 5.6(a) shows the increase in dissipation with increasing impact velocity in normalized axes, along with its approximation (Eqn. (5.17)), demonstrating that the quadratic model describes the behavior reasonably well. Figure 5.6(b) shows the increase in fraction of energy dissipated as input energy increases, with the numerical data shown in symbols, and the behavior predicted by the quadratic model (5.17) in solid curve. The quadratic law predicts the energy dissipation well for higher impact velocities, while the error at lower velocities is attributed to the fact that the quadratic approximation (5.16) is less accurate for small forces, and hence small impact velocities.

Finally, the effect of loading area on the impact response of the packing is examined by prescribing an initial velocity on multiple spheres at the center of the packing. A fixed initial velocity \(v_0\) is prescribed on a number of spheres \(W\), and the energy dissipation is computed. Figure 5.7 shows the variation of fraction of energy dissipated with input energy for different impact areas. It is observed that the energy dissipation is only a function of total input energy, and does not depend on the number of spheres impacted or the impact area for small areas of impact. Thus, for point impacts or impact loads distributed over a small number of spheres, Eqns. (5.15) and (5.17) can be used to predict the total dissipation in a granular FCC packing of any size of beads and any elastic-perfectly-plastic material subjected to an impact load, over a small area. It should be noted that this analysis holds only in the force regimes where the contact force-displacement law is valid, and cannot be extended to very high impact velocities for which the contact law breaks down.
Figure 5.7: Energy dissipation variation with loading area, with $W$ denoting the number of spheres to which an initial velocity is imparted. The response for different loading areas overlap, indicating that, for small loading areas, the amount of dissipation does not depend on the area of loading.

5.3.2 Dissipation and plastic zone size

Plastic dissipation initiates and proceeds from contacts of granules due to stress concentration. The dissipation caused by a sphere is defined to be the sum of dissipation at its contacts. The dissipation at each sphere is computed, and Fig. 5.8 shows the dissipation arranged in descending order. It is observed that though a large number of spheres undergo some plastic dissipation, the bulk of dissipation occurs in the first few beads. Similar to the observations in [52] where higher input impulse leads to more dissipation, the amount of dissipation at a sphere increases with impact velocity. As mentioned earlier and shown in Eqn. (5.17), dissipation is a quadratic function of impact velocity, and hence, although the profiles look similar in Fig. 5.8, we do not collapse them to a single curve. Since dissipation at a contact takes place only over a threshold peak force $F_y$, and as seen earlier, $F_{\text{max}}(x_i)$ is linearly dependent on impulse $I$, higher impact velocity causes more spheres to deform plastically, and hence the size of plastic zone increases with impact velocity. However, predicting the number of spheres undergoing plastic deformation for a given impulse requires knowledge of normalized peak force distribution along the packing, and this is not attempted here.
Figure 5.8: Plastic dissipation of spheres arranged in descending order for three values of the normalized impact velocity, showing that the extent of dissipation decreases rapidly as the sphere count increases.

Figure 5.9 shows the contours of dissipation at sphere in the principal and perpendicular planes. Since the dissipation at a contact is a function of peak contact force, these contours also show directionality similar to that observed in Fig. 5.3. As also observed in Fig. 5.8, most of the dissipation takes place in a few spheres close to the impact point.

5.4 Scaling laws: Continuum impact

5.4.1 Scaling laws and plastic zone volume

In this section, the scaling laws for energy dissipation with material properties for elastic-perfectly-plastic isotropic half-spaces subjected to impact are derived. The governing equation in the absence of body forces is

$$\rho \ddot{\mathbf{u}} = \nabla \cdot \mathbf{\sigma}, \quad (5.18)$$

where $\rho$, $\mathbf{u}$ and $\mathbf{\sigma}$ are the density, displacement and stress tensor respectively. Let $\sigma_y$ be the yield stress and $f(\mathbf{\sigma}, \sigma_y) = 0$ be the von Mises yield criterion.
Figure 5.9: Dissipation contours in (a) the principal plane (OAB in Fig. 5.2) and (b) a plane orthogonal to the principal plane (ODC in Fig. 5.2). The bulk of the dissipation takes place in a few spheres close to the impact point.
Let \( C \) be the elastic constitutive tensor and \((\lambda, \mu)\) be the Lamé constants. Assuming small deformations and hence a linear isotropic elastic response, for an elastic-perfectly-plastic material following the von Mises yield criterion with associative Levy-Saint Venant flow rule, the constitutive law [71] is given by \( \dot{\sigma} = C^{ep} \dot{\varepsilon} \) with

\[
C^{ep} = \begin{cases} 
  C & \text{if } \gamma = 0, \\
  \left( \lambda + \frac{2}{3} \mu \right) I \otimes I + 2 \mu (I - \frac{1}{3} I \otimes I - n \otimes n) & \text{if } \gamma > 0.
\end{cases}
\] (5.19)

where \( I, \Pi \) are the second and fourth order identity tensors, \( n = \text{dev}(\sigma) / \|\text{dev}(\sigma)\| \) and \( \gamma = n : \dot{\varepsilon} \) is the consistency parameter. The key observation in the above system is that \( C^{ep} \) is a linear function of Young’s modulus \( E \).

The half-space is impacted by a rigid impactor of diameter \( D \) with velocity \( v_0 \). The following normalizations are introduced for the independent fields \((x, t)\) and for the displacement \( u \) and stress \( \sigma \) fields:

\[
\bar{x} = \frac{x}{D},
\] (5.20)

\[
\tau = \sqrt{\frac{E \, t}{\rho \, D}},
\] (5.21)

\[
\bar{u} = \frac{u}{D} \left( \frac{E}{\sigma_y} \right),
\] (5.22)

\[
\bar{\sigma} = \frac{\sigma}{\sigma_y}.
\] (5.23)

Since the yield criterion used here is a homogeneous function of stress components, in non-dimensional variables it assumes the form \( f(\bar{\sigma}, 1) = 0 \). Substituting these nondimensional quantities in the governing equations leads to
the following normalized equations:

\[
\frac{d^2 \tilde{u}}{dt^2} = \nabla \cdot \tilde{\sigma}, \tag{5.24}
\]

\[
\frac{d\tilde{u}}{dt} = \sqrt{\rho E} \frac{du}{dt}, \tag{5.25}
\]

\[
\tilde{\varepsilon} = \frac{\tilde{\nabla} \tilde{u} + \tilde{\nabla} \tilde{u}^T}{2}, \tag{5.26}
\]

\[
\dot{\tilde{\sigma}} = \frac{1}{E} \mathbb{C}^{ep} \dot{\tilde{\varepsilon}}, \tag{5.27}
\]

\[
\tilde{\gamma} = \frac{\text{dev}(\tilde{\sigma})}{\|\text{dev}(\tilde{\sigma})\|} : \tilde{\varepsilon} \geq 0, \quad \tilde{\gamma} f(\tilde{\sigma}, 1) = 0, \quad \tilde{\gamma} f(\tilde{\sigma}, 1) = 0. \tag{5.28}
\]

Since \( \mathbb{C}^{ep} \) is a linear function of the Young’s modulus \( E \), the above system is non-dimensional and independent of material properties.

Let \( m_{\text{imp}} \) and \( v_0 \) be the mass and impact velocity of the rigid impactor. The input energy \( e_{in} \) and dissipation \( e_{\text{dis}} \) are given by

\[
e_{\text{dis}} = \int_{\Omega} \sigma_{ij} \epsilon_{ij} dV = \left( \frac{\sigma_y^2 D^3}{E} \right) \int_{\tilde{\Omega}} \tilde{\sigma}_{ij} \tilde{\epsilon}_{ij} d\tilde{V},
\]

\[
e_{in} = \frac{1}{2} m_{\text{imp}} v_0^2 = \frac{1}{2} \frac{m_{\text{imp}} \sigma_y^2}{\rho E} \tilde{v}_0^2 = \left( \frac{m_{\text{imp}} \sigma_y^2}{\rho E} \right) \frac{\tilde{v}_0^2}{2}. \tag{5.29}
\]

and they are normalized by expressing them as functions of the non-dimensional variables and material properties:

\[
\tilde{e}_{\text{dis}} = \left( \frac{E}{\sigma_y^2 D^3} \right) e_{\text{dis}} = \int_{\Omega} \tilde{\sigma}_{ij} \tilde{\epsilon}_{ij} d\tilde{V},
\]

\[
\tilde{e}_{in} = \left( \frac{E}{\sigma_y^2 D^3} \right) e_{in} = \left( \frac{m_{\text{imp}}}{\rho D^3} \right) \frac{\tilde{v}_0^2}{2}. \tag{5.30}
\]

Note that these expressions for the normalized energy components are different from those obtained for a granular packing (5.15).

Figure 5.10(a) shows the variation of fraction of energy dissipation with energy input for various material properties. It is observed that, for a fixed energy input, dissipation decreases as the yield strength increases due to the fact that, for a higher yield stress material, higher stresses and hence higher energy is required to initiate yielding at a material point. Also, dissipation increases with increasing Young’s modulus \( E \), since a stiffer material reaches yielding at a lower strain. These trends are also consistent with the scal-
Figure 5.10: Fraction of energy dissipated versus input energy for continuum systems. (a) A larger fraction of energy is dissipated for materials with high Young’s modulus and low yield strength. (b) Ratio of energy dissipation with normalized input energy. When normalized appropriately, the distinct curves collapse to a single one.
Figure 5.11: Ratio of fraction of energy dissipated with energy input for different impactor masses. The fraction of energy dissipated increases with energy input and has little sensitivity to impactor mass.

The volume of material where the von Mises stress equals the yield stress is computed from the numerical solution of the impact event, and the maximum value over time is taken to be the plastic zone volume. This volume is normalized by the volume of a sphere having diameter equal to the impactor diameter. Figure 5.12 shows the variation of normalized plastic zone volume with input energy for different impactor masses. The curves are close for distinct impactor masses, showing that the plastic zone volume is not very sensitive to the impactor mass. As apparent there, the plastic zone volume in continuum simulations is much smaller than that associated with granular
Figure 5.12: Variation of normalized plastic zone volume with input energy for distinct impactor masses. The plastic zone volume increases with input energy and has little sensitivity to impactor mass.

5.5 Comparison between granular and continuum results

In this section, a comparison of semi-infinite granular and continuum systems is performed for specific impact properties. The granular system has voids, with the FCC packing considered here having a packing density of 0.74. Since the energy normalizations (5.15), (5.30) with respect to the material properties are distinct for the two systems, one has to specify the two
Figure 5.13: Dissipation in granular and continuum systems. The granular system dissipates a higher fraction fraction of the energy input and experiences dissipation at much lower energy levels due to stress concentration at interparticle contacts.

materials for the granular and continuum systems before comparison. Here, the key features of the impact process, namely dissipated energy and plastic zone volume are studied for the two systems, both having the material properties of brass, although the general trends hold for any two materials.

Figure 5.13 shows the fraction of energy dissipated for the continuum system and granular packing with bead diameters $D = 0.01$ m and 0.02 m. The axes are normalized by $\left(\sigma_y^2 D^3 / E\right)$ so that the distinct curves for continuum systems corresponding to different material properties and impactor sizes collapse to a single curve (cf. Eqn. (5.30)). As expected, dissipation starts at much lower impact energies in granular systems due to stress concentration at contacts. Furthermore, the fraction of energy dissipation is much larger for the granular case as compared to the continuum case. Hence, for a given energy input, the granular system is more efficient in dissipating energy and has more impact resistance and lower transmitted energy.

Figure 5.14 presents the dependence of the plastic volume sizes on the impact energy for the granular and continuum systems, normalized by the volume of a sphere with diameter $D = 0.01$ m. The granular case is shown with curves, with the cumulative dissipation plotted after the spheres are
Figure 5.14: Plastic zone volumes in granular and continuum systems. Though the total plastic zone volume is very large in granular systems, the volume of material involved in causing a specific fraction of energy to dissipate is comparable for both the granular and continuum systems.

arranged in descending order of the amount of dissipation. Though the total volume of plastic dissipation is very large as discussed in the previous section, most of the dissipation takes place in a few spheres. The symbols correspond to the total plastic zone volume for the continuum system. The volume of material involved in dissipating a specific fraction of input energy is seen to be very close for continuum and granular systems, although, as evident from the legends in Fig. 5.14, the input energy required for a certain volume of material to deform plastically is much larger for continuum compared to the granular system. A fixed amount of input energy thus causes a larger volume of material to yield and a larger fraction of energy dissipation in the granular case. However, it should be noted that, in the granular case, the plastic deformation is localized at the contacts, and the entire sphere does not yield plastically. This is in contrast to the plastic volume data presented in the continuum case, which is representative of the actual volume of material undergoing plastic flow.
5.6 Conclusions

In the first part of this chapter, a systematic study of wave propagation and dissipation in large granular FCC packings has been performed. The governing equations have been non-dimensionalized and scaling laws for energy dissipation have been determined with respect to bead size, stiffness and yield strength. Dissipation was found to be a quadratic function of impact velocity and independent of loading area for a fixed input energy. The contours of plastic dissipation and size of plastic zone were also characterized, and it was observed that a small fraction of spheres contribute to most of the dissipation.

In the second part, the impact response of elasto-plastic half spaces has been studied using finite element analysis. Again, the governing equations are normalized and scaling laws for energy dissipation with respect to stiffness and yield strength of material, and size of impactor are derived and validated with numerical simulations. Finally, the total dissipation is compared for the two systems, and it is observed that dissipation starts much earlier in the granular system due to stress concentrations. Furthermore, for a fixed input energy, the dissipation in granular system is much larger than continuum systems for the range of loadings considered here. However, the volume of material involved in dissipating a fraction of energy is comparable for both systems. Since the granular and continuum systems have different normalizations, a direct comparison of normalized properties is not possible. This chapter quantifies the amount of dissipation in a granular packing and demonstrates the contribution of localized stress concentration in dissipating and mitigating impact energy.
Chapter 6

Wave tailoring by precompression in confined granular systems

In this chapter, we present a granular system whose response under an impact load can be varied from rapidly decaying to almost constant amplitude waves by an external regulator. The system consists of a granular chain of larger spheres surrounded by small spheres, confined in a hollow cylindrical tube and supporting wave propagation along the axis of the cylinder. We demonstrate using numerical simulations that the response can be controlled by applying radial precompression. These observations are then complemented by an asymptotic analysis, which shows that the decay in the leading wave is due to energy leakage to the oscillating small beads in the tail of the wave. This system has potential applications in systems requiring tuning of elastic waves.

6.1 Problem setup

Figure 6.1 shows the schematic of the configuration considered in this work. It consists of a granular chain of larger spheres in contact, with smaller spheres surrounding it. The system is densely packed under zero prestress in a long hollow cylinder. Figure 6.1(a) illustrates this dense packing with the large sphere shown by a dashed circle since it is not on the same plane as the small spheres. From geometrical considerations, the number of small spheres can vary from 6 to 9, as the surrounding spheres have to be smaller than the big spheres and large enough to be in contact with the wall. Let \( d \) and \( D \) be the diameters of the small and big beads, respectively. For closely packed structures, the following constraint condition holds, as the \( n \) smaller spheres have to lie at the vertices of a \( n \)--sided polygon:

\[
\frac{d}{\sqrt{(d + D)^2 - D^2}} = \sin \left( \frac{\pi}{n} \right),
\]

(6.1)
Figure 6.1: Top (a) and side (b) views of the granular chain confined between rigid walls. By applying a radial precompression $\delta$, the wave propagation characteristics can be controlled.

leading to the following relation between the diameters:

$$d = 2D \tan^2 \left( \frac{\pi}{n} \right).$$  \hspace{1cm} (6.2)

When an uniform radial external pressure is applied, there is contact force between the small and big spheres and between the small spheres and the wall. The hollow cylinder is assumed to be of a much stiffer material than the spheres and the walls are hence modeled as rigid surfaces during the wave propagation through the chain. The first and the last big spheres in the chain are fixed. There is no contact force between the big spheres even when there is precompression, as the contact forces due to the small spheres surrounding each big sphere balance out due to symmetry. One of the big spheres is given an impact in the direction of the axis of the cylinder and the dynamic response of the system is simulated.

All the spheres are assumed to be linearly elastic and their contact described by Hertz law with no dissipation. Let $m_b$ and $m_s$ denote the mass of the big and small beads, respectively, and let $k_{bb}$, $k_{ab}$ and $k_{bw}$ denote the elastic constants arising from the Hertzian law applied between the big sphere-big sphere, big sphere-small sphere and small sphere-wall interactions [79],
respectively. These constants are functions of geometric and material properties given by

\[ k_{sb} = \frac{4}{3} \left( 1 - \nu_s^2 \right) \left( \frac{1 - \nu_s^2}{E_s} + \frac{1 - \nu_b^2}{E_b} \right)^{-1} \sqrt{\frac{Dd}{2(D + d)}}, \quad (6.3) \]

\[ k_w = \frac{1.2E_s\sqrt{d/2}}{1 - \nu_s^2}, \quad k_{bb} = \frac{E_b\sqrt{D}}{3(1 - \nu_b^2)}, \quad (6.4) \]

where \( E \) and \( \nu \) denote the Young’s modulus and Poisson’s ratio, respectively, with the subscripts ‘s’ and ‘b’ indicating the values of these properties associated with the small and big sphere materials, respectively. Let \( w^b_i, w^s_i + 1/2 \) and \( u_{i+1/2}^s \) denote the axial displacements of the big and small beads, and the radial displacement of the small beads, respectively. Let \( \theta \) be the angle between the line joining the centers of the small and large beads and the axis of wave propagation (Fig. 6.1(b)), given by \( \cos \theta = D/(d + D) \). Under a uniform radial precompression, let \( \delta_1 \) be the relative displacement between the small spheres and the wall, \( \delta_2 \) be the radial displacement of the center of the small beads and let \( \delta = \delta_1 + \delta_2 \) be the change in the cylinder radius. From the radial equilibrium of the small beads, we have the following relation between \( \delta_1 \) and \( \delta_2 \):

\[ k_w\delta_1^{3/2} = 2k_{bb} \left[ 2\delta_2 \cos \left( \frac{\pi}{n} \right) \right]^{3/2} \sin \left( \frac{\pi}{n} \right) + 2k_{sb} \left( \delta_2 \sin \theta \right)^{3/2} \sin \theta, \]

and its solution provides the initial static configuration. Introducing

\[ [x]_+ = \begin{cases} x \quad x > 0, \\ 0 \quad x \leq 0, \end{cases} \quad (6.5) \]

and the functions

\[ \beta_{i-1/4} = \left[ w_{i-1/2}^s - w_i^b + \tan \theta(\delta_2 - u_{i-1/2}^s) \right]_+^{3/2}, \quad (6.6a) \]

\[ \beta_{i+1/4} = \left[ w_i^b - w_{i+1/2}^s + \tan \theta(\delta_2 - u_{i+1/2}^s) \right]_+^{3/2}, \quad (6.6b) \]

\[ \alpha_{i-1/2} = \left[ w_{i-1}^b - w_{i}^{b+1/2} \right]_+^{3/2}, \quad (6.6c) \]

the governing equations for the dynamic response of the configuration shown
in Fig. 6.1 are given by

\[ m_b u_i^{b,n} = k_{bb} (\alpha_i - \alpha_{i+1/2}) + n \cos \theta^{5/2} k_{sb} (\beta_i - \beta_{i+1/2}), \]  
\[ m_s w_{i+1/2} = \cos \theta^{5/2} k_{sb} (\beta_{i+1/4} - \beta_{i+3/4}), \]  
\[ m_s u_{i+1/2} = \sin \theta \cos \theta^{3/2} k_{sb} (\beta_{i+1/4} + \beta_{i+3/4}) - k_w [\delta_i + u_{i+1/2}]^{3/2}. \]  

The above system (6.7) is normalized and expressed in terms of dimensionless variables. The displacement is normalized by the diameter \( D \) of the large spheres and the nondimensional time \( \tau \) is defined in terms of the physical time \( t \) as \( \tau = t \left( k_{bb}/m_b \right)^{1/2} \). The force \( F \) and impact velocity \( v \) are normalized as \( \tilde{F} = F/E R^2 \) and \( \tilde{v} = (v/D) \sqrt{m_b/k_{bb}} \), respectively. In a monodisperse granular chain, the force scales with the impact velocity [7]. In the present system, the force is a function of two control variables: impact velocity and applied precompression, and the appropriate scaling laws are presented later, in the next section. To eliminate the remaining dimensional material parameters in (6.7), we define the following non-dimensional parameters:

\[ \epsilon = \frac{m_s}{m_b}, \quad \eta = \frac{k_{sb} \cos \theta^{5/2}}{k_{bb}}, \quad \gamma = \frac{k_w}{k_{bb}}. \]  

The precompression and distance are also normalized by the diameter of the large spheres as \( a = 2\delta/D \) and \( \tilde{x} = x/D \). Expressing the governing equations (6.7) in terms of these parameters leads to the following normalized system of equations:

\[ \ddot{w}_i^b = (\alpha_i - \alpha_{i+1/2}) + n \eta (\beta_i - \beta_{i+1/2}), \]  
\[ \epsilon \ddot{w}_{i+1/2} = \eta (\beta_{i+1/4} - \beta_{i+3/4}), \]  
\[ \epsilon \ddot{u}_{i+1/2} = \tan \theta \eta (\beta_{i+1/4} + \beta_{i+3/4}) - \gamma [\delta_i + u_{i+1/2}]^{3/2}, \]  

where, once again, \( n \) denotes the number of small interstitial spheres (6 \( \leq \) \( n \) \( \leq \) 9).

Typical values of \( \epsilon, \eta, \gamma \) corresponding to a large sphere diameter \( D = 0.015 \text{ m} \) and both spheres having Young’s modulus \( E = 115 \text{ GPa} \), Poisson’s ratio \( \nu = 0.30 \) and density \( \rho = 8500 \text{ kg/m}^3 \) for the \( n = 8 \) system are 0.04, 0.34 and 1.53, respectively. Note that the results are presented hereafter in non-dimensional form and are independent of the choice of specific material. An initial velocity \( v = 1 \text{ m/s} \) (\( \tilde{v} = 1.35 \times 10^{-4} \)) is prescribed on the tenth
big sphere in a chain composed of 300 unit cells (each unit cell having a big sphere and \( n \) small spheres) and the response of the system is observed.

6.2 Numerical results

The routine \texttt{ode45} in the MATLAB software is used to solve the system (6.9) numerically. Since the objective is to study the response in infinite chains, the final time is chosen to be before the leading wave reaches the chain end to avoid end reflections. Figure 6.2(a) shows the peak contact forces along the chain for the case of \( n = 8 \) system with different precompression levels. For zero or low precompression, the peak force of the leading wave decreases rapidly down the chain. As the static precompression level increases, the peak force decay rate along the chain decreases. The energy propagates through a unit cell along two pathways: big to big spheres and big to small to big spheres. With increasing precompression, the contact stiffness between the small and big spheres increases. This leads to more energy being transferred through the big-small-big sphere pathway for a given axial displacement of big sphere. Thus, for a fixed amount of energy, an increasing fraction goes through the second pathway with increasing precompression and hence, the peak force magnitude between the large spheres decreases. The wave velocity increases with precompression when there is negligible decay in the leading wave. The decay is associated with the energy leakage from the leading wave into the smaller spheres in the oscillating tail, which will be demonstrated later using an asymptotic analysis.

Figure 6.2(b) shows the peak contact forces for the \( n = 9 \) system. The leading wave has a very small decay rate even for low levels of precompression. As the precompression increases, the peak contact force decreases and the wave speed increases. Figure 6.3(a) shows the peak contact force down the chain for the \( n = 6 \) system, which is very similar to the \( n = 9 \) case, with the peak force decreasing and wave velocity increasing with increasing precompression. Figure 6.3(b) shows the peak contact force down the chain for the \( n = 7 \) system, which is similar to the \( n = 8 \) case. Figure 6.4 illustrates the variation of average wave speed with precompression for the four values of \( n \). It is computed using the time taken by the leading wave to traverse 150 beads. As mentioned earlier, the wave speed increases with increasing

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Figure 6.2: (Color online) Peak contact forces between the contacting big spheres. (a) Response of $n = 8$ system, showing that the peak forces decays rapidly with zero precompression but remains almost constant with increasing precompression. (b) Response of $n = 9$ system, showing that the peak force remains almost constant for all values of precompression $a$. 
Figure 6.3: (Color online) Peak contact forces between the contacting big spheres. (a) Response for $n = 6$ showing a near solitary wave like behavior, while the $n = 7$ system (b) has a decaying leading wave for low levels of precompression.
radial precompression $a$ and with the number of interstitial spheres. The wave speed scales with the effective stiffness $k_e$ and effective mass $m_e$ of a unit cell as $\sqrt{k_e/m_e}$ and the increase in wave speed with $n$ is attributed to the increase in this quantity. Furthermore, since the static contact force and thus the effective stiffness increases with precompression, the wave speed also increases although the peak force decreases. This is quite contrary to the case of axial precompression, where both the wave speed and force amplitude increase with precompression, and to the case of solitary waves in unstressed chains, where the wave speed increases with force amplitude.

Figure 6.5(a) displays the velocity variation of two large beads for two levels of precompression: $a = 0$ and $a = 1.9 \times 10^{-3}$. The large beads are at locations 100 and 150 and the curves for $a = 1.9 \times 10^{-3}$ are shifted to the left by 1200 units along the time axis to avoid overlap. The profile corresponding to $a = 1.9 \times 10^{-3}$ has very small oscillations behind the leading wave and the peak velocity remains almost constant as the wave propagates down the chain. On the other hand, the profile of $a = 0$ shows high amplitude oscillations behind the leading wave. This corresponds to energy leakage behind the leading wave and causes the peak velocity to drop as the wave progresses down the chain. Figure 6.5(b) displays the velocities of big and small beads for the same two levels of precompression for the unit cell $i = 100$. The numerical solution of velocity profiles clearly illustrate the energy leakage to the two
Figure 6.5: (Color online) Bead velocities for the $n = 8$ system. (a) At two lattice sites showing decay of leading wave and large oscillations behind it for $a = 0$ and almost no decay and small oscillations behind it for $a = 1.9 \times 10^{-3}$. The curves corresponding to $a = 1.9 \times 10^{-3}$ are shifted by 1200 time units for clarity. (b) The velocity profiles of the large and small beads for two precompression levels: $a = 0$ (solid curves) and $a = 1.9 \times 10^{-3}$ (dashed curves).
Figure 6.6: (Color online) Variation of the problem parameters with \( n \), along with the normalized force decay rate for fractional \( n \) in a fictitious system, showing that systems with \( n = 7, 8 \) have rapid decay compared to systems with \( n = 6, 9 \).

velocity components (axial and radial). For the precompressed system, the amplitude of oscillations are much smaller, and hence the decay rate of the leading wave is very small, leading to a wave profile with almost constant amplitude. We term these almost constant amplitude waves near-solitary waves, since their profile is very close to a solitary wave of monodisperse chains. As demonstrated later using an asymptotic analysis, our system does not support 'true' solitary waves with zero decay.

A systematic study of the effect of various parameters is conducted to understand the variation with number of beads. Although \( n \) can take only integer values between 6 and 9 for a real system, we solve the system (6.9) numerically for non-integer rational values over the range \([5.5, 10]\) to understand the mechanism leading to the distinct behaviors for various integer \( n \) values. Figure 6.6 shows the variation of various parameters \( n\epsilon, \eta \) and \( \gamma \) and the normalized peak force decay rate \( (1/\tilde{F})d(\tilde{F})/d\tilde{x} \) (amplified by a factor 200 to fit the scale) with \( n \). Assuming the wave to have a self-similar structure at any time instant, the decay rate \( (1/\tilde{F})d\tilde{F}/d\tilde{x} \) is constant. This quantity is seen in Fig. 6.6 to be non-monotonous, with peaks near \( n = 7, 8 \) and smaller values near \( n = 6, 9 \). Note also that the \( n\epsilon \) parameter decreases with increasing \( n \). This observation is of interest as the asymptotic analysis presented later is valid only for small \( n\epsilon \).
Since only $\epsilon$ varies significantly with the number of beads $n$ while the other parameters remain relatively constant, we focus on the effect of $\epsilon$ only (while keeping the other parameters constant) to get more insight into the complex behavior of decay rate in Fig. 6.6. To this end, we consider a fixed system (with $n = 6$) and vary only the density of the small beads to study the effect of $\epsilon$, noting that the density ratio of small to big beads is directly proportional to $\epsilon$. Figure 6.7 presents the variation of decay rate with the ratio of small to large sphere density, showing multiple peaks and troughs in the range considered. The peaks correspond to near-solitary leading waves with very little decay, while the troughs are density values for which the decay is maximum. This is similar to the trends observed in [36] where the authors studied dimer system and in [38] where wave propagation in 2-D square packing with intruders subjected to plane loading was analyzed. The peaks correspond to anti-resonances where solitary waves exist, while the troughs are nonlinear resonances where there is maximum transfer of energy from the leading longitudinal wave to the oscillations of the small beads. Thus the small decay rates in the $n = 6$ and $n = 9$ systems are due to the corresponding $\epsilon$ values being closer to anti-resonances, while the large decay values in the other two systems is due to their vicinity to a resonance condition.
We now present the scaling laws, which lead to self-similar solutions. Let the system be impacted by velocities $v_a$ and $v_b$, leading to displacement fields $w_a(x,t)$ and $w_b(x,t)$. We seek a relation between the forces $F_a(x,t)$ and $F_b(x,t)$. Substituting the condition $w_b = Cw_a$ into the governing equation $\ddot{w}_b = f(w_b)$ leads to

$$C\ddot{w}_a = C^{3/2}f(w_a) \implies C^{-1/2}\frac{d^2w_a}{d\tau_b^2} = f(w_a).$$

(6.10)

The time scales and bead velocities for the two cases are then related by $\tau_b/\tau_a = C^{-1/4}$ and $v_b/v_a = C^{5/4}$, respectively. The wave speeds $c_a$, $c_b$ are inversely related to the time scales which finally leads to

$$\left(\frac{F_b}{F_a}\right)^{2/3} = \left(\frac{v_b}{v_a}\right)^{4/5} = \frac{\delta_b}{\delta_a} = \frac{w_b}{w_a} = \left(\frac{c_b}{c_a}\right)^4.$$

(6.11)

Equation (6.11) implies that as the impact velocity increases, the precompression $\delta$ also has to increase to obtain self-similar solutions. This key scaling observation is illustrated in Fig. 6.8, which presents the spatial variation of the peak contact force normalized by the $6/5$ power of the impact velocity. If the precompression level is kept constant ($a = 4.8 \times 10^{-4}$), the curves corresponding to different impact velocities do not overlap. However, when the
Figure 6.9: (Color online) Numerical and asymptotic zeroth-order solution for the big spheres, showing a good agreement. As precompression increases, the speed of the wave increases and its amplitude decreases.

6/5 power velocity scaling of the contact force is combined with a 4/5 power velocity scaling of the imposed precompression (yielding \( a = 8.4 \times 10^{-4} \)), the curves do overlap, thus confirming the relations given in Eqn. (6.11). This scaling law is similar to the case of solitary waves in monodisperse chains, where the wave speed scales as 1/6 power of peak force. However, the difference here is that an appropriate scaling of precompression \( \delta \) is also required to obtain an equivalent system of governing equations. A single self-similar solution is obtained for the chain of monodisperse spheres [6] whereas a one-parameter family of self-similar solutions describes the force amplitude-wave speed relation for our system. The spatial width of the leading wave in our system is about five large sphere diameters, similar to the case of solitary waves in monodisperse chains.

6.3 Asymptotic analysis

In order to gain further insight into the behavior of the confined granular system subjected to precompression, an asymptotic analysis is conducted based on the following multiple time scale asymptotic expansion (for small
\[ w^b = w^b_0(\tau) + \epsilon^p w^b_1(\tau_1) + \ldots \\
\]
\[ w^s = w^s_0 + \epsilon^q w^s_1(\tau_1) + \ldots \]
\[ u^s = u^s_0 + \epsilon^q u^s_1(\tau_1) + \ldots \]

Setting \( \tau_1 = \tau / \sqrt{\epsilon}, p = 2, q = 1 \), the zeroth-order approximation for the small bead’s displacements take the form:

\[ w^s_{i+1/2,0} = \frac{w^b_{i,0} + w^b_{i+1,0}}{2} \]
\[ u^s_{i+1/2,0} = \frac{k}{1 + k \tan \theta} \left( \frac{w^b_{i,0} - w^b_{i+1,0}}{2} \right) \]
\[ k = \left( \frac{2\eta \tan \theta}{\gamma} \right)^{2/3}. \quad (6.12) \]

The zeroth-order equation for the large beads becomes

\[ \ddot{w}^b_{i,0} = [w^b_{i-1,0} - w^b_{i,0}]^{3/2} - [w^b_{i,0} - w^b_{i+1,0}]^{3/2} + \]
\[ \bar{\eta}[w^b_{i-1,0} - w^b_{i,0} + \bar{\delta}]^{3/2} - \eta[w^b_{i,0} - w^b_{i+1,0} + \bar{\delta}]^{3/2}, \quad (6.13) \]

where \( \bar{\eta} = \eta \cos \theta^{5/2} / [2(1 + k \tan \theta)]^{3/2} \) and \( \bar{\delta} = 2\delta \tan \theta (1 + k \tan \theta) \). The zeroth-order equations are thus uncoupled and the velocity components of the small beads depend only on the zeroth-order solution of the larger beads.

Following the approach adopted in [28], the solution of (6.13) can be written in a series form as a traveling wave:

\[ w^b_{0}(x, t) = w(x - ct) = w(z) = \frac{A}{1 + \exp(f)}, \quad f = \sum_{j=0}^{\infty} C_j z^j. \quad (6.14) \]

The polynomial \( f(z) \) approximates the solution and a residual is defined as

\[ R(C_n) = [w(z + 1) - w(z)]^{3/2} - [w(z) - w(z - 1)]^{3/2} + \]
\[ \eta[w(z + 1) - w(z)] - \eta[w(z) - w(z - 1)] - \bar{\delta} - \ddot{w}(z) \]

To solve for the coefficients of the function \( f(z) \), we minimize the functional

\[ \min_{C_j} \int_{-\infty}^{\infty} R^2 dt \quad \Rightarrow \quad \int_{-\infty}^{\infty} R \frac{\partial R}{\partial C_j} dt = 0. \quad (6.15) \]
The above equation leads to a system of $N$ nonlinear equations for an $N$-term series approximation, which are solved by Newton-Raphson method to get the best approximation minimizing the functional. We solve this system for a unit amplitude ($A = 1$) of the leading wave for different values of precompression $\bar{\delta}$. Figure 6.9 shows a comparison between the numerical solution to (6.13) and that obtained by the above procedure for two values of precompression. A good agreement is observed with $N = 3$ terms in the series, and we note that as the precompression increases, the wave becomes narrower. This is consistent with numerical simulations of the full system in which the wave speed is observed to increase with precompression.

The first-order equation for the smaller beads become uncoupled and are quasi-linear. The equation for $w_1^s$ is

$$
\ddot{w}_{i+1/2,1} + 3n \sqrt{\frac{w_{i,0}^b - w_{i+1,0}^b + \bar{\delta}}{2(1 + k \tan \theta)}} w_{i+1/2,1}^s = -\left(\frac{\ddot{w}_{i,0}^b + \ddot{w}_{i+1,0}^b}{2}\right), \quad (6.16)
$$

and for $u_1^s$ is

$$
\ddot{u}_{i+1/2,1} + \frac{3}{2} \sqrt{k(1 + k \tan \theta)} \sqrt{\frac{u_{i,0}^s - u_{i+1,0}^s + \bar{\delta}}{2}} u_{i+1/2,1}^s = \frac{-k}{1 + k \tan \theta} \left[\frac{\ddot{w}_{i,0}^b - \ddot{w}_{i+1,0}^b}{2}\right]. \quad (6.17)
$$

The above equations exactly resemble the case of first-order velocity in light beads in [36] and intruder beads in [38]. In those works, the first-order bead velocity vanishes after the passage of the leading wave for discrete values of mass ratio $\epsilon$. This corresponds to satisfying the anti-resonance conditions and results in solitary waves. However, for solitary waves to exist in our present problem, we require an $\epsilon$ for which the anti-resonance condition is satisfied simultaneously for both equations. This is highly unlikely for fixed material properties and hence we conclude that solitary waves are very unlikely to exist in this system. This is indeed consistent with the trends observed numerically (Fig. 6.6), where the decay rate does not decrease to zero at its minima and hence we use the term near-solitary waves in our work. The first-order system is solved by imposing a zero displacement and velocity boundary condition at $T \to \infty$ and using the zeroth-order series solution.
Figure 6.10: (Color online) Asymptotic solution for first-order axial (a) and radial (b) velocities of the small beads for three precompression levels for $n = 8$ system. As the precompression increases, the magnitude of oscillations in the tail decreases, thereby indicating a reduction in the energy leaking from the primary wave.
Figure 6.11: (Color online) Numerical force decay rate for the $n = 8$ system with precompression $a$ along with the normalized kinetic energy in the tail obtained from the asymptotic solution. The slopes are equal, demonstrating that the force decay is due to the energy leakage into the oscillating small beads behind the leading wave.

with $A = 1$. The dynamics of the small beads under precompression is well approximated by a linear law, unlike that of the big beads which are not precompressed. Hence, the asymptotic solutions for the small beads under precompression are valid even after the passage of the leading wave as it is driven by the precompression $\bar{\delta}$ and the system corresponds to a linear oscillator with frequency $\sqrt{\bar{\delta}}$.

Since the $n = 8$ system is most suited to wave tailoring, we analyze this system in detail in the remainder of this section. Figure 6.10 shows the first-order axial and radial velocities of the small beads for three levels of precompression. After the passage of the primary wave, both components of velocities decrease rapidly with increasing precompression. The energy leaking from the leading wave is directly related to the magnitude of small spheres’ velocity in the oscillating tail. This is consistent with the behavior in numerical observations, where the force decay rate decreases with increasing precompression. Similar trends are also seen in the $n = 9$ system.

Next, we predict the energy decay rate using the asymptotic solution and compare it with the force decay rate extracted from the numerical solution. Let $E$ and $K$ be the total and kinetic energy in the leading wave. Assuming a self-similar structure for the leading wave, the normalized decay rates of both
these quantities are equal, i.e., \((1/E)dE/d\tilde{x} = (1/K)dK/d\tilde{x}\). The decrease in energy of the leading wave leads to the kinetic energy of the small beads in the oscillating tail behind the primary wave. For the asymptotic solution, the energy decay rate is thus obtained by normalizing the kinetic energy in the tail with the peak kinetic energy of the big beads in the leading wave. This is given by

\[
\left(\frac{1}{E}\right) \frac{dE}{dx} = \frac{\epsilon^3}{A^2} \left[(\dot{w}_{i+1/2,1}^s)^2 + (\dot{u}_{i+1/2,1}^s)^2\right].
\] (6.18)

Figure 6.11 shows the comparison of decay rates computed from numerical solution \((1/F)dF/d\tilde{x}\) and using the asymptotic solution using Eqn. (6.18). The variation (slope) of these two quantities with precompression \(a\) are in good agreement, validating our hypothesis that the energy leakage from the leading wave to the oscillating small beads in the tail causes the decay in peak force. The mismatch in the two curves is due to the fact that the structure of the leading wave has not been utilized in computing the energy decay rate in the asymptotic solution. Indeed, for a self-similar wave, the total energy in the leading wave would be proportional to the peak kinetic energy of the big beads, which has been used for normalizing the kinetic energy decay rate, resulting in only the slopes of the two curves being in agreement. Furthermore, this result quantifies the peak force and energy decay rate with precompression in the \(n = 8\) system.

### 6.4 Conclusions

In this chapter, a confined granular system that can be used for wave tailoring by external radial precompression was investigated using numerical simulations and asymptotic analysis. Numerical simulations demonstrate that the \(n = 8\) system changes in behavior from rapidly decaying to near solitary waves under precompression. Scaling laws were derived and verified. It was also observed that, contrary to solitary waves in chains, the peak force in these waves decrease with increasing wave speed as precompression increases. Furthermore, the effect of density ratio was studied to demonstrate the phenomena of resonance and anti-resonance in this system.

In the second part, an asymptotic analysis was performed to demonstrate
that, as the precompression increases, the velocity magnitude of the oscillating small beads decrease rapidly. The energy decay is due to the leakage from the leading wave to this oscillating tail. Finally, the force decay rate extracted from the numerical simulations is compared with the kinetic energy in the tail due to the small beads and the two rates were found to be in good agreement. This work lays the foundation for designing applications involving tailoring of stress waves by an external control.
Chapter 7

Tunable granular systems with spheres and cylinders

This chapter builds upon the previous chapter where the concept of wave tailoring was demonstrated by applying external precompression. Here, based on practical considerations, we consider an alternate configuration comprising of spheres and cylinders and design systems tunable by external control. We present two designs in this chapter, whose responses under dynamic loading can be varied by an external control. In Sec. 7.1, a tunable energy filter is developed, whose impact response is controlled by external precompression. In Sec. 7.2, a system exhibiting band gaps under harmonic loading is designed, whose band gap cutoff frequencies can also be varied by applying lateral precompression. In both cases, parametric studies are conducted over the range of feasible designs and the most suitable design parameters are identified.

7.1 Impact loading: Energy filter

We develop an energy filter whose output under an impact load can be varied by applying lateral precompression. Figure 7.1 shows the schematic of the configuration and it consists of a chain of contacting spheres, with cylinders between them. The cylinders are further confined between two rigid walls. Let $D$, $d$ and $L$ denote the sphere diameter, cylinder diameter and cylinder length, respectively. The spacing between them is $H$ when all the granules are unstressed and the system is densely packed. A total precompression $\delta$ is applied on the system by moving the rigid walls toward each other and it is normalized by $a = \delta/D$. This precompression causes contact forces between the spheres and the cylinders, and between the cylinder and the walls. However, the contact forces between the spheres is zero due to symmetry of the packing. The first and last cylinders are held fixed to ensure equilibrium.
of the initial configuration under precompression. Applying precompression changes the local stiffness of the packing and thus the wave propagation in the axial direction can be varied in a controlled manner.

Let $\gamma = d/D$ and $\eta = L/D$ be the nondimensional parameters characterizing the geometry of the system. Based on geometric constraints, $\gamma$ can take values between 0.25 and 1.0. The lower limit arises because the cylinder has to be in contact with the wall and the upper limit arises because the adjacent cylinders contact each other at $\gamma = 1.0$. The spheres and cylinders are modeled as point masses with the contact between them represented by nonlinear springs, whose stiffness are derived from the Hertzian contact law. The relative displacements are the same as defined in the previous chapter. When a precompression $\delta$ is applied, it induces a precompression $\delta_1$ between the cylinder and the wall, and $\delta_2$ between the sphere and the cylinder. The contact force $F$ between the cylinder and wall with relative displacement $\alpha$ is given by the following implicit relation [79]:

$$\alpha = \frac{F}{L} \left( \frac{1 - \nu_c^2}{\pi E_c} \right) \left[ 1 - \ln \left( \frac{1 - \nu_c^2 F d}{\pi E_c 2L^3} \right) \right], \quad (7.1)$$

where $\nu_c$ and $E_c$ are the Poisson’s ratio and stiffness of the cylinder material,
Figure 7.2: Contour plot of peak force decay rate for (a) No precompression ($a = 0$) and (b) $a = 8 \times 10^{-4}$. The map has bands, indicating that any desired effect can be obtained by setting $\eta = L/D = 1$ and choosing an appropriate $\gamma = d/D$. 
respectively. Let \( m_s \) and \( m_c \) denote the mass of the sphere and cylinder, respectively, and let \( w_s, w_c \) and \( u_c \) denote the axial displacement of a sphere, axial and lateral displacement of a cylinder, respectively. Let \( \theta \) be the angle subtended by the normal at the sphere cylinder contact with the axis of the granular system. The equations of motion are

\[
m_s \ddot{w}_{s,i} = k_{bb} \left( \alpha_i - \frac{1}{2} - \alpha_i + \frac{1}{2} \right) + 2k_{cs} \left( \beta_i - \frac{1}{4} - \beta_{i+1} + \frac{3}{4} \right), \quad (7.2)
\]

\[
m_c \ddot{w}_{c,i+1/2} = k_{cs} \cos \theta \left( \beta_{i+1/4} - \beta_{i+3/4} \right), \quad (7.3)
\]

\[
m_c \ddot{u}_{c,i+1/2} = k_{cs} \sin \theta \left( \beta_{i+1/4} - \beta_{i+3/4} \right) - F_w \left( \delta_1 + u_{c,i+1/2} \right), \quad (7.4)
\]

where \( k_{bb} \) and \( k_{cs} \) are the elastic constants corresponding to sphere-sphere and sphere-cylinder interaction, respectively, and \( F_w(\xi) \) gives the wall contact force for a displacement \( \xi \) using the implicit relation (7.1).

The spheres and cylinders are chosen to be of steel \( (E_s = 210 \text{ GPa, } \nu_s = 0.3, \rho_s = 7800 \text{ kg/m}^3) \) and aluminum \( (E_c = 70 \text{ GPa, } \nu_c = 0.3, \rho_c = 2100 \text{ kg/m}^3) \), respectively. This set of material properties is based on our preliminary numerical investigations which revealed them to be the optimal choice for achieving high sensitivity to precompression, thereby making them more attractive for wave tailoring applications. Numerical simulations are conducted for a wide range of the geometric parameters \( (\gamma, \eta) \). A chain of 300 spheres in contact are considered and an initial velocity \( v = 1 \text{ m/s} \) is prescribed on the 20\textsuperscript{th} sphere along the axial direction. The system is solved with a fourth-order Runge-Kutta solver with a time step of \( 10^{-9} \text{ s} \). The key parameter of interest in our study is the decay rate of the peak force \( F_p \) with distance along the chain. The distance \( x \) is normalized by the sphere diameter as \( \tilde{x} = x/D \).

By assuming the leading wave to have a self-similar structure, the normalized peak force decay rate \( (1/F_p)dF_p/d\tilde{x} \) is constant along the chain, beyond the initial transients. Figure 7.2(a) displays the contour of peak force decay rates over the geometric parameter space \( (\gamma, \eta) \) for two values of precompression, \( a = 0 \) and \( a = 8 \times 10^{-4} \). The trends are distinct in both cases, indicating the potential to alter the response of a system by applying precompression.

For high \( \gamma \) values in the \( a = 0 \) case, the decay rate looks highly irregular and the force even seems to increase down the chain for some values. The apparent lack of regularity in the high \( \gamma \) regime is an artifact of our peak force decay rate measure, caused by a change in the structure of the leading wave, and is discussed in the forthcoming paragraph. In both the precompression
Figure 7.3: Contour plot of ratio of peak force decay rates for the case with precompression \( a = 8 \times 10^{-4} \) to the case with zero precompression.

Cases, for low \( \gamma \) values, bands of constant decay rate appear. From a design viewpoint, one can work with a fixed value of \( \eta \), i.e., fixing the length of the cylinder, and obtain any decay rate in the map (Fig. 7.2) by choosing an appropriate \( \gamma \). Figure 7.3 illustrates the ratio of the peak force with precompression \( a = 8 \times 10^{-4} \) to the peak force with zero compression. The bands observed in Fig. 7.2 are also observed in the ratio plot. To achieve the most effective wave tailoring, an operating parameter with the maximum ratio should be chosen. This will cause the granular system to have the highest sensitivity to the lateral precompression.

Figure 7.4(a) displays the peak force variation along a \( \gamma = 0.56 \) system for two levels of precompression. The system produces near solitary waves when there is no precompression, while the peak force decays rapidly when a precompression \( a = 8 \times 10^{-4} \) is applied. This is contrary to the behavior observed in the case of spheres in a cylindrical tube [80] in the previous chapter, where the decay rate always decreased with precompression. Figure 7.4(b) displays the peak force variation along a \( \gamma = 0.9 \) system for the same two levels of precompression. As the precompression increases (\( a = 8 \times 10^{-4} \)), the decay rate is small, and the peak force is almost constant down the chain. For zero precompression, the peak force decreases down the long chain non-monotonically. The structure of the leading wave is quite distinct in this

\[
\begin{align*}
\gamma &= d/D \\
\eta &= L/D \\
0.3 &< 0.4 < 0.5 < 0.6 < 0.7 < 0.8 < 0.9
\end{align*}
\]

\[
\begin{align*}
0.5 &< 0.6 < 0.7 < 0.8 < 0.9 < 1.0 \\
0.5 &< 1.1 < 1.2 < 1.3 < 1.4 < 1.5
\end{align*}
\]

\[
\begin{align*}
-10 &< -8 < -6 < -4 < -2 < 0 < 2 < 4 < 6 < 8 < 10
\end{align*}
\]
Figure 7.4: Peak contact force variation along the chain for two levels of precompression with (a) $\gamma = 0.56$ and (b) $\gamma = 0.9$. For $\gamma = 0.56$, near solitary waves are obtained for zero precompression, while the response is rapidly decaying for $a = 8 \times 10^{-4}$. For $\gamma = 0.9$, an oscillatory decaying peak force profile is obtained with $a = 0$ indicating a distinct type of leading wave.

case and it spans over 10 beads. This results in local oscillation in the peak force along the chain and hence our measure of peak force decay rate, based on the difference of forces at particular contacts, yield the wildly oscillating values observed for high $\gamma$ in Fig. 7.2(a). To estimate the decay rate in these high $\gamma$ regimes, the average peak force over multiple contacts neighboring a sphere has to be computed. We intend to work only in low $\gamma$ regimes and hence do not present this calculation here.

Figure 7.5(a) displays the variation in peak force decay rate with radius ratio $\gamma$ for distinct precompression levels. Each curve shows a highly nonlinear variation with cylinder size, having multiple peaks and valleys. Similar to the case of dimer chains [36], plane waves in periodic media [81], spheres in cylindrical tube [80], these correspond to nonlinear resonances and anti-resonances. With increasing precompression $a$, the whole curve has a trend of shifting to the right. The valleys also shift downward for low values of precompression, and then shift upward. The wave tailoring capabilities are evident by considering the change in decay rate with precompression at a fixed value of $\gamma$. This is further elucidated in Fig. 7.5(b) where the change in decay rate with precompression is illustrated for three distinct values of size ratio $\gamma$. The response is non-monotone with increasing precompression, with the decay rate increasing and then decreasing.
Figure 7.5: (a) Variation of decay rate with changing $\gamma$ for different precompression levels, with the curves shifting to the right with increasing precompression. (b) Variation of decay rate with precompression for different sizes, illustrating the tunability of the design.
Figure 7.6: Schematic showing top view of sphere-cylinder configuration for tunable band-gap in a system subjected to harmonic loading. By altering the precompression, the wave propagation downstream can be altered.

7.2 Harmonic loading: Band gap

We now consider systems subjected to long duration harmonic loadings. We present designs using granular crystals having band gaps whose cutoff frequencies can be tuned by external control. Figure 7.1 shows the schematic of the configuration under study for achieving tunable band gaps. It consists of a chain of spheres under a static precompression $F$, along with a granular tunnel in a localized region, composed of cylinders between two rigid surfaces. The rigid surfaces are movable and can impose a fixed displacement $\delta$ on the cylinders. A harmonic load $A \sin \omega t$ is applied to the first bead in the chain and response of the system downstream of the granular tunnel is studied. We illustrate by numerical simulations and theoretical analysis, the existence of a band gap due to the tunnel and that varying the precompression level $\delta$ alters the band gap cutoff frequencies of the system.

In this work, we restrict our attention to cases where $A \ll F$, i.e., the amplitude of excitation is small compared to the magnitude of the static precompression force. Let $\delta_F$ and $\delta_0$ be the precompression between the spheres outside and within the tunnel, respectively, $\delta_1$ be the precompression between the sphere and cylinder, and $\delta_2$ the precompression between the cylinder and the wall. These precompression values depend both on the axial force $F$ and the lateral precompression $\delta$ and are determined by considering the geometry, the equilibrium of the sphere at the edge of the channel and
the lateral equilibrium of a cylinder. They lead to the following relations:

\[ F = k_{bb} \delta_0^{3/2} + 2 \cos \theta k_{cs} \delta_1^{3/2}, \quad (7.5) \]

\[ 2 \sin \theta k_{cs} \delta_1^{3/2} = k_w \delta_2, \quad (7.6) \]

\[ \sqrt{(R + r - \delta_1)^2 - (R - \delta_0/2)^2} - \delta_2 = \sqrt{(R + r)^2 - R^2 - \delta/2}. \quad (7.7) \]

We determine the dispersion relation within the tunnel analytically, assuming each unit cell to be composed of a sphere and two cylinders. All the contacts are under precompression, which allows us to linearize the system of equations about the initial configuration. The linearized equations for the spheres and cylinders inside the tunnel are

\[
\begin{align*}
    m_s \ddot{w}_{s,i} &= \frac{3}{2} k_{ss} \sqrt{\delta_0} (\alpha_{i-1/2} - \alpha_{i+1/2}) + k_{cs} \sqrt{\delta_1} (\beta_{i-1/4} - \beta_{i+1/4}), \quad (7.8) \\
    m_c \ddot{w}_{c,i+1/2} &= k_{cs} \sqrt{\delta_1} (\beta_{i+1/4} - \beta_{i+3/4}), \quad (7.9) \\
    m_c \ddot{u}_{c,i+1/2} &= k_{cs} \tan \theta \sqrt{\delta_1} (\beta_{i+1/4} - \beta_{i+3/4}) - k_w u_{c,i+1/2}. \quad (7.10)
\end{align*}
\]

We assume the unit cells extend to infinity and follow the standard approach where the displacement is represented as a traveling wave solution of the form

\[
\begin{pmatrix}
    w_s \\
    u_c \\
    u_c
\end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \exp(i[kx - \omega t]).
\]

Substituting this into the linearized system leads to the following homogeneous system

\[
\begin{pmatrix}
    m_s \omega^2 + 2(\cos k - 1)k_{ss} - 2k_{cs} & 2k_{cs} \cos \frac{k}{2} & -2ik_{cs} \tan \theta \sin \frac{k}{2} \\
    2k_{cs} \cos \frac{k}{2} & m_c \omega^2 - 2k_{cs} & 0 \\
    2ik_{cs} \tan \theta \sin \frac{k}{2} & 0 & m_c \omega^2 - 2k_{cs} \tan^2 \theta - k_w
\end{pmatrix}
\begin{pmatrix}
    A \\
    B \\
    C
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

The dispersion relation is obtained by setting the determinant of the matrix in the above equation to zero. Since the unit cell has three degrees of freedom,
the dispersion curve yields three branches: one acoustic and two optical.

The problem is solved for the material parameters of brass, with Young’s modulus $E = 115$ GPa, Poisson’s ratio $\nu = 0.3$ and density $\rho = 8500$ kg/m$^3$. The diameter of the spheres and cylinders are $D = 9.525$ mm and $d = 4.76$ mm, respectively. The value of the static precompression force is $F = 1$ N and the amplitude is fixed at $A = 0.1$ N. Figure 7.7(a) shows the dispersion relation obtained by this approach, with only the acoustic branch and the lower optical branch illustrated. The higher acoustic branch has frequencies in the range of 0.3 MHz and does not play a role in wave propagation in the present configuration because the allowable range of this branch is way higher than the acoustic branch of the brass beads. Since the configuration also has beads behind the tunnel, these frequencies cannot propagate downstream and are thus not considered in our present work.

Numerical simulations were conducted on a chain of 300 spheres with 32 cylinders placed between the spheres 16 to 32. Again, a fourth-order Runge-Kutta solver is used to solve the system with a time step of $10^{-8}$ s until a final time of 5 ms for a wide range of loading frequencies $\omega$. The simulations are conducted for 3 levels of precompression and the peak velocity of the 75th sphere is shown in Fig. 7.7(b). Also shown are the numerical simulation results for the monodisperse sphere chain subjected to the same loading conditions.

Figure 7.7(a) shows the existence of a band gap for both the precompression levels. There is an upper cutoff frequency for the chain of spheres. The band gap shifts upward with increasing precompression, thus demonstrating the potential for tunable variation of band gaps. Figure 7.7(b) shows the corresponding numerical solution, where the velocity of a bead behind the tunnel is seen to vary as a function of frequency. The loading frequency is normalized using the intrinsic time scale of wave propagation through the monodisperse sphere chain with a precompression $F$ and the normalized frequency is $\tilde{\omega} = \omega/(3\sqrt{\delta F k_{bb}/2m_s})^{1/2}$. Similarly, the bead velocity is normalized as $\tilde{v} = v/\delta F (3\sqrt{\delta F k_{bb}/2m_s})^{1/2}$ and is a function of the applied static precompression force and the properties of the sphere chain outside the tunnel. For the chain of pure spheres, there is a single cutoff frequency, beyond which there is a sharp decrease in the velocity of the monitored bead. This cutoff frequency compares well with the analytically predicted value of 5.6 kHz. The cutoff frequencies predicted for the sphere cylinder system are...
Figure 7.7: (a) Dispersion curves for the monodisperse sphere chain and the sphere-cylinder system with two levels of precompression. The band gaps shift upward with increasing precompression, illustrating the tunability of the system. (b) Numerical solution of peak velocity at 75-th sphere for the same three cases. The cutoff frequencies predicted analytically are indicated with lines, showing good agreement with the numerical results.
also shown by dashed lines in Fig. 7.7(b) and they are in good agreement with the numerically observed band gap frequencies.

Having demonstrated the accuracy of analytical predictions of the cutoff frequencies, we now analyze different material configurations using Eqn. (7.12). A basic parametric study with three materials: steel, brass and aluminum, is presented for three size ratios $\gamma = d/D$. Figure 7.8(a) shows the variation of the cutoff frequencies for the combination of brass spheres with brass cylinders. As the size of cylinders decrease, both the lower and upper cutoff frequencies increase. At moderate values of $\gamma$, both the cutoff frequencies increase with precompression $a$, while, at low $\gamma$, the upper cutoff of the acoustic band decreases with precompression. It is also noted that the regime of validity of this analysis depends on two other issues. First, the presence of the homogeneous chains ahead of and behind the granular tunnel imposes an upper bound on these cutoff frequencies. It should be ensured that the desired properties of the homogeneous chain outside the tunnel should be chosen in appropriate ranges. Secondly, the analysis is carried out for a linear system behavior, under the assumption of a greater static precompression force than the loading amplitude. However, with increasing precompression $\delta$, the static precompression between spheres within the tunnel $\delta_0$ decreases, leading to a deviation from the linear behavior, i.e., nonlinear effects of the higher order terms start to become significant. Indeed, our numerical tests with higher loading amplitudes $A$ show deviation and the predicted band gap structure is not valid.

Figure 7.8(b) displays the variation of cutoff frequencies for steel spheres and aluminum cylinders, while Fig. 7.8(c) displays the variation for aluminum spheres and steel cylinders. In the former case, the band gap is small and it has higher sensitivity (varies rapidly) to precompression, while in the latter case, the band gap is larger, but with a lower sensitivity to the applied precompression. This result illustrates the wide range of behaviors that can be obtained by choosing appropriate material and geometric parameters.

### 7.3 Conclusions

In the first part of this chapter, a design for tunable energy filter has been developed, where the response can be varied from near solitary to rapidly
Figure 7.8: Cutoff frequencies with precompression for different size ratios $\gamma$. Three material combinations are presented: (a) brass spheres and brass cylinders, (b) steel spheres and aluminum cylinders, and (c) aluminum spheres and steel cylinders.
decaying waves. A parametric study over the entire range of geometric parameters has been conducted and a highly nonlinear response was observed. In the second part, a design for tunable band gaps on a system subjected to harmonic loadings has been developed. By applying precompression over a localized region, the cutoff frequencies of the band gap can be altered. The dispersion curves for the system have been obtained by linearizing the governing equations and the predicted band gap values show good agreement with the numerical solution. Finally, parametric studies over a wide range of material and geometric properties have been conducted to illustrate the range of possibilities in choosing the cutoff frequencies over a desired range.
Chapter 8

Wave tailoring by altering network topology in granular lattices

This chapter presents designs of systems involving granular lattices for wave tailoring applications and their validation with a combination of modeling, numerical simulations and experiments. The focus is on demonstrating simple and robust systems which can be built easily, in contrast with the more theoretical constructs based on precompression discussed in the previous chapter, which can be hard to control in practice. The chapter is organized as follows: Section 8.1 presents a brief description of the concept. Section 8.2 describes the experimental setup and some preliminary results. In Sec. 8.3, a model is developed to describe the experimental results. An alternate design for wave tailoring is also presented in Sec. 8.4, together with some conclusions in Sec. 8.5.

8.1 Wave tailoring concept

We develop a granular lattice where a change in the network topology in a localized region results in a significant change in wave propagation along the lattice. A schematic of the system of interest is illustrated in Fig. 8.1. It consists of a long chain of monodisperse spheres in contact. Some cylinders (four in the schematic) are placed symmetrically in a localized region of the chain. In this localized region, a unit cell is composed of a sphere and two cylinders. The cylinders are confined laterally by parallel plates placed on either side of the lattice. By moving the plates closer as in Fig. 8.1(a), there is a gap between the adjacent spheres in this localized region only, while moving the plates further apart induces gaps between the spheres and the cylinders as depicted in Fig. 8.1(b). Note that in the second configuration, adjacent spheres are in contact and the spheres are densely packed axially, while there is room for the cylinders to move within each unit cell. We will
demonstrate, using a combination of experiments, modeling and numerical simulations, that the first case results in rapidly decaying waves while the second case results in near solitary waves with little attenuation when the long sphere chain is subjected to an impact load. We remark that the design presented here requires almost zero work to change between configurations since the granular lattice has zero prestress in both the static configurations. In contrast to the design in the previous chapter involving precompression, the present design does not require any stored strain energy. The dynamics is dictated by the geometry of the granular lattice and the tunability is a result of the change in the local network topology of the lattice.

Figure 8.1: Schematic showing top view of the granular lattice for wave tailoring. Moving the plates closer (a) induces gaps between the adjacent spheres, thus altering axial wave propagation properties.
Figure 8.2: (a) Experimental setup of the impact test. The channel width can be varied by moving the center bar (label: 2). (b) Cross section of the alignment holder which holds the granular lattice. The spheres are placed in the v-groove while the cylinders are on either side.
Figure 8.3: (a) The granular lattice assembled in the holder, with the input (right) and output (left) sensors. (b) Close up of a sphere-to-sphere gap configuration with 8 cylinders where the center plate is moved inward. (c) Close up of a cylinder-to-sphere gap configuration where the center plate is moved outward to increase channel width.
8.2 Experimental demonstration

8.2.1 Setup and calibration

The experimental setup consists of a granular lattice confined between two plates and is illustrated in Fig. 8.2(a). The confinement is provided by an arrangement consisting of three aluminum bars (labeled 1, 2 and 3 in Fig. 8.2(a)) of which the outer two are clamped. The center bar is movable and is used to vary the channel width. It is moved and oriented by turning the two screws which pass through the outer bar, as illustrated in Fig. 8.2(a). The setup is assembled on an optical table. The granular lattice is assembled on an alignment holder, which has a v-groove in the center and flat surfaces on either side. The cross-section of the holder is illustrated in Fig. 8.2(b) and it is two feet long. The spheres are placed in the v-groove to ensure they remain in a straight line and the cylinders are placed on the flat surfaces on the sides of the groove in the holder. The impact is imparted to the lattice using a ramp, which is made of an acrylic half tube and is fixed on a support stand. This tube is in contact with the alignment holder and a bead is released from the top of the tube to roll down the tube, along the alignment holder and impact the chain. The released bead rolls down the v-groove of the alignment holder which helps to ensure that the impact is normal to the axis of the granular chain. Piezosensors are used to record signals by connecting them to an oscilloscope similar to the approach followed in [82, 31, 39, 26]. Two sensors are used to capture the input and output signals: one before and one after the localized zone having cylinders. The piezosensors have lead zirconate titanate based piezogauges with nickel plated electrodes. A steel sphere is cut in half, the electrodes are glued to the flat surface of a half sphere, and the half spheres are glued back to have the sensor embedded in the sphere.

The granular lattice consists of 27 spheres in contact placed in the v-groove of the alignment holder. One end of this chain has a block of aluminum, which serves as a wall boundary and also clamps the alignment holder to the optical table. The other end is free and the impacting sphere strikes this free end. The embedded sensor spheres are placed at the 6-th and 21-st location from the free end and the cylinders are placed on either side of spheres between the input and output sensor beads. A typical lattice with 8
cylinders is illustrated in Fig. 8.3(a), with the sensor embedded beads having wires coming out from them. The sensors are connected to the oscilloscope whose sampling time is set to 100 ns and the total recorded time is 1.5 ms. A trigger based on a threshold voltage level of the input sensor is used to start the signal recording. In all the experiments, the locations and number of cylinders are varied, while the number of spheres and the sensor sphere locations are kept unchanged.

Figure 8.4(a) shows the setup of a drop test used to calibrate the sensors. It consists of four Teflon guide rails along which a sphere travels as it is dropped from a specified height on a chain of spheres. The sphere with the embedded sensor is placed somewhere in the middle of the chain and the voltage data is recorded from it. The voltage data gives the average compressive force from the two contacting spheres [26, 83]. We do not place the sensor bead at the top of this chain to avoid its possible plastic deformation upon direct impact with the dropped sphere. The system is simulated numerically and compared with the measured signals for the calibration process. Figure 8.4(b) shows the collection of raw data signals illustrating a good consistency between them. The second peak corresponds to the reflected wave from the base of the system. Simulations are conducted assuming Hertzian interaction and they are matched to get the relation between force and voltage signal measured. The tests are repeated for two heights, \( h = 4.2 \) cm for the first and \( h = 3.3 \) cm for the second experiment. Good match between the set of tests and with the numerical simulations is observed.

8.2.2 Impact tests

The spheres are chosen to be of 9.5 mm diameter and made of steel with Young’s modulus \( E_s = 210 \) GPa, Poisson’s ratio \( \nu_s = 0.3 \) and density \( \rho_s = 7800 \) kg/m\(^3\). The cylinders are made of aluminum with Young’s modulus \( E_c = 70 \) GPa, Poisson’s ratio \( \nu_c = 0.3 \), density \( \rho_c = 2800 \) kg/m\(^3\) and have diameter 5.14 mm. Impact tests are conducted by releasing a steel sphere from the top of the ramp for different numbers of cylinders in the lattice. For the first case, with the gaps between the spheres, the distance between the plates varies from 15.95 to 16 mm. A closeup of the zone containing the cylinders is presented in Fig. 8.3(b), showing gaps between the adjacent
Figure 8.4: (a) Drop test setup for calibration of piezosensors. (b) Data showing calibration of sensor signals using numerical simulations for two drop heights (4.2 cm and 3.3 cm).
Figure 8.5: Experimental data for different numbers of cylinders for (a) sphere-to-sphere gap and (b) sphere-to-cylinder gap. The curves on the left correspond to input sensor forces and those on the right correspond to output sensor forces. The trends are quite different in both cases, demonstrating the wave tailoring capabilities of the setup.
spheres in the lattice. Figure 8.5(a) illustrates the collection of results for all the distinct cylinder cases, showing a clear trend in the data. The input sensor signals are the left curves while the output sensor signals are the curves on the right side in the figure. Good consistency is observed in the recorded output signals. As the number of cylinders increase, both the wave speed and the peak force decrease rapidly. There is a substantial decay compared to the case of solitary wave in a pure sphere chain.

For the second case, the plates are moved further apart to eliminate gaps between adjacent spheres, thus introducing gaps between spheres and cylinders due to the excess space available. The middle plate and the alignment holder are moved and clamped, and the channel width is now 16.5 mm. Figure 8.3(c) presents a closeup of the zone containing the cylinders, where gaps are present between the cylinders and adjacent spheres in the downstream direction. The tests are conducted by releasing spheres from the same height. Compared to the previous case, there is little decay of the incident signal as it goes through the array of cylinders. The results are also consistent for the cases of impacts indicating good repeatability of our experiments. Figure 8.5(b) illustrates the collection of results for the impact tests with different numbers of cylinders in the chain. The decay rate is very small with increasing number of cylinders in the present configuration. This variation in decay rate from high to low by moving the plates outward demonstrates the potential for wave tailoring granular chains. We remark that the experimental results are presented only for two channel widths since it is hard to reliably control the channel width and get a smooth variation in decay rate using our current experimental setup.

8.3 Modeling and simulation: Sphere-to-sphere gap configuration

We conduct numerical simulations to model the experimental results described in the previous section. Similar to our approach in the previous chapters, the spheres and cylinders are treated as point masses interacting via springs with elastic potentials. The problem is solved for the geometric configuration shown in Fig. 8.1(a). Let $H$ be the distance between the plates, then the gap $\delta_s$ between adjacent spheres where there are cylinders is given
Figure 8.6: Comparison of experimental and simulation data for the case of sphere-to-sphere configuration. The discrepancy points to the need for a model which accounts for damping associated with the frictional contact of the cylinders with the wall.

by

\[ \delta_s = \sqrt{(D + d)^2 - (H - d)^2 - D}, \]  

(8.1)

with \( D \) and \( d \) denoting the sphere and cylinder diameters, respectively. The equations of motion are the same as in the previous chapter with \( n = 2 \) as there are two cylinders in each unit cell. The only difference is the expression for the relative displacement between the spheres, which is now given by

\[ \alpha_{i+1/2} = [w_{s,i} - w_{s,i+1} - \delta_s]^{3/2} \]

with \( w_{s,i} \) denoting the axial displacement of sphere \( i \). An input velocity of 0.8 m/s is imparted to the first sphere in the numerical simulation to match the force recorded by the input sensor. Figure 8.6 shows the comparison of experimental data to the simulation for different number of cylinders for the force measured at the output sensor. Though both systems show a qualitatively similar trend of a gradual decay, the experimental data shows a much rapid decay compared to numerical simulations.

To account for this discrepancy between the numerical solution and experimental data, we introduce damping due to wall friction in our model. We choose the simple Coulomb friction model with friction coefficient \( f \) and the
equations of motion for each cylinder now become

\[ m_c \ddot{u}_c = -k u_c \max (u_c, 0) + \sin \theta (F_1 + F_2), \tag{8.2} \]
\[ m_c \ddot{w}_c = \cos \theta (F_1 - F_2) - f \text{sign}(\dot{w}) k u_c \max (u_c, 0), \tag{8.3} \]

where \( u_c, w_c \) and \( m_c \) are the axial, lateral displacements and mass of the cylinder, \( F_1 \) and \( F_2 \) are the contact forces with the sphere upstream and downstream, respectively. The contact between the wall and the cylinder is represented by a linear spring having stiffness \( k \) in compression, thus \( k u_c \max (u_c, 0) \) is the normal force between the cylinder and the wall. The simulations are conducted over a range of friction values to find a coefficient value which gives reasonable agreement with experimental results. The value \( f = 0.185 \) is found to match reasonably well with experimental data. It is also noted that other authors like Carretero-González [44] have considered a power law with two parameters to model dissipative losses in granular chains, whereas in this study we restrict to the Coulomb model, which is a simple phenomenological model with one parameter.

Figure 8.7(a) shows the comparison of numerical simulation (black curve) with the experimental data for the case of 2 cylinders. In this case, the cylinders were placed after the 14-th sphere from the free end. A good agreement of the wave profile, force amplitude and arrival time is obtained between the numerical simulation and experiments. Figure 8.7(b) shows the comparison with experimental data for 4 cylinders, placed after the 14-th and 15-th spheres in the chain. A similarly good agreement is attained for all aspects between the simulations and experimental data. Figure 8.8(a) presents the comparison for the case of 6 cylinders placed at locations 15, 16 and 17 in the chain. The amplitude is underpredicted by the numerical simulations, while a good agreement is attained for the wave profile and arrival time. Figure 8.8(b) presents the comparison for the case of 8 cylinders, with the cylinders placed at locations 14, 15, 16 and 17. A good agreement is seen for the wave profiles and the force amplitude, and the time of arrival shows some error. This difference is attributed to an error in the experiments for the 8 cylinders case, possibly due to the presence of gaps at some location. Figure 8.8(c) shows the comparison for the case of 10 cylinders, placed from locations 14 to 18 in the chain. A good agreement of the arrival time and wave profile between the results are obtained, and a reasonable agreement
Figure 8.7: Comparison of numerical simulation with experimental data for (a) 2 cylinders and (b) 4 cylinders with frictional damping (friction coefficient \( f = 0.185 \)), showing good agreement of the wave profile, force amplitude and time of arrival.
Figure 8.8: Comparison of numerical simulation with experimental data for the case with (a) 6 cylinders, (b) 8 cylinders and (c) 10 cylinders, showing good agreement of the wave profile and reasonable agreement of the force amplitude and the time of arrival. 147
with the force amplitude is obtained. The experimental results for the case with 10 cylinders show a considerable spread over the arrival time and this is attributed to the presence of gaps and asymmetry in the region having cylinders. Indeed, as the number of cylinders increases, alignment errors also increase progressively, primarily because the channel width is not uniform over the length of the channel.

8.4 Gravity assisted tunable wave propagation

In the final part of this section, we introduce another concept for wave tailoring using lattices of spheres and cylinders. Based on our insights attained in the earlier part of this section, we have the schematics of the configurations illustrated in Fig. 8.9. It attains two states based on the direction of the tilt relative to the impact direction. In the first configuration 8.9(a), the
lattice is tilted downward along the impact direction, resulting in the cylin-
ders contacting the spheres downstream and a gap between the cylinders and
the spheres upstream. In the other configuration 8.9(b), the lattice is tilted
upward along the impact direction, resulting in a gap between each cylinder
and its adjacent sphere in the downstream direction. Numerical simulations
are conducted on both the states to study their dynamic behavior under an
axial impact.

We remark here that the effect of gravity is negligible in the dynamics
considering that the weight of a typical granule is around 3 g and thus ex-
erts a force of the order of $10^{-2}$ N, which is quite small compared to the
magnitude of forces involved in wave propagation. However, gravity plays a
key role in providing a force sufficient to ensure the desired configuration in
the static assembly. Figure 8.10 illustrates the evolution of peak forces at
the 21-st sphere for different numbers of cylinders for both the states. The
behavior is clearly different for both the cases, demonstrating potential for
wave tailoring applications. In the first state, the leading wave loses energy
at each cylinder contact, resulting in progressive decay of the wave ampli-
tude with number of cylinders. Though the simulation results are presented
incorporating wall friction, we note that similar decay trends are observed in
this state even with no frictional dissipation. The energy loss is due to the
transfer of energy from the spheres to the cylinders at each contact location.
In the second state, a solitary wave propagates down the chain independent
of the number of cylinders. The spheres do not interact with the cylinders
as their displacement toward the impact direction is about 13 $\mu$m, which is
less than the gap between them and the cylinders.

8.5 Conclusions

In this chapter, two designs have been presented for wave tailoring due to im-
pact loads. They both rely on change in the network topology in a localized
region of the granular lattice. The lattice consists of a chain of spheres with
cylinders placed on either side in the localized region. In the first design,
the two configurations are attained by moving the plates inward or outward.
Low-velocity impact experiments have been conducted on both configurations
for different numbers of cylinders. Models have been developed, accounting
Figure 8.10: Forces before and after the cylinders for the two configurations showing (a) rapid decay in the case of cylinders contacting spheres upstream and (b) a solitary wave transmission in the other case.
for frictional damping and geometric configurations and their numerical simulations are compared with the experimental data. Good agreement has been achieved between them and the wave tailoring potential of the designs has been demonstrated both numerically and with experiments. Only two configurations in this design have been investigated due to the limitations of our experimental setup.

In the final part, an alternate design for wave tailoring is presented, where the change in the network topology is driven by gravity, achieved by simply tilting the lattice by a small angle. This is a binary configuration design, with only two possible states. Finally, we remark that the designs presented in this chapter have potential to be extended to systems with high impact loads, of the order of kilonewtons by preconditioning [83], and can be further optimized for desired tunability by varying the material and geometric properties of the lattice.
Chapter 9

Concluding remarks and future work

9.1 Summary

This thesis has explored the possibility of wave tailoring using granular crystals under a diverse set of conditions. Broadly, two themes have been explored: wave attenuation by plastic dissipation and tunable wave energy filters by exploiting the contact nonlinearity. At various stages, a diverse set of analytical, numerical and experimental tools were used to complement and validate each result.

In elastoplastic granules, we confined the study here to elastic-perfectly-plastic materials. For the first time, a systematic study of the role of plasticity on the dynamics of granular chains has been conducted. We did this by first constructing a unified contact law for spheres of distinct materials and sizes using finite element simulations, derived scaling laws and validated with experimental results obtained from our collaborators. This contact law has been used to conduct extensive numerical simulations and characterize the salient features of wave propagation in elastoplastic chains. We also presented new results of unified scaling behavior for elastic systems under any loading duration. Finally, we conducted a comparative study of dissipation by impact in both the granular and continuum media, showing that the dissipation is orders of magnitude more in the granular media for the loads considered here. We also derived scaling laws for both media which allow to predict the amount of energy dissipated for any elastic-perfectly-plastic material.

We then demonstrated the concept of wave tailoring by external precompression in ordered elastic granular lattice. A hollow cylinder with big and small beads was used to illustrate the idea, by numerical simulations and asymptotic analysis. We then worked with an alternate design of spheres and cylinders confined between parallel plates. We designed tunable energy
filters for impact loadings and tunable band gap systems for harmonic loadings. In both cases, the tunability is accomplished by varying the width between these plates. In the last chapter, we introduced designs for wave tailoring based on change in network topology of the granular lattice in a localized region. The designs have been validated by numerical simulations and experiments, and their feasibility for practical applications have been discussed.

9.2 Future directions

The research presented in this dissertation opens up avenues for further directions, both fundamental and applied. We briefly discuss hereafter some of these possible directions.

- Role of friction: Friction between granules exists and plays a role in dissipating energy when large motions are involved. It also causes rotation of granules and may potentially lead to new family of waves. This has potential implications in both elastic and elastoplastic systems.

- Hardening and rate dependence: We solved the problem for elastic-perfectly-plastic materials. While it is a good approximation for many materials considered in this work, several materials of interest cannot be modeled accurately by this assumption. Examples include metals like steel, which exhibit both hardening and rate dependence and polymer materials, which exhibit viscous effects.

- Effect of size randomness: In real 3D granular packing, the effect of randomness would lead to gaps thereby causing force chains. These force chains have been widely studied in the literature and influence the macroscopic properties of the granular system. The prior studies mostly consider quasistatic movements and rigid/elastic granules. Our work on ordered lattices in this dissertation has shown that the wave propagation characteristics would be very distinct for plastically deforming granules.

- Effect of particle size and high loading: The contact law is valid only until a certain load limit. Thus, for a fixed external load, the validity
regime of the contact law decreases as the granules get smaller. An open question is the effect of decreasing particle size for a fixed load, beyond the validity range of the contact law presented here. For a fixed load, as the particle size keeps decreasing, the whole granule may become plastic, and may undergo failure by comminution. This breakage of granules and movement of granules, along with friction and hardening effects will likely alter the wave propagation in the system.

- Wave tailoring devices: A conceptual framework has been presented in this dissertation, which opens up avenues for building wave tailoring devices based on specific configurations of granular lattices. A systematic design of these lattices involving both material and geometry optimization is a potential future research direction.
References


