THE PERIODIC BEHAVIOR OF A THRESHOLD MODEL ON DIRECTED GRAPHS

BY

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THESIS

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ABSTRACT

This thesis investigates a discrete-time deterministic binary threshold model over a directed graph. At each time step, each agent updates the value it holds to the value held by the majority of its incoming neighbors at the last time step. It has been proved in the literature that if the underlying graph is undirected, then for any initial condition, the solution of the threshold model will enter into a periodic solution with the period being no larger than two. Examples can be generated to show that in the cases when the underlying graph is directed, even though the solution will still be periodic, the period of the solution exhibits richer possibilities. This thesis computes the periods of all possible periodic solutions of the model over a certain class of directed graphs, including a single directed cycle and a composition of two directed cycles. It is shown that in the case when the graph is a single directed cycle, all possible periods are divisors of the size of the cycle (i.e., the number of edges). It is also shown that in the case when the graph is a composition of two directed cycles, all possible periods are common divisors of the sizes of the two cycles. The analysis used in this thesis is generalizable to more complex graphs.

Keywords: social network, periodic solution, automata

To my parents, for their love and support.

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TABLE OF CONTENTS

LIST OF ABBREVIATIONS

Eq.	Equation
gcd(a,b)	Greatest common divisor of integers a and b
m mod n	Remainder of m divided by n, also denoted as $[m]$ in this thesis
$\mathbb{Z}^+$	Set of positive integers

CHAPTER 1

INTRODUCTION

Coordination between agents and their neighbors is commonly seen in many different types of networks. For example, in social networks, people benefit from conformity to behaviors of their peers. In economic networks, firms have higher productivity if they use technology standards that are widely accepted in industry. With the rapid development of new technologies and online services, such coordinations are becoming easier, and interactions are becoming increasingly popular [1]. Such increasing coordination in a highly connected world can lead to a cascading behavior [2]. As one set of decision makers often influence the reactions of others [3], the decision of some agents can be adopted by their neighbors and from these neighbors to the rest of the network. The linear threshold model introduced by Granovetter in [4] is one of the most commonly used models for analyzing such cascading behavior, and the model is widely used to explain a variety of aggregate level behaviors, such as dynamics of opinions, diffusion of innovations, propagation of rumors and diseases, voting, spread of riots and strikes, etc.

For the discrete-time binary linear threshold model, we consider a network where each agent only has access to the information of a subset of other agents in the network, which we refer to as its *neighbors*. At the beginning, a set of agents in the network try to initiate an action which might be a demonstration. At each time step, each agent decides whether to engage in the collective action or not based on his threshold and the fraction of his neighbors who have joined the collective action at the last time step. The agent engages in the collective action if and only if the fraction is above the threshold.

1.1 Literature Review

The topic studied in this thesis is related to a considerable body of work in the literature on linear threshold models. The linear threshold model was initially proposed in [4] and [5], and was later developed in [6–15]. In [10], Morris studied the single-switch version and investigated whether there is a finite set of initial adopters such that the behavior diffuses to the entire network, and the result is applicable to the multi-switch version of the dynamics as well. In [12], a method is introduced to change the behavior of a fixed number of agents so that the spread of the behavior in the network is maximized. In [13], Watts studied conditions under which the behavior can spread to a positive fraction of the network. In [14], Lelarge determined the limits of behavior spreading in sparse and dense networks. In [15], a set of new technologies for analyzing the failure probabilities of nodes in arbitrary graphs were developed.

1.2 Contributions

As in [10], we assume that the thresholds of all agents in the network are fixed and equal. For simplicity, we use the majority threshold. Previous work [2] has investigated the period of the networks when the underlying graphs are undirected, which in fact assumed that the interactions between agents are always bidirectional. This assumption may well be justified in some situations when all agents have access to information of all their neighbors. However, it is restrictive in some other settings when agents have biased information on their neighbors, i.e., they only have access to some of their neighbors' information. In such cases, the interaction between agents will not be bidirectional. Accordingly in this thesis, we study the period of the networks when the underlying graphs are directed. Using the majority threshold model, we find that when the underlying graph is a cycle, the period can only be a divisor of the number of agents and when the underlying graph has two cycles, the period must be a common divisor of the sizes of these two cycles. The techniques developed for these two specific groups of directed graphs are expected to be useful for the analysis of directed graphs consisting of a higher number of cycles.

1.3 Organization

The rest of this thesis is organized as follows. In Chapter 2, we first provide some useful definitions and a description of the model; then we list some propositions on the characteristics of general directed graph in this model. In Chapter 3, we describe and prove the nature of the period for a cycle. In Chapter 4, we characterize the period for directed graphs which have two cycles. We first analyze the special case when there is an equal number of agents in both cycles. Then we analyze the case when the sizes of the two cycles are different, where we adopt an equivalent one-cycle model to aid the analysis. Chapter 5 concludes our work and provides some directions for future research.

CHAPTER 2

PROBLEM FORMULATION

2.1 Preliminaries

Following [16], we define a *digraph* to be an ordered pair $G = (V, E)$ where V is a finite set of nodes and E is a set of ordered pairs of distinct nodes. Elements of E are called *edges.* If $e = uv$ is an edge, we say e joins u to v; we also say that u is *adjacent to* v and that v is *adjacent from* u. The set of nodes adjacent from u is denoted by $f(u)$ and the set of nodes adjacent to u by $f^{-1}(u)$. We call $|f^{-1}(u)|$ the *indegree* $d_{in}(u)$ and $|f(u)|$ the *outdegree* $d_{out}(u)$. The *degree* of u is $d_{in}(u) + d_{out}(u)$.

A *semiwalk* is a sequence of nodes and edges $u_0e_0u_1e_1...e_{n-1}u_n$ such that for each e_i either $e_i = u_iu_{i+1}$ or $e_i = u_{i+1}u_i$; a semiwalk is *spanning* if it contains all the nodes of G, and *closed* if $u_0 = u_n$. If all the nodes (and hence all the edges) of a semiwalk are distinct, we have a *semipath.* A semiwalk for which $u_0 = u_n$ but all other nodes are distinct is a *semicycle.* A *walk* from u_0 to u_n (a $u_0 - u_n$ walk) is a semiwalk $u_0e_0...e_{n-1}u_n$ in which, for each i, $e_i = u_iu_{i+1}$; *path* and *cycle* are defined analogously. It is clear that any $u_0 - u_n$ walk contains a $u_0 - u_n$ path. If G has a symmetric pair (uv) and W is a walk containing either uv or vu we will say that W contains (uv).

2.2 The Model

Given a digraph $G = (V, E)$, each node v_i in V chooses its value a_i from the set $\{\mathbb{B}, \mathbb{W}\}$, where $\mathbb{B}$ and $\mathbb{W}$ are the only two choices a node can make, and each node must choose one of them as its value at each time step. At time step $t + 1$, each node v_i updates its value based on the choices of $f^{-1}(v_i)$ at time step t. The rule, which is applied to all nodes in G, is summarized as follows:

Value Updates Rule: If at time step t, more than half of the elements in $f^{-1}(v_i)$ have $\mathbb{B}$ as their values, v_i will choose $\mathbb{B}$ at $t+1$. In other cases, v_i will choose $\mathbb{W}$ at $t+1$.

We define the set $a(t)$ to be the collection of values of the nodes in V at time t. The union of all possible a is defined as A. A system is defined by the underlying digraph and the value updates rule. We further define a *periodic solution* to the system to be a sequence $a(t), a(t+1), ...$ where $\exists\ p \in \mathbb{Z}^+$ such that $a(t) = a(t+p)$. We say $a(t)$ is in a periodic solution if $\exists\ p$ such that $a(t) = a(t+p)$. By convention, we let p be the least positive integer for $a(t) = a(t+p)$ to hold, then p is the period of this periodic solution. We also say $a^{'}(t)$ and $a^{''}(t)$ are in the same periodic solution if $\exists\ k \geq 0$ such that $a^{''}(t) = a^{'}(t+k)$.

Proposition 1. *If at time t_1, $a(t_1)$ is in a periodic solution of period p, then for each $t > t_1$, $a(t)$ is in the same periodic solution of period p.*

Proof. For any $t > t_1$, suppose $t - t_1 = t_o$. Based on the definition of periodic solution, we know that $a(t_1) = a(t_1 + p)$. We now prove that $a(t_1 + t_o) = a(t_1 + t_o + p)$. We prove by induction on t_o.

Base: $t_o = 1$. Based on the Value Updates Rule, the only information needed for each v_i to make its choice (on having $\mathbb{B}$ or $\mathbb{W}$ as its value) on the next time step is the values of $f^{-1}(v_i)$, which is a subset of $a(t_1)$. Since $a(t_1) = a(t_1 + p)$, the values of $f^{-1}(v_i)$ at time t_1 and $t_1 + p$ are the same, which lead to the same choices of v_i at time $t_1 + 1$ and $t_1 + p + 1$. This is valid for every $v_i \in V$, so $a(t_1 + 1) = a(t_k + 1 + p)$.

Induction: Suppose that $a(t_1 + t_o) = a(t_1 + t_o + p)$ for $t_o = 1, 2, ..., m-1$; we need to show that $a(t_1 + t_o) = a(t_1 + t_o + p)$ holds for $t_o = m$. Following an argument similar to that for the base case, since $a(t_1 + m - 1) = a(t_1 + p + m - 1)$, the values of $f^{-1}(v_i)$ at time $t_1 + m - 1$ and $t_1 + p + m - 1$ will be the same, which lead to the same choices of v_i at time $t_1 + m$ and $t_1 + p + m$. This is valid for every $v_i \in V$, so $a(t_1 + m) = a(t_1 + m + p)$. □

Remark 1. *From the proof of Proposition 1, we know that for any $t_o \in \mathbb{Z}^+$*

$$a(t_1 + t_o) = a(t_1 + t_o + p) \tag{2.1}$$

Corollary 1. *By setting* $t_o = np$ *where* $n \in \mathbb{Z}^+$ *in Eq.(2.1), we have* $a(t_1 + np) = a(t_1 + (n+1)p)$. *Since this is true for all* $n \in \mathbb{Z}^+$, *we thus have*

$$a(t_1) = a(t_1 + np) \tag{2.2}$$

Proposition 2. *For any* $a(0)$, *the digraph will always be in a periodic solution after a finite number of time steps.*

Proof. Since V is a finite set and each $v_i \in V$ has only two values to choose from, a has $2^{|V|}$ possibilities, in other words, $|A| = 2^{|V|}$. So $a(2^{|V|})$ is guaranteed to repeat $a(t_o)$ for some t_o where $0 \leq t_o \leq 2^{|V|} - 1$. When $t = t_o$, there is a positive integer $p = 2^{|V|} - t_o$ such that $a(t) = a(t+p)$, which is the definition of periodic solutions. □

Proposition 3. *If* $a(t)$ *is in a periodic solution of period* p *at time* $t^{'}$, *then* $a(t^{'}), a(t^{'}+1), ..., a(t^{'}+p-1)$ *are pairwise distinct.*

Proof. We prove by contradiction. Since p is defined to be the smallest positive integer such that $a(t) = a(t+p)$, we know that $a(t) \neq a(t^{'})$ for $t^{'} < t < t^{'} + p$. Suppose that to the contrary there exists $a(t_1) = a(t_2)$ where $t^{'} < t_1 < t_2 < t^{'} + p$. Then starting from t_2, the sequence between $a(t_1)$ and $a(t_2)$ will be repeated. Since $a(t^{'}) \neq a(t)$ for $t_1 \leq t \leq t_2$, it is impossible to reach $a(t^{'} + p) = a(t^{'})$. This completes the proof. □

CHAPTER 3

ONE-CYCLE CASE

In this chapter, we consider a digraph which is a cycle. Suppose that there are n nodes, $v_1, v_2, ..., v_n$, in this digraph and each node has an indegree of 1 and an outdegree of 1. More specifically, $f(v_i) = v_{i+1}$ for $1 \leq i < n$ and $f(v_n) = v_1$. An example of one-cycle digraph is shown in Figure 3.1. In this one-cycle case, each node v_i at time $t+1$ has the same value as $f^{-1}(v_i)$ at time t, where $1 \leq i \leq n$. In other words, the values on the n nodes simply shift along the cycle as time progresses, and the number of nodes that have $\mathbb{W}$ as their values remains constant. We now investigate the period of such digraphs.

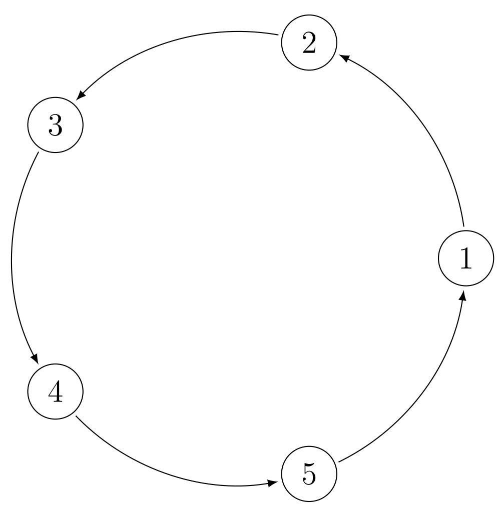

Figure 3.1: One-cycle digraph

Lemma 1. *For all $a(0)$, a digraph which is a cycle is in periodic solution from the beginning at $t = 0$.*

Proof. For any $v_i \in V$, the value of v_i at $t = 0$ will be passed back to v_i at time $t = n$ since the digraph has only one cycle with n nodes on this cycle. Since this is true for all nodes, we have $a(0) = a(n)$, which means the digraph is in periodic solution at $t = 0$. Following Proposition 1, the digraph is in periodic solution for all $t \geq 0$. □

Remark 2. *The proof of Lemma 1 also indicates that the period of such a digraph must be smaller than or equal to n.*

Lemma 2. *Let $a(t)$ be in a periodic solution of period p. Let N be an integer such that $a(t) = a(t + N)$. Then, $N = kp$ for $k \in \mathbb{Z}^+$.*

Proof. Since p is the period of this periodic solution, $a(t) = a(t+p)$. Because of Proposition 3, $a(t), a(t + 1), ..., a(t + p - 1)$ are pairwise distinct. By definition, $N \geq p$. Suppose that to the contrary p is not a divisor of N, and let r be its remainder; then $0 < r < p$ and $N = kp + r$ for $k \geq 0$. But $a(t + r) = a(t + r + p) = ... = a(t + kp + r) = a(t + N) = a(t)$, where $a(t + r) = a(t)$ leads to a contradiction. □

Lemma 3. *Suppose that p is a divisor of n. Then, there exists an initial condition $a(0)$ such that $a(0)$ (and thus $a(t)$ for $t \geq 0$) is in a periodic solution of period p.*

Proof. We prove by construction. More specifically, we assign the initial values of each node $a_i(0)$. The assignment method can be described as follows.

Initial Value Assignment for One-Cycle Case: We assign $a_{mp}(0) = \mathbb{W}$ where $m \in \mathbb{Z}^+$ and $1 \leq m \leq n/p$. All other nodes are assigned $\mathbb{B}$ as initial choice. Formally, for $1 \leq i \leq n$ and $m \in \mathbb{Z}^+$,

$$a_i(0) = \begin{cases} \mathbb{W} & \text{if } i = mp \\ \mathbb{B} & \text{if } i \neq mp \end{cases}$$

With these assignments, the nodes v_{mp} that have $\mathbb{W}$ at $t = 0$ will not choose $\mathbb{W}$ again until after p time steps when the value held by $f^{-p}(v_{mp})$ at $t = 0$ is passed to v_{mp}. This is true for every valid m. In other words, this is true for all nodes that had $\mathbb{W}$ initially. The nodes that had $\mathbb{B}$ at $t = 0$ will also have $\mathbb{B}$ at $t = p$ because, as stated before, the number of nodes that have $\mathbb{W}$ is a constant. So p is the smallest positive integer such that $a(t) = a(t + p)$, which means that p is the period of this digraph. □

Theorem 1. *In a digraph $G = (V, E)$ which is a single cycle and $|V| = n$, $p \in \mathbb{Z}^+$ can be a period of G if and only if p is a divisor of n.*

Proof. This result follows from Lemma 2 and Lemma 3, where Lemma 2 serves as the necessary condition and Lemma 3 serves as the sufficient condition. □

CHAPTER 4

TWO-CYCLE CASE

In this chapter, we consider a digraph $G = (V, E)$ consisting of two cycles while the two cycles share a walk. We use u_i, $1 \leq i \leq n_1$, to denote the nodes in one cycle and v_i, $1 \leq i \leq n_2$ to denote the nodes in the other cycle, where $i \in \mathbb{Z}^+$. $u_i, v_i \in V$. As in the one-cycle case, $f(u_i) = u_{i+1}$ for $1 \leq i \leq n_1$ and $f(u_{n_1}) = u_1$, $f(v_i) = v_{i+1}$ for $1 \leq i \leq n_2$ and $f(v_{n_2}) = v_1$. We let the two cycles share a walk which has w points, i.e. u_i and v_i refer to the same node for $1 \leq i \leq w$. In this digraph, node v_w (or equivalently, u_w) has an outdegree of 2 and indegree of 1; node v_1 (or equivalently, u_1) has an indegree of 2 and outdegree of 1. All other nodes have indegree of 1 and outdegree of 1. We also denote by $a_{u,i}$ the value held by player u_i and by $a_{v,i}$ the value held by player v_i. Following the *Value Updates Rule* described earlier, $a_{u,(w+1)}(t+1) = a_{v,(w+1)}(t+1) = a_w(t)$ and

$$a_1(t+1) = \begin{cases} \mathbb{W} & \text{if } (a_{u,n_1}(t) = \mathbb{W} \text{ or } a_{v,n_2}(t) = \mathbb{W}) \\ \mathbb{B} & \text{if } (a_{u,n_1}(t) = \mathbb{B} \text{ and } a_{v,n_2}(t) = \mathbb{B}) \end{cases}$$

Henceforth, we will only track $\mathbb{W}$'s since there are only two choices and $\mathbb{W}$ has priority over $\mathbb{B}$. We see that if v_w has $\mathbb{W}$ at some time, both u_{w+1} and v_{w+1} will have $\mathbb{W}$ at the next time step. If any of u_{n_1} or v_{n_2} has $\mathbb{W}$, then v_1 will have $\mathbb{W}$ at the next time step. We investigate the period of such digraphs in this chapter.

4.1 Two Cycles of Same Length

We first consider a special case when $n_1 = n_2$. An example of this type of digraph is given in Figure 4.1, where $n_1 = n_2 = 6$.

Lemma 4. *When the length's of the two cycles are equal, denoted by n_1, then, for all initial values $a(0)$, we have $a_{u,i}(t) = a_{v,i}(t)$ for any $1 \leq i \leq n_1$,*

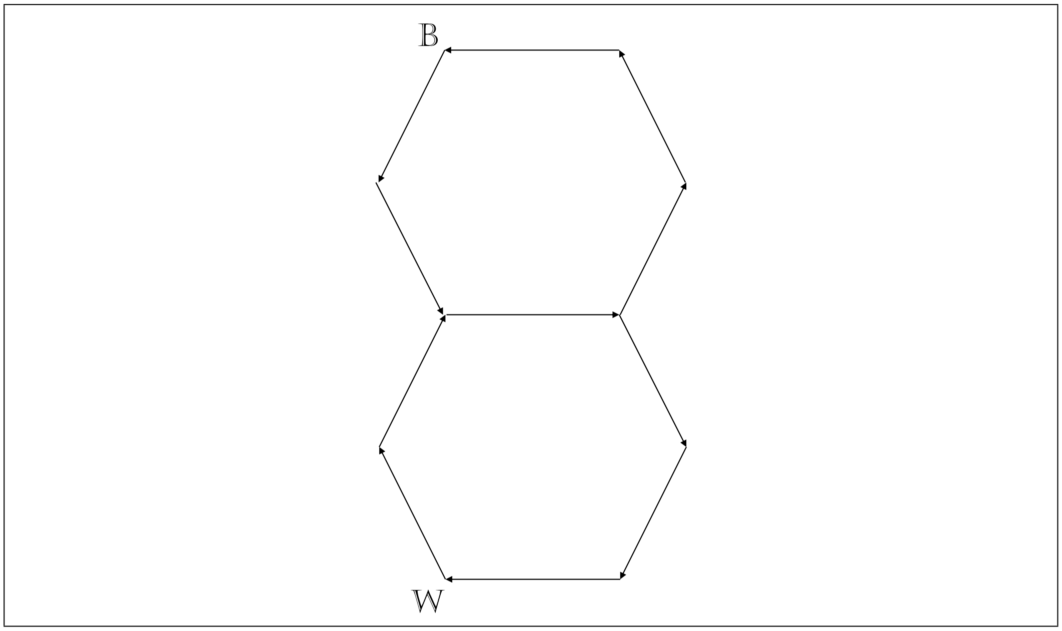

Figure 4.1: Two cycles of same length

$t > n_1$.

Proof. In the two-cycle model, $v_i = u_i$ for $1 \leq i \leq w$, so $a_{u,i}(t) = a_{v,i}(t)$ holds for $1 \leq i \leq w$ at all time steps. We suppose that to the contrary at time t_o, there exists a j such that $a_{u,j}(t_o) \neq a_{v,j}(t_o)$, where $w < j \leq n_1$ and $t_o > n_1$. For $w < j \leq n_1$, $d_{in}(u_j) = 1$ and $d_{in}(v_j) = 1$, so

$$a_{u,j}(t_o) = a_{u,j-1}(t_o - 1) = ... = a_{u,w+1}(t_o - (j - w - 1)) \tag{4.1}$$

and

$$a_{v,j}(t_o) = a_{v,j-1}(t_o - 1) = ... = a_{v,w+1}(t_o - (j - w - 1)) \tag{4.2}$$

If we extend the above equations to one earlier time step, we will have $a_{u,j}(t_o) = a_{u,w}(t_o - (j - w))$ and $a_{v,j}(t_o) = a_{v,w}(t_o - (j - w))$. Since $a_{u,j}(t_o) \neq a_{v,j}(t_o)$, it follows that $a_{u,w}(t_o - (j - w)) \neq a_{v,w}(t_o - (j - w))$. This leads to a contradiction. □

Theorem 2. *In a digraph $G = (V, E)$ which has two cycles which share a walk $u_1 - u_w$, if both cycles have n_1 nodes, then, $p \in \mathbb{Z}^+$ can be a period of G if and only if p is a divisor of n_1.*

Proof. Following Lemma 4, we know that when $t > n_1$, we must have $a_{u,n_1}(t) = a_{v,n_1}(t)$, which means that v_1, the only node that has $d_{in} = 2$, will hold the value at time $t+1$ the same as those held by both u_{n_1} and v_{n_1} at time t. When looking at one cycle, it behaves as in the one-cycle case and the existence of the other cycle has no effect on it. So both cycles can now be viewed as the one-cycle case discussed in Chapter 3. The result of Theorem 1 can be adapted here. Here we also have $a_u(t) = a_v(t)$ so the periods of both cycles must be the same p, and the period of the combined digraph remains to be p. □

4.2 Two Cycles of Different Length

Without loss of generality, we assume $n_1 > n_2$. An example of this type of digraphs is given in Figure 4.2, where $n_1 = 10$ and $n_2 = 6$.

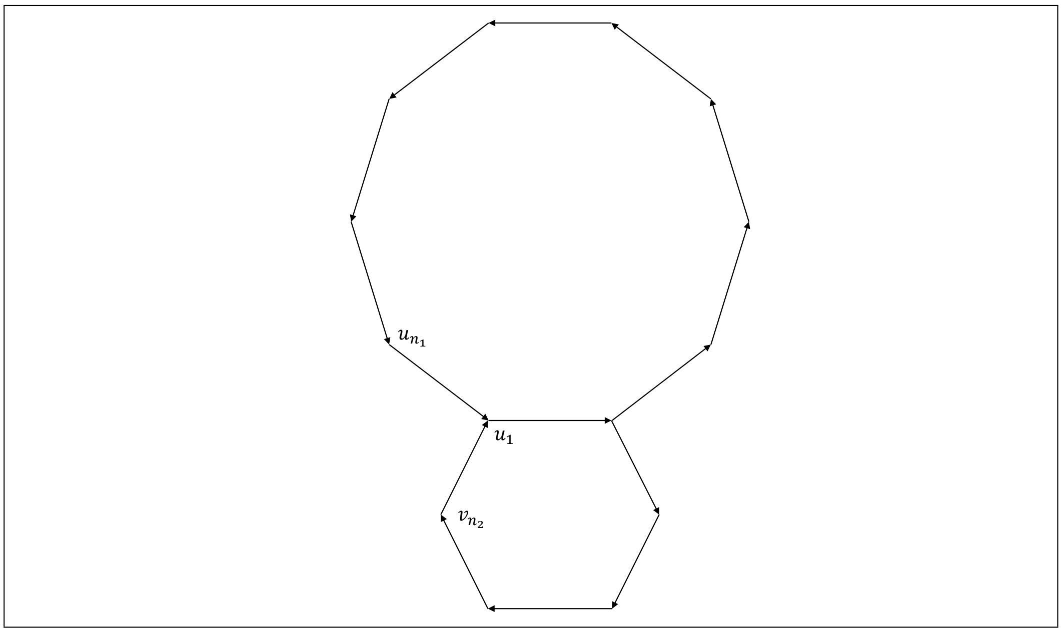

Figure 4.2: Two cycles of different length

4.2.1 Special Initial Values

We first look at the digraphs which have only one $\mathbb{W}$ at $t = 0$. An example of this type of digraph is given in Figure 4.3, where the values shown in the graph are held by the nodes at $t = 0$. We define any $\mathbb{W}$ to appear in this digraph to be a *descendant* of this original $\mathbb{W}$. We now investigate the periodicity of such digraphs.

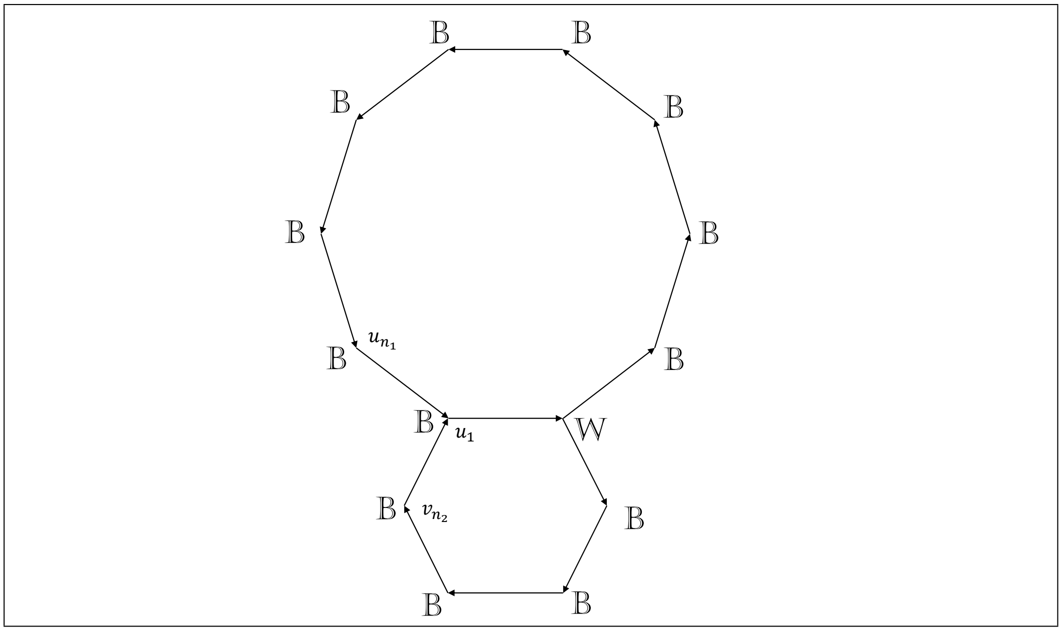

Figure 4.3: Special initial values

4.2.1.1 One-Cycle Equivalent Model

It will be shown later that two-cycle digraphs with only one $\mathbb{W}$ at the beginning are equivalent to the model to be introduced in this subsection. Here we consider an equivalent model described as follows.

In a digraph $G = (V, E)$ that is a cycle, $|V| = n$, at $t = 0$, we let a node v_i have $\mathbb{W}$, and all other nodes have $\mathbb{B}$. An example of this is given in Figure 4.4. As described in Chapter 1, this $\mathbb{W}$ will rotate around the cycle as time progresses. Suppose that at time $t = n_o$, $1 \leq n_o < n$, $n_o \in \mathbb{Z}^+$, when v_{i+n_o} is holding this $\mathbb{W}$, we force v_i to change its value from $\mathbb{B}$ to $\mathbb{W}$. Now there are two $\mathbb{W}$'s in this digraph. We continue to force the change (let v_i have $\mathbb{W}$) after every n_o time steps until no new $\mathbb{W}$'s are added to the graph. At some time $t = kn$, all nodes which have $\mathbb{W}$ can form a set $\{v_i, v_{(i+n_o) \bmod n}, v_{(i+2n_o) \bmod n}, ..., v_{(i+kn_o) \bmod n}\}$. For convenience, we use $[m]$ to refer m mod n.

Lemma 5. *For any two positive integers n_o, n, suppose $\alpha = n_o/gcd(n_o, n)$ and $\beta = n/gcd(n_o, n)$; then α and β are coprime numbers.*

Proof. We prove by contradiction. Suppose that to the contrary $gcd(\alpha, \beta) = c > 1$; then suppose $\gamma_1 = \alpha/c$ and $\gamma_2 = \beta/c$, so we have $p = \alpha \cdot gcd(p, n) = \gamma_1 c \cdot gcd(n_o, n)$ and $n = \beta \cdot gcd(n_o, n) = \gamma_2 c \cdot gcd(n_o, n)$. Now we have a common divisor of n_o and n, $c \cdot gcd(n_o, n)$, which is greater than $gcd(n_o, n)$. This leads to contradiction and completes the proof. □

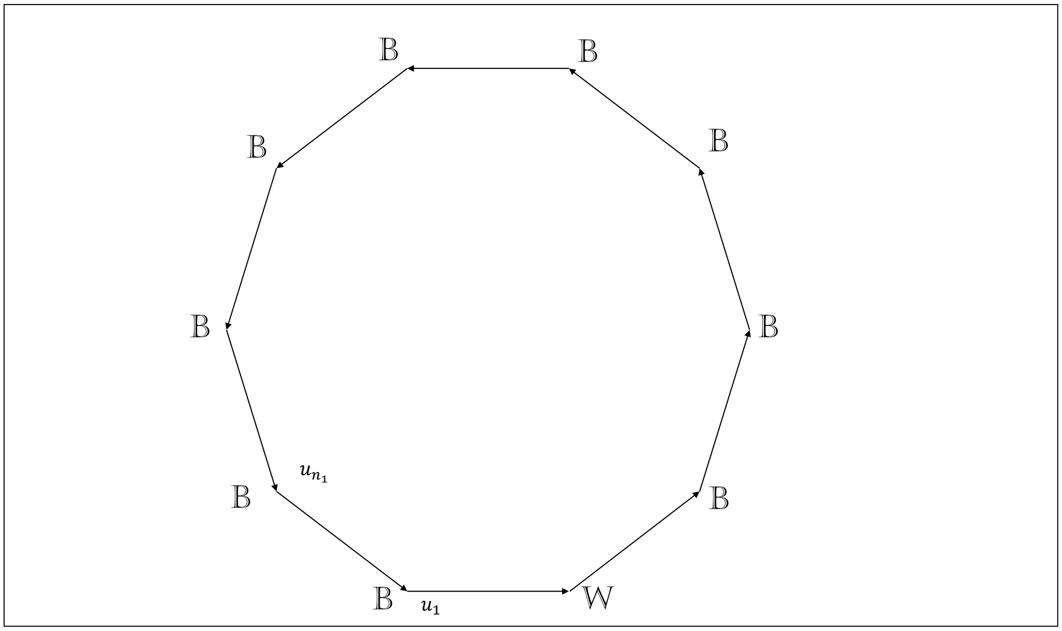

Figure 4.4: One-cycle equivalent

Lemma 6. *In a digraph $G = (V, E)$ that consists of a single cycle with $|V| = n$, let $1 \le i \le n$, $n_o < n$ and $n_o \in \mathbb{Z}^+$, $k \in \mathbb{Z}^+$. Then, letting a set of nodes $\{v_i, v_{[i+n_o]}, v_{[i+2n_o]}, \ldots, v_{[i+kn_o]}\}$ to have value $\mathbb{W}$ is equivalent to letting a set of nodes $\{v_i, v_{[i+gcd(n_o,n)]}, v_{[i+2\cdot gcd(n_o,n)]}, \ldots, v_{[i+k\cdot gcd(n_o,n)]}\}$ to have value $\mathbb{W}$.*

Proof. Since there are only n nodes in V, both sets are finite. For all i, $i+n_o = i+\frac{n_o}{gcd(n_o,n)}\cdot gcd(n_o, n)$, so every node in the first set has an equivalent representation in the second set. We now prove that for any i, there must be some k_o such that $v_{[i+k_o n_o]} = v_{[i+gcd(n_o,n)]}$, which means every node in the second set has an equivalent representation in the first set.

Suppose that as we reach $v_{[i+k_o n_o]}$, we have already gone through the cycle (passing v_i) r times, i.e. $\lfloor \frac{k_o n_o}{n} \rfloor = r$. Then we need to prove that $[i + k_o n_o] = [i + gcd(n_o, n)]$, which is equivalent to $[k_o n_o] = [gcd(n_o, n)]$. $[gcd(n_o, n)] < n$, so $[gcd(n_o, n)] = gcd(n_o, n)$. Our target becomes

$$k_o \cdot n_o - r \cdot n = gcd(n_o, n) \tag{4.3}$$

Suppose $\frac{n_o}{gcd(n_o,n)} = \alpha$ and $\frac{n}{gcd(n_o,n)} = \beta$, then Eq.(4.3) can be written as

$$k_o\alpha \cdot gcd(n_o, n) - r\beta \cdot gcd(n_o, n) = gcd(n_o, n) \tag{4.4}$$

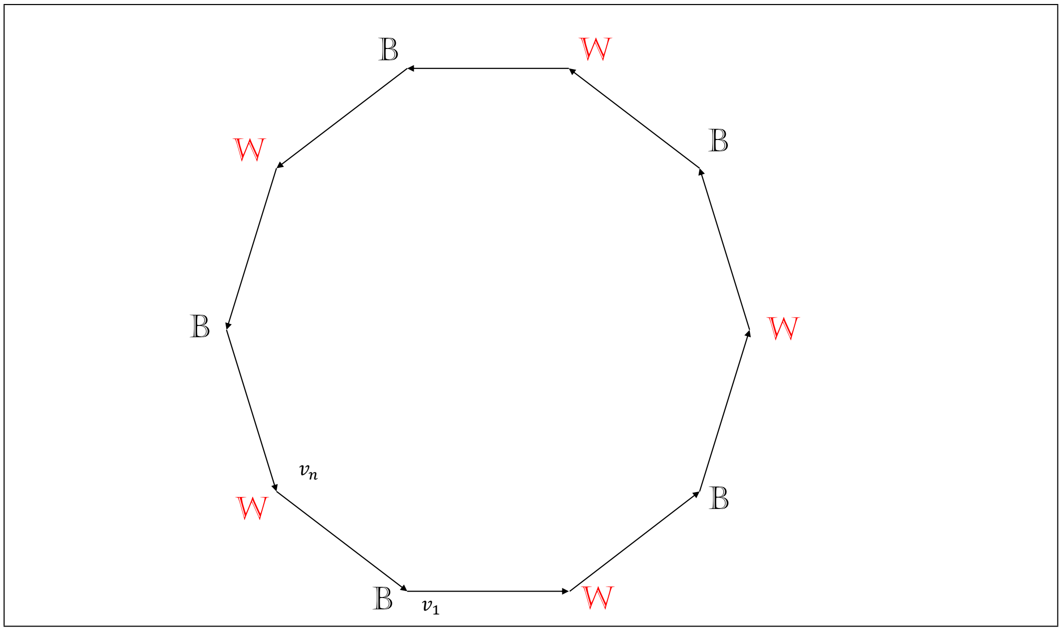

Figure 4.5: Graph for lemma 6

which is

$$k_o \cdot \alpha - r \cdot \beta = 1 \tag{4.5}$$

We know that α and β are coprime integers because of Lemma 5. Following Bézout's Lemma [17], integers k_o, r must exist such that Eq. (4.5) holds. □

An example of Lemma 6 is given in Figure 4.5. This figure shows the case when $n = 10$ and $n_o = 4$. It is clear that $\{v_i, v_{[i+n_o]}, v_{[i+2n_o]}, ..., v_{[i+kn_o]}\}$ and $\{v_i, v_{[i+gcd(n_o,n)]}, v_{[i+2\cdot gcd(n_o,n)]}, ..., v_{[i+k\cdot gcd(n_o,n)]}\}$ actually refer to the same set of nodes.

Lemma 7. *In a digraph $G = (V, E)$ that consists of a single cycle, let $1 \leq i \leq n$, $n_o < n$ and $n_o \in \mathbb{Z}^+$, $k \in \mathbb{Z}^+$. Then, there is no positive integer $d < gcd(p, n)$ such that $v_{[i+k_o n_o]} = v_{[i+d]}$.*

Proof. We prove by contradiction. Suppose that to the contrary d exists. Following the same argument as in Lemma 6, we have

$$k_o \cdot \alpha - r \cdot \beta = d_o \tag{4.6}$$

where $d_o = \frac{d}{gcd(n_o,n)} \notin \mathbb{Z}$. Since k_o, α, r, β are all integers, Eq. (4.6) cannot be true. □

4.2.1.2 In Two-Cycle Case

Back to the two-cycle case, in a digraph $G = (V, E)$ which has two cycles sharing a walk $u_1 - u_w$, one cycle has n_1 nodes (denoted by u_1, ...,u_{n_1}) and the other cycle has n_2 nodes (denoted by v_1, ..., v_{n_2}), where $n_1 > n_2$. We define the *distance* of two nodes u_i and u_j to be the time it takes for u_i to pass its value to u_j.

Lemma 8. *Let $a'(0)$ be an initial collection of the values held by the nodes in the graph where only one node in the graph has $\mathbb{W}$ initially (all others holding $\mathbb{B}$'s). Then, once the digraph enters periodic solution, $\mathbb{W}$'s will be evenly distributed over the graph, with a distance of $gcd(n_1, n_2)$ between two nearest $\mathbb{W}$'s.*

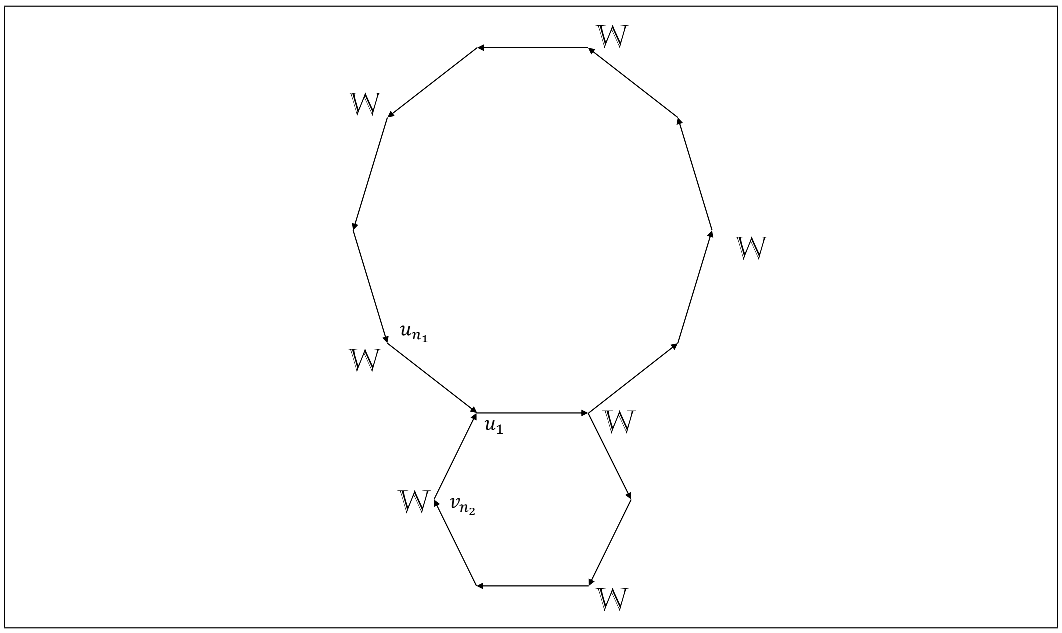

Figure 4.6: Evenly distribution in periodic solution

Proof. We first look at the initial values. Suppose node u_i (or v_i) is holding $\mathbb{W}$ at $t = 0$, and the distance between u_i (v_i) and v_w is t_o. Then this $\mathbb{W}$ will be passed to v_w after t_o time steps. So we let $a''(0)$ be the collection of values where $a_w(0) = \mathbb{W}$ and others having $\mathbb{B}$. Since $a''(0) = a'(t_o)$, they are in the same periodic solution.

Now we investigate the periodic solution reached from $a''(0)$. Following our *Value Updates Rule*, $a''_{u,w+t}(t) = a''_{v,w+t}(t) = \mathbb{W}$ for $t < n_2 - w$. So

$a''_1(n_2 - w) = a''_{u,n_2}(n_2 - w) = \mathbb{W}$. For cycle $u_1 - u_{n_1}$, this means that a new $\mathbb{W}$ is added at u_1 at time $t = n_2 - w$ when the original $\mathbb{W}$ is held by node u_{n_2}, and the distance between u_{n_2} and u_1 is $n_1 - n_2$. Following the same argument, this newly added $\mathbb{W}$ will also generate another $\mathbb{W}$ at time $t = 2n_2 - w$ and the distance between these two $\mathbb{W}$'s is also $n_1 - n_2$. Note that the original $\mathbb{W}$ is also adding a $\mathbb{W}$ after every n_2 time steps but after it first adds, the added $\mathbb{W}$ at later time steps will take place at the same $\mathbb{W}$ it added the first time, so it will have no impact at later time steps. This pattern is repeated and new $\mathbb{W}$'s are added to cycle $u_1 - u_{n_1}$ with a distance of $n_1 - n_2$ between the last added $\mathbb{W}$ and the newly added $\mathbb{W}$, until the newly added $\mathbb{W}$ is to be held by a node that will have $\mathbb{W}$ without this adding. From then on, we will say that no new $\mathbb{W}$ is to be added to the cycle and the cycle will perform as we discussed in Chapter 3.

From the discussion above, we find that when just looking at one cycle $u_1 - u_{n_1}$, the cycle performs the same way as we discussed in section 4.2.1.1. From Lemma 6 and Lemma 7, we know that after the digraph (and thus the cycle) enters a periodic solution, the distance between two nearest $\mathbb{W}$'s will be $gcd(n_1 - n_2, n_1) = gcd(n_1, n_2)$.

With similar argument, on cycle $v_1 - v_{n_2}$, instead of adding a $\mathbb{W}$ ahead of the last added $\mathbb{W}$, the newly added $\mathbb{W}$ at v_1 will be located behind the last added $\mathbb{W}$, with a distance of $n_1 - n_2$. From Lemma 6 and Lemma 7, we know that after the digraph (and thus the cycle) enters a periodic solution, the distance between two nearest $\mathbb{W}$'s will be $gcd(n_1 - n_2, n_1) = gcd(n_1, n_2)$.

With the two cycles combined, we know that when the digraph is in periodic solution, $\mathbb{W}$'s are to be evenly distributed over the graph, with a distance of $gcd(n_1, n_2)$ between two nearest $\mathbb{W}$'s. □

An example of Lemma 8 is given in Figure 4.6, where $n_1 = 10$ and $n_2 = 6$.

4.2.2 Generalized Initial Values

Theorem 3. *In a digraph $G = (V, E)$ that consists of two cycles with a shared walk, if there are n_1 nodes in one cycle and n_2 nodes in the other cycle, $p \in \mathbb{Z}^+$ can be a period of G if and only if p is a common divisor of n_1 and n_2.*

Proof. The proof is divided into two parts.

(i) Sufficient Condition

We prove the sufficient condition, i.e. if p is a common divisor of n_1 and n_2, p can be a period of G, by construction. More specifically, we assign the initial values of each node $a(0)$. The assignment method can be described as follows.

Value Assignment Method for Two Cycles: We assign $a_{u,m_1p}(0) = \mathbb{W}$ where $m_1 \in \mathbb{Z}^+$ and $1 \leq m_1 \leq n_1/p$. We also assign $a_{v,m_2p}(0) = \mathbb{W}$ where $m_2 \in \mathbb{Z}^+$ and $1 \leq m_2 \leq n_2/p$. All other nodes are assigned $\mathbb{B}$ as initial choice. Formally, for $1 \leq i \leq n$ and $m_1, m_2 \in \mathbb{Z}^+$,

$$a_{u,i}(0) = \begin{cases} \mathbb{W} & \text{if } i = m_1p \\ \mathbb{B} & \text{if } i \neq m_1p \end{cases} \quad a_{v,i}(0) = \begin{cases} \mathbb{W} & \text{if } i = m_2p \\ \mathbb{B} & \text{if } i \neq m_2p \end{cases}$$

With these assignments, the node u_{m_1p} which had $\mathbb{W}$ at $t = 0$ will not have $\mathbb{W}$ again until after p time steps when the action played by $f^{-p}(v_{m_1p})$ at $t = 0$ is passed to v_{mp}, and the node v_{m_2p} which had $\mathbb{W}$ at $t = 0$ will not have $\mathbb{W}$ again until after p time steps when the action played by $f^{-p}(v_{m_2p})$ at $t = 0$ is passed to v_{m_2p}. This is true for every valid m_1, m_2. In other words, this is true for all nodes that played $\mathbb{W}$ initially. The nodes that played $\mathbb{B}$ at $t = 0$ will also play $\mathbb{B}$ at $t = p$. So p is the smallest positive integer such that $a(t) = a(t + p)$, which means p is the period of this digraph.

(ii) Necessary Condition

Lemma 8 has shown that $p = gcd(n_1, n_2)$ if $a(0)$ has only one $\mathbb{W}$, and in periodic solution, a set of $\mathbb{W}$'s are evenly distributed over the digraph, with a distance of $gcd(n_1, n_2)$ in between. We look at the repeating sequence of values in periodic solution. There are a number of $(gcd(n_1, n_2) - 1)$ $\mathbb{B}$'s following each $\mathbb{W}$.

$$\mathbb{W}, \mathbb{B}, \mathbb{B}, ...; \mathbb{W}, \mathbb{B}, \mathbb{B}, ... \tag{4.7}$$

In the generalized case, if there is more than one $\mathbb{W}$ in the initial values, the rest of the $\mathbb{W}$'s will have a similar effect on the graph as the first one, with a time delay of the distance between them. An example of this is given in Figure 4.7. Increasing one more $\mathbb{W}$ in the initial values will lead to more $\mathbb{W}$'s being added in the periodic solution, but these newly added $\mathbb{W}$'s are also evenly distributed with a distance of $gcd(n_1, n_2)$ in between. So for

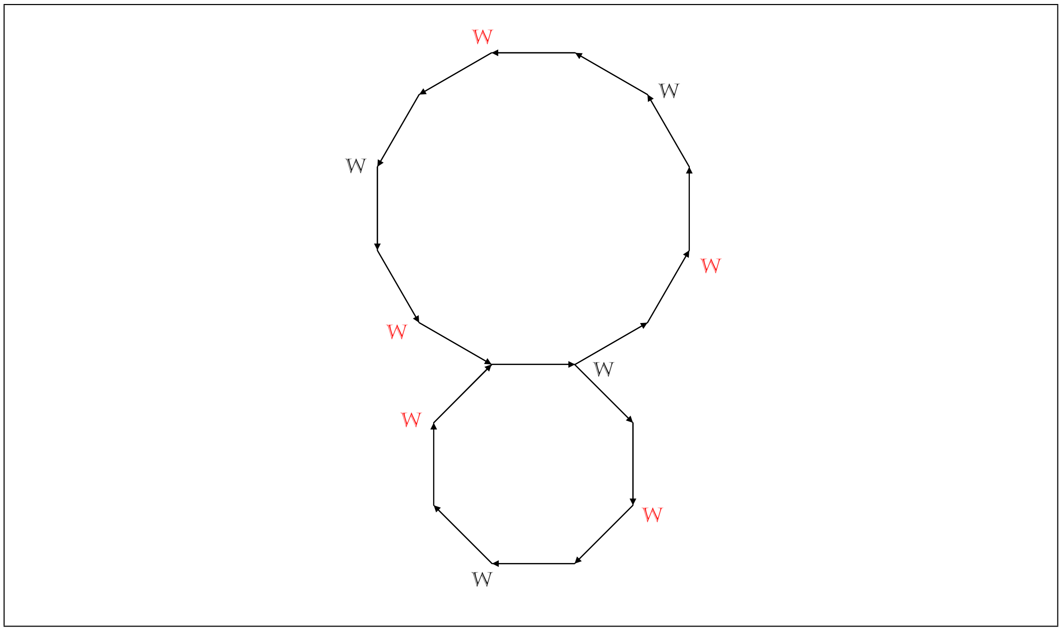

Figure 4.7: Superposition of two initial values

generalized initial values, in the periodic solution, some of the $\mathbb{B}$'s in the sequence (4.7) will be replaced by $\mathbb{W}$'s, and if one $\mathbb{B}$ is replaced with $\mathbb{W}$, the $\mathbb{B}$ sitting $gcd(n_1, n_2)$ away from the replaced node will also be replaced by $\mathbb{W}$. Thus the maximum possible period is $gcd(n_1, n_2)$ and we only need to look at one walk with values of

$$\mathbb{W}, \mathbb{B}, \mathbb{B}, ... \tag{4.8}$$

In order to form a new period which is smaller than $gcd(n_1, n_2)$, the newly added $\mathbb{W}$'s must divide the sequence (4.8) into a number of repeating sequences. In other words, the period of the digraph can only be $\frac{gcd(n_1,n_2)}{m}$ if there are $(m-1)$ $\mathbb{W}$'s added to sequence (4.8) so that it can be written as m shorter sequences $\mathbb{W}, \mathbb{B}, ...$, where $m \in \mathbb{Z}^+$ and $m \leq gcd(n_1, n_2)$.

From the above argument, we know that the period p of the two-cycle digraph must be a divisor of $gcd(n_1, n_2)$, which means it must be a common divisor of n_1 and n_2. □

CHAPTER 5

CONCLUSION

In this thesis, we have studied the periodic behavior of deterministic discrete-time majority threshold models on directed graphs. We have shown that under the *Value Updates Rule*, the period of the digraph which is a cycle must be a divisor of the length of the cycle, and each divisor can be a period of this graph. For digraphs which have two cycles sharing a set of nodes, the period must be a common divisor of the lengths of these two cycles, and each common divisor can be a period of this graph.

For future work, we will adopt the techniques that we have developed for two-cycle graphs to investigate the period for n-cycle digraphs. Another possible direction is to extend the choice set from two values to a higher number of values. Besides the digraphs consisting of cycles, we will also investigate other types such as star graphs and complete graphs.

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